LECTURES IN MATHEMATICS

Department of Mathematics
KYOTO UNIVERSITY

6

TYPICAL FORMAL GROUPS IN COMPLEX COBORDISM AND K-THEORY

BY

SHÔRÔ ARAKI

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CONTENTS

Introduction

The present lecture note consists of two parts.

Part I contains an exposition of Quillen's theory [18] of decompositions of complex cobordism theory localized at a prime p. Quillen's note [18] itself consists of two parts: the first part is connected with the proof of universality of the formal group of complex cobordism, of which detailed expositions are now available in literatures such as Adams [2], §§1-8, and Quillen [19], so I assumed these materials are known in the present lecture; the second part is the main subject of our Part I. The contents have much overlap with [2], but our exposition is given along original line of Quillen so it differs from the corresponding treatment of Adams [2] in its philosophy at least. We start with an exposition of Cartier's note [6] on the theory of (typical) curves over formal groups. This is done in §§2 and 3 in a form suitable for our purpose and restricting to one-dimensional case only. In §4 we discuss a typical formal group which is universal for typical formal groups, which turns out to be the formal group of Brown-Peterson cohomology (in §5). In §5 we prove Quillen decompositions. In §6 we discuss generators of $U^*(pt)$ and $BP^*(pt)$ in a form related with formal group. I believe this section contains some new results. Finally in §7 we discuss Landweber-Novikov type operations in Brown-Peterson cohomology.

In Part II we treat typical formal groups in (complex) K-theory and their relation to Adams' idempotent decomposition of K-theory localized at a prime p [1]. The results here were announced in [5].

These lecture notes came out of my lectures in Kyushu University, December 1972, Osaka City University, February and May 1973, and Kyoto University, July 1973. I acknowledge to Professors T.Kudo and H.Toda for their organizing my lectures in Kyushu University and Kyoto University, particularly to the latter for his arrangement to publish the present lecture notes as a part of "Lectures in mathematics, Department of Mathematics, Kyoto University".

Part I

§1. Formal groups

1.1. Let R be a commutative ring with unity. By a (one-dimensional commutative) formal group, or a group law, we understand a formal power series F in two variables over R satisfying

(1.1) $F(0, X) = X$,

(1.2) $F(X, Y) = F(Y, X)$,

(1.3) $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

Then F can be expressed as

(1.4) $F(X, Y) = X + Y + XY\bar{F}(X, Y)$

with $\bar{F}(X, Y) \in R[[X, Y]]$.

We are mainly interested in formal groups associated with cohomology theories which are complex oriented in the sense of [8], [19] (cf., 5.1). In such a case R is graded, i.e., $R = \sum R^i$, and F satisfies

(1.5) $\dim F(X, Y) = 2$ if $\dim X = \dim Y = 2$, i.e., if we put

$$F(X, Y) = \sum_{i,j} a_{ij} X^i Y^j$$

then $a_{ij} \in R^{2(1-i-j)}$ (cf., 5.2).

1.2. Let F and F' be formal groups over R, and ψ a formal power series over R in one variable without constant term satisfying

(1.6) $\psi(F(X, Y)) = F'(\psi(X), \psi(Y))$,

then we call ψ a homomorphism,

$$\psi : F \longrightarrow F',$$

of formal groups.

When $\psi : F \longrightarrow F'$ and $\varphi : F' \longrightarrow F''$, then $\varphi \circ \psi : F \longrightarrow F''$, where $\varphi \circ \psi$ is the composition of formal power series. Thus formal groups over R and their homomorphisms form a category, which will be denoted by $\mathcal{F}(R)$. When $\psi : F \longrightarrow F'$ and ψ is invertible with respect to composition, then

$$\psi^{-1} : F' \longrightarrow F$$

such that $\psi^{-1} \circ \psi = 1_F$ and $\psi \circ \psi^{-1} = 1_{F'}$, where $1_F(T) = 1_{F'}(T) = T$. Thus ψ is an <u>isomorphism</u> in the category $\mathcal{F}(R)$, denoted by

$$\psi : F \underset{\rightarrow}{\sim} F'.$$

In particular, when

$$\psi(T) = T + \text{higher terms},$$

we call ψ a <u>strict isomorphism</u> which we denote by

$$\psi : F \underset{\rightarrow}{\simeq} F'.$$

We denote the set of all homomorphisms $F \longrightarrow F'$ by $\mathrm{Hom}_R(F, F')$ and put $\mathrm{End}_R(F) = \mathrm{Hom}_R(F, F)$.

<u>1.3.</u> Let $\theta : R \longrightarrow S$ be a homomorphism of commutative rings with unity. Let $\theta_* : R[[X, Y]] \longrightarrow S[[X, Y]]$ and $\theta_* : R[[T]] \longrightarrow S[[T]]$

be the homomorphisms of rings of formal power series induced by coefficient map θ, i.e., $\theta_*(\sum a_{ij}X^iY^j) = \sum\theta(a_{ij})X^iY^j$ and $\theta_*(\sum a_iT^j) = \sum\theta(a_i)T^i$. Since θ_* preserves also compositions, we see that, if $F \in \mathrm{obj}\,\mathcal{F}(R)$, then $\theta_*F \in \mathrm{obj}\,\mathcal{F}(S)$, and if $\psi \in \mathrm{Hom}_R(F, F')$, then $\theta_*\psi \in \mathrm{Hom}_S(\theta_*F, \theta_*F')$ and $\theta_*(\varphi \circ \psi) = \theta_*\varphi \circ \theta_*\psi$, i.e., $\theta_* ; \mathcal{F}(R) \longrightarrow \mathcal{F}(S)$ is a covariant functor.

Thus we obtained, roughly speaking, a functor $\mathcal{F}$ defined on the category of commutative rings with unity with values in a category whose objects are categories of formal groups and morphisms are covariant functors ($\mathcal{F}(\theta) = \theta_*$). Later we meet often with needs to restrict this functor either restricting the domain to a subcategory or the range, or both.

1.4. We recall some known results without proof.

A formal group F_U defined over a ring U is called universal if for any ring R and for any formal group F on R there exists a unique homomorphism $u : U \longrightarrow R$ of rings with unity such that $u_*F_U = F$. The existence of a universal formal group and the structure of the ring U was first established by Lazard [15]. The uniqueness of (F_U, U) up to equivalence follows by the general nature of "universality". The structure theorem of U says:

[Lazard's Theorem] $\quad U = \mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots]$,

a polynomial ring over integers with countable indeterminates $x_1, x_2, \ldots,$.

We call the ring U Lazard ring. For the benefit of topologists we mention that the Lazard ring U can be given as a graded ring, graded by non-positive even dimensions so that F_U satisfies the condition (1.5). In this case $\dim x_n = -2n$. Cf., also [2], §§5 and 7.

1.5. A formal group G_a given by

$$G_a(X, Y) = X + Y$$

is called additive. Such a formal group is defined over any ring R.

Let F be a formal group. A strict isomorphism

$$\ell_F : F \underset{\to}{\simeq} G_a$$

is called a logarithm of F.

[existence of logarithm] Let F be a formal group defined over a $\mathbb{Q}$-algebra R. There exists a unique logarithm $\ell_F : F \underset{\to}{\simeq} G_a$.

For the proofs, cf., [13], [15] and [9], p.69. The existence is essential, and the uniqueness is easy.

Let F be a formal group defined over a ring R and suppose that R is of characteristic zero, i.e., every prime is not a zero divisor in R. Then $R \subset R \otimes \mathbb{Q}$ and we can regard F as a formal group over $R \otimes \mathbb{Q}$ by extending the domain of coefficients. Now we have a unique logarithm

$$\ell_F : F \underset{\to}{\simeq} G_a \qquad \text{over } R \otimes \mathbb{Q}.$$

We often denote as $\ell_F = \log_F$, and call it the logarithm of F for simplicity. If we express as

$$\log_F T = \sum_{k \geq 0} m_k T^{k+1}, \qquad m_0 = 1,$$

then it is known that

$$(k + 1)\, m_k \in R.$$

For topologists this is familiar by Mischenko series in case $F = F_U$, and the general case follows by functoriality (cf., §2).

§2. Modules of curves

We describe here modules of curves on formal groups according to Cartier [6].

2.1. Let R be a commutative ring with unity. The ring of formal power series in one variable T, $R[[T]]$, is filtered by degrees, i.e.,

$$R[[T]] = R[[T]]_0 \supset R[[T]]_1 \supset \cdots \supset R[[T]]_n \supset \cdots ,$$

where $R[[T]]_n = \{f(T) = \sum f_i T^i \in R[[T]] : f_0 = \cdots = f_{n-1} = 0\}$. $R[[T]]$ is complete and Hausdorff with respect to this filtration topology. $R[[T]]_1$ is the submodule of $R[[T]]$ consisting of all power series without constant terms.

Let $F \in \operatorname{obj} \mathcal{F}(R)$. For $\gamma, \gamma' \in R[[T]]_1$ we define their sum $\gamma \overset{F}{+} \gamma'$ with respect to F by

(2.1) $$(\gamma \overset{F}{+} \gamma')(T) = F(\gamma(T), \gamma'(T)).$$

Proposition 2.1. $R[[T]]_1$ *with the sum* $\overset{F}{+}$ *is an abelian group.*

Proof. By (1.2) and (1.3) it follows the commutativity and associativity. Zero power series $0(T) = 0$ is the zero element by (1.1). There exists a unique power series

(2.2) $$\iota_F \in R[[T]]_1 \quad \text{satisfying} \quad F(T, \iota_F(T)) = 0.$$

Then, for any $\gamma \in R[[T]]_1$

$$\overset{F}{-} \gamma = \iota_F \circ \gamma$$

is the inverse of γ with respect to the addition $+^F$.

q. e. d.

Put $\mathcal{C}_F = (R[[T]]_1, +^F)$, the above additive group. We call an element of $\mathcal{C}_F$ a <u>curve</u> over F. Thus $\mathcal{C}_F$ is the additive group of curves over F. The curve γ_0, defined by $\gamma_0(T) = T$, plays an important role and will be called the <u>identity curve</u> (over F).

<u>2.2.</u> We remark that

(2.3) $$\iota_F \in \mathrm{End}_R(F).$$

This is proved by observing that there exists a unique power series $\gamma(X, Y)$ satisfying $F(F(X, Y), \gamma(X, Y)) = 0$ and that both $\iota_F(F(X, Y))$ and $F(\iota_F(X), \iota_F(Y))$ satisfy the property of $\gamma(X, Y)$.

We embed $\mathrm{Hom}_R(F', F)$ into $\mathcal{C}_F$ canonically. Then we see easily that $\mathrm{Hom}_R(F', F)$ is a subgroup of $\mathcal{C}_F$. And the map

$$\mathrm{Hom}_R(F', F) \times \mathcal{C}_{F'} \longrightarrow \mathcal{C}_F$$

is bi-additive. Thus $\mathrm{End}_R(F)$ is an additive subgroup of $\mathcal{C}_F$ and is a ring with composition as multiplication and with γ_0 as unity (non-commutative in general). Furthermore $\mathcal{C}_F$ is a left $\mathrm{End}_R(F)$-module.

<u>2.3.</u> There exists a unique homomorphism of additive groups

$$[\ ]_F : \mathbb{Z} \longrightarrow \mathcal{C}_F$$

such that $[\ ]_F(1) = \gamma_0$. We write $[\ ]_F(n) = [n]_F$ for any integer n.

We have

(2.4) $[1]_F = \gamma_0$, $[-1]_F = \iota_F$ and $[0]_F = 0$,

(2.5) $[n]_F(T) = F(T, [n-1]_F(T)) = F(\iota_F(T), [n+1]_F(T))$.

Remark that

$$\iota_F(T) = -T + \text{higher terms},$$

which follows from (1.4) and (2.2). Then we see by (2.5) that

(2.6) $[n]_F(T) = nT + \text{higher terms}.$

By (2.3) and (2.5) we see that

(2.7) $$[n]_F \in \mathrm{End}_R(F),$$

i.e., the map $[\ ]_F : \mathbb{Z} \longrightarrow \mathcal{C}_F$ factorizes as

$$[\ ]_F : \mathbb{Z} \longrightarrow \mathrm{End}_R(F) \subset \mathcal{C}_F.$$

In fact, the first factor of $[\ ]_F$ is a ring homomorphism because $[nm]_F = [n]_F \circ [m]_F$ as is easily seen; and the $\mathbb{Z}$-module structure of $\mathcal{C}_F$ is the same as that given by this ring homomorphism, i.e.,

(2.7)' $[n]_F \circ \gamma = n \cdot \gamma$, n times of γ in $\mathcal{C}_F$.

2.4. Let d be an integer which is a unit in R. By (2.6) $[d]_F$ is invertible. We define as

$$[1/d]_F = [d]_F^{-1} \in \mathrm{End}_R(F).$$

Suppose that R is a Λ-algebra, where Λ is a ring such that $Z \subset \Lambda \subset \mathbb{Q}$. For any $\lambda \in \Lambda$ express λ as a fraction $\lambda = a/b$ such that $(a, b) = 1$, then b is a unit in R and we define as

$$[\lambda]_F = [a]_F \circ [b]_F^{-1} \in \mathrm{End}_R(F).$$

This extends the ring homomorphism $[\ ]_F : \mathbb{Z} \longrightarrow \mathrm{End}_R(F)$ to the ring homomorphism

$$[\ ]_F : \Lambda \longrightarrow \mathrm{End}_R(F)$$

And we obtain

Proposition 2.2. _When_ R _is a_ Λ-_algebra with a ring_ Λ _such that_ $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$, _then_ $\mathcal{C}_F$ _is a left_ Λ-_module by_

$$\lambda \cdot \gamma = [\lambda]_F \circ \gamma$$

for $\lambda \in \Lambda$ _and_ $\gamma \in \mathcal{C}_F$.

_2.5._ Let F be a formal group over a ring R. We define three kinds of operators on $\mathcal{C}_F$.

i) $([a]\gamma)(T) = \gamma(aT)$, $a \in R$,

ii) $(\mathbb{v}_n\gamma)(T) = \gamma(T^n)$, $n \geq 1$,

iii) $(\mathbb{f}_n\gamma)(T) = \sum^F_{1 \leq i \leq n} \gamma(\zeta_i\, T^{1/n})$, $n \geq 1$,

where $\sum^F_{1 \leq i \leq n}$ is the summation in $\mathcal{C}_F$ and $\zeta_1, \ldots, \zeta_n$ are n-th roots of unity. $\mathbb{f}_n\gamma$ lies in $R[\zeta_1, \ldots, \zeta_n][[T^{1/n}]]$ in first glance. Since F is commutative, each coefficient of $\mathbb{f}_n\gamma$ is a symmetric polynomial of

$\zeta_1, \ldots, \zeta_n$, hence a polynomial of elementary symmetric polynomials $\sigma_1(\zeta)$, $\ldots, \sigma_n(\zeta)$ of $\zeta_1, \ldots, \zeta_n$. Put

$$(\mathfrak{f}_n\gamma)(T) = \sum_{d\geq 1} g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta))T^{d/n},$$

then $g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta))$ is a polynomial of homogeneous degree d with $\deg \sigma_i(\zeta) = i$. Now $\zeta_1, \ldots, \zeta_n$ are n-th roots of unity, whence

$$\sigma_1(\zeta) = \ldots = \sigma_{n-1}(\zeta) = 0, \qquad \sigma_n(\zeta) = (-1)^{n-1}.$$

Thus

$$g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) = 0 \qquad\qquad \text{if} \quad d \not\equiv 0 \pmod{n},$$

$$g_{nk}(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) = g_{nk}(0, \ldots, 0, (-1)^{n-1}) \in R,$$

and $\mathfrak{f}_n\gamma$ is a well-defined curve in $\mathcal{C}_F$.

Operators $[a]$ are called homotheties, $\mathfrak{v}_n$ are called shifting operators and $\mathfrak{f}_n$ are called Frobenius operators. Among three kinds of operators Frobenius operators may be regarded as the most important ones and are the only ones defined essentially depending on the formal group F, so we write sometimes as $\mathfrak{f}_n = \mathfrak{f}_{n,F}$ to clarify on what formal group they are considered.

We used notations $[\ ]$ and $[\ ]_F$ to mean entirely different objects (with or without suffix !). I hope there arises no confusion.

Proposition 2.3. Operators $[a]$, $\mathfrak{v}_n$ and $\mathfrak{f}_n$ are additive.

Proof follows from routine calculations.

Thus $\mathcal{C}_F$ is an operator-module. These operators satisfy certain universal relations (cf., Proposition 2.9 below).

2.6. Let $F, G \in \operatorname{obj} \mathcal{F}(R)$ and $\psi : F \longrightarrow G$ in $\mathcal{F}(R)$. We define

$$\psi_\# : \mathcal{C}_F \longrightarrow \mathcal{C}_G$$

by $(\psi_\# \ \gamma)(T) = (\psi \circ \gamma)(T)$.

Proposition 2.4. $\psi_\#$ is linear and commutes with operators $[a]$, $\mathfrak{v}_n$ and $\mathfrak{f}_n$, i.e., a homomorphism of operator-modules.

Proof follows by routine calculations.

In particular, operators $[a]$, $\mathfrak{v}_n$ and $\mathfrak{f}_n$ commute with operations of $\operatorname{End}_R(F)$ on $\mathcal{C}_F$, And we obtain

Proposition 2.5. When R is a Λ-algebra such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$, then operators $[a]$, $\mathfrak{v}_n$ and $\mathfrak{f}_n$ are endomorphisms of Λ-module $\mathcal{C}_F$, i.e., $\mathcal{C}_F$ is an operator-Λ-module, and $\psi_\# : \mathcal{C}_F \longrightarrow \mathcal{C}_G$, $\psi \in \operatorname{Hom}_R(F, G)$. is a homomorphism of operator-Λ-modules.

Now it is clear that " $F \longmapsto \mathcal{C}_F$, $\psi \longmapsto \psi_\#$ " is a covaiant functor on $\mathcal{F}(R)$ with values in the category of operator-modules. We denote this functor by $\mathcal{C}(R)$.

2.7. Let $\theta : R \longrightarrow S$ be a homomorphism of commutative rings with unity, and $F \in \operatorname{obj} \mathcal{F}(R)$.

Proposition 2.6. $\theta_* : \mathcal{C}_F \longrightarrow \mathcal{C}_{\theta_* F}$ is linear and commutes with operators $[a]$, v_n and $\mathfrak{f}_n$ in the sense that $\theta_* \circ [a] = [\theta(a)] \circ \theta_*$ and $\theta_* \circ \mathfrak{f}_{n,F} = \mathfrak{f}_{n,\theta_* F} \circ \theta_*$, i.e., θ_* is a homomorphism of operator-modules. When R is a Λ- algebra such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$, then θ_* is a homomorphism of operator-Λ- modules.

Proof follows again by routine calculations.

Remark also the commutativity

$$\begin{tikzcd} \mathcal{C}_F \arrow[r, "\psi_\#"] \arrow[d, "\theta_*"] & \mathcal{C}_G \arrow[d, "\theta_*"] \\ \mathcal{C}_{\theta_* F} \arrow[r, "(\theta_*\psi)_\#"] & \mathcal{C}_{\theta_* G} \end{tikzcd}$$

for $\theta : R \longrightarrow S$ and $\psi : F \longrightarrow G$. Thus θ_* is a natural homomorphism of functors : $\mathcal{C}(R) \longrightarrow \mathcal{C}(S) \circ \theta_*$.

2.8. Let R be a commutative ring with unity and $F \in \operatorname{obj} \mathfrak{F}(R)$. Put $\mathcal{C}_n = R[[T]]_n$ for $n \geq 1$. By definition we see immediately that $\mathcal{C}_n$ are subgroups of $\mathcal{C}_F = \mathcal{C}_1$. Thus we have a filtration of $\mathcal{C}_F$:

$$\mathcal{C}_F = \mathcal{C}_1 \supset \mathcal{C}_2 \supset \ldots\ldots \supset \mathcal{C}_n \supset \ldots\ldots .$$

We say that, for two power series $f, g \in R[[T]]$, $f \equiv g \mod \deg n$ iff $f - g \in R[[T]]_n$, i.e., they have the same terms of degree $< n$.

