$\mathbb{R}$-MOTIVIC STABLE STEMS

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ABSTRACT. We compute some $\mathbb{R}$-motivic stable homotopy groups. For $s-w \leq 11$, we describe the motivic stable homotopy groups $\pi_{s,w}$ of a completion of the $\mathbb{R}$-motivic sphere spectrum. We apply the ρ-Bockstein spectral sequence to obtain $\mathbb{R}$-motivic Ext groups from the $\mathbb{C}$-motivic Ext groups, which are well-understood in a large range. These Ext groups are the input to the $\mathbb{R}$-motivic Adams spectral sequence. We fully analyze the Adams differentials in a range, and we also analyze hidden extensions by ρ, 2, and η. As a consequence of our computations, we recover Mahowald invariants of many low-dimensional classical stable homotopy elements.

1. Introduction

The goal of this article is to compute the stable homotopy groups of the $\mathbb{R}$-motivic sphere spectrum in a range. These stable homotopy groups are the most fundamental invariants of the $\mathbb{R}$-motivic stable homotopy category, and thus lead to a deeper understanding of many of the computational aspects of $\mathbb{R}$-motivic homotopy theory. More specifically, we work in cellular $\mathbb{R}$-motivic stable homotopy theory, completed appropriately at 2 and η so that the $\mathbb{R}$-motivic Adams spectral sequence converges.

Our main tool is the $\mathbb{R}$-motivic Adams spectral sequence, which takes the form

$$E_2 = \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2) \implies \pi_{**}.$$

Here $\mathcal{A}$ is the $\mathbb{R}$-motivic Steenrod algebra, $\mathbb{M}_2$ is the $\mathbb{R}$-motivic cohomology of a point, and $\pi_{*,*}$ is the bigraded homotopy groups of the $\mathbb{R}$-motivic sphere (completed at 2 and η). We obtain complete results about $\pi_{s,w}$ for $s-w \leq 11$. This approach follows [11], which computed $\pi_{s,w}$ for $s-w \leq 3$.

See [7] for large-scale $\mathbb{R}$-motivic Adams charts. These charts are an essential companion to this manuscript. In a sense, this manuscript consists of a series of arguments for the computational facts displayed in the Adams charts.

1.1. The ρ-Bockstein spectral sequence.

The first step in an Adams spectral sequence program is to obtain the algebraic E_2-page. We study this computation in Sections 5, 6, and 7. We use the ρ-Bockstein spectral sequence, which takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

Here $\mathcal{A}^{\mathbb{C}}$ is the $\mathbb{C}$-motivic Steenrod algebra, and $\mathbb{M}_2^{\mathbb{C}}$ is the $\mathbb{C}$-motivic cohomology of a point.

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The ρ-Bockstein spectral sequence is a tool that passes from $\mathbb{C}$-motivic Ext groups to $\mathbb{R}$-motivic Ext groups. We discuss the general properties of this spectral sequence in Section 5, and we describe an unexpectedly effective strategy for computing differentials. The key idea is to compute the ρ-periodic groups $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)[\rho^{-1}]$ in advance. Then naive combinatorial considerations force a very large number of Bockstein differentials. We discuss specific Bockstein differential computations in Section 6.

Having obtained the E_∞-page of the ρ-Bockstein spectral sequence, we do not yet have a complete knowledge of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. It remains to resolve extensions that are hidden by the ρ-Bockstein filtration. There is an unmanageable quantity of hidden extensions, so we do not attempt to analyze them completely, not even in a range. Nevertheless, we do analyze all extensions by h_0 and h_1 in the range under consideration. These computations are carried out in Section 7.

1.2. **The $\mathbb{R}$-motivic Adams spectral sequence.** Having obtained the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence, the next step is to determine Adams differentials. We carry out these computations in Section 8. These differentials can be obtained by a variety of techniques. One important technique is the use of the Moss Convergence Theorem 8.2 to compute Toda brackets, which determine that certain elements are permanent cycles. Another technique is comparison to previously established computations in the $\mathbb{C}$-motivic and classical computations. See Section 1.3 for more discussion of these comparisons.

After computing Adams differentials and obtaining the Adams E_∞-page, there are once again hidden extensions to resolve. As in the algebraic case, there are too many extensions to study exhaustively, but we do consider all extensions by ρ, h, and η exhaustively (where ρ, h, and η are stable homotopy elements detected by ρ, h_0, and h_1 respectively). These computations are carried out in Section 9. Once again, the key techniques are shuffling relations involving Toda brackets and comparison to the $\mathbb{C}$-motivic and classical cases.

1.3. **Comparison of homotopy theories.** An essential ingredient in our computations is comparison between the $\mathbb{R}$-motivic, $\mathbb{C}$-motivic, C_2-equivariant, and classical stable homotopy theories, as depicted in the diagram

$$\begin{array}{ccc} \mathbb{R}\text{-motivic} & \xrightarrow{\text{realization}} & C_2\text{-equivariant} \\ {\scriptstyle \text{extension of scalars}}\big\downarrow & & \big\downarrow{\scriptstyle \text{forgetful}} \\ \mathbb{C}\text{-motivic} & \xrightarrow[\text{realization}]{} & \text{classical.} \end{array} \tag{1.1}$$

The horizontal arrows labelled "realization" refer to the Betti realization functors that take a variety over $\mathbb{C}$ (resp., over $\mathbb{R}$) to the space (resp., C_2-equivariant space) of $\mathbb{C}$-valued points. The vertical arrow labelled "extension of scalars" refers to the functor that takes a variety over $\mathbb{R}$ and views it as a variety over $\mathbb{C}$. The vertical arrow labelled "forgetful" refers to the functor that takes a C_2-equivariant object to its underlying non-equivariant object.

Our philosophy in this article is to accept computational information about the $\mathbb{C}$-motivic and classical stable homotopy groups as given, and to use this information to study the $\mathbb{R}$-motivic stable homotopy groups. See [18] for an extensive summary of computational information about the $\mathbb{C}$-motivic and classical Adams spectral sequences. The presence of the C_2-equivariant stable homotopy category in this diagram is relevant for our consideration of Mahowald invariants, to be discussed below in Section 1.4.

There is a surprising connection between ℂ-motivic and ℝ-motivic that enables many of our detailed computations. Namely, Theorem 3.4 shows that the ℂ-motivic stable homotopy groups are isomorphic to the ℝ-motivic homotopy groups of the cofiber S/ρ of ρ. This means that the structure of ℂ-motivic stable homotopy groups governs both the cokernel and the kernel of multiplication by ρ. This allows us to deduce many ℝ-motivic computational facts with relative ease from known ℂ-motivic information.

1.4. **Mahowald invariants.** Let α be a non-zero classical stable homotopy element. The Mahowald invariant (or root invariant) $R(\alpha)$ is a non-zero equivalence class of classical stable homotopy elements in a stem that is higher than the stem of α. One source of interest in Mahowald invariants is that $R(\alpha)$ appears to have greater chromatic complexity than α. Thus one can construct more exotic stable homotopy elements out of elements that are better understood [20].

Bruner and Greenlees reformulated the definition of the Mahowald invariant in terms of C_2-equivariant stable homotopy groups [9]. Although we do not study C_2-equivariant homotopy groups directly, we have indirectly obtained information about them because the ℝ-motivic and C_2-equivariant stable homotopy groups are isomorphic in a range [6]. In Section 4, we show how many Mahowald invariants can be immediately deduced from our ℝ-motivic computations. While these results only recover previously known Mahowald invariants [20] [4], we believe that our techniques can be extended into uncharted territory without much more effort.

Theorem 1.5. *Table 1 gives some values of the Mahowald invariant.*

Table 1: Some Mahowald invariants

stem	α	$R(\alpha)$	indeterminacy
0	2	η	
0	4	η^2	
0	8	η^3	
1	η	ν	2ν, 4ν
2	η^2	ν^2	
3	ν	σ	2σ, 4σ, 8σ
3	2ν	$\eta\sigma$	ϵ
3	4ν	$\eta^2\sigma$	$\eta\epsilon$
6	ν^2	σ^2	κ
7	σ	σ^2	
7	2σ	η_4	$\eta\rho_{15}$
7	4σ	$\eta\eta_4$	$\nu\kappa$, $\eta^2\rho_{15}$
8	$\eta\sigma$	ν_4	$2\nu_4$, $4\nu_4$
8	ϵ	$\overline{\sigma}$	
9	$\eta^2\sigma$	$\nu\nu_4$	$\eta\overline{\kappa}$

Proof. Theorem 4.10 reduces the computation to an ℝ-motivic Mahowald invariant, as defined in Section 4.3. Table 3 gives the values of the ℝ-motivic Mahowald invariant. Finally, Table 17 gives the Betti realizations of the $\mathbb{R}$-motivic Mahowald invariants. □

See Examples 4.9 and 4.11 for detailed illustrations of how this technique plays out in practice.

We have computed the Mahowald invariant of most, but not every, α through the 11-stem. In particular, we do not compute the Mahowald invariants of 2^k for $k \geq 4$, 8σ, $\eta\epsilon$, μ_9, $\eta\mu_9$, nor ζ_{11} and its multiples. In these cases, the problem is that the inequality of Theorem 4.10 does not apply, so our $\mathbb{R}$-motivic computations do not determine C_2-equivariant behavior.

2. Notation

We write $\mathbb{M}_2$ for the $\mathbb{R}$-motivic homology of a point with coefficients in $\mathbb{F}_2$. Recall that $\mathbb{M}_2$ is isomorphic to $\mathbb{F}_2[\rho, \tau]$, where ρ and τ have degrees $(-1, -1)$ and $(0, -1)$ respectively [26].

We write $\mathcal{A}$ for the $\mathbb{R}$-motivic dual Steenrod algebra. Recall that $\mathcal{A}$ is described by the equations

$$\mathcal{A} = \mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_k^2 = \tau\xi_{k+1} + \rho\tau_{k+1} + \rho\tau_0\xi_{k+1})$$

$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0, \quad \eta_L(\rho) = \eta_R(\rho) = \rho$$

$$\Delta(\tau_k) = \tau_k \otimes 1 + \sum \xi_{k-i}^{2^i} \otimes \tau_i$$

$$\Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i,$$

where τ_i and ξ_k have degrees $(2^{i+1} - 1, 2^i - 1)$ and $(2^{i+1} - 2, 2^i - 1)$ respectively [27].

We write $\mathbb{M}_2^{\mathbb{C}}$ for the $\mathbb{C}$-motivic homology of a point with coefficients in $\mathbb{F}_2$, and we write $\mathcal{A}_*^{\mathbb{C}}$ for the $\mathbb{C}$-motivic dual Steenrod algebra. These objects are easily described in terms of $\mathbb{M}_2$ and $\mathcal{A}$. Namely, they are the result of setting ρ equal to zero.

We write $\mathcal{A}_*^{\mathrm{cl}}$ for the classical dual Steenrod algebra, which can be obtained from $\mathcal{A}$ by setting ρ and τ to be 0 and 1 respectively.

We write Ext or $\mathrm{Ext}_{\mathbb{R}}$ for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$, i.e., the cohomology of the $\mathbb{R}$-motivic Steenrod algebra. We write $\mathrm{Ext}_{\mathbb{C}}$ and $\mathrm{Ext}_{\mathrm{cl}}$ for the cohomologies of the $\mathbb{C}$-motivic and classical Steenrod algebras respectively.

We write $\pi_{p,q}$ or $\pi_{p,q}^{\mathbb{R}}$ for the stable homotopy groups of the $\mathbb{R}$-motivic sphere spectrum. Similarly, we write $\pi_{p,q}^{\mathbb{C}}$ for the stable homotopy groups of the $\mathbb{C}$-motivic sphere spectrum. We adopt the usual motivic grading convention, so that $\pi_{p,q}X$ denotes maps out of $S^{p,q}$, where $S^{p,q}$ is the smash product of $p - q$ copies of the simplicial sphere and q copies of $\mathbb{A}^1 - 0$.

We write $\pi_{p,q}^{C_2}$ for the stable homotopy groups of the C_2-equivariant sphere spectrum. We use an equivariant grading convention that is compatible with the motivic grading convention, so that $\pi_{p,q}X$ denotes maps out of $S^{p,q}$, where $S^{p,q}$ is the one-point compactification of $\mathbb{R}^p$, with C_2 acting by negating the last q coordinates. Betti realization takes $\mathbb{R}$-motivic $S^{p,q}$ to C_2-equivariant $S^{p,q}$.

We write π_p for the classical stable homotopy groups.

All stable homotopy groups are suitably completed so that Adams spectral sequences converge. Classically, this means completion at 2. In the motivic cases, this means completion at 2 and η [17].

Grading conventions. Following [18] and [11], we use the following grading convention for the motivic Adams spectral sequence: s denotes the stem, f denotes the Adams filtration, and w denotes the motivic weight. Then the internal degree is $s+f$. In this grading, Adams differentials take the form

$$d_r : E_r^{s,f,w} \to E_r^{s-1,f+r,w}.$$

The *coweight* of an element in degree (s, f, w) is defined to be $s - w$. Note that ρ has coweight 0. In particular, an element x and its ρ-multiple ρx lie in the same coweight. This makes coweights particularly useful in the ρ-Bockstein perspective that we adopt.

2.1. **Stable homotopy elements.** We adopt conventional notation, as used (for example) in [18] [19], for the names of elements in the classical stable homotopy groups π_* and the $\mathbb{C}$-motivic stable homotopy groups $\pi^{\mathbb{C}}_{*,*}$.

Table 9 gives the notation that we use for elements of $\pi^{\mathbb{R}}_{*,*}$. We define these elements in terms of the elements of the Adams E_∞-page that detect them. These definitions have indeterminacy parametrized by elements of the Adams E_∞-page in higher Adams filtration. As a general rule, this indeterminacy does not matter to our computations. It is possible to use Toda brackets, or geometric constructions (see [10]), to eliminate the indeterminacy in many cases.

Remark 2.2. We use the symbol h to denote an element of $\pi_{0,0}$ that is detected by h_0. The symbol stands for "hyperbolic" because it corresponds to the hyperbolic plane in the Grothendieck-Witt group interpretation of $\pi_{0,0}$ [22, Remark 6.4.2]. (Alternatively, it can also stand for "Hopf", since h is the zeroth Hopf map.) Beware that h does not equal 2; in fact, $2 = \mathsf{h} + \rho\eta$.

Remark 2.3. The element σ requires more discussion. We write σ for an element of $\pi_{7,4}$ that is detected by h_3. There are 256 possible choices for σ, because of the presence of elements in higher Adams filtration. One such element in higher filtration is ρc_0. Lemma 7.19 shows that $\tau^2 h_2 \cdot \rho c_0$ equals $\rho^4 d_0$. Therefore, some possible choices of σ have the property that $\tau^2\nu\cdot\sigma$ is detected by $\rho^4 d_0$ in $\pi_{10,4}$, while other possible choices of σ have the property that $\tau^2\nu\cdot\sigma$ is zero. (The elements $\tau h_1 \cdot \tau P h_1$ and $\rho h_1 \cdot \tau h_1 \cdot \tau P h_1$ are not relevant, by comparison to kq as in Remark 8.15.)

We will need to use the relation $\tau^2\nu\cdot\sigma = 0$ in later computations, so we must assume that our choice of σ satisfies this condition.

Remark 2.4. In some cases, we have chosen names for elements of $\pi^{\mathbb{R}}_{*,*}$ that reflect the values of the extension of scalars functor given in Table 17. For example, we write $\tau\sigma^2$ for an element of $\pi^{\mathbb{R}}_{14,7}$ that is detected by ρh_4, since this element maps to $\tau\sigma^2$ in $\pi^{\mathbb{C}}_{14,7}$.

Remark 2.5. Beware that our use of the symbol $\overline{\kappa}$ is inconsistent with its usage in [18]. In this manuscript, $\tau\overline{\kappa}$ refers to a non-zero element of $\pi^{\mathbb{C}}_{20,11}$ that is detected by τg. The symbol $\overline{\kappa}$ is used in [18] for the same element.

Remark 2.6. Occasionally we refer to stable homotopy elements that have no standard name. In these cases, we use the symbol $\{x\}$ to indicate a stable homotopy element that is detected by an element x of an Adams E_∞-page.

3. Comparison between $\mathbb{R}$-motivic and $\mathbb{C}$-motivic homotopy

We first discuss the relationship between $\mathbb{R}$-motivic and $\mathbb{C}$-motivic stable homotopy theory. We will use these ideas frequently in later sections to obtain $\mathbb{R}$-motivic information from known $\mathbb{C}$-motivic information.

