explain in the next section, we can analyze the singular terms in (4.2) in terms of these subquotient theories.

THEOREM 4.3. Assume that k_J^* is e-invariant for every proper subquotient J of G and let $Y = \tilde{E}\mathscr{P}$.

(i) If G is not elementary Abelian, then $k_G^*(Y; \tilde{E}G) = 0$.
(ii) If $G = (\mathbb{Z}/p)^r$, then $k_G^*(Y; \tilde{E}G)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1} k_{G/G}^*(S^0)$.

Warning: the nonequivariant theory $k_{G/G}^*$ is usually quite different from the underlying nonequivariant theory $k^* = k_e^*$.

As we shall explain in Section 6, we can use Adams spectral sequences to analyze the free terms in (4.2).

THEOREM 4.4. Assume that k_G is split and k is bounded below and let $Y = \tilde{E}\mathscr{P}$.

(i) If G is not elementary Abelian, then $k_G^*(Y; EG_+) = 0$.
(ii) If $G = (\mathbb{Z}/p)^r$ and $H^*(k)$ is finite dimensional, then $k_G^*(Y; EG_+)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^r k^*(S^0)$.

The hypothesis that $H^*(k)$ be finite dimensional in (ii) is extremely restrictive, although it is satisfied trivially when k is the sphere spectrum. The hypothesis is actually necessary. We shall see in Section 7 that the theories $\pi_G^*(\cdot; B_G\Pi_+)$ are e-invariant for finite groups Π. They satisfy all other hypotheses of our theorems, but here k^* and $k_{G/G}^*$ are different. In such cases, the calculation of $k_G^*(Y; EG_+)$ falls out from the e-invariance, which must be proven differently, and (4.2).

Carlsson's reduction is now the case π_G of the following immediate inductive consequence of the first parts of Theorems 4.3 and 4.4.

THEOREM 4.5. Suppose that G is not elementary Abelian. Assume

(i) k_J^* is e-invariant for all elementary Abelian subquotients J;
(ii) k_J is split and $k_{K/K}$ is bounded below for all non-elementary Abelian subquotients $J = H/K$.

Then k_J^* is e-invariant for all subquotients J, including $J = G$.

Returning to cohomotopy and the proof of the Segal conjecture, it only remains to prove that the map δ in (4.2) is an isomorphism when $G = (\mathbb{Z}/p)^r$. We assume that the result has been proven for $1 \leq q < r$. Comparing Theorems 4.3 and 4.4, we see that the map δ in (4.2) is a map between free π^*-modules on the same number of generators. It suffices to show that δ is a bijection on generators, which means that it is an isomorphism in degree $r-1$. Here δ is a map between free modules on the same number of generators over the p-adic integers $\mathbb{Z}_p^{\wedge}$, so that it will be an isomorphism if it is a monomorphism when reduced mod p.

To prove this, let $k_G = F(EG_+, H\underline{\mathbb{F}}_p)$, where $H\mathbb{F}_p$ is the Eilenberg-MacLane G-spectrum associated to the "constant Mackey functor" at $\mathbb{F}_p$ that we obtain from IX.4.3. This theory, like any other theory represented by a function spectrum $F(EG_+, \cdot)$, is e-invariant. Since $\pi_0^G(H\underline{\mathbb{F}}_p) = \mathbb{F}_p$, we have a unit map $S_G \longrightarrow H\underline{\mathbb{F}}_p$, and we compose with $\varepsilon : H\underline{\mathbb{F}}_p \longrightarrow k_G$ to obtain $\eta : S_G \longrightarrow k_G$. There is an induced map $S = S_{G/G} \longrightarrow k_{G/G}$, and a little calculation shows that it sends the unit in $\pi^0(S)$ to an element that is non-zero mod p. We can also check that the subquotient theories k_J^* are all e-invariant. By the naturality of (4.2), we have the commutative diagram

$$\begin{array}{ccc} \pi_G^{r-1}(Y; \tilde{E}G) & \xrightarrow{\delta} & \pi_G^r(Y; EG_+) \\ \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \eta_*} \\ k_G^{r-1}(Y; \tilde{E}G) & \xrightarrow[\delta]{} & k_G^r(Y; EG_+). \end{array}$$

The bottom map δ is an isomorphism since $k_G^*(Y) = 0$. The left map η_* is the sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}\eta_*$, $\eta_* : \pi^0(S) \longrightarrow \pi^0(k_{G/G})$, and is therefore a monomorphism mod p. Thus the top map δ is a monomorphism mod p, and this concludes the proof.

J. F. Adams, J. H. Gunawardena, and H. Miller. The Segal conjecture for elementary Abelian p-groups-I. Topology 24(1985), 435-460.

G. Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. Annals Math. 120(1984), 189-224.

J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups-II. Topology 26(1987), 413-433.

5. Approximations of singular subspaces of G-spaces

Let SX denote the singular set of a G-space X, namely the set of points with non-trivial isotropy group. The starting point of the proof of Theorem 4.3 is the space level observation that the inclusions

$$SX \longrightarrow X \quad \text{and} \quad S^0 \longrightarrow \tilde{E}G$$

induce bijections

$$[X, \tilde{E}G \wedge X']_G \longrightarrow [SX, \tilde{E}G \wedge X']_G \longleftarrow [SX, X']_G.$$

We may represent theories on finite G-CW complexes via colimits of space level homotopy classes of maps. The precise formula is not so important. What is important is that, when calculating $k_G^*(X; \tilde{E}G)$, we get a colimit of terms of the general form $[SW, Z]_G$. We can replace S here by other functors T on spaces that satisfy appropriate axioms and still get a cohomology theory in X, called $k_G^*(X; T)$. Such functors are called "S-functors". Natural transformations $T \longrightarrow T'$ induce maps of theories, contravariantly. We have a notion of a cofibration of S-functors, and cofibrations give rise to long exact sequences. In sum, we have something like a cohomology theory on S-functors T.

We construct a filtered S-functor A that approximates the singular functor S. Let $\mathscr{A} = \mathscr{A}(G)$ be the partially ordered set of non-trivial elementary Abelian subgroups of G, thought of as a G-category with a map $A \longrightarrow B$ when $B \subset A$, with G acting by conjugation. If $G \neq e$, the classifying space $B\mathscr{A}$ is G-contractible. In fact, if C is a central subgroup of order p, then the diagram $A \longleftarrow AC \longrightarrow C$ displays the values on an object A of three G-equivariant functors on $\mathscr{A}$ together with two equivariant natural transformations between them; these induce a G-homotopy from the identity to the constant G-map at the vertex C.

We can parametrize $\mathscr{A}$ by points of SX. Precisely, we construct a topological G-category $\mathscr{A}[X]$ whose objects are pairs (A, x) such that $x \in X^A$; there is a morphism $(A, x) \longrightarrow (B, y)$ if $B \subset A$ and $y = x$, and G acts by $g(A, x) = (gAg^{-1}, gx)$. Projection on the X-coordinate gives a functor $\mathscr{A}[X] \longrightarrow SX$, where SX is a category in the trivial way, and $B\mathscr{A}[X] \longrightarrow BSX = SX$ is a G-homotopy equivalence. The subspace $B\mathscr{A}[*]$ of $B\mathscr{A}[X]$ is G-contractible. Let $AX = B\mathscr{A}[X]/B\mathscr{A}[*]$. We still have a G-homotopy equivalence $AX \longrightarrow SX$, but now A is an S-functor and our equivalences give a map of S-functors. For any space Y, we have

$$k_G^*(Y; \tilde{E}G) \cong k_G^*(Y; S) \cong k_G^*(Y; A).$$

The functor A arises from geometric realizations of simplicial spaces and carries the simplicial filtration F_qA; here $F_{-1}A = *$ and $F_{r-1}A = A$, where $r = \operatorname{rank}(G)$. Inspection of definitions shows that the successive subquotients satisfy

$$(F_qA/F_{q-1}A)(X) = \bigvee \Sigma^q(G_+ \wedge_{H(\omega)} X^{A(\omega)}).$$

Here ω runs over the G-conjugacy classes of strictly ascending chains $(A_0, \cdots, A_q)$ of non-trivial elementary Abelian subgroups of G, $H(\omega)$ is the isotropy group of ω, namely $\{g | gA_ig^{-1} = A_i,\ 0 \leq i \leq q\}$, and $A(\omega) = A_q$. For each normal subgroup K of a subgroup H of G, there is an S-functor $C(K, H)$ whose value on X is $G_+ \wedge_H X^K$, and, as S-functors,

$$(F_qA/F_{q-1}A) = \bigvee \Sigma^q C(A(\omega), H(\omega)). \tag{5.1}$$

By direct inspection of definitions, we find that, for any space Y,

$$k^*_G(Y; C(K, H)) \cong k^*_{H/K}(Y^K). \tag{5.2}$$

This is why the Φ-fixed point functors enter into the picture.

To prove Theorem 4.3, we restrict attention to $Y = \tilde{E}\mathscr{P}$. If G is not elementary Abelian, then Y^K is contractible and the subquotients H/K are proper for all pairs (K, H) that appear in (5.1). If $G = (\mathbb{Z}/p)^r$, and $q \leq r - 2$, this is still true. All these terms vanish by hypothesis. If $G = (\mathbb{Z}/p)^r$, we are left with the case $q = r - 1$. Here $A(\omega) = H(\omega) = G$ for all chains ω, there are $p(p-1)/2$ chains ω, and $Y^G = S^0$. Using (5.2), Theorem 4.3 follows.

G. Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. Annals Math. 120(1984), 189-224.

J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups-II. Topology 26(1987), 413-433.

6. An inverse limit of Adams spectral sequences

We turn to the proof of Theorem 4.4. Its hypothesis that k_G is split allows us to reduce the problem to a nonequivariant one, and the hypothesis that the underlying nonequivariant spectrum k is bounded below ensures the convergence of the relevant Adams spectral sequences. We prove Theorem 4.4 by use of a particularly convenient model Y for $\tilde{E}\mathscr{P}$, namely the union of the G-spheres S^{nV}, where V is the reduced regular complex representation of G. It is a model since $V^G = \{0\}$ and $V^H \neq 0$ for $H \in \mathscr{P}$.

In general, for any representation V, there is a Thom spectrum BG^{-V}. Here we may think of $-V$ as the negative of the representation bundle $EG \times_G V \longrightarrow BG$, regarded as a map $-V : BG \longrightarrow BO \times \mathbb{Z}$. If V is suitably oriented, for example if V is complex, there is a Thom isomorphism showing that $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_v of degree $-n$, where n is the (real) dimension of V. We take cohomology with mod p coefficients. For $V \subset W$, there is a map $f : BG^{-W} \longrightarrow BG^{-V}$ such that $f^* : H^*(BG^{-V}) \longrightarrow H^*(BG^{-W})$ carries

ι_v to $\chi(W-V)\iota_w$. Here $\chi(V) \in H^*(BG)$ is the Euler class of V, which is the Euler class of its representation bundle. For a split G-spectrum k_G we have an isomorphism

$$k^G_*(S^V; EG_+) \cong k_*(BG^{-V}).$$

For $V \subset W$, the map $f_* : k_*(BG^{-W}) \longrightarrow k_*(BG^{-V})$ corresponds under the isomorphisms to the map induced by $e : S^V \longrightarrow S^W$. (The paper of mine cited at the end gives details on all of this.) With our model Y for $\tilde{E}\mathscr{P}$, we now see that

$$k_G^{-q}(Y; EG_+) = k_q^G(Y; EG_+) \cong \lim k_q(BG^{-nV}).$$

Remember that we are working p-adically; we complete spectra at p without change of notation. The inverse limit E_r of Adams spectral sequences of an inverse sequence $\{X_n\}$ of bounded below spectra of finite type over the p-adic integers $\mathbb{Z}_p$ converges from

$$E_2 = \mathrm{Ext}_A(\mathrm{colim}\, H^*(X_n), \mathbb{F}_p)$$

to $\lim \pi_*(X_n)$. With $X_n = k \wedge BG^{-nV}$, this gives an inverse limit of Adams spectral sequences that converges from

$$E_2 = \mathrm{Ext}_A(H^*(k) \otimes \mathrm{colim}\, H^*(BG^{-nV}), \mathbb{F}_p)$$

to $k^*_G(Y; EG_+)$. The colimit is taken with respect to the maps

$$\chi(V) : H^*(BG^{-nV}) \longrightarrow H^*(BG^{-(n+1)V}).$$

Since $V^H \neq \{0\}$, $\chi(V)$ restricts to zero in $H^*(BH)$ for all $H \in \mathscr{P}$. A theorem of Quillen implies that $\chi(V)$ must be nilpotent if G is not elementary Abelian, and this implies that $E_2 = 0$. This proves part (i) of Theorem 4.4.

Now assume that $G = (\mathbb{Z}/p)^r$. Let $L = \chi(V) \in H^{2(p^r-1)}(BG)$. Then

$$\mathrm{colim}\, H^*(BG^{-nV}) = H^*(BG)[L^{-1}].$$

It is easy to write L down explicitly, and the heart of part (ii) is the following purely algebraic calculation of Adams, Gunawardena, and Miller, which gives the E_2 term of our spectral sequence.

THEOREM 6.1. Let $St = H^*(BG)[L^{-1}] \otimes_A \mathbb{F}_p$, and regard St as a trivial A-module. Then St is concentrated in degree $-r$ and has dimension $p^{r(r-1)/2}$. The quotient homomorphism $\varepsilon : H^*(BG)[L^{-1}] \longrightarrow St$ induces an isomorphism

$$\mathrm{Ext}_A(K \otimes St, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A(K \otimes H^*(BG)[L^{-1}], \mathbb{F}_p)$$

for any finite dimensional A-module K.

The notation "St" stands for Steinberg: $GL(r, \mathbb{F}_p)$ acts naturally on everything in sight, and St is the classical Steinberg representation.

Let W be the wedge of $p^{r(r-1)/2}$ copies of S^{-r}. It follows by convergence that there is a compatible system of maps $W \longrightarrow BG^{-nV}$ that induces an isomorphism

$$k_*(W) = \pi_*(k \wedge W) \longrightarrow \lim \pi_*(k \wedge BG^{-nV}) \cong k_G^{-*}(Y; EG_+).$$

This gives Theorem 4.4(ii). It also implies the following remarkable corollary, which has had many applications.

COROLLARY 6.2. The wedge of spheres W is equivalent to the homotopy limit, $BG^{-\infty V}$, of the Thom spectra BG^{-nV}. In particular, with $G = \mathbb{Z}/2$, S^{-1} is equivalent to the spectrum $\operatorname{holim} \mathbb{R}P^{\infty}_{-i}$.

J. F. Adams, J. H. Gunawardena, and H. Miller. The Segal conjecture for elementary Abelian p-groups-I. Topology 24(1985), 435-460.
J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups-II. Topology 26(1987), 413-433.
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7. Further generalizations; maps between classifying spaces

Even before the Segal conjecture was proven, Lewis, McClure, and I showed that it would have the following implication. Let G and Π be finite groups and let $A(G, \Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. Observe that $A(G, \Pi)$ is an $A(G)$-module and let I be the augmentation ideal of $A(G)$.

THEOREM 7.1. There is a canonical isomorphism

$$\alpha_I^\wedge : A(G, \Pi)_I^\wedge \longrightarrow [\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+].$$

The map $\alpha : A(G, \Pi) \longrightarrow [\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+]$ can be described explicitly in terms of transfer maps and classifying maps (and the paper of mine cited at the end gives more about the relationship between the algebra on the left and the topology on the right). A Π-free $(G \times \Pi)$-set T determines a principal Π-bundle

$$EG \times_G T \longrightarrow EG \times_G T/\Pi,$$

which is classified by a map $\xi(T) : EG\times_G T/\Pi \longrightarrow B\Pi$. It also determines a (not necessarily connected) finite cover

$$EG\times_G T/\Pi \longrightarrow EG\times_G \{*\} = BG,$$

which has a stable transfer map $\tau(T) : BG_+ \longrightarrow (EG\times_G T/\Pi)_+$. Both ξ and τ are additive in T, and α is the unique homomorphism such that

$$\alpha(T) = \xi(T)\circ\tau(T).$$

In principle, this reduces to pure algebra the problem of computing stable maps between the classifying spaces of finite groups. Many authors have studied the relevant algebra — Nishida, Martino and Priddy, Harris and Kuhn, Benson and Feshback, and Webb, among others — and have obtained a rather good understanding of such maps. We shall not go into these calculations. Rather, we shall place the result in a larger context and describe some substantial generalizations.

Recall that we interpreted the consequences of the Sullivan conjecture for maps between classifying spaces as statements about equivariant classifying spaces. Analogously, Theorem 7.1 is a consequence of a result about the suspension G-spectra of equivariant classifying spaces.

THEOREM 7.2. The cohomology theory $\pi_G^*(\cdot;\Sigma^\infty(B_G\Pi)_+)\hat{_I}$ is e-invariant. Therefore the map $EG\longrightarrow *$ induces an isomorphism

$$\pi_G^*(S^0;\Sigma^\infty(B_G\Pi)_+)\hat{_I} \longrightarrow \pi_G^*(EG_+;\Sigma^\infty(B_G\Pi)_+) \cong \pi^*(BG_+;\Sigma^\infty B\Pi_+).$$

The isomorphism on the right comes from XVI.2.4. In degree zero, this is Theorem 7.1. The description of the map α of that result is obtained by describing the map of Theorem 7.2 in nonequivariant terms, using the splitting theorem for $(B_G\Pi)^G$ of VII.2.7, the splitting theorem for the homotopy groups of suspension spectra of XIX.1.2, and some diagram chasing.

We next point out a related consequence of the generalization of the Segal conjecture to families. In it, we let Π be a normal subgroup of a finite group Γ.

THEOREM 7.3. The projection $E(\Pi;\Gamma)\longrightarrow *$ induces an isomorphism

$$A(\Gamma)\hat{_{I\mathscr{F}(\Pi;\Gamma)}} \longrightarrow \pi_\Gamma^0(E(\Pi;\Gamma)_+) \cong \pi_G^0(B(\Pi;\Gamma)_+).$$

This is just the degree zero part of Theorem 2.5 for the family $\mathscr{F}(\Pi;\Gamma)$ in the group Γ; the last isomorphism is a consequence of XVI.5.4. With the Burnside ring replaced by the representation ring, a precisely analogous result holds in K-theory, but in that context the result generalizes to an arbitrary extension of compact Lie groups. Of course, these may be viewed as calculations of equivariant characteristic classes. It is natural to ask if Theorems 7.1 and 7.3 admit a common generalization or, better, if the completion theorems of which they are special cases admit a common generalization.

A result along these lines was proven by Snaith, Zelewski, and myself. Here, for the first time in our discussion, we let compact Lie groups enter into the picture. We consider finite groups G and J and a compact Lie group Π. Let $A(G\times J,\Pi)$ be the Grothendieck group of principal $(G\times J,\Pi)$-bundles over finite $(G\times J)$-sets. This is an $A(G\times J)$-module, and we can complete it at the ideal $I\mathscr{F}_G(J)$. As in VII§1, $\mathscr{F}_G(J)$ is the family of subgroups H of $G\times J$ such that $H\cap J=e$.

THEOREM 7.4. There is a canonical isomorphism

$$\alpha\hat{_{I\mathscr{F}_G(J)}} : A(G\times J,\Pi)\hat{_{I\mathscr{F}_G(J)}} \longrightarrow [\Sigma^\infty B_G J_+, \Sigma^\infty B_G\Pi_+]_G.$$

Again, the map $\alpha : A(G\times J,\Pi) \longrightarrow [\Sigma^\infty B_G J_+, \Sigma^\infty B_G\Pi_+]_G$ is given on principal $(G\times J,\Pi)$-bundles as composites of equivariant classifying maps and equivariant transfer maps. Although the derivation is not quite immediate, this result is a consequence of an invariance result exactly analogous to the version of the Segal conjecture given in Theorem 3.2.

THEOREM 7.5. Let Π be a normal subgroup of a compact Lie group Γ with finite quotient group G. Let S be a multiplicatively closed subset of $A(G)$ and let I be an ideal in $A(G)$. Then the cohomology theory $S^{-1}\boldsymbol{\pi}_G^*(\cdot; B(\Pi;\Gamma)_+)\hat{_I}$ is $\mathscr{H}$-invariant, where

$$\mathscr{H} = \bigcup\{\mathrm{Supp}(P) | P\cap S=\emptyset \text{ and } P\supset I\}.$$

The statement is identical with that of Theorem 3.2, except that we have substituted $B(\Pi;\Gamma)_+$ for S^0 as the second variable of our bitheory. We could generalize a bit further by substituting $E(\Pi;\Gamma)_+\wedge_\Pi X$ for any finite Γ-CW complex X. What other G-spaces can be substitutedΓ The elementary p-group case of the proof of the Segal conjecture makes it clear that one cannot substitute an arbitrary G-space. In fact, very little more than what we have already stated is known.

Theorem 7.5 specializes to give the analog of Theorem 3.1.

THEOREM 7.6. Let $\mathscr{F}$ be a family in G, where $G=\Gamma/\Pi$. The map $E\mathscr{F}\longrightarrow *$ induces an isomorphism

$$\boldsymbol{\pi}_G^*(S^0; B(\Pi;\Gamma)_+)\hat{_{I\mathscr{F}}} \longrightarrow \boldsymbol{\pi}_G^*(E\mathscr{F}_+; \Sigma^\infty B(\Pi;\Gamma)_+).$$

We can restate this in Mackey functor form, as in Theorem 2.5, and then deduce a conceptual formulation generalizing Theorem 1.10.

THEOREM 7.7. For every family $\mathscr{F}$ in G, the map

$$\xi^* : F(K(I\mathscr{F}), \Sigma^\infty B(\Pi;\Gamma)_+) \longrightarrow F(E\mathscr{F}_+, \Sigma^\infty B(\Pi;\Gamma)_+)$$

is an equivalence of G-spectra.

This extends the calculational consequences to the $RO(G)$-graded represented theories. Exactly as in Sections 1–3, all of these theorems reduce to the following special case.

THEOREM 7.8. Let Π be a normal subgroup of a compact Lie group Γ such that the quotient group G is a finite p-group. Then the theory $\boldsymbol{\pi}_G^*(\cdot; B(\Pi;\Gamma)_+)^\wedge_p$ is e-invariant.

The proof is a bootstrap argument starting from the Segal conjecture. When Γ is finite, the result can be deduced from the generalized splitting theorem of XIX.2.1 and the case of the Segal conjecture for Γ that deals with the family of subgroups of Γ that are contained in Π. When Γ is a finite extension of a torus, the result is then deduced by approximating Γ by an expanding sequence of finite groups; this part of the argument entails rather rather elaborate duality and colimit arguments, together with several uses of the generalized Adams isomorphisms XVI.5.4. Finally, the general case is deduced by a transfer argument.

As is discussed in my paper with Snaith and Zelewski, and more extensively in the survey of Lee and Minami, these results connect up with and expands what is known about the Segal conjecture for compact Lie groups.

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CHAPTER XXI

Generalized Tate cohomology

by J. P. C. Greenlees and J. P. May

In this chapter, we will describe some joint work on the generalization of the Tate cohomology of a finite group G with coefficients in a G-module V to the Tate cohomology of a compact Lie group G with coefficients in a G-spectrum k_G. There has been a great deal of more recent work in this area, with many calculations and applications. We shall briefly indicate some of the main directions.

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J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoirs Amer. Math. Soc. No 543. 1995.

1. Definitions and basic properties

Tate cohomology has long played a prominent role in finite group theory and its applications. For a finite group G and a G-module V, the Tate cohomology $\hat{H}^*_G(V)$ is obtained as follows. One starts with a free resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}[G]$-modules, dualizes it to obtain a resolution

$$0 \longrightarrow \mathbb{Z} \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots,$$

renames $P_i^* = P_{-i-1}$, and splices the two sequences together to obtain a $\mathbb{Z}$-graded exact complex P of finitely generated free $\mathbb{Z}[G]$-modules with a factorization $P_0 \longrightarrow \mathbb{Z} \longrightarrow P_{-1}$ of d_0. The complex P is called a "complete resolution of $\mathbb{Z}$", and $\hat{H}^*_G(V)$ is defined to be the cohomology of the cochain complex $\mathrm{Hom}_G(P, V)$. There results a "norm exact sequence" that relates $\hat{H}^*_G(V)$, $H^G_*(V)$, and $H^*_G(V)$.

In connection with Smith theory, Swan generalized this algebraic theory to a cohomology theory $\hat{H}^*_G(X;V)$ on G-spaces X, using $\mathrm{Hom}(P\otimes C_*(X),V)$. (Swan took X to be a G-simplicial complex, but singular chains could be used.) When $G=S^1$ or $G=S^3$ and X is a CW-complex with a cellular action by G, there is a closely analogous theory that is obtained by replacing P by $\mathbb{Z}[u,u^{-1}]$, where u has degree -2 or -4. Here $\mathrm{Hom}(P\otimes C_*(X),V)$ has differential

$$d(p\otimes x)=p\otimes d(x)+pu\otimes i\cdot x,$$

where $i\in C_1(S^1)$ or $i\in C_3(S^3)$ is the fundamental class. For S^1, this is periodic cyclic cohomology theory.