Lemma 2.7. Let $\gamma_1, \gamma_2 \in \mathcal{C}_F$. $\gamma_1 \equiv \gamma_2 \mod \mathcal{C}_n$ iff $\gamma_1 \equiv \gamma_2 \mod \deg n$.

Proof. Suppose that $\gamma_1 \equiv \gamma_2 \mod \mathcal{C}_n$. There exists a curve $\gamma' \in \mathcal{C}_n$

such that $\gamma_1 +^F \gamma' = \gamma_2$. Then

$$\begin{aligned}\gamma_2(T) &= F(\gamma_1(T), \gamma'(T)) \\ &\equiv \gamma_1(T) + \gamma'(T) \quad \text{mod deg } n+1 \\ &\equiv \gamma_1(T) \quad \text{mod deg } n.\end{aligned}$$

The converse will be proved by induction. The case $n = 1$ is trivial. Assume it is true for $n - 1$, and suppose $\gamma_1 \equiv \gamma_2$ mod deg n. Then $\gamma_1 \equiv \gamma_2$ mod deg $n - 1$, hence $\gamma_0' = \gamma_1 -^F \gamma_2 \in \mathcal{C}_{n-1}$ by assumption. Now

$$\begin{aligned}\gamma_1(T) &= F(\gamma'(T), \gamma_2(T)) \\ &\equiv \gamma'(T) + \gamma_2(T) \quad \text{mod deg } n \\ &\equiv a_{n-1} \cdot T^{n-1} + \gamma_2(T) \quad \text{mod deg } n,\end{aligned}$$

where $\gamma'(T) = a_{n-1}T^{n-1}$ + higher terms. Since γ_1 and γ_2 have the same terms of degree $n - 1$, we conclude that $a_{n-1} = 0$ and $\gamma' \in \mathcal{C}_n$.

q. e. d.

By the above lemma we conclude the following

Proposition 2.8. $\mathcal{C}_F$ <u>is complete and Hausdorff with respect to the above filtration topology</u>.

By definition we have

$$[a](\mathcal{C}_m) \subset \mathcal{C}_m, \quad \mathbf{v}_n(\mathcal{C}_m) \subset \mathcal{C}_{nm}, \quad \mathbf{f}_n(\mathcal{C}_m) \subset \mathcal{C}_{[m-1/n]+1}.$$

Thus all three kinds of operators are continuous with respect to the filtration topology of $\mathcal{C}_F$.

2.9. Let R be a commutative ring with unity and $F \in \mathrm{obj}\,\mathcal{F}(R)$.

Proposition 2.9. Among three kinds of operators on $\mathcal{C}_F$ there hold the following relations.

i) $[a][b] = [ab]$, $a, b \in R$,

ii) $\mathbf{v}_n \mathbf{v}_m = \mathbf{v}_{nm}$, $n \geq 1, m \geq 1$,

iii) $\mathbf{f}_n \mathbf{f}_m = \mathbf{f}_{nm}$, $n \geq 1, m \geq 1$,

iv) $\mathbf{v}_n[a^n] = [a]\mathbf{v}_n$, $n \geq 1, a \in R$,

v) $\mathbf{f}_n[a] = [a^n]\mathbf{f}_n$, $n \geq 1, a \in R$,

vi) $\mathbf{f}_n \mathbf{v}_n = n \cdot \mathrm{id}_{\mathcal{C}_F}$, $[1] = \mathbf{v}_1 = \mathbf{f}_1 = \mathrm{id}_{\mathcal{C}_F}$,

vii) if $(n, m) = 1$ then $\mathbf{f}_n \mathbf{v}_m = \mathbf{v}_m \mathbf{f}_n$,

viii) $[a] + [b] = \sum_{n \geq 1} \mathbf{v}_n[s_n(a,b)]\, \mathbf{f}_n$.

In the relation viii) of this proposition $s_n(X, Y)$ are symmetric polynomials of degree n over integers, which are defined recursively by the formula

$$X^n + Y^n = \sum_{d|n} d \cdot s_d(X, Y)^{n/d}.$$

The right hand side of viii) means an operator which sends each curve $\gamma \in \mathcal{C}_F$ to

$$\Big(\sum_{n \geq 1} \mathbf{v}_n[s_n(a, b)]\, \mathbf{f}_n\Big)\gamma = \sum_{n \geq 1}{}^{F}\, \mathbf{v}_n[s_n(a, b)]\, \mathbf{f}_n\gamma,$$

which is a Cauchy series in $\mathcal{C}_F$, hence convergent to a curve in $\mathcal{C}_F$ by

Proposition 2.8.

Proof. Relations i), ii), iii), iv) and v) follow by routine calculations.

For any $\gamma \in \mathcal{C}_F$, we have

$$\mathbf{f}_n \mathbf{v}_n \gamma = [n]_F \circ \gamma,$$

then by (2.7)' it follows the relation vi).

Suppose $(m, n) = 1$ and $\{\zeta_1, \ldots, \zeta_n\}$ be n-th roots of unity. Then $\{\zeta_1, \ldots, \zeta_n\} = \{\zeta_1^m, \ldots, \zeta_n^m\}$, by which follows the relation vii).

It remains only the proof of the relation viii). First we prove the relation for additive group laws, i.e., suppose $F = G_a$, an additive formal group. Remark that for $\gamma(T) = \sum_{i \geq 1} c_i T^i \in \mathcal{C}_{G_a}$ we have

$$\begin{aligned} ([a]\gamma)(T) &= \sum_{i \geq 1} c_i a^i T^i, & a \in R, \\ (\mathbf{v}_n \gamma)(T) &= \sum_{i \geq 1} c_i T^{ni}, & n \geq 1, \\ (\mathbf{f}_{n,G_a} \gamma)(T) &= n \sum_{i \geq 1} c_{ni} T^i, & n \geq 1, \end{aligned} \tag{2.8}$$

and the addition in $\mathcal{C}_{G_a}$ is the ordinary addition of formal power series. Then

$$\begin{aligned} ([a]\gamma + [b]\gamma)(T) &= \sum_{n \geq 1} c_n (a^n + b^n) T^n \\ &= \sum_{n \geq 1} c_n \Big(\sum_{n=dm} d \cdot s_d(a, b)^m \Big) T^n \\ &= \sum_{d \geq 1, m \geq 1} c_{dm}\, d \cdot s_d(a, b)^m T^{md} \\ &= \sum_{d \geq 1} v_d \Big(\sum_{m \geq 1} d\, c_{dm} \cdot s_d(a, b)^m T^m \Big) \end{aligned}$$

$$= \sum_{d\geq 1} v_d[s_d(a, b)](\sum_{m\geq 1} d\, c_{dm}T^m)$$
$$= \sum_{d\geq 1} (v_d[s_d(a, b)]\, f_d\gamma)(T),$$

i.e., the relation is proved for $F = G_a$.

Next suppose that R is of characteristic zero. Then there exists a unique logarithm

$$\ell_F : F \underset{\to}{\simeq} G_a \qquad \text{over} \quad R \otimes \mathbb{Q}$$

by 1.5. $\ell_{F\#} : \mathcal{C}_F \underset{\to}{\simeq} \mathcal{C}_{G_a}$ (over $R \otimes \mathbb{Q}$) is a topological isomorphism of operator-modules by Proposition 2.4. Thus $\ell_{F\#}^{-1}$ preserves the relations among operators and the relation viii) holds in $\mathcal{C}_F$ over $R \otimes \mathbb{Q}$. Since coefficients extension $R \subset R \otimes \mathbb{Q}$ embeds $\mathcal{C}_F$ over R into $\mathcal{C}_F$ over $R \otimes \mathbb{Q}$ as operator-module, the relation viii) is true in $\mathcal{C}_F$ when R is of characteristic zero.

The Lazard ring U is of characteristic zero (by Lazard's Theorem). We consider the universal formal group F_U over $U \otimes \mathbb{Z}[t, u]$ extending coefficients domain, where t and u are indeterminates. The relation viii) is true in $\mathcal{C}_{F_U}$ over $U \otimes \mathbb{Z}[t, u]$ by the above arguments. Let F be a formal group over an arbitrary ring R. By universality there exists a homomorphism $\theta : U \longrightarrow R$ such that $\theta_* F_U = F$. Let $a, b \in R$. Extend θ to a homomorphism $\tilde{\theta} : U \otimes \mathbb{Z}[t, u] \longrightarrow R$ by $\tilde{\theta}(t) = a$ and $\tilde{\theta}(u) = b$. Clearly $\tilde{\theta}_* F_U = F$. Now

$$\theta_* : \mathcal{C}_{F_U} \text{ over } U \otimes \mathbb{Z}[t, u] \longrightarrow \mathcal{C}_F$$

is a homomorphism of operator-modules by Proposition 2.6, and hence sends the relation viii) in $\mathcal{C}_{F_U}$ for the pair (t, u) to the relation viii) in $\mathcal{C}_F$ for the pair (a, b). Thus the proof is complete.

<u>2.10.</u> Let $F \in \mathrm{obj}\,\mathcal{F}(R)$.

Proposition 2.10. <u>Every curve</u> γ <u>over</u> F <u>can be expressed uniquely as a Cauchy series</u>

$$\gamma = \sum_{n \geq 1}^{F} \mathbb{V}_n[c_{n-1}]\gamma_0, \quad c_{n-1} \in R, \quad (\text{i.e., } \gamma(T) = \sum_{n \geq 1}^{F}(c_{n-1}T^n)).$$

Proof. Let

$$\gamma(T) = c_0 T + \text{higher terms.}$$

and put

$$\gamma_1 = \gamma -^F [c_0]\gamma_0.$$

Then by definitions we see easily that $\gamma_1 \in \mathcal{C}_2$. Now let

$$\gamma_1(T) = c_1 T^2 + \text{higher terms}$$

and put

$$\gamma_2 = \gamma_1 -^F \mathbb{V}_2[c_1]\gamma_0.$$

Then we obtain that $\gamma_2 \in \mathcal{C}_3$. By a recursive construction we obtain $\gamma_n \in \mathcal{C}_{n+1}$ and $\gamma_{n+1} = \gamma_n -^F \mathbb{V}_{n+1}[c_n]\gamma_0 \in \mathcal{C}_{n+2}$, and so on. Thereby we obtain a Cauchy series $\sum_{n \geq 1}^{F} \mathbb{V}_n[c_{n-1}]\gamma_0$ which converges to γ. The uniqueness is obvious by construction. q. e. d.

2.11. Let $F \in \mathrm{obj}\,\mathcal{F}(R)$, $\gamma \in R[[T]]_1$ and invertible. We put

$$F^{\gamma}(X, Y) = \gamma^{-1} \circ F(\gamma(X), \gamma(Y)).$$

Then we see easily that $F^{\gamma} \in \mathrm{obj}\,\mathcal{F}(R)$ and

$$\gamma : F^{\gamma} \xrightarrow{\sim} F.$$

We call F^{γ} the transpose of F by γ. Since $\gamma \in \mathrm{Hom}_R(F^{\gamma}, F) \subset \mathcal{C}_F$, it is natural to regard γ as an (invertible) curve over F when we consider the transpose of F by γ.

§3. Typical curves and formal groups

Let I be a set of primes. We use the notation I only to denote such a set of primes. The following special cases are the most important: $I = (p)$, the set of all primes except p; $I = [p]$, the set consisting of the single prime p.

We denote by $\mathbb{Z}_I$ the following subring of $\mathbb{Q}$:

$$\mathbb{Z}_I = Z[\frac{1}{q} : q \in I].$$

Thus $\mathbb{Z}_{(p)}$ = integers localized at the prime p,

$\mathbb{Z}_{[p]}$ = the ring consisting of rationals of the form a/p^k.

<u>3.1.</u> Let R be a commutative ring with unity and F a formal group over R.

A curve γ over F is called I-<u>typical</u> iff $\mathbf{f}_q\gamma = 0$ for all $q \in I$. F is called I-<u>typical</u> iff the identity curve γ_0 over F is I-<u>typical</u>. When $I = (p)$, we call simply <u>typical</u> in place of (p)-typical. Typical curves or formal groups are usually observed when R is a $\mathbb{Z}_{(p)}$-algebra.

Denote by $\mathcal{C}T_{F,I}$ the set of all I-typical curves over F. Clearly it is a subgroup of $\mathcal{C}_F$ and stable under operators $[a]$, $a \in R$, $\mathbf{v}_n$ and $\mathbf{f}_n$ such that $(n,q) = 1$ for all $q \in I$ by Proposition 2.9. We regard these operators as <u>allowable</u> operators on $\mathcal{C}T_{F,I}$. Then $\mathcal{C}T_{F,I}$ is an operator-module over allowable operators.

When $I = (p)$ we write simply $\mathcal{C}T_{F,(p)} = \mathcal{C}T_F$. In this case allowable operators are generated by $[a]$, $a \in R$, $\mathbb{V}_p$ and $\mathbb{f}_p$.

<u>3.2.</u> Suppose R is a $\mathbb{Z}_I$-algebra and define operators

$$e_q = e_{q,F} : \mathcal{C}_F \longrightarrow \mathcal{C}_F, \quad q \in I,$$

by $e_q(\gamma) = \gamma \overset{F}{-} (\frac{1}{q}) \mathbb{V}_q \mathbb{f}_q \gamma$. By Propositions 2.5 and 2.9 we see easily that e_q's are idempotents and mutually commutative. Moreover $e_q\gamma \equiv \gamma \mod \mathcal{C}_q$. Thus the product

$$\varepsilon_I = \varepsilon_{I,F} = \prod_{q \in I} e_q \tag{3.1}$$

is convergent and well-defined operator on $\mathcal{C}_F$. We have also a Cauchy sum expansion

$$\varepsilon_I \gamma = \sum_{n \text{ rel } I}^{F} \left(\frac{\mu(n)}{n}\right) \mathbb{V}_n \mathbb{f}_n \gamma \tag{3.2}$$

for $\gamma \in \mathcal{C}_F$, where $\mu(n)$ is the Möbius function and the summation runs over all natural numbers n of which every prime factor belongs to I (including $n = 1$).

Proposition 3.1. ε_I <u>is an idempotent and projects</u> $\mathcal{C}_F$ <u>onto the subgroup</u> $\mathcal{C}T_{F,I}$.

The proof is straight-forward if we remark that $\mathbb{f}_q\, e_q = 0$.

We call the operator ε_I <u>Cartier operator</u> over F. In particular the curve

$$\xi_I = \xi_{I,F} = \varepsilon_I \gamma_0 \tag{3.3}$$

is I-typical, which we regard as the canonical I-typical curve over F. By (3.2) we have

$$\xi_I(T) = T + \text{higher terms}.$$

Thus

(3.4) $$\xi_I : F^{\xi_I} \underset{\to}{\cong} F.$$

(cf., 2.11 for definition). By Proposition 2.4 $\xi_{I\#}$ maps I-typical curves to I-typical curves and vice-versa. Then, since $\xi_{I\#}\gamma_0 = \xi_I$, I-typical over F, we obtain

Proposition 3.2. F^{ξ_I} is an I-typical formal group which is strictly isomorphic to F.

We regard F^{ξ_I} as the I-typical formal group canonically associated to F.

3.3. Let R be a $\mathbb{Z}_I$-algebra. Let $\mathbb{N}$ be the set of all natural numbers 1, 2, ..., and put

$$\mathbb{N}_I'' = \{k \in \mathbb{N} \ ; \ (k, q) = 1 \quad \text{for all} \quad q \in I\},$$

$$\mathbb{N}_I' = \mathbb{N} - \mathbb{N}_I''.$$

First consider a curve

$$\gamma(T) = \sum_{k \geq 1} \gamma_{k-1} T^k$$

over G_a, the additive group law over R. By (2.8) we obtain

$$(e_{q,G_a}\gamma)(T) = \sum_{(k,q)=1} \gamma_{k-1} T^k$$

for $q \in I$. Thus

$$(3.5) \qquad (\varepsilon_{I,G_a}\gamma)(T) = \sum_{k \in \mathbb{N}''_I} \gamma_{k-1} T^k .$$

Next assume that R is of characteristic zero and $F \in \operatorname{obj} \mathcal{F}(R)$.

Proposition 3.3. <u>Let</u> ℓ <u>and</u> ℓ_I <u>be the logarithms of</u> F <u>and</u> F^{ξ_I} <u>respectively over</u> $R \otimes \mathbb{Q}$. <u>Put</u>

$$\ell(T) = \sum_{k \geq 1} m_{k-1} T^k ,$$

<u>then</u>

$$\ell_I(T) = \sum_{k \in \mathbb{N}''_I} m_{k-1} T^k .$$

Proof. By (3.4)

$$\ell \circ \xi_I : F^{\xi_I} \xrightarrow{\simeq} G_a .$$

Then, by the uniqueness of logarithm we have

$$\ell_I = \ell \circ \xi_I ,$$

and

$$\ell_I(T) = \ell \circ \xi_I(T) = \ell_\#(\varepsilon_{I,F}\gamma_0)(T)$$

$$= \varepsilon_{I,G_a}(\ell_\#\gamma_0)(T) = (\varepsilon_{I,G_a}\ell)(T).$$

Now by (3.5) the proof follows.

<u>3.4.</u> Let $F \in \operatorname{obj} \mathcal{F}(R)$ and consider $\mathcal{C}T_{F,I}$. Since Frobenius

operators are linear and continuous, we see easily that $\mathcal{C}T_{F,I}$ is closed in $\mathcal{C}_F$. Thus $\mathcal{C}T_{F,I}$ is complete and Hausdorff with respect to the induced filtrations $\mathcal{C}T_{F,I} \cap \mathcal{C}_n$.

Now suppose that R is a $\mathbb{Z}_I$-algebra.

Lemma 3.4. Let $\gamma \in \mathcal{C}_F$ such that

$$\gamma(T) = a \cdot T^k + \text{higher terms}, \qquad a \neq 0.$$

If γ is I-typical, then $k \in \mathbb{N}_I''$.

Proof. For any $q \in I$ we obtain

$$(\mathbf{f}_q\gamma)(T) = a \cdot (\zeta_1^k + \dots + \zeta_q^k) T^{k/q} + \text{higher terms},$$

where $\zeta_1, \dots, \zeta_q$ are q-th roots of 1. Since γ is I-typical, we have

$$a \cdot (\zeta_1^k + \dots + \zeta_q^k) = 0.$$

If $q|k$, then

$$\zeta_1^k + \dots + \zeta_q^k = q,$$

which is invertible and contradicts to the assumption. Thus $(q, k) = 1$.

q. e. d.

Lemma 3.5. Let F be I-typical. Then, for any $k \in \mathbb{N}_I''$, $a \neq 0$ in R, we have

$$\mathbf{v}_k[a]\gamma_0 \in \mathcal{C}T_{F,I}.$$

Proof. For any $q \in I$, $(q, k) = 1$. Thus $\mathbf{f}_q \mathbf{v}_k = \mathbf{v}_k \mathbf{f}_q$ by

Proposition 2.9, vii). And

$$\mathbf{f}_q \mathbf{v}_k[a]\gamma_0 = \mathbf{v}_k[a^q]\mathbf{f}_q\gamma_0 = 0.$$

q. e. d.

Theorem 3.6. <u>Let</u> R <u>be a</u> $\mathbb{Z}_I$<u>-algebra and</u> F <u>an</u> I-<u>typical formal group over</u> R. <u>A curve</u> γ <u>over</u> F <u>is</u> I-<u>typical iff it can be expressed as</u>

$$\gamma = \sum_{k\in\mathbb{N}''_I}^{F} \mathbf{v}_k[c_{k-1}]\gamma_0 \qquad (\text{or}\quad \gamma(T) = \sum_{k\in\mathbb{N}''_I}^{F} (c_{k-1}T^k))$$

<u>with</u> $c_{k-1} \in R$. <u>The expression is unique</u>.