Consider the cofiber sequence

$$S^{-1,-1} \xrightarrow{\rho} S^{0,0} \longrightarrow S/\rho.$$

The cofiber S/ρ of ρ is a 2-cell complex whose structure governs multiplication by ρ in the $\mathbb{R}$-motivic stable homotopy groups, in a sense to be made precise in this section. In addition, we will draw an unexpected connection between the $\mathbb{R}$-motivic homotopy groups of S/ρ and $\mathbb{C}$-motivic stable homotopy groups.

As shown in diagram (1.1), there is an extension of scalars functor from $\mathbb{R}$-motivic stable homotopy theory to $\mathbb{C}$-motivic stable homotopy theory, and a Betti realization functor from $\mathbb{C}$-motivic stable homotopy theory to classical stable homotopy theory. These functors take Eilenberg-Mac Lane spectra to Eilenberg-Mac Lane spectra, and thus interact nicely with Adams spectral sequences. In particular, they induce highly structured morphisms of Adams spectral sequences. We will frequently use these comparison functors to deduce information about the $\mathbb{R}$-motivic Adams spectral sequence from already known information about the $\mathbb{C}$-motivic and classical Adams spectral sequences. See [18] for an extensive summary of computational information about the $\mathbb{C}$-motivic and classical Adams spectral sequences.

Extension of scalars takes the element ρ of $\pi_{-1,-1}$ to zero. In particular, it induces the map $\mathbb{M}_2 \to \mathbb{M}_2^{\mathbb{C}}$ that takes ρ to zero, and it similarly induces the map $\mathcal{A} \to \mathcal{A}_*^{\mathbb{C}}$ that takes ρ to zero.

For an $\mathbb{R}$-motivic spectrum, we write $\mathrm{Ext}_{\mathbb{R}}(X)$ for the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence that converges to $\pi_{*,*}(X)$, i.e., for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, H^{*,*}(X))$, and similarly for $\mathrm{Ext}_{\mathbb{C}}(X)$.

Extension of scalars induces a diagram

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{Ext}_{\mathbb{R}}(S^{-1,-1}) & \xrightarrow{\rho} & \mathrm{Ext}_{\mathbb{R}}(S^{0,0}) & \longrightarrow & \mathrm{Ext}_{\mathbb{R}}(S/\rho) & \longrightarrow & \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & \mathrm{Ext}_{\mathbb{C}}(S^{-1,-1}) & \xrightarrow{0} & \mathrm{Ext}_{\mathbb{C}}(S^{0,0}) & \longrightarrow & \mathrm{Ext}_{\mathbb{C}}(S^{0,0} \vee S^{-2,-1}) & \longrightarrow & .
\end{array}$$

Because ρ becomes zero after extension of scalars, the bottom row of the diagram splits. The map $\mathrm{Ext}_{\mathbb{R}}(S/\rho) \to \mathrm{Ext}_{\mathbb{C}}(S^{0,0} \vee S^{-2,-1})$ lifts to a map $\mathrm{Ext}_{\mathbb{R}}(S/\rho) \to \mathrm{Ext}_{\mathbb{C}}(S^{0,0})$ that makes the diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{\mathbb{R}}(S^{0,0}) & \longrightarrow & \mathrm{Ext}_{\mathbb{R}}(S/\rho) \\
\downarrow & \swarrow & \\
\mathrm{Ext}_{\mathbb{C}}(S^{0,0}) & &
\end{array}$$

commute.

Proposition 3.1. *The map* $\mathrm{Ext}_{\mathbb{R}}(S/\rho) \to \mathrm{Ext}_{\mathbb{C}}(S^{0,0})$ *is an isomorphism.*

Proof. Let $C^*_{\mathbb{R}}$ and $C^*_{\mathbb{C}}$ be the cobar complexes for $\mathrm{Ext}_{\mathbb{R}}(S^{0,0})$ and $\mathrm{Ext}_{\mathbb{C}}(S^{0,0})$ respectively. Note that $C^*_{\mathbb{C}}$ is isomorphic to $C^*_{\mathbb{R}}/\rho$. Because multiplication by ρ is injective on $C^*_{\mathbb{R}}$, this is also isomorphic to the cobar complex that computes $\mathrm{Ext}_{\mathbb{R}}(S/\rho)$. □

Remark 3.2. Because of the isomorphism of Proposition 3.1, the object $\mathrm{Ext}_{\mathbb{C}}$ is a module over $\mathrm{Ext}_{\mathbb{R}}$. By careful inspection of definitions, this module action is easy to describe. Using the ρ-Bockstein spectral sequence notation from Section 5, a typical element of $\mathrm{Ext}_{\mathbb{R}}$ is of the form $\rho^k x$, where x belongs to $\mathrm{Ext}_{\mathbb{C}}$. The $\mathrm{Ext}_{\mathbb{R}}$-module action on $\mathrm{Ext}_{\mathbb{C}}$ is described by

$$\rho^k x \cdot y = \begin{cases} 0 & \text{if } k > 0 \\ xy & \text{if } k = 0, \end{cases}$$

where the last expression xy is to be interpreted as the usual Yoneda product of elements in $\mathrm{Ext}_{\mathbb{C}}$.

Remark 3.3. Proposition 3.1 implies that there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_{\mathbb{R}} \xrightarrow{\rho} \mathrm{Ext}_{\mathbb{R}} \xrightarrow{i} \mathrm{Ext}_{\mathbb{C}} \xrightarrow{p} \mathrm{Ext}_{\mathbb{R}} \xrightarrow{\rho} \mathrm{Ext}_{\mathbb{R}} \longrightarrow \cdots$$

of $\mathrm{Ext}_{\mathbb{R}}$-module maps, where $\mathrm{Ext}_{\mathbb{C}}$ is an $\mathrm{Ext}_{\mathbb{R}}$-module as in Remark 3.2. If x is a permanent cycle in the ρ-Bockstein spectral sequence, then the map i takes x in $\mathrm{Ext}_{\mathbb{R}}$ to the element of $\mathrm{Ext}_{\mathbb{C}}$ of the same name.

Now consider the diagram

$$\begin{array}{ccccc} \pi^{\mathbb{R}}_{*+1,*+1} & \xrightarrow{\rho} & \pi^{\mathbb{R}}_{*,*} & \longrightarrow & \pi^{\mathbb{R}}_{*,*}(S/\rho) \\ & & \downarrow & \swarrow & \\ & & \pi^{\mathbb{C}}_{*,*}, & & \end{array} \tag{3.1}$$

in which the diagonal arrow exists because ρ maps to zero in $\pi^{\mathbb{C}}_{*,*}$.

Theorem 3.4. *The map* $\pi^{\mathbb{R}}_{*,*}(S/\rho) \to \pi^{\mathbb{C}}_{*,*}$ *is an isomorphism.*

Proof. Proposition 3.1 shows that there is an isomorphism of E_2-pages of Adams spectral sequences, so the targets of the spectral sequences are also isomorphic. □

Corollary 3.5. *Let* α *be an element of* $\pi^{\mathbb{R}}_{*,*}$. *Extension of scalars takes* α *to zero in* $\pi^{\mathbb{C}}_{*,*}$ *if and only if* α *is divisible by* ρ.

Proof. Chase the diagram (3.1), using that the diagonal map is an isomorphism. □

Remark 3.6. Corollary 3.5 has a C_2-equivariant analogue, as stated later in Proposition 4.2.

Remark 3.7. The isomorphism of Theorem 3.4 can be strengthened to an equivalence of categories [5, Corollary 8.6]. Namely, the 2-complete ℂ-motivic cellular stable homotopy category is equivalent to the homotopy category of S/ρ-modules in the 2-complete ℝ-motivic cellular stable homotopy category.

Corollary 3.8. *There is a long exact sequence*

$$\cdots \longrightarrow \pi^{\mathbb{R}}_{s+1,w+1}(S) \xrightarrow{\rho} \pi^{\mathbb{R}}_{s,w}(S) \longrightarrow \pi^{\mathbb{C}}_{s,w}(S) \longrightarrow \pi^{\mathbb{R}}_{s,w+1}(S) \longrightarrow \cdots .$$

Proof. This is the long exact sequence in homotopy for the fiber sequence

$$S \xrightarrow{\rho} S \longrightarrow S/\rho$$

in $\mathbb{R}$-motivic spectra, after applying the identification in Theorem 3.4. □

4. Mahowald invariants

The goal of this section is to use $\mathbb{R}$-motivic computations to recompute some Mahowald invariants. See [4, Section 4] for a careful discussion of the definition, using Lin's theorem that $\mathbb{R}P^{\infty}_{-\infty}$ is equivalent to S^{-1}.

4.1. **C_2-equivariant homotopy theory and Mahowald invariants.** Using C_2-equivariant homotopy theory, Bruner and Greenlees [9] gave an alternative definition of the Mahowald invariant. We will summarize this definition, but first we need some background on C_2-equivariant homotopy theory.

Let $S^{a,b}$ be the one-point compactification of $\mathbb{R}^a$, where C_2 acts by negating the last b coordinates. Then $\rho : S^{0,0} \to S^{1,1}$ is the inclusion of fixed points. Note that the cofiber of this map is $\Sigma(C_2)_+$, i.e., the suspension of the based free C_2-space.

We use the same notation ρ for the map $S^{-1,-1} \to S^{0,0}$ in the C_2-equivariant stable homotopy group $\pi^{C_2}_{-1,-1}$. The identification of the cofiber of ρ leads immediately to the following proposition, whose short proof appears in [12, Proposition 11.2].

Proposition 4.2. *Let α be a C_2-equivariant stable homotopy element. The underlying classical stable homotopy element $U(\alpha)$ of α is zero if and only if α is divisible by ρ.*

Geometric fixed points gives a map $\pi^{C_2}_{a,b} \to \pi_{a-b}$, and this map takes ρ to 1. The ρ-periodic groups $\pi^{C_2}_{*,*}[\rho^{-1}]$ are isomorphic to $\pi_* \otimes \mathbb{Z}[\rho^{\pm 1}]$, i.e., to the classical stable homotopy groups with ρ and ρ^{-1} adjoined [8, Proposition] [2, Proposition 7.0].

With this background on C_2-equivariant stable homotopy groups, we now give the Bruner-Greenlees definition of the Mahowald invariant. Start with a classical stable homotopy element α in π_n, which we identify with the obvious element of $\pi_* \otimes \mathbb{Z}[\rho^{\pm 1}]$ in degree $(0, -n)$. Using the isomorphism

$$\pi_* \otimes \mathbb{Z}[\rho^{\pm 1}] \cong \pi^{C_2}_{*,*}[\rho^{-1}],$$

write $\alpha = \rho^k \beta$ for some β in $\pi^{C_2}_{*,*}$ and some integer k, with k maximal. Finally, the Mahowald invariant $R(\alpha)$ is the underlying classical stable homotopy element $U(\beta)$ of β.

Note that the Mahowald invariant is not strictly defined; it is a set of classical stable homotopy elements. While the choice of k is unique, the choice of β is not. Different choices of β can lead to different values of $U(\beta)$.

Also note that $U(\beta)$ is necessarily non-zero by Proposition 4.2. The point is that β is not divisible by ρ, since k was chosen to be maximal.

4.3. **$\mathbb{R}$-motivic homotopy theory and Mahowald invariants.** We will now adapt the framework of Bruner and Greenlees [9] from the C_2-equivariant to the $\mathbb{R}$-motivic settings. In order to carry this out, we need to observe some key $\mathbb{R}$-motivic properties.

First, the ρ-periodic groups $\pi^{\mathbb{R}}_{*,*}[\rho^{-1}]$ are isomorphic to $\pi_* \otimes \mathbb{Z}[\rho^{\pm 1}]$, i.e., to the classical stable homotopy groups with ρ and ρ^{-1} adjoined [11]. See also [3] for a more structured version of this isomorphism. Second, Corollary 3.5 relates ρ-divisibility to the kernel of the extension of scalars map.

Definition 4.4. Let α be a classical stable homotopy element in π_n. The $\mathbb{R}$-motivic Mahowald invariant $R^{\mathbb{R}}(\alpha)$ is defined as follows. Identify α with the obvious element of

$$\pi_* \otimes \mathbb{Z}[\rho^{\pm 1}] \cong \pi^{\mathbb{R}}_{*,*}[\rho^{-1}]$$

in degree $(0, -n)$. Write $\alpha = \rho^k \beta$ for some β in $\pi^{\mathbb{R}}_{*,*}$ and some integer k, with k maximal. Define $R^{\mathbb{R}}(\alpha)$ in $\pi^{\mathbb{C}}_{*,*}$ to be the extension of scalars of β.

Remark 4.5. As for the traditional Mahowald invariant, the $\mathbb{R}$-motivic Mahowald invariant is not strictly defined. Different choices of β can have different values in $\pi^{\mathbb{C}}_{*,*}$ under extension of scalars.

Remark 4.6. As for the traditional Mahowald invariant, the $\mathbb{R}$-motivic Mahowald invariant is always non-zero by Corollary 3.5. The point is that β is not divisible by ρ, since k was chosen to be maximal.

Remark 4.7. See [24] [25] for a different consideration of Mahowald invariants in the motivic context. Our construction does not compare directly.

Theorem 4.8. *Some values of the $\mathbb{R}$-motivic Mahowald invariant are given in Table 3.*

Proof. This follows immediately from the computations carried out later in the article. In particular, one needs the values of the extension of scalars map, as shown in Table 17 and discussed in Section 10 □

Example 4.9. We illustrate Theorem 4.8 by describing the computation of $M^{\mathbb{R}}(\sigma)$. The element σ in π_7 is identified with the element α of $\pi^{\mathbb{R}}_{*,*} \otimes \mathbb{Z}[\rho^{\pm 1}]$ in degree $(0, -7)$ that is detected by $\rho^{15} h_4$. Then α equals $\rho^{14}\beta$, where β is detected by ρh_4. Finally, Table 17 shows that the realization of β is $\tau\sigma^2$ in $\pi^{\mathbb{C}}_{14,7}$.

In general, the relationship between $R(\alpha)$ and $R^{\mathbb{R}}(\alpha)$ is not obvious. The choices involved in the definitions are not necessarily compatible. For example, it is possible that an element β in $\pi^{\mathbb{R}}_{*,*}$ is not divisible by ρ, while its realization in $\pi^{C_2}_{*,*}$ is divisible by ρ.

The main result of [6] tells us that the $\mathbb{R}$-motivic and C_2-equivariant stable homotopy groups agree in a range. In this range, $R(\alpha)$ and $R^{\mathbb{R}}(\alpha)$ are easier to compare.

Theorem 4.10. *Let $R^{\mathbb{R}}(\alpha)$ belong to $\pi^{\mathbb{C}}_{s,w}$, and Suppose that $2w - s < 4$. Then $R(\alpha)$ equals the Betti realization of $R^{\mathbb{R}}(\alpha)$.*

Proof. The isomorphism between $\mathbb{R}$-motivic and C_2-equivariant stable homotopy groups [6] implies that the choice of β in the definition of $R^{\mathbb{R}}(\alpha)$ realizes to the choice of β in the definition of $R(\alpha)$. By the commutativity of the diagram (1.1), the realization of $R^{\mathbb{R}}(\alpha)$ equals $R(\alpha)$. □

Example 4.11. We showed in Example 4.9 that $R^{\mathbb{R}}(\sigma)$ equals $\tau\sigma^2$ in $\pi^{\mathbb{C}}_{14,7}$. The numerical condition of Theorem 4.10 is satisfied. It follows that $R(\sigma)$ equals σ^2 in π_{14}, since σ^2 is the realization of $\tau\sigma^2$.

Remark 4.12. Theorem 4.10, together with our computations of $\mathbb{R}$-motivic stable homotopy groups, can be used to compute the Mahowald invariants $R(\alpha)$ for most α up to the 11-stem. The exceptions are 2^k for $k \geq 4$, 8σ, $\eta\epsilon$, μ_9, $\eta\mu_9$, and ζ_{11} and its multiples. In these cases, $R^{\mathbb{R}}(\alpha)$ can still be computed as shown in Table 3. However, the numerical condition of Theorem 4.10 does not hold, so we cannot draw a conclusion about $R(\alpha)$ in these cases.

5. The ρ-Bockstein spectral sequence

We briefly recall some background on the ρ-Bockstein spectral sequence that computes the cohomology of the $\mathbb{R}$-motivic Steenrod algebra. See [16] and [11] for additional details.

Begin with the observation that the $\mathbb{C}$-motivic cohomology of a point $\mathbb{M}_2^{\mathbb{C}}$ equals $\mathbb{M}_2/\rho$, and the $\mathbb{C}$-motivic dual Steenrod algebra $\mathcal{A}_*^{\mathbb{C}}$ equals $\mathcal{A}/\rho$. Then filter the cobar complex by powers of ρ to obtain the ρ-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}^{**}_{\mathcal{A}_*^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}^{**}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2). \tag{5.1}$$

Our goal is to analyze the ρ-Bockstein spectral sequence (5.1) in computational detail in a range of degrees. We recall some structural results about this spectral sequence from [11].

