5. $RO(G)$-graded homology and cohomology

We shall be precise about how to define $RO(G)$-graded homology and cohomology theories in XIII§1. Here we give an intuitive description. The basic idea is that if we understand G-spheres to be representation spheres S^V, then we must understand the suspension axiom to allow suspension by such spheres. This forces us to grade on representations. However, the standard term "$RO(G)$-grading" is a technical misnomer since the real representation ring $RO(G)$ is defined in terms of isomorphism classes of representations, and this is too imprecise to allow the control of "signs" (which must be interpreted as units in the Burnside ring of G).

Thus, intuitively, a reduced $RO(G)$-graded homology theory $\tilde{E}^G_*$ defined on based G-spaces X consists of functors $\tilde{E}^G_\alpha : \bar{h}G\mathscr{T} \longrightarrow \mathscr{A}b$ for all $\alpha \in RO(G)$ together with suitably compatible natural suspension isomorphisms

$$\tilde{E}^G_\alpha(X) \cong \tilde{E}^G_{\alpha+V}(\Sigma^V X)$$

for all G-representations V. We require each $\tilde{E}^G_\alpha$ to carry cofibration sequences $A \longrightarrow X \longrightarrow X/A$ of based G-spaces to three term exact sequences and to carry wedges to direct sums. We have combined the homotopy and weak equivalence axioms in the statement that the $\tilde{E}^G_\alpha$ are defined on $\bar{h}G\mathscr{T}$.

For each representation V with $V^G = 0$, it follows by use of the suspension isomorphism for S^1 that the groups $\{\tilde{E}^G_{V+n} | n \in \mathbb{Z}\}$ give a reduced $\mathbb{Z}$-graded homology theory in the sense that the evident equivariant analogs of the Eilenberg-Steenrod axioms, other than the dimension axiom, are satisfied. Taking $V = 0$, this gives the underlying $\mathbb{Z}$-graded homology theory of the given $RO(G)$-graded theory. We could elaborate by defining unreduced theories, showing how to construct unreduced theories from reduced ones by adjoining disjoint basepoints and defining appropriate relative groups, and showing that unreduced theories give rise to reduced ones in the usual fashion. However, we concentrate on the essential new feature, which is the suspension axiom for general representations V.

Of course, we have a precisely similar definition of an $RO(G)$-graded cohomology theory. There are two quite different philosophies about these $RO(G)$-graded theories. One may view them as the right context in which to formulate calculations. For example, there are calculations of Lewis that show that the cohomology of a space may have an elegant algebraic description in $RO(G)$-graded cohomology that is completely obscured when one looks only at the $\mathbb{Z}$-graded part of the relevant theory. In contrast, one may view $RO(G)$-gradability as a tool for the study of the $\mathbb{Z}$-graded parts of theories. Our proof of the Conner conjecture in the next section will be a direct application of that philosophy.

When can the $\mathbb{Z}$-graded cohomology theory with coefficients in a coefficient system M be extended to an $RO(G)$-graded cohomology theory? If we are given such an extension, then the transfer maps $\tau(G/H) : S^V \longrightarrow G/H_+ \wedge S^V$ of (3.4) will induce transfer homomorphisms

$$\begin{array}{c} \tilde{H}^n_H(X; M|H) \cong \tilde{H}^{v+n}_G(\Sigma^V(G/H_+ \wedge X); M) \\ \downarrow \\ \tilde{H}^n_G(X; M) \cong \tilde{H}^{v+n}_G(\Sigma^V X; M). \end{array} \tag{5.1}$$

Taking $n = 0$ and $X = S^0$, we obtain a transfer homomorphism $M(G/H) \longrightarrow M(G/G)$. An elaboration of this argument shows that the coefficient system M must extend to a Mackey functor. It is a pleasant fact that this necessary condition is sufficient.

THEOREM 5.2. Let G be a compact Lie group and let M and N be a contravariant and a covariant coefficient system The ordinary cohomology theory $\tilde{H}^*_G(-; M)$ extends to an $RO(G)$-graded cohomology theory if and only if M extends to a Mackey functor. The ordinary homology theory $\tilde{H}^G_*(-; N)$ extends to an $RO(G)$-graded homology theory if and only if N extends to a coMackey functor.

We shall later explain two very different proofs. Waner will describe a chain level construction in terms of G-CW(V) complexes in the next chapter. I will describe a spectrum level construction of the representing Eilenberg-Mac Lane G-spectra in XIII§4.

L. G. Lewis, Jr. The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions. Springer Lecture Notes in Mathematics Vol. 1361, 1988, 53-122.

6. The Conner conjecture

To illustrate the force of $RO(G)$-gradability, we explain how the results stated in the previous two sections directly imply the following conjecture of Conner.

THEOREM 6.1 (CONNER CONJECTURE). Let G be a compact Lie group and let X be a finite dimensional G-space with finitely many orbit types. Let A be any Abelian group. If $\tilde{H}^*(X; A) = 0$, then $\tilde{H}^*(X/G; A) = 0$.

This was first proven by Oliver, using Čech cohomology and wholly different techniques. It was known early on that the conjecture would hold if one could construct a suitable transfer map.

THEOREM 6.2. Let X be any G-space and let $\pi : X/H \longrightarrow X/G$ be the natural projection, where $H \subset G$. For $n \geq 0$, there is a natural transfer homomorphism

$$\tau : \tilde{H}^n(X/H; A) \longrightarrow \tilde{H}^n(X/G; A)$$

such that $\tau \circ \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$.

PROOF. Tensoring the Mackey functor $\underline{\mathbb{Z}}$ of Proposition 4.3 with A, we obtain a Mackey functor $\underline{A}$ whose underlying coefficient system is constant at A. The map $\underline{A}(G/H) \longrightarrow \underline{A}(G/G)$ associated to the stable transfer map $G/G_+ \longrightarrow G/H_+$ is multiplication by $\chi(G/H)$. As we observed in our first treatment of Smith theory (IV§1), ordinary G-cohomology with coefficients in a constant coefficient system is the same as ordinary nonequivariant cohomology on orbit spaces:

$$H^n(X/H; A) \cong H^n_H(X; \underline{A}|H) \quad \text{and} \quad H^n(X/G; R) \cong H^n_G(X; \underline{A}).$$

Taking $M = \underline{A}$, (5.1) already displays the required transfer map. The formula for $\tau \circ \pi^*$ follows formally, but it can also be derived from the fact that the equivariant Euler characteristic

$$S^V \longrightarrow G/H_+ \wedge S^V \longrightarrow S^V,$$

regarded as a nonequivariant map, has degree $\chi(G/H)$. □

How does the Conner conjecture follow? Conner himself proved it when G is a finite extension of a torus, the methods being induction and use of Smith theory — one proves that both X^G and X/G are A-acyclic. For example, the result for a torus reduces immediately to the result for a circle. Here the "finitely many orbit types" hypothesis implies that $X^G = X^C$ for C cyclic of large enough order, so that we really are in the realm where Smith theory can be applied. Assuming that the result holds when G is a finite extension of a torus, let N be the normalizer of a maximal torus in G. Then N is a finite extension of a torus and $\chi(G/N) = 1$. The composite

$$\tau \circ \pi^* : \tilde{H}^n(X/G; A) \longrightarrow \tilde{H}^n(X/N; A) \longrightarrow \tilde{H}^n(X/G; A)$$

is the identity, and that's all there is to it.

P. Conner. Retraction properties of the orbit space of a compact topological transformation group. Duke Math. J. 27(1960), 341-357.

G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. Bulletin Amer. Math. Soc. 4(1981), 208-212.

R. Oliver. A proof of the Conner conjecture. Annals of Math. 103(1976), 637-644.

CHAPTER X

G-CW(V) complexes and $RO(G)$-graded cohomology

by Stefan Waner

1. Motivation for cellular theories based on representations

If a compact Lie group G acts smoothly on a smooth manifold M then the action is locally orthogonal. That is, for each $x \in M$ there is a G_x-invariant neighborhood U of x diffeomorphic to the open unit disc in a representation V of G_x. Moreover, writing G_x as H, if $L(H)$ is the tangent representation of H at $eH \in G/H$, then $L(H)$ is a summand of V. (Of course, $L(H) = 0$ if G is finite.) It follows that the G-orbit of x has a neighborhood diffeomorphic to $G \times_H D(V - L(H))$, where $V - L(H)$ is the orthogonal complement of $L(H)$ in V.

The above remarks seem to suggest that one ought to consider G-complexes modeled by cells of this form. On the other hand, it has been established by Bredon and others that ordinary G-CW complexes seem to suffice for practical purposes. These are G-complexes with "cells" of the form $G/H \times D^n$, where G acts trivially on D^n. Basically, the local neighborhoods $G \times_H D(V - L(H))$ can be G-triangulated into cells of the above form, so it would seem that there is no need to consider anything more elaborate than G-CW complexes. But there are some theoretical difficulties:

(1) Duality doesn't work. That is, the cellular chains obtained from G-CW structures on smooth G-manifolds do not exhibit Poincaré duality. The geometric reason for this is that the dual of an n-dimensional G-cell $G/H \times D^n$ is not a G-cell. The dual cell to a zero dimensional cell G/H is defined as its star in the first barycentric subdivision, while the duals of higher dimensional cells are

intersections of such stars. In general, the dual of a G-cell $G/H \times D^n$ has the form $G \times_H D(V - L(H) - \mathbb{R}^n)$, where V is the local representation at eH. This really forces our hand.

(2) One has the result, due to various authors (Lewis, May, McLure, Waner) that, if M is a Mackey functor, then Bredon cohomology with coefficients in M extends to an $RO(G)$-graded cohomology theory. This will be treated from the stable homotopy category point of view later in the book. The question then is: what is the geometric representation of the cells in dimension V? In particular, can we write the Vth cohomology group in terms of the cohomology of a cellular cochain complex?

The purpose of this chapter is to outline the basic theory of cell complexes modeled on representations of G, and to use them to construct explicit models of ordinary $RO(G)$-graded cohomology in which Poincaré duality holds for certain classes of G-manifolds. For reasons of clarity, only complexes modeled on a single representation V of G will be discussed. The more elaborate theory in which V is allowed to vary is already completed as joint work with Costenoble and May, and some of it has appeared in papers of Costenoble and myself. Roughly speaking, whatever works for a single representation generalizes to the more elaborate case.

When G is not finite, there appear to be two theories of G-CW(V) complexes. The one that I will concentrate on will be the one that is *not* dual to the usual G-CW theory (on suitable G-manifolds), but that does work as a cellular theory and gives rise to ordinary $RO(G)$-graded cohomology. To make amends, we will very briefly indicate the present state of the variant that gives the true dual theory.

S. R. Costenoble and S. Waner. The equivariant Thom isomorphism theorem. Pacific J. Math. 152(1992), 21-39.

S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J. 39(1992).

2. G-CW(V) complexes

Let V be a fixed given orthogonal representation of G and write dim $V = |V|$. To understand the definitions that follow, it is useful to keep in mind the following observation, whose easy inductive proof will be left to the reader.

LEMMA 2.1. Let $H_n \subset H_{n-1} \subset \cdots \subset H_0 = G$ be a strictly increasing chain of subgroups of G such that each H_i occurs as the isotropy subgroup of some point in V (the point 0 having isotropy group G). Then, as a representation of H_n, V contains a trivial representation of dimension n.

For $H \subseteq G$, we let $V(H)$ denote the orthogonal complement of V^H in V. If W is an H-module, we let $D(W)$ and $S(W)$ denote the unit disc and sphere in W.

DEFINITION 2.2. A G-CW(V) complex is a G-space X with a decomposition $X = \operatorname{colim}_n X^n$ such that X^0 is a disjoint union of G-orbits of the form G/H, where H acts trivially on V, and X^n is obtained from X^{n-1} by attaching "cells" $G \times_H D(V(H) \oplus \mathbb{R}^t)$, where $|V(H)| + t = n$, along attaching G-maps

$$G \times_H S(V + \mathbb{R}^t) \longrightarrow X^{n-1}.$$

A map $f : X \longrightarrow Y$ between G-CW(V) complexes is cellular if $f(X^n) \subset Y^n$ for all n, and the notions of skeleta, dimension, subcomplex, relative G-CW(V) complex, and so on are defined as one would expect from the classical case $V = 0$.

REMARKS 2.3. (i) Although imprecise, it is convenient to think of $V(H) \oplus \mathbb{R}^t$ as $V + \mathbb{R}^s$, where $|V| + s = n$ and thus $|V^H| + s = t$; here s may be negative, but then the definition implies that $|V^H| \geq -s$ for all subgroups H occurring in the decomposition.
(ii) The stipulation on the dimension implies that the cell $G \times_H D(V(H) \oplus \mathbb{R}^t)$ is an $(n + dim\, G/H)$-dimensional G-manifold.

The last observation explains why the definition does not give the true dual theory when G has positive dimension. The following variant rectifies this. However, this theory has not yet been worked out thoroughly or extended to deal with varying representations, although we suspect that all works well.

VARIANT 2.4. Let G be an infinite compact Lie group. There is a variant definition of a G-CW(V) complex which differs from the definition given in that we require X^0 to be a disjoint union of finite orbits G/H such that H acts trivially on V and we attach cells of the form $G \times_H D((V - L(H)) + \mathbb{R}^s)$, where $|V| + s = n$, when constructing X^n from X^{n-1}. Here $L(H)$ is the tangent representation of G/H at eH, and the definition implies that $L(H)$ is contained in $V|_H$ for all subgroups H occurring in the decomposition. With these stipulations on dimensions, the n-cells that we attach are n-dimensional G-manifolds.