We shall give a very simple definition of a common generalization of these variants of Tate theory. In fact, as part of a general "norm cofibration sequence", we shall associate a Tate G-spectrum $t(k_G)$ to any G-spectrum k_G, where G is any compact Lie group. The construction is closely related to the "stable homotopy limit problem" and to nonequivariant stable homotopy theory.

We have the cofiber sequence

$$EG_+\longrightarrow S^0\longrightarrow \tilde{E}G, \tag{1.1}$$

and the projection $EG_+\longrightarrow S^0$ induces the canonical map of G-spectra

$$\varepsilon: k_G=F(S^0,k_G)\longrightarrow F(EG_+,k_G). \tag{1.2}$$

Taking the smash product of the cofibering (1.1) with the map (1.2), we obtain the following map of cofiberings of G-spectra:

$$\begin{array}{ccccc} k_G\wedge EG_+ & \longrightarrow & k_G & \longrightarrow & k_G\wedge \tilde{E}G\\ \downarrow{\scriptstyle \varepsilon\wedge\mathrm{id}} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon\wedge\mathrm{id}}\\ F(EG_+,k_G)\wedge EG_+ & \longrightarrow & F(EG_+,kG) & \longrightarrow & F(EG_+,kG)\wedge\tilde{E}G. \end{array} \tag{1.3}$$

We have seen most of the ingredients of this diagram in our discussion of the Segal conjecture. We introduce abbreviated notations for these spectra. Define

$$f(k_G)=k_G\wedge EG_+. \tag{1.4}$$

We call $f(k_G)$ the free G-spectrum associated to k_G. It represents the appropriate generalized version of the Borel homology theory $H_*(EG\times_G X)$. Precisely, if

k_G is split with underlying nonequivariant spectrum k, then, by XVI.2.4,

$$f(k_G)_*(X) \cong k_*(EG_+ \wedge_G \Sigma^{Ad(G)} X). \tag{1.5}$$

We refer to the homology theories represented by G-spectra of the form $f(k_G)$ as Borel homology theories. We refer to the cohomology theories represented by the $f(k_G)$ simply as f-cohomology theories. Define

$$f'(k_G) = F(EG_+, k_G) \wedge EG_+. \tag{1.6}$$

It is clear that the map $\varepsilon \wedge \mathrm{Id} : f(k_G) \longrightarrow f'(k_G)$ is always an equivalence, so that the G-spectra $f(k_G)$ and $f'(k_G)$ can be used interchangeably. We usually drop the notation f', preferring to just use f. Define

$$f^-(k_G) = k_G \wedge \tilde{E}G. \tag{1.7}$$

We call $f^-(k_G)$ the singular G-spectrum associated to k_G.

Define

$$c(k_G) = F(EG_+, k_G). \tag{1.8}$$

We call $c(k_G)$ the geometric completion of k_G. The problem of determining the behavior of $\varepsilon : k_G \longrightarrow c(k_G)$ on G-fixed point spectra is the "stable homotopy limit problem". We have already discussed this problem in several cases, and we have seen that it is best viewed as the equivariant problem of comparing the geometric completion $c(k_G)$ with the algebraic completion $(k_G)\hat{}_I$ of k_G at the augmentation ideal of the Burnside ring or of some other ring more closely related to k_G. As one would expect, $c(k_G)$ represents the appropriate generalized version of Borel cohomology $H^*(EG \times_G X)$. Precisely, if k_G is a split G-spectrum with underlying nonequivariant spectrum k, then, by XVI.2.4,

$$c(k_G)^*(X) \cong k^*(EG_+ \wedge_G X). \tag{1.9}$$

We therefore refer to the cohomology theories represented by G-spectra $c(k_G)$ as Borel cohomology theories. We refer to the homology theories represented by the $c(k_G)$ as c-homology theories.

Finally, define

$$t(k_G) = F(EG_+, k_G) \wedge \tilde{E}G = f^- c(k_G). \tag{1.10}$$

We call $t(k_G)$ the Tate G-spectrum associated to k_G. It is the singular part of the geometric completion of k_G. Our primary focus will be on the theories represented by the $t(k_G)$. These are our generalized Tate homology and cohomology theories.

With this cast of characters, and with the abbreviation of $\varepsilon\wedge\mathrm{id}$ to ε, the diagram (1.3) can be rewritten in the form

$$\begin{array}{ccccc} f(k_G) & \longrightarrow & k_G & \longrightarrow & f^-(k_G) \\ {\scriptstyle\varepsilon}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ f'(k_G) & \longrightarrow & c(k_G) & \longrightarrow & t(k_G). \end{array} \tag{1.11}$$

The bottom row is the promised "norm cofibration sequence". The theories represented by the spectra on this row are all e-invariant.

The definition implies that if X is a free G-spectrum, then

$$t(k_G)_*(X) = 0 \quad\text{and}\quad t(k_G)^*(X) = 0.$$

Similarly, if X is a nonequivariantly contractible G-spectrum, then

$$c(k_G)^*(X) = 0 \quad\text{and}\quad f(k_G)_*(X) = 0.$$

By definition, Tate homology is a special case of c-homology,

$$t(k_G)_n(X) = c(k_G)_n(\tilde{E}G \wedge X). \tag{1.12}$$

The two vanishing statements imply that Tate cohomology is a special case of f-cohomology,

$$t(k_G)^n(X) \cong f(k_G)^{n+1}(\tilde{E}G \wedge X). \tag{1.13}$$

In fact, on the spectrum level, the vanishing statements imply the remarkable equivalence

(1.14)
$$t(k_G) \equiv F(EG_+, k_G) \wedge \tilde{E}G \simeq F(\tilde{E}G, \Sigma EG_+ \wedge k_G) \equiv F(\tilde{E}G, \Sigma f(k_G)).$$

It is a consequence of the definition that $t(k_G)$ is a ring G-spectrum if k_G is a ring G-spectrum, and then $t(k_G)^G$ is a ring spectrum.

Much of the force of our definitional framework comes from the fact that (1.11) is a diagram of genuine and conveniently explicit G-spectra indexed on representations, so that all of the $\mathbb{Z}$-graded cohomology theories in sight are $RO(G)$-gradable. The $RO(G)$-grading is essential to the proofs of many of the results discussed below. Nevertheless, it is interesting to give a naive reinterpretation of the fixed point cofibration sequence associated to the norm sequence.

With our definitions, the Tate homology of X is

$$t(k_G)_*(X) = \pi_*((t(k_G) \wedge X)^G).$$

Since any k_G is e-equivalent to j_G for a naive G-spectrum j_G and Tate theory is e-invariant, we may as well assume that $k_G = i_* j_G$. Provided that X is a finite G-CW complex, the spectrum $(t(k_G) \wedge X)^G$ is then equivalent to the cofiber of an appropriate transfer map

$$\begin{array}{c} (j_G \wedge \Sigma^{Ad(G)} X)_{hG} \equiv (j_G \wedge EG_+ \wedge \Sigma^{Ad(G)} X)/G \\ \downarrow \\ (j_G \wedge X)^{hG} \equiv F(EG_+, j_G \wedge X)^G. \end{array}$$

A description like this was first written down by Adem, Cohen, and Dwyer. When G is finite, $X = S^0$, and j_G is a nonequivariant spectrum k given trivial action by G, this reduces to

$$k \wedge BG_+ \longrightarrow F(BG_+, k).$$

The interpretation of Tate theory as the third term in a long sequence whose other terms are Borel k-homology and Borel k-cohomology is then transparent.

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2. Ordinary theories; Atiyah-Hirzebruch spectral sequences

Let M be a Mackey functor and V be the $\pi_0(G)$-module $M(G/e)$. The norm sequence of HM depends only on V: if M and M' are Mackey functors for which $M(G/e) \cong M'(G/e)$ as $\pi_0(G)$-modules, then the norm cofibration sequences of HM and HM' are equivalent. We therefore write

(2.1) $$\hat{H}^G_*(X;V) = t(HM)_*(X) \quad \text{and} \quad \hat{H}^*_G(X;V) = t(HM)^*(X).$$

For finite groups G, this recovers the Tate-Swan cohomology groups, as the notation anticipates. We sketch the proof. The simple objects to the eyes of ordinary cohomology are cells, and the calculation depends on an analogue of the skeletal filtration of a CW complex that mimics the construction of a complete resolution. The idea is to splice the skeletal filtration of EG_+ with its Spanier-Whitehead dual. More precisely, we define an integer graded filtration on $\tilde{E}G$, or rather on its suspension spectrum, by letting

$$F^i\tilde{E}G = \begin{cases} \tilde{E}G^{(i)} = S^0 \cup C(EG_+^{(i-1)}) & \text{for } i \geq 1 \\ S^0 & \text{for } i = 0 \\ D(\tilde{E}G^{(-i)}) & \text{for } i \leq -1. \end{cases}$$

The ith subquotient of this filtration is a finite wedge of spectra $S^i \wedge G_+$, and the E^1 term of the spectral sequence that is obtained by applying ordinary nonequivariant integral homology is a complete resolution of $\mathbb{Z}$. Therefore, if one takes the smash product of this filtration with the skeletal filtration of X and applies an equivariant cohomology theory $k_G^*(\cdot)$, one obtains the "Atiyah-Hirzebruch-Tate" spectral sequence

$$E_2^{p,q} = \hat{H}_G^p(X; k^q) \Longrightarrow t(k)_G^{p+q}(X). \tag{2.2}$$

Here k is the underlying nonequivariant spectrum of k_G, and $k^q = \pi_{-q}(k)$ regarded as a G-module. To see that the target is Tate cohomology as claimed, note that the "cohomological" description (1.14) of the Tate spectrum gives

$$t(k)_G^*(X) = [\tilde{E}G \wedge X, k \wedge \Sigma EG_+]_G^*.$$

There are compensating shifts of grading in the identifications of the E_2 terms and of the target, so that the grading works out as indicated in (2.2).

When $k_G = HM$, the spectral sequence collapses at the E_2-term by the dimension axiom, and this proves that $t(HM)_G^*(X)$ is the Tate-Swan cohomology of X. In general, we have a whole plane spectral sequence, but it converges strongly to $t(k_G)^*(X)$ provided that there are not too many non-zero higher differentials. When k_G is a ring spectrum, it is a spectral sequence of differential algebras.

With a little care about the splice point and the model of EG used, we can apply part of this construction to compact Lie groups G of dimension $d > 0$. In this case, there is a "gap" in the appropriate filtration of $\tilde{E}G$:

$$F^i\tilde{E}G = \begin{cases} \tilde{E}G^{(i)} = S^0 \cup C(EG_+^{(i-1)}) & \text{for } i \geq 1 \\ S^0 & \text{for } -d \leq i \leq 0 \\ D(\tilde{E}G^{(-i)}) & \text{for } i < -d. \end{cases}$$

The gap is dictated by the fact that the Spanier-Whitehead dual of G_+ is $G_+ \wedge S^{-d}$.

In the case of Eilenberg-MacLane spectra, this gives an explicit chain level calculation of the coefficient groups $\hat{H}_*^G(V) \equiv \hat{H}_*^G(S^0; V)$ in terms of the ordinary (unreduced) homology and cohomology groups of the classifying space BG:

$$\hat{H}^n_G(V) = t(HM)^n \cong \begin{cases} H^n(BG;V) & \text{if } 0 \le n \\ 0 & \text{if } -d \le n < 0 \\ H_{-n-1-d}(BG;V) & \text{if } n \le -d-1. \end{cases} \tag{2.3}$$

However, we would really like a chain complex for calculating the ordinary Tate cohomology of G-CW complexes X, and for groups of positive dimension it is not obvious how to make one. At present, we only have such descriptions for $G = S^1$ and $G = S^3$. In these cases, we can exploit the obvious cell structure on G and the standard models $S(\mathbb{C}^\infty)$ and $S(\mathbb{H}^\infty)$ for EG to put a cunning G-CW structure on $EG_+ \wedge X$ and to derive an appropriate filtration of $\tilde{E}G \wedge X$ when G acts cellularly on X. In the case of S^1, the resulting chain complex is a cellular version of Jones' complex for cyclic cohomology, and this proves that $t(H\underline{\mathbb{Z}})^*_{S^1}(X)$ is the periodic cyclic cohomology $\hat{H}^*_{S^1}(X)$, as defined by Jones in terms of the singular complex of X. There is a precisely analogous identification in the case of S^3. In general, the problem of giving $\tilde{E}G \wedge X$ an appropriate filtration appears to be intractable, although a few other small groups are under investigation.

Despite this difficulty, we still have spectral sequences of the form (2.2) for general compact Lie groups G, where $k^q = \pi_{-q}(k)$ is now regarded as a $\pi_0(G)$-module. However, in the absence of a good filtration of $\tilde{E}G \wedge X$, we construct the spectral sequences by using a Postnikov filtration of k_G. In this generality, the ordinary Tate groups $\hat{H}^*_G(X;V)$ used to describe the E_2 terms are not familiar ones, and systematic techniques for their calculation do not appear in the literature. One approach to their calculation is to use the skeletal filtration of X together with (2.3) and change of groups. More systematic approaches involve the construction of spectral sequences that converge to $\hat{H}^*_G(X;V)$, and there are several sensible candidates. This is an area that needs further investigation, and we shall say no more about it here.

We have similar and compatible spectral sequences for Borel and f-cohomology, and in these cases too the E_2-terms depend only on the graded $\pi_0(G)$-module k^*, as one would expect from the e-invariance of the bottom row of Diagram (1.11). This very weak dependence on k_G makes the bottom row much more calculationally accessible than the top row.

3. Cohomotopy, periodicity, and root invariants

For finite groups G, the Segal conjecture directly implies the determination of the Tate spectrum associated to the sphere spectrum S_G. Indeed, we have

$$t(S_G) = F(EG_+, S^0) \wedge \tilde{E}G \simeq (S_G)^{\wedge}_I \wedge \tilde{E}G \simeq (\Sigma^\infty \tilde{E}G)^{\wedge}_I. \tag{3.1}$$

For instance, if G is a p-group, then

$$t(S_G) \simeq (\Sigma^\infty \tilde{E}G)^{\wedge}_p, \tag{3.2}$$

and we may calculate from the splitting theorem XIX.1.1 that, after completion,

$$t(S_G)^G_*(X) = \bigoplus_{(H)\neq(1)} \pi_*(EW_G(H)_+ \wedge_{W_G(H)} X^H). \tag{3.3}$$

With $X = S^0$, the summand for $H = G$ is $\pi_*(S^0)$, and it follows that, for each G, the Atiyah-Hirzebruch-Tate spectral sequence defines a "root invariant" on the stable stems. Its values are cosets in the Tate cohomology group $\hat{H}^*(G; \pi_*(S^0))$. Essentially, the root invariant assigns to an element $\alpha \in \pi_*(S^0)$ all elements of E^2 of the appropriate filtration that project to the image of α in the E^∞ term of the spectral sequence.

These invariants have not been much investigated beyond the classical case of $G = C_p$, the cyclic group of order p. In this case, our construction agrees with earlier constructions of the root invariant. Indeed, this is a consequence of the observations that, if $G = C_2$ and $k_G = i_*k$ is the G-spectrum associated to a non-equivariant spectrum k, then

$$t(k_G)^G \simeq \operatorname{holim}(\mathbb{R}P^\infty_{-i} \wedge \Sigma k) \tag{3.4}$$

and, if $G = C_p$ for an odd prime p and $k_G = i_*k$, then

$$t(k_G)^G \simeq \operatorname{holim}(L^\infty_{-i} \wedge \Sigma k), \tag{3.5}$$

where L^∞_{-i} is the lens space analog of $\mathbb{R}P^\infty_{-i}$. Taking $k = S$, there results a spectral sequence that agrees with our Atiyah-Hirzebruch-Tate spectral sequence and was used in the classical definition of the root invariant.

Similarly, if G is the circle group and $k_G = i_*k$, then

$$t(k_G)^G \simeq \operatorname{holim}(\mathbb{C}P^\infty_{-i} \wedge \Sigma^2 k). \tag{3.6}$$

These are all special cases of a phenomenon that occurs whenever G acts freely on the unit sphere of a representation V, and this phenomenon is the source of periodic behavior in Tate theory. The point is that the union of the S^{nV} is then a model for $\tilde{E}G$, and we can use this model to evaluate the right side as a homotopy limit in the equivalence (1.14). This immediately gives (3.4)–(3.6). These equivalences allow us to apply nonequivariant calculations of Davis, Mahowald, and others of spectra on the right sides to study equivariant theories. We will say a little more about this in Section 6. It also gives new insight into the nonequivariant theories. In particular, if k is a ring spectrum, then $t(k_G)^G$ is a ring spectrum. Looking nonequivariantly at the right sides, this is far from clear.

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D. M. Davis, D. C. Johnson, J. Klippenstein, M. Mahowald and S. Wegmann. The spectrum $(P \wedge BP\langle 2\rangle)_{-\infty}$. Trans. American Math. Soc. 296(1986) 95-110.

4. The generalization to families

The theory described above is only part of the story: it admits a generalization in which the universal free G-space EG is replaced by the universal $\mathscr{F}$-space $E\mathscr{F}$ for any family $\mathscr{F}$ of subgroups of G. The definitions above deal with the case $\mathscr{F} = \{e\}$, and there is a precisely analogous sequence of definitions for any other family. We have the cofibering

$$E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}, \tag{4.1}$$

and the projection $E\mathscr{F}_+ \longrightarrow S^0$ induces a G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{4.2}$$

Taking the smash product of the cofibering (4.1) with the map (4.2), we obtain the following map of cofiberings of G-spectra:

$$\begin{array}{ccccc} k_G \wedge E\mathscr{F}_+ & \longrightarrow & k_G & \longrightarrow & k_G \wedge \tilde{E} \\ \downarrow{\scriptstyle \varepsilon\wedge \mathrm{id}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon\wedge \mathrm{id}} \\ F(E\mathscr{F}_+, kG) \wedge E\mathscr{F}_+ & \longrightarrow & F(E\mathscr{F}_+, kG) & \longrightarrow & F(E\mathscr{F}_+, kG) \wedge \tilde{E}\mathscr{F}. \end{array} \tag{4.3}$$

Define the $\mathscr{F}$-free G-spectrum associated to k_G to be

$$f_{\mathscr{F}}(k_G) = k_G \wedge E\mathscr{F}_+. \tag{4.4}$$

We refer to the homology theories represented by G-spectra $f_{\mathscr{F}}(k_G)$ as $\mathscr{F}$-Borel homology theories. Define

$$f'_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G) \wedge E\mathscr{F}_+. \tag{4.5}$$

Again, $\varepsilon \wedge \mathrm{Id} : f_{\mathscr{F}}(k_G) \longrightarrow f'_{\mathscr{F}}(k_G)$ is an equivalence, hence we usually use the notation $f_{\mathscr{F}}$. Define the $\mathscr{F}$-singular G-spectrum associated to k_G to be

$$f^{-}_{\mathscr{F}}(k_G) = k_G \wedge \tilde{E}\mathscr{F}. \tag{4.6}$$

Define the geometric $\mathscr{F}$-completion of kG to be

$$c_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G). \tag{4.7}$$

We refer to the cohomology theories represented by G-spectra $c_{\mathscr{F}}(k_G)$ as $\mathscr{F}$-Borel cohomology theories. The map $\varepsilon : k_G \longrightarrow c_{\mathscr{F}}(k_G)$ of (4.2) is the object of study of such results as the generalized Atiyah-Segal completion theorem and the generalized Segal conjecture of Adams-Haeberly-Jackowski-May. As in these results, one version of the $\mathscr{F}$-homotopy limit problem is the equivariant problem of comparing the geometric $\mathscr{F}$-completion $c_{\mathscr{F}}(k_G)$ with the algebraic completion $(k_G)\hat{}_{I\mathscr{F}}$ of k_G at the ideal $I\mathscr{F}$ of the Burnside ring or at an analogous ideal in a ring more closely related to k_G. Observe that we usually do not have analogs of (1.5) and (1.9) for general families $\mathscr{F}$; the Adams isomorphism XVI.5.4 and the discussion around it are relevant at this point.

Define

$$t_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G) \wedge \tilde{E}\mathscr{F} = f^{-}_{\mathscr{F}} c_{\mathscr{F}}(k_G). \tag{4.8}$$

We call $t_{\mathscr{F}}(k_G)$ the $\mathscr{F}$-Tate G-spectrum associated to k_G. These G-spectra represent $\mathscr{F}$-Tate homology and cohomology theories. With this cast, and with the abbreviation of $\varepsilon \wedge \mathrm{id}$ to ε, the diagram (4.3) can be rewritten in the form

$$\begin{array}{ccccc} f_{\mathscr{F}}(k_G) & \longrightarrow & k_G & \longrightarrow & f^{-}_{\mathscr{F}}(k_G) \\ {\scriptstyle\varepsilon}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ f'_{\mathscr{F}}(k_G) & \longrightarrow & c_{\mathscr{F}}(k_G) & \longrightarrow & t_{\mathscr{F}}(k_G). \end{array} \tag{4.9}$$

We call the bottom row the "$\mathscr{F}$-norm cofibration sequence". The theories represented by the spectra on this row are all $\mathscr{F}$-invariant.

The diagram leads to a remarkable and illuminating relationship between the Tate theories and the $\mathscr{F}$-homotopy limit problem. Recall that $I\mathscr{F} \subset A(G)$ is the intersection of the kernels of the restrictions $A(G) \longrightarrow A(H)$ for $H \in \mathscr{F}$.

THEOREM 4.10. *The spectra $c_{\mathscr{F}}(k_G)$ are $I\mathscr{F}$-complete. The spectra $f_{\mathscr{F}}(kG)$ and $t_{\mathscr{F}}(k_G)$ are $I\mathscr{F}$-complete if k_G is bounded below.*

We promised in XX§1 to relate the questions of when

$$\xi^* : (k_G)\hat{_{I\mathscr{F}}} = F(K(I\mathscr{F}), k_G) \longrightarrow F(E\mathscr{F}_+, k_G) = c_{\mathscr{F}}(k_G)$$

and

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F})$$

are equivalences. The answer is rather surprising.

THEOREM 4.11. Let k_G be a ring G-spectrum, where G is finite. Then ξ_* is an equivalence if and only if ξ^* is an equivalence and $t_{\mathscr{F}}(k_G)$ is rational.

The proof is due to the first author and will be discussed in XXIV§8. We shall turn to relevant examples in the next section.

When G is finite and k_G is an Eilenberg-MacLane G-spectrum HM, the $\mathscr{F}$-Tate G-spectrum $t_{\mathscr{F}}(HM)$ represents the generalization to homology and cohomology theories on G-spaces and G-spectra of certain "Amitsur-Dress-Tate cohomology theories" $\hat{H}^*_{\mathscr{F}}(M)$ that figure prominently in induction theory. We again obtain generalized Atiyah-Hirzebruch-Tate spectral sequences in the context of families. These vastly extend the web of symmetry relations relating equivariant theory with the stable homotopy groups of spheres. In particular, for a finite p-group G, if we use the family $\mathscr{P}$ of all proper subgroups of G, we obtain a spectral sequence whose E_2-term is $\hat{H}^{\mathscr{P}}_*(\underline{\pi}^G_*)$ and which converges to $(\pi_*)\hat{_p}$. We have moved the groups $\pi_*(BWH_+)$ from the target to ingredients in the calculation of E_2. In this spectral sequence the "root invariant" of an element $\alpha \in \pi_q$ lies in degree at least $q(|G|-1)$. The root invariant measures where elements are detected in E^2 of the spectral sequence, and the dependence on the order of G indicates an increasing dependence of lower degree homotopy groups of spheres on higher degree homotopy groups of classifying spaces.

More generally, if G is any finite group, we use the family $\mathscr{P}$ to obtain two related spectral sequences, both of which converge to the completion of the nonequivariant stable homotopy groups of spheres at $n(\mathscr{P})$, where $n(\mathscr{P})$ is the product of those primes p such that $\mathbb{Z}/p\mathbb{Z}$ is a quotient of G. For example, if G is a nonabelian group of order pq, $p < q$, then $n(\mathscr{P}) = p$ and the spectral sequences provide a mechanism for the prime q to affect stable homotopy groups at the prime p. One of the spectral sequences is the Atiyah-Hirzebruch-Tate spectral sequence whose E_2-term is the Amitsur-Dress-Tate homology $\hat{H}^{\mathscr{P}}_*(\underline{\pi}^G_*)$. The other comes from a filtration of $\tilde{E}G$ in terms of the regular representation of G. These spectral sequences lead to new equivariant root invariants, and the basic Bredon-Jones-Miller root invariant theorem generalizes to the spectral sequence constructed by use of the regular representation.