Proposition 5.1. [11, Lemma 3.4] *If $d_r(x)$ is nontrivial in the ρ-Bockstein spectral sequence, then x and $d_r(x)$ are both ρ-torsion free on the E_r-page.*

Recall that $\mathcal{A}_*^{\mathrm{cl}}$ is the classical dual Steenrod algebra.

Proposition 5.2. [11, Theorem 4.1] *There is an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}_*^{cl}}(\mathbb{F}_2, \mathbb{F}_2)[\rho^{\pm 1}] \cong \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)[\rho^{-1}]$$

that takes elements of degree (s, f) in $\mathrm{Ext}_{\mathcal{A}_*^{cl}}(\mathbb{F}_2, \mathbb{F}_2)$ to elements of degree $(2s + f, f, s+f)$ in $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. In particular, the classical element h_n corresponds to the $\mathbb{R}$-motivic element h_{n+1}. Moreover, the isomorphism is highly structured, i.e., preserves products and Massey products.

The point of Proposition 5.2 is that we a priori know the elements of $\mathrm{Ext}_{\mathbb{R}}$ that are ρ-periodic, in the sense that they support infinitely many non-zero multiplications by ρ. In the range considered in this manuscript, these ρ-periodic elements are h_1, h_2, h_3, h_4, c_1, $h_2 g$, $h_3 g$, as well as products of these elements. This corresponds to the fact that through the 11-stem, $\mathrm{Ext}_{\mathrm{cl}}$ is generated by the classical elements h_0, h_1, h_2, h_3, c_0, Ph_1, and Ph_2. We may effectively ignore these ρ-periodic elements when analyzing the ρ-Bockstein spectral sequence, since they can be neither source nor target of any ρ-Bockstein differential.

Let $\{x_i\}$ be an $\mathbb{F}_2$-linear basis for $\mathrm{Ext}_{\mathbb{C}}$, i.e., an $\mathbb{F}_2[\rho]$-linear basis for the ρ-Bockstein E_1-page, excluding the ρ-periodic permanent cycles described in the previous paragraph. For every i, either x_i supports a differential, or $\rho^r x_i$ is the target of the d_r differential for some r. In other words, the set $\{x_i\}$ may be partitioned into pairs (x_i, x_j) such that $d_r(x_i) = \rho^r x_j$ for some j. Actually, one must be somewhat careful about the choice of basis in situations where two or more elements of the basis have the same degree. Nevertheless, it is always possible to change basis so that the basis elements can be partitioned into pairs.

The Bockstein differential $d_r : E_r^{s,f,w} \to E_r^{s-1,f+1,w}$ preserves the quantity $s + f - w$, and ρ lies in a degree satisfying $s + f - w = 0$. Thus we may consider one value of $s + f - w$ at a time when analyzing the ρ-Bockstein spectral sequence.

We exploit this structure in the following strategy for analyzing the ρ-Bockstein spectral sequence.

Strategy 5.3.

(1) Fix a value $N = s + f - w$.
(2) Find an $\mathbb{F}_2[\rho]$-basis B_N for the part of the ρ-Bockstein E_1-page in degrees (s, f, w) satisfying $N = s + f - w$.
(3) Remove elements from B_N that detect ρ-periodic elements of $\mathrm{Ext}_{\mathbb{R}}$.
(4) Use a variety of techniques, to be described below, to identify some differential $d_r(x_i) = \rho^r x_j$, where x_i and x_j belong to B_N.
(5) Remove x_i and x_j from B_N.
(6) Repeat steps (4) and (5) until B_N is empty.

For this strategy to be effective, we need to know that the basis B_N chosen in step 2 is finite. Lemma 5.4 establishes this fact.

Lemma 5.4. *Let N be fixed. In degrees (s, f, w) satisfying $N = s + f - w$, the ρ-Bockstein E_1-page is a finitely generated $\mathbb{F}_2[\rho]$-module.*

Proof. Recall that $\mathrm{Ext}_{\mathbb{C}}$ is non-zero only in degrees (s, f, w) satisfying $s+f-2w \geq 0$ [18, Remark 2.20]. This inequality can be rewritten in the form

$$s + f - w \geq \frac{1}{2}(s + f).$$

In other words, we only need consider the part of $\mathrm{Ext}_{\mathbb{C}}$ in total degree at most $2N$. □

One consequence of our strategy is that we do not compute the Bockstein differentials d_r in order of increasing r. Rather, we obtain all differentials as part of the same process.

Step (4) is the limiting factor in the practical effectiveness of our algorithm. The ad hoc arguments required to establish specific differentials become more difficult as the value of N increases. However, these difficulties increase at a surprisingly slow rate, and we are able to carry out the computation remarkably far without much difficulty.

Our goal is to compute the ρ-Bockstein spectral sequence through coweight 13. Unfortunately, infinitely many values of N in Step 1 are relevant in this range. For example, consider the elements h_1^k of coweight 0, which belong to degrees satisfying $s + f - w = k$.

Similarly, any h_1-periodic sequence of elements $h_1^k x$ of $\mathrm{Ext}_{\mathbb{C}}$ lies in degrees for which $s + f - w$ is unbounded. Fortunately, it is only these h_1-periodic families that are problematic.

Lemma 5.5. *Let x be a non-zero element of $\mathrm{Ext}_{\mathbb{C}}$ of degree (s, f, w) whose coweight is at most k. Then:*

(1) x is an h_1-periodic element, in the sense that $h_1^i x$ is non-zero for all $i \geq 0$; or
(2) $s + f - w \leq 3k + 3$.

Proof. If $2f - s \geq 4$, then x is h_1-periodic [14]. So we may assume that $2f - s < 4$.

By [18, Remark 2.20], we also have the inequality $s + f - 2w \geq 0$. Combining with the assumption $s - w \leq k$, we conclude that

$$s + f - w = (2f - s) - (s + f - 2w) + 3(s - w) < 4 + 0 + 3k = 3k + 4.$$

□

As we wish to consider elements up to coweight 13, Lemma 5.5 suggests we need to look at degrees satisfying the inequality $s + f - w \leq 42$, in addition to studying h_1-periodic elements. However, inspection of elements in $\mathrm{Ext}_{\mathbb{C}}$ shows that $s + f - w \leq 28$ for all elements that are relevant in our range.

The h_1-periodic elements of $\mathrm{Ext}_{\mathbb{C}}$ are well-understood [13]. Up to coweight 13, all such elements are of the form 1, $P^k h_1$, $P^k c_0$, $P^k d_0$, $P^k e_0$, $P^k c_0 d_0$, d_0^2, or $c_0 e_0$, as well as the h_1-multiples of these elements. Lemma 5.5 indicates that the behavior of the ρ-Bockstein spectral sequence on these elements must be studied separately. See Proposition 6.2 for the analysis of these h_1-periodic elements.

6. ρ-Bockstein differentials

The goal of this section is to describe a variety of methods for determining ρ-Bockstein differentials. These methods are applied in Step (4) of Strategy 5.3. Taken together, these methods allow us to determine all ρ-Bockstein differentials through coweight 13.

We begin with a result that describes all ρ-Bockstein differentials on the elements of Adams filtration zero.

Proposition 6.1. [11, Proposition 3.2]

(1) $d_1(\tau) = \rho h_0$.
(2) $d_{2^k}(\tau^{2^k}) = \rho^{2^k} \tau^{2^{k-1}} h_k$ *for* $k \geq 1$.

Next we consider h_1-periodic elements. These elements must be treated as special cases because of Case (1) of Lemma 5.5.

Proposition 6.2. *Table 4 gives some Bockstein differentials that are non-zero after inverting h_1. Through coweight 13, these are the only h_1-periodic ρ-Bockstein differentials.*

For legibility, we have not included powers of ρ in the values of the Bockstein differentials in Table 4. For example, the first row of the table is to be interpreted as $d_3(Ph_1) = \rho^3 h_1^3 c_0$.

Proof. The differentials in the h_1-periodic ρ-Bockstein spectral sequence are completely known [15]. For each h_1-periodic element x, this determines $d_r(h_1^k x)$ for large values of k. However, it is possible that the elements $h_1^k x$ support shorter differentials for small values of k. By inspection, no such shorter differentials occur. □

Remark 6.3. The phenomenon considered at the end of the proof of Proposition 6.2 turns out not to occur through coweight 13. However, it does occur in higher coweights.

The following examples are representative arguments for establishing ρ-Bockstein differentials. In many situations, more than one argument leads to the same result.

Example 6.4. Table 2 summarizes the analysis of Bockstein differentials in degrees (s, f, w) satisfying $s + f - w = 6$. In these degrees, the E_1-page consists of ρ multiples of twenty elements. The first part of Table 2 lists the two elements that are ρ-periodic, as in Proposition 5.2. They correspond to the classical elements h_0^6 and $h_0^2 h_2$.

The second section of Table 2 lists some differentials that are easily deduced from Proposition 6.1 and the Leibniz rule.

At this point, only the elements $\tau^4 h_1^2$ and c_0 remain unaccounted. The third section of Table 2 gives the only possibility.

Table 2: Bockstein differentials for $s + f - w = 6$

coweight	(s, f, w)	x	d_r	$d_r(x)$
0	$(6, 6, 6)$	h_1^6		
3	$(9, 3, 6)$	$h_1^2 h_3$		
6	$(0, 0, -6)$	τ^6	d_2	$\tau^5 h_1$
5	$(0, 1, -5)$	$\tau^5 h_0$	d_1	$\tau^4 h_0^2$
3	$(0, 1, -3)$	$\tau^3 h_0^3$	d_1	$\tau^2 h_0^4$
1	$(0, 1, -1)$	τh_0^5	d_1	h_0^6
4	$(3, 2, -1)$	$\tau^3 h_0 h_2$	d_1	$\tau^3 h_1^3$
5	$(7, 1, 2)$	$\tau^2 h_3$	d_2	$\tau h_1 h_3$
4	$(7, 2, 3)$	$\tau h_0 h_3$	d_1	$h_0^2 h_3$
5	$(3, 1, -2)$	$\tau^4 h_2$	d_4	$\tau^2 h_2^2$
4	$(2, 2, -2)$	$\tau^4 h_1^2$	d_7	c_0

Example 6.5. In some situations, a more careful analysis of multiplicative structure establishes a differential. For example, $d_1(f_0)$ cannot equal $\rho h_1 e_0$ because $h_1 f_0 = 0$ but $\rho h_1^2 e_0$ is not zero.

For a slightly more complicated example, consider the relation $h_0 \cdot \tau g = \tau \cdot h_0 g$. This implies that

$$h_0 \cdot d_1(\tau g) = d_1(\tau) \cdot h_0 g = \rho h_0^2 g,$$

so $d_1(\tau g)$ must equal $\rho h_0 g$.

Example 6.6. Sometimes, the multiplicative structure and an already known differential imply that a certain element is killed by ρ^k. Then that element must be killed by a differential d_r with $r \leq k$. For example, the element $\tau^4 h_1^2 h_3 = (\tau^2 h_2)^2 h_2$ is a permanent cycle because it is a product of permanent cycles. There are two possible differentials that could hit a ρ-multiple of it: $d_4(\tau^6 h_2^2)$ or $d_8(\tau^8 h_1^2)$. Note that $\tau^4 h_1^2 h_3$ is killed by ρ^4 because of the differential $d_4(\tau^4) = \rho^4 \tau^2 h_2$. Therefore, $\rho^4 \tau^4 h_1^2 h_3$ must be hit by a d_r differential with $r \leq 4$. The only possibility is that $d_4(\tau^6 h_2^2) = \rho^4 \tau^2 h_1^2 h_3$.

This differential can be obtained another way using the Leibniz rule, the multiplicative relation $\tau^6 h_2^2 = \tau^4 \cdot \tau^2 h_2 \cdot h_2$, and the differential $d_4(\tau^4) = \rho^4 \tau^2 h_2$.

Example 6.7. Sometimes one must look ahead to larger values of $s + f - w$ in order to use multiplicative relations to rule out differentials. For example, in order to show that $d_4(i) = \rho^4 h_1 c_0 e_0$ (in degrees satisfying $s + f - w = 18$), we first use other techniques to rule out possible differentials until it suffices to eliminate the possibility that $d_{11}(\tau^4 P c_0)$ might equal $\rho^{11} h_1 c_0 e_0$. But this would imply that

$d_{11}(\tau^4 P h_1 c_0)$ equals $h_1^2 c_0 e_0$ (in degrees satisfying $s+f-w=19$), and this contradicts the h_1-periodic differential $d_3(Pe_0) = \rho^3 h_1^2 c_0 e_0$ from Table 4.

Example 6.8. The Leibniz rule implies that certain elements survive at least to a certain page of the spectral sequence. For example, the element $\tau^6 h_3^2$ cannot be hit by a differential, so it must support a differential. There are two possibilities: $d_4(\tau^6 h_3^2)$ might equal $\rho^4 \tau^4 h_1^2 h_4$, or $d_6(\tau^6 h_3^2)$ might equal $\rho^6 \tau^3 c_1$. The Leibniz rule and the relation $\tau^6 h_3^2 = \tau^4 \cdot \tau^2 h_3^2$ imply that

$$d_4(\tau^6 h_3^2) = d_4(\tau^4) \cdot \tau^2 h_3^2 = \rho^4 \tau^2 h_2 \cdot \tau^2 h_3^2 = 0.$$

Therefore, $d_6(\tau^6 h_3^2)$ must equal $\rho^6 \tau^3 c_1$.

Example 6.9. The multiplicative structure implies that certain elements do not support any differentials because they are the product of elements that do not support any differentials.

Extending Example 6.6, sometimes the Massey product structure of $\mathrm{Ext}_{\mathbb{R}}$ implies that some element $\rho^k x$ must be zero. Then $\rho^k x$ must be the target of a Bockstein d_r differential for $r \leq k$. Through coweight 12, we apply this method only once in the following Lemma 6.10. However, we anticipate that this approach will become more and more important in higher coweights. Massey products in $\mathrm{Ext}_{\mathbb{R}}$ are discussed below in Section 7 and Table 6.

Lemma 6.10. $d_2(\tau^2 g) = \rho^2 h_2 f_0$.

Proof. Table 6 shows that $h_2 f_0$ equals the Massey product $\langle \tau h_1, h_1^4, h_4 \rangle$ in $\mathrm{Ext}_{\mathbb{R}}$. Shuffle to obtain

$$\rho^2 \langle \tau h_1, h_1^4, h_4 \rangle = \langle \rho^2, \tau h_1, h_1^4 \rangle h_4,$$

which equals zero because the last bracket is zero. Therefore, $\rho^2 h_2 f_0$ is hit by a d_1 or d_2 differential, and the only possibility is that $d_2(\tau^2 g) = \rho^2 h_2 f_0$. □

Theorem 6.11 summarizes the results of the analysis of ρ-Bockstein differentials.

Theorem 6.11. *Table 5 lists some values of the ρ-Bockstein d_r differentials on multiplicative generators of the E_r-page. Through coweight 13, the d_r differential vanishes on all other multiplicative generators of the E_r-page.*

For legibility, we have not included powers of ρ in the values of the Bockstein differentials in Table 5. For example, the first row of the table is to be interpreted as $d_1(\tau) = \rho h_0$.

7. Hidden extensions in the ρ-Bockstein spectral sequence

Section 6 explains how to obtain the E_∞-page of the ρ-Bockstein spectral sequence through coweight 12. As usual, this E_∞-page is an associated graded object of $\mathrm{Ext}_{\mathbb{R}}$.

We abuse notation and use the same name for generators of the ρ-Bockstein E_∞-page and elements of $\mathrm{Ext}_{\mathbb{R}}$ that they represent. A generator of the ρ-Bockstein E_∞-page can represent more than one element in $\mathrm{Ext}_{\mathbb{R}}$, where the indeterminacy is parametrized by elements of the E_∞-page in higher filtration. For example, the element $\tau^2 h_2$ of the E_∞-page represents two elements of $\mathrm{Ext}_{\mathbb{R}}$ whose difference is $\rho^4 h_3$.

We adopt the following convention in selecting generators in $\mathrm{Ext}_{\mathbb{R}}$. We always choose an element of $\mathrm{Ext}_{\mathbb{R}}$ that is annihilated by the same power of ρ as its representative in the E_∞-page. For example, $\tau^2 h_2$ is annihilated by ρ^4 in the E_∞-page. Therefore, we write $\tau^2 h_2$ for the (unique) element of $\mathrm{Ext}_{\mathbb{R}}$ that is annihilated by ρ^4. (The other possible choice is ρ-periodic.)