Part of our motivation comes from consideration of G-manifolds that are locally modeled on a single representation.

Definition 2.5. A smooth G-manifold M has dimension V if, for each $x \in M$, there is a G_x-invariant neighborhood U of x that is diffeomorphic to the open unit disc in the restriction of V to G_x. It follows that $L(H)$ embeds in $V|_H$ and the orbit Gx has a neighborhood of the form $G \times_H D(V - (L(H))$. Any smooth G-manifold M each of whose fixed point sets is non-empty and connected must have dimension V, where V is the tangent representation at any G-fixed point. More generally, M has dimension $V - i$ for a positive integer $i \le |V|$ if, for each $x \in M$, G_x acts on V with an i-dimensional trivial summand and there is a G_x-invariant neighborhood U of x that is G_x-diffeomorphic to the open unit disc in $V - \mathbb{R}^i$. Thus, if M has dimension V, then ∂M has dimension $V - 1$. For example, $D(V)$ is a V-dimensional manifold and $S(V)$ is a $(V - 1)$-dimensional manifold.

When G is finite, G-manifolds of dimension V and their bordism theories were first discussed by Pulikowski and Kosniowski; I later carried the study further. By a theorem of Stong, if G is finite of odd order, then any G-manifold is cobordant to a sum of G-manifolds of the form $G \times_H N$, where N has dimension W for some H-module W.

The classical theory of dual cell decompositions of smooth manifolds (for which see Seifert and Threlfall) generalizes to V-manifolds. We shall not go into the definitions needed to make this precise. The intuition comes from equivariant Spanier-Whitehead and Atiyah duality, which will be discussed in XVI§§7-8. If a closed smooth G-manifold M embeds in V, then M_+ is V-dual to the Thom space $T\nu$ of the normal bundle of the embedding. In the case $M = G/H$, this normal bundle is $T\nu = G_+ \wedge_H S^{V-L(H)}$.

Proposition 2.6. If G is finite, then we obtain a G-CW(V) structure on a $(V - i)$-dimensional manifold M by passage to dual cells from an ordinary G-CW structure. With the variant definition of a G-CW(V) complex, the statement remains true for general compact Lie groups G.

From now on, we restrict attention to our first definition of a G-CW(V) complex.

Lemma 2.7. If X is a G-CW complex, then $X \times D(V)$ has the structure of a G-CW(V) complex under the usual product structure. Therefore, for any V, any G-CW complex is G-homotopy equivalent to a G-CW(V) complex.

Proposition 2.8. For any V, a G-space has the G-homotopy type of a G-CW complex if and only if it has the G-homotopy type of a G-CW(V) complex.

The lemma gives the forward implication in the case of finite G. The case for general compact Lie groups is harder, and we need to use the equivariant version of Brown's construction to give a brute force weak G-approximation by a G-CW(V) complex. That this approximation is in fact a G-homotopy equivalence then follows from the converse and the G-Whitehead theorem. For the converse, if X is a G-CW(V) complex, then X is a colimit of spaces of the G-homotopy type of G-CW complexes, and thus X is also such a homotopy type by a telescope argument and the homotopy invariance of colimits.

PROPOSITION 2.9. If X and Y have, respectively, a G-CW(V) and G-CW(W) structure, then $X \times Y$ has a G-CW($V \oplus W$) structure.

C. Kosniowski. A note on $RO(G)$-graded G-bordism. Quart J. Math. Oxford 26(1975), 411-419.

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H. Seifert and W. Threlfall. A Textbook of Topology (translation). Academic Press. 1980.

R. E. Stong. Unoriented bordism and actions of finite groups. Memoirs A.M.S. No. 103. 1970.

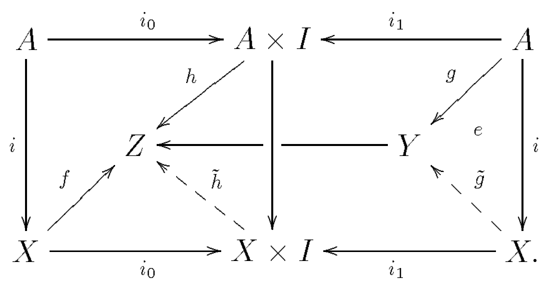
Equivariant $RO(G)$-graded bordism theories. Topology and its Applications. 17(1984), 1-26.

3. Homotopy theory of G-CW(V) complexes

We now do a little homotopy theory. Since we are using representations to define attaching maps, it is reasonable to consider the homotopy groups that were defined in terms of representations in IX.1.1.

DEFINITION 3.1. A G-space X is V-connected if X^H is $|V^H|$-connected for each closed subgroup $H \subseteq G$. Let $e : X \longrightarrow Y$ be a G-map and let n be an integer. Then e is a $(V+n)$-equivalence if, for each $H \subset G$ and each choice of basepoint in X^H, $e_* : \pi^H_{V(H)+q}(X) \longrightarrow \pi^H_{V(H)+q}(Y)$ is an isomorphism if $q \leq |V^H| + n - 1$ and an epimorphism if $q \leq |V^H| + n$.

THEOREM 3.2 (HELP). Let $e : Y \longrightarrow Z$ be a $(V+n)$-equivalence and let (X, A) be a relative G-CW(V) complex of dimension $\leq |V| + n$. Then we can complete the following homotopy extension and lifting diagram:



Sketch of Proof. We extend the G-maps g and h cell-by-cell and work inductively. This reduces the problem to the special case where $A = G \times_H S(W)$ and $X = G \times_H D(W)$. The pair (X, A) then has the structure of a relative G-CW complex with G-cells of the form $G/K \times D^r$ with $r \le |W^K| \le |V^K| + n$ and K subconjugate to H. Since e^K is a $(|V^K| + n)$-equivalence, this allows us to apply the HELP theorem of ordinary G-homotopy theory to complete the proof. □

Theorem 3.3 (G-CW(V) Whitehead). Let $e : Y \longrightarrow Z$ be a $(V + n)$-equivalence and let X be a G-CW(V) complex. Then $e_* : [X, Y]_G \longrightarrow [X, Z]_G$ (unbased G-homotopy classes) is an isomorphism if $\dim X < n + |V|$ and an epimorphism if $\dim X = n + |V|$. Moreover the conclusion remains true if $n = \infty$.

Proof. As usual, apply HELP to the pair $(X, \emptyset)$ for surjectivity and to the pair $(X \times I, X \times \partial I)$ for injectivity. □

Theorem 3.4 (Cellular Approximation). Every G-map $f : X \longrightarrow Y$ of G-CW(V) complexes is G-homotopic to a cellular map. If f is already cellular on a subcomplex A, then the homotopy can be taken relative to A.

Sketch of proof. One easily shows that the inclusion $i : Y^n \longrightarrow Y$ is a $(V + n - |V|)$-equivalence, and HELP then applies inductively to push X^n into Y^n and give the required homotopy. □

Theorem 3.5. For any G-space X, there is a G-CW(V) complex $?X$ and a weak equivalence $\gamma : ?X \longrightarrow X$.

Sketch of proof. In view of Proposition 2.8, this follows directly from the analog for ordinary G-CW complexes. □

4. Ordinary $RO(G)$-graded homology and cohomology

Recall the discussion of stable coefficient systems, alias Mackey and coMackey functors, from IX§4. The algebra of stable coefficient systems works in the same way as the algebra of coefficient systems discussed in I§3. The categories of Mackey functors and of coMackey functors are Abelian. If M and N are, respectively, Mackey and coMackey functors, we have the coend or tensor product $M \otimes_{\mathscr{B}_G} N$. If M and M' are Mackey functors, we have the group of natural transformations $\operatorname{Hom}_{\mathscr{B}_G}(M, M')$.

Observe that, for any based G-spaces X and Y, we have a Mackey functor $\underline{\{X,Y\}}_G$ with values

$$\underline{\{X,Y\}}_G(G/H) = \{G/H_+ \wedge X, Y\}_G.$$

The contravariant functoriality is given by composition in the evident way.

Definition 4.1. Let X be a G-CW(V) complex. Define a chain complex $\underline{C}^V_*(X)$ in the Abelian category of Mackey functors as follows. Let

$$\underline{C}^V_n(X) = \underline{\left\{S^{V-|V|+n}, X^n/X^{n-1}\right\}}_G.$$

This is the stable H-homotopy group of X^n/X^{n-1} in dimension $V - |V| + n$. Let

$$d_n : \underline{C}^V_n(X) \longrightarrow \underline{C}^V_{n-1}(X)$$

be the stable connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}).

Observe that X^n/X^{n-1} is the wedge over the n-cells of X of G-spaces of the form $G/H_+ \wedge S^{V-|V|+n}$ and that $\underline{C}^V_n(X)$ is the direct sum of corresponding free Mackey functors represented by the objects G/H.

Definition 4.2. Let X be a G-CW(V) complex. For a Mackey functor M, define the ordinary cohomology of X with coefficients in M to be

$$H^{V+n}_G(X; M) = H^{|V|+n} \operatorname{Hom}_{\mathscr{B}_G}(\underline{C}^V_*(X), M).$$

For a coMackey functor N, define the ordinary homology of X with coefficients in N to be

$$H^G_{V+n}(X; N) = H_{|V|+n}(\underline{C}^V_*(X) \otimes_{\mathscr{B}_G} N).$$

Precisely similar definitions apply to give relative homology and cohomology groups for relative G-CW(V) complexes (X,A). In the special case when A is a subcomplex of X, $\underline{C}^V_*(X,A)$ is isomorphic to $\underline{C}^V_*(X)/\underline{C}^V_*(A)$, and we obtain the expected long exact sequences. If $* \in X$ is a G-fixed basepoint and $(X,*)$ is a relative G-CW(V) complex, we define the reduced homology and cohomology of X by

$$\tilde{H}_G^{V+n}(X;M) = H_G^{V+n}(X,*;M) \quad \text{and} \quad \tilde{H}^G_{V+n}(X;N) = H^G_{V+n}(X,*;N).$$

Observe, however, that $*$ cannot be a vertex of X unless G acts trivially on V, by our limitation on the orbits G/H that are allowed in the zero skeleta of G-CW(V) complexes.

Using cellular approximation, homology and cohomology are seen to be functorial on the homotopy category of G-CW(V) complexes. We extend the definition to arbitrary G-spaces by using approximations by weakly equivalent G-CW(V) complexes. The definitions for pairs extend similarly. Finally, we extend the grading to all of $RO(G)$ by setting

$$\tilde{H}_G^{W-V+n}(X;M) = \tilde{H}_G^{W+n}(\Sigma^V X;M)$$

and

$$\tilde{H}^G_{W-V+n}(X;N) = \tilde{H}^G_{W+n}(\Sigma^V X;N).$$

We easily deduce from a relative version of Proposition 2.9 that, for a relative G-CW(W) complex $(X,*)$ and any representation V, $(\Sigma^V X,*)$ inherits a structure of relative G-CW($V \oplus W$) complex such that the W-cellular chain complex of $(X,*)$ is isomorphic to the $(V \oplus W)$-cellular chain complex of $(\Sigma^V X,*)$, with an appropriate shift of dimensions. This gives isomorphisms

$$\tilde{H}_G^{W+n}(X;M) \cong \tilde{H}_G^{V\oplus W+n}(\Sigma^V X)$$

and

$$\tilde{H}^G_{W+n}(X;M) \cong \tilde{H}^G_{V\oplus W+n}(\Sigma^V X).$$

It is quite tedious, but not difficult, to verify the precise axioms for $RO(G)$-graded homology and cohomology theories from the definitions just indicated. The alternative construction by stable homotopy category techniques in XIII§4 is less tedious, but perhaps less intuitive.

REMARKS 4.3. (1) There is a twisted version of the theory, where the twisting is taken over the fundamental groupoid of X.

(2) As already indicated, this theory also extends to a theory graded on representations of the fundamental groupoids of G-spaces. Roughly, such a representation assigns a representation to each component of each fixed point set in an appropriately coherent fashion. We also have a twisted version of this fancier theory.

(3) In the untwisted theory given above, Poincaré duality and the Thom isomorphism theorem hold for oriented V-manifolds. These are V-manifolds whose tangent bundles admit orientations in the geometric sense. They possess fundamental classes in dimension V.

(4) There is also a version of the Hurewicz theorem, which Lewis will discuss in the next chapter.

(5) There is an unpublished theory of equivariant Chern classes which live in off-integral dimensions, but this theory is not yet well-understood.

(6) The cohomology of a point is highly nontrivial, since there is no dimension axiom away from integer gradings. Indeed, among other applications related to ordinary cohomology, I have a curious result to the effect that if you localize the cohomology of a point by inverting a Chern class in dimension $V - |V|$, where V contains a free G-orbit, then you get the cohomology of BG.

REMARK 4.4. The chain level construction just sketched has applications to manifold theory. Since Poincaré duality works for this theory (V-manifolds have fundamental classes in the twisted theory), Costenoble and I have been able to use it to obtain a workable definition of Poincaré duality spaces and to prove a $\pi - \pi$ theorem for such spaces, giving a criterion for a G-CW complex to have the G-homotopy type of a G-manifold in the presence of suitable "gap hypotheses" on the homotopy groups of its fixed point spaces. We have also extended this to the case of simple G-homotopy theory, since it turns out that Poincaré duality is given by a simple chain equivalence, just as in the nonequivariant case. Thus we can say when a G-CW complex has the simple G-homotopy type of a G-manifold.