Proof. Suppose γ is I-typical and express γ as a Cauch series

$$\gamma = \sum_{k\geq 1}^{F} \mathbf{v}_k[c_{k-1}]\gamma_0$$

in $\mathcal{C}_F$ by Proposition 2.10. Let c_{n-1} be the first non-zero coefficient in this expression. Then

$$\gamma(T) = c_{n-1}T^n + \text{higher terms},$$

and $n \in \mathbb{N}''_I$ by Lemma 3.4. Since $\mathbf{v}_n[c_{n-1}]\gamma_0$ is I-typical by Lemma 3.5 we see that

$$\gamma_1 = \gamma -^F \mathbf{v}_n[c_{n-1}]\gamma_0 = \sum_{k>n}^{F} \mathbf{v}_k[c_{k-1}]\gamma_0$$

is I-typical. Now apply the same argument to γ_1 and repeat. We see that $c_{k-1} = 0$ unless $k \in \mathbb{N}''_I$. Thus we obtain the desired expression.

The converse follows by Lemma 3.5 and the completeness of $\mathcal{C}T_{F,I}$.

The uniqueness of the expression follows by the uniqueness of Proposition 2.10.

§4. Universal typical formal groups

4.1. Let U be the Lazard ring and F_U the universal formal group over U. We regard U as the graded ring by non-positive even dimensions so that F_U satisfies the condition (1.5).

Let I be a set of primes and put $U_I = U \otimes \mathbb{Z}_I$, $F_{U,I} = F_U$ over U_I by coefficients extension. By the universality of F_U it follows immediately the universality of $F_{U,I}$ for formal groups over $\mathbb{Z}_I$-algebras.

Now we want to construct an I-typical formal group which is universal for I-typical formal groups over $\mathbb{Z}_I$-algebras (by restricting the range of the functor $\mathcal{F}$).

Let R be a $\mathbb{Z}_I$-algebras and $F \in \operatorname{obj} \mathcal{F}(R)$. There exists a unique homomorphism

$$\theta : U_I \longrightarrow R$$

of $\mathbb{Z}_I$-algebras such that $\theta_* F_{U,I} = F$. By Proposition 2.6 and the definition of Cartier operators we see that

$$\theta_* \varepsilon_{I,U} = \varepsilon_{I,F} \theta_*$$

where $\varepsilon_{I,U}$ denotes the Cartier operator over $F_{U,I}$. (Similar conventions apply also for other notations).

Put

$$\xi_{I,U}(T) = (\varepsilon_{I,U}\gamma_0)(T) = \sum_{k \geq 1}^{F_U} (\xi_{k-1} T^k).$$

By definitions we see that

$$(4.1) \qquad \xi_s \in U^{-2s} \quad \text{and} \quad \xi_0 = 1.$$

Since F is I-typical iff $\varepsilon_{I,F}\gamma_0 = \gamma_0$, and since

$$\theta_*\xi_{I,U} = \theta_*\varepsilon_{I,U}\gamma_0 = \varepsilon_{I,F}\gamma_0,$$

we obtain

Proposition 4.1. F <u>is</u> I-<u>typical iff</u>

$$\{\xi_1, \xi_2, \ldots, \xi_{n-1}, \ldots\} \subset \operatorname{Ker} \theta.$$

<u>4.2.</u> Put $\tilde{U} = U \otimes \mathbb{Q}$, then

$$U \subset U_I \subset \tilde{U}.$$

Put

$$\log_U T = \sum_{k \geq 1} m_{k-1} T^k, \qquad m_0 = 1,$$

where $\log_U$ is the logarithm of F_U (and of course of $F_{U,I}$) over $\tilde{U}$.

Then

$$(4.2) \qquad \tilde{U} = \mathbb{Q}[m_1, m_2, \ldots, m_k, \ldots], \qquad \dim m_k = -2k,$$

as is well known.

Put

$$\mu_{U,I} = \text{the transpose of } F_{U,I} \text{ by } \xi_{I,U}.$$

$\mu_{U,I}$ is I-typical by Proposition 3.2. (Hereafter we use the letter " μ "

to denote typical (or I-typical) formal groups in general). Let

$$u_I : U_I \longrightarrow U_I$$

be the unique homomorphism of $\mathbb{Z}_I$-algebras such that

$$u_{I*}F_{U,I} = \mu_{U,I}$$

and

$$\tilde{u}_I : \tilde{U} \longrightarrow \tilde{U}$$

the homomorphism of $\tilde{U}$ obtained from u_I by coefficient extension.

Apply $\tilde{u}_{I*}$ to the strict isomorphism

$$\log_U : F_U \overset{\approx}{\to} G_a \qquad \text{over } \tilde{U}$$

and obtain

$$\tilde{u}_{I*}(\log_U) : \mu_{U,I} \overset{\approx}{\to} G_a$$

by the functoriality. Thus

(4.3) $$\tilde{u}_{I*}(\log_U) = \log_{\mu_{U,I}}$$

by the uniqueness of logarithm. Then, by Proposition 3.3 we obtain

Proposition 4.2. $\tilde{u}_I(m_{k-1}) = 0$ <u>if</u> $k \in \mathbb{N}'_I$

$= m_{k-1}$ <u>if</u> $k \in \mathbb{N}''_I$.

Corollary 4.3. u_I <u>and</u> $\tilde{u}_I$ <u>are idempotents,</u>

$$\operatorname{Ker} \tilde{u}_I = \text{the ideal } (m_{\ell-1} \; ; \; \ell \in \mathbb{N}'_I)$$

and

$$\mathrm{Im}\ \tilde{u}_I = \mathbb{Q}[m_{k-1} : k \in \mathbb{N}''_I,\quad k \neq 1].$$

4.3. Put

$$\hat{U} = \mathbb{Z}[m_1, m_2, \ldots, m_k, \ldots].$$

As is well known

$$U \subset \hat{U} \subset \tilde{U}$$

and $\hat{U}$ is the minimal extension of U over which F_U becomes isomorphic to G_a.

Since

$$\log_U \circ \xi_{I,U} : \mu_{U,I} \overset{\cong}{\to} G_a$$

we obtain

$$\log_{\mu_{U,I}} = \log_U \circ \xi_{I,U} \tag{4.4}$$

by the uniqueness of logarithm.

Now we compute

$$\begin{aligned}\log_U \circ \xi_{I,U}(T) &= \log_{U\#}\Big(\sum_{i\geq 1}^{F_U}(\xi_{i-1}T^i)\Big) \\ &= \sum_{i\geq 1}\sum_{j\geq 1} m_{j-1}\xi_{i-1}^j\ T^{ij} \\ &= \sum_{k\geq 1}\Big(\sum_{ij=k} m_{j-1}\xi_{i-1}^j\Big)T^k.\end{aligned}$$

On the other hand, by Proposition 3.3 we have

$$\log_{\mu_{U,I}} T = \sum_{k\in\mathbb{N}''_I} m_{k-1}T^k. \tag{4.5}$$

Thus, by (4.4), comparing the coefficients of T^k we see the relations

(4.6) $$\xi_{k-1} + \sum_{\substack{ij=k \\ 1<i<k}} m_{j-1}\xi_{i-1}^{j} = 0 \qquad \text{for} \quad k \in \mathbb{N}_I' - \{1\},$$

(4.7) $$m_{k-1} + \xi_{k-1} + \sum_{\substack{ij=k \\ 1<i<k}} m_{j-1}\xi_{i-1}^{j} = 0 \qquad \text{for} \quad k \in \mathbb{N}_I'.$$

By (4.6), inductively on k, we see that

(4.8) $$\xi_{k-1} = 0 \qquad \text{for} \quad k \in \mathbb{N}_I'' - \{1\},$$

and

(4.9) $$\xi_{I,U}(T) = \sum_{k\in\mathbb{N}_I'\cup\{1\}}^{F_U} (\xi_{k-1}T^k).$$

By (4.7), again inductively on k we obtain

(4.10) $$\xi_{k-1} \in \hat{U} \cap U_I \qquad \underline{\text{for}} \quad k \in \mathbb{N}_I'$$

and

(4.11) $$\xi_{k-1} \equiv - m_{k-1} \quad \text{mod decomposables } \underline{\text{in}} \ \hat{U} \ \underline{\text{for}} \ k \in \mathbb{N}_I'.$$

Thus

(4.12) $$\hat{U} = \mathbb{Z}[\xi_{\ell-1}, \ \ell \in \mathbb{N}_I'] \otimes \mathbb{Z}[m_{k-1}, \ k \in \mathbb{N}_I'' - \{1\}].$$

<u>4.4.</u> Let $x_1, x_2, \ldots, x_n, \ldots$ be a polynomial basis of U, $\dim x_n = -2n$. Then

$$U_I = \mathbb{Z}_I[x_1, x_2, \ldots, x_n, \ldots].$$

Observe the inclusion

$$U_I \subset \hat{U}_I = \hat{U} \otimes \mathbb{Z}_I.$$

For $k \in \mathbb{N}'_I$ by (4.11)

$$\xi_{k-1} \not\equiv 0 \quad \text{mod decomposables and mod } q$$

for all $q \notin I$ in $\hat{U}_I$. The same must be true also in U_I. Hence we can use $\xi_{\ell-1}$, $\ell \in \mathbb{N}'_I$, as a part of the polynomial basis of U_I and we obtain

Proposition 4.4.

$$U_I = \mathbb{Z}_I[\xi_{\ell-1}, \ell \in \mathbb{N}'_I] \otimes \mathbb{Z}_I[x_{k-1}, k \in \mathbb{N}''_I - \{1\}].$$

By Proposition 4.1 we have

$$\{\xi_{k-1}, k \in \mathbb{N}'_I\} \subset \operatorname{Ker} u_I.$$

On the other hand, putting $\bar{x}_{k-1} = u_I(x_{k-1})$ for $k \in \mathbb{N}''_I - \{1\}$ we have

$$x_{k-1} \equiv \bar{x}_{k-1} \equiv c_{k-1}\, m_{k-1} \quad \text{mod decomposables}$$

in $\tilde{U}$ with $c_{k-1} \neq 0$. Thus $\bar{x}_{k-1}$, $k \in \mathbb{N}''_I - \{1\}$, are algebraically independent and u_I maps $\mathbb{Z}_I[x_{k-1}, k \in \mathbb{N}''_I - \{1\}]$ isomorphically onto $\mathbb{Z}_I[\bar{x}_{k-1}, k \in \mathbb{N}''_I - \{1\}]$. Hence we obtain

Proposition 4.5. i) $\operatorname{Ker} u_I = (\xi_{\ell-1}, \ell \in \mathbb{N}'_I)$,

ii) $U_I/\operatorname{Ker} u_I \cong \mathbb{Z}_I[\bar{x}_{k-1}, k \in \mathbb{N}''_I - \{1\}] \subset U_I$,

iii) $U_I = \mathbb{Z}_I[\xi_{\ell-1}, \ell \in \mathbb{N}'_I] \otimes \mathbb{Z}_I[\bar{x}_{k-1}, k \in \mathbb{N}''_I - \{1\}]$, where $\bar{x}_{k-1} = u_I(x_{k-1})$.

4.5. Put

$$(4.13) \qquad BP_I = \operatorname{Im} u_I \subset U_I.$$

Then

$$BP_I = \mathbb{Z}_I[\bar{x}_{k-1}, k \in \mathbb{N}_I'' - \{1\}] \cong U_I/\operatorname{Ker} u_I.$$

by Proposition 4.5. Since $\mu_{U,I} = u_{I*}F_U$ we see that all coefficients of $\mu_{U,I}(X, Y)$ belong to BP_I, and $\mu_{U,I}$ determines a formal group

$$\mu_{BP,I} \in \operatorname{obj} \mathcal{F}(BP_I)$$

which extends to $\mu_{U,I}$ by extension of the domain of coefficients $BP_I \subset U_I$.

Theorem 4.6. $\mu_{BP,I}$ is I-typical and universal for I-typical formal groups over $\mathbb{Z}_I$-algebras.

Proof. Clearly $\mu_{BP,I}$ is I-typical by definition because $\mu_{U,I}$ is I-typical.

Let R be a $\mathbb{Z}_I$-algebra and μ an I-typical formal group over R. There exists a unique homomorphism $\theta : U_I \longrightarrow R$ such that $\theta_* F_{U,I} = \mu$. By Proposition 4.1 $\operatorname{Ker} \theta \supset \operatorname{Ker} u_I$. Thus θ factorizes to

$$U_I \xrightarrow{u_I} BP_I \xrightarrow{\theta_I} R.$$

Since $u_{I*}F_{U,I} = \mu_{BP,I}$, we have

$$\theta_{I*}\mu_{BP,I} = \theta_{I*}u_{I*}F_{U,I} = \theta_* F_{U,I} = \mu .$$

The uniqueness of θ_I follows by the uniqueness of θ.

q. e. d.

Thereby we obtained also the following

Corollary 4.7. _Let_ R _be a_ $\mathbb{Z}_I$_-algebra and_ μ _be an I-typical formal group over_ R. _The homomorphism_ $\theta : U \longrightarrow R$ _such that_ $\theta_* F_U = \mu$ _factorizes to_ $\theta = \theta_I \circ u_I$, $\theta_I : BP_I \longrightarrow R$ _such that_ $\theta_{I*}\mu_{BP,I} = \mu$.

_4.6._ Let I and J be sets of primes such that $I \subset J$. Let $\xi_{I,J}$ be the canonical J-typical curve over $\mu_{U,I}$ (over U_J). Since $\mu_{U,I}$ is I-typical we have

$$\xi_{I,J} = \varepsilon_{J,\mu}\gamma_0 = \prod_{q\in J-I} e_{q,\mu}\gamma_0$$

where $\mu = \mu_{U,I}$. Then

$$\begin{aligned}\xi_{I,U} \circ \xi_{I,J} &= \xi_{I,U\#}\Big(\prod_{q\in J-I} e_{q,\mu}\gamma_0\Big)\\ &= \prod_{q\in J-I} e_{q,U}\xi_{I,U},\end{aligned}$$

i.e.,

$$\xi_{I,U} \circ \xi_{I,J} = \xi_{J,U}. \tag{4.14}$$

Since $\xi_{I,J}$ is of course I-typical, we have the homomorphism

$$u_{I,J} : BP_I \otimes \mathbb{Z}_J \longrightarrow BP_I \otimes \mathbb{Z}_J$$

of $\mathbb{Z}_J$-algebras such that $u_{I,J*}\mu_{BP,I} = \mu_{BP,I}^{\xi_{I,J}}$. Using $\log_{BP,I}$ instead of

$\log_U$, by the same arguments as Proposition 4.2 we see that

$$(4.15)\qquad \tilde{u}_{I,J}(m_{k-1}) = 0 \qquad \text{if} \qquad k \in \mathbb{N}''_I - \mathbb{N}''_J$$
$$= m_{k-1} \qquad \text{if} \qquad k \in \mathbb{N}''_J,$$

where $\tilde{u}_{I,J}$ is the $\mathbb{Q}$-extension of $u_{I,J}$. In particular $u_{I,J}$ is an idempotent of $BP_I \otimes \mathbb{Z}_J$ and we can expect a decomposition of $BP_I \otimes \mathbb{Z}_J$.

By Proposition 4.2 and (4.15) we see that

$$(4.16)\qquad u_J = u_{I,J} \circ u_I$$

regarded as the map : $U_J \longrightarrow U_J$. Thus

$$(4.17)\qquad \operatorname{Im} u_{I,J} = BP_J \subset BP_I \otimes \mathbb{Z}_J.$$

Next we express $\xi_{I,J}$ as

$$\xi_{I,J}(T) = \sum_{k \in \mathbb{N}''_I}{}^{\mu_{BP,I}} (\xi'_{k-1} T^k)$$

by Theorem 3.6. By (4.4) and (4.14) we see that

$$\log_{BP_I} \circ \xi_{I,J} = \log_{BP_J} ;$$

then by parallel arguments to (4.7) and (4.8) we see that

$$\xi'_{k-1} = 0 \qquad \text{for} \qquad k \in \mathbb{N}''_J - \{1\}$$

and

$$\xi'_{k-1} \in \widehat{BP}_I \cap BP_I \otimes \mathbb{Z}_J,$$

$$\xi'_{k-1} + m_{k-1} \equiv 0 \quad \text{mod decomposables in } \widehat{BP}_I$$

for $k \in \mathbb{N}''_I - \mathbb{N}''_J$. Thus

$$(4.18) \qquad \xi_{I,J}(T) = \sum_{k \in (\mathbb{N}''_I - \mathbb{N}''_J) \cup \{1\}}^{\mu_{BP,I}} (\xi'_{k-1} T^k),$$

and by the same arguments as Propositions 4.4 and 4.5 we obtain

Proposition 4.8.

i) $\quad BP_I \otimes \mathbb{Z}_J = \mathbb{Z}_J[\xi'_{k-1},\ k \in \mathbb{N}''_I - \mathbb{N}''_J] \otimes BP_J,$

ii) $\quad \mathrm{Ker}\ u_{I,J} = (\xi'_{k-1},\ k \in \mathbb{N}''_I - \mathbb{N}''_J).$

§5. Quillen decomposition

5.1. Let h^* be a multiplicative cohomology theory defined on finite CW-complexes. We assume that the multiplication in h^* is commutative (in graded sense) and associative, and that the Euler class $e^h(L)$ is defined for any complex line bundle L over a complex X such that i) it is natural for bundle maps, ii) $e^h(L) \in h^2(X)$, and iii) $h^*(CP_n)$ is the truncated polynomial algebra over $h^*(pt)$ generated by the Euler class x of the canonical line bundle over CP_n, truncated by x^{n+1}. Then we can define Chern classes and multiplicative Thom classes in h^* for complex vector bundles. Cf., Dold [8] for details. We call such a cohomology theory h^* complex oriented by a terminology of Quillen [19].

In complex cobordism Thom classes and hence Euler classes for complex vector bundles are canonically defined [7]. Hence complex cobordism is one of the typical examples of complex oriented cohomology theories. We denote by $e^U(L)$ the Euler classes of line bundles in complex cobordism.

We recall the following well-known universality of complex cobordism for complex oriented cohomology theories.

[Universality of complex cobordism] *Let* h^* *be a complex oriented cohomology theory defined on finite* CW-*complexes. There exists a unique cohomology transformation*

$$\theta : U^* \longrightarrow h^*$$

which is i) linear, ii) degree-preserving, iii) multiplicative ($\theta(1) = 1$ for $1 \in U^0(pt)$), and iv) $\theta(e^U(L)) = e^h(L)$ for complex line bundle L.

For proofs we refer to [8], [19].

This universality is actually true also for complex oriented h^* defined on " arbitrary " CW-complexes if we assume h^* to be " additive " [8]. And we can expect to develop Quillen decomposition theory for arbitrary CW-complexes. But in that case we need in certain places to discuss convergences with respect to filtrations by finite subcomplexes. To avoid this complexity we shall be content with limiting our discussions only to finite CW-complexes.

5.2. Let h^* be a complex oriented cohomology theory. For complex line bundles L_1 and L_2 we have

$$e^h(L_1 \otimes L_2) = \sum a_{ij}\ e^h(L_1)^i\ e^h(L_2)^j$$

with $a_{ij} \in h^{2(1-i-j)}(pt)$. By naturality the coefficients a_{ij} do not depend on the choices of L_1 and L_2 and we have a well-determined formal power series

$$F_h(X, Y) = \sum a_{ij}\ X^i Y^j$$

of two variables over $h^*(pt)$. By commutativity and associativity of tensor products, and naturality of Euler classes, we see that F_h satisfies (1.1), (1.2) and (1.3), i.e., is a formal group. Moreover F_h satisfies the condition (1.5) by our choice of dimension of Euler classes.

Of course this formal group F_h depends on the complex orientation of h^*(i.e., the choice of Euler classes). So that we may have several formal groups associated with the same cohomology theory h^* depending on various choices of Euler classes.