This convention concerning annihilation by powers of ρ eliminates much of the ambiguity in passing from the E_∞-page to $\mathrm{Ext}_{\mathbb{R}}$. In some cases, our convention does not eliminate all ambiguities. However, the remaining ambiguities make little practical difference.

In order to recover the full structure of $\mathrm{Ext}_{\mathbb{R}}$ from the ρ-Bockstein E_∞-page, we must determine hidden multiplicative extensions. We adopt the precise definition of a hidden extension given in [18, Section 4.1.1]. In this section, we will analyze all hidden extensions by h_0 and h_1 through coweight 12.

The ρ-Bockstein spectral sequence has numerous hidden extensions by other elements. There are so many examples that it is not practical to enumerate them exhaustively. In practice, these other hidden extensions are occasionally useful, and we treat them on an ad hoc basis as necessary.

Definition 7.1. A hidden a extension from x to y is decomposable if there exists a hidden a extension from u to v, and there exists z such that $x = zu$ and $y = zv$ in the E_∞-page.

Example 7.2. There is a hidden h_0 extension from τh_1 to $\rho\tau h_1^2$. Multiplication by τh_1 gives the decomposable hidden h_0 extension from $\tau^2 h_1^2$ to $\rho\tau^2 h_1^3$.

Definition 7.1 allows us to focus only on the hidden extensions that are most significant. In practice, decomposable hidden extensions are easy to understand, once the indecomposable hidden extensions have been studied.

Remark 7.3. The structure of the ρ-Bockstein spectral sequence guarantees that there are no hidden extensions by ρ. For degree reasons, if there is a possible hidden ρ extension from x to y, then in fact y is a multiple of ρ. According to the definition of a hidden extension [18, Section 4.1.1], this means that y cannot be the target of a hidden ρ extension.

7.4. **Massey products.** Our main tool for establishing hidden extensions is the May Convergence Theorem [21, Theorem 4.1], restated here for convenience.

Theorem 7.5 (May Convergence Theorem)**.** *Let α_0, α_1, and α_2 be elements of $\mathrm{Ext}_{\mathbb{R}}$ such that the Massey product $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. For each i, let a_i be a permanent cycle in the Bockstein E_r-page that detects α_i. Suppose further that:*

(1) there exist elements a_{01} and a_{12} in the Bockstein E_r-page such that $d_r(a_{01})$ equals $a_0 a_1$ and $d_r(a_{12})$ equals $a_1 a_2$;

(2) if either a_{01} or a_{12} has degree (s, f, w) and ρ-Bockstein degree m, and x is an element in degree (s, f, w) and ρ-Bockstein degree m' such that $m' \leq m$, then $d_t(x) = 0$ for all t such that $m' + t > (m - m') + r$.

Then $a_0 a_{12} + a_{01} a_2$ is a permanent cycle in the ρ-Bockstein spectral sequence, and it detects an element of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ in $\mathrm{Ext}_{\mathbb{R}}$.

We will often use Theorem 7.5 in the situation when a_{01} has ρ-Bockstein degree 0 and a_{12} has negative ρ-Bockstein degree. Since the ρ-Bockstein spectral sequence is zero in negative ρ-Bockstein degrees, condition (2) of Theorem 7.5 simplifies to the condition that no element in the same degree as a_{01} with ρ-Bockstein degree 0 supports a longer differential.

Proposition 7.6. *Table 6 lists some Massey products in* $\mathrm{Ext}_{\mathbb{R}}$.

Proof. Most of these Massey products are straightforward applications of the May Convergence Theorem 7.5. In those cases, the sixth column of Table 6 gives the ρ-Bockstein differential that is relevant for computing the Massey product.

In some cases, the Massey products follow by comparison to the $\mathbb{C}$-motivic case. This is denoted by the word "$\mathbb{C}$-motivic" in the sixth column of Table 6. However, this only determines the Massey product up to multiples of ρ. These ambiguities can typically be eliminated by the multiplicative structure. In particular, if the Massey product $\langle x, y, z\rangle$ is defined and $\rho^a x$ and $\rho^b z$ are both zero, then

$$\rho^{a+b}\langle x, y, z\rangle = \rho^b \langle \rho^a, x, y\rangle z = 0.$$

The indeterminacies can be computed by inspection. □

Table 6 is not meant to be an exhaustive list of Massey products. It merely provides an assortment of Massey products that are needed for various specific computations throughout the manuscript.

7.7. **Hidden h_0 extensions.**

Proposition 7.8. *Table 7 lists all indecomposable hidden h_0 extensions in the ρ-Bockstein spectral sequence, through coweight* 12.

Proof. All of the hidden h_0 extensions in Table 7 are proved using a single technique, which was introduced in the proof of [11, Lemma 6.2]. To illustrate this technique, we will show that there is a hidden h_0 extension from $\tau^2 h_1 c_0$ to $\rho^2 P h_2$.

First we show that the product $h_0 \cdot \tau^2 h_1 c_0$ is nonzero in $\mathrm{Ext}_{\mathbb{R}}$. If not, then the Massey product $\langle \rho, h_0, \tau^2 h_1 c_0\rangle$ would be defined in $\mathrm{Ext}_{\mathbb{R}}$. The May Convergence Theorem 7.5, together with the ρ-Bockstein differential $d_1(\tau) = \rho h_0$, would then imply that $\tau^3 h_1 c_0$ is a permanent cycle. But this contradicts the ρ-Bockstein differential $d_3(\tau^3 h_1 c_0) = \rho^3 P h_2$.

This shows that there must be a hidden h_0 extension on $\tau^2 h_1 c_0$. The target of this hidden extension can only be $\rho^2 P h_2$ or $\tau P h_1$. But the target must have higher ρ-Bockstein filtration than the source, which rules out $\tau P h_1$.

In some cases, one needs to use multiplicative relations to rule out possible hidden h_0 extensions. For example, the target of a hidden h_0 extension cannot support a ρ multiplication, since $\rho h_0 = 0$ in $\mathrm{Ext}_{\mathbb{R}}$.

We must also show that many elements do not support hidden h_0 extensions. In all cases through coweight 12, the non-existence follows from simple multiplicative relations. For example, if x is already known to not support an h_0 extension, then the product xy cannot support an h_0 extension. Similarly, if $h_1 y$ or ρy is non-zero, then y cannot be the target of a hidden extension because of the relations $h_0 h_1 = 0$ and $\rho h_0 = 0$ in $\mathrm{Ext}_{\mathbb{R}}$. □

7.9. **Hidden h_1 extensions.**

Proposition 7.10. *Table 8 lists all indecomposable hidden h_1 extensions in the ρ-Bockstein spectral sequence, through coweight* 12.

Proof. Many of the extensions are established using the map

$$\mathrm{Ext}_{\mathbb{C}} \xrightarrow{p} \mathrm{Ext}_{\mathbb{R}}$$

of Remark 3.3. To illustrate this technique, we will show that there is a hidden h_1 extension from $\tau^2 h_1 c_0$ to $\rho P h_2$. The relation $h_1 \cdot \tau^3 c_0 = \tau^3 h_1 c_0$ in $\mathrm{Ext}_{\mathbb{C}}$ implies that $h_1 \cdot p(\tau^3 c_0) = p(\tau^3 h_1 c_0)$. Observe that $p(\tau^3 c_0) = \rho\tau h_1 \cdot \tau c_0$ and $p(\tau^3 h_1 c_0) = \rho^2 P h_2$. This shows that there is a hidden h_1 extension from $\rho\tau^2 h_1 c_0$ to $\rho^2 P h_2$, and it follows that there is also a hidden h_1 extension from $\tau^2 h_1 c_0$ to $\rho P h_2$.

Several more difficult cases are established in the following lemmas.

We must also show that many elements do not support hidden h_1 extensions. In most cases through coweight 12, the non-existence follows from simple multiplicative relations. For example, if x is already known to not support an h_1 extension, then the product xy cannot support an h_1 extension. Similarly, if $h_0 y$ is nonzero, then y cannot be the target of a hidden h_1 extension because of the relation $h_0 h_1 = 0$ in $\mathrm{Ext}_{\mathbb{R}}$.

Additionally, the map $p : \mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_{\mathbb{R}}$ can be used to detect the absence of some h_1 extensions. □

Remark 7.11. The first three extensions in Table 8 were established in [11].

Lemma 7.12. *There is a hidden h_1 extension from $\tau^3 h_2^3$ to $\rho^4 d_0$.*

Proof. The element $\tau^3 h_2^3$ of the ρ-Bockstein E_∞-page detects the element $\tau^2 h_2 \cdot \tau h_2^2$ in $\mathrm{Ext}_{\mathbb{R}}$. Table 8 shows that $h_1 \cdot \tau h_2^2 = \rho c_0$, and $h_1^2 \cdot \tau^2 h_2 = \rho^3 c_0$. Therefore,

$$h_1^3 \cdot \tau^2 h_2 \cdot \tau h_2^2 = \rho^3 c_0 \cdot \rho c_0 = \rho^4 h_1^2 d_0.$$

It follows that $h_1 \cdot \tau^2 h_2 \cdot \tau h_2^2$ equals $\rho^4 d_0$. □

Lemma 7.13. *There is a hidden h_1 extension from $\tau^2 f_0$ to $\rho^2 \tau^2 h_1 g$.*

Proof. Table 6 shows that $\tau^2 f_0$ belongs to the Massey product $\langle \tau^2 h_2, h_3, h_0^2 h_3 \rangle$. Table 8 shows that there is a hidden h_1 extension from $\tau^2 h_2$ to $\rho^2 \tau h_2^2$. Therefore, we have

$$h_1 \langle \tau^2 h_2, h_3, h_0^2 h_3 \rangle = \langle \rho^2 \tau h_2^2, h_3, h_0^2 h_3 \rangle = \rho^2 \langle \tau h_2^2, h_3, h_0^2 h_3 \rangle ,$$

where the equalities follow from inspection of indeterminacies. Table 6 shows that the element $\tau^2 h_1 g$ of the Bockstein E_∞-page detects both elements of the Massey product $\langle \tau h_2^2, h_3, h_0^2 h_3 \rangle$, so $\rho^2 \tau^2 h_1 g$ is the target of the hidden h_1 extension. □

Lemma 7.14.

(1) *There is a hidden h_1 extension from $\tau^8 h_1 c_0$ to $\rho\tau^6 P h_2$.*
(2) *There is a hidden h_1 extension from $\tau^6 P h_2$ to $\rho^2 \tau^5 h_0^2 d_0$.*
(3) *There is a hidden h_1 extension from $\tau^4 P h_1 c_0$ to $\rho\tau^2 P^2 h_2$.*
(4) *There is a hidden h_1 extension from $\tau^2 P^2 h_2$ to $\rho^2 \tau P h_0^2 d_0$.*

Proof. We will show that $h_1^3 \cdot \tau^8 c_0$ equals $\rho^3 \tau^5 h_0^2 d_0$. This will establish the first two extensions simultaneously.

Table 6 shows that $h_1 \cdot \tau^8 c_0$ equals the Massey product $\langle \tau h_1 \cdot \tau^5 c_0, \tau h_1, \rho^2 \rangle$. By inspection of indeterminacies,

$$h_1^2 \langle \tau h_1 \cdot \tau^5 c_0, \tau h_1, \rho^2 \rangle = h_1 \langle h_1 \cdot \tau h_1 \cdot \tau^5 c_0, \tau h_1, \rho^2 \rangle .$$

This expression equals $h_1 \langle \rho\tau^4 Ph_2, \tau h_1, \rho^2 \rangle$, since Table 8 shows that there is a hidden h_1 extension from $\tau^6 h_1 c_0$ to $\rho\tau^4 Ph_2$. By inspection of indeterminacies again, this also equals $\rho h_1 \langle \tau^4 Ph_2, \tau h_1, \rho^2 \rangle$.

Now shuffle to obtain

$$\rho h_1 \langle \tau^4 Ph_2, \tau h_1, \rho^2 \rangle = \rho^3 \langle h_1, \tau^4 Ph_2, \tau h_1 \rangle .$$

Finally, Table 6 shows that $\langle h_1, \tau^4 Ph_2, \tau h_1 \rangle$ equals $\tau^5 h_0^2 d_0$. This establishes the first two extensions.

The argument for the last two extensions is essentially identical. The Massey product $\langle \tau h_1 \cdot \tau Pc_0, \tau h_1, \rho^2 \rangle$ equals $h_1 \cdot \tau^4 Pc_0$. We have

$$h_1^2 \langle \tau h_1 \cdot \tau Pc_0, \tau h_1, \rho^2 \rangle = h_1 \langle h_1 \cdot \tau h_1 \cdot \tau Pc_0, \tau h_1, \rho^2 \rangle ,$$

which equals

$$h_1 \langle \rho P^2 h_2, \tau h_1, \rho^2 \rangle = \rho h_1 \langle P^2 h_2, \tau h_1, \rho^2 \rangle .$$

Finally, shuffle to obtain

$$\rho h_1 \langle P^2 h_2, \tau h_1, \rho^2 \rangle = \rho^3 \langle h_1, P^2 h_2, \tau h_1 \rangle = \rho^3 \tau P h_0^2 d_0.$$

□

Lemma 7.15. *There is a hidden h_1-extension from $\tau^3 c_1$ to $\rho^2\tau^2 h_2 c_1$.*

Proof. Table 6 shows that $\tau^3 c_1$ is contained in the Massey product $\langle \rho^2, \tau h_1, \tau c_1 \rangle$. Shuffle to obtain

$$\langle \rho^2, \tau h_1, \tau c_1 \rangle h_1 = \rho^2 \langle \tau h_1, \tau c_1, h_1 \rangle .$$

Table 6 shows that the element $\tau^2 h_2 c_1$ of the Bockstein E_∞-page detects both elements of $\langle \tau h_1, \tau c_1, h_1 \rangle$, so $\rho^2\tau^2 h_2 c_1$ is the target of the hidden h_1 extension. □

Lemma 7.16.

(1) There is a hidden h_1 extension from $\tau^3 h_2^2 e_0$ to $\rho^2 j$.
(2) There is a hidden h_1 extension from j to ρd_0^2.

Proof. Table 8 shows that $h_1 \cdot \tau h_2^2 = \rho c_0$, and $h_1^3 \cdot \tau^2 e_0 = h_1 \cdot \rho\tau h_2^2 \cdot d_0 = \rho^2 c_0 d_0$. Therefore,

$$h_1^4 \cdot \tau h_2^2 \cdot \tau^2 e_0 = \rho^3 c_0^2 d_0 = \rho^3 h_1^2 d_0^2.$$

Both hidden extensions are immediate consequences. □

7.17. **Miscellaneous relations.** We briefly consider a few other types of hidden extensions.

In the Bockstein E_∞-page, we have the relation $h_1^2 \cdot \tau^4 h_3 + (\tau^2 h_2)^2 h_2 = 0$. However, in $\mathrm{Ext}_\mathbb{R}$, it is possible that the sum $h_1^2 \cdot \tau^4 h_3 + (\tau^2 h_2)^2 h_2$ equals a non-zero element that is detected in higher ρ-Bockstein filtration. Lemma 7.18 demonstrates that this does in fact occur. It provides one additional piece of information about the multiplicative structure of $\mathrm{Ext}_\mathbb{R}$.

Lemma 7.18. *In $\mathrm{Ext}_\mathbb{R}$ we have the relation*

$$h_1^2 \cdot \tau^4 h_3 + (\tau^2 h_2)^2 h_2 = \rho^5 \tau h_0 h_3^2.$$

Proof. This follows by comparison along the map $p : \mathrm{Ext}_\mathbb{C} \to \mathrm{Ext}_R$ of Remark 3.3. The relation $h_1 \cdot \tau^8 h_1 = \tau^8 h_1^2$ in $\mathrm{Ext}_\mathbb{C}$ implies that $h_1 \cdot p(\tau^8 h_1) = p(\tau^8 h_1^2)$ in $\mathrm{Ext}_\mathbb{R}$. Observe that $p(\tau^8 h_1) = \rho^7 \tau^4 h_1 h_3$ and $p(\tau^8 h_1^2) = \rho^{12} \tau h_0 h_3^2$. This shows that there is a hidden h_1 extension from $\rho^7\tau^4 h_1 h_3$ to $\rho^{12}\tau h_0 h_3^2$, which implies the desired relation. □

Lemma 7.19. *There is a hidden $\tau^2 h_2$ extension from c_0 to $\rho^3 d_0$.*

Proof. Table 8 shows that there are hidden h_1 extensions from τh_2^2 to ρc_0, and from $\tau^3 h_2^2$ to $\rho^4 d_0$. Therefore,

$$\tau^2 h_2 \cdot \rho c_0 = \tau^2 h_2 \cdot h_1 \cdot \tau h_2^2 = \rho^4 d_0.$$

□

Lemma 7.20. *There is a hidden h_2 extension from $h_2 f_0$ to $\rho h_1^2 h_4 c_0$.*

Proof. We use the map $p : \mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_{\mathbb{R}}$ of Remark 3.3. The relation $h_2 \cdot \tau^2 g = \tau^2 h_2 g$ in $\mathrm{Ext}_{\mathbb{C}}$ implies that $h_2 \cdot p(\tau^2 g) = p(\tau^2 h_2 g)$. Observe that $p(\tau^2 g) = \rho h_2 f_0$, and $p(\tau^2 h_2 g) = \rho^2 h_1^2 h_4 c_0$.