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5. Equivariant K-theory

Our most interesting calculation shows that, for any finite group G, $t(K_G)$ is a rational G-spectrum, namely

$$t(K_G) \simeq \bigvee K(\hat{J} \otimes \mathbb{Q}, 2i), \tag{5.1}$$

where $\hat{J}$ is the Mackey functor of completed augmentation ideals of representation rings and i ranges over the integers. In this case, the relevant Atiyah-Hirzebruch-Tate spectral sequence is rather amazing. Its E_2-term is torsion, being annihilated by multiplication by the order of G. If G is cyclic, then $E_2 = E_\infty$ and the spectral sequence certainly converges strongly. In general, the E_2-term depends solely on the classical Tate cohomology of G and not at all on its representation ring, whereas $t(K_G)^*$ depends solely on the representation ring and not at all on the Tate cohomology. Needless to say, the proof of (5.1) is not based on use of the spectral sequence.

In fact, and the generalization is easier to prove than the special case, $t_{\mathscr{F}}(K_G)$ turns out to be rational for every family $\mathscr{F}$. Again, there results an explicit calculation of $t_{\mathscr{F}}(K_G)$ as a wedge of Eilenberg-MacLane spectra. Let $J\mathscr{F}$ be the intersection of the kernels of the restrictions $R(G) \longrightarrow R(H)$ for $H \in \mathscr{F}$. It is clear by character theory that

$$J\mathscr{F} = \{\chi | \chi(g) = 0 \text{ if the group generated by } g \text{ is in } \mathscr{F}\},$$

and we define a rationally complementary ideal $J'\mathscr{F}$ by

$$J'\mathscr{F} = \{\chi | \chi(g) = 0 \text{ if the group generated by } g \text{ is not in } \mathscr{F}\}.$$

Then (5.1) generalizes to

$$t_{\mathscr{F}}(K_G) \simeq \bigvee K((R/J'\mathscr{F})\hat{_{J\mathscr{F}}} \otimes \mathbb{Q}, 2i), \tag{5.2}$$

where $(R/J'\mathscr{F})_{J\mathscr{F}}^{\wedge}$ denotes the Mackey functor whose value at G/H is the completion at the ideal $J(\mathscr{F}|_H)$ of the quotient $R(H)/J'(\mathscr{F}|H)$. This is consistent with (5.1) since, when $\mathscr{F} = \{e\}$, $J'(\mathscr{F}|H)$ is a copy of $\mathbb{Z}$ generated by the regular representation of H and JH maps isomorphically onto $R(H)/\mathbb{Z}$. It follows in all cases that the completions $t_{\mathscr{F}}(K_G)_{I\mathscr{F}}^{\wedge}$ are contractible.

The following folklore result is proven in our paper on completions at ideals of the Burnside ring. On passage to π_0^G, the unit $S_G \longrightarrow K_G$ induces the homomorphism $A(G) \longrightarrow R(G)$ that sends a finite set X to the permutation representation $\mathbb{C}[X]$. We regard $R(G)$-modules as $A(G)$-modules by pullback.

THEOREM 5.3. The completion of an $R(G)$-module M at the ideal $J\mathscr{F}$ of $R(G)$ is isomorphic to the completion of M at the ideal $I\mathscr{F}$ of the Burnside ring $A(G)$.

In fact, the proof shows that the ideals $I\mathscr{F}R(G)$ and $J\mathscr{F}$ of $R(G)$ have the same radical. Therefore the generalized completion theorem of Adams-Haeberly-Jackowski-May discussed in XIV.6.1 implies that

$$\xi^* : (K_G)_{I\mathscr{F}}^{\wedge} \longrightarrow F(E\mathscr{F}_+, K_G)$$

is an equivalence. By (5.2) and Theorem 4.11, this in turn implies that

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F})$$

is an equivalence. In fact, the latter result was proven by the first author before the implication was known; we shall explain his argument and discuss the algebra behind it in Chapter XXIV.

As a corollary of the calculation of $t(K_G)$, we obtain a surprisingly explicit calculation of the nonequivariant K-homology of the classifying space BG:

$$(5.4) \qquad K_0(BG) \cong \mathbb{Z} \ \text{ and } \ K_1(BG) \cong J(G)_{J(G)}^{\wedge} \otimes (\mathbb{Q}/\mathbb{Z}).$$

In fact, (5.1) and (5.4) both follow easily once we know that $t(K_G)$ is rational. Given that, we have the exact sequence

$$\cdots \to K_*^G(EG_+) \otimes \mathbb{Q} \to K_G^*(EG_+) \otimes \mathbb{Q} \to t(K)_G^* \to \cdots ,$$

which turns out to be short exact. The Atiyah-Segal theorem shows that

$$K_G^*(EG_+) \otimes \mathbb{Q} \cong R(G)_J^{\wedge}[\beta, \beta^{-1}] \otimes \mathbb{Q},$$

where β is the Bott element. Rationally, the K-homology of EG_+ is a summand of K_*^G, and in fact $K_*^G(EG_+) \otimes \mathbb{Q} \cong \mathbb{Q}[\beta, \beta^{-1}]$. It is not hard to identify the maps and conclude that

$$t(K)_G^* = \{R(G)/\mathbb{Z}\}_J^\wedge[\beta, \beta^{-1}] \otimes \mathbb{Q}.$$

Since, as explained in XIX§5, all rational G-spectra split, this gives the exact equivariant homotopy type claimed in (5.1). Now we can deduce (5.4) by analysis of the *integral* norm sequence, using the Atiyah-Segal completion theorem to identify $K_G^*(EG_+)$.

We must still say something about why $t(K_G)$ and all other $t_{\mathscr{F}}(K_G)$ are rational. An inductive scheme reduces the proof to showing that $t_{\mathscr{F}}(K_G) \wedge \tilde{E}\mathscr{P}$ is rational, where $\mathscr{P}$ is the family of proper subgroups of G. If V is the reduced regular complex representation of V, then $S^{\infty V}$ is a model for $\tilde{E}\mathscr{P}$. It follows that, for any K_G-module spectrum M and any spectrum X, $(M \wedge \tilde{E}\mathscr{P})_*^G(X)$ is the localization of $M_*^G(X)$ away from the Euler class (which is the total exterior power) $\lambda(V) \in R(G)$. Since $\lambda(V)$ is in $J\mathscr{P}$, it restricts to zero in all proper subgroups. Since the product over the cyclic subgroups C of G of the restrictions $R(G) \longrightarrow R(C)$ is an injection, $\lambda(V) = 0$ and the conclusion holds trivially unless G is cyclic. In that case, the Atiyah-Hirzebruch-Tate spectral sequence for $t_{\mathscr{P}}(K_G)_*(X)$ gives that primes that do not divide the order n of G act invertibly since n annihilates the E^2-term. An easy calculational argument in representation rings handles the remaining primes.

The evident analogs of all of these statements for real K-theory are also valid.

In the case of connective K-theory, we do not have the same degree of periodicity to help, and the calculations are harder. Results of Davis and Mahowald give the following result.

THEOREM 5.5. If $G = C_p$ for a prime p, then

$$t(ku_G) \simeq \prod_{n \in \mathbb{Z}} \Sigma^{2n} H(\hat{J}),$$

and similarly for connective real K-theory.

This result led us to the overoptimistic conjecture that its conclusion would generalize to arbitrary finite groups. However, Bayen and Bruner have shown that the conjecture fails for both real and complex connective K-theory.

Finally, we must point out that the restriction to finite groups in the discussion above is essential; even for $G = S^1$ something more complicated happens since in that case $t(K_G)^G$ is a homotopy inverse limit of wedges of even suspensions of

K and each even degree homotopy group of $t(K_G)^G$ is isomorphic to $\mathbb{Z}[[\chi]][\chi^{-1}]$, where $1-\chi$ is the canonical irreducible one-dimensional representation of G. In particular, $t(K_G)$ is certainly not rational. Similarly, still taking $G = S^1$, each even degree homotopy group of $t(k_G)^G$ is isomorphic to $\mathbb{Z}[[\chi]]$. In this case, we can identify the homotopy type of the fixed point spectrum:

$$t(ku_{S^1})^{S^1} \simeq \prod_{n\in\mathbb{Z}} \Sigma^{2n} ku_{S^1}. \tag{5.6}$$

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6. Further calculations and applications

Philosophically, one of the main differences between the calculation of the Tate K-theory for finite groups and for the circle group is that the Krull dimension of $R(G)$ is one in the case of finite groups and two in the case of the circle group. Quite generally, the complexity of the calculations increases with the Krull dimension of the coefficient ring. It is relevant that the Krull dimension of $R(G)$ for a compact connected Lie group G is one greater than its rank.

For finite groups, most calculations that have been carried out to date concern ring G-spectra k_G, like those that represent K-theory, that are so related to cobordism as to have Thom isomorphisms of the general form

$$k_*^G(\Sigma^V X) \cong k_*^G(\Sigma^{|V|} X) \tag{6.1}$$

for all complex representations V. Let $e(V): S^0 \longrightarrow S^V$ be the inclusion. Applying $e(V)^*$ to the element $1 \in k_G^0(S^0) \cong k_G^V(S^V)$, we obtain an element of $k_G^V(S^0) = k_{-V}^G(S^0)$. The Thom isomorphism yields an isomorphism between this group and the integer coefficient group $k_{-|V|}^G$, and there results an Euler class $\chi(V) \in k_{-|V|}^G$. As in our indication of the rationality of $t(K_G)$, localizations and other algebraic constructions in terms of such Euler classes can often lead to explicit calculations.

This works particularly well in cases, such as p-groups, where G acts freely on a product of unit spheres $S(V_1) \times \cdots \times S(V_n)$ for some representations $V_1, \ldots, V_n$. This implies that the smash product $S(\infty V_1)_+ \wedge \cdots \wedge S(\infty V_n)_+$ is a model for EG_+, and there results a filtration of $\tilde{E}G$ that has subquotients given by wedges of smash products of spheres. This gives rise to a different spectral sequence for the computation of $t(k_G)^*_G(X)$. When $X = S^0$, the E_2-term can be identified as the "Čech cohomology $\check{H}^*_{J'}(k^*(BG))$ of the k^*_G-module $k^*(BG_+)$ with respect to the ideal $J' = (\chi(V_1), \cdots, \chi(V_n)) \subset k^*_G$". The relevant algebraic definitions will be given in Chapter XXIV. These groups depend only on the radical of J', and, when k^*_G is Noetherian, it turns out that J' has the same radical as the augmentation ideal $J = \mathrm{Ker}(k^*_G \longrightarrow k^*)$.

The interesting mathematics begins with the calculation of the E_2-term, where the nature of the Euler classes for the particular theory becomes important. In fact, this spectral sequence collapses unusually often because the complexity is controlled by the Krull dimension of the coefficients. In cases where one can calculate the coefficients $t(k)^*_G$, one can often also deduce the homotopy type of the fixed point spectrum $t(k_G)^G$ because $t(k_G)^G$ is a module spectrum over k. However, the periodic and connective cases have rather different flavors. In the periodic case the algebra of the coefficients has a field-like appearance and is more often enough to determine the homotopy type of the fixed point spectrum $t(k_G)^G$. In the connective case the algebra of the coefficients in the answer has the appearance of a complete local ring and some sort of Adams spectral sequence argument seems to be necessary to deduce the topology from the algebra. In very exceptional circumstances, such as the use of rationality in the case of K_G, one can go on to deduce the equivariant homotopy type of $t(k_G)$.

In the discussion that follows, we consider equivariant forms k_G of some familiar nonequivariant theories k. We may take k_G to be i_*k, but any split G-spectrum with underlying nonequivariant spectrum k could be used instead. Technically, it is often best to use $F(EG_+, i_*k)$. This has the advantage that its coefficients can often be calculated, and it can be thought of as a geometric completion of any other candidate (and an algebraic completion of any candidate for which a completion theorem holds).

The most visible feature of the calculations to date is that the Tate construction tends to decrease chromatic periodicity. We saw this in the case of K_G, where the periodicity reduced from one to zero. This appears in especially simple form in a theorem of Greenlees and Sadofsky: if $K(n)$ is the nth Morava K-theory spectrum, whose coefficient ring is the graded field

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}], \quad \deg v_n = 2p^n - 2,$$

then

$$t(K(n)_G) \simeq *. \tag{6.2}$$

In fact, this is a quite easy consequence of Ravenel's result that $K(n)^*(BG_+)$ is finitely generated over $K(n)^*$. Another example of this nature is a calculation of Fajstrup, which shows that if the spectrum KR that represents K-theory with reality is regarded as a C_2-spectrum, then the associated Tate spectrum is trivial.

These calculations illustrate another phenomenon that appears to be general: it seems that the Tate construction reduces the Krull dimension of periodic theories. More precisely, the Krull dimension of $t(k_G)_G^0$ is usually less than that of k_G^0. In the case of Morava K-theory, one deduces from Ravenel's result that $K(n)_G^0$ is finite over $K(n)^0$ and thus has dimension 0. The contractibility of $t(K(n)_G)$ can then be thought of as a degenerate form of dimension reduction. More convincingly, work of Greenlees and Sadofsky shows that for many periodic theories for which k_G^0 is one dimensional, $t(k_G)_G^0$ is finite dimensional over a field. The higher dimensional case is under consideration by Greenlees and Strickland.

This reduction of Krull dimension is reflected in the E_2-term of the spectral sequence cited above. When k is v_n-periodic for some n, one typically first proves that some v_i, $i < n$ is invertible on $t(k_G)$ and then uses the localisation of the norm sequence

$$\cdots \to k_*^G(EG_+)\left[v_i^{-1}\right] \to k_G^*(EG_+)\left[v_i^{-1}\right] \to t(k_G)_G^* \to \cdots$$

to assist calculations. For example, consider the spectra $E(n)$ with coefficient rings

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}].$$

Since there is a cofiber sequence $E(2)/p \xrightarrow{v_1} E(2)/p \to K(2)$, we deduce from (6.2) that v_1 is invertible on $t((E(2)/p)_G)$. More generally v_{n-1} is invertible on a suitable completion of $t(E(n)_G)$.

The intuition that the Tate construction lowers Krull dimension is reflected in the following conjecture about the spectra $BP\langle n\rangle$ with coefficient rings

$$BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n].$$

Conjecture 6.3 (Davis-Johnson-Klippenstein-Mahowald-Wegmann).

$$t(BP\langle n\rangle_{C_p})^{C_p} \simeq \prod_{n\in\mathbb{Z}} \Sigma^{2n} BP\langle n-1\rangle_p^\wedge.$$

The cited authors proved the case $n = 2$; the case $n = 1$ was due to Davis and Mahowald. Since $BP\langle n\rangle_*$ has Krull dimension $n+1$, the depth of the conjecture increases with n.

We end by pointing the reader to what is by far the most striking application of generalized Tate cohomology. In a series of papers, Madsen, Bökstedt, Hesselholt, and Tsalidis have used the case of S^1 and its subgroups to carry out fundamentally important calculations of the topological cyclic homology and thus of the algebraic K-theory of number rings. It would take us too far afield to say much about this. Madsen has given two excellent surveys. In another direction, Hesselholt and Madsen have calculated the coefficient groups of the S^1-tate spectrum associated to the periodic J-theory spectrum at an odd prime. The calculation is consistent with the following conjecture.

Conjecture 6.4 (Hesselholt-Madsen).

$$t(J_G)^{S^1} \simeq K'(1) \vee \Sigma K'(1) \vee (\prod_{n\in\mathbb{Z}} \Sigma^{2n+1} K)/(\bigvee_{n\in\mathbb{Z}} \Sigma^{2n+1} K),$$

where $K'(1)$ is the Adams summand of p-complete K-theory with homotopy groups concentrated in degrees $\equiv 0 \bmod 2(p-1)$.

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CHAPTER XXII

Brave new algebra

1. The category of S-modules

Let us return to the introductory overview of the stable homotopy category given in XII§1. As said there, Elmendorf, Kriz, Mandell, and I have gone beyond the foundations of Chapter XII to the construction of a new category of spectra, the category of "S-modules", that has a smash product that is symmetric monoidal (associative, commutative, and unital up to coherent natural isomorphisms) on the point-set level. The complete treatment is given in [EKMM], and an exposition has been given in [EKMM′]. The latter emphasizes the logical development of the foundations. Here, instead, we will focus more on the structure and applications of the theory. Working nonequivariantly in this chapter, we will describe the new categories of rings, modules, and algebras and summarize some of their more important applications. All of the basic theory generalizes to the equivariant context and, working equivariantly, we will return to the foundations and outline the construction of the category of S-modules in the next chapter. We begin work here by summarizing its properties.

An S-module is a spectrum (indexed on some fixed universe U) with additional structure, and a map of S-modules is a map of spectra that preserves the additional structure. The sphere spectrum S and, more generally, any suspension spectrum $\Sigma^\infty X$ has a canonical structure of S-module. The category of S-modules is denoted $\mathscr{M}_S$. It is symmetric monoidal with unit object S under a suitable smash product, which is denoted $\wedge_S$, and it also has a function S-module functor, which is denoted F_S. The expected adjunction holds:

$$\mathscr{M}_S(M \wedge_S N, P) \cong \mathscr{M}_S(M, F_S(N, P)).$$

Moreover, for based spaces X and Y, there is a natural isomorphism of S-modules

$$\Sigma^\infty X \wedge_S \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y).$$

When regarded as a functor from spaces to S-modules, rather than as a functor from spaces to spectra, Σ^∞ is *not* left adjoint to the zeroth space functor Ω^∞; rather, we have an adjunction

$$\mathscr{M}_S(\Sigma^\infty X, M) \cong \mathscr{T}(X, \mathscr{M}_S(S, M)).$$

Here the space of maps $\mathscr{M}_S(S, M)$ is not even equivalent to $\Omega^\infty M$. As observed by Hastings and Lewis, this is intrinsic to the mathematics: since $\mathscr{M}_S$ is symmetric monoidal, $\mathscr{M}_S(S, S)$ is a commutative topological monoid, and it therefore cannot be equivalent to the space $QS^0 = \Omega^\infty S$.

For an S-module M and a based space X, the smash product $M \wedge X$ is an S-module and

$$M \wedge X \cong M \wedge_S \Sigma^\infty X.$$

Cylinders, cones, and suspensions of S-modules are defined by smashing with I_+, I, and S^1. A homotopy between maps $f, g : M \longrightarrow N$ of S-modules is a map $M \wedge I_+ \longrightarrow N$ that restricts to f and g on the ends of the cylinder. The function spectrum $F(X, M)$ is not an S-module; $F_S(\Sigma^{\Sigma t y} X, M)$ is the appropriate substitute and must be used when defining cocylinder, path, and loop S-modules.

The category $\mathscr{M}_S$ is cocomplete (has all colimits), its colimits being created in $\mathscr{S}$. That is, the colimit in $\mathscr{S}$ of a diagram of S-modules is an S-module that is the colimit of the given diagram in $\mathscr{M}_S$. It is also complete (has all limits). The limit in $\mathscr{S}$ of a diagram of S-modules is not quite an S-module, but it takes values in a category $\mathscr{S}[\mathbb{L}]$ of "$\mathbb{L}$-spectra" that lies intermediate between spectra and S-modules. Limits in $\mathscr{S}[\mathbb{L}]$ are created in $\mathscr{S}$, and the forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ has a right adjoint that creates the limits in $\mathscr{M}_S$. We shall explain this scaffolding in XXIII§2. For pragmatic purposes, what matters is that limits exist and have the same weak homotopy types as if they were created in $\mathscr{S}$.

There is a "free S-module functor" $\mathbb{F}_S : \mathscr{S} \longrightarrow \mathscr{M}_S$. It is not quite free in the usual sense since its right adjoint $\mathbb{U}_S : \mathscr{M}_S \longrightarrow \mathscr{S}$ is not quite the evident forgetful functor. This technicality reflects the fact that the forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ is a left rather than a right adjoint. Again, for pragmatic purposes, what matters is that $\mathbb{U}_S$ is naturally weakly equivalent to the evident forgetful functor.

We define sphere S-modules by

$$S^n_S = \mathbb{F}_S S^n.$$

We define the homotopy groups of an S-module to be the homotopy groups of the underlying spectrum and find by the adjunction cited in the previous paragraph that they can be computed as

$$\pi_n(M) = h\mathscr{M}_S(S^n_S, M).$$

From here, we develop the theory of cell and CW S-modules precisely as we developed the theory of cell and CW spectra, taking the spheres S^n_S as the domains of attaching maps of cells CS^n_S. We construct the "derived category of S-modules", denoted $\mathscr{D}_S$, by adjoining formal inverses to the weak equivalences and find that $\mathscr{D}_S$ is equivalent to the homotopy category of CW S-modules. The following fundamental theorem then shows that no homotopical information is lost if we replace the stable homotopy category $\bar{h}\mathscr{S}$ by the derived category $\mathscr{D}_S$.

Theorem 1.1. The following conclusions hold.

(i) The free functor $\mathbb{F}_S : \mathscr{S} \longrightarrow \mathscr{M}_S$ carries CW spectra to CW S-modules.
(ii) The forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}$ carries S-modules of the homotopy types of CW S-modules to spectra of the homotopy types of CW spectra.
(iii) Every CW S-module M is homotopy equivalent as an S-module to $\mathbb{F}_S E$ for some CW spectrum E.

The free functor and forgetful functors establish an adjoint equivalence between the stable homotopy category $\bar{h}\mathscr{S}$ and the derived category $\mathscr{D}_S$. This equivalence of categories preserves smash products and function objects. Thus

$$\mathscr{D}_S(\mathbb{F}_S E, M) \cong \bar{h}\mathscr{S}(E, M),$$

$$\mathbb{F}_S : \bar{h}\mathscr{S}(E, E') \xrightarrow{\cong} \mathscr{D}_S(\mathbb{F}_S E, \mathbb{F}_S E'),$$

$$\mathbb{F}_S(E \wedge E') \simeq (\mathbb{F}_S E) \wedge_S (\mathbb{F}_S E'),$$

and

$$\mathbb{F}_S(F(E, E')) \simeq F_S(\mathbb{F}_S E, \mathbb{F}_S E').$$

We can describe the equivalence in the language of (closed) model categories in the sense of Quillen, but we shall say little about this. Both $\mathscr{S}$ and $\mathscr{M}_S$ are model categories whose weak equivalences are the maps that induce isomorphisms of homotopy groups. The q-cofibrations (or Quillen cofibrations) are the retracts of inclusions of relative cell complexes (that is, cell spectra or cell S-modules). The q-fibrations in $\mathscr{S}$ are the Serre fibrations, namely the maps that satisfy the covering homotopy property with respect to maps defined on the cone spectra

$\Sigma_q^\infty CS^n$, where $q \geq 0$ and $n \geq 0$. The q-fibrations in $\mathscr{M}_S$ are the maps $M \longrightarrow N$ of S-modules whose induced maps $\mathbb{U}_S M \longrightarrow \mathbb{U}_S N$ are Serre fibrations of spectra.

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2. Categories of R-modules

Let us think about S-modules algebraically. There is a perhaps silly analogy that I find illuminating. Algebraically, it is of course a triviality that Abelian groups are essentially the same things as $\mathbb{Z}$-modules. Nevertheless, these notions are conceptually different. Thinking of brave new algebra in stable homotopy theory as analogous to classical algebra, I like to think of spectra as analogues of Abelian groups and S-modules as analogues of $\mathbb{Z}$-modules. While it required some thought and work to figure out how to pass from spectra to S-modules, now that we have done so we can follow our noses and mimic algebraic definitions word for word in the category of S-modules, thinking of $\wedge_S$ as analogous to $\otimes_{\mathbb{Z}}$ and F_S as analogous to $\mathrm{Hom}_{\mathbb{Z}}$.

We think of rings as $\mathbb{Z}$-algebras, and we define an S-algebra R by requiring a unit $S \longrightarrow R$ and product $R \wedge_S R \longrightarrow R$ such that the evident unit and associativity diagrams commute. We say that R is a commutative S-algebra if the evident commutativity diagram also commutes. We define a left R-module similarly, requiring a map $R \wedge_S M \longrightarrow M$ such that the evident unit and associativity diagrams commute.

For a right R-module M and left R-module N, we define an S-module $M \wedge_R N$ by the coequalizer diagram

$$M \wedge_S R \wedge_S N \underset{\mathrm{Id} \wedge_S \nu}{\overset{\mu \wedge_S \mathrm{Id}}{\rightrightarrows}} M \wedge_S N \longrightarrow M \wedge_R N,$$

where μ and ν are the given actions of R on M and N. Similarly, for left R-modules M and N, we define an S-module $F_R(M,N)$ by an appropriate equalizer diagram. We then have adjunctions exactly like those relating $\otimes_R$ and Hom_R in algebra.