Here we recall that Quillen identified $U^*(pt)$ with the Lazard ring U, whereby he identified the formal group of complex cobordism with the universal formal group F_U, i.e., we have

$$e^U(L_1 \otimes L_2) = F_U(e^U(L_1), e^U(L_2))$$

for complex line bundles L_1 and L_2. Cf., [18], [19] and [2], §8.

Now let h^* be complex oriented and

$$\theta : U^* \longrightarrow h^*$$

the unique cohomology transformation by the universality of complex cobordism. Since θ is linear, multiplicative and preserves Euler classes we see readily that

(5.1) $$\theta(pt)_* F_U = F_h.$$

<u>5.3.</u> Let I be a set of primes. The assignment

$$(X, A) \longmapsto U^*(X, A)_I = U^*(X, A) \otimes \mathbb{Z}_I$$

is a multiplicative cohomology, denoted by $U^*(\ )_I$.

Using the power series $\xi_{I,U}$ we define

$$\xi_{I,U}^{-1}(e^U(L)) \in U^2(X)_I$$

as Euler class of a line bundle L over X for $U^*(\ )_I$. Thus $U^*(\ )_I$ is complex oriented. Since

$$\xi_{I,U}^{-1}(e^U(L_1 \otimes L_2)) = \mu_{U,I}(\xi_{I,U}^{-1}(e^U(L_1)),\ \xi_{I,U}^{-1}(e^U(L_2)))$$

the corresponding formal group is $\mu_{U,I}$.

By the universality of complex cobordism we have a cohomology transformation

$$U^* \longrightarrow U^*(\ )_I$$

which sends $e^U(L)$ to $\xi_{I,U}^{-1}(e^U(L))$. Extending this $\mathbb{Z}_I$-linearly we obtain the cohomology transformation

$$\bar{\xi}_I : U^*(\ )_I \longrightarrow U^*(\ )_I \tag{5.2}$$

which is $\mathbb{Z}_I$-linear, degree-preserving, multiplicative and $\bar{\xi}_I(e^U(L)) = \xi_{I,U}^{-1}(e^U(L))$. Then

$$\bar{\xi}_I(pt)_* F_{U,I} = \mu_{U,I}$$

by (5.1), i.e.,

$$\bar{\xi}_I(pt) = u_I : U^*(pt)_I \longrightarrow U^*(pt)_I. \tag{5.3}$$

In particular $\bar{\xi}_I(pt)$ is an idempotent of $U^*(pt)_I$ by Corollary 4.3.

5.4. We want to show that $\bar{\xi}_I$ is an idempotent of the cohomology theory $U^*(\ )_I$. To this end we use Landweber-Novikov operations [14], [17] in a modified form.

Let $\mathfrak{t} = (t_1, t_2, \ldots, t_n, \ldots)$ be a sequence of indeterminates t_n with $\dim t_n = -2n$. For each finite CW-pair (X, A) we put

$$U^*(X, A)[\mathfrak{t}] = U^*(X, A) \otimes \mathbb{Z}[t_1, t_2, \ldots, t_n, \ldots].$$

Obviously $U^*(\ )[\mathfrak{t}]$ is a multiplicative cohomology theory.

Put

$$\phi_{\mathfrak{t}}(T) = \sum_{k\geq 1}^{F_U}(t_{k-1}T^k), \qquad t_0 = 1, \tag{5.4}$$

and assign

$$\phi_{\mathfrak{t}}^{-1}(e^U(L)) \in U^2(X)[\mathfrak{t}]$$

as Euler class of a line bundle L over X. Thus $U^*(\ )[\mathfrak{t}]$ is complex oriented and its formal group is $F_U^{\phi_{\mathfrak{t}}}$. Then by the universality of complex cobordism we have a cohomology transformation

$$\tilde{s}_{\mathfrak{t}} : U^* \longrightarrow U^*(\ )[\mathfrak{t}]$$

which is linear, degree-preserving, multiplicative and

$$\tilde{s}_{\mathfrak{t}}(e^U(L)) = \phi_{\mathfrak{t}}^{-1}(e^U(L)) \tag{5.5}$$

for a line bundle L. And

$$\tilde{s}_{\mathfrak{t}}(pt)_*F_U = F_U^{\phi_{\mathfrak{t}}}. \tag{5.6}$$

This is parallel to Quillen's presentation [19] of Landweber-Novikov operations but not the same. After certain polynomial changes of indeterminates over $U^*(pt)$ our $\tilde{s}_{\mathbb{t}}$ could be identified with Quillen's $s_{\mathbb{t}}$.

Put

$$\tilde{s}_{\mathbb{t}}(x) = \sum_{\alpha} \tilde{s}_{\alpha}(x)\mathbb{t}^{\alpha}$$

for any $x \in U^*(X)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ is a sequence of non-negative integers such that all α_n but finite are zero, and $\mathbb{t}^{\alpha}$ denotes the monomial

$$\mathbb{t}^{\alpha} = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_n^{\alpha_n} \ldots .$$

Then we get linear cohomology operations

$$\tilde{s}_{\alpha} : U^* \longrightarrow U^*$$

of degree $2|\alpha|$ for each sequence α, where $|\alpha| = \sum n \cdot \alpha_n$. These are our modified Landweber-Novikov operations and can be expressed as linear combinations of Landweber-Novikov operations over $U^*(pt)$.

By the property of $\tilde{s}_{\mathbb{t}}$ it follows that

(5.7) $$\tilde{s}_0 = id, \quad \underline{\text{where}} \quad 0 = (0, 0, \ldots, 0, \ldots),$$

(5.8) $$\tilde{s}_{\alpha}(x, y) = \sum_{\beta+\gamma=\alpha} \tilde{s}_{\beta}(x) \cdot \tilde{s}_{\gamma}(y)$$

for internal and external multiplications.

5.5. Let

$$\rho : U^*(\ )[\mathfrak{t}] \longrightarrow U^*(\ )_I$$

be a cohomology transformation defined by $\rho(t_j) = \xi_j$, $j \geq 1$, and $\rho(x) = x$ for $x \in U^*(X)$, where ξ_j are coefficients of $\xi_{I,U}(T)$ in the expression (4.9), whence $\xi_{k-1} = 0$ if $k \subset \mathbb{N}_I''$ and $k \neq 1$.

Clearly ρ is linear, degree preserving and multiplicative, and $\rho \circ \tilde{s}_{\mathfrak{t}}$ sends $e^U(L)$ to $\xi_{I,U}^{-1}(e^U(L))$. Hence, by the uniqueness of cohomology transformation obtained by the universality of complex cobordism we see that

$$(5.9) \qquad \bar{\xi}_I = \rho \circ \tilde{s}_{\mathfrak{t}}.$$

Theorem 5.1. $\bar{\xi}_I$ <u>is an idempotent of</u> $U^*(\ )_I$.

Proof. By (5.3) and Proposition 4.5 it follows that

$$\bar{\xi}_I(\xi^\alpha) = 0 \qquad \text{if } \alpha \neq 0,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots$. Now for any $x \in U^*(X, A)_I$ we have

$$\bar{\xi}_I(x) = \sum_\alpha \tilde{s}_\alpha(x)\xi^\alpha$$

by (5.9). Then

$$\begin{aligned}\bar{\xi}_I(\bar{\xi}_I(x)) &= \sum_\alpha \bar{\xi}_I(\tilde{s}_\alpha(x)) \cdot \bar{\xi}_I(\xi^\alpha) \\ &= \bar{\xi}_I(\tilde{s}_0(x)) = \bar{\xi}_I(x),\end{aligned}$$

i.e., $\bar{\xi}_I$ is an idempotent.

Corollary 5.2. <u>There holds natural stable direct sum decomposition</u>

$$U^*(X, A)_I = \operatorname{Im} \bar{\xi}_I(X, A) \oplus \operatorname{Ker} \bar{\xi}_I(X, A).$$

For any $x \in \operatorname{Ker} \bar{\xi}_I(X, A)$ we have

$$x + \sum_{\alpha \neq 0} \tilde{s}_\alpha(x)\xi^\alpha = \bar{\xi}_I(x) = 0,$$

i.e.,

$$x = -\sum_{\alpha \neq 0} \tilde{s}_\alpha(x)\xi^\alpha$$

and we obtain

Corollary 5.3. $\operatorname{Ker} \bar{\xi}_I(X, A) = (\operatorname{Ker} \bar{\xi}_I(pt)) \circ U^*(X, A)_I$.

<u>5.6.</u> Put

(5.10) $$BP_I^*(X, A) = \operatorname{Im} \bar{\xi}_I(X, A)$$

for any finite CW-pair (X, A). By (5.3) and (4.13)

$$BP_I^*(pt) = BP_I,$$

and by Corollary 5.2 the assignment

$$(X, A) \longmapsto BP_I^*(X, A)$$

is a cohomology theory. Moreover it is multiplicative because $\bar{\xi}_I$ is multiplicative, that is,

Proposition 5.4. BP_I^* <u>is a multiplicative cohomology theory such that</u> $BP_I^*(pt) = BP_I$, <u>a $\mathbb{Z}_I$-algebra</u>.

By definition (5.10) we have canonical cohomology transformations

(5.11) $\pi_I : U^*(\ )_I \longrightarrow BP_I^*$, natural surjection,

(5.12) $i_I : BP_I^* \longrightarrow U^*(\ )_I$, natural injection,

such that

(5.13) $i_I \circ \pi_I = \bar{\xi}_I$.

Using coefficients $\xi_{\ell-1}$ of $\xi_{I,U}(T)$ in the expression (4.9) we put

$$V_I^*(X, A) = \mathbb{Z}_I[\xi_{\ell-1}, \ \ell \in \mathbb{N}_I'] \otimes BP_I^*(X, A)$$

for finite CW-pairs (X, A). Then V_I^* is a multiplicative cohomology theory. Define

$$\Theta_I : V_I^* \longrightarrow U^*(\ )_I$$

by $\Theta_I(\xi^\alpha \otimes x) = \xi^\alpha \cdot i_I(x)$ for $x \in BP_I^*(X, A)$. As is easily seen Θ_I is a linear multiplicative cohomology transformation, and $\Theta_I(pt)$ is an isomorphism by Proposition 4.5. Hence Θ_I is an isomorphism of cohomology theories by the comparison theorem of cohomology theories over finite CW-pairs. Thus

Theorem 5.5. i_I _induces the natural isomorphism_

$$U^*(\ )_I \cong \mathbb{Z}_I[\xi_{\ell-1}, \ \ell \in \mathbb{N}_I'] \otimes BP_I^*$$

of cohomology theories.

_5.7._ Let p be a specified prime and put

(5.14) $BP^* = BP_{(p)}^*$

for $I = (p)$. This is called <u>Brown-Peterson cohomology theory</u>.

In this case the isomorphism of Theorem 5.5 takes the form

$$(5.15) \qquad U^*(\ )_{(p)} = \mathbb{Z}_{(p)}[\xi_{\ell-1};\ \ell \neq p^s] \otimes BP^*.$$

This is the <u>Quillen decomposition</u> of complex cobordism localized at the prime p.

<u>5.8.</u> Let I be a set of primes and put

$$(5.16) \qquad e^{BP,I}(L) = \pi_I(e^U(L))$$

for a line bundle L. By the decomposition Theorem 5.5 we see easily that $e^{BP,I}(L)$ satisfies the required properties of Euler classes. Hence BP_I^* is complex oriented. By definition of $\bar{\xi}_I$ and (5.13) we have

$$(5.17) \qquad i_I(e^{BP,I}(L)) = \xi_I^{-1}(e^U(L)).$$

Then, by (5.1), (5.3) and the definition of $\mu_{BP,I}$ we obtain

<u>Theorem</u> 5.6. BP_I^* <u>is complex oriented and its associated formal group is</u> $\mu_{BP,I}$.

<u>5.9.</u> Our next purpose is to give a decomposition of $BP_I^*(\ ) \otimes \mathbb{Z}_J$ into BP_J^* $(I \subset J)$ which extends the decomposition of Proposition 4.8 to cohomology theory. For this purpose we start with introducing Landweber-Novikov type operations in BP_I^*.

Let I be a set of primes and

$$t_I = \{t_{k-1};\ k \in \mathbb{N}_I'' - \{1\}\}$$

the subsequence of $\mathfrak{t}$. We consider multiplicative cohomology

$$BP_I^*(\ )[\mathfrak{t}_I] = BP_I^*(\ ) \otimes \mathbb{Z}[t_{k-1};\ k \in \mathbb{N}_I'' - \{1\}].$$

Putting

$$(5.18) \qquad \phi_{\mathfrak{t},I}(T) = \sum_{k \in N_I''}{}^{\mu_{BP,I}}(t_{k-1}T^k), \qquad t_0 = 1,$$

we assign

$$\phi_{\mathfrak{t},I}^{-1}(e^{BP,I}(L)) \in BP_I^2(X)[\mathfrak{t}_I]$$

as Euler class of a line bundle L over X. Thus $BP_I^*(\ )[\mathfrak{t}_I]$ is complex oriented and its associated formal group is $\mu_{BP,I}^{\phi_{\mathfrak{t},I}}$. By the universality of complex cobordism we have a cohomology transformation

$$\tilde{s}_{\mathfrak{t},I} : \ U^* \longrightarrow BP_I^*(\ )[\mathfrak{t}_I]$$

which is linear, degree-preserving, multiplicative and

$$\tilde{s}_{\mathfrak{t},I}(e^U(L)) = \phi_{\mathfrak{t},I}^{-1}(e^{BP,I}(L))$$

for a line bundle L. Then

$$\tilde{s}_{\mathfrak{t},I}(pt)_* F_U = \mu_{BP,I}^{\phi_{\mathfrak{t},I}}.$$

Here we remark that $\mu_{BP,I}$ is I-typical and $\phi_{\mathfrak{t},I}$ is an I-typical curve over $\mu_{BP,I}$ (extending the domain of coefficients to $BP_I \otimes \mathbb{Z}[\mathfrak{t}_I]$) by Theorem 3.6. Thus $\mu_{BP,I}^{\phi_{\mathfrak{t},I}}$ is I-typical. Then, by Proposition 4.1

$$\{\xi_{k-1},\ k \in \mathbb{N}_I'\} \subset \operatorname{Ker} \tilde{s}_{\mathfrak{t},I}(pt)$$

(extending $\tilde{s}_{\mathfrak{t},I}$ over U_I^* by $\mathbb{Z}_I$-linearity), where ξ_{k-1} are coefficients of $\xi_{I,U}(T)$ in expression (4.9). Now by Corollary 5.3 we have a factorization of $\tilde{s}_{\mathfrak{t},I}$:

(5.19)
$$\begin{tikzcd} U^*(\ )_I \arrow[r, "\pi_I"] \arrow[rr, bend right, "\tilde{s}_{\mathfrak{t},I}"'] & BP_I^*(\ ) \arrow[r, "r_{\mathfrak{t},I}"] & BP_I^*(\ )[\mathfrak{t}_I]. \end{tikzcd}$$

By construction it is clear that $r_{\mathfrak{t},I}$ is a linear cohomology transformation which is degree-preserving, multiplicative and

(5.20)
$$r_{\mathfrak{t},I}(e^{BP,I}(L)) = \phi_{\mathfrak{t},I}^{-1}(e^{BP,I}(L))$$

for a line bundle L. Then

(5.21)
$$r_{\mathfrak{t},I}(pt)_*\ \mu_{BP,I} = \mu_{BP,I}^{\phi_{\mathfrak{t},I}}.$$

If we take the coefficients of monomials (of $\mathfrak{t}_I$) in $r_{\mathfrak{t},I}$ we get Landweber-Novikov type operations in BP_I^*. We discuss their properties for $I = (p)$ later in §7.

<u>5.10.</u> Let I and J be sets of primes such that $I \subset J$. Take the canonical J-typical curve $\xi_{I,J}$ over $\mu_{PB,I}$ (over $BP_I \otimes \mathbb{Z}_J$). Let ξ'_{k-1}, $k \in \mathbb{N}''_I$, be the coefficients of $\xi_{I,J}(T)$ in expression (4.18) and define cohomology transformation

$$\rho' :\ BP_I^*(\ )[\mathfrak{t}] \longrightarrow BP_I^*(\ ) \otimes \mathbb{Z}_J$$

by $\rho'(t_{k-1}) = \xi'_{k-1}$, $k \in \mathbb{N}''_I$, and $\rho'(x) = x$ for $x \in BP_I^*(X)$. Put

(5.22)
$$\bar{\xi}_{I,J} = \rho' \circ \tilde{s}_{\mathfrak{t},I} :\ BP_I^*(\ ) \otimes \mathbb{Z}_J \longrightarrow BP_I^*(\ ) \otimes \mathbb{Z}_J.$$

This is a linear cohomology transformation which is degree-preserving, multiplicative and

$$\overline{\xi}_{I,J}(e^{BP,I}(L)) = \xi^{-1}_{I,J}(e^{BP,I}(L)).$$

Thus

$$(5.23) \qquad \overline{\xi}_{I,J}(pt)_* \mu_{BP,I} = \mu^{\xi_{I,J}}_{BP,I} = \mu_{BP,J}.$$

Therefore

$$(5.24) \qquad \overline{\xi}_{I,J}(pt) = u_{I,J} : BP_I \otimes \mathbb{Z}_J \longrightarrow BP_I \otimes \mathbb{Z}_J.$$

In particular $\overline{\xi}_{I,J}(pt)$ is an idempotent of $BP_I \otimes \mathbb{Z}_J$. Now by a parallel argument to Theorem 5.1 we obtain

Proposition 5.7. $\overline{\xi}_{I,J}$ <u>is an idempotent of</u> $BP^*_I(\ ) \otimes \mathbb{Z}_J$.

Corollary 5.8. i) $\operatorname{Im} \overline{\xi}_{I,J} = BP^*_J$.

ii) $\operatorname{Ker} \overline{\xi}_{I,J}(X,A) = (\operatorname{Ker} \overline{\xi}_{I,J}(pt)) \cdot BP^*_I(X,A) \otimes \mathbb{Z}_J$.

iii) $BP^*_I(\ ) \otimes \mathbb{Z}_J = \mathbb{Z}_J[\xi'_{k-1},\ k \in \mathbb{N}''_I - \mathbb{N}''_J] \otimes BP^*_J$.

In particular, when $p \notin I$ and $J = (p)$ we have the decomposition

$$(5.25) \qquad BP^*_I(\ ) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\xi'_{k-1},\ k \in \mathbb{N}''_I,\ k \neq p^s] \otimes BP^*,$$

which we call the <u>Quillen decomposition</u> of BP^*_I localized at the prime p.

<u>5.11.</u> Let $\Omega^*(\ )$ be the oriented cobordism theory. Here we consider the Quillen decomposition of $\Omega^*(\ )_{[2]} = \Omega^*(\ ) \otimes \mathbb{Z}[\frac{1}{2}]$. Let

$$S : U^*(\ ) \longrightarrow \Omega^*(\ )$$

be the forgetful functor of complex structures. Clearly S is a multiplicative cohomology transformation.

In $\Omega^*(\ )$ Euler classes are canonically defined for real oriented vector bundles. Every complex line bundle L determines canonically an oriented 2-plane bundle $L_{\mathbb{R}}$. We define

$$e^{SO}(L) = \Omega^*\text{-Euler class of } L_{\mathbb{R}}.$$

Thus $\Omega^*(\ )$ is complex oriented. We denote its associated formal group by F_{SO}. We see easily that

$$S(pt)_* F_U = F_{SO}.$$

Remark that, if we change the orientation of a real oriented vector bundle, then Ω^*-Euler class changes to its negative. Thus

$$e^{SO}(\overline{L}) = - e^{SO}(L)$$

and

$$F_{SO}(- T, T) = 0.$$

Now we have

$$(\mathbf{f}_{2,SO}\gamma_0)(T) = F_{SO}(- T^{1/2}, T^{1/2}) = 0,$$

i. e.,

Proposition 5.9. F_{SO} <u>is</u> [2]-<u>typical</u>.