Therefore, there is a hidden h_2 extension from $\rho h_2 f_0$ to $\rho^2 h_1^2 h_4 c_0$, and also a hidden h_2 extension from $h_2 f_0$ to $\rho h_1^2 h_4 c_0$. □

8. Adams differentials

Sections 6 and 7 describe how to compute $\mathrm{Ext}_{\mathbb{R}}$, which serves as the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence. We now proceed to analyze Adams differentials. We remind the reader of the notation for stable homotopy elements discussed in Section 2.1 and Table 9.

Recall from Section 3 that extension of scalars induces a map from the $\mathbb{R}$-motivic Adams spectral sequence to the $\mathbb{C}$-motivic Adams spectral sequence. We will frequently use these comparison functors to deduce information about the $\mathbb{R}$-motivic Adams spectral sequence from already known information about the $\mathbb{C}$-motivic and classical Adams spectral sequences. See [18] for an extensive summary of computational information about the $\mathbb{C}$-motivic and classical Adams spectral sequences.

8.1. Toda brackets.

The Moss Convergence Theorem 8.2 is a key tool for determining Toda brackets [23] [18, Section 3.1]. We restate a version of the theorem here for convenience.

Theorem 8.2 (Moss Convergence Theorem)**.** *Let α_0, α_1, and α_2 be elements of the $\mathbb{R}$-motivic stable homotopy groups such that the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. Let a_i be a permanent cycle on the Adams E_r-page that detects α_i for each i. Suppose further that:*

1. *the Massey product $\langle a_0, a_1, a_2 \rangle_{E_r}$ is defined (in $\mathrm{Ext}_{\mathbb{R}}$ when $r = 2$, or using the Adams d_{r-1} differential when $r \geq 3$).*
2. *if (s, f, w) is the degree of either $a_0 a_1$ or $a_1 a_2$; $f' < f - r + 1$; $f'' > f$; and $t = f'' - f'$; then every Adams differential $d_t : E_t^{s+1,f',w} \to E_t^{s,f'',w}$ is zero.*

Then $\langle a_0, a_1, a_2 \rangle_{E_r}$ contains a permanent cycle that detects an element of the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.

Theorem 8.3. *Table 10 lists some Toda brackets in $\pi_{*,*}$.*

Proof. Most of these Toda brackets are straightforward applications of the Moss Convergence Theorem 8.2. When a Massey product appears in the fifth column of Table 10, the Toda bracket follows from the Moss Convergence Theorem 8.2 with $r = 2$. When an Adams differential appears in the fifth column of Table 10, the Toda bracket follows from the Moss Convergence Theorem 8.2 with $r > 2$, and the given Adams differential is relevant for computing the Toda bracket.

In some cases, the Toda brackets follow by comparison along the extension of scalars functor to the $\mathbb{C}$-motivic case. This is denoted by the word "$\mathbb{C}$-motivic" in the fifth column of Table 10.

One slightly different case is handled below in Lemma 8.4. □

Table 10 is not meant to be exhaustive in any sense. It merely provides the Toda brackets that are needed for various specific computations. Beware that these brackets have non-trivial indeterminacies, although we have not specified the indeterminacies because they are not generally relevant to our specific needs.

Beware that some of the Toda brackets in Table 10 require knowledge of Adams differentials that are established below in Section 8.5.

Lemma 8.4. *The Toda bracket $\langle \rho^2, \tau\eta, \nu_4 \rangle$ is detected by $\tau^2 h_2 \cdot h_4$.*

Proof. Table 6 shows that $\tau^2 h_2$ is contained in the Massey product $\langle \rho^2, \tau h_1, h_2 \rangle$. By inspection of indeterminacies,

$$\tau^2 h_2 \cdot h_4 = \langle \rho^2, \tau h_1, h_2 \rangle h_4 = \langle \rho^2, \tau h_1, h_2 h_4 \rangle .$$

The Moss Convergence Theorem 8.2 implies that $\tau^2 h_2 \cdot h_4$ detects the corresponding Toda bracket. □

8.5. **Adams d_2 differentials.** We now proceed to analyze Adams differentials.

Theorem 8.6. *Table 12 lists some values of the $\mathbb{R}$-motivic Adams d_2 differential. Through coweight 12, the d_2 differential is zero on all other multiplicative generators of the $\mathbb{R}$-motivic Adams E_2-page.*

Proof. The multiplicative structure rules out many possible differentials. For example, $d_2(\tau^5 h_1)$ cannot equal $\tau^4 h_0 \cdot h_0^2$ because $h_0^2 \cdot \tau^5 h_1 = 0$, while $\tau^4 h_0 \cdot h_0^4$ is non-zero.

Other multiplicative generators are known to be permanent cycles, because the Moss Convergence Theorem 8.2 shows that they must survive to detect various Toda brackets. These instances are shown in Table 11. In one case, the element $h_4 \cdot \tau c_0$ must survive to detect the product $\sigma \cdot \tau\eta_4$, by comparison to the $\mathbb{C}$-motivic stable homotopy groups.

Many non-zero differentials follow by comparison to the $\mathbb{C}$-motivic or classical Adams spectral sequences.

Several more difficult cases are established in the following lemmas. □

Remark 8.7. Table 11 shows that $\tau^4 h_3$ is a permanent cycle because it detects the Toda bracket $\langle \rho^4, \tau^2\nu, \sigma \rangle$. We give an alternative proof that is geometrically interesting, following the method of [11, Lemma 7.3].

There is a functor from classical homotopy theory to $\mathbb{R}$-motivic homotopy theory that takes the sphere S^p to $S^{p,0}$. Let $\sigma_{\mathrm{top}} : S^{15,0} \to S^{8,0}$ be the image of the classical Hopf map $\sigma : S^{15} \to S^8$ under this functor.

The cohomology of the cofiber of σ_{top} is free on two generators x and y of degrees $(8, 0)$ and $(16, 0)$, satisfying $\mathrm{Sq}^8(x) = \tau^4 y$ and $\mathrm{Sq}^{16}(x) = \rho^8 y$. The proof of these formulas is essentially identical to the proof of [11, Lemma 7.4].

This shows that $\tau^4 h_3 + \rho^8 h_4$ is a permanent cycle in the Adams spectral sequence, since it detects the stabilization of σ_{top} in $\pi_{7,0}$. Also, $\rho^8 h_4$ is a permanent cycle because there are no possible values for differentials. Therefore, $\tau^4 h_3$ is a permanent cycle.

Lemma 8.8. $d_2(\tau h_0 h_3^2) = \rho^2 h_1 d_0$.

Proof. Table 12 shows that $d_2(e_0) = h_1^2 d_0$. Therefore,

$$d_2(h_1 \cdot \tau h_0 h_3^2) = d_2(\rho^2 e_0) = \rho^2 h_1^2 d_0.$$

It follows that $d_2(\tau h_0 h_3^2)$ equals $\rho^2 h_1 d_0$. □

Lemma 8.9. $d_2(f_0) = h_0^2 e_0$.

Proof. Comparison to the $\mathbb{C}$-motivic or classical case shows that $d_2(f_0)$ equals either $h_0^2 e_0$ or $h_0^2 e_0 + \rho^2 h_1^2 e_0$. But $h_1 \cdot f_0 = 0$ in the E_2-page, while $h_1(h_0^2 e_0 + \rho^2 h_1^2 e_0)$ is non-zero. The only possibility is that $d_2(f_0)$ equals $h_0^2 e_0$. □

Lemma 8.10. $d_2(\tau^2 f_0) = h_0^2 \cdot \tau^2 e_0 + \rho^3 \tau h_2^2 \cdot d_0$.

Proof. The $\mathbb{C}$-motivic differential $d_2(\tau^2 f_0) = \tau^2 h_0^2 e_0$ implies that $d_2(\tau^2 f_0)$ equals either $h_0^2 \cdot \tau^2 e_0$ or $h_0^2 \cdot \tau^2 e_0 + \rho^3 \tau h_2^2 \cdot d_0$. We rule out the first possibility by noting that $(h_0^2 + \rho^2 h_1^2) \cdot \tau^2 f_0 = 0$ in $\mathrm{Ext}_{\mathbb{R}}$ whereas $(h_0^2 + \rho^2 h_1^2) \cdot \tau^2 h_0^2 e_0 = \rho^6 h_1 c_0 d_0$. □

Lemma 8.11. $d_2(\tau^2 h_1 g) = \rho^2 c_0 d_0$.

Proof. Table 8 shows that $h_1 \cdot \tau^2 h_1 g = \rho \tau h_2^2 \cdot e_0$. Therefore,

$$h_1 \cdot d_2(\tau^2 h_1 g) = \rho \tau h_2^2 \cdot d_2(e_0) = \rho \tau h_2^2 \cdot h_1^2 d_0,$$

which equals $\rho^2 h_1 c_0 d_0$ because Table 8 shows that $h_1 \cdot \tau h_2^2 = \rho c_0$. □

8.12. **Higher Adams differentials.** Theorem 8.6 completely describes the Adams d_2 differential through coweight 12. From this information, one can compute the Adams E_3-page in a range. We now proceed to analyze higher differentials.

Theorem 8.13. *Table 13 lists some values of the $\mathbb{R}$-motivic Adams d_3 differential for $r \geq 3$. Through coweight 12, the d_3 differential is zero on all other multiplicative generators of the $\mathbb{R}$-motivic Adams E_3-page. Moreover, through coweight 12, there are no higher differentials, and the $\mathbb{R}$-motivic Adams E_4-page equals the $\mathbb{R}$-motivic Adams E_∞-page.*

Proof. As in the proof of Theorem 8.6, many multiplicative generators cannot support differentials because there are no possible targets. Comparison to the $\mathbb{C}$-motivic and classical cases also determines some differentials. For example, $d_3(h_1 h_4)$ cannot equal $h_1 d_0$.

Other multiplicative generators are known to be permanent cycles, because the Moss Convergence Theorem 8.2 shows that they must survive to detect various Toda brackets. These instances are shown in Table 11.

The multiplicative structure rules out additional cases. For example $d_3(\rho h_4)$ cannot equal ρd_0 because of the relation $h_1 \cdot \rho h_4 = \rho \cdot h_1 h_4$, together with the fact that $d_3(h_1 h_4)$ is already known to be zero.

The harder cases are established in the following lemmas. □

Lemma 8.14. $d_3(\rho^6 e_0) = 0$.

Proof. If $d_3(\rho^6 e_0)$ equaled $\rho h_1 \cdot \tau h_1 \cdot \tau P h_1$, then $\rho^7 e_0$ would be a permanent cycle that detected an element α of $\pi_{10,3}$, and α could not be divisible by ρ. Therefore, by Corollary 3.5, α would map to a non-zero element β in $\pi^{\mathbb{C}}_{10,3}$. Then β would have to be detected by $\tau^3 P h_1^2$, so $\eta\beta$ would also have to be non-zero in $\pi^{C}_{11,4}$.

But $\eta\alpha$ would be detected by $\rho^7 h_1 e_0$ and would be divisible by ρ, so it would map to zero in $\pi^{\mathbb{C}}_{11,4}$. This contradicts that $\eta\beta$ is non-zero. □

Remark 8.15. Lemma 8.14 can also be proved using the $\mathbb{R}$-motivic spectrum kq, which is the very effective slice cover of the Hermitian K-theory spectrum KQ [1]. The cohomology of kq is isomorphic to $\mathcal{A}//\mathcal{A}(1)$, where $\mathcal{A}(1)$ is the $\mathbb{M}_2$-subalgebra of the $\mathbb{R}$-motivic Steenrod algebra that is generated by Sq^1 and Sq^2.

By a change-of-rings isomorphism, the homotopy of kq is computed by an Adams spectral sequence whose E_2-page is $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{M}_2, \mathbb{M}_2)$. This E_2-page was computed in [16], and also in [12, Section 6].

The element $\rho\tau h_1 \cdot \tau P h_1 \cdot h_1$ maps to a non-zero permanent cycle in

$$\mathrm{Ext}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2),$$

so it cannot be the target of a differential.

Lemma 8.16. $d_3(h_0h_4) = h_0d_0 + \rho h_1 d_0$

Proof. The classical differential $d_3(h_0h_4) = h_0d_0$ implies that in the $\mathbb{R}$-motivic case, $d_3(h_0h_4)$ equals either h_0d_0 or $h_0d_0 + \rho h_1 d_0$.

Note that $\tau h_1 \cdot h_0 d_0 = \rho\tau h_1 \cdot h_1 d_0$ is non-zero on the E_3-page, but $\tau h_1 \cdot h_0 h_4 = \rho\tau h_1 \cdot h_1 h_4$ is a permanent cycle, as shown in Table 11. Therefore, $d_3(h_0h_4)$ cannot equal h_0d_0. □

Lemma 8.17.

(1) $d_3(\tau h_2^2 \cdot \tau^2 e_0) = \rho\tau P h_1 \cdot d_0$.
(2) $d_3(\rho j) = \tau P h_1 \cdot h_1 d_0$.

Proof. Let α be an element of $\pi_{24,13}$ that is represented by $\tau P h_1 \cdot h_1 d_0$. By comparison of Adams spectral sequences, extension of scalars must take α to zero in $\pi^{\mathbb{C}}_{24,13}$. Moreover, $\tau P h_1 \cdot h_1 d_0$ cannot be the target of a hidden ρ extension. Therefore, by Corollary 3.5, $\tau P h_1 \cdot h_1 d_0$ must be the target of an $\mathbb{R}$-motivic Adams differential, and there is only one possible such differential. This establishes the second formula.

The first formula follows immediately from the second one, using the relation $h_1 \cdot \tau h_2^2 \cdot \tau^2 e_0 = \rho c_0 \cdot \tau^2 e_0$. □

9. Hidden extensions in the Adams spectral sequence

We have now obtained the Adams E_∞-page through coweight 11. It remains to determine hidden extensions that are hidden in the $\mathbb{R}$-motivic Adams spectral sequence. As in Section 7, we use the precise definition of a hidden extension given in [18, Section 4.1.1]. We will analyze all hidden extensions by ρ, h, and η through coweight 11.

We begin by analyzing all hidden extensions by ρ. The main tools are Corollaries 3.5 and 3.8.

Proposition 9.1. *Table 14 lists all hidden ρ extensions in the Adams spectral sequence, through coweight* 11.

Proof. The long exact sequence of Corollary 3.8 gives short exact sequences

$$0 \to (\operatorname{coker} \rho)_{s,w} \to \pi^{\mathbb{C}}_{s,w} \to (\ker \rho)_{s,w+1} \to 0.$$

The rank of $\pi^{\mathbb{C}}_{s,w}$, which is entirely known in our range [18] [19], severely constrains the possible ranks of $\operatorname{coker} \rho$ and $\ker \rho$. From these constraints, we can generally deduce the presence and absence of hidden ρ extensions, and there is typically only one possibility in each case in the range under consideration. The only exception is considered below in Lemma 9.2. □

Lemma 9.2. *There is a hidden ρ extension from $\tau h_1 c_0 d_0$ to $Ph_0 d_0$.*

Proof. Table 16 shows that there is a hidden η extension from $\rho\tau c_0 \cdot d_0$ to $Ph_0 d_0$. Therefore, there must be a hidden ρ extension from $h_1 \cdot \tau c_0 \cdot d_0$ to $Ph_0 d_0$. □

Theorem 9.3. *Table 15 lists all hidden h extensions in the $\mathbb{R}$-motivic Adams spectral sequence, through coweight 11.*

Proof. The long exact sequence of Corollary 3.8 gives short exact sequences

$$0 \to (\operatorname{coker} \rho)_{s,w} \to \pi^{\mathbb{C}}_{s,w} \to (\ker \rho)_{s,w+1} \to 0.$$

Some of the extensions can be determined via these short exact sequences, using known 2 extensions in $\pi^{C}_{*,*}$. For example, the element $\rho^6 e_0$ in the $\mathbb{R}$-motivic Adams E_∞-page lies in $(\operatorname{coker} \rho)_{11,4}$, and it maps to the element $\tau^2\zeta_{11}$ in $\pi^{\mathbb{C}}_{11,4}$ that is detected by $\tau^2 Ph_2$. But $2\tau^2\zeta_{11}$ is non-zero in $\pi^{\mathbb{C}}_{11,4}$, so $\mathsf{h}\alpha$ must also be non-zero. It follows that $\rho^6 e_0$ supports a hidden h extension.