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S. R. Costenoble and S. Waner. The equivariant Spivak normal bundle and equivariant surgery. Michigan Math. J. To appear.

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CHAPTER XI

The equivariant Hurewicz and Suspension Theorems

by L. Gaunce Lewis, Jr.

1. Background on the classical theorems

We begin by recalling the statements of two basic theorems in nonequivariant homotopy theory. The first of these is the very familiar Hurewicz Theorem.

THEOREM A. If Y is a simply connected space and $n \geq 2$, then the following are equivalent:

(i) $H_k(Y;\mathbb{Z}) = 0$ for all $k < n$.

(ii) $\pi_k Y = 0$ for all $k < n$.

Moreover, either of these implies that the Hurewicz homomorphism

$$h : \pi_n Y \rightarrow H_n(Y;\mathbb{Z})$$

is an isomorphism.

There is, of course, an extension of this theorem that describes the relation between $\pi_1 Y$ and $H_1(Y;\mathbb{Z})$, but we shall here restrict attention to the simply connected case, in both nonequivariant and equivariant homotopy theory, to avoid some unpleasant technicalities that obscure the central issues. The Hurewicz theorem is important because it describes the basic connection between the two most commonly used functors in algebraic topology. It allows us to convert information about homology groups, which are relatively easy to compute, into information about homotopy groups, which are much harder to compute but also much more useful.

The second theorem is the Freudenthal suspension theorem.

THEOREM B. Let Y be an n-connected space, where $n \geq 1$, and let X be a finite CW complex. Then the suspension map

$$\sigma : [X, Y] \to [\Sigma X, \Sigma Y]$$

is surjective if $dim\, X \leq 2n + 1$ and bijective if $dim\, X \leq 2n$.

Historically, this result grew out of Freudenthal's study of the homotopy groups of spheres. His original version of this result merely gave conditions on m and n under which the suspension map

$$\sigma : \pi_n S^m \to \pi_{n+1} S^{m+1}$$

was surjective or bijective. This initial result was rather quickly extended to one giving conditions under which the suspension map

$$\sigma : \pi_n Y \to \pi_{n+1} \Sigma Y$$

was surjective or bijective. Eventually, the result was generalized to Theorem B. As with the Hurewicz Theorem, this result allows us to compare a well-behaved object that we have some hope of understanding with an apparently less well-behaved one. The point here is that $[\Sigma X, \Sigma Y]$ is a group and, if we suspend it once more, it becomes an abelian group. On the other hand, $[X, Y]$ need only be a pointed set. As a vague general principle, which will be made more precise later, the more we suspend a space, the more algebraic tools (like group structures) we gain for the study of the space. The Freudenthal result allows us to convert information that we obtain working in the more structured setting of objects that have been repeatedly suspended into information about the original, unsuspended, objects.

These two basic theorems are actually quite closely related. If one constructs homology using Eilenberg-Mac Lane spaces, then the Hurewicz theorem follows directly from the suspension theorem and the simple observation that the Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$ in dimension n associated to the group $\mathbb{Z}$ has a CW structure in which the bottom cell is a sphere in dimension n and in which there are no $(n+1)$-cells. The Hurewicz map itself is derived from the inclusion of this bottom cell. If one thinks of homology in terms of the Eilenberg-Mac Lane spectrum $K\mathbb{Z}$ associated to the group $\mathbb{Z}$, then the Hurewicz theorem follows even more directly from the suspension theorem and the observation that $K\mathbb{Z}$ has a CW structure in which the bottom cell is a copy of the zero sphere and in which there are no 1-cells.

We shall discuss the equivariant analogues of these two theorems in this chapter. Full details and more general versions of the results are given in the first two of the following three papers; we shall occasionally refer to these papers by number, and a little guide to them is given in a scholium at the end of the chapter.

[L1] L. G. Lewis, Jr., Equivariant Eilenberg-Mac Lane spaces and the equivariant Seifert-van Kampen and suspension theorems. Topology and its Applications 48 (1992), 25-61.

[L2] L. G. Lewis, Jr., The equivariant Hurewicz map. Trans. Amer. Math. Soc., 329 (1992), 433-472.

[L3] L. G. Lewis, Jr., Change of universe functors in equivariant stable homotopy theory. Fund. Math. To appear.

2. Formulation of the problem and counterexamples

Throughout the chapter, we assume that G is a compact Lie group and that the spaces considered are left G-spaces. There are two issues that come up immediately when one starts thinking about generalizing these basic theorems to the equivariant context. The first is how one should measure the connectivity of G-spaces. There are two solutions to this problem. The first is the notion of V-connectivity that Stefan Waner introduced in the previous chapter. This notion focuses on a single G-representation V and measures the connectivity of a G-space Y as seen through the "eyes" of that representation. The other notion of equivariant connectivity is less dependent on individual representations and somewhat less exotic in its definition. It too has already been introduced earlier, but we recall the definition.

Definition 2.1. (a) A *dimension function* ν is a function from the set of conjugacy classes of subgroups of G to the integers ≥ -1. Write n^* for the dimension function that takes the value n at each H. Associated to any G-representation V is the dimension function $|V^*|$ whose value at K is the real dimension of the K-fixed subspace V^K of V.

(b) Let ν be a dimension function. Then a G-space Y is *G-ν-connected* if, for each subgroup K of G, the fixed point space Y^K is $\nu(K)$-connected. The based G-space Y is *homologically G-ν-connected* if, for every subgroup K of G and every integer m with $0 \leq m \leq \nu(K)$, the equivariant homology group $\tilde{H}^K_m Y$ is zero. A G-space Y is *G-connected* if it is G-0^*-connected. A G-space is *simply G-connected* if it is G-1^*-connected. The prefix "G-" will be deleted from the notation whenever the omission should not lead to confusion.

(c) Define the connectivity function c^*Y of a G-space Y by letting c^KY be the connectivity of the space Y^K for each subgroup K of G. Define $c^KY = -1$ if Y^K is not path connected.

A basic result of Waner indicates that the two rather different measures of equivariant connectivity that we have described are intimately related.

LEMMA 2.2. Let Y be a G-space and V be a G-representation. Then the space Y is V-connected if and only if it is $|V^*|$-connected.

Because of this lemma, we will use the terms V-connected and $|V^*|$-connected interchangeably.

The second issue that comes up immediately is what sort of suspensions one wishes to allow in the equivariant context and, intimately tied to that, how one grades equivariant homotopy and homology groups. The point here is that one may define ΣY to be $Y \wedge S^1$. Therefore, in the equivariant context, if V is a G-representation and S^V is its one-point compactification (with G acting trivially on the point at infinity, which is taken to be the basepoint), then it is natural to think of $Y \wedge S^V$ as the suspension $\Sigma^V Y$ of Y by V. With this viewpoint, it is natural to want an equivariant suspension theorem which describes the map

$$\sigma_V : [X, Y]_G \to [\Sigma^V X, \Sigma^V Y]_G.$$

Moreover, since, in the nonequivariant context, $\pi_n Y$ is just $[S^n, Y]$, it is natural to regard $[S^V, Y]_G$ as the V^{th} homotopy group (or set) $\pi_V^G Y$. Thus, we would like to have a V^{th} homology group $H_V^G Y$, an equivariant Hurewicz map

$$h : \pi_V^G Y \to H_V^G Y,$$

and an equivariant Hurewicz theorem that tells us when this map is an isomorphism. The previous chapter has already given one construction of $H_V^G Y$, and Chapter XIII will give another. The precise definition of the map h is given in [L2], but it should become apparent from the discussion of the relationship between equivariant spectra and equivariant homology to be given later.

We must still resolve the issue of what coefficients should be used for this homology group since it is very important in the nonequivariant Hurewicz Theorem that integral coefficients be used. Burnside ring coefficients turn out to be the appropriate ones, essentially because the equivariant zero stem is the Burnside ring.

It should be fairly clear that the sort of equivariant suspension theorem that we would like to have would be something along the lines of:

"THEOREM". Let Y be a simply G-connected space, X be a finite G-CW complex, and V be a G-representation. Then the suspension map

$$\sigma_V \; : \; [X, Y]_G \; \to \; [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if, for every subgroup K of G, $dim\, X^K \leq 2c^K Y + 1$ and is bijective if, for every subgroup K, $dim\, X^K \leq 2c^K Y$.

Unfortunately, this result is wildly false. For example, let $G = \mathbb{Z}/2$, $n \geq 3$, and V be the real one-dimensional sign representation of G. Then our proposed "Theorem" would require that the maps

$$\sigma_V \; : \; [S^n, S^n]_G \; \to \; [S^{n+V}, S^{n+V}]_G$$

and

$$\sigma_V \; : \; [S^{n+V}, \Sigma^{n+V} G_+]_G \; \to \; [S^{n+2V}, \Sigma^{n+2V} G_+]_G$$

be isomorphisms. However, simple calculations give that

$$[S^n, S^n]_G = \mathbb{Z} \qquad \text{and} \qquad [S^{n+V}, S^{n+V}]_G = \mathbb{Z}^2,$$

$$[S^{n+V}, \Sigma^{n+V} G_+]_G = \mathbb{Z}^2 \qquad \text{and} \qquad [S^{n+2V}, \Sigma^{n+2V} G_+]_G = \mathbb{Z}.$$

Thus, the first of the two maps above can't be surjective and the second can't be injective. In fact, calculations for arbitrary groups G and low-dimensional nontrivial G-representations V and W suggest that the suspension map

$$\sigma_W \; : \; [S^V, S^V]_G \; \to \; [S^{V+W}, S^{V+W}]_G$$

is almost never an isomorphism. The restriction of "low dimension" is essential here because, as we have seen in IX.2.3, if G is finite and V contains enough copies of the regular representation of G, then σ_W is an isomorphism for any G-representation W. Similar calculations of equivariant homotopy and homology groups suggest rather quickly that there is no simple generalization of the Hurewicz theorem to the equivariant context.

One way to save the equivariant suspension theorem is to insert additional hypotheses, as in IX.1.4. The inequalities required there between the dimension of Y^H and the connectivity of Y^K when $K \subseteq H$ with $V^K \neq V^H$ tend to be quite restrictive and hard to verify. Thus, what we intend to discuss is another approach to generalizing the Hurewicz and suspension theorems to the equivariant context. For this alternative approach, we must revert to the earlier form of the suspension theorem which deals only with the suspension of homotopy groups.

3. An oversimplified description of the results

Hereafter, in discussing the suspension map

$$\sigma_W : [X, Y]_G \to [\Sigma^W X, \Sigma^W Y]_G,$$

we will consider only the case in which $X = S^V$ for some G-representation V. As a matter of convenience, we will assume that the representation V contains at least two copies of the one-dimensional trivial G-representation. This ensures that the set $\pi_V^G Y$ is an abelian group. The motivation for the alternative approach is that, even though the suspension map

$$\sigma_W : [S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G$$

is rather badly behaved, we can, at least in theory, compute exactly what it does. Thus, it is reasonable to ask if our understanding of this map can be used to shed some light on the suspension map

$$\sigma_W : \pi_V^G Y = [S^V, Y]_G \to [S^{V+W}, \Sigma^W Y]_G = \pi_{V+W}^G \Sigma^W Y$$

for any suitably connected G-space Y.

A feeling for the sort of result that we should expect is best conveyed by a slight oversimplification of the actual result. The set $[S^V, S^V]_G$ is a ring under composition. Here the right distributivity law depends on the fact that V contains two copies of $\mathbb{R}$ and uses IX.1.4, which ensures that every element of $[S^V, S^V]_G$ is a suspension. Moreover,

$$\sigma_W : [S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G$$

is a ring homomorphism. For any based G-space Y, the abelian groups $\pi_V^G Y$ and $\pi_{V+W}^G \Sigma^W Y$ may be regarded as modules over $[S^V, S^V]_G$ and $[S^{V+W}, S^{V+W}]_G$, respectively. If $\pi_{V+W}^G \Sigma^W Y$ is regarded as a $[S^V, S^V]_G$-module via the ring homomorphism

$$[S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G,$$

then the map

$$\sigma_W : \pi_V^G Y \to \pi_{V+W}^G \Sigma^W Y$$

is a $[S^V, S^V]_G$-module homomorphism. The usual change of rings functor converts the $[S^V, S^V]_G$-module $\pi_V^G Y$ into the $[S^{V+W}, S^{V+W}]_G$-module

$$\pi_V^G Y \otimes_{[S^V, S^V]_G} [S^{V+W}, S^{V+W}]_G.$$

The homomorphism σ_W induces an $[S^{V+W}, S^{V+W}]_G$-module homomorphism

$$\widehat{\sigma}_W : \ \pi_V^G Y \otimes_{[S^V,S^V]_G} [S^{V+W}, S^{V+W}]_G \ \ \to \ \ \pi_{V+W}^G \Sigma^W Y.$$

The alternative suspension theorem should, in this oversimplified form, assert that the map $\widehat{\sigma}_W$, rather than σ_W, is an isomorphism or epimorphism.