If R is commutative, then $M \wedge_R N$ and $F_R(M,N)$ are R-modules, the category $\mathscr{M}_R$ of R-modules is symmetric monoidal with unit R, and we have the expected adjunction relating $\wedge_R$ and F_R. We can go on to define (R,R')-bimodules and to derive a host of formal relations involving smash products and function modules over varying rings, all of which are exactly like their algebraic counterparts.

For a left R-module M and a based space X, $M \wedge X \cong M \wedge_S \Sigma^\infty X$ and $F_S(\Sigma^\infty X, M)$ are left R-modules. If K is an S-module, then $M \wedge_S K$ is a left and $F_S(M,K)$ is a right R-module. We have theories of cofiber and fiber sequences of R-modules exactly as for spectra. We define the free R-module generated by a spectrum X to be

$$\mathbb{F}_R X = R \wedge_S \mathbb{F}_S X.$$

Again the right adjoint $\mathbb{U}_R$ of this functor is naturally weakly equivalent to the forgetful functor from R-modules to spectra. We define sphere R-modules by

$$S^n_R = \mathbb{F}_R S^n = R \wedge_S S^n_S$$

and find that

$$\pi_n(M) = h\mathscr{M}_R(S^n_R, M).$$

There is also a natural weak equivalence of R-modules $\mathbb{F}_R S \longrightarrow R$.

We develop the theory of cell and CW R-modules exactly as we developed the theory of cell and CW spectra, using the spheres S^n_R as the domains of attaching maps. However, the CW theory is only of interest when R is connective ($\pi_n(R) = 0$ for $n < 0$) since otherwise the cellular approximation theorem fails. We construct the derived category $\mathscr{D}_R$ from the category $\mathscr{M}_R$ of R-modules by adjoining formal inverses to the weak equivalences and find that $\mathscr{D}_R$ is equivalent to the homotopy category of cell R-modules.

Brown's representability theorem holds in the category $\mathscr{D}_R$: a contravariant set-valued functor k on $\mathscr{D}_R$ is representable in the form $kM \cong \mathscr{D}_R(M,N)$ if and only if k converts wedges to products and converts homotopy pushouts to weak pullbacks. However, as recently observed by Neeman in an algebraic context, Adams' variant for functors defined on finite cell R-modules only holds under a countability hypothesis on $\pi_*(R)$.

The category $\mathscr{M}_R$ is a model category. The weak equivalences and q-fibrations are the maps of R-modules that are weak equivalences and q-fibrations when regarded as maps of S-modules. The q-cofibrations are the retracts of relative cell R-modules. It is also a tensored and cotensored topological category. That is, its

Hom sets are based topological spaces, composition is continuous, and we have adjunction homeomorphisms

$$\mathscr{M}_R(M \wedge X, N) \cong \mathscr{T}(X, \mathscr{M}_R(M, N)) \cong \mathscr{M}_R(M, F_S(\Sigma^\infty X, N)).$$

Recently, Hovey, Palmieri, and Strickland have axiomatized the formal properties that a category ought to have in order to be called a "stable homotopy category". The idea is to abstract those properties that are independent of any underlying point-set level foundations and see what can be derived from that starting point. Our derived categories $\mathscr{D}_R$ provide a wealth of examples.

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3. The algebraic theory of R-modules

The categories $\mathscr{D}_R$ are both tools for the the study of classical algebraic topology, and interesting new subjects of study in their own right. In particular, they subsume much of classical algebra. The Eilenberg-MacLane spectrum HR associated to a (commutative) discrete ring R is a (commutative) S-algebra, and the Eilenberg-MacLane spectrum HM associated to an R-module is an HR-module. Moreover, the derived category $\mathscr{D}_{HR}$ is equivalent to the algebraic derived category $\mathscr{D}_R$ of chain complexes over R, and the equivalence converts derived smash products and function modules in topology to derived tensor products and Hom functors in algebra. In algebra, the homotopy groups of derived tensor product and Hom functors compute Tor and Ext, and we have natural isomorphisms

$$\pi_n(HM \wedge_{HR} HN) \cong Tor_n^R(M, N)$$

for a right R-module M and left R-module N and

$$\pi_{-n}(F_{HR}(HM, HN)) \cong Ext_R^n(M, N)$$

for left R-modules M and N, where HM is taken to be a CW HR-module.

Now return to the convention that R is an S-algebra. By the equivalence of $\bar{h}\mathscr{S}$ and $\mathscr{D}_S$, we see that homology and cohomology theories on spectra are subsumed as homotopy groups of smash products and function modules over S. Precisely, for a CW S-module M and an S-module N,

$$\pi_n(M \wedge_S N) = M_n(N)$$

and

$$\pi_{-n}(F_S(M, N)) = N^n(M).$$

These facts suggest that we should think of the homotopy groups of smash product and function R-modules ambiguously as generalizations of both Tor and Ext groups and homology and cohomology groups. Thus, for a right cell R-module M and a left R-module N, we define

$$Tor^R_n(M,N) = \pi_n(M \wedge_R N) = M^R_n(N) \tag{3.1}$$

and, for a left cell R-module M and a left R-module N, we define

$$Ext^n_R(M,N) = \pi_{-n}(F_R(M,N)) = N^n_R(M). \tag{3.2}$$

We assume that M is a cell module to ensure that these are well-defined derived category invariants.

These functors enjoy many properties familiar from both the algebraic and topological settings. For example, assuming that R is commutative, we have a natural, associative, and unital system of pairings of R^*-modules ($R^n = \pi_{-n}(R)$)

$$\mathrm{Ext}^*_R(M,N) \otimes_{R^*} \mathrm{Ext}^*_R(L,M) \longrightarrow \mathrm{Ext}^*_R(L,N).$$

Similarly, setting $D_R M = F_R(M,R)$, a formal argument in duality theory implies a natural isomorphism

$$Tor^R_n(D_R M, N) \cong \mathrm{Ext}^{-n}_R(M,N)$$

for finite cell R-modules M and arbitrary R-modules N. Thought of homologically, this isomorphism can be interpreted as Spanier-Whitehead duality: for a finite cell R-module M and any R-module N,

$$N^R_n(D_R M) \cong N^{-n}_R(M).$$

There are spectral sequences for the computation of these invariants. As usual, for a spectrum E, we write $E_n = \pi_n(E) = E^{-n}$.

Theorem 3.3. For right and left R-modules M and N, there is a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(M_*, N_*) \Longrightarrow \mathrm{Tor}^R_{p+q}(M,N);$$

For left R-modules M and N, there is a spectral sequence

$$E^{p,q}_2 = \mathrm{Ext}^{p,q}_{R^*}(M^*, N^*) \Longrightarrow \mathrm{Ext}^{p+q}_R(M,N).$$

If R is commutative, these are spectral sequences of differential R_*-modules, and the second admits pairings converging from the evident Yoneda pairings on the E_2 terms to the natural pairings on the limit terms.

Setting $M = \mathbb{F}_R X$ in these two spectral sequences, we obtain universal coefficient spectral sequences.

THEOREM 3.4 (UNIVERSAL COEFFICIENT). For an R-module N and any spectrum X, there are spectral sequences of the form

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), N_*) \Longrightarrow N_*(X)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), N^*) \Longrightarrow N^*(X).$$

Replacing R and N by Eilenberg-Mac Lane spectra HR and HN for a discrete ring R and R-module N, we obtain the classical universal coefficient theorems. Replacing N by $\mathbb{F}_R Y$ and by $F_R(\mathbb{F}_R Y, R)$ in the two universal coefficient spectral sequences, we obtain Künneth spectral sequences.

THEOREM 3.5 (KÜNNETH). For any spectra X and Y, there are spectral sequences of the form

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), R_*(Y)) \Longrightarrow R_*(X \wedge Y)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), R^*(Y)) \Longrightarrow R^*(X \wedge Y).$$

Under varying hypotheses, the Künneth theorem in homology generalizes to an Eilenberg-Moore type spectral sequence. Here is one example.

THEOREM 3.6. Let E and R be commutative S-algebras and M and N be R-modules. Then there is a spectral sequence of differential $E_*(R)$-modules of the form

$$\mathrm{Tor}^{E_*(R)}_{p,q}(E_*(M), E_*(N)) \Longrightarrow E_{p+q}(M \wedge_R N).$$

4. The homotopical theory of R-modules

Thinking of the derived category of R-modules as an analog of the stable homotopy category, we have the notion of an R-ring spectrum, which is just like the classical notion of a ring spectrum in the stable homotopy category.

DEFINITION 4.1. An R-ring spectrum A is an R-module A with unit $\eta : R \longrightarrow A$ and product $\phi : A \wedge_R A \longrightarrow A$ in $\mathscr{D}_R$ such that the following left and right unit diagram commutes in $\mathscr{D}_R$:

$$\begin{array}{ccccc} R \wedge_R A & \xrightarrow{\eta\wedge\mathrm{id}} & A \wedge_R A & \xleftarrow{\mathrm{id}\wedge\eta} & A \wedge_R R \\ & \searrow^{\lambda} & \downarrow{\scriptstyle\phi} & \swarrow_{\lambda\tau} & \\ & & A. & & \end{array}$$

A is associative or commutative if the appropriate diagram commutes in $\mathscr{D}_R$. If A is associative, then an A-module spectrum M is an R-module M with an action $\mu : A \wedge_R M \longrightarrow M$ such that the evident unit and associativity diagrams commute in $\mathscr{D}_R$.

LEMMA 4.2. If A and B are R-ring spectra, then so is $A \wedge_R B$. If A and B are associative or commutative, then so is $A \wedge_R B$.

When $R = S$, S-ring spectra and their module spectra are equivalent to classical ring spectra and their module spectra. By neglect of structure, an R-ring spectrum A is an S-ring spectrum and thus a ring spectrum in the classical sense; its unit is the composite of the unit of R and the unit of A and its product is the composite of the product of A and the canonical map

$$A \wedge A \simeq A \wedge_S A \longrightarrow A \wedge_R A.$$

If A is commutative or associative as an R-ring spectrum, then it is commutative and associative as an S-ring spectrum and thus as a classical ring spectrum. The R-ring spectra and their module spectra play a role in the study of $\mathscr{D}_R$ analogous to the role played by ring and module spectra in classical stable homotopy theory. Moreover, the new theory of R-ring and module spectra provides a powerful constructive tool for the study of the classical notions. The point is that, in $\mathscr{D}_R$, we have all of the internal structure, such as cofiber sequences, that we have in the stable homotopy category.

This can make it easy to construct R-ring spectra and modules in cases when a direct proof that they are merely classical ring spectra and modules is far more difficult, if it can be done at all. We assume that R is a commutative S-algebra and illustrate by indicating how to construct M/IM and $M[Y^{-1}]$ for an R-module M, where I is the ideal generated by a sequence $\{x_i\}$ of elements of R_* and Y is a countable multiplicatively closed set of elements of R_*. We shall also state some results about when these modules have R-ring structures and when such structures are commutative or associative.

We have isomorphisms

$$M_n \cong h\mathscr{M}_R(S^n_R, M).$$

The suspension $\Sigma^n M$ is equivalent to $S^n_R \wedge_R M$ and, for $x \in R_n$, the composite map of R-modules

$$S^n_R \wedge_R M \xrightarrow{x\wedge\mathrm{id}} R \wedge_R M \xrightarrow{\lambda} M \tag{4.3}$$

is a module theoretic version of the map $x\cdot : \Sigma^n M \longrightarrow M$.

Definition 4.4. Define M/xM to be the cofiber of the map (4.3) and let $\rho : M \longrightarrow M/xM$ be the canonical map. Inductively, for a finite sequence $\{x_1, \dots, x_n\}$ of elements of R_*, define

$$M/(x_1, \dots, x_n)M = N/x_nN, \quad \text{where} \quad N = M/(x_1, \dots, x_{n-1})M.$$

For a sequence $X = \{x_i\}$, define $M/XM = \mathrm{tel}\, M/(x_1, \dots, x_n)M$, where the telescope is taken with respect to the successive canonical maps ρ.

Clearly we have a long exact sequence

$$\cdots \longrightarrow \pi_{q-n}(M) \xrightarrow{x\cdot} \pi_q(M) \xrightarrow{\rho_*} \pi_q(M/xM) \longrightarrow \pi_{q-n-1}(M) \longrightarrow \cdots.$$

If x is regular for $\pi_*(M)$ ($xm = 0$ implies $m = 0$), then ρ_* induces an isomorphism of R_*-modules

$$\pi_*(M)/x\cdot\pi_*(M) \cong \pi_*(M/xM).$$

If $\{x_1, \dots, x_n\}$ is a regular sequence for $\pi_*(M)$, in the sense that x_i is regular for $\pi_*(M)/(x_1, \dots, x_{i-1})\pi_*(M)$ for $1 \le i \le n$, then

$$\pi_*(M)/(x_1, \dots, x_n)\pi_*(M) \cong \pi_*(M/(x_1, \dots, x_n)M),$$

and similarly for a possibly infinite regular sequence $X = \{x_i\}$. The following result implies that M/XM is independent of the ordering of the elements of the set X. We write R/X instead of R/XR.

Lemma 4.5. For a set X of elements of R_*, there is a natural weak equivalence

$$(R/X) \wedge_R M \longrightarrow M/XM.$$

In particular, for a finite set $X = \{x_1, \dots, x_n\}$,

$$R/(x_1, \dots, x_n) \simeq (R/x_1) \wedge_R \cdots \wedge_R (R/x_n).$$

If I denotes the ideal generated by X, then it is reasonable to define

$$M/IM = M/XM.$$

However, this notation must be used with caution since, if we fail to restrict attention to regular sequences X, the homotopy type of M/XM will depend on the set X and not just on the ideal it generates. For example, quite different modules are obtained if we repeat a generator x_i of I in our construction.

To construct localizations, let $\{y_i\}$ be any sequence of elements of Y that is cofinal in the sense that every $y \in Y$ divides some y_i. If $y_i \in R_{n_i}$, we may represent y_i by an R-map $S_R^0 \longrightarrow S_R^{-n_i}$, which we also denote by y_i. Let $q_0 = 0$ and, inductively, $q_i = q_{i-1} + n_i$. Then the R-map

$$y_i \wedge \mathrm{id} : S_R^0 \wedge_R M \longrightarrow S_R^{-n_i} \wedge_R M$$

represents multiplication by y_i. Smashing over R with $S_R^{-q_{i-1}}$, we obtain a sequence of R-maps

$$S_R^{-q_{i-1}} \wedge_R M \longrightarrow S_R^{-q_i} \wedge_R M. \tag{4.6}$$

DEFINITION 4.7. Define the localization of M at Y, denoted $M[Y^{-1}]$, to be the telescope of the sequence of maps (4.6). Since $M \cong S_R^0 \wedge_R M$ in $\mathscr{D}_R$, we may regard the inclusion of the initial stage $S_R^0 \wedge_R M$ of the telescope as a natural map $\lambda : M \longrightarrow M[Y^{-1}]$.

Since homotopy groups commute with localization, we see immediately that λ induces an isomorphism of R_*-modules

$$\pi_*(M[Y^{-1}]) \cong \pi_*(M)[Y^{-1}].$$

As in Lemma 4.5, the localization of M is the smash product of M with the localization of R.

LEMMA 4.8. For a multiplicatively closed set Y of elements of R_*, there is a natural equivalence

$$R[Y^{-1}] \wedge_R M \longrightarrow M[Y^{-1}].$$

Moreover, $R[Y^{-1}]$ is independent of the ordering of the elements of Y. For sets X and Y, $R[(X \cup Y)^{-1}]$ is equivalent to the composite localization $R[X^{-1}][Y^{-1}]$.

The behavior of localizations with respect to R-ring structures is now immediate.

Proposition 4.9. Let Y be a multiplicatively closed set of elements of R_*. If A is an R-ring spectrum, then so is $A[Y^{-1}]$. If A is associative or commutative, then so is $A[Y^{-1}]$.

Proof. It suffices to observe that $R[Y^{-1}]$ is an associative and commutative R-ring spectrum with unit λ and product the equivalence

$$R[Y^{-1}] \wedge_R R[Y^{-1}] \simeq R[Y^{-1}][Y^{-1}] \simeq R[Y^{-1}]. \quad \square$$

This doesn't work for quotients since $(R/X)/X$ is not equivalent to R/X. However, we can analyze the problem by analyzing the deviation, and, by Lemma 4.5, we may as well work one element at a time. We have a necessary condition for R/x to be an R-ring spectrum that is familiar from classical stable homotopy theory.

Lemma 4.10. Let A be an R-ring spectrum. If A/xA admits a structure of R-ring spectrum such that $\rho : A \longrightarrow A/xA$ is a map of R-ring spectra, then $x : A/xA \longrightarrow A/xA$ is null homotopic as a map of R-modules.

Thus, for example, the Moore spectrum $S/2$ is not an S-ring spectrum since the map $2 : S/2 \longrightarrow S/2$ is not null homotopic. We have the following sufficient condition for when R/x does have an R-ring spectrum structure.

Theorem 4.11. Let $x \in R_m$, where $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Then R/x admits a structure of R-ring spectrum with unit $\rho : R \longrightarrow R/x$. Therefore, for every R-ring spectrum A and every sequence X of elements of R_* such that $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$ if $x \in X$ has degree m, A/XA admits a structure of R-ring spectrum such that $\rho : A \longrightarrow A/XA$ is a map of R-ring spectra.

For an R-ring spectrum A and an element x as in the theorem, we give $A/xA \simeq (R/x) \wedge_R A$ the product induced by one of our constructed products on R/x and the given product on A. We refer to any such product as a "canonical" product on A/xA. We also have sufficient conditions for when the canonical product is unique and when a canonical product is commutative or associative.

Theorem 4.12. Let $x \in R_m$, where $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Let A be an R-ring spectrum and assume that $\pi_{2m+2}(A/xA) = 0$. Then there is a unique canonical product on A/xA. If A is commutative, then A/xA is commutative. If A is associative and $\pi_{3m+3}(A/xA) = 0$, then A/xA is associative.

This leads to the following conclusion.

THEOREM 4.13. Assume that $R_i = 0$ if i is odd. Let X be a sequence of non zero divisors in R_* such that $\pi_*(R/X)$ is concentrated in degrees congruent to zero mod 4. Then R/X has a unique canonical structure of R-ring spectrum, and it is commutative and associative.

This is particularly valuable when applied with $R = MU$. The classical Thom spectra arise in nature as E_∞ ring spectra and give rise to equivalent commutative S-algebras. In fact, inspection of the prespectrum level definition of Thom spectra in terms of Grassmannians first led to the theory of E_∞ ring spectra and therefore of S-algebras. Of course,

$$MU_* = \mathbb{Z}[x_i | deg\, x_i = 2i]$$

Thus the results above have the following immediate corollary.

THEOREM 4.14. Let X be a regular sequence in MU_*, let I be the ideal generated by X, and let Y be any sequence in MU_*. Then there is an MU-ring spectrum $(MU/X)[Y^{-1}]$ and a natural map of MU-ring spectra (the unit map)

$$\eta : MU \longrightarrow (MU/X)[Y^{-1}]$$

such that

$$\eta_* : MU_* \longrightarrow \pi_*((MU/X)[Y^{-1}])$$

realizes the natural homomorphism of MU_*-algebras

$$MU_* \longrightarrow (MU_*/I)[Y^{-1}].$$

If MU_*/I is concentrated in degrees congruent to zero mod 4, then there is a unique canonical product on $(MU/X)[Y^{-1}]$, and this product is commutative and associative.

In comparison with earlier constructions of this sort based on the Baas-Sullivan theory of manifolds with singularities or on Landweber's exact functor theorem (where it applies), we have obtained a simpler proof of a substantially stronger result since an MU-ring spectrum is a much richer structure than just a ring spectrum and commutativity and associativity in the MU-ring spectrum sense are much more stringent conditions than mere commutativity and associativity of the underlying ring spectrum.

5. Categories of R-algebras

In the previous section, we considered R-ring spectra, which are homotopical versions of R-algebras. We also have a pointwise definition of R-algebras that is just like the definition of S-algebras. That is, R-algebras and commutative R-algebras A are defined via unit and product maps $R \longrightarrow A$ and $A \wedge_R A \longrightarrow A$ such that the appropriate diagrams commute in the symmetric monoidal category $\mathscr{M}_R$. All of the standard formal properties of algebras in classical algebra carry over directly to these brave new algebras. For example, a commutative R-algebra A is the same thing as a commutative S-algebra together with a map of S-algebras $R \longrightarrow A$ (the unit map), and the smash product $A \wedge_R A'$ of commutative R-algebras A and A' is their coproduct in the category of commutative R-algebras.

Some of the most subtantive work in [EKMM] concerns the understanding of the categories $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ of R-algebras and commutative R-algebras. The crucial point is to be able to compute the homotopical behavior of formal constructions in these categories. Technically, what is involved is the homotopical understanding of the forgetful functors from $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ to $\mathscr{M}_R$. Although not in itself enough to answer these questions, the context of enriched model categories is essential to give a framework in which they can be addressed. We shall indicate some of the main features here, but this material is addressed to the relatively sophisticated reader who has some familiarity with enriched category and model category theory. It provides the essential technical underpinning for the applications to Bousfield localization and topological Hochschild homology that are summarized in the following two sections.

Both $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ are tensored and cotensored topological categories. In fact, they are topologically complete and cocomplete, which means that they have not only the usual limits and colimits but also "indexed" limits and colimits. Limits are created in the category of R-modules, but colimits are less obvious constructions. In the absence of basepoints in their Hom sets, these categories are enriched over the category $\mathscr{U}$ of unbased spaces. The cotensors in both cases are the function S-algebras $F_S(\Sigma^\infty X_+, A)$ with the R-algebra structure induced from the diagonal on X and the product on A. The tensors are less familiar. They are denoted $A \otimes_{\mathscr{A}_R} X$ and $A \otimes_{\mathscr{C}\mathscr{A}_R} X$. These are different constructions in the two cases, but we write $A \otimes X$ when the context is understood. We have adjunctions

$$(5.1) \qquad \mathscr{A}_R(A \otimes X, B) \cong \mathscr{U}(X, \mathscr{A}_R(A, B)) \cong \mathscr{A}_R(A, F_S(\Sigma^\infty X_+, B)),$$

and similarly in the commutative case. Some idea of the structure and meaning of tensors is given by the following result. For R-algebras A and B and a space X, we say that a map $f : A \wedge X_+ \longrightarrow B$ of R-modules is a pointwise map of R-algebras if each composite $f \circ i_x : A \longrightarrow B$ is a map of R-algebras, where, for $x \in X$, $i_x : A \longrightarrow A \wedge X_+$ is the map induced by the evident inclusion $\{x\}_+ \longrightarrow X_+$.

PROPOSITION 5.2. For R-algebras A and spaces X there is a natural map of R-modules

$$\omega : A \wedge X_+ \longrightarrow A \otimes X$$

such that a pointwise map $f : A \wedge X_+ \longrightarrow B$ of R-algebras uniquely determines a map $\tilde{f} : A \otimes X \longrightarrow B$ of R-algebras such that $f = \tilde{f} \circ \omega$. The same statement holds for commutative R-algebras.

More substantial results tell how to compute tensors when X is the geometric realization of a simplicial set or simplicial space. These results are at the heart of the development and understanding of model category structures on the categories $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$. In both categories, the weak equivalences and q-fibrations are the maps of R-algebras that are weak equivalences or q-fibrations of underlying R-modules. It follows that the q-cofibrations are the maps of R-algebras that satisfy the left lifting property with respect to the acyclic q-fibrations. (The LLP is recalled in VI§5.) However, the q-cofibrations admit a more explicit description as retracts of relative "cell R-algebras" or "cell commutative R-algebras". Such cell algebras are constructed by using free algebras generated by sphere spectra as the domains of attaching maps and mimicking the construction of cell R-modules, using coproducts, pushouts, and colimits in the relevant category of R-algebras.

The question of understanding the homotopical behavior of the forgetful functors from $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ to $\mathscr{M}_R$ now takes the form of understanding the homotopical behavior of q-cofibrant algebras (retracts of cell algebras) with respect to these forgetful functors. However, the formal properties of model categories have nothing to say about this homotopical question.

In what follows, let R be a fixed q-cofibrant commutative R-algebra. Since R is the initial object of $\mathscr{A}_R$ and of $\mathscr{C}\mathscr{A}_R$, it is q-cofibrant both as an R-algebra and as a commutative R-algebra. However, it is not q-cofibrant as an R-module. Therefore the most that one could hope of the underlying R-module of a q-cofibrant R-algebra is the conclusion of the following result.

THEOREM 5.3. If A is a q-cofibrant R-algebra, then A is a retract of a cell R-module relative to R. That is, the unit $R \longrightarrow A$ is a q-cofibration of R-modules.

The conclusion fails in the deeper commutive case. The essential reason is that the free commutative R-algebra generated by an R-module M is the wedge of the symmetric powers M^j/Σ_j, and passage to orbits obscures the homotopy type of the underlying R-module. The following technically important result at least gives the homotopy type of the underlying spectrum.