We must also show that many elements do not support hidden h extensions. In most of the cases through coweight 11, the non-existence follows from simple multiplicative relations. For example, if x is a multiple of ρ or of h_1, then x cannot support a hidden h extension because of the relations $\rho\mathsf{h} = 0$ and $\mathsf{h}\eta = 0$. Similarly, if $h_1 y$ or ρy is non-zero, then y cannot be the target of a hidden h extension.

The following lemmas handle a few additional more complicated cases. □

Lemma 9.4. *There is a hidden h extension from $h_2 f_0$ to $\rho c_0 d_0$.*

Proof. Table 10 shows that $h_2 f_0$ detects the Toda bracket $\langle \rho, \{h_2 e_0\}, \eta\rangle$. Shuffle to obtain

$$\langle \rho, \{h_2 e_0\}, \eta\rangle\, \mathsf{h} = \rho \,\langle \{h_2 e_0\}, \eta, \mathsf{h}\rangle\,.$$

Table 10 shows that $c_0 d_0$ detects the latter bracket. □

Lemma 9.5. *There is no hidden h extension on $\tau h_2^2 \cdot h_4$.*

Proof. The only possible target is $\rho\tau c_0 \cdot d_0$. Table 16 shows that $\rho\tau c_0 \cdot d_0$ supports a hidden η extension, so it cannot be the target of a hidden h extension. □

Lemma 9.6. *There is a hidden h extension from $\tau c_0 \cdot d_0$ to $Ph_0 d_0$.*

Proof. Let α be an element of $\pi_{8,4}$ that is detected by τc_0, so $\tau c_0 \cdot d_0$ detects $\alpha\kappa$. Table 14 shows that there is a hidden ρ extension from $h_1 \cdot \tau c_0 \cdot d_0$ to $Ph_0 d_0$, so $Ph_0 d_0$ detects $\rho\eta\alpha\kappa$. But $(\mathsf{h} + \rho\eta)\kappa$ is zero, so $(\mathsf{h} + \rho\eta)\alpha\kappa$ must also be zero. This implies that $\mathsf{h}\alpha\kappa$ is also detected by $Ph_0 d_0$. □

Lemma 9.7. *There is no hidden h extension on $h_4 c_0$.*

Proof. By comparison to the $\mathbb{C}$-motivic (or classical) case, $h_4 c_0$ detects the product $\sigma\eta_4$. By inspection, $\mathsf{h}\eta_4$ is zero in $\pi_{16,9}$. □

Theorem 9.8. *Table 16 lists some hidden η extensions in the $\mathbb{R}$-motivic Adams spectral sequence, through coweight 11.*

Proof. The long exact sequence of Corollary 3.8 gives short exact sequences

$$0 \to (\operatorname{coker} \rho)_{s,w} \to \pi^{\mathbb{C}}_{s,w} \to (\ker \rho)_{s,w+1} \to 0.$$

Many of these extensions can be obtained by comparison to the $\mathbb{C}$-motivic case, using these short exact sequences, as in the proof of Theorem 9.3. For example, the element $\rho\tau h_1 \cdot \tau P c_0$ detects an element α in $(\ker \rho)_{16,7}$. The pre-image β of α in $\pi^{\mathbb{C}}_{16,6}$ is detected by $\tau^3 P c_0$. There is a $\mathbb{C}$-motivic hidden η extension from $\tau^3 h_0^3 h_4$ to $\tau^3 P c_0$, so β is divisible by η. This implies that α is also divisible by η, and that there is an $\mathbb{R}$-motivic hidden η extension from $\tau^2 h_0 \cdot h_0^3 h_4$ to $\rho\tau h_1 \cdot \tau P c_0$.

We must also show that many elements do not support hidden η extensions. In all cases through coweight 11, the non-existence follows from simple multiplicative relations. For example, if x is a multiple of h_0, then x cannot support a hidden η extension because of the relation $\mathsf{h}\eta = 0$. Similarly, if $h_0 y$ is non-zero, then y cannot be the target of a hidden η extension. □

Lemma 9.9. *There is no hidden η extension on $\tau^2 h_3^2$.*

Proof. Table 10 shows that $\tau^2 h_3^2$ detects the Toda bracket $\langle \tau^2\nu, \sigma, \nu \rangle$. Shuffle to obtain

$$\langle \tau^2\nu, \sigma, \nu \rangle \eta = \tau^2 \nu \langle \sigma, \nu, \eta \rangle .$$

The latter bracket is zero. □

Lemma 9.10. *There is no hidden η extension on τc_1.*

Proof. The possible target $\rho h_2 f_0$ is ruled out by the fact that $\rho h_2 f_0$ supports an h_2 extension, as shown in Lemma 7.20. The possible target $\tau h_2^2 \cdot d_0$ is ruled out by comparison to the $\mathbb{C}$-motivic case. □

10. Extension of scalars

We will now study the values of the extension of scalars map $\pi^{\mathbb{R}}_{*,*} \to \pi^{\mathbb{C}}_{*,*}$. Corollary 3.5 tells us exactly which elements of $\pi^{\mathbb{R}}_{*,*}$ have non-trivial images in $\pi^{\mathbb{C}}_{*,*}$. This information about extension of scalars is essential to our approach to the Mahowald invariant described in Section 4.

For the most part, the extension of scalars map is detected by the map from the $\mathbb{R}$-motivic Adams E_∞-page to the $\mathbb{C}$-motivic Adams E_∞-page. For example, the element $(\tau\eta)^2$ of $\pi^{\mathbb{R}}_{2,0}$ is detected by τh_1^2 in the $\mathbb{R}$-motivic Adams E_∞-page, so its image in $\pi^{\mathbb{C}}_{2,0}$ must be $\tau^2\eta^2$, which is detected by $\tau^2 h_1^2$ in the $\mathbb{C}$-motivic Adams E_∞-page.

However, there are a few values that are hidden by the Adams spectral sequence. In other words, there exist elements α in $\pi^{\mathbb{R}}_{*,*}$ such that the Adams filtration of α is strictly less than the Adams filtration of its image in $\pi^{\mathbb{C}}_{*,*}$.

Theorem 10.1. *Through coweight* 11*, Table 17 lists all hidden values of the extension of scalars map $\pi^{\mathbb{R}}_{*,*} \to \pi^{\mathbb{C}}_{*,*}$.*

Proof. We inspect all elements of the $\mathbb{R}$-motivic Adams E_∞-page that are not targets of ρ extensions. Most of these elements map non-trivially to the $\mathbb{C}$-motivic Adams E_∞-page. For example, $(\tau h_1)^2$ maps to $\tau^2 h_1^2$.

A few elements map to zero in the $\mathbb{C}$-motivic Adams E_∞-page. We treat these elements individually. In some cases, there is only one possible target in sufficiently high Adams filtration. The remaining cases are handled by the following lemmas. □

Lemma 10.2. *Extension of scalars takes elements detected by ρh_4 to elements detected by τh_3^2.*

Proof. Table 10 shows that ρh_4 detects the Toda bracket $\langle \rho, \mathsf{h}, \sigma^2 \rangle$. Extension of scalars takes $\langle \rho, \mathsf{h}, \sigma^2 \rangle$ in $\pi^{\mathbb{R}}_{14,7}$ to $\langle 0, 2, \sigma^2 \rangle$ in $\pi^{\mathbb{C}}_{14,7}$, which equals $\{0, \tau\sigma^2\}$. The only non-zero value is $\tau\sigma^2$, which is detected by τh_3^2. □

Lemma 10.3. *Extension of scalars takes elements detected by ρf_0 to elements detected by $\tau h_2 d_0$.*

Proof. Table 10 shows that ρf_0 detects the Toda bracket $\langle \rho, \mathsf{h}, \nu\kappa \rangle$. Extension of scalars takes $\langle \rho, \mathsf{h}, \nu\kappa \rangle$ in $\pi^{\mathbb{R}}_{17,9}$ to $\langle 0, 2, \nu\kappa \rangle$ in $\pi^{\mathbb{C}}_{17,9}$, which equals $\{0, \tau\nu\kappa\}$. The only non-zero value is $\tau\nu\kappa$, which is detected by $\tau h_2 d_0$. □

Lemma 10.4. *Extension of scalars takes elements detected by $\rho^3\tau^2 f_0$ to elements detected by $\tau^4 h_1 d_0$.*

Proof. The long exact sequence of Corollary 3.8 gives a short exact sequence

$$0 \to (\operatorname{coker} \rho)_{15,5} \to \pi^{\mathbb{C}}_{15,5} \to (\ker \rho)_{15,6} \to 0.$$

The group $\pi^{\mathbb{C}}_{15,5}$ is generated by an element of order 32, detected by $\tau^3 h_0^3 h_4$, and an element of order 2, detected by $\tau^4 h_1 d_0$. Also $(\ker \rho)_{15,6}$ is generated by an element of order 32, detected by $\tau^2 h_0 \cdot h_0^3 h_4$. It follows that $(\operatorname{coker} \rho)_{15,5}$ maps onto an element of order 2 that is detected by $\tau^4 h_1 d_0$. □

11. Tables

Table 3: Some values of the $\mathbb{R}$-motivic Mahowald invariant

s	α	$M^{\mathbb{R}}(\alpha)$	indeterminacy
0	2^k	η^k	
1	η	ν	2ν, 4ν
2	η^2	ν^2	
3	ν	σ	2σ, 4σ, 8σ
3	2ν	$\eta\sigma$	ϵ
3	4ν	$\eta^2\sigma$	$\eta\epsilon$
6	ν^2	σ^2	κ
7	σ	$\tau\sigma^2$	
7	2σ	η_4	$\eta\rho_{15}$
7	4σ	$\eta\eta_4$	$\eta^2\rho_{15}$, $\nu\kappa$
7	8σ	$\eta^2\eta_4$	$\eta^3\rho_{15}$
8	$\eta\sigma$	ν_4	$2\nu_4$, $4\nu_4$
8	ϵ	$\overline{\sigma}$	
9	$\eta^2\sigma$	$\nu\nu_4$	$\tau\eta\overline{\kappa}$
9	$\eta\epsilon$	$\nu\overline{\sigma}$	$\tau\eta^2\overline{\kappa}$
9	μ_9	$\nu\overline{\kappa}$	$2\nu\overline{\kappa}$, $4\nu\overline{\kappa}$
10	$\eta\mu_9$	$\nu \cdot \nu\overline{\kappa}$	
11	ζ_{11}	$\tau\nu^2\overline{\kappa}$	$\eta^3\rho_{23}$
11	$2\zeta_{11}$	$\{h_1 h_3 g\}$	$\eta^5\rho_{23}$
11	$4\zeta_{11}$	$\eta\{h_1 h_3 g\}$	$\eta^6\rho_{23}$

Table 4: h_1-periodic Bockstein differentials

coweight	(s,f,w)	x	d_r	$d_r(x)$
4	$(9,5,5)$	Ph_1	d_3	$h_1^3c_0$
7	$(16,7,9)$	Pc_0	d_3	$h_1^4d_0$
8	$(17,9,9)$	P^2h_1	d_7	$h_1^6e_0$
10	$(22,8,12)$	Pd_0	d_3	$h_1^2c_0d_0$
11	$(25,8,14)$	Pe_0	d_3	$h_1^2c_0e_0$
12	$(25,13,13)$	P^3h_1	d_3	$P^2h_1^3c_0$
13	$(30,11,17)$	Pc_0d_0	d_3	$h_1^4d_0^2$

Table 5: Bockstein differentials

coweight	(s,f,w)	x	d_r	$d_r(x)$
1	$(0,0,-1)$	τ	d_1	h_0
2	$(0,0,-2)$	τ^2	d_2	τh_1
4	$(0,0,-4)$	τ^4	d_4	τ^2h_2
4	$(1,1,-3)$	τ^4h_1	d_6	τh_2^2
4	$(2,2,-2)$	$\tau^4h_1^2$	d_7	c_0
4	$(7,4,3)$	$\tau h_0^3h_3$	d_4	$h_1^2c_0$
4	$(9,5,5)$	Ph_1	d_3	$h_1^3c_0$
5	$(6,2,1)$	$\tau^3h_2^2$	d_3	τc_0
6	$(7,4,1)$	$\tau^3h_0^3h_3$	d_3	τPh_1
6	$(9,4,3)$	$\tau^3h_1c_0$	d_3	Ph_2
7	$(8,3,1)$	τ^4c_0	d_7	d_0
7	$(11,5,4)$	τ^2Ph_2	d_6	$h_1^2d_0$
7	$(14,6,7)$	$\tau h_0^2d_0$	d_4	$h_1^3d_0$
7	$(16,7,9)$	Pc_0	d_3	$h_1^4d_0$
8	$(0,0,-8)$	τ^8	d_8	τ^4h_3
8	$(2,2,-6)$	$\tau^8h_1^2$	d_{13}	$\tau h_0h_3^2$
8	$(3,3,-5)$	$\tau^8h_1^3$	d_{15}	e_0
8	$(7,4,-1)$	$\tau^5h_0^3h_3$	d_{12}	h_1e_0
8	$(9,5,1)$	τ^4Ph_1	d_{11}	$h_1^2e_0$
8	$(15,8,7)$	$\tau h_0^7h_4$	d_8	$h_1^5e_0$
8	$(17,9,9)$	P^2h_1	d_7	$h_1^6e_0$
9	$(3,1,-6)$	τ^8h_2	d_{12}	$\tau^2h_3^2$
9	$(14,3,5)$	$\tau^3h_0h_3^2$	d_5	f_0
9	$(14,6,5)$	$\tau^3h_0^2d_0$	d_3	τPc_0
9	$(20,4,11)$	τg	d_1	h_0g
10	$(6,2,-4)$	$\tau^8h_2^2$	d_{14}	τc_1
10	$(9,3,-1)$	$\tau^7h_1^2h_3$	d_9	τ^2e_0
10	$(14,4,4)$	τ^4d_0	d_5	$\tau^2h_1e_0$
10	$(15,8,5)$	$\tau^3h_0^7h_4$	d_3	τP^2h_1
10	$(17,8,7)$	$\tau^3Ph_1c_0$	d_3	P^2h_2
10	$(20,4,10)$	τ^2g	d_2	τh_1g
10	$(22,8,12)$	Pd_0	d_3	$h_1^2c_0d_0$

Table 5: Bockstein differentials

coweight	(s,f,w)	x	d_r	$d_r(x)$
11	$(8,2,-3)$	$\tau^8 h_1 h_3$	d_{12}	$\tau^2 c_1$
11	$(14,3,3)$	$\tau^5 h_0 h_3^2$	d_5	$\tau^2 f_0$
11	$(17,4,6)$	$\tau^4 e_0$	d_5	$\tau^2 h_1 g$
11	$(20,6,9)$	$\tau^3 h_0 h_2 e_0$	d_6	$c_0 e_0$
11	$(23,5,12)$	$\tau^2 h_2 g$	d_3	$h_1^2 h_4 c_0$
11	$(23,7,12)$	i	d_4	$h_1 c_0 e_0$
11	$(25,8,14)$	Pe_0	d_3	$h_1^2 c_0 e_0$
12	$(7,4,-5)$	$\tau^9 h_0^3 h_3$	d_5	$\tau^6 P h_2$
12	$(9,5,-3)$	$\tau^8 P h_1$	d_6	$\tau^5 h_0^2 d_0$
12	$(10,6,-2)$	$\tau^8 P h_1^2$	d_7	$\tau^4 P c_0$
12	$(14,2,2)$	$\tau^6 h_3^2$	d_6	$\tau^3 c_1$
12	$(15,8,3)$	$\tau^5 h_0^7 h_4$	d_5	$\tau^2 P^2 h_2$
12	$(17,9,5)$	$\tau^4 P^2 h_1$	d_6	$\tau P h_0^2 d_0$
12	$(18,10,6)$	$\tau^4 P^2 h_1^2$	d_7	$P^2 c_0$
12	$(23,12,11)$	$\tau h_0^5 i$	d_4	$P^2 h_1^2 c_0$
12	$(25,13,13)$	$P^3 h_1$	d_3	$P^2 h_1^3 c_0$
13	$(14,3,1)$	$\tau^7 h_0 h_3^2$	d_7	$\tau^4 g$
13	$(17,4,4)$	$\tau^6 e_0$	d_5	$\tau^4 h_1 g$
13	$(18,5,5)$	$\tau^6 h_1 e_0$	d_6	$\tau^3 h_0 h_2 g$
13	$(20,6,7)$	$\tau^5 h_0 h_2 e_0$	d_7	j
13	$(22,10,9)$	$\tau^3 P h_0^2 d_0$	d_3	$\tau P^2 c_0$
13	$(23,7,10)$	$\tau^2 i$	d_6	d_0^2
13	$(25,8,12)$	$\tau^2 P e_0$	d_5	$h_1 d_0^2$