Next we observe the cohomology transformation

$$S_{[2]} = S \otimes \mathbb{Z}_{[2]} \ : \ U^*(\ )_{[2]} \longrightarrow \Omega^*(\ )_{[2]}.$$

By Propositions 4.1 and 5.9 we see that

$$\operatorname{Ker} S_{[2]}(pt) \supset \operatorname{Ker} \bar{\xi}_{[2]}(pt).$$

Then, by Corollary 5.3 and multiplicativity of $S_{[2]}$ we see that

$$\operatorname{Ker} S_{[2]}(X, A) \supset \operatorname{Ker} \bar{\xi}_{[2]}(X, A)$$

for any (X,A), i. e., $S_{[2]}$ factorizes to

$$\begin{tikzcd} U^*(\ )_{[2]} \arrow[r, "\pi_{[2]}"] \arrow[rr, bend right, "S_{[2]}"'] & BP^*_{[2]} \arrow[r, "\Phi"] & \Omega^*(\ )_{[2]} \end{tikzcd}$$

By Proposition 4.5 we know that $\operatorname{Ker} \pi_{[2]}(pt)$ is the ideal generated by all elements of dimensions $\equiv 2 \pmod 4$. But $\operatorname{Ker} S_{[2]}(pt)$ is also the same by Stong [21], p.178. Thus $\Phi(pt)$ is injective. On the other hand $S_{[2]}(pt)$ is surjective by [21], p.180. Hence $\Phi(pt)$ is isomorphic and by the comparison theorem of cohomology theories we obtain

Theorem 5.10. *The forgetful functor* S *of complex structures induces an isomorphism*

$$\Phi \ : \ BP^*_{[2]} \cong \Omega^*(\ )_{[2]}$$

of cohomology theories.

Let p be an odd prime. By Quillen decomposition (5.25) we obtain the following decomposition of $\Omega^*(\ )_{(p)}$:

(5.26) $$\Omega^*(\ )_{(p)} \cong \mathbb{Z}_{(p)}[\xi'_{k-1},\ k \text{ odd and } k \neq p^s] \otimes BP^*.$$

§6. Generators of $U^*(pt)$ and $BP^*(pt)$

<u>6.1.</u> *Let p be a prime. Putting*

$$(6.1) \qquad (\mathbf{f}_{p,U}\gamma_0)(T) = \sum_{n\geq 1}^{F_U}(v^{(p)}_{pn-1}T^n)$$

we see that $v_s = v_s^{(p)} \in U^{-2s}$ by definitions. Compute $\log_U \mathbf{f}_{p,U}\gamma_0$ in two ways :

$$\begin{aligned}(\log_U \mathbf{f}_{p,U}\gamma_0)(T) &= \log_{U\#}(\sum_{n\geq 1}^{F_U}(v_{pn-1}T^n)) \\ &= \sum_{i\geq 1}\sum_{j\geq 1} m_{j-1}v^j_{pi-1}T^{ij} \\ &= \sum_{n\geq 1}(\sum_{ij=n} m_{j-1}v^j_{pi-1})T^n\end{aligned}$$

and

$$\begin{aligned}(\log_U \mathbf{f}_{p,U}\gamma_0)(T) &= (\mathbf{f}_{p,G_a}\log_U)(T) \\ &= p\cdot \sum_{n\geq 1} m_{pn-1}T^n,\end{aligned}$$

(cf.,(2.8)). Then compare the coefficients of T^n, and we obtain

$$(6.2) \quad p\cdot m_{pn-1} = \sum_{ij=n} m_{j-1}v^j_{pi-1} = v^{(p)}_{pn-1} + \sum_{\substack{ij=n \\ 1\leq i<n}} m_{j-1}v^j_{pi-1}$$

for all $n \geq 1$.

Let s_n denote the Chern number corresponding to the power sum $\sum t_i^n$. Remark that

$$s_n : U^{-2n}(pt)\otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

is a linear map such that

$$s_n(\text{decomposable element}) = 0.$$

By Mischenko series we have

$$m_{k-1} = \frac{[CP_{k-1}]}{k} \quad .$$

Thus

$$s_{k-1}(m_{k-1}) = 1$$

for all $k > 1$. Now apply s_{pn-1} to (6.2) we obtain

$$s_{pn-1}(v^{(p)}_{pn-1}) = p. \tag{6.3}$$

By a well known theorem of Milnor, if a sequence $\{ x_1, x_2, \ldots, x_n, \ldots \}$ of elements of $U^*(pt)$, $\dim x_n = -2n$, satisfies

$$\begin{aligned} s_n(x_n) &= p \qquad \text{if } n = p^s - 1 \text{ for some prime } p, \\ &= 1 \qquad \text{otherwise,} \end{aligned} \tag{6.4}$$

then it is a polynomial basis of $U^*(pt)$. Such a sequence is called a <u>Milnor basis</u> of $U^*(pt)$. Then (6.3) shows that

(6.5) $\{ v^{(p)}_{p-1}, v^{(p)}_{p^2-1}, \ldots, v^{(p)}_{p^k-1}, \ldots \}$ <u>forms a part of a Milnor basis.</u>

Let k be an integer > 1 which is not a prime power. Let p and q be different prime factors of k. Since $s_{k-1}(v^{(p)}_{k-1}) = p$ and $s_{k-1}(v^{(q)}_{k-1}) = q$ we find integers a and b such that

$$s_{k-1}(a\, v^{(p)}_{k-1} + b\, v^{(q)}_{k-1}) = 1.$$

Thus for each dimension $-2n$ we can find an element x_n satisfying (6.4) as a linear combination of our elements $v^{(p)}_n$'s, and we obtain

Proposition 6.1. _The elements_ $v^{(p)}_{pn-1}$ _defined by_ (6.1), _for all_ $n \geq 1$ _and all primes_ p, _generate_ $U^*(pt)$.

6.2. Let p be a fixed prime. In (6.2) we put $n = p^{k-1}$, then

$$(6.6)\qquad p\cdot m_{p^k-1} = v^{(p)}_{p^k-1} + \sum_{i=1}^{k-1} m_{p^{k-i}-1}(v^{(p)}_{p^i-1})^{p^{k-i}} \qquad \text{for all } k \geq 1.$$

Apply $\tilde{u}_{(p)}$ on both sides of (6.6). By Proposition 4.2 we have

$$(6.7)\qquad p\cdot m_{p^k-1} = \overline{v}^{(p)}_{p^k-1} + \sum_{i=1}^{k-1} m_{p^{k-i}-1}(\overline{v}^{(p)}_{p^i-1})^{p^{k-i}} \qquad \text{for all } k \geq 1,$$

where $\overline{v}^{(p)}_{p^i-1} = u_{(p)}(v^{(p)}_{p^i-1})$. Comparing the two recursive formulas (6.6) and (6.7) we obtain

$$(6.8)\qquad u_{(p)}(v^{(p)}_{p^k-1}) = v^{(p)}_{p^k-1} \qquad \text{for all } k \geq 1.$$

Now by Proposition 4.5, (6.5) and (6.8) we obtain

Theorem 6.2. $BP^*(pt) = \mathbb{Z}_{(p)}[v^{(p)}_{p-1}, v^{(p)}_{p^2-1}, \ldots, v^{(p)}_{p^k-1}, \ldots]$.

Apply $\pi_{(p)*}$ to both sides of (6.1) and obtain

$$(\mathbf{f}_{p,BP}\,\gamma_0)(T) = \sum_{n\geq 1}{}^{\mu_{BP}}(\overline{v}^{(p)}_{pn-1}T^n),$$

where $\overline{v}^{(p)}_{pn-1} = \pi_{(p)}(v^{(p)}_{pn-1})$. Since $\mathbf{f}_{p,BP}\,\gamma_0$ is a typical curve over μ_{BP} we see by Theorem 3.6 that

$$(6.9)\qquad u_{(p)}(v^{(p)}_{pn-1}) = 0 \qquad \text{for } n \neq p^{\ell},$$

and using (6.8) we obtain

Theorem 6.3. $(\mathbf{f}_{p,BP}\,\gamma_0)(T) = \sum_{i\geq 1}{}^{\mu_{BP}}(v^{(p)}_{p^i-1}T^{p^{i+1}})$,

where the coefficients are the polynomial basis of $BP^*(pt)$ _of Theorem 6.2._

Remark. i) Our polynomial basis of $BP^*(pt)$ of Theorem 6.2 satisfies the recursive formula (6.6) which is the same as the corresponding formula of Hazewinkel [12]. Hence our generators are the same as those of Hazewinkel. In case $p = 2$ a similar recursive formula is obtained also by Liulevicius [16].

ii) By our method it is already clear that the generators $\{v^{(p)}_{p^k-1}\}$ of $BP^*(pt)$ are integrable, i.e., elements of $U^*(pt)$. This fact was observed also by Alexander [4].

<u>6.3.</u> Let p be a fixed prime and q be another prime. Since $\mathfrak{f}_{q,BP}\gamma_0 = 0$, applying $\pi_{(p)}(pt)_*$ to $\mathfrak{f}_{q,U}\gamma_0$ expressed in the form (6.1), we see that

$$u_{(p)}(v^{(q)}_{qn-1}) = 0 \tag{6.10}$$

for all $n \geq 1$. By (6.8), (6.9), (6.10) and Proposition 6.1 we see

Proposition 6.4 (Integrity of $u_{(p)}$). $u_{(p)}(U) = \mathbb{Z}[v_{p^i-1}, i \geq 1] \subset U$, i.e., U <u>is stable under the idempotent</u> $u_{(p)}$ <u>and by restriction</u> $u_{(p)}$ <u>determines an idempotent of</u> U.

<u>6.4.</u> Let p be a prime and put

$$[p]_U(T) = \sum_{n\geq 1}^{F_U}(w^{(p)}_{n-1}T^n). \tag{6.11}$$

Then $w_{n-1} = w^{(p)}_{n-1} \in U^{-2(n-1)}$ and $w_0 = p$. Now compute $\log_U \circ [p]_U$ and obtain

$$p \cdot m_{n-1} = \sum_{ij=n} m_{j-1} w_{i-1}^{j}$$

$$= p^{n} m_{n-1} + w_{n-1} + \sum_{\substack{ij=n \\ 1<i<n}} m_{j-1} w_{i-1}^{j}$$

In particular we obtain

$$w_{p^k-1}^{(p)} = (p - p^{p^k}) m_{p^k-1} - \sum_{i=1}^{k-1} m_{p^{k-i}-1} w_{p^i-1}^{p^{k-i}} \tag{6.12}$$

for $k \geq 1$. Then we see that

$$s_{p^k-1}(w_{p^k-1}^{(p)}) = p - p^{p^k},$$

hence $\{w_{p-1}^{(p)}, w_{p^2-1}^{(p)}, \ldots, w_{p^k-1}^{(p)}, \ldots\}$ forms a part of a polynomial basis of $U^*(pt)_{(p)}$. On the other hand applying $u_{(p)}$ to (6.12) we obtain

$$u_{(p)}(w_{p^k-1}^{(p)}) = w_{p^k-1}^{(p)}.$$

Again apply $\pi_{(p)*}$ to $[p]_U$ and remark that $[p]_{BP}$ is a typical curve. Thus we obtain

Theorem 6.5. Putting

$$[p]_{BP}(T) = \sum_{k\geq 0}^{\mu_{BP}} (w_{p^k-1}^{(p)} T^{p^k}), \qquad w_0^{(p)} = p,$$

we obtain

$$BP^*(pt) = \mathbb{Z}_{(p)}[w_{p-1}^{(p)}, w_{p^2-1}^{(p)}, \ldots, w_{p^k-1}^{(p)}, \ldots].$$

These generators are also integrable.

§7. Operations in Brown-Peterson cohomology.

7.1. Fix a prime p. In 5.9 we defined the cohomology transformation

(7.1) $$r_{\mathfrak{t}} = r_{\mathfrak{t},(p)} : BP^* \longrightarrow BP^*(\ )[\mathfrak{t}_{(p)}]$$

which is $\mathbb{Z}_{(p)}$-linear, degree-preserving, multiplicative and

(7.2) $$r_{\mathfrak{t}}(e^{BP}(L)) = \phi^{-1}_{\mathfrak{t},(p)}(e^{BP}(L)),$$

where $\mathfrak{t}_{(p)} = (t_1, t_2, \ldots, t_n, \ldots)$ is a sequence of indeterminates with $\dim t_k = -2(p^k - 1)$ (we replace here the letter t_{p^k-1} by t_k for simplicity), $\phi_{\mathfrak{t},(p)}(T) = \sum\limits_{k\geq 0}^{\mu_{BP}} (t_k T^{p^k})$ with $t_0 = 1$, and $e^{BP}(L) \in BP^2(X)$, BP^*-Euler class of a line bundle L over X.

Put

$$r_{\mathfrak{t}}(x) = \sum_{\alpha} r_\alpha(x)\, \mathfrak{t}^{\alpha}_{(p)}, \qquad x \in BP^*(X, A),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a sequence of non-negative integers such that all α_k but a finite are zero and

$$\mathfrak{t}^{\alpha}_{(p)} = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k} \ldots$$

is a monomial of t_k's. Then we get linear stable operations

(7.3) $$r_\alpha : BP^*(\ ) \longrightarrow BP^*(\ )$$

with

(7.4) $$\deg r_\alpha = 2 \sum_i \alpha_i (p^i - 1)$$

for each sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$. These are Landweber-Novikov type operations in Brown-Peterson cohomology, and we call them <u>Quillen operations</u>.

After identifying

$$BP^*(\ )[\mathfrak{t}_{(p)}] = (BP \wedge BP)^*(\ )$$

by making use of Brown-Peterson spectrum, we can see that every operation in BP^* can be expressed uniquely as an infinite sum

$$\sum_\alpha u_\alpha r_\alpha, \qquad u_\alpha \in BP^*(pt),$$

as in [2], [14], [17]. But we will not discuss this point here, but rather we observe certain properties of these Quillen operations.

First of all it is clear by definition and properties of $r_{\mathfrak{t}}$ that

(7.5) $$r_0 = \mathrm{id}, \qquad 0 = (0, 0, \ldots, 0, \ldots),$$

and

(7.6) $$r_\alpha(x \cdot y) = \sum_{\beta+\gamma=\alpha} r_\beta(x) \cdot r_\gamma(y)$$

for internal and external multiplications.

<u>7.2.</u> Putting $\mu = \mu_{BP}$ and $\mu' = \mu^{\phi_{\mathfrak{t},(p)}}$, we have

(7.7) $$r_{\mathfrak{t}}(pt)_* \mu_{BP} = \mu',$$

(cf., (5.21)). Let

$$\log_\mu : \mu \xrightarrow{\cong} G_a \quad \text{and} \quad \log_{\mu'} ; \mu' \xrightarrow{\cong} G_a,$$

logarithms over $\mathbb{Q}$-extensions. Then

(7.8) $$r_{\mathfrak{t}}(pt)_* \log_\mu = \log_{\mu'}$$

by functionality and the uniqueness of logarithm. On the other hand, since $\phi_{\mathfrak{t},(p)} : \mu' \cong \mu$, by the uniqueness of logarithm we obtain

(7.9) $$\log_{\mu'} = \log_\mu \circ \phi_{\mathfrak{t},(p)}.$$

We compute $\log_{\mu'}$ in two ways by (7.8) and (7.9) :

since

(7.10) $$\log_\mu T = \sum_{k\geq 0} m_{p^k-1} T^{p^k}$$

by Proposition 3.3, where m_{k-1} is the coefficient of T^k in $\log_U T$, we see by (7.8) that

$$\log_{\mu'} T = \sum_{k\geq 0} r_{\mathfrak{t}}(pt)(m_{p^k-1})T^{p^k} ;$$

on the other hand by (7.9) we see that

$$\begin{aligned} \log_{\mu'} T &= \log_{\mu\#}(\phi_{\mathfrak{t},(p)}(T)) \\ &= \sum_{j\geq 0} \log_\mu(t_j T^{p^j}) \\ &= \sum_{k\geq 0} \Big(\sum_{h=0}^{k} m_{p^h-1} t_{k-h}^{p^h}\Big)T^{p^k}. \end{aligned}$$

Comparing the coefficients of T^{p^n} we obtain

(7.11) $$r_{\mathfrak{t}}(pt)(m_{p^n-1}) = \sum_{h=0}^{n} m_{p^h-1} t_{n-h}^{p^h}$$

after extending $r_{\mathfrak{t}}(pt)$ over $BP^*(pt) \otimes \mathbb{Q}$ by $\mathbb{Q}$-linearity.

Since $m_{p^h-1} = \dfrac{[CP_{p^h-1}]}{p^h}$ we obtain

Theorem 7.1 (Theorem 5,(iii) of [18]).

$$r_{\sharp}(pt)([CP_{p^n-1}]) = \sum_{h=0}^{n} p^{n-h}[CP_{p^h-1}]t_{n-h}^{p^h}.$$

This theorem describes the actions of r_α on $BP^*(pt)$ at least theoretically.

7.3. I feel it is better to formulate a general theorem of which the operation $r_{\sharp}$ is obtained by a specialization.

Theorem 7.2. *Let* h^* *be a complex-oriented cohomology theory such that* $h^*(pt)$ *is a* $\mathbb{Z}_{(p)}$*-algebra and its associated formal group is typical. Then there exists a unique cohomology transformation*

$$\theta_{(p)} \ : \ BP^* \longrightarrow h^*$$

which is $\mathbb{Z}_{(p)}$*-linear, degree-preserving, multiplicative and*

$$\theta_{(p)}(e^{BP}(L)) = e^h(L)$$

for a complex line bundle L. *It results also*

$$\theta_{(p)}(pt)_* \mu_{BP} = \mu_h,$$

the typical formal group of h^*.

Thus we may say that BP^* is universal for cohomology theories with typical formal groups. $\theta_{(p)}$ can be obtained by factorizing the unique cohomology transformation

$$\theta \ : \ U^* \longrightarrow h^*$$

which follows by the universality of complex bordism. Our necessary arguments for the proof are quite parallel to those in 5.9 to establish $r_{\mathfrak{t}}$, so I will omit them.

7.4. Next we discuss compositions $r_\alpha \circ r_\beta$ of Quillen operations. Consider the diagram

$$BP^*(\ ) \xrightarrow{r_{\mathfrak{t}}} BP^*(\ )[\mathfrak{t}_{(p)}] \xrightarrow{r_{\mathfrak{s}} \otimes 1} BP^*(\ )[\mathfrak{s}_{(p)}, \mathfrak{t}_{(p)}],$$

where $\mathfrak{s}_{(p)} = (s_1, s_2, \ldots, s_k, \ldots)$ is another sequence of indeterminates such that $\dim s_\ell = -2(p^\ell - 1)$ and $r_{\mathfrak{s}} \otimes 1$ is an extension of $r_{\mathfrak{s}} : BP^*(\ ) \longrightarrow BP^*(\ )[\mathfrak{s}_{(p)}]$ such that $(r_{\mathfrak{s}} \otimes 1)(t_j) = t_j$, $j \geq 1$.