THEOREM 5.4. Let R be a q-cofibrant commutative S-algebra. If M is a cell R-module, then the projection

$$\pi : (E\Sigma_j)_+ \wedge_{\Sigma_j} M^j \longrightarrow M^j/\Sigma_j$$

is a homotopy equivalence of spectra.

The following theorem provides a workable substitute for Theorem 5.3. It shows that the derived smash product is represented by the point-set level smash product on a large class $\bar{\mathscr{E}}_R$ of R-modules, one that in particular includes the underlying R-modules of all q-cofibrant R-algebras and commutative R-algebras.

THEOREM 5.5. There is a collection $\bar{\mathscr{E}}_R$ of R-modules of the underlying homotopy types of CW spectra that is closed under wedges, pushouts, colimits of countable sequences of cofibrations, homotopy equivalences, and finite smash products over R and that contains all q-cofibrant R-modules and the underlying R-modules of all q-cofibrant R-algebras and all q-cofibrant commutative R-algebras. Moreover, if $M_1, \cdots, M_n$ are R-modules in $\bar{\mathscr{E}}_R$ and $\gamma_i : N_i \longrightarrow M_i$ are weak equivalences, where the N_i are cell R-modules, then

$$\gamma_1 \wedge_R \cdots \wedge_R \gamma_n : N_1 \wedge_R \cdots \wedge_R N_n \longrightarrow M_1 \wedge_R \cdots \wedge_R M_n$$

is a weak equivalence. Therefore the cell R-module $N_1 \wedge_R \cdots \wedge_R N_n$ represents $M_1 \wedge_R \cdots \wedge_R M_n$ in the derived category $\mathscr{D}_R$.

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6. Bousfield localizations of R-modules and algebras

Bousfield localization is a basic tool in the study of classical stable homotopy theory, and the construction generalizes readily to the context of brave new algebra. In fact, using our model category structures, this context leads to a smoother treatment than can be found in the classical literature. More important, as we shall sketch, any brave new algebraic structure is preserved by Bousfield localization.

Let R be an S-algebra and E be a cell R-module. A map $f : M \longrightarrow N$ of R-modules is said to be an E-equivalence if

$$\mathrm{id} \wedge_R f : E \wedge_R M \longrightarrow E \wedge_R N$$

is a weak equivalence. An R-module W is said to be E-acyclic if $E \wedge_R W \simeq *$, and a map f is an E-equivalence if and only if its cofiber is E-acyclic. We say that an R-module L is E-local if $f^* : \mathscr{D}_R(N, L) \longrightarrow \mathscr{D}_R(M, L)$ is an isomorphism for any E-equivalence f or, equivalently, if $\mathscr{D}_R(W, L) = 0$ for any E-acyclic R-module W. Since this is a derived category criterion, it suffices to test it when W is a cell R-module. A localization of M at E is a map $\lambda : M \longrightarrow M_E$ such that λ is an E-equivalence and M_E is E-local. The formal properties of such localizations discussed by Bousfield carry over verbatim to the present context. There is a model structure on $\mathscr{M}_R$ that implies the existence of E-localizations of R-modules.

THEOREM 6.1. The category $\mathscr{M}_R$ admits a new structure as a topological model category in which the weak equivalences are the E-equivalences and the cofibrations are the q-cofibrations in the standard model structure, that is, the retracts of the inclusions of relative cell R-modules.

We call the fibrations in the new model structure E-fibrations. They are determined formally as maps that satisfy the right lifting property with respect to the E-acyclic q-cofibrations, namely the q-cofibrations that are E-equivalences. (The RLP is recalled in VI§5.) One can characterize the E-fibrations more explicitly, but the following result gives all the relevant information. Say that an R-module L is E-fibrant if the trivial map $L \longrightarrow *$ is an E-fibration.

THEOREM 6.2. An R-module is E-fibrant if and only if it is E-local. Any R-module M admits a localization $\lambda : M \longrightarrow M_E$ at E.

In fact, one of the standard properties of a model category shows that we can factor the trivial map $M \longrightarrow *$ as the composite of an E-acyclic q-cofibration $\lambda : M \longrightarrow M_E$ and an E-fibration $M_E \longrightarrow *$, so that the first statement implies the second. The following complement shows that the localization of an R-module at a spectrum (not necessarily an R-module) can be constructed as a map of R-modules.

PROPOSITION 6.3. Let K be a CW-spectrum and let E be the R-module $\mathbb{F}_R K$. Regarded as a map of spectra, a localization $\lambda : M \longrightarrow M_E$ of an R-module M at E is a localization of M at K.

The result generalizes to show that, for an R-algebra A, the localization of an A-module at an R-module E can be constructed as a map of A-modules.

PROPOSITION 6.4. Let A be a q-cofibrant R-algebra, let E be a cell R-module, and let F be the A-module $A \wedge_R E$. Regarded as a map of R-modules, a localization $\lambda : M \longrightarrow M_F$ of an A-module M at F is a localization of M at E.

Restrict R to be a q-cofibrant commutative S-algebra in the rest of this section. We then have the following fundamental theorem about localizations of R-algebras.

THEOREM 6.5. For a cell R-algebra A, the localization $\lambda : A \longrightarrow A_E$ can be constructed as the inclusion of a subcomplex in a cell R-algebra A_E. Moreover, if $f : A \longrightarrow B$ is a map of R-algebras into an E-local R-algebra B, then f lifts to a map of R-algebras $\tilde{f} : A_E \longrightarrow B$ such that $\tilde{f} \circ \lambda = f$; if f is an E-equivalence, then $\tilde{f}$ is a weak equivalence. The same statements hold for commutative R-algebras.

The idea is to replace the category $\mathscr{M}_R$ by either the category $\mathscr{A}_R$ or the category $\mathscr{C}\mathscr{A}_R$ in the development just sketched. That is, we attempt to construct new model category structures on $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ in such a fashion that a factorization of the trivial map $A \longrightarrow *$ as the composite of an E-acyclic q-cofibration and a q-fibration in the appropriate category of R-algebras gives a localization of the underlying R-module of A. The argument doesn't quite work to give a model structure because the module level argument uses vitally that a pushout of an E-acyclic q-cofibration of R-modules is an E-equivalence. There is no reason to believe that this holds for q-cofibrations of R-algebras. However, we can use Theorems 5.3–5.5 to prove that it does hold for pushouts of inclusions of subcomplexes in cell R-algebras along maps to cell R-algebras. This gives enough information to prove the theorem.

The theorem implies in particular that we can construct the localization of R at E as the unit $R \longrightarrow R_E$ of a q-cofibrant commutative R-algebra. This leads to a new perspective on localizations in classical stable homotopy theory. To see this, we compare the derived category $\mathscr{D}_{R_E}$ to the stable homotopy category $\mathscr{D}_R[E^{-1}]$ associated to the model structure on $\mathscr{M}_R$ that is determined by E. Thus $\mathscr{D}_R[E^{-1}]$ is obtained from $\mathscr{D}_R$ by inverting the E-equivalences and is equivalent to the full subcategory of $\mathscr{D}_R$ whose objects are the E-local R-modules. Observe that, for a cell R-module M, we have the canonical E-equivalence

$$\xi = \eta \wedge \mathrm{id} : M \cong R \wedge_R M \longrightarrow R_E \wedge_R M.$$

The following observation is the same as in the classical case.

LEMMA 6.6. If M is a finite cell R-module, then $R_E \wedge_R M$ is E-local and therefore ξ is the localization of M at E.

We say that localization at E is smashing if, for all cell R-modules M, $R_E \wedge_R M$ is E-local and therefore ξ is the localization of M at E. The following observation is due to Wolbert.

PROPOSITION 6.7 (WOLBERT). If localization at E is smashing, then the categories $\mathscr{D}_R[E^{-1}]$ and $\mathscr{D}_{R_E}$ are equivalent.

These categories are closely related even when localization at E is not smashing, as the following elaboration of Wolbert's result shows. Remember that R is assumed to be commutative.

THEOREM 6.8. The following three categories are equivalent.

(i) The category $\mathscr{D}_R[E^{-1}]$ of E-local R-modules.
(ii) The full subcategory $\mathscr{D}_{R_E}[E^{-1}]$ of $\mathscr{D}_{R_E}$ whose objects are the R_E-modules that are E-local as R-modules.
(iii) The category $\mathscr{D}_{R_E}[F^{-1}]$ of F-local R_E-modules, where $F = R_E \wedge_R E$.

This implies that the question of whether or not localization at E is smashing is a question about the category of R_E-modules, and it leads to the following factorization of the localization functor. In the case $R = S$, this shows that the commutative S-algebras S_E and their categories of modules are intrinsic to the classical theory of Bousfield localization.

THEOREM 6.9. Let $F = R_E \wedge_R E$. The E-localization functor

$$(\cdot)_E : \mathscr{D}_R \longrightarrow \mathscr{D}_R[E^{-1}]$$

is equivalent to the composite of the extension of scalars functor

$$R_E \wedge_R (\cdot) : \mathscr{D}_R \longrightarrow \mathscr{D}_{R_E}$$

and the F-localization functor

$$(\cdot)_F : \mathscr{D}_{R_E} \longrightarrow \mathscr{D}_{R_E}[F^{-1}].$$

COROLLARY 6.10. Localization at E is smashing if and only if all R_E-modules are E-local as R-modules, so that

$$\mathscr{D}_R[E^{-1}] \approx \mathscr{D}_{R_E} \approx \mathscr{D}_{R_E}[F^{-1}].$$

We illustrate the constructive power of Theorem 6.5 by showing that the algebraic localizations of R considered in Section 4 actually take R to commutative R-algebras on the point set level and not just on the homotopical level (as given by Proposition 4.9). Thus let Y be a countable multiplicatively closed set of elements of R_*. Using Lemma 4.8, we see that localization of R-modules at $R[Y^{-1}]$ is smashing and is given by the canonical maps

$$\lambda = \lambda \wedge_R \mathrm{id} : M \cong R \wedge_R M \longrightarrow R[Y^{-1}] \wedge_R M.$$

THEOREM 6.11. The localization $R \longrightarrow R[Y^{-1}]$ can be constructed as the unit of a cell R-algebra.

By multiplicative infinite loop space theory and our model category structure on the category of S-algebras, the spectra ko and ku that represent real and complex connective K-theory can be taken to be q-cofibrant commutative S-algebras. The spectra that represent periodic K-theory can be reconstructed up to homotopy by inverting the Bott element $\beta_O \in \pi_8(ko)$ or $\beta_U \in \pi_2(ku)$. That is,

$$KO \simeq ko[\beta_O^{-1}] \qquad \text{and} \qquad KU \simeq ku[\beta_U^{-1}].$$

We are entitled to the following result as a special case of the previous one.

THEOREM 6.12. The spectra KO and KU can be constructed as commutative ko and ku-algebras.

In particular, KO and KU are commutative S-algebras, but it seems very hard to prove this directly. Wolbert has studied the algebraic structure of the derived categories of modules over the connective and periodic versions of the real and complex K-theory S-algebras.

REMARK 6.13. For finite groups G, Theorem 6.12 applies with the same proof to construct the periodic spectra KO_G and KU_G of equivariant K-theory as commutative ko_G and ku_G-algebras. As we shall discuss in Chapter XXIV, this leads to an elegant proof of the Atiyah-Segal completion theorem in equivariant K-cohomology and of its analogue for equivariant K-homology.

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7. Topological Hochschild homology and cohomology

As another application of brave new algebra, we describe the topological Hochschild homology of R-algebras with coefficients in bimodules. We assume familiarity with the classical Hochschild homology of algebras (as in Cartan and Eilenberg, for example). The study of this topic and of topological cyclic homology, which takes topological Hochschild homology as its starting point and involves equivariant considerations, is under active investigation by many people. We shall just give a brief introduction.

We assume given a q-cofibrant commutative S-algebra R and a q-cofibrant R-algebra A. If A is commutative, we require it to be q-cofibrant as a commutative R-algebra. We define the enveloping R-algebra of A by

$$A^e = A \wedge_R A^{op},$$

where A^{op} is defined by twisting the product on A, as in algebra. If A is commutative, then $A^E \cong A \wedge_R A$ and the product $A^e \longrightarrow A$ is a map of R-algebras. We also assume given an (A, A)-bimodule M; it can be viewed as either a left or a right A^e-module.

DEFINITION 7.1. Working in derived categories, define topological Hochschild homology and cohomology with values in $\mathscr{D}_R$ by

$$THH^R(A; M) = M \wedge_{A^e} A \qquad \text{and} \qquad THH_R(A; M) = F_{A^e}(A, M).$$

If A is commutative, then these functors take values in the derived category $\mathscr{D}_{A^e}$. On passage to homotopy groups, define

$$THH^R_*(A; M) = \mathrm{Tor}^{A^e}_*(M, A) \quad \text{and} \quad THH^*_R(A; M) = \mathrm{Ext}^*_{A^e}(A, M).$$

When $M = A$, we delete it from the notations.

Since we are working in derived categories, we are implicitly taking M to be a cell A^e-module in the definition of $THH^R(A; M)$ and approximating A by a weakly equivalent cell A^e-module in the definition of $THH_R(A; M)$.

PROPOSITION 7.2. If A is a commutative R-algebra, then $THH^R(A)$ is isomorphic in $\mathscr{D}_{A^e}$ to a commutative A^e-algebra.

The module structures on $THH^R(A;M)$ have the following implication.

PROPOSITION 7.3. If either R or A is the Eilenberg-Mac Lane spectrum of a commutative ring, then $THH^R(A;M)$ is a product of Eilenberg-Mac Lane spectra.

We have spectral sequences that relate algebraic and topological Hochschild homology. For a commutative graded ring R_*, a graded R_*-algebra A_* that is flat as an R_*-module, and a graded (A_*, A_*)-bimodule M_*, we define

$$HH^{R_*}_{p,q}(A_*;M_*) = \mathrm{Tor}^{(A_*)^e}_{p,q}(M_*, A_*) \quad \text{and} \quad HH^{p,q}_{R^*}(A^*;M^*) = \mathrm{Ext}^{p,q}_{(A^*)^e}(A^*, M^*),$$

where p is the homological degree and q is the internal degree. (This algebraic definition would not be correct in the absence of the flatness hypothesis.) When $M_* = A_*$, we delete it from the notation. If A_* is commutative, then $HH^{R_*}_{*,*}(A_*)$ is a graded A_*-algebra. Observe that $(A^{op})_* = (A_*)^{op}$.

In view of Theorem 5.5, the spectral sequence of Theorem 3.2 specializes to give the following spectral sequences relating algebraic and topological Hochschild homology.

THEOREM 7.4. There are spectral sequences of the form

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(A_*, A^{op}_*) \Longrightarrow (A^e)_{p+q},$$

$$E^2_{p,q} = \mathrm{Tor}^{(A^e)_*}_{p,q}(M_*, A_*) \Longrightarrow THH^R_{p+q}(A;M),$$

and

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{(A^e)^*}(A^*, M^*) \Longrightarrow THH_R^{p+q}(A;M).$$

If A_* is a flat R_*-module, so that the first spectral sequence collapses, then the initial terms of the second and third spectral sequences are, respectively,

$$HH^{R_*}_{*,*}(A_*;M_*) \quad \text{and} \quad HH^{*,*}_{R^*}(A^*;M^*).$$

This is of negligible use in the absolute case $R = S$, where the flatness hypothesis on A_* is unrealistic. However, in the relative case, it implies that algebraic Hochschild homology and cohomology are special cases of topological Hochschild homology and cohomology.

THEOREM 7.5. Let R be a (discrete, ungraded) commutative ring, let A be an R-flat R-algebra, and let M be an (A, A)-bimodule. Then

$$HH^R_*(A;M) \cong THH^{HR}_*(HA;HM)$$

and

$$HH_R^*(A;M) \cong THH^*_{HR}(HA;HM).$$

If A is commutative, then $HH_*^R(A) \cong THH_*^{HR}(HA)$ as A-algebras.

We concentrate on homology henceforward. In the absolute case $R = S$, it is natural to approach $THH_*^S(A;M)$ by first determining the ordinary homology of $THH^S(A;M)$, using the case $E = H\mathbb{F}_p$ of the following spectral sequence, and then using the Adams spectral sequence. A spectral sequence like the following one was first obtained by Bökstedt. Under flatness hypotheses, there are variants in which E need only be a commutative ring spectrum, e.g. Theorem 7.12 below.

THEOREM 7.6. Let E be a commutative S-algebra. There are spectral sequence of differential $E_*(R)$-modules of the forms

$$E^2_{p,q} = \mathrm{Tor}^{E_*R}_{p,q}(E_*A, E_*(A^{op})) \Longrightarrow E_{p+q}(A^e)$$

and

$$E^2_{p,q} = \mathrm{Tor}^{E_*(A^e)}_{p,q}(E_*(M), E_*(A)) \Longrightarrow E_{p+q}(THH^R(A;M)).$$

There is an alternative description of topological Hochschild homology in terms of the brave new algebra version of the standard complex for the computation of Hochshild homology. Write A^p for the p-fold $\wedge_R$-power of A, and let

$$\phi : A \wedge_R A \longrightarrow A \quad \text{and} \quad \eta : R \longrightarrow A$$

be the product and unit of A. Let

$$\xi_\ell : A \wedge_R M \longrightarrow M \quad \text{and} \quad \xi_r : M \wedge_R A \longrightarrow M$$

be the left and right action of A on M. We have cyclic permutation isomorphisms

$$\tau : M \wedge_R A^p \wedge_R A \longrightarrow A \wedge_R M \wedge_R A^p.$$

The topological analogue of passage from a simplicial k-module to a chain complex of k-modules is passage from a simplicial spectrum E_* to its spectrum level geometric realization $|E_*|$; this construction is studied in [EKMM].

DEFINITION 7.7. Define a simplicial R-module $thh^R(A;M)_*$ as follows. Its R-module of p-simplices is $M \wedge_R A^p$. Its face and degeneracy operators are

$$d_i = \begin{cases} \xi_r \wedge (\mathrm{id})^{p-1} & \text{if } i = 0 \\ \mathrm{id} \wedge (\mathrm{id})^{i-1} \wedge \phi \wedge (\mathrm{id})^{p-i-1} & \text{if } 1 \le i < p \\ (\xi_\ell \wedge (\mathrm{id})^{p-1}) \circ \tau & \text{if } i = p \end{cases}$$

and $s_i = \mathrm{id} \wedge (\mathrm{id})^i \wedge \eta \wedge (\mathrm{id})^{p-i}$. Define

$$thh^R(A;M) = |thh^R(A;M)_*|;$$

When $M = A$, we delete it from the notation, writing $thh^R(A)_*$ and $|thh^R(A)_*|$.

PROPOSITION 7.8. Let A be a commutative R-algebra. Then $thh^R(A)$ is a commutative A-algebra and $thh^R(A;M)$ is a $thh^R(A)$-module.

As in algebra, the starting point for a comparison of definitions is the relative two-sided bar construction $B^R(M,A,N)$. It is defined for a commutative S-algebra R, an R-algebra A, and right and left A-modules M and N. Its R-module of p-simplices is $M \wedge_R A^p \wedge N$. There is a natural map

$$\psi : B^R(A,A,N) \longrightarrow N$$

of A-modules that is a homotopy equivalence of R-modules. More generally, there is a natural map of R-modules

$$\psi : B^R(M,A,N) \longrightarrow M \wedge_A N$$

that is a weak equivalence of R-modules when M is a cell A-module. The relevance of the bar construction to thh is shown by the following observation, which is the same as in algebra. We write

$$B^R(A) = B^R(A,A,A);$$

$B^R(A)$ is an (A,A)-bimodule; on the simplicial level, $B_0^R(A) = A^e$.

PROPOSITION 7.9. For (A,A)-bimodules M, there is a natural isomorphism

$$thh^R(A;M) \cong M \wedge_{A^e} B^R(A).$$

Therefore, for cell A^e-modules M, the natural map

$$thh^R(A;M) \cong M \wedge_{A^e} B^R(A) \xrightarrow{\mathrm{id}\wedge\psi} M \wedge_{A^e} A = THH^R(A;M)$$

is a weak equivalences of R-modules, or of A^e-modules if A is commutative.

While we assumed that M is a cell A^e-module in our derived category level definition of THH, we are mainly interested in the case $M = A$ of our point-set level construction thh, and A is not of the A^e-homotopy type of a cell A^e-module except in trivial cases. However, Theorem 5.5 leads to the following result.

THEOREM 7.10. Let $\gamma : M \longrightarrow A$ be a weak equivalence of A^e-modules, where M is a cell A^e-module. Then the map

$$thh^R(\mathrm{id};\gamma) : thh^R(A;M) \longrightarrow thh^R(A;A) = thh^R(A)$$

is a weak equivalence of R-modules, or of A^e-modules if A is commutative. Therefore $THH^R(A;M)$ is weakly equivalent to $thh^R(A)$.

COROLLARY 7.11. In the derived category $\mathscr{D}_R$, $THH^R(A) \cong thh^R(A)$.

Use of the standard simplicial filtration of the standard complex gives us the promised variant of the spectral sequence of Theorem 7.6. For simplicity, we restrict attention to the absolute case $R = S$.

THEOREM 7.12. Let E be a commutative ring spectrum, A be an S-algebra, and M be a cell A^e-module. If $E_*(A)$ is E_*-flat, there is a spectral sequence of the form

$$E^2_{p,q} = HH^{E_*}_{p,q}(E_*(A); E_*(M)) \Longrightarrow E_{p+q}(thh^S(A; M)).$$

If A is commutative and $M = A$, this is a spectral sequence of differential $E_*(A)$-algebras, the product on E^2 being the standard product on Hochschild homology.

McClure, Schwänzl, and Vogt observed that, when A is commutative, as we assume in the rest of the section, there is an attractive conceptual reinterpretation of the definition of $thh^R(A)$. Recall that the category $\mathscr{C}\mathscr{A}_R$ of commutative R-algebras is tensored over the category of unbased spaces. By writing out the standard simplicial set S^1_* whose realization is the circle and comparing faces and degeneracies, it is easy to check that there is an identification of simplicial commutative R-algebras

$$thh^R(A)_* \cong A \otimes S^1_*. \tag{7.13}$$

Passing to geometric realization and identifying S^1 with the unit complex numbers, we obtain the following consequence.

THEOREM 7.14 (MCCLURE, SCHWÄNZL, VOGT). For commutative R-algebras A, there is a natural isomorphism of commutative R-algebras

$$thh^R(A) \cong A \otimes S^1.$$

The product of $thh^R(A)$ is induced by the codiagonal $S^1 \amalg S^1 \longrightarrow S^1$. The unit $\zeta : A \longrightarrow thh^R(A)$ is induced by the inclusion $\{1\} \to S^1$.

The adjunction (5.1) that defines tensors implies that the functor $thh^R(A)$ preserves colimits in A, something that is not at all obvious from the original definition. The theorem and the adjunction (5.1) imply much further structure on $thh^R(A)$.

COROLLARY 7.15. The pinch map $S^1 \longrightarrow S^1 \vee S^1$ and trivial map $S^1 \longrightarrow *$ induce a (homotopy) coassociative and counital coproduct and counit

$$\psi : thh^R(A) \longrightarrow thh^R(A) \wedge_A thh^R(A) \quad \text{and} \quad \varepsilon : thh^R(A) \longrightarrow A$$

that make $thh^R(A)$ a homotopical Hopf A-algebra.

The product on S^1 gives rise to a map

$$\alpha : (A \otimes S^1) \otimes S^1 \cong A \otimes (S^1 \times S^1) \longrightarrow A \otimes S^1.$$

COROLLARY 7.16. For an integer r, define $\phi^r : S^1 \longrightarrow S^1$ by $\phi^r(e^{2\pi it}) = e^{2\pi irt}$. The ϕ^r induce power operations

$$\Phi^r : thh^R(A) \longrightarrow thh^R(A).$$

These are maps of R-algebras such that

$$\Phi^0 = \zeta\varepsilon, \quad \Phi^1 = \text{id}, \quad \Phi^r \circ \Phi^s = \Phi^{rs},$$

and the following diagrams commute:

$$\begin{array}{ccc} thh^R(A) \otimes S^1 & \xrightarrow{\alpha} & thh^R(A) \\ {\scriptstyle \Phi^r \otimes \phi^s} \downarrow & & \downarrow {\scriptstyle \Phi^{r+s}} \\ thh^R(A) \otimes S^1 & \xrightarrow[\alpha]{} & thh^R(A). \end{array}$$

Consider naive S^1-spectra and let S^1 act trivially on R and A. Via the adjunction (5.1), the map α gives rise to an action of S^1 on $thh^R(A)$.

COROLLARY 7.17. $thh^R(A)$ is a naive commutative S^1-R-algebra. If B is a naive commutative S^1-R-algebra and $f : A \longrightarrow B$ is a map of commutative R-algebras, then there is a unique map $\tilde{f} : thh^R(A) \longrightarrow B$ of naive commutative S^1-R-algebras such that $\tilde{f} \circ \zeta = f$.