We would also like to obtain an equivariant Hurewicz theorem along the same lines. Again, to convey some intuition for what we hope to prove, we begin with an oversimplified version of the desired theorem. If one has a sufficiently slick definition of the homology group $H_V^G Y$, then it is obvious that this group is a module over the ring $[S^V, S^V]_G$. Moreover, there is an equivariant Hurewicz map

$$h \ : \ \pi_V^G Y \ \to \ H_V^G Y$$

that is a $[S^V, S^V]_G$-module homomorphism. However, the group $H_V^G Y$ carries a far richer structure than just that of a $[S^V, S^V]_G$-module. For any G-representation W, there is a homology suspension isomorphism $H_V^G Y \cong H_{V+W}^G \Sigma^W Y$. Here, our assumption that V contains at least two copies of the trivial representation removes the need to worry about reduced and unreduced homology. This isomorphism indicates that $H_V^G Y$ actually carries the structure of a $[S^{V+W}, S^{V+W}]_G$-module. A bit of fiddling with the definitions indicates that the $[S^V, S^V]_G$-module structure on $H_V^G Y$ is just that obtained by restricting the $[S^{V+W}, S^{V+W}]_G$-module structure along the ring homomorphism

$$\sigma_W : \ [S^V, S^V]_G \ \ \to \ \ [S^{V+W}, S^{V+W}]_G.$$

Since this is true for every G-representation W, what we have on $H_V^G Y$ is a coherent family of $[S^{V+W}, S^{V+W}]_G$-module structures for all possible representations W. This suggests that we introduce a new ring in which we let W go to infinity. This ring ought to be defined as some sort of colimit of the rings $[S^{V+W}, S^{V+W}]_G$, where W ranges over all possible finite-dimensional representations of G.

As was explained in IX§§3,4, we use a complete G-universe U to make this colimit precise. With the notations there, the ring structure on $B_G = \{S^0, S^0\}_G$ is that inherited from the ring structures on the $[S^V, S^V]_G$. Since U is complete, it contains a copy of every representation V. Selecting one of these copies, we obtain a ring homomorphism

$$\sigma_\infty : \ [S^V, S^V]_G \ \ \to \ \ B_G.$$

It can be shown that σ_∞ is actually independent of the choice of the copy of V in U. It follows from our observation about the module structures on $H_V^G Y$ that $H_V^G Y$ carries the structure of a B_G-module. Moreover, its natural $[S^V, S^V]_G$-module structure is just that obtained by restricting the B_G-module structure along σ_∞. The Hurewicz map

$$h : \pi_V^G Y \to H_V^G Y$$

induces a map

$$\widehat{h} : \pi_V^G Y \otimes_{[S^V, S^V]_G} B_G \to H_V^G Y$$

of B_G-modules. In this oversimplified outline form, our equivariant Hurewicz theorem gives conditions under which the map $\widehat{h}$, rather than the map h, is an isomorphism.

The proposed equivariant suspension and Hurewicz theorems may seem more reasonable if one considers the nonequivariant Hurewicz theorem in dimension 1. This result asserts that, if Y is connected, then the map $h : \pi_1 Y \to H_1 Y$ induces an isomorphism between $H_1 Y$ and the abelianization of $\pi_1 Y$. We are encountering the same sort of phenomenon in the equivariant context—that is, we are trying to compare two objects which carry rather different structures. The two objects become isomorphic when we modify the less well-structured one to have the same sort of structure as that carried by the nicer object.

4. The statements of the theorems

The oversimplification in the introduction to our two theorems comes from the fact that, in order to understand the maps

$$\sigma_W : \pi_V^G Y \to \pi_{V+W}^G \Sigma^W Y$$

and

$$h : \pi_V^G Y \to H_V^G Y$$

fully, one must look not only at the group $\pi_V^G Y$, but also at the groups $\pi_V^K Y$ for all the subgroups K of G. The maps $\widehat{\sigma}_W$ and $\widehat{h}$ constructed in the rough sketch of our results do not take into account the influence that the groups $\pi_V^K Y$ have on the maps σ_W and h. In order to take this influence into account, we must replace the ring $[S^V, S^V]_G$ with a small Ab-category $\mathscr{B}_G(V)$ and replace the module $\pi_V^G Y$ with a contravariant additive functor $\underline{\pi}_V^G Y$ from $\mathscr{B}_G(V)$ into the category Ab of abelian groups. The category $\mathscr{B}_G(V)$ and the functor $\underline{\pi}_V^G Y$ should be regarded as bookkeeping devices that allow us to keep track of the influence of the groups $\pi_V^K Y$ on the maps σ_W and h.

Recall the definitions of the Burnside category $\mathscr{B}_G$ and of Mackey functors from IX.4.1 and IX.4.2.

DEFINITION 4.1. (a) Let V be a finite-dimensional representation of G that contains at least two copies of the trivial representation. The *V-Burnside category* $\mathscr{B}_G(V)$ has as its objects the orbits G/K. The set of morphisms from G/K to G/J in $\mathscr{B}_G(V)$ is $[\Sigma^V G/K_+, \Sigma^V G/J_+]_G$. Note that the morphism sets of $\mathscr{B}_G(V)$ are abelian groups.

(b) If V and W are G-representations of G, then suspension gives a functor

$$s : \mathscr{B}_G(V) \rightarrow \mathscr{B}_G(V+W)$$

that is the identity on objects. Moreover, any inclusion of V into the G-universe U gives a functor

$$s_\infty : \mathscr{B}_G(V) \rightarrow \mathscr{B}_G$$

that is also the identity on objects. It can be shown that the functor s_∞ is independent of the choice of the copy of V in U.

Motivated by the interpretation of contravariant additive functors $\mathscr{B}_G \longrightarrow \mathscr{A}b$ as Mackey functors, we refer to contravariant additive functors $\mathscr{B}_G(V) \longrightarrow \mathscr{A}b$ as V-Mackey functors for any compact Lie group G and G-representation V. The category of V-Mackey functors and natural transformations between such is denoted $\mathscr{M}_G(V)$. The category of Mackey functors is denoted $\mathscr{M}_G$.

EXAMPLES 4.2. (a) If V is a representation of G that contains at least two copies of the trivial representation and Y is a G-space, then the homotopy group $\pi_V^G Y$ can be extended to a V-Mackey functor $\underline{\pi}_V^G Y$. For $K \leq G$, we define $(\underline{\pi}_V^G Y)(G/K)$ to be the group

$$[\Sigma^V G/K_+, Y]_G \cong [S^V, Y]_K = \pi_V^K Y.$$

The effect of a morphism f in $\mathscr{B}_G(V)(G/K, G/J) = [\Sigma^V G/K_+, \Sigma^V G/J_+]_G$ on $(\underline{\pi}_V^G Y)(G/J)$ is just that of precomposition by f.

(b) If V is a G-representation and Y is a G-space, then the homology group $H_V^G Y$ can be extended to a Mackey functor $\underline{H}_V^G Y$. If $K \leq G$, then

$$(\underline{H}_V^G Y)(G/K) = H_V^K Y.$$

The functoriality of $\underline{H}_V^G Y$ on $\mathscr{B}_G$ will be apparent from the spectrum level construction of XIII§4.

Our actual equivariant suspension and Hurewicz theorems describe the relations among the functors $\underline{\pi}_V^G Y$, $\underline{\pi}_{V+W}^G \Sigma^W Y$, and $\underline{H}_V^G Y$. In order to state these theorems, we must introduce the change of category functors that replace the change of ring functors that were used in the intuitive presentation of our results.

Definition 4.3. (a) Precomposition by the functors s and s_∞ of Definition 4.1 gives functors

$$s^* : \mathscr{M}_G(V+W) \rightarrow \mathscr{M}_G(V)$$

and

$$s_\infty^* : \mathscr{M}_G \rightarrow \mathscr{M}_G(V).$$

These functors have left adjoints

$$s_* : \mathscr{M}_G(V) \rightarrow \mathscr{M}_G(V+W)$$

and

$$s_*^\infty : \mathscr{M}_G(V) \rightarrow \mathscr{M}_G$$

that are given categorically by left Kan extension.

(b) The suspension maps

$$\sigma_W^K : \ \pi_V^K Y \rightarrow \pi_{V+W}^K \Sigma^W Y,$$

as K varies over the subgroups of G, fit together to form a natural transformation

$$\sigma_W : \ \underline{\pi}_V^G Y \rightarrow s^* \underline{\pi}_{V+W}^G \Sigma^W Y.$$

The adjoint of this map under the (s_*, s^*)-adjunction is denoted

$$\widetilde{\sigma}_W : \ s_* \underline{\pi}_V^G Y \rightarrow \underline{\pi}_{V+W}^G \Sigma^W Y.$$

(c) The Hurewicz maps

$$h_W^K : \ \pi_V^K Y \rightarrow H_V^K Y,$$

as K varies over the subgroups of G, fit together to form a natural transformation

$$h : \ \underline{\pi}_V^G Y \rightarrow s_\infty^* \underline{H}_V^G Y.$$

The adjoint of this map under the (s_*^∞, s_∞^*)-adjunction is denoted

$$\widetilde{h} : \ s_*^\infty \underline{\pi}_V^G Y \rightarrow \underline{H}_V^G Y.$$

It is the maps $\widetilde{\sigma}_W$ and $\widetilde{h}$ that play the role in the precise statements of our Hurewicz and suspension theorems that was played by the maps $\widehat{\sigma}_W$ and $\widehat{h}$ in our intuitive sketch of these results.

Theorem 4.4 (Hurewicz). Let Y be a based G-CW complex and let V be a representation of G that contains at least two copies of the trivial representation. Then the following two conditions are equivalent.

(i) Y is $|(V-1)^*|$-connected.

(ii) Y is simply G-connected and homologically $|(V-1)^*|$-connected.

Moreover, if W is any representation of G such that $2^* \leq |W^*| \leq |V^*|$, then either of these conditions implies that the map

$$\widetilde{h} : s_*^\infty \underline{\pi}_W^G Y \to \underline{H}_W^G Y$$

is an isomorphism and that both $\underline{\pi}_W^G Y$ and $\underline{H}_W^G Y$ are zero if $|W^*| < |V^*|$.

THEOREM 4.5 (FREUDENTHAL SUSPENSION). Let V and W be representations of G and let Y be a based G-CW complex. If V contains at least two copies of the trivial representation and Y is $|(V-1)^*|$-connected, then the suspension map

$$\widetilde{\sigma}_W : \; s_* \underline{\pi}_V^G Y \;\; \to \;\; \underline{\pi}_{V+W}^G \Sigma^W Y$$

is an isomorphism.

There are several ways in which these two theorems are a bit disappointing. One of the most obvious is that, in our anticipated applications, we expect to be able to compute $\underline{H}_W^G Y$ and $\underline{\pi}_{V+W}^G \Sigma^W Y$, and we want to derive information about $\underline{\pi}_V^G Y$ from these computations. The presence of the functors s_*^∞ and s_* would seem to make it difficult to learn much about $\underline{\pi}_V^G Y$ in this fashion. However, the following lemma ensures that we can, at least, detect the vanishing of $\underline{\pi}_V^G Y$ with these two theorems.

LEMMA 4.6. Let V be a representation of G that contains at least two copies of the trivial representation and M be a V-Mackey functor. Then the following are equivalent:

(i) $M = 0$.
(ii) $s_* M = 0$ for any representation W of G.
(iii) $s_*^\infty M = 0$.

Moreover, the explicit descriptions of the functors s_* and s_*^∞ given in [L1, L2] can be used to extract some information about $\underline{\pi}_V^G Y$ from a knowledge of $s_* \underline{\pi}_V^G Y$ or $s_*^\infty \underline{\pi}_V^G Y$ even in the cases where $\underline{\pi}_V^G Y$ does not vanish.

A second disappointment in these two theorems is that they say nothing about the case in which V contains only one copy of the trivial representation. In this context, $\pi_V^G Y$ need not be an abelian group, but one would expect generalizations of our two theorems which relate the abelianization of $\pi_V^G Y$ to $H_W^G Y$ and $\pi_{V+W}^G \Sigma^W Y$ (or more precisely, to $\underline{H}_W^G Y$ and $\underline{\pi}_{V+W}^G \Sigma^W Y$). Generalizations of this form are given in [L1]. They are omitted here because including them would require introducing some unpleasant technicalities that would only obscure the central ideas.

A third disappointment is that, in our suspension theorem, Y is required to be $|(V-1)^*|$-connected, whereas one would expect that connectivity on the order of $|V^*|/2$ would suffice. There are counterexamples (see [L1]) which show that there is no simple way to weaken this connectivity condition on Y. The source of this problem is that the functor s_* is not exact. It is therefore able to capture the effects of suspension only in the lowest dimensions. There is, however, a spectral sequence whose E^2-term is formed from the homotopy groups of Y. This spectral sequence converges to the homotopy groups of $\Sigma^W Y$ in the range of dimensions that one would expect based on the connectivity restrictions in Theorem B; see [L3].

A further disappointing aspect of our suspension theorem is that it applies only to the homotopy groups $\pi_V^G Y$ and not to the set $[X,Y]_G$ of G-homotopy classes of G-maps out of an arbitrary space X. This restriction seems to be unavoidable in the equivariant context.

5. Sketch proofs of the theorems

We turn now to the matter of proving our two theorems. The equivariant Hurewicz theorem follows almost trivially from the equivariant suspension theorem if one is willing to use a little equivariant stable homotopy theory. We will devote our attention to the proof of the suspension theorem. The best way to gain insight into the proof is to look at a rather nonstandard proof of a special case of the corresponding nonequivariant result. This nonstandard proof uses nothing more than two rather simple facts about Eilenberg-Mac Lane spaces and a simple lemma from category theory.