The composition $r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}}$ is a cohomology transformation which is linear, degree-preserving and multiplicative. Moreover, putting $\mu'_{\mathfrak{s}} = \mu^{\phi_{\mathfrak{s},(p)}}$ and $\psi_{\mathfrak{t},(p)}(T) = \sum_{i \geq 0}^{\mu'_{\mathfrak{s}}} (t_i T^{p^i})$, we have

$$(7.12) \qquad (r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}})(e^{BP}(L)) = \psi^{-1}_{\mathfrak{t},(p)}(\phi^{-1}_{\mathfrak{s},(p)}(e^{BP}(L)))$$

for a line bundle L. This formula can be seen as follows : remark that

$$r_{\mathfrak{s}}(pt)_*\mu = \mu'_{\mathfrak{s}}$$

by (7.7). Then

$$(r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}})(e^{BP}(L)) = (r_{\mathfrak{s}} \otimes 1)(\phi^{-1}_{\mathfrak{t},(p)}(e^{BP}(L)))$$

$$= (r_{\mathfrak{s}}(pt)_*\phi_{\mathfrak{t},(p)})^{-1}(\phi^{-1}_{\mathfrak{s},(p)}(e^{BP}(L))),$$

and

$$(r_{\mathfrak{s}}(pt)_* \phi_{\mathfrak{t},(p)})(T) = r_{\mathfrak{s}}(pt)_* \sum_{j\geq 0}^{\mu}(t_j T^{p^j})$$
$$= \sum_{j\geq 0}^{\mu'_{\mathfrak{s}}}(t_j T^{p^j}) = \psi_{\mathfrak{t},(p)}(T).$$

We give complex-orientation of the cohomology theory $BP^*[\mathfrak{s}_{(p)}, \mathfrak{t}_{(p)}]$ by assigning $(\phi_{\mathfrak{s},(p)} \circ \psi_{\mathfrak{t},(p)})^{-1}(e^{BP}(L))$ as Euler class of a line bundle L. Then the associated formal group μ'' is the transpose of $\mu'_{\mathfrak{s}}$ by $\psi_{\mathfrak{t},(p)}$. Now $\psi_{\mathfrak{t},(p)}$ is a typical curve over $\mu'_{\mathfrak{s}}$. Hence μ'' is typical. Hence

$$r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}} \ : \ BP^*(\ ) \longrightarrow BP^*(\ )[\mathfrak{s}_{(p)}, \mathfrak{t}_{(p)}]$$

is the unique cohomology transformation of Theorem 7.2.

By our construction it is clear that

$$(7.13) \qquad (r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}})(x) = \sum_{\alpha,\beta} r_\beta(r_\alpha(x))\, \mathfrak{t}^\alpha\, \mathfrak{s}^\beta,$$

where $\mathfrak{t}^\alpha$ and $\mathfrak{s}^\beta$ are monomials of $(t_1, t_2, \ldots)$ and $(s_1, s_2, \ldots)$.

<u>7.5.</u> Remark that $\phi_{\mathfrak{s},(p)} \circ \psi_{\mathfrak{t},(p)}$ is a typical curve over μ, the extension of μ_{BP} over $BP^*(pt)[\mathfrak{s}_{(p)}, \mathfrak{t}_{(p)}]$. Hence we have a unique expression

$$(7.14) \qquad (\phi_{\mathfrak{s},(p)} \circ \psi_{\mathfrak{t},(p)})(T) = \sum_{j\geq 0}^{\mu}(u_j T^{p^j}), \qquad u_0 = 1,$$

where $u_j = u_j(s_1,\ldots,s_j, t_1, \ldots,t_j)$ are polynomials of $s_1,\ldots, s_j, t_1, \ldots, t_j$ over $BP^*(pt)$ such that $\dim u_j = -2(p^j - 1)$.

We want to obtain these polynomials if possible. Since

$$\phi_{\mathfrak{s},(p)} \circ \psi_{\mathfrak{t},(p)}(T) = \phi_{\mathfrak{s},(p)\#}(\psi_{\mathfrak{t},(p)}(T))$$

$$= \sum_{j\geq 0}^{\mu} \sum_{i\geq 0}^{\mu}(s_i t_j^{p^i} T^{p^{i+j}}),$$

applying $\log_{\mu\#}$ on both sides of (7.14) we obtain

$$\sum_{i,j,\ell} m_{p^\ell -1} s_i^{p^\ell} t_j^{p^{i+\ell}} T^{p^{i+j+\ell}} = \sum_{k,\ell} m_{p^\ell -1} u_k^{p^\ell} T^{p^{\ell+k}}.$$

Comparing the coefficients of T^{p^n} we obtain

$$\sum_{\ell=0}^{n} m_{p^\ell -1} \sum_{i+j=n-\ell} s_i^{p^\ell} t_j^{p^{i+\ell}} = \sum_{\ell=0}^{n} m_{p^\ell -1} u_{n-\ell}^{p^\ell}.$$

or, since the terms of $\ell = n$ of both sides are the same $m_{p^n -1}$, we see that

$$\text{(7.15)} \qquad \sum_{\ell=0}^{n-1} m_{p^\ell -1} \sum_{i+j=n-\ell} s_i^{p^\ell} t_j^{p^{i+\ell}} = \sum_{\ell=0}^{n-1} m_{p^\ell -1} u_{n-\ell}^{p^\ell}.$$

This is a recursive formula to determine u_n over $\mathbb{Q}$-extensions. Multiplying p^n to both sides of (7.15) we obtain

$$\text{(7.16)} \qquad \sum_{\ell=0}^{n-1} p^{n-\ell}[CP_{p^\ell -1}] \sum_{i+j=n-\ell} s_i^{p^\ell} t_j^{p^{i+\ell}} = \sum_{\ell=0}^{n-1} p^{n-\ell}[CP_{p^\ell -1}] u_{n-\ell}^{p^\ell}.$$

This is a recursive formula over $BP^*(pt)$.

By (7.16) we see easily that u_j is a polynomial of $s_1, \ldots, s_j, t_1, \ldots, t_j$. But it seems to be very difficult to write these polynomials completely.

7.6. Let $\mathfrak{u}_{(p)} = (u_1, u_2, \ldots, u_k, \ldots)$ be a sequence of indeterminates such that $\dim u_j = -2(p^j - 1)$, and

$$\lambda : BP^*(\ )[\mathfrak{u}_{(p)}] \longrightarrow BP^*(\ )[\mathfrak{s}_{(p)}, \mathfrak{t}_{(p)}]$$

be a cohomology transformation such that $\lambda(x) = x$ for $x \in BP^*(X)$ and $\lambda(u_j) = u_j(s_1, \ldots, s_j, t_1, \ldots, t_j)$, the polynomials determined by (7.16).

λ is linear, degree-preserving and multiplicative ; and by (7.14) we see that both $\lambda \circ r_{\mathfrak{u}}$ and $r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}}$ send $e^{BP}(L)$ to $e^{\mu''}(L)$. Thus, by the uniqueness of Theorem 7.2 we obtain

$$\lambda \circ r_{\mathfrak{u}} = r_{\mathfrak{s}} \otimes 1 \circ r_{\mathfrak{t}},$$

or

$$(7.17) \qquad \sum_{\alpha,\beta} r_\beta(r_\alpha(x))\ \mathfrak{t}^\alpha_{(p)}\ \mathfrak{s}^\beta_{(p)} = \sum_\gamma r_\gamma(x)\ \mathfrak{u}^\gamma_{(p)},$$

for $x \in BP^*(X)$, where $\mathfrak{u} = (u_1(s_1, t_1), u_2(s_1, s_2, t_1, t_2), \ldots, u_j(s_1, \ldots, s_j, t_1, \ldots, t_j), \ldots)$ is the sequence of polynomials determined by (7.16). If we write monomials $\mathfrak{u}^\gamma_{(p)}$ as polynomials

$$(7.18) \qquad \mathfrak{u}^\gamma_{(p)} = \sum a^\gamma_{\alpha,\beta}\ \mathfrak{t}^\alpha_{(p)}\ \mathfrak{s}^\beta_{(p)}$$

over $BP^*(pt)$, then we get

$$(7.19) \qquad r_\beta \circ r_\alpha = \sum_\gamma a^\gamma_{\alpha,\beta}\ r_\gamma,$$

the formulas to express compositions $r_\beta \circ r_\alpha$ as linear combinations of r_γ over $BP^*(pt)$ (Theorem 5, (iv) of [18]).

Part II

§8. Typical formal groups in complex K-theory

8.1. Let K be the complex K-functor over finite CW-complexes. For any complex vector bundle E we use

$$e^K(E) = \lambda_{-1}(E) = \sum_i (-1)^i \lambda^i(E)$$

as its K-theoretic Euler class. Thus

$$e^K(L) = 1 - L$$

for a line bundle L. Then

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1) \cdot e^K(L_2)$$

for line bundles L_1 and L_2, i.e., the associated formal group F_K of K-functor is given by

$$F_K(X, Y) = X + Y - XY = 1 - (1 - X)(1 - Y). \tag{8.1}$$

n fold multiplication with respect to F_K is defined by

$$F_K(X_1, \ldots, X_n) = F_K(X_1, F_K(X_2, \ldots, X_n))$$

recursively. Then

$$F_K(X_1, \ldots, X_n) = 1 - (1 - X_1) \ldots (1 - X_n). \tag{8.2}$$

Thus

$$[n]_K(T) = 1 - (1 - T)^n$$

for any positive integer n. More generally, over any ring Λ such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$,

(8.3) $$[a]_K(T) = 1 - (1 - T)^a$$

for any $a \in \Lambda$, where

$$(1 - T)^a = 1 - a\cdot T + \frac{a(a-1)}{2} T^2 - \dots + (-1)^k \binom{a}{k} T^k + \dots$$

(cf., 2.4).

The Frobenius operator $\mathbf{f}_{n,K}$, $n \geq 1$, applied to the identity curve γ_0, is computed as follows.

$$\begin{aligned}(\mathbf{f}_{n,K}\gamma_0)(T) &= (-1)^{n-1}(\zeta_1 T^{1/n}) \dots (\zeta_n T^{1/n}) \\ &= T,\end{aligned}$$

where $\zeta_1, \dots, \zeta_n$ are n-th roots of 1, i.e.,

(8.4) $$\mathbf{f}_{n,K}\gamma_0 = \gamma_0$$

for any $n \geq 1$.

Over $\mathbb{Q}$ the logarithm

$$\log_K : F_K \overset{\simeq}{\to} G_a$$

is described by

(8.5) $$\log_K T = -\log(1 - T) = \sum_{n \geq 1} \frac{1}{n} T^n,$$

where log is the usual natural logarithm.

8.2. Let p be a fixed prime. Over $\mathbb{Z}_{(p)} = K(pt)_{(p)}$ the canonical typical curve $\xi_K = \xi_{K,(p)}$ can be computed by (8.2), (8.3) and (8.4), and we obtain

$$(8.6) \qquad \xi_K(T) = (\varepsilon_K\gamma_0)(T) = \sum_{(m,p)=1}^{F_K} (1 - (1 - T^m)^{\frac{\mu(m)}{m}})$$

$$= 1 - P(1 - T),$$

where $\mu(m)$ is the Möbius function and

$$(8.7) \qquad P(1 - T) = \prod_{(m,p)=1} (1 - T^m)^{\frac{\mu(m)}{m}}$$

is the Artin-Hasse series. (Cf., [10].)

Let $\mu_K = F_K^{\xi_K}$, the typical formal group canonically associated to F_K. Then

$$(8.8) \qquad \log_{\mu_K} = \log_K \circ \xi_K$$

over $\mathbb{Q}$, and by Proposition 3.3 and (8.5) we have

$$(8.9) \qquad \log_{\mu_K} T = \sum_{k \geq 0} \frac{1}{p^k} T^{p^k} = L(1 - T)$$

using a notation $L(1 - T)$ of Hasse [10]. Now

$$\begin{aligned} \log_{\mu_K} T &= \log_K(\xi_K(T)) && \text{by (8.8)} \\ &= -\log(1 - \xi_K(T)) && \text{by (8.5)} \\ &= -\log P(1 - T) && \text{by (8.6).} \end{aligned}$$

Thus

$$(8.10) \qquad L(1 - T) = -\log P(1 - T)$$

by (8.9). This was observed also in [10].

8.3. Next we observe formal groups of periodic K-cohomology $K^*(\ )$. Its coefficient object is $K^*(pt) = \mathbb{Z}[u, u^{-1}]$, where $u \in K^{-2}(pt)$ is the Bott periodic element. To make K^* complex oriented we define K^*-Euler class of a line bundle L over X by

$$e^{K^*}(L) = u^{-1} \cdot e^K(L) \in K^2(X).$$

Then its associated formal group is

$$F_{K^*} = F_K^{[u]\gamma_0},$$

i.e.,

$$(8.1)^* \qquad F_{K^*}(X, Y) = X + Y - u \cdot XY = u^{-1}(1 - (1 - uX)(1 - uY)).$$

Thus n fold multiplication with respect to F_{K^*} is

$$(8.2)^* \qquad F_{K^*}(X_1, \ldots, X_n) = u^{-1}(1 - (1 - uX_1)\ldots(1 - uX_n))$$

and

$$(8.3)^* \qquad [a]_{K^*}(T) = u^{-1}(1 - (1 - uT)^a)$$

for any $a \in \Lambda$ over $K^*(pt) \otimes \Lambda$, where $Z \subset \Lambda \subset \mathbb{Q}$.

The Frobenius operator $\mathbf{f}_{n,K^*}$, $n \geq 1$, applied to the identity curve γ_0, is

$$(8.4)^* \qquad \mathbf{f}_{n,K^*} \gamma_0 = [u^{n-1}] \gamma_0.$$

Finally, over $K^*(pt) \otimes \mathbb{Q}$ the logarithm

$$\log_{K^*} : F_{K^*} \overset{\simeq}{\to} G_a$$

is described by

$$(8.5)^* \qquad \log_{K^*} T = -u^{-1}\cdot\log(1 - u\cdot T) = \sum_{n\geq 1} \frac{u^{n-1}}{n}\cdot T^n.$$

8.4. Let p be a fixed prime. Over $K^*(pt)_{(p)} = K^*(pt) \otimes \mathbb{Z}_{(p)}$ the canonical typical curve ξ_{K^*} can be computed by (8.2)*, (8.3)* and (8.4)* and we obtain

$$(8.6)^* \qquad \xi_{K^*}(T) = (\varepsilon_{K^*}\gamma_0)(T) = u^{-1}(1 - P(1 - uT)).$$

Let $\mu_{K^*} = F_{K^*}^{\xi_{K^*}}$, the typical formal group canonically associated to F_{K^*}. Then

$$(8.8)^* \qquad \log_{\mu_{K^*}} = \log_{K^*} \circ \xi_{K^*}$$

over $K^*(pt) \otimes \mathbb{Q}$, and by Proposition 3.3 and (8.5)* we have

$$(8.9)^* \qquad \log_{\mu_{K^*}} T = u^{-1}L(1 - uT) = \sum_{k\geq 0} \frac{u^{p^k-1}}{p^k} T^{p^k}.$$

§9. Adams decompositions and typical formal groups

<u>9.1.</u> Let p be a fixed prime. Adams [1] and Sullivan [22] gave a decomposition of complex K-theory localized at p into p-1 factors. Since Adams' decomposition is more explicit we shall observe his decomposition.

Let $K(\ )_{(p)} = K(\ ) \otimes \mathbb{Z}_{(p)}$. Adams [1] gave linear idempotents of this functor

$$E_s \ : \ K(\ )_{(p)} \longrightarrow K(\ )_{(p)}$$

for each $s \in \mathbb{Z}$ by

$$(9.1) \qquad E_s(L) = \frac{1}{p-1} \sum_{m=1}^{p-1} \omega^{-ms} (1 - e^K(L))^{\omega^m}$$

for a line bundle L, where ω is a primitive (p-1)-th root of 1 as a p-adic integer. Even though $\omega \in \hat{\mathbb{Z}}_p$ all coefficients of $E_s(L)$ (as a power series of $e^K(L)$) lie actually in $\mathbb{Z}_{(p)}$ so that (9.1) is a well-defined formula. The formula (9.1) determines $\mathbb{Z}_{(p)}$-linear natural transformations E_s completely by splitting principle.

Following [1] we list basic properties of E_s quickly.

$$(9.2) \qquad E_s = E_{s'} \qquad \underline{\text{if}} \quad s \equiv s' \mod p-1.$$

Thus these natural transformations are defined actually for elements $\alpha = \{s\} \in \mathbb{Z}/(p-1)\mathbb{Z}$. Then

$$(9.3) \qquad E_\alpha^2 = E_\alpha \qquad \text{(idempotent)},$$

$$(9.4) \qquad E_\alpha E_\beta = 0 \qquad \underline{\text{if}} \quad \alpha \neq \beta \quad \underline{\text{in}} \quad \mathbb{Z}/(p-1)\mathbb{Z},$$

(9.5) $$\sum_{\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}} E_\alpha = 1.$$

By (9.3), (9.4) and (9.5) we have a natural decomposition

(9.6) $$K(\ )_{(p)} = E_0K(\ )_{(p)} \oplus E_1K(\ )_{(p)} \oplus \cdots \oplus E_{p-2}K(\ )_{(p)}$$

of the functor $K(\ )_{(p)}$ as a direct sum. Next

(9.7) $$E_\alpha(xy) = \sum_{\beta+\gamma=\alpha} E_\beta(x)E_\gamma(y)$$

for internal and external multiplications. In particular we see that, if $x \in E_\beta K(X)_{(p)}$ and $y \in E_\gamma K(X)_{(p)}$, then $xy \in E_{\beta+\gamma}K(X)_{(p)}$, and

(9.8) $E_0K(\ )_{(p)}$ is a multiplicative functor.

9.2. Let L_1 be a line bundle over S^2. Since $e^K(L_1)^2 = 0$, we have

$$\begin{aligned} E_1(L_1) &= \frac{1}{p-1}\sum_{m=1}^{p-1} \omega^{-m}(1 - e^K(L_1))^{\omega^m} \\ &= \frac{1}{p-1}\sum_{m=1}^{p-1} \omega^{-m}(1 - \omega^m e^K(L_1)) \\ &= \frac{1}{p-1}\left(\sum_{m=1}^{p-1} \omega^{-m}\right) - e^K(L_1) \\ &= -e^K(L_1). \end{aligned}$$

Here we used the fact that

$$\sum_{m=1}^{p-1} \omega^{ms} = \begin{cases} 0 & \text{if } s \not\equiv 0 \mod p-1 \\ p-1 & \text{if } s \equiv 0 \mod p-1, \end{cases}$$

which will be used later freely. In particular

$$E_1(1) = 0.$$

Thus

$$E_1(e^K(L_1)) = e^K(L_1).$$

Since $\tilde{K}(S^2)_{(p)}$ is generated by $e^K(L_1)$ (by choosing L_1 as the canonical line bundle) we see that

(9.9)
$$E_1\tilde{K}(S^2)_{(p)} = \tilde{K}(S^2)_{(p)},$$
$$E_\alpha\tilde{K}(S^2)_{(p)} = 0 \quad \underline{\text{if}} \quad \alpha \neq 1 \quad \underline{\text{in}} \quad \mathbb{Z}/(p-1)\mathbb{Z}.$$

Apply (9.7) to the smash product $S^2 \wedge \ldots \wedge S^2 = S^{2n}$ and obtain

(9.10)
$$E_n\tilde{K}(S^{2n})_{(p)} = \tilde{K}(S^{2n})_{(p)},$$
$$E_s\tilde{K}(S^{2n})_{(p)} = 0 \quad \underline{\text{if}} \quad s \not\equiv n \mod p-1.$$

Let $\phi : \tilde{K}(X) \xrightarrow{\cong} \tilde{K}(S^2 \wedge X)$ be the Bott isomorphism. By (9.7) and (9.9) we have the commutativity

$$\begin{array}{ccc} \tilde{K}(X)_{(p)} & \xrightarrow{\phi} & \tilde{K}(S^2 \wedge X)_{(p)} \\ \downarrow{\scriptstyle E_\alpha} & & \downarrow{\scriptstyle E_{\alpha+1}} \\ \tilde{K}(X)_{(p)} & \xrightarrow{\phi} & \tilde{K}(S^2 \wedge X)_{(p)} \end{array}$$

and ϕ induces an isomorphism

(9.11) $$\phi_\alpha : E_\alpha\tilde{K}(X)_{(p)} \xrightarrow{\cong} E_{\alpha+1}\tilde{K}(S^2 \wedge X)_{(p)}$$

for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$.