Finally, the description of tensors in Proposition 5.2 leads to the following result.

COROLLARY 7.18. There is a natural S^1-equivariant map of R-modules

$$\omega : A \wedge S^1_+ \longrightarrow thh^R(A)$$

such that if B is a commutative R-algebra and $f : A \wedge S^1_+ \longrightarrow B$ is a map of R-modules that is a pointwise map of R-algebras, then f uniquely determines a map of R-algebras $\tilde{f} : thh^R(A) \longrightarrow B$ such that $f = \tilde{f} \circ \omega$.

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H. Cartan and S. Eilenberg. Homological Algebra. Princeton Univ. Press. 1956.
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CHAPTER XXIII

Brave new equivariant foundations

by A. D. Elmendorf, L. G. Lewis, Jr., and J. P. May

1. Twisted half-smash products

We here give a quick sketch of the basic constructions behind the work of the last chapter. Although the basic source, [EKMM], is written nonequivariantly, it applies verbatim to the equivariant context in which we shall work in this chapter. We shall take the opportunity to describe some unpublished perspectives on the role of equivariance in the new theory.

The essential starting point is the twisted half-smash product construction from [LMS]. Although we have come this far without mentioning this construction, it is in fact central to equivariant stable homotopy theory. Before describing it, we shall motivate it in terms of the main theme of this chapter, which is the construction of the category of $\mathbb{L}$-spectra. As we shall see, this is the main step in the construction of the category of S-modules.

Fix a compact Lie group G and a G-universe U and consider the category $G\mathscr{S}U$ of G-spectra indexed on U. Write U^j for the direct sum of j copies of U. Recall that we have an external smash product $\wedge : G\mathscr{S}U \times G\mathscr{S}U \longrightarrow G\mathscr{S}U^2$ and an internal smash product $f_* \circ \wedge : G\mathscr{S}U^2 \longrightarrow G\mathscr{S}U$ for each G-linear isometry $f : U^2 \longrightarrow U$. The external smash product is suitably associative, commutative, and unital on the point set level, hence we may iterate and form an external smash product $\wedge : (G\mathscr{S}U)^j \longrightarrow G\mathscr{S}U^j$ for each $j \geq 1$, the first external smash power being the identity functor. For each G-linear isometry $f : U^j \longrightarrow U$, we have an associated internal smash product $f_* \circ \wedge : G\mathscr{S}U^j \longrightarrow G\mathscr{S}U$. We allow the case $j = 0$; here $G\mathscr{S}\{0\} = G\mathscr{T}$, the only linear isometry $\{0\} \longrightarrow U$ is the inclusion i, and i_* is the suspension G-spectrum functor. At least if we restrict attention to tame G-spectra, the functors induced by varying f are all equivalent (see Theorem 1.5 below). Thus varying G-linear isometries $f : U^j \longrightarrow U$ parametrize equivalent internal smash products.

There is a language for the discussion of such parametrized products in various mathematical contexts, namely the language of "operads" that was introduced for the study of iterated loop space theory in 1972. Let $\mathscr{L}(j)$ denote the space $\mathscr{I}(U^j, U)$ of linear isometries $U^j \longrightarrow U$. Here we allow all linear isometries, not just the G-linear ones, and G acts on $\mathscr{L}(j)$ by conjugation. Thus the fixed point space $\mathscr{L}(j)^G$ is the space of G-linear isometries $U^j \longrightarrow U$. The symmetric group Σ_j acts freely from the right on $\mathscr{L}(j)$, and the actions of G and Σ_j commute. The equivariant homotopy type of $\mathscr{L}(j)$ depends on U. If U is complete, then, for $\Lambda \subset G \times \Sigma_j$, $\mathscr{L}(j)^\Lambda$ is empty unless $\Lambda \cap \Sigma_j = e$ and contractible otherwise. That is, $\mathscr{L}(j)$ is a universal (G, Σ_j)-bundle. We have maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \cdots + j_k)$$

defined by

$$\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \oplus \cdots \oplus f_k).$$

These data are interrelated in a manner codified in the definition of an operad, and $\mathscr{L}$ is called the "linear isometries G-operad" of the universe U. When U is complete, $\mathscr{L}$ is an E_∞ G-operad.

There is a "twisted half-smash product"

$$\mathscr{L}(j) \ltimes (E_1 \wedge \cdots \wedge E_j) \tag{1.1}$$

into which we can map each of the j-fold internal smash products $f_*(E_1 \wedge \cdots \wedge E_j)$. Moreover, if we restrict attention to tame G-spectra, then each of these maps into the twisted half-smash product (1.1) is an equivalence. The twisted half-smash products $\mathscr{L}(1) \ltimes E$ and $\mathscr{L}(2) \ltimes E \wedge E'$ are the starting points for the construction of the category of $\mathbb{L}$-spectra and the definition of its smash product. We shall return to this point in the next section, after saying a little more about twisted smash products of G-spectra.

Suppose given G-universes U and U', and let $\mathscr{I}(U, U')$ be the G-space of linear isometries $U \longrightarrow U'$, with G acting by conjugation. Let A be an (unbased) G-space together with a given G-map $\alpha : A \longrightarrow \mathscr{I}(U, U')$. We then have a twisted half-smash product functor

$$\alpha \ltimes (\cdot) : G\mathscr{S}U \longrightarrow G\mathscr{S}U'.$$

When A has the homotopy type of a G-CW complex and $E \in G\mathscr{S}U$ is tame, different choices of α give homotopy equivalent G-spectra $\alpha \ltimes E$. For this reason, and because we often have a canonical choice of α in mind, we usually abuse notation by writing $A \ltimes E$ instead of $\alpha \ltimes E$. Thus we think of A as a space over $\mathscr{I}(U, U')$.

When A is a point, α is a choice of a G-linear isometry $f : U \longrightarrow U'$. In this case, the twisted half-smash functor is just the change of universe functor $f_* : G\mathscr{S}U \longrightarrow G\mathscr{S}U'$ (see XII.3.1–3.2). Intuitively, one may think of $\alpha \ltimes E$ as obtained by suitably topologizing and giving a G-action to the union of the nonequivariant spectra $\alpha(a)_*(E)$ as a runs through A. Another intuition is that the twisted half-smash product is a generalization to spectra of the "untwisted" functor $A_+ \wedge X$ on based G-spaces X. This intuition is made precise by the following "untwisting formula" that relates twisted half-smash products and shift desuspensions.

PROPOSITION 1.2. For a G-space A over $\mathscr{I}(U, U')$ and an isomorphism $V \cong V'$ of indexing G-spaces, where $V \subset U$ and $V' \subset U'$, there is an isomorphism of G-spectra

$$A \ltimes \Sigma^\infty_V X \cong A_+ \wedge \Sigma^\infty_{V'} X$$

that is natural in G-spaces A over $\mathscr{I}(U, U')$ and based G-spaces X.

The twisted-half smash product functor enjoys essentially the same formal properties as the space level functor $A_+ \wedge X$. For example, we have the following properties, whose space level analogues are trivial to verify.

PROPOSITION 1.3. The following statements hold.

(i) There is a canonical isomorphism $\{\mathrm{id}_U\} \ltimes E \cong E$.

(ii) Let $A \to \mathscr{I}(U, U')$ and $B \to \mathscr{I}(U', U'')$ be given and give $B \times A$ the composite structure map

$$B \times A \longrightarrow \mathscr{I}(U', U'') \times \mathscr{I}(U, U') \xrightarrow{\circ} \mathscr{I}(U, U'').$$

Then there is a canonical isomorphism

$$(B \times A) \ltimes E \cong B \ltimes (A \ltimes E).$$

(iii) Let $A \to \mathscr{I}(U_1, U_1')$ and $B \to \mathscr{I}(U_2, U_2')$ be given and give $A \times B$ the composite structure map

$$A \times B \longrightarrow \mathscr{I}(U_1, U_1') \times \mathscr{I}(U_2, U_2') \xrightarrow{\oplus} \mathscr{I}(U_1 \oplus U_2, U_1' \oplus U_2').$$

Let E_1 and E_2 be G-spectra indexed on U_1 and U_2 respectively. Then there is a canonical isomorphism

$$(A \times B) \ltimes (E_1 \wedge E_2) \cong (A \ltimes E_1) \wedge (B \ltimes E_2).$$

(iv) For $A \to \mathscr{I}(U, U')$, $E \in G\mathscr{S}U$, and a based G-space X, there is a canonical isomorphism

$$A \ltimes (E \wedge X) \cong (A \ltimes E) \wedge X.$$

The functor $A \ltimes (\bullet)$ has a right adjoint twisted function spectrum functor

$$F[A, \cdot) : G\mathscr{S}U' \longrightarrow G\mathscr{S}U,$$

which is the spectrum level analog of the function G-space $F(A_+, X)$. Thus

$$G\mathscr{S}U'(A \ltimes E, E') \cong G\mathscr{S}U(E, F[A, E')). \tag{1.4}$$

The functor $A \ltimes E$ is homotopy-preserving in E, and it therefore preserves homotopy equivalences in the variable E. However, it only preserves homotopies over $\mathscr{I}(U, U')$ in A. Nevertheless, it very often preserves homotopy equivalences in the variable A. The following central technical result is an easy consequence of Proposition 1.2 and XII.9.2. It explains why all j-fold internal smash products are equivalent to the twisted half-smash product (1.1).

THEOREM 1.5. *Let $E \in G\mathscr{S}U$ be tame and let A be a G-space over $\mathscr{I}(U, U')$. If $\phi : A' \longrightarrow A$ is a homotopy equivalence of G-spaces, then $\phi \ltimes \mathrm{id} : A' \ltimes E \longrightarrow A \ltimes E$ is a homotopy equivalence of G-spectra.*

Since $A \ltimes E$ is a G-CW spectrum if A is a G-CW complex and E is a G-CW spectrum, this has the following consequence.

COROLLARY 1.6. *Let $E \in G\mathscr{S}U$ have the homotopy type of a G-CW spectrum and let A be a G-space over $\mathscr{I}(U, U')$ that has the homotopy type of a G-CW complex. Then $A \ltimes E$ has the homotopy type of a G-CW spectrum.*

[LMS, Chapter VI]
J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Volume 271. 1972.

2. The category of $\mathbb{L}$-spectra

Return to the twisted half-smash product of (1.1). We think of it as a canonical j-fold internal smash product. However, if we are to take this point of view seriously, we must take note of the difference between E and its "1-fold smash product" $\mathscr{L}(1) \ltimes E$. The space $\mathscr{L}(1)$ is a monoid under composition, and the formal properties of twisted half-smash products imply a natural isomorphism

$$\mathscr{L}(1) \ltimes (\mathscr{L}(1) \ltimes E) \cong (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E,$$

where, on the right, $\mathscr{L}(1) \times \mathscr{L}(1)$ is regarded as a G-space over $\mathscr{L}(1)$ via the composition product. This product induces a map

$$\mu : (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E \longrightarrow \mathscr{L}(1) \ltimes E,$$

and the inclusion $\{1\} \longrightarrow \mathscr{L}(1)$ induces a map $\eta : E \longrightarrow \mathscr{L}(1) \ltimes E$. The functor $\mathbb{L}$ given by $\mathbb{L}E = \mathscr{L}(1) \ltimes E$ is a monad under the product μ and unit η. We therefore have the notion of a G-spectrum E with an action $\xi : \mathbb{L}E \longrightarrow E$ of $\mathbb{L}$; the evident associativity and unit diagrams are required to commute.

Definition 2.1. An $\mathbb{L}$-spectrum is a G-spectrum M together with an action of the monad $\mathbb{L}$. Let $G\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra.

The formal properties of $G\mathscr{S}[\mathbb{L}]$ are virtually the same as those of $G\mathscr{S}$; since $\mathscr{L}(1)$ is a contractible G-space, so are the homotopical properties. For tame G-spectra E, we have a natural equivalence $E = \mathrm{id}_*E \longrightarrow \mathbb{L}E$. For $\mathbb{L}$-spectra M that are tame as G-spectra, the action $\xi : \mathbb{L}M \longrightarrow M$ is a weak equivalence. Taking the $\mathbb{L}S^n$ as sphere $\mathbb{L}$-modules, we obtain a theory of G-CW $\mathbb{L}$-spectra exactly like the theory of G-CW spectra. The functor $\mathbb{L}$ preserves G-CW spectra. We let $\bar{h}G\mathscr{S}[\mathbb{L}]$ be the category that is obtained from the homotopy category $hG\mathscr{S}[\mathbb{L}]$ by formally inverting the weak equivalences and find that it is equivalent to the homotopy category of G-CW $\mathbb{L}$-spectra. The functor $\mathbb{L} : G\mathscr{S} \longrightarrow G\mathscr{S}[\mathbb{L}]$ and the forgetful functor $G\mathscr{S}[\mathbb{L}] \longrightarrow G\mathscr{S}$ induce an adjoint equivalence between the stable homotopy category $\bar{h}G\mathscr{S}$ and the category $\bar{h}G\mathscr{S}[\mathbb{L}]$.

Via the untwisting isomorphism $\mathscr{L}(1) \ltimes \Sigma^\infty X \cong \mathscr{L}(1)_+ \wedge \Sigma^\infty X$ and the obvious projection $\mathscr{L}(1)_+ \longrightarrow S^0$, we obtain a natural action of $\mathbb{L}$ on suspension spectra. However, even when X is a G-CW complex, $\Sigma^\infty X$ is not of the homotopy type of a G-CW $\mathbb{L}$-spectrum, and it is the functor $\mathbb{L} \circ \Sigma^\infty$ and not the functor Σ^∞ that is left adjoint to the zeroth space functor $G\mathscr{S}[\mathbb{L}] \longrightarrow \mathscr{T}$.

The reason for introducing the category of $\mathbb{L}$-spectra is that it has a well-behaved "operadic smash product", which we define next. Via instances of the structural maps γ of the operad $\mathscr{L}$, we have both a left action of the monoid $\mathscr{L}(1)$ and a right action of the monoid $\mathscr{L}(1) \times \mathscr{L}(1)$ on $\mathscr{L}(2)$. These actions commute with each other. If M and N are $\mathbb{L}$-spectra, then $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the left on the external smash product $M \wedge N$ via the map

$$\xi : (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \cong (\mathscr{L}(1) \ltimes M) \wedge (\mathscr{L}(1) \ltimes N) \xrightarrow{\xi \wedge \xi} M \wedge N.$$

To form the twisted half smash product on the left, we think of $\mathscr{L}(1) \times \mathscr{L}(1)$ as mapping to $\mathscr{I}(U^2, U^2)$ via direct sum of linear isometries. The smash product over $\mathscr{L}$ of M and N is simply the balanced product of the two $\mathscr{L}(1) \times \mathscr{L}(1)$-actions.

Definition 2.2. Let M and N be $\mathbb{L}$-spectra. Define the operadic smash product $M \wedge_{\mathscr{L}} N$ to be the coequalizer displayed in the diagram

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \underset{\mathrm{id} \ltimes \xi}{\overset{\gamma \ltimes \mathrm{id}}{\rightrightarrows}} \mathscr{L}(2) \ltimes (M \wedge N) \longrightarrow M \wedge_{\mathscr{L}} N.$$

Here we have implicitly used the isomorphism

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \cong \mathscr{L}(2) \ltimes [(\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N)]$$

given by Proposition 1.4(ii). The left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$ induces a left action of $\mathscr{L}(1)$ on $M \wedge_{\mathscr{L}} N$ that gives it a structure of $\mathbb{L}$-spectrum.

We may mimic tensor product notation and write

$$M \wedge_{\mathscr{L}} N = \mathscr{L}(2) \ltimes_{\mathscr{L}(1) \times \mathscr{L}(1)} (M \wedge N).$$

This smash product is commutative, and a special property of the linear isometries operad, first noticed by Hopkins, implies that it is also associative. There is a function $\mathbb{L}$-spectrum functor $F_{\mathscr{L}}$ to go with $\wedge_{\mathscr{L}}$; it is constructed from the external and twisted function spectra functors, and we have the adjunction

$$G\mathscr{S}[\mathbb{L}](M \wedge_{\mathscr{L}} M', M'') \cong G\mathscr{S}[\mathbb{L}](M, F_{\mathscr{L}}(M, M')). \tag{2.3}$$

The smash product $\wedge_{\mathscr{L}}$ is not unital. However, there is a natural map

$$\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$$

of $\mathbb{L}$-spectra that is always a weak equivalence of spectra. It is not usually an isomorphism, but another special property of the linear isometries operad implies that it is an isomorphism if $M = S$ or if $M = S \wedge_{\mathscr{L}} N$ for any $\mathbb{L}$-spectrum N. Thus any $\mathbb{L}$-spectrum is weakly equivalent to one whose unit map is an isomorphism. This makes sense of the following definition, in which we understand S to mean the sphere G-spectrum indexed on our fixed chosen G-universe U.

DEFINITION 2.4. An S-module is an $\mathbb{L}$-spectrum M such that $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is an isomorphism. The category $G\mathscr{M}_S$ of S-modules is the full subcategory of $G\mathscr{S}[\mathbb{L}]$ whose objects are the S-modules. For S-modules M and M', define

$$M \wedge_S M' = M \wedge_{\mathscr{L}} M' \quad \text{and} \quad F_S(M, M') = S \wedge_{\mathscr{L}} F_{\mathscr{L}}(M, M').$$

Although easy to prove, one surprising formal feature of the theory is that the functor $S \wedge_{\mathscr{L}} (\cdot) : G\mathscr{S}[\mathbb{L}] \longrightarrow G\mathscr{M}_S$ is right and not left adjoint to the forgetful functor; it is left adjoint to the functor $F_{\mathscr{L}}(S, \cdot)$. This categorical situation dictates our definition of function S-modules. It also dictates that we construct limits of S-modules by constructing limits of their underlying $\mathbb{L}$-spectra and then applying the functor $S \wedge_{\mathscr{L}} (\bullet)$, as indicated in XXII§1. The free S-module functor $\mathbb{F}_S : G\mathscr{S} \longrightarrow G\mathscr{M}_S$ is defined by

$$\mathbb{F}_S(E) = S \wedge_{\mathscr{L}} \mathbb{L}E.$$

It is left adjoint to the functor $F_{\mathscr{L}}(S, \cdot) : G\mathscr{M}_S \longrightarrow G\mathscr{S}$, and this is the functor that we denoted by $\mathbb{U}_S$ in XXII§1. From this point, the properties of the category of S-modules that we described in XXII§1 are inherited directly from the good properties of the category of $\mathbb{L}$-spectra.

3. A_∞ and E_∞ ring spectra and S-algebras

We defined S-algebras and their modules in terms of structure maps that make the evident diagrams commute in the symmetric monoidal category of S-modules. There are older notions of A_∞ and E_∞ ring spectra and their modules that May, Quinn, and Ray introduced nonequivariantly in 1972; the equivariant generalization was given in [LMS]. Working equivariantly, an A_∞ ring spectrum is a spectrum R together with an action by the linear isometries G-operad $\mathscr{L}$. Such an action is given by G-maps

$$\theta_j : \mathscr{L}(j) \ltimes R^j \longrightarrow R, \ j \geq 0,$$

such that appropriate associativity and unity diagrams commute. If the θ_j are Σ_j-equivariant, then R is said to be an E_∞ ring spectrum. Similarly a left module M over an A_∞ ring spectrum R is defined in terms of maps

$$\mu_j : \mathscr{L}(j) \ltimes R^{j-1} \wedge M \longrightarrow M, \ j \geq 1;$$

in the E_∞ case, we require these maps to be Σ_{j-1}-equivariant. It turns out that the higher θ_j and μ_j are determined by the θ_j and μ_j for $j \leq 2$. That is, we have the following result, which might instead be taken as a definition.

THEOREM 3.1. An A_∞ ring spectrum is an $\mathbb{L}$-spectrum R with a unit map $\eta : S \longrightarrow R$ and a product $\phi : R \wedge_{\mathscr{L}} R \to R$ such that the following diagrams commute:

$$\begin{array}{ccccc} S \wedge_{\mathscr{L}} R & \xrightarrow{\eta\wedge\mathrm{id}} & R \wedge_{\mathscr{L}} R & \xleftarrow{\mathrm{id}\wedge\eta} & R \wedge_{\mathscr{L}} S \\ & \searrow^{\lambda} & \downarrow{\scriptstyle\phi} & \swarrow_{\lambda\tau} & \\ & & R & & \end{array}$$

and

$$\begin{array}{ccc} R \wedge_{\mathscr{L}} R \wedge_{\mathscr{L}} R & \xrightarrow{\mathrm{id}\wedge\phi} & R \wedge_{\mathscr{L}} R \\ {\scriptstyle\phi\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\phi} \\ R \wedge_{\mathscr{L}} R & \xrightarrow[\phi]{} & R; \end{array}$$

R is an E_∞ ring spectrum if the following diagram also commutes:

$$\begin{array}{ccc} R \wedge_{\mathscr{L}} R & \xrightarrow{\tau} & R \wedge_{\mathscr{L}} R \\ & \searrow^{\phi} \quad \swarrow_{\phi} & \\ & R. & \end{array}$$

A module over an A_∞ or E_∞ ring spectrum R is an $\mathbb{L}$-spectrum M with a map $\mu : R \wedge_{\mathscr{L}} M \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} S \wedge_{\mathscr{L}} M & \xrightarrow{\eta\wedge\mathrm{id}} & R \wedge_{\mathscr{L}} M \\ & \searrow^{\lambda} & \downarrow{\scriptstyle\mu} \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R \wedge_{\mathscr{L}} R \wedge_{\mathscr{L}} M & \xrightarrow{\mathrm{id}\wedge\mu} & R \wedge_{\mathscr{L}} M \\ {\scriptstyle\phi\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\mu} \\ R \wedge_{\mathscr{L}} M & \xrightarrow{\mu} & M. \end{array}$$

This leads to the following description of S-algebras.

COROLLARY 3.2. An S-algebra or commutative S-algebra is an A_∞ or E_∞ ring spectrum that is also an S-module. A module over an S-algebra or commutative S-algebra R is a module over the underlying A_∞ or E_∞ ring spectrum that is also an S-module.

In particular, we have a functorial way to replace A_∞ and E_∞ ring spectra and their modules by weakly equivalent S-algebras and commutative S-algebras and their modules.

COROLLARY 3.3. For an A_∞ ring spectrum R, $S \wedge_{\mathscr{L}} R$ is an S-algebra and $\lambda : S \wedge_{\mathscr{L}} R \longrightarrow R$ is a weak equivalence of A_∞ ring spectra, and similarly in the E_∞ case. If M is an R-module, then $S \wedge_{\mathscr{L}} M$ is an $S \wedge_{\mathscr{L}} R$-module and

$\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is a weak equivalence of R-modules and of modules over $S \wedge_{\mathscr{L}} R$ regarded as an A_∞ ring spectrum.

Thus the earlier definitions are essentially equivalent to the new ones, and earlier work gives a plenitude of examples. Thom G-spectra occur in nature as E_∞ ring G-spectra. For finite groups G, multiplicative infinite loop space theory works as it does nonequivariantly; however, the details have yet to be fully worked out and written up: that is planned for a later work. This theory gives that the Eilenberg-Mac Lane G-spectra of Green functors, the G-spectra of connective real and complex K-theory, and the G-spectra of equivariant algebraic K-theory are E_∞ ring spectra. As observed in XXII.6.13, it follows that the G-spectra of periodic real and complex K-theory are also E_∞ ring G-spectra. Nonequivariantly, many more examples are known due to recent work, mostly unpublished, of such people as Hopkins, Miller, and Kriz.

J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Volume 577. 1977.

J. P. May. Multiplicative infinite loop space theory. J. Pure and Applied Algebra, 26(1982), 1–69.

4. Alternative perspectives on equivariance

We have developed the theory of $\mathbb{L}$-spectra and S-modules starting from a fixed given G-universe U. However, there are alternative perspectives on the role of the universe and of equivariance that shed considerable light on the theory. Much of this material does not appear in the literature, and we give proofs in Section 6 after explaining the ideas here. Let S_U denote the sphere G-spectrum indexed on a G-universe U. The essential point is that while the categories $G\mathscr{S}U$ of G-spectra indexed on U vary as U varies, the categories $G\mathscr{M}_{S_U}$ of S_U-modules do not: all such categories are actually isomorphic. These isomorphisms preserve homotopies and thus pass to ordinary homotopy categories. However, they do not preserve weak equivalences and therefore do not pass to derived categories, which do vary with U. This observation first appeared in a paper of Elmendorf and May, but we shall begin with a different explanation than the one we gave there.