Recall that, if n is a positive integer and M is an abelian group, then the Eilenberg-Mac Lane space $K(M,n)$ is a CW-complex such that $\pi_n K(M,n) = M$ and $\pi_j K(M,n) = 0$ for $j \neq n$. This property characterizes $K(M,n)$ up to homotopy. The first fact that we need about Eilenberg-Mac Lane spaces is that, for any positive integer n and any abelian group M, $\Omega K_{n+1} M \simeq K(M,n)$. This fact follows immediately from a computation of the homotopy groups of $\Omega K(M,n+1)$. If X is any based space, then taking n^{th} homotopy groups gives a map

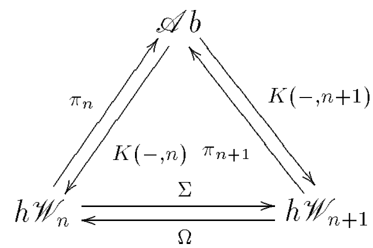
$$\pi : [X, K(M,n)] \longrightarrow \hom(\pi_n X, \pi_n K(M,n)) = \hom(\pi_n X, M)$$

from the set $[X, K(M,n)]$ of based homotopy classes of maps from X into $K(M,n)$ to the set $\hom(\pi_n X, M)$ of group homomorphisms from $\pi_n X$ to M. Since the Eilenberg-Mac Lane space $K(M,n)$ represents cohomology in dimension n with

M coefficients, the set $[X, K(M,n)]$ is just $H^n(X;M)$. It follows easily from the nonequivariant Hurewicz theorem and the universal coefficient theorem that the map π is an isomorphism if X is an $(n-1)$-connected CW-complex. Homotopy theorists use this observation on a regular basis.

For our proof of the nonequivariant suspension theorem, we need a categorical interpretation of this result. Let $\mathscr{W}_n$ be the category of $(n-1)$-connected based spaces that have the homotopy types of CW-complexes, and let $h\mathscr{W}_n$ be the associated (based) homotopy category. Then the assignment of the Eilenberg-Mac Lane space $K(M,n)$ to the abelian group M gives a functor $K(-,n)$ from the category Ab of abelian groups to the category $h\mathscr{W}_n$. On the other hand, taking n^{th} homotopy groups gives a functor π_n from $h\mathscr{W}_n$ to Ab. Our assertion that the map π above is an isomorphism when X is $(n-1)$-connected translates formally into the categorical assertion that the functor $K(-,n)$ is right adjoint to the functor π_n. This adjunction is the second fact about Eilenberg-Mac Lane spaces that we need.

Now consider the diagram of categories and functors



The functor Σ is left adjoint to the functor Ω. Thus, we have two functors, $\Omega K(-,n+1)$ and $K(-,n)$, from Ab to $h\mathscr{W}_n$ with left adjoints $\pi_{n+1}\circ\Sigma$ and π_n, respectively. The homotopy equivalences $\Omega K_{n+1}M \simeq K(M,n)$ fit together to give a natural isomorphism between the functors $\Omega K(-,n+1)$ and $K(-,n)$. The following easy lemma from category theory allows us to convert this natural isomorphism into a nonequivariant suspension theorem.

LEMMA 5.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, $R_1, R_2 : \mathscr{C} \to \mathscr{D}$ be functors from $\mathscr{C}$ to $\mathscr{D}$, and $L_1, L_2 : \mathscr{D} \to \mathscr{C}$ be functors from $\mathscr{D}$ to $\mathscr{C}$ such that L_i is left adjoint to R_i. Then there is a one-to-one correspondence between natural transformations $\tau : R_1 \to R_2$ and natural transformations $\tilde{\tau} : L_2 \to L_1$. Moreover, the natural tranformation $\tau : R_1 \to R_2$ is a natural isomorphism if and only if the associated natural transformation $\tilde{\tau} : L_2 \to L_1$ is a natural isomorphism.

The lemma gives us a natural isomorphism $\pi_n Y \longrightarrow \pi_{n+1}\Sigma Y$ for $(n-1)$-connected spaces Y of the homotopy types of CW-complexes. By examining the proof of the lemma and chasing a few diagrams, it is possible to see that this isomorphism is, in fact, the usual suspension map $\sigma : \pi_n Y \longrightarrow \pi_{n+1}\Sigma Y$.

This nonequivariant suspension theorem is, of course, substantially weaker than Theorem B because it requires much more connectivity of Y and because it applies only to the homotopy group $\pi_n Y$ rather than to an arbitrary set $[X, Y]$ of homotopy classes of maps. However, counterexamples exist which show that limitations of this sort are an essential part of an equivariant suspension theorem. Thus, our alternative approach to proving the nonequivariant suspension theorem is an ideal approach to proving the equivariant theorem.

Let V be a representation of G that contains at least two copies of the trivial representation. Let $\mathscr{W}_G(V)$ be the category of based $|(V-1)^*|$-connected G-spaces that have the G-homotopy types of G-CW complexes, and let $h\mathscr{W}_G(V)$ be the associated homotopy category; its morphisms are based G-homotopy classes of based G-maps.

To prove our equivariant suspension theorem, we must associate an Eilenberg-Mac Lane space $K_G(M, V)$ to each V-Mackey functor M in such a way that we obtain a functor from $\mathscr{M}_G(V)$ to $h\mathscr{W}_G(V)$. We must show that this functor is right adjoint to the functor $\underline{\pi}_V^G : h\mathscr{W}_G(V) \to \mathscr{M}_G(V)$. Then we must demonstrate that, if N is a $(V+W)$-Mackey functor, there is a G-homotopy equivalence $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$. Here, the functor s^* enters in a way that no analogous functor appears in the nonequivariant case because, in the equivariant case, the functors $\underline{\pi}_V^G$ and $\underline{\pi}_{V+W}^G$ land in different categories, whereas the functors π_n and π_{n+1} both produce abelian groups in the nonequivariant case. Now consider the diagram

$$\begin{array}{ccc}
\mathscr{M}_G(V) & \underset{s^*}{\overset{s_*}{\rightleftarrows}} & \mathscr{M}_G(V+W) \\
{\scriptstyle \underline{\pi}_V^G}\uparrow\downarrow{\scriptstyle K_G(-,V)} & & {\scriptstyle \underline{\pi}_{V+W}^G}\uparrow\downarrow{\scriptstyle K_G(-,V+W)} \\
h\mathscr{W}_G(V) & \underset{\Omega^W}{\overset{\Sigma^W}{\rightleftarrows}} & h\mathscr{W}_G(V+W)
\end{array}$$

of categories and functors. The composites $\Omega^W K_G(-, V+W)$ and $K_G(s^*-, V)$ have left adjoints $\underline{\pi}_{V+W}^G\Sigma^W$ and $s_*\underline{\pi}_V^G$ respectively. Thus the natural isomorphism $\Omega^W K_G(-, V+W) \longrightarrow K_G(s^*-, V)$ that is derived from our G-homotopy equivalences $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$ implies a natural isomorphism $s_*\underline{\pi}_V^G \longrightarrow \underline{\pi}_{V+W}^G\Sigma^W$. Again, a bit of diagram chasing confirms that this isomorphism is just the standard suspension map

$$\widetilde{\sigma}_W : s_* \underline{\pi}_V^G \longrightarrow \underline{\pi}_{V+W}^G \Sigma^W.$$

It is easy enough to say what a V-Eilenberg-Mac Lane space ought to be.

DEFINITION 5.2. Let V be a representation of G that contains at least two copies of the trivial representation and M be a V-Mackey functor. A *V-Eilenberg-Mac Lane space* $K_G(M,V)$ is a based, $|(V-1)^*|$-connected G-space $K_G(M,V)$ of the G-homotopy type of a G-CW complex such that $\underline{\pi}_V^G K_G(M,V) = M$ and $\underline{\pi}_{V+k}^G K_G(M,V) = 0$ for $k > 0$.

The problem is to show that such spaces exist, that the assignment of $K_G(M,V)$ to M gives a functor from $\mathscr{M}_G(V)$ to $h\mathscr{W}_G(V)$, and that this functor is right adjoint to $\underline{\pi}_V^G$. In order to fill in these details, we utilize a variant of the G-CW(V) complexes that Waner described in the previous chapter. Waner worked with unbased complexes and adjoined his cells using unbased maps. The variant with which we must work is that of based complexes formed using based attaching maps. We take our cells to be the cones on spheres of the form $\Sigma^{V+k} G/K_+$, where $k \geq -1$ and K runs over the (closed) subgroups of G. A based G-CW(V) complex is then a G-space Y together with a sequence $\{Y^k\}_{k \geq -1}$ of closed subspaces such that Y^{-1} is a point, Y^{k+1} is the cofibre of a based map $\lambda : \bigvee_{j \in J_k} \Sigma^{V+k} G/K_j \to Y^k$ for some indexing set J_k and some collection $\{K_j\}_{j \in J_k}$ of subgroups of G, and Y is the colimit of the Y^k.

There is a general theory of abstract CW complexes that applies to spaces constructed in this form. This theory ensures that G-CW(V) complexes have all the nice properties that one might expect. For us, their most important properties are that they have the homotopy types of G-CW complexes, that they are $|(V-1)^*|$-connected, and that they can be used to approximate, up to weak G-equivalence, any G-space that is $|(V-1)^*|$-connected. Using G-CW(V) complexes, one can construct a V-Eilenberg-Mac Lane space $K_G(M,V)$ for any V-Mackey functor M by attaching cells of the form $C\Sigma^{V+k} G/K_+$ in exactly the same way that one constructs ordinary, nonequivariant, Eilenberg-Mac Lane spaces by attaching ordinary cells.

As in the nonequivariant context, there is a map

$$\pi : [X, K_G(M,V)]_G \longrightarrow \hom(\underline{\pi}_V^G X, \underline{\pi}_V^G K(M,n)) = \hom(\underline{\pi}_V^G X, M)$$

given by taking V^{th} homotopy "groups". Here, hom means the set of natural transformations between two functors in $\mathscr{M}_G(V)$. If X is $|(V-1)^*|$-connected, then it can be approximated by a G-CW(V) complex. This approximation can be used to show that the map π is an isomorphism. We proved the analogous result in the nonequivariant context using the Hurewicz theorem and the universal coefficient theorem. It can, however, be just as easily proved by using a CW approximation to X and arguing inductively up the skeleton of the approximation.

From here, the second approach to the nonequivariant result generalizes without any trouble to the equivariant context. The fact that π is an isomorphism when X is $|(V-1)^*|$-connected can be used to show that the assignment of $K_G(M,V)$ to M gives a functor and that this functor is right adjoint to $\underline{\pi}_V^G$. It can also be used to construct a G-homotopy equivalence between $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$ for any $(V+W)$-Mackey functor N. This completes the proof of the equivariant suspension theorem.

Scholium 5.3. This presentation has been an overview of the papers [L1] and [L2]. Reference [L1] provides full details on everything that has been said here about the equivariant suspension theorem. It includes a careful treatment of based G-CW(V) complexes and of V-Eilenberg-Mac Lane spaces. In that paper, V is assumed to have at least one copy, rather than at least two copies, of the trivial representation. Thus the theorems in [L1] are more general in that they describe the effects of the presence of a nontrivial fundamental group on the suspension and Hurewicz maps. However, this extra generality necessitates several unpleasant technical complications in the arguments that obscure the basic simplicity of the ideas. Reference [L2] is an older paper and in some respects obsolete. Its most important results, the absolute and relative unstable Hurewicz theorems (Theorems 1.7 and 1.8), are restated in a better and more general form as Theorems 2.8 and 2.9 of [L1]. The improved versions of these theorems take into account the results in [L1] dealing with the case in which V contains only one copy of the trivial representation. On the positive side, [L2] contains a description of the structure of the categories $\mathscr{B}_G(V)$ and of the functors s_* and s_*^∞. It contains the proof of Lemma 2.2 above on the equivalence of V- and $|V^*|$-connectivity in the case when G is a compact Lie group; Waner proved this result only for finite groups. Lemma 4.6 above on the vanishing of s_*M and $s_*^\infty M$ is also proved in [L2]. The definitions of the absolute and relative stable and unstable Hurewicz maps are contained in [L2]. The proof of the stable Hurewicz isomorphism theorem in section 2 of [L2] is a simple application of some of the basic techniques in equivariant stable homotopy theory that will be covered in later chapters. Going over that argument is a good way to solidify one's grasp on these basic tricks. Reference [L2] also contains a description of the process for deriving the relative Hurewicz theorem from the absolute Hurewicz theorem. All of the other arguments in [L2], and especially those in sections 5 and 6, are correct but obsolete. I developed them before I became aware of the basic connection between equivariant Eilenberg-Mac Lane spaces and the equivariant suspension theorem. The results presented in section 6 of [L2] are presented in a better and more general form in [L3], which is, essentially, an extension of [L1] from the realm of equivariant unstable homotopy theory to that of equivariant stable homotopy theory.

CHAPTER XII

The Equivariant Stable Homotopy Category

1. An introductory overview

Let us start nonequivariantly. As the home of stable phenomena, the subject of stable homotopy theory includes all of homology and cohomology theory. Over thirty years ago, it became apparent that very significant benefits would accrue if one could work in an additive triangulated category whose objects were "stable spaces", or "spectra", a central point being that the translation from topology to algebra through such tools as the Adams spectral sequence would become far smoother and more structured. Here "triangulated" means that one has a suspension functor that is an equivalence of categories, together with cofibration sequences that satisfy all of the standard properties.