<u>9.3.</u> From the above idempotents we define linear idempotents

$$E_s^* \;:\; K^*(\;)_{(p)} \longrightarrow K^*(\;)_{(p)}$$

of K-cohomology localized at p for each $s \in \mathbb{Z}$. Define

$$E_s^{2i} \;:\; K^{2i}(\;)_{(p)} \longrightarrow K^{2i}(\;)_{(p)}$$

by requirement that the following diagram

$$\begin{tikzcd}
K^{2i}(X)_{(p)} \arrow[r, "\beta^i"] \arrow[d, "E_s^{2i}"] & K(X)_{(p)} \arrow[d, "E_{s+i}"] \\
K^{2i}(X)_{(p)} \arrow[r, "\beta^i"] & K(X)_{(p)}
\end{tikzcd}$$

commutes, where β is the Bott periodicity, i.e., the multiplication with $u \in K^{-2}(pt)$. Define

$$E_s^{2i+1} \;:\; K^{2i+1}(\;)_{(p)} \longrightarrow K^{2i+1}(\;)_{(p)}$$

by requirement that the following diagram

$$\begin{tikzcd}
K^{2i+1}(X)_{(p)} = \tilde{K}^{2i+1}(X)_{(p)} \arrow[r, "\sigma"] \arrow[d, "E_s^{2i+1}"] & \tilde{K}^{2i+2}(S^1 \wedge X)_{(p)} \arrow[d, "E_s^{2i+2}"] \\
K^{2i+1}(X)_{(p)} = \tilde{K}^{2i+1}(X)_{(p)} \arrow[r, "\sigma"] & \tilde{K}^{2i+2}(S^1 \wedge X)_{(p)}
\end{tikzcd}$$

commutes, where σ is the suspension isomorphism. Then by (9.11) we see that $E_s^* = \{E_s^i,\ i \in \mathbb{Z}\}$ commutes with suspensions and is a well-defined $\mathbb{Z}_{(p)}$-linear, degree-preserving idempotents of the cohomology theory $K^*(\;)_{(p)}$.

Basic properties of these idempotents follow by the corresponding properties of E_s's. First of all

(9.2)* $$E_s^* = E_{s'}^*, \qquad \underline{\text{if}} \quad s \equiv s' \mod p-1.$$

Thus these idempotents are defined actually for elements $\alpha = \{s\} \in \mathbb{Z}/(p-1)\mathbb{Z}$. Then, by (9.3), (9.4) and (9.5) we obtain

(9.3)* $$(E_\alpha^*)^2 = E_\alpha^*,$$

(9.4)* $$E_\alpha^* E_\beta^* = 0 \qquad \underline{\text{if}} \quad \alpha \neq \beta \quad \underline{\text{in}} \quad \mathbb{Z}/(p-1)\mathbb{Z},$$

(9.5)* $$\sum_{\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}} E_\alpha^* = 1.$$

Thus we have a natural decomposition

(9.6)* $$K^*(\ )_{(p)} = E_0^* K^*(\ )_{(P)} \oplus E_1^* K^*(\ )_{(p)} \oplus \cdots \oplus E_{p-2}^* K^*(\ )_{(p)}$$

of the cohomology $K^*(\ )_{(p)}$ as a direct sum and each direct factor $E_\alpha^* K^*(\ )_{(p)}$ itself is a cohomology theory.

Next by (9.7) and the definition of E_α^* we obtain

(9.7)* $$E_\alpha^*(xy) = \sum_{\beta+\gamma=\alpha} E_\beta^*(x) E_\gamma^*(y)$$

for internal and external multiplications. In particular we see that, if $x \in E_\beta^* K^*(X)_{(p)}$ and $y \in E_\gamma^* K^*(X)_{(p)}$, then $xy \in E_{\beta+\gamma}^* K^*(X)_{(p)}$, and

(9.8)* $E_0^* K^*(\ )_{(p)}$ is a multiplicative cohomology theory.

9.4. Put

(9.12) $$G^*(\ ) = E_0^* K^*(\ )_{(p)}.$$

This is a multiplicative cohomology theory inheriting its multiplicative structure from $K^*(\ )$. By definitions we see that

(9.13) $$G^*(pt) = \mathbb{Z}_{(p)}[u_1, u_1^{-1}], \quad u_1 = u^{p-1},$$

i.e., $G^*(\ )$ is a periodic cohomology theory of period $2(p-1)$ with u_1 as the periodicity element.

We give complex orientation of $K^*(\ )_{(p)}$ by assigning $\xi_{K^*}^{-1}(e^{K^*}(L))$ as Euler class of a line bundle L. Then its associated formal group is the typical group law μ_{K^*}. We denote as

$$e^{\mu_{K^*}}(L) = \xi_{K^*}^{-1}(e^{K^*}(L)).$$

Theorem 9.1. $e^{\mu_{K^*}}(L) \in G^2(X)$ <u>for any line bundle L over</u> X.

Proof. Using the notations of [10] we put

$$1 - \tau = P(1 - T) \quad \text{and} \quad 1 - T = Q(1 - \tau).$$

Then

$$\tau = \xi_K(T)$$

by (8.6). As is well known

(9.14) $$(1 - \tau)^{\omega^m} = P(1 - \omega^m T),$$

where ω is the primitive $(p-1)$-th root of 1 in $\hat{\mathbb{Z}}_p$ and $m \in \mathbb{Z}$, which can be seen by taking $-\log$ of both sides and by easy computaions.

Putting

$$\xi_K(T) = \sum_{j\geq 0} \xi_j^K T^{j+1},$$

we obtain

$$(1-\tau)^{\omega^m} = 1 - \sum_{j\geq 0} \xi_j^K(\omega^m T)^{j+1}$$

by (9.14) and (8.6). Now compute, for $s \in \mathbb{Z}$,

$$\frac{1}{p-1}\sum_{m=1}^{p-1} \omega^{-ms}(1-\tau)^{\omega^m}$$

$$= \frac{1}{p-1}\Big(\sum_{m=1}^{p-1} \omega^{-ms} - \sum_{j\geq 0} \xi_j^K\Big(\sum_{m=1}^{p-1} \omega^{m(j+1-s)}\Big)T^{j+1}\Big)$$

$$= \begin{cases} 1 - \sum\limits_{p-1 \mid j+1} \xi_j^K T^{j+1} & \text{if } s \equiv 0 \mod p\text{-}1 \\ - \sum\limits_{p-1 \mid j+1-s} \xi_j^K T^{j+1} & \text{if } s \not\equiv 0 \mod p\text{-}1. \end{cases}$$

Putting $\tau = e^K(L)$, by (9.1) we obtain

$$(9.15) \qquad \begin{aligned} E_0(L) &= 1 - \sum_{k\geq 1} \xi^K_{k(p-1)-1}(e^{\mu_K}(L))^{k(p-1)}, \\ E_s(L) &= - \sum_{k\geq 0} \xi^K_{k(p-1)+s-1}(e^{\mu_K}(L))^{k(p-1)+s} \end{aligned}$$

for $1 \leq s \leq p-2$. (Remark that $e^{\mu_K}(L) = \xi_K^{-1}(e^K(L))$.)

Put

$$\xi_{K^*}(T) = \sum_{j\geq 0} \xi_j^{K^*} T^{j+1},$$

then by (8.6) and (8.6)* we see that

$$\xi_j^{K^*} = \xi_j^K \cdot u^j$$

for $j \geq 0$. Since $e^{K^*}(L) = u^{-1} \cdot e^{K}(L)$ for a line bundle L over X, we have

$$e^{\mu_{K^*}}(L) = \xi_{K^*}^{-1}(e^{K^*}(L)) = u^{-1} \cdot e^{\mu_K}(L).$$

Now put

$$A(L) = \sum_{k \geq 0} \xi^{K^*}_{k(p-1)} (e^{\mu_{K^*}}(L))^{k(p-1)+1}.$$

Then

$$A(L) = u^{-1} \cdot \sum_{k \geq 0} \xi^{K}_{k(p-1)} (e^{\mu_K}(L))^{k(p-1)+1}$$
$$= u^{-1} \cdot E_1(L)$$

by (9.15). Hence by definition we see that

$$A(L) \in E_0^2 K^2(X)_{(p)} = G^2(X).$$

Remark that $A(L)$ is an invertible power series of $e^{\mu_{K^*}}(L)$, and all possible non-zero coefficients are

$$\xi^{K^*}_{k(p-1)} = \xi^{K}_{k(p-1)} \cdot (u^{p-1})^k \in G^*(pt).$$

Thus $e^{\mu_{K^*}}(L)$ can be solved as a power series of $A(L)$ with coefficients in $G^*(pt)$. Hence

$$e^{\mu_{K^*}}(L) \in G^2(X), \qquad \text{q. e. d.}$$

9.5. The above theorem implies that all coefficients of $\mu_{K^*}(X, Y)$ lie in $G^*(pt)$ and μ_{K^*} determines a typical formal group μ_{G^*} over

$G^*(pt)$ by restricting the domain of coefficients to $G^*(pt)$. The corresponding complex orientation of $G^*(\ )$ is given by assigning

$$e^{G^*}(L) = e^{\mu_{K^*}}(L) \in G^2(X)$$

as Euler class of a line bundle L over X. Its logarithm

$$\log_{\mu_{G^*}} : \mu_{G^*} \overset{\cong}{\to} G_a$$

over $G^*(pt) \otimes \mathbb{Q}$ is given by

$$(9.16) \qquad \log_{\mu_{G^*}} T = \sum_{k \geq 0} \frac{1}{p^k} u_1^{1+p+\dots+p^{k-1}} \cdot T^{p^k}.$$

(Cf., (8.9)*).

Identifying by periodic isomorphisms in $G^*(\ )$ we obtain a multiplicative cohomology $G^{\#}(\ )$ graded in $\mathbb{Z}/2(p-1)\mathbb{Z}$. (Remark that the difference of notations from the usual convention in K-theory !) $G^{\#}$ is complex oriented by assigning

$$e^G(L) = e^{G^*}(L)$$

as Euler classes. Its associated formal group is a typical formal group μ_G over $G^{\#}(pt) = \mathbb{Z}_{(p)}$ with

$$(9.16') \qquad \log_{\mu_G} T = \sum_{k \geq 0} \frac{1}{p^k} T^{p^k}.$$

9.6. By the universality of complex cobordism (cf., 5.1) we have a unique cohomology transformation

(9.17) $$td : U^* \longrightarrow K^*$$

which is linear, degree-preserving, multiplicative and

(9.18) $$td(e^U(L)) = e^{K^*}(L)$$

for a line bundle L. This is essentially the same as the td-map of Conner-Floyd [7], and we have

(9.19) $$td(x_{2n}) = Td(x_{2n})\cdot u^n$$

for $x_{2n} \in U^{-2n}(pt)$, where $Td(x_{2n})$ denotes the Todd genus of the weakly complex manifold representing x_{2n}. Remark the difference of signs from the corresponding formula of [7]. This point is adjusted by a choice of Bott-periodicity element u (among $\pm u$).

By (9.18) we see that

$$td(pt)_* F_U = F_{K^*} \quad \text{and} \quad td(pt)_* \xi_{U,(p)} = \xi_{K^*,(p)}$$

after localized at a prime p. Hence

(9.20) $$td(e^{BP}(L)) = e^{\mu_{K^*}}(L) = e^{G^*}(L)$$

for a line bundle L. This implies that

$$td(BP^*(X)) \subset G^*(X)$$

(cf., Theorem 7.2). And we obtain

Theorem 9.2. *By restricting* td *to* BP*( ) *we obtain a cohomology* transformation

$$\widetilde{td} : BP^*(\ ) \longrightarrow G^*(\ )$$

which is $\mathbb{Z}_{(p)}$-linear, degree-preserving, multiplicative and

$$\widetilde{td}(e^{BP}(L)) = e^{G^*}(L)$$

for a line bundle L.

§10. Coefficients of curves

Practically we need some calculus of coefficients of curves modulo some ideals. Here we collect some propositions necessary for these purposes.

<u>10.1.</u> Let R be a commutative ring with unity and I an ideal of R. Let $f(T) = \sum f_i T^i$ and $g(T) = \sum g_i T^i$ be formal power series over R. We say that

$$f \equiv 0 \quad \mod I$$

iff $f_i \in I$ for all $i \geq 0$, and

$$f \equiv g \quad \mod I$$

iff $f - g \equiv 0 \mod I$.

Lemma 10.1. *Let* f, f', g *and* g' *be formal power series over* R. *If* $f \equiv f' \mod I$ *and* $g \equiv g' \mod I$, *then*

$$f + g \equiv f' + g' \quad \mod I, \qquad f\,g \equiv f'g' \quad \mod I$$

and, when g *and* g' *are without constant terms,*

$$f \circ g \equiv f' \circ g' \quad \mod I.$$

Proof follows by routine arguments.

Let F be a formal group over R.

Lemma 10.2. *Let* γ_1, γ_1', γ_2 *and* γ_2' *be curves over* F. *If*

$\gamma_1 \equiv \gamma_1'$ mod I <u>and</u> $\gamma_2 \equiv \gamma_2'$ mod I, <u>then</u>

$$\gamma_1 +^F \gamma_2 \equiv \gamma_1' +^F \gamma_2' \quad \text{mod } I.$$

Proof follows by definition and the above Lemma.

Proposition 10.3. <u>Let</u> $\{\gamma_1, \gamma_2, \dots\}$ <u>and</u> $\{\gamma_1', \gamma_2', \dots\}$ <u>be</u> <u>Canchy sequences in</u> $\mathcal{C}_F$ <u>such that</u>

$$\gamma_i \equiv \gamma_i' \quad \text{mod } I$$

<u>for all</u> $i \geq 1$, <u>then</u>

$$\sum_{i\geq 1}{}^F \gamma_i \equiv \sum_{i\geq 1}{}^F \gamma_i' \quad \text{mod } I.$$

Proof. By the above Lemma the congruence is true for every finite partial sums. Since every coefficient of infinite sums can be found as a coefficient of suitable finite sum,the Proposition is true.

Lemma 10.4. <u>Let</u> γ_1 <u>and</u> γ_2 <u>be curves over</u> F <u>such that</u> $\gamma_1 \equiv 0$ mod I <u>and</u> $\gamma_2 \equiv 0$ mod I, <u>then</u>

$$\gamma_1 +^F \gamma_2 \equiv \gamma_1 + \gamma_2 \quad \text{mod } I^2.$$

Proof. By (1.4)

$$F(X, Y) = X + Y + XY\bar{F}(X, Y).$$

Thus

$$(\gamma_1 +^F \gamma_2)(T) = \gamma_1(T) + \gamma_2(T) + \gamma_1(T)\gamma_2(T)\bar{F}(\gamma_1(T), \gamma_2(T))$$

$$\equiv \gamma_1(T) + \gamma_2(T) \mod I^2.$$

q. e. d.

Proposition 10.5. <u>Let</u> $\{\gamma_1, \gamma_2, \ldots\}$ <u>be a Canchy sequence in</u> $\mathcal{C}_F$ <u>such that</u>

$$\gamma_i \equiv 0 \mod I$$

<u>for all</u> $i \geq 1$, <u>then</u>

$$\sum_{i\geq 1}{}^F \gamma_i \equiv \gamma_1 + \gamma_2 + \ldots + \gamma_n + \ldots \mod I^2.$$

Proof. The congruence is true for every finite partial sums by Lemmas 10.1 and 10.4, whence true for infinite sums.

Corollary 10.6. <u>Let</u> γ <u>be a curve over</u> F <u>such that</u> $\gamma \equiv 0 \mod I$. <u>Express as</u>

$$\gamma(T) = \sum_{k\geq 1}{}^F (c_{k-1}T^k),$$

<u>then</u>

$$c_k \in I$$

<u>for all</u> $k \geq 0$, <u>and</u>

$$\gamma(T) \equiv c_0 T + c_1 T^2 + \ldots + c_{k-1}T^k + \ldots \mod I^2.$$

In particular, when R is a $\mathbb{Z}_{(p)}$-algebra, μ is a typical formal group over R and γ is a typical curve over μ, expressing γ as $\gamma(T) = \sum_{k\geq 0}{}^{\mu}(c_k T^{p^k})$, if $\gamma \equiv 0 \mod I$, then $c_k \in I$ for all $k \geq 0$ and

$$\gamma(T) \equiv c_0T + c_1T^p + \dots + c_kT^{p^k} + \dots \mod I^2.$$

10.2. Let R, I, F be as above.

Lemma 10.7. *Let* γ *be a curve over* F *such that* $\gamma \equiv 0 \mod I$. *Then*

$$([c]\gamma_0 +^F \gamma)(T) \equiv cT + \frac{\partial F}{\partial Y}(cT, 0) \cdot \gamma(T) \mod I^2$$

for $c \in R$, *where* γ_0 *is the identity curve over* R.

Proof.
$$\begin{aligned}([c]\gamma_0 +^F \gamma)(T) &= F(cT, \gamma(T)) \\ &= \sum a_{ij}(cT)^i\gamma(T)^j \\ &\equiv cT + \sum_{n>0} a_{n1}(cT)^n \cdot \gamma(T) \mod I^2 \\ &\equiv cT + \frac{\partial F}{\partial Y}(cT, 0) \cdot \gamma(T) \mod I^2.\end{aligned}$$

q. e. d.

Next suppose R is of characteristic zero. Differentiating the relation

$$\log_F F(X, Y) = \log_F X + \log_F Y$$

with respect to Y, we obtain

$$\frac{\partial F}{\partial Y}(T, 0) \cdot \log_F' T = 1,$$

where $\log_F'$ is the derivative of $\log_F$. This shows firstly that $\log_F'$ is a power series defined over R, i.e., if we put

(10.1) $$\log_F T = \sum_{k\geq 1} \frac{n_k}{k+1} T^{k+1}, \quad n_0 = 1,$$

then all $n_k \in R$, and secondly that all coefficients of $\frac{\partial F}{\partial Y}(T, 0)$ are integral polynomials of coefficients n_k of $\log_F^!$.

Proposition 10.8. *Let R be of characteristic zero, and I an ideal of R containing all coefficients n_k of positive degrees of $\log_F^!$. Let γ be a curve over F, and expressing as* $\gamma(T) = \sum_{k\geq 1}^{F}(c_{k-1}\, T^k)$, *suppose that $c_k \in I$ for all $k > 0$. Then*

$$\gamma(T) \equiv c_0T + c_1T^2 + \dots + c_{k-1}T^k + \dots \mod I^2.$$

Proof. Put

$$\gamma_1(T) = \sum_{k>1}^{F}(c_{k-1}\, T^k),$$

then $\gamma_1 \equiv 0 \mod I$ and by Proposition 10.5

$$\gamma_1(T) \equiv c_1T^2 + \dots + c_{k-1}T^k + \dots \mod I^2.$$

Now by Lemma 10.7

$$\begin{aligned}\gamma(T) &= ([c_0]\gamma_0 +^F \gamma_1)(T) \\ &\equiv c_0T + \frac{\partial F}{\partial Y}(c_0T, 0) \cdot \gamma_1(T) \mod I^2.\end{aligned}$$

By the remark above the proposition we have

$$\frac{\partial F}{\partial Y}(c_0T, 0) - 1 \equiv 0 \mod I.$$

Thus

$$\begin{aligned}\gamma(T) &\equiv c_0T + \gamma_1(T) \mod I^2 \\ &\equiv c_0T + c_1T^2 + \dots + c_{k-1}T^k + \dots \mod I^2.\end{aligned}$$

q. e. d.