We shall explain matters by describing the categories of G-spectra and of $\mathbb{L}$-G-spectra indexed on varying universes U in terms of algebras over monads defined on the ground category $\mathscr{S} = \mathscr{S}\mathbb{R}^\infty$ of nonequivariant spectra indexed on $\mathbb{R}^\infty$. Abbreviate notation by writing L for the monoid $\mathscr{L}(1) = \mathscr{I}(\mathbb{R}^\infty, \mathbb{R}^\infty)$. Any G-universe U is isomorphic to $\mathbb{R}^\infty$ with an action by G through linear isometries. The action may be written in the form $gx = f(g)(x)$ for $x \in \mathbb{R}^\infty$, where $f : G \longrightarrow L$ is a homomorphism of monoids. To fix ideas, we shall write $\mathbb{R}^\infty_f$ for the G-universe determined by such a homomorphism f. For a spectrum E, we then define

$$\mathbb{G}_f E = G \ltimes E,$$

where the twisted half-smash product is determined by the map f. The multiplication and unit of G determine maps $\mu : \mathbb{G}_f\mathbb{G}_f E \longrightarrow \mathbb{G}_f E$ and $\eta : E \longrightarrow \mathbb{G}_f E$ that give $\mathbb{G}_f$ a structure of monad in $\mathscr{S}$. As was observed in [LMS], the category $G\mathscr{S}\mathbb{R}^\infty_f$ of G-spectra indexed on $\mathbb{R}^\infty_f$ is canonically isomorphic to the category $\mathscr{S}[\mathbb{G}_f]$ of algebras over the monad $\mathbb{G}_f$. Of course, we also have the monad $\mathbb{L}$ in $\mathscr{S}$ with $\mathbb{L}E = L \ltimes E$; by definition, a nonequivariant $\mathbb{L}$-spectrum is an algebra over this monad.

Proposition 4.1. The following statements about the monads $\mathbb{L}$ and $\mathbb{G}_f$ hold for any homomorphism of monoids $f : G \longrightarrow L = \mathscr{I}(\mathbb{R}^\infty, \mathbb{R}^\infty)$.

(i) $\mathbb{L}$ restricts to a monad in the category $\mathscr{S}[\mathbb{G}_f]$ of G-spectra indexed on $\mathbb{R}^\infty_f$.
(ii) $\mathbb{G}_f$ restricts to a monad in the category $\mathscr{S}[\mathbb{L}]$ of $\mathbb{L}$-spectra indexed on $\mathbb{R}^\infty$.
(iii) The composite monads $\mathbb{L}\mathbb{G}_f$ and $\mathbb{G}_f\mathbb{L}$ in $\mathscr{S}$ are isomorphic.

Moreover, up to isomorphism, the composite monad $\mathbb{L}\mathbb{G}_f$ is independent of f.

Corollary 4.2. The category $G\mathscr{S}\mathbb{R}^\infty_f[\mathbb{L}] = \mathscr{S}[G_f][\mathbb{L}]$ of $\mathbb{L}$-G-spectra indexed on $\mathbb{R}^\infty_f$ is isomorphic to the category $\mathscr{S}[\mathbb{L}][\mathbb{G}_f]$ of G-$\mathbb{L}$-spectra indexed on $\mathbb{R}^\infty_f$. Up to isomorphism, this category is independent of f.

The isomorphisms that we shall obtain preserve spheres and operadic smash products and so restrict to give isomorphisms between categories of S-modules.

Corollary 4.3. Up to isomorphism, the category $G\mathscr{M}_{S_U}$ of S_U-modules is independent of the G-universe U.

Thus a structure of $S_{\mathbb{R}^\infty}$-module on a naive G-spectrum is so rich that it encompasses an S_U-action on a G-spectrum indexed on U for any universe U. This richness is possible because the action of G on U can itself be expressed in terms of the monoid L.

There is another way to think about these isomorphisms, which is given in Elmendorf and May and which we now summarize. It is motivated by the definition of the operadic smash product.

DEFINITION 4.4. Fix universes U and U', write $\mathbb{L}$ and $\mathbb{L}'$ for the respective monads in $G\mathscr{S}U$ and $G\mathscr{S}U'$ and write $\mathscr{L}$ and $\mathscr{L}'$ for the respective G-operads. For an $\mathbb{L}$-spectrum M, define an $\mathbb{L}'$-spectrum $I_U^{U'}M$ by

$$I_U^{U'}M = \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} M.$$

That is, $I_U^{U'}M$ is the coequalizer displayed in the diagram

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \underset{\mathrm{id}\ltimes\xi}{\overset{\gamma\ltimes\mathrm{id}}{\rightrightarrows}} \mathscr{I}(U,U') \ltimes M \longrightarrow I_U^{U'}M.$$

Here $\xi : \mathscr{I}(U,U) \ltimes M \longrightarrow M$ is the given action of $\mathbb{L}$ on M. We regard the product $\mathscr{I}(U,U') \times \mathscr{I}(U,U)$ as a space over $\mathscr{I}(U,U')$ via the composition map

$$\gamma : \mathscr{I}(U,U') \times \mathscr{I}(U,U) \longrightarrow \mathscr{I}(U,U');$$

Proposition 1.3(ii) gives a natural isomorphism

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \cong (\mathscr{I}(U,U') \times \mathscr{I}(U,U)) \ltimes M.$$

This makes sense of the map $\gamma \ltimes \mathrm{id}$ in the diagram. The required left action of $\mathscr{I}(U',U')$ on $I_U^{U'}M$ is induced by the composition product

$$\gamma : \mathscr{I}(U',U') \times \mathscr{I}(U,U') \longrightarrow \mathscr{I}(U,U'),$$

which induces a natural map of coequalizer diagrams on passage to twisted half-smash products.

PROPOSITION 4.5. Let U, U', and U'' be G-universes. Consider the functors

$$I_U^{U'} : G\mathscr{S}U[\mathbb{L}] \longrightarrow G\mathscr{S}U'[\mathbb{L}'] \quad \text{and} \quad \Sigma_U^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U[\mathbb{L}].$$

(i) $I_U^{U'} \circ \Sigma_U^\infty$ is naturally isomorphic to $\Sigma_{U'}^\infty$.
(ii) $I_{U'}^{U''} \circ I_U^{U'}$ is naturally isomorphic to $I_U^{U''}$.
(iii) I_U^U is naturally isomorphic to the identity functor.

Therefore the functor $I_U^{U'}$ is an equivalence of categories with inverse $I_{U'}^U$. Moreover, the functor $I_U^{U'}$ is continuous and satisfies $I_U^{U'}(M \wedge X) \cong (I_U^{U'}M) \wedge X$ for $\mathbb{L}$-spectra M and based G-spaces X. In particular, it is homotopy preserving, and $I_U^{U'}$ and $I_{U'}^U$ induce inverse equivalences of homotopy categories.

Now suppose that $U = \mathbb{R}_f^\infty$ and $U' = \mathbb{R}_{f'}^\infty$. Since the coequalizer defining $I_U^{U'}$ is the underlying nonequivariant coequalizer with a suitable action of G, we see that, with all group actions ignored, the functor $I_U^{U'}$ is naturally isomorphic to the identity functor on $\mathscr{S}[\mathbb{L}]$. In this case, the equivalences of categories of the previous result are natural isomorphisms and, tracing through the definitions, one can check that they agree with the equivalences given by the last statement of Corollary 4.2. Therefore the following result, which applies to any pair of universes U and U', is an elaboration of Corollary 4.3.

Proposition 4.6. The following statements hold.

(i) $I_U^{U'} S_U$ is canonically isomorphic to $S_{U'}$.

(ii) For $\mathbb{L}$-spectra M and N, there is a natural isomorphism

$$\omega : I_U^{U'}(M \wedge_{\mathscr{L}} N) \cong (I_U^{U'} M) \wedge_{\mathscr{L}'} (I_U^{U'} N).$$

(iii) The following diagram commutes for all $\mathbb{L}$-spectra M:

$$\begin{array}{ccc} I_U^{U'}(S_U \wedge_{\mathscr{L}} M) & \xrightarrow{\omega} & S_{U'} \wedge_{\mathscr{L}'} (I_U^{U'} M) \\ & {\scriptstyle I_U^{U'}\lambda} \searrow \quad \swarrow {\scriptstyle \lambda} & \\ & I_U^{U'} M. & \end{array}$$

(iv) M is an S_U-module if and only if $I_U^{U'} M$ is an $S_{U'}$-module.

Therefore the functors $I_U^{U'}$ and $I_{U'}^{U}$ restrict to inverse equivalences of categories between $G\mathscr{M}_{S_U}$ and $G\mathscr{M}_{S_{U'}}$ that induce inverse equivalences of categories between $hG\mathscr{M}_{S_U}$ and $hG\mathscr{M}_{S_{U'}}$.

This has the following consequence, which shows that, on the point-set level, our brave new equivariant algebraic structures are independent of the universe in which they are defined.

Theorem 4.7. The functor $I_U^{U'} : G\mathscr{M}_{S_U} \longrightarrow G\mathscr{M}_{S_{U'}}$ is monoidal. If R is an S_U-algebra and M is an R-module, then $I_U^{U'} R$ is an $S_{U'}$-algebra and $I_U^{U'} M$ is an $I_U^{U'} R$-module.

The ideas of this section are illuminated by thinking model theoretically. We focus attention on the category $G\mathscr{M}_{\mathbb{R}^\infty}$, where G acts trivially on $\mathbb{R}^\infty$. We can reinterpret our results as saying that the model categories of S_U-modules for varying universes U are all isomorphic to the category $G\mathscr{M}_{\mathbb{R}^\infty}$, but given a model structure that depends on U. Indeed, for any $U = \mathbb{R}^\infty_f$, we have the isomorphism of categories $I_U^{\mathbb{R}^\infty} : G\mathscr{M}_U \longrightarrow G\mathscr{M}_{\mathbb{R}^\infty}$, and we can transport the model category structure of $G\mathscr{M}_U$ to a new model category structure on $G\mathscr{M}_{\mathbb{R}^\infty}$, which we call the U-model structure on $G\mathscr{M}_{\mathbb{R}^\infty}$.

The essential point is that $I_U^{\mathbb{R}^\infty}$ does not carry the cofibrant sphere S_U-modules $S^n_{S_U} = S_U \wedge_{\mathscr{L}} \mathbb{L}S^n$ to the corresponding cofibrant sphere $S_{\mathbb{R}^\infty}$-modules. The weak equivalences in the U-model structure are the maps that induce isomorphisms on homotopy classes of $S_{\mathbb{R}^\infty}$-module maps out of the "U-spheres" $G/H_+ \wedge I_U^{\mathbb{R}^\infty} S^n_{S_U}$. We define U-cell and relative U-cell $S_{\mathbb{R}^\infty}$-modules by using these U-spheres as the domains of their attaching maps. The U-cofibrations are the retracts of the relative U-cell $S_{\mathbb{R}^\infty}$-modules, and the U-fibrations are then determined as the maps that satisfy the right lifting property with respect to the acyclic U-cofibrations.

A. D. Elmendorf and J. P. May. Algebras over equivariant sphere spectra. Preprint, 1995.

5. The construction of equivariant algebras and modules

The results of the previous section are not mere esoterica. They lead to homotopically well-behaved constructions of brave new equivariant algebraic structures from brave new nonequivariant algebraic structures. The essential point is to understand the homotopical behavior of point-set level constructions that have desirable formal properties. We shall explain the solutions to two natural problems in this direction.

First, suppose given a nonequivariant S-algebra R and an R-module M; for definiteness, we suppose that these spectra are indexed on the fixed point universe U^G of a complete G-universe U. Is there an S_G-algebra R_G and an R_G-module M_G whose underlying nonequivariant spectra are equivalent to R and M in a way that preserves the brave new algebraic structuresΓ In this generality, the only obvious candidates for R_G and M_G are i_*R and i_*M, where $i : U^G \longrightarrow U$ is the inclusion. In any case, we want R_G and M_G to be equivalent to i_*R and i_*M. However, the change of universe functor i_* does not preserve brave new algebraic structures. Thus the problem is to find a functor that does preserve such structures and yet is equivalent to i_*. A very special case of the solution of this problem has been used by Benson and Greenlees to obtain calculational information about the ordinary cohomology of classifying spaces of compact Lie groups.

Second, suppose given an S_G-algebra R_G with underlying nonequivariant S-algebra R and suppose given an R-module M. Can we construct an R_G-module M_G whose underlying nonequivariant R-module is MΓ Note in particular that the problem presupposes that, up to equivalence, the underlying nonequivariant spectrum of R_G is an S-algebra, and similarly for modules. We are thinking of MU_G and MU, and the solution of this problem gives equivariant versions as MU_G-modules of all of the spectra, such as the Brown-Peterson and Morava K-theory spectra, that can be constructed from MU by killing some generators and inverting others.

The following homotopical result of Elmendorf and May combines with Theorem 4.7 to solve the first problem. In fact, it shows more generally that, up to isomorphism in derived categories, any change of universe functor preserves brave new algebraic structures. Observe that, for a linear isometry $f : U \longrightarrow U'$ and S_U-modules $M \in G\mathscr{M}_{S_U}$, we have a composite natural map

$$\alpha : f_*M \longrightarrow \mathscr{I}(U,U') \ltimes M \longrightarrow I_{U'}^U M$$

of G-spectra indexed on U', where the first arrow is induced by the inclusion $\{f\} \longrightarrow \mathscr{I}(U,U')$ and the second is the evident quotient map.

THEOREM 5.1. Let $f : U \longrightarrow U'$ be a G-linear isometry. Then for sufficiently well-behaved S_U-modules $M \in G\mathscr{M}_{S_U}$ (those in the collection $\bar{\mathscr{E}}_{S_U}$ of XXII.5.5), the natural map $\alpha : f_*M \longrightarrow I_U^{U'}M$ is a homotopy equivalence of G-spectra indexed on U'

Remember that $\bar{\mathscr{E}}_{S_U}$ includes the q-cofibrant objects in all of our categories of brave new algebras and modules. We are entitled to conclude that, up to equivalence, the change of universe functor f_* preserves brave new algebras and modules. The most important case is the inclusion $i : U^G \longrightarrow U$. If we start from any nonequivariant q-cofibrant brave new algebraic structure, then, up to equivalence, the change of universe functor i_* constructs from it a corresponding equivariant brave new algebraic structure.

Turning to the second problem that we posed, we give a result (due to May) that interrelates brave new algebraic structures in $G\mathscr{M}_U$ and $\mathscr{M}_{U^G}$. Its starting point is the idea of combining the operadic smash product with the functors $I_{U'}^U$. We think of U as the basic universe of interest in what follows.

DEFINITION 5.2. Let U, U', and U'' be G-universes. For an $\mathbb{L}'$-spectrum M and an $\mathbb{L}''$-spectrum N, define an $\mathbb{L}$-spectrum $M \wedge_{\mathscr{L}} N$ by

$$M \wedge_{\mathscr{L}} N = I_{U'}^U M \wedge_{\mathscr{L}} I_{U''}^U N.$$

The formal properties of this product can be deduced from those of the functors $I_{U'}^U$ together with those of the operadic smash product for the fixed universe U. In particular, since the functor $I_{U'}^U$ takes $S_{U'}$-modules to S_U-modules and the smash product over S_U is the restriction to S_U-modules of the smash product over $\mathscr{L}$, we have the following observation.

LEMMA 5.3. The functor $\wedge_{\mathscr{L}} : G\mathscr{S}U'[\mathbb{L}'] \times G\mathscr{S}U''[\mathbb{L}''] \longrightarrow G\mathscr{S}U[\mathbb{L}]$ restricts to a functor

$$\wedge_{S_U} : G\mathscr{M}_{S_{U'}} \times G\mathscr{M}_{S_{U''}} \longrightarrow G\mathscr{M}_{S_U}.$$

This allows us to define modules indexed on one universe over algebras indexed on another.

DEFINITION 5.4. Let $R \in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra and let $M \in G\mathscr{M}_{S_{U'}}$. Say that M is a right R-module if it is a right $I_{U''}^{U'}R$-module, and similarly for left modules.

To define smash products over R in this context, we use the functors $I_{U'}^{U}$ to index everything on our preferred universe U and then take smash products there.

DEFINITION 5.5. Let $R \in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra, let $M \in G\mathscr{M}_{S_{U'}}$ be a right R-module and let $N \in G\mathscr{M}_{U'''}$ be a left R-module. Define

$$M \wedge_R N = I_{U'}^{U} M \wedge_{I_{U''}^{U} R} I_{U'''}^{U} N.$$

These smash products inherit good formal properties from those of the smash products of R-modules, and their homotopical properties can be deduced from the homotopical properties of the smash product of R-modules and the homotopical properties of the functors $I_{U'}^{U}$, as given by Theorem 5.1.

Now specialize to consideration of $U^G \subseteq U$. Write S_G for the sphere G-spectrum indexed on U and S for the nonequivariant sphere spectrum indexed on U^G. We take S_G-modules to be in $G\mathscr{M}_U$ and S-modules to be in $\mathscr{M}_{U^G}$ in what follows.

THEOREM 5.6. Let R_G be a commutative S_G-algebra and assume that R_G is split as an algebra with underlying nonequivariant S-algebra R . Then there is a monoidal functor $R_G \wedge_R (\cdot) : \mathscr{M}_R \longrightarrow G\mathscr{M}_{R_G}$. If M is a cell R-module, then $R_G \wedge_R M$ is split as a module with underlying nonequivariant R-module M. The functor $R_G \wedge_R (\cdot)$ induces a derived monoidal functor $\mathscr{D}_R \longrightarrow G\mathscr{D}_{R_G}$. Therefore, if M is an R-ring spectrum (in the homotopical sense), then $R_G \wedge_R M$ is an R_G-ring G-spectrum.

The terms "split as an algebra" and "split as a module" are a bit technical, and we will explain them in a moment. However, we have the following important example; see XV§2 for the definition of MU_G.

PROPOSITION 5.7. The G-spectrum MU_G that represents stable complex cobordism is a commutative S_G-algebra, and it is split as an algebra with underlying nonequivariant S-algebra MU.

We shall return to this point and say something about the proof of the proposition in XXV§7. We conclude that, for any compact Lie group G and any MU-module M, we have a corresponding split MU_G-module $M_G \equiv MU_G \wedge_{MU} M$. This allows us to transport the nonequivariant constructions of XXII§4 into the equivariant world. For example, taking $M = BP$ or $M = K(n)$, we obtain equivariant Brown-Peterson and Morava K-theory MU_G-modules BP_G and $K(n)_G$. Moreover, if M is an MU-ring spectrum, then M_G is an MU_G-ring G-spectrum, and M_G is associative or commutative if M is so.

We must still explain our terms and sketch the proof of Theorem 5.6. The notion of a split G-spectrum was a homotopical one involving the change of universe functor i_*, and neither that functor nor its right adjoint i^* preserves brave new algebraic structures. We are led to the following definitions.

Definition 5.8. A commutative S_G-algebra R_G is split as an algebra if there is a commutative S-algebra R and a map $\eta : I_{U^G}^{U} R \longrightarrow R_G$ of S_G-algebras such that η is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*R \longrightarrow I_{U^G}^{U} R$ is an (equivariant) equivalence of G-spectra. We call R the (or, more accurately, an) underlying nonequivariant S-algebra of R_G.

Since the composite $\eta \circ \alpha$ is a nonequivariant equivalence and the natural map $R \longrightarrow i^*i_*R$ is a weak equivalence (provided that R is tame), R is weakly equivalent to i^*R_G with G-action ignored. Thus R is a highly structured version of the underlying nonequivariant spectrum of R_G. Clearly R_G is split as a G-spectrum with splitting map $\eta \circ \alpha$.

We have a parallel definition for modules.

Definition 5.9. Let R_G be a commutative S_G-algebra that is split as an algebra with underlying S-algebra R and let M_G be an R_G-module. Regard M_G as an $I_{U^G}^{U} R$-module by pullback along η. Then M_G is split as a module if there is an R-module M and a map $\chi : I_{U^G}^{U} M \longrightarrow M_G$ of $I_{U^G}^{U} R$-modules such that χ is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*M \longrightarrow I_{U^G}^{U} M$ is an (equivariant) equivalence of G-spectra. We call M the (or, more accurately, an) underlying nonequivariant R-module of M_G.

Again, M is a highly structured version of the underlying nonequivariant spectrum of M_G, and M_G is split as a G-spectrum with splitting map $\chi \circ \alpha$. The ambiguity that we allow in the notion of an underlying object is quite useful: it allows us to use Theorem 5.1 and q-cofibrant approximation (of S-algebras and of R-modules) to arrange the condition on α in the definitions if we have succeeded in arranging the other conditions.

For the proof of Theorem 5.6, Definition 5.5 specializes to give the required functor $R_G \wedge_R (\cdot)$, and it is clearly monoidal. We may as well assume that our given underlying nonequivariant S-algebra R is q-cofibrant as an S-algebra. Let

M be a cell R-module. By Theorem 5.1, the condition on α in the definition of an underlying R-module is satisfied. Define

$$\chi = \eta \wedge \mathrm{id} : I_{UG}^U M \cong I_{UG}^U R \wedge_{I_{UG}^U R} I_{UG}^U M \longrightarrow R_G \wedge_{I_{UG}^U R} I_{UG}^U M = M_G.$$

Clearly χ is a map of $I_{UG}^U R$-modules, and it is not hard to prove that it is an equivalence of spectra. Thus M_G is split as a module with underlying nonequivariant R-module M. That is the main point, and the rest follows without difficulty.

D. J. Benson and J. P. C. Greenlees. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. Preprint. 1995.
A. D. Elmendorf and J. P. May. Algebras over equivariant sphere spectra. Preprint, 1995.
J. P. May. Equivariant and nonequivariant module spectra. Preprint, 1995.

6. Comparisons of categories of $\mathbb{L}$-G-spectra

We prove Proposition 4.1 and Corollary 4.2 here. The proof of Proposition 4.1 is based on the comparison of certain monoids constructed from the monoids G and L and the homomorphism $f : G \longrightarrow L$. Thus let $G \ltimes_f L$ and $L \rtimes_f G$ be the left and right semidirect products of G and L determined by f. As spaces,

$$G \ltimes_f L = G \times L \quad \text{ and } \quad L \rtimes_f G = L \times G,$$

and their multiplications are specified by

$$(g, m)(g', m') = (gg', f({g'}^{-1}) m f(g') m')$$

and

$$(m, g)(m', g') = (m f(g) m' f(g^{-1}), gg').$$

There is an isomorphism of monoids

$$\tau : G \ltimes_f L \longrightarrow L \rtimes_f G$$

specified by

$$\tau(g, m) = (f(g) m f(g^{-1}), g);$$

there is also an isomorphism of monoids

$$\zeta : G \ltimes_f L \longrightarrow G \times L$$

specified by

$$\zeta(g, m) = (g, f(g) m);$$

its inverse takes (g, m) to $(g, f(g^{-1}) m)$. Let

$$\pi : G \times L \longrightarrow L$$

be the projection. We regard $G \times L$ as a monoid over L via π and we regard $G \ltimes_f L$ and $L \times_f G$ as monoids over L via the composites $\pi \circ \zeta$ and $\pi \circ \zeta \circ \tau^{-1}$, so that ζ and τ are isomorphisms over L. Using Proposition 1.3, we see that, for spectra $E \in \mathscr{S}$, the map τ induces a natural isomorphism

$$(6.1) \qquad \tau : \mathbb{G}_f \mathbb{L} E \cong (G \ltimes_f L) \ltimes E \longrightarrow (L \rtimes_f G) \ltimes E \cong \mathbb{L}\mathbb{G}_f E$$

and the map ζ induces a natural isomorphism

$$(6.2) \qquad \zeta : \mathbb{G}_f \mathbb{L} E \cong (G \ltimes_f L) \ltimes E \longrightarrow (G \times L) \ltimes E \cong G_+ \wedge \mathbb{L} E.$$

In the domains and targets here, the units and products of the given monoids determine natural transformations η and μ that give the specified composite monad structures to the displayed functors $\mathscr{S} \longrightarrow \mathscr{S}$. Elementary diagram chases on the level of monoids imply that the displayed natural transformations are well-defined isomorphisms of monads. If f is the trivial homomorphism that sends all of G to $1 \in L$, then $G \ltimes_f L = G \times L$. Thus in (6.2) we are comparing the monad for the G-universe $\mathbb{R}^\infty_f$ to the monad determined by $\mathbb{R}^\infty$ regarded as a trivial G-universe. The conclusions of Proposition 4.1 follow, and Corollary 4.2 follows as a matter of category theory.

The following two lemmas in category theory may or may not illuminate what is going on. The first is proven in [EKMM] and shows why Corollary 4.2 follows from Proposition 4.1. The second dictates exactly what "elementary diagram chases" are needed to complete the proof of Proposition 4.1.