Table 6: Some Massey products in $\mathrm{Ext}_{\mathbb{R}}$

coweight	(s,f,w)	bracket	contains	indeterminacy	proof	used in
3	$(3,1,0)$	$\langle \rho^2, \tau h_1, h_2 \rangle$	$\tau^2 h_2$	$\rho^4 h_3$	$d_2(\tau^2) = \rho^2 \tau h_1$	$\langle \rho^2, \tau\eta, \nu \rangle$, Lemma 8.4
4	$(8,3,4)$	$\langle c_0, h_0, \rho \rangle$	τc_0	$\rho\tau h_1 \cdot h_1 h_3$	$d_1(\tau) = \rho h_0$	$\langle \epsilon, \mathsf{h}, \rho \rangle$
7	$(7,1,0)$	$\langle \rho^4, \tau^2 h_2, h_3 \rangle$	$\tau^4 h_3$	$\rho^8 h_4$	$d_4(\tau^4) = \rho^4 \tau^2 h_2$	$\langle \rho^4, \tau^2\nu, \sigma \rangle$
9	$(21,5,12)$	$\langle \tau h_1, h_1^4, h_4 \rangle$	$h_2 f_0$	0	$\mathbb{C}$-motivic	Lemma 6.10
9	$(21,5,12)$	$\langle \rho, h_2 e_0, h_1 \rangle$	$h_2 f_0$	$\rho^2 h_2 g$	$d_1(\tau g) = \rho h_2 e_0$	$\langle \rho, \{h_2 e_0\}, \eta \rangle$
10	$(18,4,8)$	$\langle \tau^2 h_2, h_3, h_0^2 h_3 \rangle$	$\tau^2 f_0$	$\tau^2 h_2 \cdot h_0^2 h_4,\ \rho^5 h_4 c_0$	$\mathbb{C}$-motivic	Lemma 7.13
10	$(21,5,11)$	$\langle \tau h_2^2, h_3, h_0^2 h_3 \rangle$	$\tau^2 h_1 g$	$\rho^3 h_1 h_4 c_0$	$\mathbb{C}$-motivic	Lemma 7.13
11	$(3,1,-8)$	$\langle \rho^2, \tau^9 h_1, h_2 \rangle$	$\tau^{10} h_2$	0	$d_2(\tau^{10}) = \rho^2 \tau^9 h_1$	$\langle \rho^2, \tau^9\eta, \nu \rangle$
11	$(9,4,-2)$	$\langle \tau h_1 \cdot \tau^5 c_0, \tau h_1, \rho^2 \rangle$	$h_1 \cdot \tau^8 c_0$	0	$d_2(\tau^2) = \rho^2 \tau h_1$	Lemma 7.14
11	$(11,5,0)$	$\langle \rho^2, \tau^5 h_1, P h_2 \rangle$	$\tau^6 P h_2$	$\rho^{16} h_3 g$	$d_2(\tau^6) = \rho^2 \tau^5 h_1$	$\langle \rho^2, \tau^5\eta, \zeta_{11} \rangle$
11	$(14,6,3)$	$\langle h_1, \tau^4 P h_2, \tau h_1 \rangle$	$\tau^5 h_0^2 d_0$	0	$\mathbb{C}$-motivic	Lemma 7.14
11	$(17,8,6)$	$\langle \tau h_1 \cdot \tau P c_0, \tau h_1, \rho^2 \rangle$	$h_1 \cdot \tau^4 P c_0$	0	$d_2(\tau^2) = \rho^2 \tau h_1$	Lemma 7.14
11	$(19,3,8)$	$\langle \rho, h_0, \tau^2 c_1 \rangle$	$\tau^3 c_1$	$\rho^2 \tau^2 h_2 \cdot h_2 h_4$	$d_1(\tau) = \rho h_0$	$\langle \rho, \mathsf{h}, \tau^2 \overline{\sigma} \rangle$
11	$(19,3,8)$	$\langle \rho^2, \tau h_1, \tau c_1 \rangle$	$\tau^3 c_1$	$\rho^2 \tau^2 h_2 \cdot h_2 h_4$	$d_2(\tau^2) = \rho^2 \tau h_1$	Lemma 7.15
11	$(19,9,8)$	$\langle \rho^2, \tau h_1, P^2 h_2 \rangle$	$\tau^2 P^2 h_2$	0	$d_2(\tau^2) = \rho^2 \tau h_1$	$\langle \rho^2, \tau\eta, \zeta_{19} \rangle$
11	$(22,4,11)$	$\langle \tau h_1, \tau c_1, h_1 \rangle$	$h_2 \cdot \tau^2 c_1$	$\rho h_4 \cdot \tau c_0$	$\mathbb{C}$-motivic	Lemma 7.15
11	$(22,10,11)$	$\langle h_1, P^2 h_2, \tau h_1 \rangle$	$\tau P h_0^2 d_0$	0	$\mathbb{C}$-motivic	Lemma 7.14
12	$(20,4,8)$	$\langle \rho, \tau^2 h_0, \rho, h_2 e_0 \rangle$	$\tau^4 g$	$\rho^2 h_2 \cdot \tau^3 c_1$	$d_1(\tau^3) = \rho\tau^2 h_0$, $d_1(\tau g) = \rho h_2 e_0$	$\langle \rho, \tau^2 \mathsf{h}, \rho, \{h_2 e_0\} \rangle$

Table 7: Hidden h_0 extensions in the ρ-Bockstein spectral sequence

coweight	(s,f,w)	source	target
1	$(1,1,0)$	τh_1	$\rho\tau h_1^2$
3	$(3,3,0)$	$\tau^2 h_0^2 h_2$	$\rho^6 h_1 c_0$
3	$(7,4,4)$	$h_0^3 h_3$	$\rho^3 h_1^2 c_0$
4	$(6,2,2)$	$\tau^2 h_2^2$	$\rho^2 \tau c_0$
4	$(8,3,4)$	τc_0	$\rho\tau h_1 c_0$
5	$(1,1,-4)$	$\tau^5 h_1$	$\rho\tau^5 h_1^2$
5	$(7,4,2)$	$\tau^2 h_0^3 h_3$	$\rho^2 \tau P h_1$
5	$(9,4,4)$	$\tau^2 h_1 c_0$	$\rho^2 P h_2$
5	$(9,5,4)$	$\tau P h_1$	$\rho\tau P h_1^2$
6	$(6,2,0)$	$\tau^4 h_2^2$	$\rho^3 \tau^3 h_2^3$
6	$(14,6,8)$	$h_0^2 d_0$	$\rho^3 h_1^3 d_0$
7	$(3,3,-4)$	$\tau^6 h_0^2 h_2$	$\rho^{14} e_0$
7	$(7,4,0)$	$\tau^4 h_0^3 h_3$	$\rho^{11} h_1 e_0$
7	$(11,7,4)$	$\tau^2 P h_0^2 h_2$	$\rho^{10} h_1^4 e_0$
7	$(15,8,8)$	$h_0^7 h_4$	$\rho^7 h_1^5 e_0$
8	$(8,3,0)$	$\tau^5 c_0$	$\rho\tau^5 h_1 c_0$
8	$(14,3,6)$	$\tau^2 h_0 h_3^2$	$\rho^4 f_0$
8	$(14,6,6)$	$\tau^2 h_0^2 d_0$	$\rho^2 \tau P c_0$
8	$(16,7,8)$	$\tau P c_0$	$\rho\tau P h_1 c_0$
9	$(1,1,-8)$	$\tau^9 h_1$	$\rho\tau^9 h_1^2$
9	$(7,4,-2)$	$\tau^6 h_0^3 h_3$	$\rho^2 \tau^5 P h_1$
9	$(9,3,0)$	$\tau^6 h_1^2 h_3$	$\rho^8 \tau^2 e_0$
9	$(9,4,0)$	$\tau^6 h_1 c_0$	$\rho^2 \tau^4 P h_2$
9	$(9,5,0)$	$\tau^5 P h_1$	$\rho\tau^5 P h_1^2$
9	$(15,8,6)$	$\tau^2 h_0^7 h_4$	$\rho^2 \tau P^2 h_1$
9	$(17,8,8)$	$\tau^2 P h_1 c_0$	$\rho^2 P^2 h_2$
9	$(17,9,8)$	$\tau P^2 h_1$	$\rho\tau P^2 h_1^2$
10	$(14,3,4)$	$\tau^4 h_0 h_3^2$	$\rho^4 \tau^2 f_0$
10	$(18,5,8)$	$\tau^2 h_0 f_0$	$\rho^5 \tau h_2^2 e_0$
10	$(20,6,10)$	$\tau^2 h_0 h_2 e_0$	$\rho^5 c_0 e_0$
11	$(3,3,-8)$	$\tau^{10} h_0^2 h_2$	$\rho^6 \tau^8 h_1 c_0$
11	$(7,4,-4)$	$\tau^8 h_0^3 h_3$	$\rho^4 \tau^6 P h_2$
11	$(11,7,0)$	$\tau^6 P h_0^2 h_2$	$\rho^6 \tau^4 P h_1 c_0$
11	$(15,8,4)$	$\tau^4 h_0^7 h_4$	$\rho^4 \tau^2 P^2 h_2$
11	$(19,3,8)$	$\tau^3 c_1$	$\rho^3 \tau^2 h_2 c_1$
11	$(19,11,8)$	$\tau^2 P^2 h_0^2 h_2$	$\rho^6 P^2 h_1 c_0$
11	$(23,12,12)$	$h_0^5 i$	$\rho^3 P^2 h_1^2 c_0$
12	$(6,2,-6)$	$\tau^{10} h_2^2$	$\rho^2 \tau^9 c_0$
12	$(8,3,-4)$	$\tau^9 c_0$	$\rho\tau^9 h_1 c_0$
12	$(14,3,2)$	$\tau^6 h_0 h_3^2$	$\rho^6 \tau^4 g$
12	$(14,6,2)$	$\tau^6 h_0^2 d_0$	$\rho^2 \tau^5 P c_0$
12	$(16,7,4)$	$\tau^5 P c_0$	$\rho\tau^5 P h_1 c_0$
12	$(18,5,6)$	$\tau^6 h_0 f_0$	$\rho^5 \tau^3 h_2^2 e_0$
12	$(20,6,8)$	$\tau^4 h_0^2 g$	$\rho^6 j$

Table 7: Hidden h_0 extensions in the ρ-Bockstein spectral sequence

coweight	(s, f, w)	source	target
12	$(22, 10, 10)$	$\tau^2 P h_0^2 d_0$	$\rho^2 \tau P^2 c_0$
12	$(24, 11, 12)$	$\tau P^2 c_0$	$\rho \tau P^2 h_1 c_0$
12	$(26, 9, 14)$	$h_0^2 j$	$\rho^4 h_1^2 d_0^2$

Table 8: Hidden h_1 extensions in the ρ-Bockstein spectral sequence

coweight	(s, f, w)	source	target	proof
2	$(0, 1, -2)$	$\tau^2 h_0$	$\rho \tau^2 h_1^2$	
3	$(3, 1, 0)$	$\tau^2 h_2$	$\rho^2 \tau h_2^2$	
3	$(6, 2, 3)$	τh_2^2	ρc_0	
5	$(9, 4, 4)$	$\tau^2 h_1 c_0$	$\rho P h_2$	
6	$(0, 1, -6)$	$\tau^6 h_0$	$\rho \tau^6 h_1^2$	
6	$(9, 3, 3)$	$\tau^3 h_2^3$	$\rho^4 d_0$	Lemma 7.12
7	$(14, 3, 7)$	$\tau h_0 h_3^2$	$\rho^2 e_0$	
9	$(9, 3, 0)$	$\tau^6 h_1^2 h_3$	$\rho^7 \tau^2 e_0$	
9	$(9, 4, 0)$	$\tau^6 h_1 c_0$	$\rho \tau^4 P h_2$	
9	$(17, 8, 8)$	$\tau^2 P h_1 c_0$	$\rho P^2 h_2$	
9	$(18, 5, 9)$	$\tau^2 h_1 e_0$	$\rho \tau h_2^2 d_0$	
10	$(0, 1, -10)$	$\tau^{10} h_0$	$\rho \tau^{10} h_1^2$	
10	$(14, 2, 4)$	$\tau^4 h_3^2$	$\rho^4 \tau^2 c_1$	
10	$(18, 4, 8)$	$\tau^2 f_0$	$\rho^2 \tau^2 h_1 g$	Lemma 7.13
10	$(19, 3, 9)$	$\tau^2 c_1$	$\rho^2 \tau h_2 c_1$	
11	$(3, 1, -8)$	$\tau^{10} h_2$	$\rho^2 \tau^9 h_2^2$	
11	$(6, 2, -5)$	$\tau^9 h_2^2$	$\rho \tau^8 c_0$	
11	$(9, 4, -2)$	$\tau^8 h_1 c_0$	$\rho \tau^6 P h_2$	Lemma 7.14
11	$(11, 5, 0)$	$\tau^6 P h_2$	$\rho^2 \tau^5 h_0^2 d_0$	Lemma 7.14
11	$(14, 6, 3)$	$\tau^5 h_0^2 d_0$	$\rho \tau^4 P c_0$	
11	$(17, 8, 6)$	$\tau^4 P h_1 c_0$	$\rho \tau^2 P^2 h_2$	Lemma 7.14
11	$(19, 3, 8)$	$\tau^3 c_1$	$\rho^2 \tau^2 h_2 c_1$	Lemma 7.15
11	$(19, 9, 8)$	$\tau^2 P^2 h_2$	$\rho^2 \tau P h_0^2 d_0$	Lemma 7.14
11	$(22, 10, 11)$	$\tau P h_0^2 d_0$	$\rho P^2 c_0$	
12	$(21, 5, 9)$	$\tau^4 h_1 g$	$\rho \tau^3 h_2^2 e_0$	
12	$(22, 9, 10)$	$\tau^2 P h_0 d_0$	$\rho \tau^2 P h_1^2 d_0$	
12	$(23, 6, 11)$	$\tau^3 h_2^2 e_0$	$\rho^2 j$	Lemma 7.16
12	$(26, 7, 14)$	j	ρd_0^2	Lemma 7.16

Table 9: Multiplicative generators of $\pi^{\mathbb{R}}_{*,*}$

coweight	(s,w)	element	detected by
0	$(-1,-1)$	ρ	ρ
0	$(0,0)$	h	h_0
0	$(1,1)$	η	h_1
1	$(1,0)$	$\tau\eta$	τh_1
1	$(3,2)$	ν	h_2
2	$(0,-2)$	$\tau^2\mathsf{h}$	$\tau^2 h_0$
3	$(3,0)$	$\tau^2\nu$	$\tau^2 h_2$
3	$(6,3)$	$\tau\nu^2$	τh_2^2
3	$(7,4)$	σ	h_3
3	$(8,5)$	ϵ	c_0
4	$(0,-4)$	$\tau^4\mathsf{h}$	$\tau^4 h_0$
4	$(8,4)$	$\tau\epsilon$	τc_0
5	$(1,-4)$	$\tau^5\eta$	$\tau^5 h_1$
5	$(9,4)$	$\tau\mu_9$	$\tau P h_1$
5	$(11,6)$	ζ_{11}	$P h_2$
6	$(0,-6)$	$\tau^6\mathsf{h}$	$\tau^6 h_0$
6	$(14,8)$	κ	d_0
7	$(7,0)$	$\tau^4\sigma$	$\tau^4 h_3$
7	$(11,4)$	$\tau^2\zeta_{11}$	$\rho^6 e_0$
7	$(14,7)$	$\tau\sigma^2$	ρh_4
7	$(15,8)$	ρ_{15}	$h_0^3 h_4$
7	$(16,9)$	η_4	$h_1 h_4$
8	$(0,-8)$	$\tau^8\mathsf{h}$	$\tau^8 h_0$
8	$(8,0)$	$\tau^5\epsilon$	$\tau^5 c_0$
8	$(14,6)$	$\tau^2\sigma^2$	$\tau^2 h_3^2$
8	$(16,8)$	$\tau\eta_4$	$\tau h_1 \cdot h_4$
8	$(17,9)$	$\tau\nu\kappa$	ρf_0
8	$(18,10)$	ν_4	$h_2 h_4$
8	$(19,11)$	$\bar{\sigma}$	c_1
8	$(20,12)$	$\{h_2 e_0\}$	$h_2 e_0$
9	$(1,-8)$	$\tau^9\eta$	$\tau^9 h_1$
9	$(9,0)$	$\tau^5\mu_9$	$\tau^5 P h_1$
9	$(11,2)$	$\tau^4\zeta_{11}$	$\tau^4 P h_2$
9	$(15,6)$	$\tau^3\eta\kappa$	$\rho^2\tau^2 e_0$
9	$(17,8)$	$\tau\mu_{17}$	$\tau P^2 h_1$
9	$(19,10)$	$\tau\overline{\sigma}$	τc_1
9	$(19,10)$	ζ_{19}	$P^2 h_2$
9	$(21,12)$	$\tau\eta\overline{\kappa}$	$h_2 f_0$
9	$(23,14)$	$\nu\overline{\kappa}$	$h_2 g$
10	$(0,-10)$	$\tau^{10}\mathsf{h}$	$\tau^{10} h_0$
10	$(15,5)$	$\tau^4\eta\kappa$	$\rho^3\tau^2 f_0$
10	$(18,8)$	$\tau^2\nu_4$	$\tau^2 h_2 \cdot h_4$
10	$(19,9)$	$\tau^2\overline{\sigma}$	$\tau^2 c_1$
10	$(20,10)$	$\tau^2\mathsf{h}\overline{\kappa}$	$h_2 \cdot \tau^2 e_0$