<u>10.3.</u> Let R be a $\mathbb{Z}_{(p)}$-algebra and μ a typical formal group over R. Let $\mathbf{t} = (t_1, t_2, \ldots)$ be a sequence of indeterminates and put

$$\phi_{\mathbf{t}}(T) = \sum_{k \geq 0}^{\mu} (t_k T^{p^k}), \qquad t_0 = 1,$$

which is a typical curve over μ (extending the domain of coefficients to $R[\mathbf{t}] = R[t_1, t_2, \ldots])$.

$$\mu' = \mu^{\phi_{\mathbf{t}}}$$

is a typical formal group over $R[\mathbf{t}]$. Since

$$\phi_{\mathbf{t}}^{-1} : \mu \cong \mu'$$

and μ is typical, $\phi_{\mathbf{t}}^{-1}$ is a typical curve over μ'. Put

$$\phi_{\mathbf{t}}^{-1}(T) = \sum_{k \geq 0}^{\mu'} (s_k T^{p^k}), \qquad s_0 = 1,$$

then $s_j = s_j(t_1, t_2 \ldots) \in R[\mathbf{t}]$. Here we put $\mathbf{t} = 0 = (0, \ldots, 0, \ldots)$, then $\phi_0 = \gamma_0$ so $\phi_0^{-1} = \gamma_0$ and

$$s_j(0, \ldots, 0, \ldots) = 0 \qquad \text{for } j > 0,$$

i.e.,

(10.2) $$s_j \in I = (t_1, t_2, \ldots) \qquad \text{for all } j > 0.$$

Proposition 10.9. <u>Under the above situation let</u> $I = (t_1, t_2, \ldots)$, the ideal generated by $t_1, t_2, \ldots,$ in $R[\mathbf{t}]$. Then

$$s_k + t_k \equiv 0 \mod I^2$$

for all $k > 0$.

Proof.
$$\begin{aligned} T &= \phi_{\mathbf{t}} \circ \phi_{\mathbf{t}}^{-1}(T) \\ &= \phi_{\mathbf{t}\#}\left(\sum_{j\geq 0}{}^{\mu'} s_j T^{p^j}\right) \\ &= \sum_{i\geq 0, j\geq 0} \left(t_i s_j^{p^i} T^{p^{i+j}}\right) \\ &= T +^{\mu} \sum_{i+j>0}{}^{\mu}\left(t_i s_j^{p^i} T^{p^{i+j}}\right). \end{aligned}$$

Thus

$$\sum_{i+j>0}{}^{\mu}\left(t_i s_j^{p^i} T^{p^{i+j}}\right) = 0.$$

Then, by Proposition 10.5

$$\begin{aligned} 0 = \sum_{i+j>0}{}^{\mu}\left(t_i s_j^{p^i} T^{p^{i+j}}\right) &\equiv \sum_{i+j>0} t_i s_j^{p^i} T^{p^{i+j}} \mod I^2 \\ &\equiv \sum_{k>0} (t_k + s_k) T^{p^k} \mod I^2. \end{aligned}$$

Hence

$$t_k + s_k \equiv 0 \mod I^2$$

for all $k > 0$.

§11. Stong-Hattori Theorem

In this section we prove Stong-Hattori Theorem [11], [20] in our version based on formal group materials.

<u>11.1.</u> Here we put $\mu = \mu_{G^*}$. Let $\mathfrak{t} = (t_1, t_2, \ldots)$ be a sequence of indeterminates with $\dim t_j = -2(p^j - 1)$, and put

$$\phi_{\mathfrak{t}}(T) = \sum_{j\geq 0}^{\mu} (t_j T^{p^j}), \qquad t_0 = 1.$$

$\phi_{\mathfrak{t}}$ is a typical curve over μ by extending the domain of coefficients to $G^*(pt)[\mathfrak{t}] = G^*(pt)[t_1, t_2, \ldots]$.

$$\mu' = \mu^{\phi_{\mathfrak{t}}}$$

is a typical formal group over $G^*(pt)[\mathfrak{t}]$. We give the complex orientation of the cohomology theory $G^*(\ )[\mathfrak{t}]$ by assigning $\phi_{\mathfrak{t}}^{-1}(e^{G^*}(L))$ as Euler class of a line bundle L. Then its associated formal group is the typical μ'. Hence by Theorem 7.2 there exists a unique multiplicative cohomology transformation

$$h : BP^*(\ ) \longrightarrow G^*(\ )[\mathfrak{t}]$$

such that

$$h(e^{BP}(L)) = \phi_{\mathfrak{t}}^{-1}(e^{G^*}(L))$$

for a line bundle L, and

(11.1) $$h(pt)_* \mu_{BP} = \mu'.$$

Put

$$\bar{h} = h(pt)$$

for simplicity. By a standard argument (cf., [2]) we can identify $\bar{h}$ with the Boardman map

$$\pi_*(BP) \longrightarrow \pi_*(G \wedge BP),$$

and the Stong-Hattori map

$$\pi_*(MU) \longrightarrow \pi_*(K \wedge MU)$$

decomposes as direct sum of copies of $\bar{h}$ after localized at the prime p. Thus we can state Stong-Hattori Theorem in our version as

Theorem 11.1. $\bar{h} = h(pt) : BP^*(pt) \longrightarrow G^*(pt)[\mathfrak{t}]$ <u>is a split monomorphism</u>.

Stong-Hattori Theorem in this form is proved also in [3] by a different method.

<u>11.2.</u> Before going into the proof of Theorem 11.1 we compute some materials of μ. $\log_\mu$ is already given in (9.16).

We compute $[p]_\mu$:

$$\begin{aligned}\log_\mu [p]_\mu(T) &= p \cdot \log_\mu T \\ &= pT + \sum_{k\geq 0} \frac{1}{p^k} u_1^{1+p+\ldots+p^k} T^{p^{k+1}} \\ &= pT + \sum_{k\geq 0} \frac{1}{p^k} u_1^{1+p+\ldots+p^{k-1}} (u_1 T^p)^{p^k}\end{aligned}$$

$$= \log_\mu(\exp_\mu(pT)) + \log_\mu(u_1 T^p),$$

where $\exp_\mu = \log_\mu^{-1}$. Hence

$$[p]_\mu(T) = \exp_\mu(pT) +^\mu (u_1 T^p). \tag{11.2}$$

Since

$$\exp_\mu : G_a \underset{=}{\sim} \mu \qquad \text{over } G^*(pt) \otimes \mathbb{Q}$$

and G_a is additive, whence typical, we see that $\exp_\mu$ is a typical curve over μ (over $G^*(pt) \otimes \mathbb{Q}$). Put

$$\exp_\mu T = \sum_{i\geq 0}^{\mu} (e_i T^{p^i}), \qquad e_0 = 1,$$

with $\dim e_i = -2(p^i - 1)$. Then

$$\begin{aligned} T &= \log_\mu(\exp_\mu T) \\ &= \log_{\mu\#}\Big(\sum_{i\geq 0}^{\mu} (e_i T^{p^i}) \Big) \\ &= \sum_{i\geq 0} \sum_{j\geq 0} \frac{1}{p^i} u_1^{1+p+\ldots+p^{j-1}} e_i^{p^j} T^{p^{i+j}} . \end{aligned}$$

Therefore

$$\sum_{j=0}^{k} \frac{1}{p^j} u_1^{1+p+\ldots+p^{j-1}} e_{k-j}^{p^j} = 0$$

for all $k > 0$. Or, multiplying p^{p^k} to this formula we obtain

$$\sum_{j=0}^{k} \frac{1}{p^j} u_1^{1+p+\ldots+p^{j-1}} (p^{p^{k-j}} e_{k-j})^{p^j} = 0 \quad \text{for } k > 0. \tag{11.3}$$

Lemma 11.2. $p^{p^k} e_k \in p \cdot G^*(pt)$ <u>for all</u> $k \geq 0$.

Proof. For $k = 0$: since $e_0 = 1$ we have

$$p^{p^0} e_0 = p \in p \cdot G^*(pt).$$

We prove the Lemma by induction on k. Assume it is proved until $k - 1$.

Then by (11.3)

$$-p^{p^k} e_k = \sum_{j=1}^{k} \frac{1}{p^j} u_1^{1+p+\ldots+p^{j-1}} (p^{p^{k-j}} e_{k-j})^{p^j}.$$

Here

$$p^{p^{k-j}} e_{k-j} \in p \cdot G^*(pt)$$

for $1 \leq j \leq k$ by induction hypothesis. Then, since $j < p^j$ for $1 \leq j \leq k$ we have

$$\frac{1}{p^j}(p^{p^{k-j}} e_{k-j})^{p^j} \in p \cdot G^*(pt)$$

for $1 \leq j \leq k$. Hence

$$-p^{p^k} e_k \in p \cdot G^*(pt).$$

q. e. d.

Now

$$\exp_\mu(pT) = \sum_{k \geq 0}^{\mu} (p^{p^k} e_k T^{p^k}).$$

Then by Proposition 10.5 and the above Lemma, putting $I = p \cdot G^*(pt)$ we obtain

Lemma 11.3. $\exp_\mu(pT)$ _is a power series over_ $G^*(pt)$ _and_

$$\exp_\mu(pT) \equiv 0 \mod p \cdot G^*(pt).$$

Corollary 11.4. $[p]_\mu(T) \equiv u_1 T^p \mod p \cdot G^*(pt).$

This follows by Lemma 10.2, (11.2) and Lemma 11.3.

<u>11.3.</u> We compute $\mathbb{f}_{p,\mu}\gamma_0$:

$$\begin{aligned}
\xi_{K*\#}(\mathbb{f}_{p,\mu_{K*}}\gamma_0) &= \mathbb{f}_{p,K*}(\xi_{K*}) \\
&= \mathbb{f}_{p,K*}\varepsilon_{K*}\gamma_0 = \varepsilon_{K*}\ \mathbb{f}_{p,K*}\gamma_0 \\
&= \varepsilon_{K*}([u^{p-1}]\gamma_0) \quad \text{by} \quad (8.4)^* \\
&= [u^{p-1}]\varepsilon_{K*}\gamma_0 \quad \text{by Proposition 2.9} \\
&= [u^{p-1}]\xi_{K*\#}\gamma_0 \\
&= \xi_{K*\#}[u^{p-1}]\gamma_0 \quad \text{by Proposition 2.4.}
\end{aligned}$$

Since $\xi_{K*\#} : \mathcal{C}_{\mu_{K*}} \cong \mathcal{C}_{F_{K*}}$, an isomorphism, we obtain

$$\mathbb{f}_{p,\mu_{K*}}\gamma_0 = [u^{p-1}]\gamma_0 .$$

Hence

(11.4) $$\mathbb{f}_{p,\mu}\gamma_0 = [u_1]\gamma_0 .$$

Then we compute $\mathbb{f}_{p,\mu}(t_j T^{p^j})$:

for $j = 0$, since $t_0 = 1$ we have

$$\begin{aligned}
\mathbb{f}_{p,\mu}(t_0 T) &= (\mathbb{f}_{p,\mu}\gamma_0)(T) \\
&= u_1 T \qquad \text{by (11.4) ;}
\end{aligned}$$

for $j > 0$,

$$\begin{aligned}
\mathbb{f}_{p,\mu}(t_j T^{p^j}) &= (\mathbb{f}_{p,\mu}\mathbb{V}_p\mathbb{V}_{p^{j-1}}[t_j]\gamma_0)(T) \\
&= [p]_\mu(\mathbb{V}_{p^{j-1}}[t_j]\gamma_0)(T) \qquad \text{by Proposition 2.9}
\end{aligned}$$

$$= [p]_\mu(t_j T^{p^{j-1}})$$

$$\equiv u_1 t_j^p T^{p^j} \qquad \text{mod} \quad p\cdot G^*(pt)[\mathfrak{t}]$$

by Corollary 11.4. Thus we obtained

Lemma 11.5. i) $\quad \mathfrak{f}_{p,\mu}(t_0 T) = u_1 T,$

ii) $\quad \mathfrak{f}_{p,\mu}(t_j T^{p^j}) \equiv u_1 t_j^p T^{p^j} \quad \text{mod} \quad p\cdot G^*(pt)[\mathfrak{t}]$

for $j > 0$.

11.4. Proof of Theorem 11.1. Put

$$\overline{v}_i = \overline{h}(v^{(p)}_{p^i-1})$$

for $i \geq 1$, where $v^{(p)}_{p^i-1}$'s are the polynomial basis of $BP^*(pt)$ (Theorem 6.2). Then by Theorem 6.3 we obtain

$$(11.5) \qquad (\mathfrak{f}_{p,\mu}, \gamma_0)(T) = \overline{h}_*(\mathfrak{f}_{p,BP}\gamma_0)(T) = \sum_{i\geq 1}^{\mu'}(\overline{v}_i T^{p^{i-1}}).$$

Since $\phi_{\mathfrak{t}} : \mu' \cong \mu$, we have

$$\phi_{\mathfrak{t}} \circ (\mathfrak{f}_{p,\mu}, \gamma_0) = \phi_{\mathfrak{t}\#}(\mathfrak{f}_{p,\mu}, \gamma_0)$$

$$= f_{p,\mu}(\phi_{\mathfrak{t}\#}\gamma_0)$$

$$= f_{p,\mu}\phi_{\mathfrak{t}}.$$

Thus

$$(11.6) \qquad (\mathfrak{f}_{p,\mu}, \gamma_0)(T) = \phi_{\mathfrak{t}}^{-1}\Big(\sum_{j\geq 0}^{\mu} \mathfrak{f}_{p,\mu}(t_j T^{p^j})\Big)$$

$$\equiv \phi_{\mathfrak{t}}^{-1}\Big(\sum_{j\geq 0}^{\mu} (u_1 t_j^p T^{p^j})\Big) \qquad \text{mod} \quad p\cdot G^*(pt)[\mathfrak{t}]$$

by Lemma 10.1, Proposition 10.3 and Lemma 11.5. Comparing the lowest terms (deg 1) of both sides of (11.6) we obtain

$$\bar{v}_1 \equiv u_1 \qquad \bmod \quad p\cdot G^*(pt)[\mathfrak{t}]. \tag{11.7}$$

Put

$$\phi_{\mathfrak{t}}^{-1}(T) = \sum_{j\geq 0}^{\mu'}(s_j T^{p^j}), \qquad s_0 = 1,$$

as in 10.3, and put

$$I = (t_1, t_2, \ldots),$$

the ideal generated by $t_1, t_2, \ldots$ in $G^*(pt)[\mathfrak{t}]$. Then

$$s_j + t_j \equiv 0 \qquad \bmod \quad I^2$$

for $j > 0$ by Proposition 10.9. Hence we can use $s_1, s_2, \ldots$ as a polynomial basis of $G^*(pt)[\mathfrak{t}]$, i.e.,

$$G^*(pt)[\mathfrak{t}] = G^*(pt)[s_1, s_2, \ldots].$$

By (11.6), using Lemma 10.1 and Proposition 10.3, we obtain

$$(\mathfrak{f}_{p,\mu}{}^{\prime}\gamma_0)(T) \equiv \sum_{j\geq 0}^{\mu'}(u_1^{p^j} s_j T^{p^j}) \qquad \bmod \quad p\cdot G^*(pt)[\mathfrak{t}] + I^2,$$

whence by (11.5) we see that

$$\bar{v}_j \equiv u_1^{p^{j-1}} s_{j-1} \qquad \bmod \quad p\cdot G^*(pt)[\mathfrak{t}] + I^2 \tag{11.8}$$

for $j > 1$.

To prove Theorem 11.1, it is sufficient to prove that "$\bar{h}$ mod p" is injective. Since u_1 is invertible we can use $\{u_1^{p^j} s_j, j \geq 1\}$ as a polynomial basis of $G^*(pt)[\mathfrak{t}]$. Then by (11.8)

$$G^*(pt)[\mathfrak{t}] \otimes F_p = G^*(pt)[\overline{v}_2, \overline{v}_3, \ldots] \otimes F_p,$$

where $F_p = \mathbb{Z}/p\cdot\mathbb{Z}$, which contains $F_p[u_1, \overline{v}_2, \overline{v}_3, \ldots]$ as a subalgebra. Finally by (11.7) we see that $\overline{v}_1, \overline{v}_2, \ldots$ are algebraically independent over F_p. Therefore " $\overline{h}$ mod p" is injective.

§12. Conner-Floyd Theorem

Conner-Floyd [7] proved the natural isomorphism

$$U^*(X) \otimes_{U^*(pt)} \mathbb{Z} = K^*(X)$$

regarding both sides as $\mathbb{Z}_2$-graded. Here we shall see a corresponding relation holds between BP^* and G^*.

12.1. Here we write the polynomial basis $v^{(p)}_{p^k-1}$ of $BP^*(pt)$ (Theorem 6.2) by v_k for simplicity. Compute $Td(v_k)$ by the recursive formula (6.6). Remarking that

$$Td(m_{p^k-1}) = \frac{1}{p^k},$$

by an induction on k we obtain

$$(12.1) \qquad \begin{aligned} Td(v_1) &= 1 \\ Td(v_k) &= 0 \qquad\qquad \text{for} \quad k > 1. \end{aligned}$$

12.2. Using notations of 9.4 we map a line bundle L over X to

$$1 - \sum_{k\geq 1} \xi^K_{k(p-1)-1}\, v_1^k (e^{BP}(L))^{k(p-1)} \in BP^0(X)$$

where we regard as $\xi^K_{j-1} \in \mathbb{Z}_{(p)}$. By splitting principle this extends to a natural map

$$\chi' \;:\; K(X) \longrightarrow BP^0(X).$$

For a line bundle L we have

$$\begin{aligned} td \circ \chi'(L) &= 1 - \sum_{k\geq 1} \xi^K_{k(p-1)-1}\, u_1^k (e^{\mu_{K^*}}(L))^{k(p-1)} \\ &= 1 - \sum_{k\geq 1} \xi^K_{k(p-1)-1} u^{k(p-1)} (u^{-1} e^{\mu_K}(L))^{k(p-1)} \\ &= 1 - \sum_{k\geq 1} \xi^K_{k(p-1)-1}\, e^{\mu_K}(L)^{k(p-1)} \\ &= E_0(L) \end{aligned}$$

by (9.15). Thus

(12.2) $$td \circ \chi' = E_0.$$

Remark that

$$G^0(X) = E_0 K(X) \subset K(X)$$

and define

$$\chi^0 \;:\; G^0(X) \longrightarrow BP^0(X)$$

by a restriction of χ'. Since E_0 is an idempotent, by (12.2) we see that

$$\widetilde{td} \circ \chi^0 = 1.$$

For negative integers s such that $-2(p-1) < s < 0$ we define

$$\chi^s \;:\; G^s(X) \longrightarrow BP^s(X)$$

by requiring they commute with suspensions and χ^0. χ^s is uniquely defined by this requirement. Since $\widetilde{td}$ also commutes with suspensions we see that

(12.3) $$\widetilde{td} \circ \chi^s = 1$$

for $-2(p-1) < s \leq 0$.

12.3. Make BP^* $\mathbb{Z}/2(p-1)\mathbb{Z}$-graded by

$$BP^{\alpha}(X) = \sum_{s \equiv \alpha \bmod 2(p-1)} BP^{s}(X)$$

for $\alpha \in \mathbb{Z}/2(p-1)\mathbb{Z}$. We denote this cohomology by $BP^{\#}$. Then td induces multiplicative cohomology transformation

$$\widetilde{td}^{\#} \; : \; BP^{\#}(\;) \longrightarrow G^{\#}(\;)$$

such that

$$\widetilde{td}^{\#}(pt) = Td \; : \; BP^{\#}(pt) \longrightarrow \mathbb{Z}_{(p)}.$$

Thus $\mathbb{Z}_{(p)}$ is a $BP^{\#}(pt)$-module. Now we can state

Theorem 12.1. There exists natural isomorphism

$$BP^{\#}(X) \otimes_{BP^{\#}(pt)} \mathbb{Z}_{(p)} = G^{\#}(X).$$

For the proof of this theorem the most basic thing is the existence of natural degree-preserving map

$$\chi^{\#} \; : \; G^{\#}(X) \longrightarrow BP^{\#}(X)$$

such that

(12.4) $$\widetilde{td}^{\#} \circ \chi^{\#} = 1.$$

This is defined by $\chi^{\#} = \{\chi^{s} : -2(p-1) < s \leq 0\}$ and proved by (12.3).

The rest of the proof is completely parallel to the proof of [7], Theorem (10.1), p.60. The proof is devided into three steps as in [7].

At each step Quillen decomposition and the use of corresponding facts of complex cobordism are helpful. Details are left to readers.

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