The essential point is to have a smash product that is associative, commutative, and unital up to coherent natural isomorphisms, with unit the sphere spectrum S. A category with such a product is said to be "symmetric monoidal". This structure allows one to transport algebraic notions such as ring and module into stable homotopy theory. Thus, in the stable homotopy category of spectra — which we shall denote by $\bar{h}\mathscr{S}$ — a ring is just a spectrum R together with a product $\phi : R \wedge R \longrightarrow R$ and unit $\eta : S \longrightarrow R$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{array}{ccccc} S \wedge R & \xrightarrow{\eta \wedge 1} & R \wedge R & \xleftarrow{1 \wedge \eta} & R \wedge S \\ & {\scriptstyle\simeq}\searrow & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle\simeq} & \\ & & R & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{1 \wedge \phi} & R \wedge R \\ {\scriptstyle \phi \wedge 1}\downarrow & & \downarrow{\scriptstyle \phi} \\ R \wedge R & \xrightarrow{\phi} & R. \end{array}$$

The unlabelled isomorphisms are canonical isomorphisms giving the unital property, and we have suppressed associativity isomorphisms in the second diagram. Similarly, there is a transposition isomorphism $\tau : E \wedge F \longrightarrow F \wedge E$ in $\bar{h}\mathscr{S}$, and R is said to be commutative if the following diagram commutes in $\bar{h}\mathscr{S}$:

$$\begin{array}{ccc} R \wedge R & \xrightarrow{\ \tau\ } & R \wedge R \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & R. & \end{array}$$

A left R-module is a spectrum M together with a map $\mu : R \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{array}{ccc} S \wedge M & \xrightarrow{\eta \wedge 1} & R \wedge M \\ & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle \mu} \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{1 \wedge \mu} & R \wedge M \\ {\scriptstyle \phi \wedge 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ R \wedge M & \xrightarrow{\ \mu\ } & M. \end{array}$$

Over twenty years ago, it became apparent that it would be of great value to have more precisely structured notions of ring and module, with good properties before passage to homotopy. For example, when one is working in $\bar{h}\mathscr{S}$ it is not even true that the cofiber of a map of R-modules is an R-module, so that one does not have a triangulated category of R-modules. More deeply, when R is commutative, one would like to be able to construct a smash product $M \wedge_R N$ of R-modules. Quinn, Ray, and I defined such structured ring spectra in 1972. Elmendorf and I, and independently Robinson, defined such structured module spectra around 1983. However, the problem just posed was not fully solved until after the Alaska conference, in work of Elmendorf, Kriz, Mandell, and myself. We shall return to this later.

For now, let us just say that the technical problems focus on the construction of an associative and commutative smash product of spectra. Before June of 1993, I would have said that it was not possible to construct such a product on a category that has all colimits and limits and whose associated homotopy category is equivalent to the stable homotopy category. We now have such a construction, and it actually gives a point-set level symmetric monoidal category.

However, it is not a totally new construction. Rather, it is a natural extension of the approach to the stable category $\bar{h}\mathscr{S}$ that Lewis and I developed in the early 1980's. Even from the viewpoint of classical nonequivariant stable homotopy theory, this approach has very significant advantages over any of its predecessors.

What is especially relevant to us is that it is the only approach that extends effortlessly to the equivariant context, giving a good stable homotopy category of G-spectra for any compact Lie group G. Moreover, for a great deal of the homotopical theory, the new point-set level construction offers no advantages over the original Lewis-May theory: the latter is by no means rendered obsolete by the new theory.

From an expository point of view this raises a conundrum. The only real defect of the Lewis-May approach is that the only published account is in the general equivariant context, with emphasis on those details that are special to that setting. Therefore, despite the theme of this book, I will first outline some features of the theory that are nearly identical in the nonequivariant and equivariant contexts, returning later to a discussion of significant equivariant points. I will follow in part an unpublished exposition of the Lewis-May category due to Jim McClure. A comparison with earlier approaches and full details of definitions and proofs may be found in the encyclopedic first reference below. The second reference contains important technical refinements of the theory, as well as the new theory of highly structured ring and module spectra. The third reference gives a brief general overview of the theory that the reader may find helpful. We shall often refer to these as [LMS], [EKMM], and [EKMM′].

General References

[LMS] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics. Vol. 1213. 1986.
[EKMM] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Preprint, 1995.
[EKMM′] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 213-254.

2. Prespectra and spectra

The simplest relevant notion is that of a *prespectrum* E. The naive version is a sequence of based spaces E_n, $n \geq 0$, and based maps

$$\sigma_n : \Sigma E_n \longrightarrow E_{n+1}.$$

A map $D \longrightarrow E$ of prespectra is a sequence of maps $D_n \longrightarrow E_n$ that commute with the structure maps σ_n. The structure maps have adjoints

$$\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1},$$

and it is customary to say that E is an Ω-spectrum if these maps are equivalences. While this is the right kind of spectrum for representing cohomology theories on spaces, we shall make little use of this concept. By a *spectrum*, we mean a prespectrum for which the adjoints $\tilde{\sigma}_n$ are *homeomorphisms*. (The insistence on homeomorphisms goes back to a 1969 paper of mine that initiated the present approach to stable homotopy theory.) In particular, for us, an "Ω-spectrum" need not be a spectrum: henceforward, we use the more accurate term Ω-prespectrum for this notion.

One advantage of our definition of a spectrum is that the obvious forgetful functor from spectra to prespectra — call it ℓ — has a left adjoint spectrification functor L such that the canonical map $L\ell E \longrightarrow E$ is an isomorphism. Thus there is a formal analogy between L and the passage from presheaves to sheaves, which is the reason for the term "prespectrum". The category of spectra has limits, which are formed in the obvious way by taking the limit for each n separately. It also has colimits. These are formed on the prespectrum level by taking the colimit for each n separately; the spectrum level colimit is then obtained by applying L.

The central technical issue that must be faced in any version of the category of spectra is how to define the smash product of two prespectra $\{D_n\}$ and $\{E_n\}$. Any such construction must begin with the naive bi-indexed smash product $\{D_m \wedge E_n\}$. The problem arises of how to convert it back into a singly indexed object in some good way. It is an instructive exercise to attempt to do this directly. One quickly gets entangled in permutations of suspension coordinates. Let us think of a circle as the one-point compactification of $\mathbb{R}$ and the sphere S^n as the one-point compactification of $\mathbb{R}^n$. Then the iterated structure maps $\Sigma^n E_m = E_m \wedge S^n \longrightarrow E_{m+n}$ seem to involve $\mathbb{R}^n$ as the last n coordinates in $\mathbb{R}^{m+n}$. This is literally true if we consider the sphere prespectrum $\{S^n\}$ with identity structural maps. This suggests that our entanglement really concerns changes of basis. If so, then we all know the solution: do our linear algebra in a coordinate-free setting, choosing bases only when it is convenient and avoiding doing so when it is inconvenient.

Let $\mathbb{R}^\infty$ denote the union of the $\mathbb{R}^n$, $n \geq 0$. This is a space whose elements are sequences of real numbers, all but finitely many of which are zero. We give it the evident inner product. By a universe U, we mean an inner product space isomorphic to $\mathbb{R}^\infty$. If V is a finite dimensional subspace of U, we refer to V as an indexing space in U, and we write S^V for the one-point compactification of V, which is a based sphere. We write $\Sigma^V X$ for $X \wedge S^V$ and $\Omega^V X$ for $F(S^V, X)$.

By a prespectrum indexed on U, we mean a family of based spaces EV, one for each indexing space V in U, together with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} EV \longrightarrow EW$$

whenever $V \subset W$, where $W - V$ denotes the orthogonal complement of V in W. We require $\sigma_{V,V} = \mathrm{Id}$, and we require the evident transitivity diagram to commute for $V \subset W \subset Z$:

$$\begin{array}{ccc} \Sigma^{Z-W}\Sigma^{W-V} EV & \longrightarrow & \Sigma^{Z-W} EW \\ \cong \downarrow & & \downarrow \\ \Sigma^{Z-V} EV & \longrightarrow & EZ. \end{array}$$

We call E a spectrum indexed on U if the adjoints

$$\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$$

of the structural maps are homeomorphisms. As before, the forgetful functor ℓ from spectra to prespectra has a left adjoint spectrification functor L that leaves spectra unchanged. We denote the categories of prespectra and spectra indexed on U by $\mathscr{P}U$ and $\mathscr{S}U$. When U is fixed and understood, we abbreviate notation to $\mathscr{P}$ and $\mathscr{S}$.

If $U = \mathbb{R}^\infty$ and E is a spectrum indexed on U, we obtain a spectrum in our original sense by setting $E_n = E\mathbb{R}^n$. Conversely, if $\{E_n\}$ is a spectrum in our original sense, we obtain a spectrum indexed on U by setting $EV = \Omega^{\mathbb{R}^n - V} E_n$, where n is minimal such that $V \subset \mathbb{R}^n$. It is easy to work out what the structural maps must be. This gives an isomorphism between our new category of spectra indexed on U and our original category of sequentially indexed spectra.

More generally, it often happens that a spectrum or prespectrum is naturally indexed on some other cofinal set $\mathscr{A}$ of indexing spaces in U. Here cofinality means that every indexing space V is contained in some $A \in \mathscr{A}$; it is convenient to also require that $\{0\} \in \mathscr{A}$. We write $\mathscr{P}\mathscr{A}$ and $\mathscr{S}\mathscr{A}$ for the categories of prespectra and spectra indexed on $\mathscr{A}$. On the spectrum level, all of the categories $\mathscr{S}\mathscr{A}$ are isomorphic since we can extend a spectrum indexed on $\mathscr{A}$ to a spectrum indexed on all indexing spaces V in U by the method that we just described for the case $\mathscr{A} = \{\mathbb{R}^n\}$.

J. P. May. Categories of spectra and infinite loop spaces. Springer Lecture Notes in Mathematics Vol. 99. 1969, 448–479.

3. Smash products

We can now define a smash product. Given prespectra E and E' indexed on universes U and U', we form the collection $\{EV \wedge E'V'\}$, where V and V' run through the indexing spaces in U and U', respectively. With the evident structure maps, this is a prespectrum indexed on the set of indexing spaces in $U \oplus U'$ that are of the form $V \oplus V'$. If we start with spectra E and E', we can apply the functor L to get to a spectrum indexed on this set, and we can then extend the result to a spectrum indexed on all indexing spaces in $U \oplus U'$. We thereby obtain the "external smash product" of E and E',

$$E \wedge E' \in \mathscr{S}(U \oplus U').$$

Thus, if $U = U'$, then two-fold smash products are indexed on U^2, three-fold smash products are indexed on U^3, and so on.

This external smash product is associative up to isomorphism,

$$(E \wedge E') \wedge E'' \cong E \wedge (E' \wedge E'').$$

This is evident on the prespectrum level. It follows on the spectrum level by a formal argument of a sort that pervades the theory. One need only show that, for prespectra D and D',

$$L(\ell L(D) \wedge D') \cong L(D \wedge D') \cong L(D \wedge \ell L(D')).$$

Conceptually, these are commutation relations between functors that are left adjoints, and they will hold if and only if the corresponding commutation relations are valid for the right adjoints. We shall soon write down the right adjoint function spectra functors. They turn out to be so simple and explicit that it is altogether trivial to check the required commutation relations relating them and the right adjoint ℓ.

The external smash product is very nearly commutative, but to see this we need another observation. If $f : U \longrightarrow U'$ is a linear isometric isomorphism, then we obtain an isomorphism of categories $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$ via

$$(f^*E')(V) = E'(fV).$$

Its inverse is $f_* = (f^{-1})^*$. If $\tau : U \oplus U' \longrightarrow U' \oplus U$ is the transposition, then the commutativity isomorphism of the smash product is

$$E' \wedge E \cong \tau_*(E \wedge E').$$

Analogously, the associativity isomorphism implicitly used the obvious isomorphism of universes $(U \oplus U') \oplus U'' \cong U \oplus (U' \oplus U'')$.

What about unity? We would like $E \wedge S$ to be isomorphic to E, but this doesn't make sense on the face of it since these spectra are indexed on different universes. However, for a based space X and a prespectrum E, we have a prespectrum $E \wedge X$ with

$$(E \wedge X)(V) = EV \wedge X.$$

If we start with a spectrum E and apply L, we obtain a spectrum $E \wedge X$. It is quite often useful to think of based spaces as spectra indexed on the universe $\{0\}$. This makes good sense on the face of our definitions, and we have $E \wedge S^0 \cong E$, where S^0 means the space S^0.

Of course, this is not adequate, and we have still not addressed our original problem about bi-indexed smash products: we have only given it a bit more formal structure. To solve these problems, we go back to our "change of universe functors" $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$. Clearly, to define f^*, the map $f : U \longrightarrow U'$ need only be a linear isometry, not necessarily an isomorphism. While a general linear isometry f need not be an isomorphism, it is a monomorphism. For a prespectrum $E \in \mathscr{P}U$, we can define a prespectrum $f_*E \in U'$ by

$$(3.1) \qquad (f_*E)(V') = EV \wedge S^{V'-fV}, \quad \text{where} \quad V = f^{-1}(V' \cap f(U)).$$

Its structure maps are induced from those of E via the isomorphisms

$$(3.2) \qquad EV \wedge S^{V'-fV} \wedge S^{W'-V'} \cong EV \wedge S^{W-V} \wedge S^{W'-fW}.$$

As usual, we use the functor L to extend to a functor $f_* : \mathscr{S}U \longrightarrow \mathscr{S}U'$. As is easily verified on the prespectrum level and follows formally on the spectrum level, the inverse isomorphisms that we had in the case of isomorphisms generalize to adjunctions in the case of isometries:

$$(3.3) \qquad \mathscr{S}U'(f_*E, E') \cong \mathscr{S}U(E, f^*E').$$

How does this help us? Let $\mathscr{I}(U, U')$ denote the set of linear isometries $U \longrightarrow U'$. If V is an indexing space in U, then $\mathscr{I}(V, U')$ has an evident metric topology, and we give $\mathscr{I}(U, U')$ the topology of the union. It is vital — and not hard to prove — that $\mathscr{I}(U, U')$ is in fact a contractible space. As we shall explain later, this can be used to prove a version of the following result (which is slightly misstated for clarity in this sketch of ideas).