Table 9: Multiplicative generators of $\pi^{\mathbb{R}}_{*,*}$

coweight	(s, w)	element	detected by
10	$(21, 11)$	$\tau\nu\nu_4$	$\tau h_2^2 \cdot h_4$
11	$(3, -8)$	$\tau^{10}\nu$	$\tau^{10}h_2$
11	$(6, -5)$	$\tau^9\nu^2$	$\tau^9 h_2^2$
11	$(8, -3)$	$\tau^8\epsilon$	$\tau^8 c_0$
11	$(11, 0)$	$\tau^6\zeta_{11}$	$\tau^6 Ph_2$
11	$(15, 4)$	$\tau^4\rho_{15}$	$\tau^4 h_0^3 h_4$
11	$(17, 6)$	$\tau^4\nu\kappa$	$\tau^2 h_0 \cdot \tau^2 e_0$
11	$(19, 8)$	$\tau^3\overline{\sigma}$	$\tau^3 c_1$
11	$(19, 8)$	$\tau^2\zeta_{19}$	$\tau^2 P^2 h_2$
11	$(23, 12)$	ρ_{23}	$h_0^2 i$
11	$(26, 15)$	$\tau\nu^2\overline{\kappa}$	$\rho h_3 g$
11	$(28, 17)$	$\{h_1 h_3 g\}$	$h_1 h_3 g$

Table 10: Some Toda brackets in $\pi_{*,*}$

coweight	(s,w)	bracket	detected by	proof	used in
3	$(3,0)$	$\langle \rho^2, \tau\eta, \nu \rangle$	$\tau^2 h_2$	$\langle \rho^2, \tau h_1, h_2 \rangle$	Table 11
4	$(8,4)$	$\langle \epsilon, \mathsf{h}, \rho \rangle$	τc_0	$\langle c_0, h_0, \rho \rangle$	Table 11
7	$(7,0)$	$\langle \rho^4, \tau^2\nu, \sigma \rangle$	$\tau^4 h_3$	$\langle \rho^4, \tau^2 h_2, h_3 \rangle$	Table 11
7	$(14,7)$	$\langle \rho, \mathsf{h}, \sigma^2 \rangle$	ρh_4	$d_2(h_4) = h_0 h_3^2$	Lemma 10.2
8	$(8,0)$	$\langle \tau^5\eta, \mathsf{h}\nu, \nu \rangle$	$\tau^5 c_0$	$\mathbb{C}$-motivic	Table 11
8	$(14,6)$	$\langle \tau^2\nu, \sigma, \nu \rangle$	$\tau^2 h_3^2$	$\mathbb{C}$-motivic	Table 11, Lemma 9.9
8	$(16,8)$	$\langle \sigma^2, 2, \tau\eta \rangle$	$\tau h_1 \cdot h_4$	$d_2(h_4) = (h_0 + \rho h_1) h_3^2$	Table 11
8	$(16,8)$	$\langle \tau\mu_9, \mathsf{h}\nu, \nu \rangle$	$\tau P c_0$	$\mathbb{C}$-motivic	Table 11
8	$(17,9)$	$\langle \rho, \mathsf{h}, \nu\kappa \rangle$	ρf_0	$d_2(f_0) = h_0^2 e_0$	Lemma 10.3
8	$(18,10)$	$\langle \nu, \sigma, \mathsf{h}\sigma \rangle$	$h_2 h_4$	$d_2(h_4) = h_0 h_3^2$	Table 11
9	$(15,6)$	$\langle \rho, \rho\tau\eta, \tau\eta \cdot \kappa \rangle$	$\rho^2 \tau^2 e_0$	$d_2(\tau^2 e_0) = \tau^2 h_1^2 d_0$	Table 11
9	$(21,12)$	$\langle \rho, \{h_2 e_0\}, \eta \rangle$	$h_2 f_0$	$\langle \rho, h_2 e_0, h_1 \rangle$	Lemma 9.4
9	$(21,13)$	$\langle \{h_2 e_0\}, \eta, \mathsf{h} \rangle$	$c_0 d_0$	$\mathbb{C}$-motivic	Lemma 9.4
10	$(18,8)$	$\langle \rho^2, \tau\eta, \nu_4 \rangle$	$\tau^2 h_2 \cdot h_4$	Lemma 8.4	Table 11
10	$(19,9)$	$\langle \tau^2\nu, \eta\sigma, \sigma \rangle$	$\tau^2 c_1$	$\mathbb{C}$-motivic	Table 11
11	$(3,-8)$	$\langle \rho^2, \tau^9\eta, \nu \rangle$	$\tau^{10} h_2$	$\langle \rho^2, \tau^9 h_1, h_2 \rangle$	Table 11
11	$(11,0)$	$\langle \rho^2, \tau^5\eta, \zeta_{11} \rangle$	$\tau^6 P h_2$	$\langle \rho^2, \tau^5 h_1, P h_2 \rangle$	Table 11
11	$(19,8)$	$\langle \rho^2, \tau\eta, \zeta_{19} \rangle$	$\tau^2 P^2 h_2$	$\langle \rho^2, \tau h_1, P^2 h_2 \rangle$	Table 11
11	$(19,8)$	$\langle \rho, \mathsf{h}, \tau^2\bar{\sigma} \rangle$	$\tau^3 c_1$	$\langle \rho, h_0, \tau^2 c_1 \rangle$	Table 11
12	$(8,-4)$	$\langle \tau^9\eta, \mathsf{h}\nu, \nu \rangle$	$\tau^9 c_0$	$\mathbb{C}$-motivic	Table 11
12	$(16,4)$	$\langle \sigma^2, 2, \tau^5\eta \rangle$	$\tau^5 h_1 \cdot h_4$	$d_2(h_4) = (h_0 + \rho h_1) h_3^2$	Table 11
12	$(16,4)$	$\langle \tau^5\mu_9, \mathsf{h}\nu, \nu \rangle$	$\tau^5 P c_0$	$\mathbb{C}$-motivic	Table 11
12	$(20,8)$	$\langle \rho, \tau^2\mathsf{h}, \rho, \{h_2 e_0\} \rangle$	$\tau^4 g$	$\langle \rho, \tau^2 h_0, \rho, h_2 e_0 \rangle$	Table 11
12	$(24,12)$	$\langle \tau\mu_{17}, \mathsf{h}\nu, \nu \rangle$	$\tau P^2 c_0$	$\mathbb{C}$-motivic	Table 11

Table 11: Some permanent cycles in the $\mathbb{R}$-motivic Adams spectral sequence

coweight	(s,f,w)	element	proof
3	$(3,1,0)$	$\tau^2 h_2$	$\langle \rho^2, \tau\eta, \nu \rangle$
4	$(8,3,4)$	τc_0	$\langle \epsilon, \mathsf{h}, \rho \rangle$
7	$(7,1,0)$	$\tau^4 h_3$	$\langle \rho^4, \tau^2\nu, \sigma \rangle$
7	$(11,4)$	$\rho^6 e_0$	Lemma 8.14
8	$(8,3,0)$	$\tau^5 c_0$	$\langle \tau^5\eta, \mathsf{h}\nu, \nu \rangle$
8	$(14,6)$	$\tau^2 h_3^2$	$\langle \tau^2\nu, \sigma, \nu \rangle$
8	$(16,7,8)$	$\tau P c_0$	$\langle \tau\mu_9, \mathsf{h}\nu, \nu \rangle$
8	$(16,2,8)$	$\tau h_1 \cdot h_4$	$\langle \sigma^2, 2, \tau\eta \rangle$
8	$(18,2,10)$	$h_2 h_4$	$\langle \nu, \sigma, \mathsf{h}\sigma \rangle$
9	$(15,4,6)$	$\rho^2\tau^2 e_0$	$\langle \rho, \rho\tau\eta, \tau\eta \cdot \kappa \rangle$
10	$(18,2,8)$	$\tau^2 h_2 \cdot h_4$	$\langle \rho^2, \tau\eta, \nu_4 \rangle$
10	$(19,3,9)$	$\tau^2 c_1$	$\langle \tau^2\nu, \eta\sigma, \sigma \rangle$
11	$(3,1,-8)$	$\tau^{10} h_2$	$\langle \rho^2, \tau^9\eta, \nu \rangle$
11	$(11,5,0)$	$\tau^6 P h_2$	$\langle \rho^2, \tau^5\eta, \zeta_{11} \rangle$
11	$(19,3,8)$	$\tau^3 c_1$	$\langle \rho, \mathsf{h}, \tau^2\bar{\sigma} \rangle$
11	$(19,9,8)$	$\tau^2 P^2 h_2$	$\langle \rho^2, \tau\eta, \zeta_{19} \rangle$
11	$(23,4,12)$	$h_4 \cdot \tau c_0$	$\sigma \cdot \tau\eta_4$
12	$(8,3,-4)$	$\tau^9 c_0$	$\langle \tau^9\eta, \mathsf{h}\nu, \nu \rangle$
12	$(16,2,4)$	$\tau^5 h_1 \cdot h_4$	$\langle \sigma^2, 2, \tau^5\eta \rangle$
12	$(16,7,4)$	$\tau^5 P c_0$	$\langle \tau^5\mu_9, \mathsf{h}\nu, \nu \rangle$
12	$(20,4,8)$	$\tau^4 g$	$\langle \rho, \tau^2 h_0, \rho, h_2 e_0 \rangle$
12	$(24,11,12)$	$\tau P^2 c_0$	$\langle \tau\mu_{17}, \mathsf{h}\nu, \nu \rangle$

Table 12: Adams d_2 differentials

coweight	(s,f,w)	x	$d_2(x)$	proof
7	$(15,1,8)$	h_4	$h_0 h_3^2$	classical
7	$(17,4,10)$	e_0	$h_1^2 d_0$	classical
7	$(14,3,7)$	$\tau h_0 h_3^2$	$\rho^2 h_1 d_0$	Lemma 8.8
8	$(18,4,10)$	f_0	$h_0^2 e_0$	Lemma 8.9
9	$(17,4,8)$	$\tau^2 e_0$	$(\tau h_1)^2 d_0$	classical
10	$(18,4,8)$	$\tau^2 f_0$	$\tau^2 h_0^2 e_0 + \rho^3 \tau h_2^2 \cdot d_0$	Lemma 8.10
10	$(21,5,11)$	$\tau^2 h_1 g$	$\rho^2 c_0 d_0$	Lemma 8.11
11	$(23,8,12)$	$h_0 i$	$P h_0^2 d_0$	classical
11	$(27,5,16)$	$h_3 g$	$h_1^3 h_4 c_0$	$\mathbb{C}$-motivic
12	$(26,7,14)$	j	$P h_2 \cdot d_0$	classical

Table 13: Adams d_3 differentials

coweight	(s,f,w)	x	$d_r(x)$	proof
7	$(15,2,8)$	h_0h_4	$h_0d_0+\rho h_1d_0$	Lemma 8.16
12	$(23,6,11)$	$\tau h_2^2\cdot\tau^2e_0$	$\rho\tau Ph_1\cdot d_0$	Lemma 8.17
12	$(25,7,13)$	$c_0\cdot\tau^2e_0$	$\tau Ph_1\cdot h_1d_0$	Lemma 8.17

Table 14: Hidden ρ extensions in the ℝ-motivic Adams spectral sequence

coweight	(s,f,w)	source	target
7	$(15,4,8)$	$h_0^3h_4$	$\rho^4h_1e_0$
7	$(17,5,10)$	h_2d_0	$\tau h_1\cdot h_1d_0$
8	$(15,2,7)$	$\rho\tau h_1\cdot h_4$	$h_0\cdot\tau^2h_3^2$
8	$(15,4,7)$	ρ^3f_0	$\tau^2h_0\cdot d_0$
10	$(15,2,5)$	$\rho^3\tau^2h_2\cdot h_4$	$\tau^4h_3\cdot h_0h_3$
10	$(15,4,5)$	$\rho^3\tau^2f_0$	$\tau^4h_0\cdot d_0$
10	$(23,8,13)$	$h_1\cdot\tau c_0\cdot d_0$	Ph_0d_0
11	$(15,4,4)$	$\tau^4h_0\cdot h_0^2h_4$	$\tau^5h_0^2d_0$
11	$(17,5,6)$	$\tau^2h_0\cdot\tau^2e_0$	$\tau^5h_1\cdot h_1d_0$
11	$(18,5,7)$	$\rho^3f_0\cdot\tau^2h_2$	$h_0\cdot\tau^2h_0\cdot\tau^2e_0$
11	$(23,9,12)$	h_0^2i	$\tau Ph_0^2d_0$

Table 15: Hidden h extensions in the ℝ-motivic Adams spectral sequence

coweight	(s,w)	source	target
7	$(11,4)$	ρ^6e_0	$\tau^2h_0\cdot Ph_2$
9	$(21,12)$	h_2f_0	ρc_0d_0
9	$(23,14)$	h_0h_2g	$h_1c_0d_0$
10	$(22,12)$	$\tau c_0\cdot d_0$	Ph_0d_0
11	$(23,12)$	$\tau^2h_0\cdot h_2g$	$\tau Ph_1\cdot d_0$

Table 16: Hidden η extensions in the ℝ-motivic Adams spectral sequence

coweight	(s,f,w)	source	target
7	$(15,4,8)$	$h_0^3h_4$	$\rho^3h_1^2e_0$
9	$(15,5,6)$	$\tau^2h_0\cdot h_0^3h_4$	$\rho\tau h_1\cdot\tau Pc_0$
9	$(21,5,12)$	h_2f_0	c_0d_0
10	$(20,5,10)$	$h_2\cdot\tau^2e_0$	$\rho\tau c_0\cdot d_0$
10	$(21,7,11)$	$\rho\tau c_0\cdot d_0$	Ph_0d_0
11	$(15,4,4)$	$\tau^4h_0\cdot h_0^2h_4$	τ^4Pc_0
11	$(23,9,12)$	h_0^2i	P^2c_0

Table 17: Hidden values of extension by scalars

coweight	(s, f, w)	source	target
7	$(11, 4, 4)$	$\rho^6 e_0$	$\tau^2 P h_2$
7	$(14, 1, 7)$	ρh_4	τh_3^2
7	$(16+k, 6+k, 9+k)$	$\rho^3 h_1^{k+2} e_0$	$P h_1^k c_0$
8	$(17, 4, 9)$	ρf_0	$\tau h_2 d_0$
9	$(15, 4, 6)$	$\rho^2 \tau^2 e_0$	$\tau^3 h_1 d_0$
10	$(15, 4, 5)$	$\rho^3 \tau^2 f_0$	$\tau^4 h_1 d_0$
10	$(22, 7, 12)$	$\tau c_0 \cdot d_0$	$P d_0$
10	$(23, 8, 13)$	$h_1 \cdot \tau c_0 \cdot d_0$	$P h_1 d_0$
11	$(20, 5, 9)$	$\tau^2 h_2 \cdot \rho f_0$	$\tau^3 h_0^2 g$
11	$(26, 5, 15)$	$\rho h_3 g$	$\tau h_2^2 g$

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