LEMMA 6.3. Let $\mathbb{S}$ be a monad in a category $\mathscr{C}$ and let $\mathbb{T}$ be a monad in the category $\mathscr{C}[\mathbb{S}]$ of $\mathbb{S}$-algebras. Then the category $\mathscr{C}[\mathbb{S}][\mathbb{T}]$ of $\mathbb{T}$-algebras in $\mathscr{C}[\mathbb{S}]$ is isomorphic to the category $\mathscr{C}[\mathbb{T}\mathbb{S}]$ of algebras over the composite monad $\mathbb{T}\mathbb{S}$ in $\mathscr{C}$.

Here the unit of $\mathbb{T}\mathbb{S}$ is the composite $\mathrm{id} \longrightarrow \mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$ given by the units of $\mathbb{S}$ and $\mathbb{T}$ and the product on $\mathbb{T}\mathbb{S}$ is the composite $\mathbb{T}\mathbb{S}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$, where the second map is given by the product of $\mathbb{T}$ and the first is obtained by applying $\mathbb{T}$ to the action $\mathbb{S}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$ given by the fact that $\mathbb{T}$ is a monad in $\mathscr{C}[\mathbb{S}]$. In our applications, we are taking $\mathbb{T}$ to be the restriction to $\mathscr{C}[\mathbb{S}]$ of a monad in $\mathscr{C}$. This requires us to start with monads $\mathbb{S}$ and $\mathbb{T}$ that commute with one another.

LEMMA 6.4. Let $\mathbb{S}$ and $\mathbb{T}$ be monads in $\mathscr{C}$. Suppose there is a natural isomorphism $\tau : \mathbb{S}\mathbb{T} \longrightarrow \mathbb{T}\mathbb{S}$ such that the following diagrams commute:

$$\begin{CD} \mathbb{S}\mathbb{S}\mathbb{T} @>{\mu}>> @. \mathbb{S}\mathbb{T} \\ @V{\mathbb{S}\tau}VV @. @VV{\tau}V \\ \mathbb{S}\mathbb{T}\mathbb{S} @>>{\tau}> \mathbb{T}\mathbb{S}\mathbb{S} @>>{\mathbb{T}\mu}> \mathbb{T}\mathbb{S} \end{CD} \qquad \text{and} \qquad \begin{array}{ccc} & \mathbb{T} & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \mathbb{T}\eta} \\ \mathbb{S}\mathbb{T} & \xrightarrow[\tau]{} & \mathbb{T}\mathbb{S}. \end{array}$$

Then $\mathbb{T}$ restricts to a monad in $\mathscr{C}[\mathbb{S}]$ to which the previous lemma applies. Suppose further that these diagrams with the roles of $\mathbb{S}$ and $\mathbb{T}$ reversed also commute, as do the following diagrams:

$$\begin{CD} \mathbb{S}\mathbb{T}\mathbb{S}\mathbb{T} @>{\mathbb{S}\tau^{-1}}>> \mathbb{S}\mathbb{S}\mathbb{T}\mathbb{T} @>{\mathbb{S}\mathbb{S}\mu}>> \mathbb{S}\mathbb{S}\mathbb{T} @>{\mu}>> \mathbb{S}\mathbb{T} \\ @V{\tau\circ\mathbb{S}\mathbb{T}\tau}VV @. @. @VV{\tau}V \\ \mathbb{T}\mathbb{S}\mathbb{T}\mathbb{S} @>>{\mathbb{T}\tau}> \mathbb{T}\mathbb{T}\mathbb{S}\mathbb{S} @>>{\mathbb{T}\mathbb{T}\mu}> \mathbb{T}\mathbb{T}\mathbb{S} @>>{\mu}> \mathbb{T}\mathbb{S} \end{CD} \qquad \text{and} \qquad \begin{CD} \mathrm{id} @>{\eta}>> \mathbb{T} @>{\eta}>> \mathbb{S}\mathbb{T} \\ @| @. @VV{\tau}V \\ \mathrm{id} @>>{\eta}> \mathbb{S} @>>{\eta}> \mathbb{T}\mathbb{S}. \end{CD}$$

Then $\tau : \mathbb{S}\mathbb{T} \longrightarrow \mathbb{T}\mathbb{S}$ is an isomorphism of monads. Therefore the categories $\mathscr{C}[\mathbb{S}][\mathbb{T}]$ and $\mathscr{C}[\mathbb{T}][\mathbb{S}]$ are both isomorphic to the category $\mathscr{C}[\mathbb{S}\mathbb{T}] \cong \mathscr{C}[\mathbb{T}\mathbb{S}]$.

Here, for the first statement, if X is an $\mathbb{S}$-algebra with action ξ, then the required action of $\mathbb{S}$ on $\mathbb{T}X$ is the composite $\mathbb{S}\mathbb{T}X \xrightarrow{\tau} \mathbb{T}\mathbb{S}X \xrightarrow{\mathbb{T}\xi} \mathbb{T}X$.

CHAPTER XXIV

Brave New Equivariant Algebra

by J. P. C. Greenlees and J. P. May

1. Introduction

We shall explain how useful it is to be able to mimic commutative algebra in equivariant topology. Actually, the nonequivariant specializations of the constructions that we shall describe are also of considerable interest, especially in connection with the chromatic filtration of stable homotopy theory. We have discussed this in an expository paper [GM1], and that paper also says more about the relevant algebraic constructions than we shall say here. We shall give a connected sequence of examples of brave new analogues of constructions in commutative algebra. The general pattern of how the theory works is this. We first give an algebraic definition. We next give its brave new analogue. The homotopy groups of the brave new analogue will be computable in terms of a spectral sequence that starts with the relevant algebraic construction computed on coefficient rings and modules. The usefulness of the constructions is that they are often related by a natural map to or from an analogous geometric construction that one wishes to compute. Localization and completion theorems say when such maps are equivalences.

The Atiyah-Segal completion theorem and the Segal conjecture are examples of this paradigm that we have already discussed. However, very special features of those cases allowed them to be handled without explicit use of brave new algebra: the force of Bott periodicity in the case of K-theory and the fact that the sphere G-spectrum acts naturally on the stable homotopy category in the case of cohomotopy. We shall explain how brave new algebra gives a coherent general framework for the study of such completion phenomena in cohomology and analogous localization phenomena in homology. We have given another exposition of these matters in [GM2], which says more about the basic philosophy. We shall describe the results in a little greater generality here and so clarify the application to K-theory. We shall also explain the relationship between localization theorems and Tate theory, which we find quite illuminating.

[GM1] J. P. C. Greenlees and J. P. May. Completions in algebra and topology. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 255-276.
[GM2] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 277-324.

2. Local and Čech cohomology in algebra

Suppose given a ring R, which may be graded and which need not be Noetherian, and suppose given a finitely generated ideal $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. If R is graded the α_i are required to be homogeneous.

For any element α, we may consider the stable Koszul cochain complex

$$K^\bullet(\alpha) = \big(R \to R[\alpha^{-1}]\big)$$

concentrated in codegrees 0 and 1. Notice that we have a fiber sequence

$$K^\bullet(\alpha) \longrightarrow R \longrightarrow R[\alpha^{-1}]$$

of cochain complexes.

We may now form the tensor product

$$K^\bullet(\alpha_1, \ldots, \alpha_n) = K^\bullet(\alpha_1) \otimes \ldots \otimes K^\bullet(\alpha_n).$$

It is clear that this complex is unchanged if we replace some α_i by a power, and it is not hard to check the following result.

Lemma 2.1. If $\beta \in I$, then $K^\bullet(\alpha_1, \ldots, \alpha_n)[\beta^{-1}]$ is exact. Up to quasi-isomorphism, the complex $K^\bullet(\alpha_1, \ldots, \alpha_n)$ depends only on the ideal I.

Therefore, up to quasi-isomorphism, $K^\bullet(\alpha_1, \ldots, \alpha_n)$ depends only on the radical of the ideal I, and we henceforth write $K^\bullet(I)$ for it.

Following Grothendieck, we define the local cohomology groups of an R-module M by

$$H^*_I(R; M) = H^*(K^\bullet(I) \otimes M). \qquad (2.2)$$

It is easy to see that $H^0_I(R;M)$ is the submodule

$$\Gamma_I(M) = \{m \in M | I^k m = 0 \text{ for some positive integer } k\}$$

of I-power torsion elements of M. If R is Noetherian it is not hard to prove that $H^*_I(R;\cdot)$ is effaceable and hence that local cohomology calculates the right derived functors of $\Gamma_I(\cdot)$. It is clear that the local cohomology groups vanish above codegree n; in the Noetherian case Grothendieck's vanishing theorem shows that they are actually zero above the Krull dimension of R. Observe that if $\beta \in I$ then $H^*_I(R;M)[\beta^{-1}] = 0$; this is a restatement of the exactness of $K^\bullet(I)[\beta^{-1}]$.

The Koszul complex $K^\bullet(\alpha)$ comes with a natural map $\varepsilon : K^\bullet(\alpha) \longrightarrow R$; the tensor product of such maps gives an augmentation $\varepsilon : K^\bullet(I) \longrightarrow R$. Define the Čech complex $\check{C}^\bullet(I)$ to be $\Sigma(\mathrm{Ker}\ \varepsilon)$. (The name is justified in [GM1].) By inspection, or as an alternative definition, we then have the fiber sequence of cochain complexes

$$K^\bullet(I) \longrightarrow R \longrightarrow \check{C}^\bullet(I). \tag{2.3}$$

We define the Čech cohomology groups of an R-module M by

$$\check{C}H^*_I(R;M) = H^*(\check{C}^\bullet(I) \otimes M). \tag{2.4}$$

We often delete R from the notation for these functors. The fiber sequence (2.3) gives rise to long exact sequences relating local and Čech cohomology, and these reduce to exact sequences

$$0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow \check{C}H^0_I(M) \longrightarrow H^1_I(M) \longrightarrow 0$$

together with isomorphisms

$$H^i_I(M) \cong \check{C}H^{i-1}_I(M).$$

A. Grothendieck (notes by R.Hartshorne). Local cohomology. Springer Lecture notes in mathematics, Vol. 42. 1967.

3. Brave new versions of local and Čech cohomology

Turning to topology, we fix a compact Lie group G and consider G-spectra indexed on a complete G-universe U. We let S_G be the sphere G-spectrum, and we work in the category of S_G-modules. Fix a commutative S_G-algebra R and consider R-modules M. We write

$$M^G_n = \pi^G_n(M) = M^{-n}_G.$$

Thus R^G_* is a ring and M^G_* is an R^G_*-module.

Mimicking the algebra, for $\alpha \in R^G_*$ we define the Koszul spectrum $K(\alpha)$ by the fiber sequence

$$K(\alpha) \longrightarrow R \longrightarrow R[\alpha^{-1}].$$

Here, suppressing notation for suspensions, $R[\alpha^{-1}] = \mathrm{hocolim}(R \xrightarrow{\alpha} R \xrightarrow{\alpha} \dots)$; it is an R-module and the inclusion of R is a module map; therefore $K(\alpha)$ is an R-module. Analogous to the filtration at the chain level, we obtain a filtration of $K(\alpha)$ by viewing it as $\Sigma^{-1}(R[1/\alpha] \cup CR)$.

Next we define the Koszul spectrum of a sequence $\alpha_1, \dots, \alpha_n$ by

$$K(\alpha_1, \dots, \alpha_n) = K(\alpha_1) \wedge_R \dots \wedge_R K(\alpha_n).$$

Using the same proof as in the algebraic case we conclude that, up to equivalence, $K(\alpha_1, \dots, \alpha_n)$ depends only on the radical of $I = (\alpha_1, \dots, \alpha_n)$; we therefore denote it $K(I)$. We then define the homotopy I-power torsion (or local cohomology) module of an R-module M by

$$\Gamma_I(M) = K(I) \wedge_R M. \tag{3.1}$$

In particular, $\Gamma_I(R) = K(I)$.

To calculate the homotopy groups of $\Gamma_I(M)$ we use the product of the filtrations of the $K(\alpha_i)$ given above. Since the filtration models the algebra precisely, there results a spectral sequence of the form

$$E^2_{s,t} = H^{-s}_I(R^G_*; M^G_*)_t \Rightarrow \pi^G_{s+t}(\Gamma_I(M)) \tag{3.2}$$

with differentials $d^r : E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$.

REMARK 3.3. In practice it is often useful to use the fact that the algebraic local cohomology $H^*_I(R; M)$ is essentially independent of R. Indeed if the generators of I come from a ring R_0 (in which they generate an ideal I_0) via a ring homomorphism $\theta : R_0 \to R$, then $H^*_{I_0}(R_0; M) = H^*_I(R; M)$. In practice we often use this if the ideal I of R^G_* may be radically generated by elements of degree 0. This holds for any ideal of S^G_* since the elements of positive degree in S^G_* are nilpotent.

Similarly, we define the Čech spectrum of I by the cofiber sequence of R-modules

$$K(I) \longrightarrow R \longrightarrow \check{C}(I). \tag{3.4}$$

We think of $\check{C}(I)$ as analogous to $\tilde{E}G$. We then define the homotopical localization (or Čech cohomology) module associated to an R-module M by

$$M[I^{-1}] = \check{C}(I) \wedge_R M. \tag{3.5}$$

In particular, $R[I^{-1}] = \check{C}(I)$. Again, we have a spectral sequence of the form

$$E^2_{s,t} = \check{C}H_I^{-s}(R^G_*; M^G_*)_t \Rightarrow \pi^G_{s+t}(M[I^{-1}]) \tag{3.6}$$

with differentials $d^r : E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$.

The "localization" $M[I^{-1}]$ is generally not a localization of M at a multiplicatively closed subset of R_*. However, the term is justified by the following theorem from [GM1, §5]. Recall the discussion of Bousfield localization from XXII§6.

Theorem 3.7. For any finitely generated ideal $I = (\alpha_1, \ldots, \alpha_n)$ of R^G_*, the map $M \longrightarrow M[I^{-1}]$ is Bousfield localization with respect to the R-module $R[I^{-1}]$ or, equivalently, with respect to the wedge of the R-modules $R[\alpha_i^{-1}]$.

Observe that we have a natural cofiber sequence

$$\Gamma_I(M) \longrightarrow M \longrightarrow M[I^{-1}] \tag{3.8}$$

relating our I-power torsion and localization functors.

4. Localization theorems in equivariant homology

For an R-module M, we have the fundamental cofiber sequence of R-modules

$$EG_+ \wedge M \longrightarrow M \longrightarrow \tilde{E}G \wedge M. \tag{4.1}$$

Such sequences played a central role in our study of the Segal conjecture and Tate cohomology, for example, and we would like to understand their homotopical behavior. In favorable cases, the cofiber sequence (3.8) models this sequence and so allows computations via the spectral sequences of the previous section. The relevant ideal is the augmentation ideal

$$I = \mathrm{Ker}(\mathrm{res}^G_1 : R^G_* \to R_*).$$

In order to apply the constructions of the previous section, we need an assumption. It will be satisfied automatically when R^G_* is Noetherian.

Assumption 4.2. Up to taking radicals, the ideal I is finitely generated. That is, there are elements $\alpha_1, \ldots, \alpha_n \in I$ such that

$$\sqrt{(\alpha_1, \ldots, \alpha_n)} = \sqrt{I}.$$

Under Assumption (4.2), it is reasonable to let $K(I)$ denote $K(\alpha_1, \ldots, \alpha_n)$. The canonical map $\varepsilon : K(I) \longrightarrow R$ is then a nonequivariant equivalence. Indeed, this is a special case of the following observation, which is evident from our constructions.

LEMMA 4.3. *Let $H \subseteq G$, let $\beta_i \in R_*^G$, and let $\gamma_i = \mathrm{res}_H^G(\beta_i) \in R_*^H$. Then, regarded as a module over the S_H-algebra $R|_H$,*

$$K(\beta_1, \cdots, \beta_n)|_H = K(\gamma_1, \cdots, \gamma_n).$$

Therefore, if $\beta_i \in \mathrm{Ker}\,\mathrm{res}_H^G$, then the natural map $K(\beta_1, \cdots, \beta_n) \longrightarrow R$ is an H-equivalence.

Here the last statement holds since $K(0) = R$. If we take the smash product of ε with the identity map of EG_+, we obtain a G-equivalence of R-modules $EG_+ \wedge K(I) \longrightarrow EG_+ \wedge R$. Working in the derived category $G\mathscr{D}_R$, we may invert this map and compose with the map

$$EG_+ \wedge K(I) \longrightarrow S^0 \wedge K(I) = K(I)$$

induced by the projection $EG_+ \longrightarrow S^0$ to obtain a map of R-modules over R

$$\kappa : EG_+ \wedge R \longrightarrow K(I). \tag{4.4}$$

Passing to cofibers we obtain a compatible map

$$\tilde{\kappa} : \tilde{E}G \wedge R \longrightarrow \check{C}(I). \tag{4.5}$$

Finally, taking the smash product over R with an R-module M, there results a natural map of cofiber sequences

$$\begin{array}{ccccc} EG_+ \wedge M & \longrightarrow & M & \longrightarrow & \tilde{E}G \wedge M \\ \downarrow{\scriptstyle \kappa} & & \| & & \downarrow{\scriptstyle \tilde{\kappa}} \\ \Gamma_I(M) & \longrightarrow & M & \longrightarrow & M[I^{-1}]. \end{array} \tag{4.6}$$

Clearly κ is an equivalence if and only if $\tilde{\kappa}$ is an equivalence. When the latter holds, it should be interpreted as stating that the 'topological' localization of M away from its free part is equivalent to the 'algebraic' localization of M away from I. We adopt this idea in a definition. Recall the homotopical notions of an R-ring spectrum A and of an A-module spectrum from XXII.4.1; we tacitly assume throughout the chapter that all given R-ring spectra are associative and commutative.

DEFINITION 4.7. The 'localization theorem' holds for an R-ring spectrum A if

$$\tilde{\kappa}_A = \tilde{\kappa} \wedge \mathrm{id} : \tilde{E}G \wedge A = \tilde{E}G \wedge R \wedge_R A \longrightarrow \check{C}(I) \wedge_R A$$

is a weak equivalence of R-modules, that is, if it is an isomorphism in $G\mathscr{D}_R$. It is equivalent that

$$\kappa_A = \kappa \wedge \mathrm{id} : EG_+ \wedge A = EG_+ \wedge R \wedge_R A \longrightarrow K(I) \wedge_R A$$

be an isomorphism in $G\mathscr{D}_R$.

In our equivariant context, we define the A-homology of an R-module M by

$$A_n^{G,R}(M) = \pi_n^G(M \wedge_R A); \tag{4.8}$$

compare XXII.3.1. This must not be confused with $A_n^G(X) = \pi_n^G(X \wedge A)$, which is defined on all G-spectra X. When $A = R$, $A_*^{G,R}$ is the restriction of A_*^G to R-modules. When $R = S_G$, A_*^{G,S_G} is A_*^G thought of as a theory defined on S_G-modules. In general, for G-spectra X, we have the relation

$$A_*^G(X) \cong A_*^{G,R}(\mathbb{F}_R X), \tag{4.9}$$

where the free R-module $\mathbb{F}_R X$ is weakly equivalent to the spectrum $X \wedge R$. The localization theorem asserts that κ is an $A_*^{H,R}$-isomorphism for all subgroups H of G and thus that the cofiber $C\kappa$ is $A_*^{H,R}$-acyclic for all H. Observe that the definition of κ implies that $C\kappa$ is equivalent to $\tilde{E}G \wedge K(I)$. We are mainly interested in the case $A = R$, but we shall see in the next section that the localization theorem holds for K_G regarded as an S_G-ring spectrum, although it fails for S_G itself. The conclusion of the localization theorem is inherited by arbitrary A-modules.

LEMMA 4.10. If the localization theorem holds for the R-ring spectrum A, then the maps

$$EG_+ \wedge M \longrightarrow \Gamma_I(M) \quad \text{and} \quad \tilde{E}G \wedge M \longrightarrow M[I^{-1}]$$

of (4.6) are isomorphisms in $G\mathscr{D}_R$ for all A-modules M.

PROOF. $C\kappa \wedge_R M$ is trivial since it is a retract in $G\mathscr{D}_R$ of $C\kappa \wedge_R A \wedge_R M$. □

When this holds, we obtain the isomorphism

$$M_*^G(EG_+) = \pi_*^G(EG_+ \wedge M) \cong \pi_*^G(\Gamma_I(M))$$

on passage to homotopy groups. Here, in favorable cases, the homotopy groups on the right can be calculated by the spectral sequence (3.2). When M is split and G is finite, the homology groups on the left are the (reduced) homology groups

$M_*(BG_+)$ defined with respect to the underlying nonequivariant spectrum of M; see XVI§2. We also obtain the isomorphism

$$M_*^G(\tilde{E}G) = \pi_*^G(\tilde{E}G \wedge M) \cong \pi_*^G(M[I^{-1}]);$$

the homotopy groups on the right can be calculated by the spectral sequence (3.5).

More generally, it is valuable to obtain a localization theorem about $EG_+ \wedge_G X$ for a general based G-space X, obtaining the result about BG_+ by taking X to be S^0. To obtain this, we simply replace M by $M \wedge X$ in the first equivalence of the previous lemma. If M is split, we conclude from XVI§2 that

$$\pi_*^G(\Sigma^{-Ad(G)}(EG_+ \wedge X \wedge M)) \cong M_*(EG_+ \wedge_G X),$$

where $Ad(G)$ is the adjoint representation of G. Thus we have the following implication.

Corollary 4.11. If the localization theorem holds for A and M is an A-module spectrum that is split as a G-spectrum, then

$$\Gamma_I(\Sigma^{-Ad(G)} M \wedge X)_*^G \cong M_*(EG_+ \wedge_G X)$$

for any based G-space X. Therefore there is a spectral sequence of the form

$$E^2_{s,t} = H_I^{-s}(R_*^G; M_*^G(\Sigma^{-Ad(G)}\Sigma^\infty X))_t \Rightarrow M_{s+t}(EG_+ \wedge_G X).$$

5. Completions, completion theorems, and local homology

The localization theorem also implies a completion theorem. In fact, applying the functor $F_R(\cdot, M)$ to the map κ, we obtain a cohomological analogue of Lemma 4.10. To give the appropriate context, we define the completion of an R-module M at a finitely generated ideal I by

$$M_I^\wedge = F_R(K(I), M). \tag{5.1}$$

We shall shortly return to algebra and define certain "local homology groups" $H_I^*(R; M)$ that are closely related to the I-adic completion functor. In the topological context, it will follow from the definitions that the filtration of $K(I)$ gives rise to a spectral sequence of the form

$$E_2^{s,t} = H^I_{-s}(R_G^*; M_G^*)^t \Rightarrow \pi^G_{-s-t}(M_I^\wedge) \tag{5.2}$$

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. Here, if R_G^* is Noetherian and M_G^* is finitely generated, then

$$\pi^G_{-*}(M_I^\wedge) = (M_G^*)^\wedge_I.$$

Again, a theorem from [GM1, §5] gives an interpretation of the completion functor as a Bousfield localization.

THEOREM 5.3. For any finitely generated ideal $I = (\alpha_1, \dots, \alpha_n)$ of R^G_*, the map $M \longrightarrow M^\wedge_I$ is Bousfield localization in the category of R-modules with respect to the R-module $K(I)$ or, equivalently, with respect to the smash product of the R-modules R/α_i.

Returning to the augmentation ideal I, we have the promised cohomological implication of the localization theorem; the case $M = A$ is called the 'completion theorem' for A.

LEMMA 5.4. If the localization theorem holds for the R-ring spectrum A, then the map

$$M^\wedge_I = F_R(K(I), M) \longrightarrow F_R(EG_+ \wedge R, M) \cong F(EG_+, M)$$

is an isomorphism in $G\mathscr{D}_R$ for all A-module spectra M.

PROOF. $F_R(C\kappa, M)$ is trivial since any map $C\kappa \longrightarrow M$ factors as a composite

$$C\kappa \longrightarrow C\kappa \wedge_R A \longrightarrow M \wedge_R A \longrightarrow M,$$

and similarly for suspensions of $C\kappa$. □

When this holds, we obtain the isomorphism

$$\pi^G_{-*}(M^\wedge_I) \cong M^*_G(EG_+)$$

on passage to homotopy groups. If M is split, the cohomology groups on the right are the (reduced) cohomology groups $M^*(BG_+)$ defined with respect to the underlying nonequivariant spectrum of M; see XVI§2.

To obtain a completion theorem about $EG_+ \wedge_G X$ for a based G-space X, we replace M by $F(X, M)$ in the previous lemma. If M is split, then

$$\pi^G_*(F(EG_+ \wedge X, M)) \cong M^*(EG_+ \wedge_G X).$$

COROLLARY 5.5. If the localization theorem holds for A and M is an A-module spectrum that is split as a G-spectrum, then

$$(F(X, M)^\wedge_I)^*_G \cong M^*(EG_+ \wedge_G X)$$

for any based G-space X. Therefore there is a spectral sequence of the form

$$E_2^{s,t} = H^I_{-s}(R^*_G; M^*_G(X))^t \Rightarrow M^{s+t}(EG_+ \wedge_G X).$$