THEOREM 3.4. Any two linear isometries $U \longrightarrow U'$ induce canonically and coherently weakly equivalent functors $\mathscr{S}U \longrightarrow \mathscr{S}U'$.

We have not yet defined weak equivalences, nor have we defined the stable category. A map $f : D \longrightarrow E$ of spectra is said to be a weak equivalence if each of its component maps $DV \longrightarrow EV$ is a weak equivalence. Since the smash product of a spectrum and a space is defined, we have cylinders $E \wedge I_+$ and thus a notion of homotopy in $\mathscr{S}U$. We let $h\mathscr{S}U$ be the resulting homotopy category, and we let $\bar{h}\mathscr{S}U$ be the category that is obtained from $h\mathscr{S}U$ by adjoining formal inverses to the weak equivalences. We shall be more explicit later.

This is our stable category, and we proceed to define its smash product. We choose a linear isometry $f : U^2 \longrightarrow U$. For spectra E and E' indexed on U, we define an internal smash product $f_*(E \wedge E') \in \mathscr{S}U$. Up to canonical isomorphism in the stable category $\bar{h}\mathscr{S}U$, $f_*(E \wedge E')$ is independent of the choice of f. For associativity, we have

$$f_*(E\wedge f_*(E'\wedge E'')) \cong (f(1\oplus f))_*(E\wedge E'\wedge E'') \simeq (f(f\oplus 1))_* \cong f_*(f_*(E\wedge E')\wedge E'').$$

Here we write $\cong$ for isomorphisms that hold on the point-set level and $\simeq$ for isomorphisms in the category $\bar{h}\mathscr{S}U$. For commutativity,

$$f_*(E' \wedge E) \cong f_*\tau_*(E \wedge E') \cong (f\tau)_*(E \wedge E') \simeq f_*(E \wedge E').$$

For a space X, we have a suspension prespectrum $\{\Sigma^V X\}$ whose structure maps are identity maps. We let $\Sigma^\infty X = L\{\Sigma^V X\}$. In this case, the construction of L is quite concrete, and we find that

$$(3.5) \qquad \Sigma^\infty X = \{Q\Sigma^V X\}, \quad \text{where} \quad QY = \bigcup \Omega^W \Omega^W Y.$$

This gives the suspension spectrum functor $\Sigma^\infty : \mathscr{T} \longrightarrow \mathscr{S}U$. It has a right adjoint Ω^∞ which sends a spectrum E to the space $E_0 = E\{0\}$:

$$(3.6) \qquad \mathscr{S}U(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E).$$

The functor Q is the same as $\Omega^\infty\Sigma^\infty$. For a linear isometry $f : U \longrightarrow U'$, we have

$$(3.7) \qquad f_*\Sigma^\infty X \cong \Sigma^\infty X$$

since, trivially, $\Omega^\infty f^* E' = E'_0 = \Omega^\infty E'$. A space equivalent to E_0 for some spectrum E is called an infinite loop space.

Remember that we can think of the category $\mathscr{T}$ of based spaces as the category $\mathscr{S}\{0\}$ of spectra indexed on the universe $\{0\}$. With this interpretation, Ω^∞ coincides with i^*, where $i : \{0\} \longrightarrow U$ is the inclusion. Therefore, by the uniqueness of adjoints, $\Sigma^\infty X$ is isomorphic to $i_* X$. Let $i_1 : U \longrightarrow U^2$ be the inclusion of U as the first summand in $U \oplus U$. The unity isomorphism of the smash product is the case $X = S^0$ of the following isomorphism in $\bar{h}\mathscr{S}U$:

(3.8)

$$f_*(E \wedge \Sigma^\infty X) \cong f_*(i_1)_*(E \wedge X) \cong (f \circ i_1)_*(E \wedge X) \simeq 1_*(E \wedge X) = E \wedge X.$$

We conclude that, up to natural isomorphisms that are implied by Theorem 3.4 and elementary inspections, the stable category $\bar{h}\mathscr{S}U$ is symmetric monoidal with respect to the internal smash product $f_*(E \wedge E')$ for any choice of linear isometry $f : U^2 \longrightarrow U$. It is customary, once this has been proven, to write $E \wedge E'$ to mean this internal smash product, relying on context to distinguish it from the external product.

4. Function spectra

We must define the function spectra that give the right adjoints of our various kinds of smash products. For a space X and a spectrum E, the function spectrum $F(X, E)$ is given by

$$F(X, E)(V) = F(X, EV).$$

Note that this is a spectrum as it stands, without use of the functor L. We have the isomorphism

$$F(E \wedge X, E') \cong F(E, F(X, E'))$$

and the adjunction

$$\mathscr{S}U(E \wedge X, E') \cong \mathscr{T}(X, \mathscr{S}U(E, E')) \cong \mathscr{S}U(E, F(X, E')), \tag{4.1}$$

where the set of maps $E \longrightarrow E'$ is topologized as a subspace of the product over all indexing spaces V of the spaces $F(EV, E'V)$. As an example of the use of right adjoints to obtain information about left adjoints, we have isomorphisms

$$(\Sigma^\infty X) \wedge Y \cong \Sigma^\infty(X \wedge Y) \cong X \wedge (\Sigma^\infty Y). \tag{4.2}$$

For the first, the two displayed functors of X both have right adjoint

$$F(Y, E)_0 = F(Y, E_0).$$

More generally, for universes U and U' and for spectra $E' \in \mathscr{S}U'$ and $E'' \in \mathscr{S}(U \oplus U')$, we define an external function spectrum

$$F(E', E'') \in \mathscr{S}U$$

as follows. For an indexing space V in U, define $E''[V] \in \mathscr{S}U'$ by

$$E''[V](V') = E''(V \oplus V').$$

The structural homeomorphisms are induced by some of those of E'', and others give a system of isomorphisms $E''[V] \longrightarrow \Omega^{W-V} E''[W]$. Define

$$F(E', E'')(V) = \mathscr{S}U'(E', E''[V]).$$

We have the adjunction

$$\mathscr{S}(U \oplus U')(E \wedge E', E'') \cong \mathscr{S}U(E, F(E', E'')). \tag{4.3}$$

When $E' = \Sigma^\infty Y$, $\mathscr{S}U'(E', E''[V]) \cong \mathscr{T}(Y, E''(V))$. Thus, if $i_1 : U \longrightarrow U \oplus U'$ is the inclusion, then

$$F(\Sigma^\infty Y, E'') \cong F(Y, (i_1)^* E'').$$

By adjunction, this implies the first of the following two isomorphisms:

$$(i_1)_*((\Sigma^\infty X) \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y \cong (i_2)_*(X \wedge (\Sigma^\infty Y)). \tag{4.4}$$

When $U = U'$ and $f : U^2 \longrightarrow U$ is a linear isometry, we obtain the internal function spectrum $F(E', f^*E) \in \mathscr{S}U$ for spectra $E, E' \in \mathscr{S}U$. Up to canonical isomorphism in $\bar{h}\mathscr{S}U$, it is independent of the choice of f. For spectra all indexed on U, we have the composite adjunction

$$\mathscr{S}U(f_*(E \wedge E'), E'') \cong \mathscr{S}U(E, F(E', f^*E'')). \tag{4.5}$$

Again, it is customary to abuse notation by also writing $F(E', E)$ for the internal function spectrum, relying on the context for clarity. By combining the three isomorphisms (3.7), (4.2), and (4.4) — all of which were proven by trivial inspections of right adjoints — we obtain the following non-obvious isomorphism for internal smash products.

$$\Sigma^\infty(X \wedge Y) \cong (\Sigma^\infty X) \wedge (\Sigma^\infty Y). \tag{4.6}$$

Generalized a bit, this will be seen to determine the structure of smash products of CW spectra.

5. The equivariant case

We now begin working equivariantly, and we have a punch line: we were led to the framework above by nonequivariant considerations about smash products, and yet the framework is ideally suited to equivariant considerations. Let G be a compact Lie group and recall the discussion of G-spheres and G-universes from IX§§1,2. On the understanding that every space in sight is a G-space and every map in sight is a G-map, the definitions and results above apply verbatim to give the basic definitions and properties of the stable category of G-spectra. For a given G-universe U, we write $G\mathscr{S}U$for the resulting category of G-spectra, $hG\mathscr{S}U$ for its homotopy category, and $\bar{h}G\mathscr{S}U$ for the stable homotopy category that results by adjoining inverses to the weak equivalences.

The only caveat is that $\mathscr{I}(U,U')$ is understood to be the G-space of linear isometries, with G acting by conjugation, and not the space of G-linear isometries. If the G-universes U and U' are isomorphic — which means that they contain the same irreducible representations — then $\mathscr{I}(U,U')$ is G-contractible, and therefore its subspace $\mathscr{I}(U,U')^G$ of G-linear isometries is contractible.

We already see something new in the equivariant context: we have different stable categories of G-spectra depending on the isomorphism type of the underlying universe. This fact will play a vital role in the theory. Remember that a G-universe U is said to be complete if it contains every irreducible representation and trivial if it contains only the trivial irreducible representation. We sometimes refer to G-spectra indexed on a complete G-universe U as genuine G-spectra. We always refer to G-spectra indexed on a trivial G-universe, such as U^G, as naive G-spectra; they are essentially just spectra $\{E_n\}$ of the sort we first defined, but with each E_n a G-space and each structure map a G-map. We have concomitant notions of genuine and naive infinite loop G-spaces. The inclusion $i : U^G \longrightarrow U$ gives us an adjoint pair of functors relating naive and genuine G-spectra:

$$G\mathscr{S}U(i_*E, E') \cong G\mathscr{S}U^G(E, i^*E'). \tag{5.1}$$

We will see that naive G-spectra represent $\mathbb{Z}$-graded cohomology theories, whereas genuine G-spectra represent cohomology theories graded over the real representation ring $RO(G)$. Before getting to this, however, we must complete our development of the stable category.

6. Spheres and homotopy groups

We have deliberately taken a more or less geodesic route to smash products and function spectra, and we have left aside a number of other matters that must be considered. At the risk of obscuring the true simplicity of the nonequivariant theory, we work with G-spectra indexed on a fixed G-universe U from now on in this chapter. We write $G\mathscr{S}$ for $G\mathscr{S}U$. Since G will act on everything in sight, we often omit the prefix, writing spectra for G-spectra and so on.

We shall shortly define G-CW spectra in terms of sphere spectra and their cones, which provide cells. We shall deduce properties of G-CW spectra, such as HELP, by reducing to the case of a single cell and there applying an adjunction to reduce to the G-space level. For this, spheres must be defined in terms of suitable left adjoint functors from spaces to spectra. For $n \geq 0$, there is no problem: we take $\underline{S}^n = \Sigma^\infty S^n$. We shall later write S^n ambiguously for both the sphere space and the sphere spectrum, relying on context for clarity, but we had better be pedantic at first.

We also need negative dimensional spheres. We will define them in terms of shift desuspension functors, and these functors will also serve to clarify the relationship between spectra and their component spaces. Generalizing Ω^∞, define a functor

$$\Omega^\infty_V : G\mathscr{S} \longrightarrow G\mathscr{T}$$

by $\Omega^\infty_V = EV$ for an indexing space V in U. The functor Ω^∞_V has a left adjoint shift desuspension functor

$$\Sigma^\infty_V : G\mathscr{S} \longrightarrow G\mathscr{T}.$$

The spectrum $\Sigma^\infty_V X$ is $L\{\Sigma^{W-V}X\}$. Here the prespectrum $\{\Sigma^{W-V}X\}$ has Wth space Σ^{W-V} if $V \subset W$ and a point otherwise; if $V \subset W \subset Z$, then the corresponding structure map is the evident identification

$$\Sigma^{Z-W}\Sigma^{W-V}X \cong \Sigma^{Z-V}X.$$

The Vth space of $\Sigma^\infty_V X$ is the zeroth space QX of $\Sigma^\infty X$. It is easy to check the prespectrum level version of the claimed adjunction, and the spectrum level adjunction follows:

$$G\mathscr{S}(\Sigma^\infty_V X, E) \cong G\mathscr{T}(X, \Omega^\infty_V E). \tag{6.1}$$

Exactly as in (4.2) and (4.6), we have natural isomorphisms

$$(\Sigma^\infty_V X) \wedge Y \cong \Sigma^\infty_V (X \wedge Y) \cong X \wedge (\Sigma^\infty_V Y) \tag{6.2}$$

and, for the internal smash product,

$$\Sigma^\infty_{V+W}(X \wedge Y) \cong \Sigma^\infty_V X \wedge \Sigma^\infty_W Y \quad \text{if} \quad V \cap W = \{0\}. \tag{6.3}$$

Another check of right adjoints gives the relation

$$\Sigma^\infty_V X \cong \Sigma^\infty_W \Sigma^{W-V} X \quad \text{if} \quad V \subset W. \tag{6.4}$$

It is not hard to see that any spectrum E can be written as the colimit of the shift desuspensions of its component spaces. That is,

$$E \cong \operatorname{colim} \Sigma^\infty_V EV, \tag{6.5}$$

where the colimit is taken over the maps

$$\Sigma^\infty_W \sigma : \Sigma^\infty_V EV \cong \Sigma^\infty_W(\Sigma^{W-V} EV) \longrightarrow \Sigma^\infty_W EW.$$

Let us write U in the form $U = U^G \oplus U'$ and fix an identification of U^G with $\mathbb{R}^\infty$. We abbreviate notation by writing Ω^∞_n and Σ^∞_n when $V = \mathbb{R}^n$. Now define $\underline{S}^{-n} = \Sigma^\infty_n S^0$ for $n > 0$. The reader will notice that we can generalize our definitions to obtain sphere spectra $\underline{S}^V$ and $\underline{S}^{-V}$ for any indexing space V. We can even define spheres $\underline{S}^{V-W} = \Sigma^\infty_W S^V$. We shall need such generality later. However, in developing G-CW theory, it turns out to be appropriate to restrict attention to the spheres $\underline{S}^n$ for integers n. Theorem 6.8 will explain why.

In view of our slogan that orbits are the equivariant analogues of points, we also consider all spectra

$$\underline{S}^n_H \equiv G/H_+ \wedge \underline{S}^n, \; H \subset G \quad \text{and} \quad n \in \mathbb{Z}, \tag{6.6}$$

as spheres. By (6.2), $\underline{S}^n_H \cong \Sigma^\infty(G/H_+ \wedge S^n)$ if $n \geq 0$ and $\underline{S}^n_H \cong \Sigma^\infty_n G/H_+$ if $n < 0$. We shall be more systematic about change of groups later, but we prefer to minimize such equivariant considerations in this section. We define the homotopy group systems of G-spectra by setting

$$\pi^H_n(E) = \underline{\pi}_n(E)(G/H) = hG\mathscr{S}(\underline{S}^n_H, E). \tag{6.7}$$

Let $\mathscr{B}_G U$ be the homotopy category of orbit spectra $\underline{S}^0_H = \Sigma^\infty G/H_+$; we generally abbreviate the names of its objects to G/H. This is an additive category, as will become clear shortly, and $\underline{\pi}_n(E)$ is an additive contravariant functor $\mathscr{B}_G U \longrightarrow \mathscr{A}b$. Recall from IX§4 that such functors are called Mackey functors when the universe U is complete. They play a fundamentally important role in equivariant theory, both in algebra and topology, and we shall return to them later. For now, however, we shall concentrate on the individual homotopy groups $\pi_n^H(E)$. We shall later reinterpret these as homotopy groups $\pi_n(E^H)$ of fixed point spectra, but that too can wait.

The following theorem should be viewed as saying that a weak equivalence of G-spectra really is a weak equivalence of G-spectra. Recall that we defined a weak equivalence $f : D \longrightarrow E$ to be a G-map such that each space level G-map $fV : DV \longrightarrow EV$ is a weak equivalence. In setting up CW-theory, which logically should precede the following theorem, one must mean a weak equivalence to be a map that induces an isomorphism on all of the homotopy groups π_n^H of (6.7).

Theorem 6.8. *Let $f : E \longrightarrow E'$ be a map of G-spectra. Then each component map $fV : EV \longrightarrow E'V$ is a weak equivalence of G-spaces if and only if $f_* : \pi_n^H E \longrightarrow \pi_n^H E'$ is an isomorphism for all $H \subset G$ and all integers n.*

By our adjunctions, we have

(6.9) $$\pi_n^H(E) \cong \pi_n((E_0)^H) \text{ if } n \geq 0 \quad\text{and}\quad \pi_n^H(E) \cong \pi_0((E\mathbb{R}^n)^H) \text{ if } n < 0.$$

Therefore, nonequivariantly, the theorem is a tautological triviality. Equivariantly, the forward implication is trivial but the backward implication says that if each $E\mathbb{R}^n \longrightarrow E'\mathbb{R}^n$ is a weak equivalence, then each $EV \longrightarrow E'V$ is also a weak equivalence. Thus it says that information at the trivial representations in U is somehow capturing information at all other representations in U. Its validity justifies the development of G-CW theory in terms of just the sphere spectra of integral dimensions.

We sketch the proof, which goes by induction. We want to prove that each map $f_* : \pi_*(EV)^H \longrightarrow \pi_*(E'V)^H$ is an isomorphism. Since G contains no infinite descending chains of (closed) subgroups, we may assume that f_* is an isomorphism for all proper subgroups of H. An auxiliary argument shows that we may assume that $V^H = \{0\}$. We then use the cofibration $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$, where $S(V)$ and $D(V)$ are the unit sphere and unit ball in V and thus $D(V)_+ \simeq S^0$. Applying $f : F(\cdot, EV)^H \longrightarrow F(\cdot, E'V)^H$ to this cofibration, we obtain a comparison of fibration sequences. On one end, this is

$$f_0 : (\Omega^V EV)^H = (E_0)^H \longrightarrow (E_0')^H = (\Omega^V E'V)^H,$$

which is given to be a weak equivalence. On the other end, we can triangulate $S(V)$ as an H-CW complex with cells of orbit type H/K, where K is a proper subgroup of H. We can then use change of groups and the inductive hypothesis to deduce that f induces a weak equivalence on this end too. Modulo an extra argument to handle π_0, we can conclude that the middle map $f : (EV)^H \longrightarrow (E'V)^H$ is a weak equivalence.

7. G-CW spectra

Before getting to CW theory, we must say something about compactness, which plays an important role. A compact spectrum is one of the form $\Sigma^\infty_V X$ for some indexing space V and compact space X. Since a map of spectra with domain $\Sigma^\infty_V X$ is determined by a map of spaces with domain X, facts about maps out of compact spaces imply the corresponding facts about maps out of compact spectra. For example, if E is the union of an expanding sequence of subspectra E_i, then

$$G\mathscr{S}(\Sigma^\infty_V X, E) \cong \operatorname{colim} G\mathscr{S}(\Sigma^\infty_V X, E_i). \tag{7.1}$$

The following lemma clarifies the relationship between space level and spectrum level maps. Recall the isomorphisms of (6.4).

LEMMA 7.2. Let $f : \Sigma^\infty_V X \longrightarrow \Sigma^\infty_W Y$ be a map of G-spectra, where X is compact. Then, for a large enough indexing space Z, there is a map $g : \Sigma^{Z-V} X \longrightarrow \Sigma^{Z-W} Y$ of G-spaces such that f coincides with

$$\Sigma^\infty_Z g : \Sigma^\infty_V X \cong \Sigma^\infty_Z(\Sigma^{Z-V} X) \longrightarrow \Sigma^\infty_Z(\Sigma^{Z-W} Y) \cong \Sigma^\infty_W Y.$$

This result shows how to calculate the full subcategory of the stable category consisting of those G-spectra of the form $\Sigma^\infty_V X$ for some indexing space V and finite G-CW complex X in space level terms. It can be viewed as giving an equivariant reformulation of the Spanier-Whitehead S-category. In particular, we have the following consistency statement with the definitions of IX§2.

PROPOSITION 7.3. For a finite based G-CW complex X and a based G-space Y,

$$\{X, Y\}_G \cong [\Sigma^\infty X, \Sigma^\infty Y]_G.$$

From here, the development of CW theory is essentially the same equivariantly as nonequivariantly, and essentially the same on the spectrum level as on the space level. The only novelty is that, because we have homotopy groups in negative degrees, we must use two filtrations. Older readers may see more novelty. In contrast with earlier treatments, our CW theory is developed on the spectrum level and has nothing whatever to do with any possible cell structures on the component spaces of spectra. I view the use of space level cell structures in this context as an obsolete historical detour that serves no useful mathematical purpose.

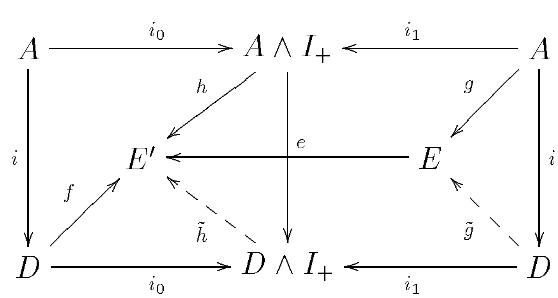
Let $CE = E \wedge I$ denote the cone on a G-spectrum E.

Definition 7.4. A G-cell spectrum is a spectrum $E \in G\mathscr{S}$ that is the union of an expanding sequence of subspectra E_n, $n \geq 0$, such that E_0 is the trivial spectrum (each of its component spaces is a point) and E_{n+1} is obtained from E_n by attaching G-cells $C\underline{S}^q_H \cong G/H_+ \wedge C\underline{S}^q$ along attaching G-maps $\underline{S}^q_H \longrightarrow E_n$. Cell subspectra, or "subcomplexes", are defined in the evident way. A G-CW spectrum is a G-cell spectrum each of whose attaching maps $\underline{S}^q_H \longrightarrow E_n$ factors through a subcomplex that contains only cells of dimension at most q. The n-skeleton E^n is then defined to be the union of the cells of dimension at most n.

Lemma 7.5. A map from a compact spectrum to a cell spectrum factors through a finite subcomplex. Any cell spectrum is the union of its finite subcomplexes.

The filtration $\{E_n\}$ is called the sequential filtration. It records the order in which cells are attached, and it can be chosen in many different ways. In fact, using the lemma, we see that by changing the sequential filtration on the domain, any map between cell spectra can be arranged to preserve the sequential filtration. Using this filtration, we find that the inductive proofs of the following results that we sketched on the space level work in exactly the same way on the spectrum level. We leave it to the reader to formulate their more precise "dimension ν" versions.

Theorem 7.6 (HELP). Let A be a subcomplex of a G-CW spectrum D and let $e : E \longrightarrow E'$ be a weak equivalence. Suppose given maps $g : A \longrightarrow E$, $h : A \wedge I_+ \longrightarrow E'$, and $f : D \longrightarrow E'$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:



Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

THEOREM 7.7 (WHITEHEAD). Let $e : E \longrightarrow E'$ be a weak equivalence and D be a G-CW spectrum. Then $e_* : hG\mathscr{S}(D, E) \longrightarrow hG\mathscr{S}(D, E')$ is a bijection.

COROLLARY 7.8. If $e : E \longrightarrow E'$ is a weak equivalence between G-CW spectra, then e is a G-homotopy equivalence.

THEOREM 7.9 (CELLULAR APPROXIMATION). Let (D, A) and (E, B) be relative G-CW spectra, (D', A') be a subcomplex of (D, A), and $f : (D, A) \longrightarrow (E, B)$ be a G-map whose restriction to (D', A') is cellular. Then f is homotopic rel $D' \cup A$ to a cellular map $g : (D, A) \longrightarrow (E, B)$.

COROLLARY 7.10. Let D and E be G-CW spectra. Then any G-map $f : D \longrightarrow E$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.

THEOREM 7.11. For any G-spectrum E, there is a G-CW spectrum $?E$ and a weak equivalence $\gamma : ?E \longrightarrow E$.

Exactly as on the space level, it follows from the Whitehead theorem that ? extends to a functor $hG\mathscr{S} \longrightarrow hG\mathscr{C}$, where $G\mathscr{C}$ is here the category of G-CW spectra and cellular maps, and the morphisms of the stable category $\bar{h}G\mathscr{S}$ can be specified by

$$\bar{h}G\mathscr{S}(E, E') = hG\mathscr{S}(?E, ?E') = hG\mathscr{C}(?E, ?E'). \tag{7.12}$$

From now on, we shall write $[E, E']_G$ for this set. Again, ? gives an equivalence of categories $\bar{h}G\mathscr{S} \longrightarrow hG\mathscr{C}$.

We should say something about the transport of functors F on $G\mathscr{S}$ to the category $\bar{h}G\mathscr{S}$. All of our functors preserve homotopies, but not all of them preserve weak equivalences. If F does not preserve weak equivalences, then, on the stable category level, we understand F to mean the functor induced by the composite $F \circ ?$, a functor which preserves weak equivalences by converting them to genuine equivalences.

For this and other reasons, it is quite important to understand when functors preserve CW-homotopy types and when they preserve weak equivalences. These questions are related. In a general categorical context, a left adjoint preserves CW-homotopy types if and only if its right adjoint preserves weak equivalences. When these equivalent conditions hold, the induced functors on the categories obtained by inverting the weak equivalences are again adjoint.

For example, since Ω_V^∞ preserves weak equivalences (with the correct logical order, by Theorem 6.8), Σ_V^∞ preserves CW homotopy types. Of course, since our left adjoints preserve colimits and smash products with spaces, their behavior on CW spectra is determined by their behavior on spheres. Since Σ_n^∞ clearly preserves spheres, it carries G-CW based complexes (with based attaching maps) to G-CW spectra. This focuses attention on a significant difference between the equivariant and nonequivariant contexts. In both, a CW spectrum is the colimit of its finite subcomplexes. Nonequivariantly, Lemma 7.2 implies that any finite CW spectrum is isomorphic to $\Sigma_n^\infty X$ for some n and some finite CW complex X. Equivariantly, this is only true up to homotopy type. It would be true up to isomorphism if we allowed non-trivial representations as the domains of attaching maps in our definitions of G-CW complexes and spectra. We have seen that such a theory of "G-CW(V)-complexes" is convenient and appropriate on the space level, but it seems to serve no useful purpose on the spectrum level.

Along these lines, we point out an important consequence of (6.3). It implies that the smash product of spheres $\underline{S}_H^m$ and $\underline{S}_J^n$ is $(G/H \times G/J)_+ \wedge \underline{S}^{m+n}$. When G is finite, we can use double cosets to describe $G/H \times G/J$ as a disjoint union of orbits G/K. This allows us to deduce that the smash product of G-CW spectra is a G-CW spectrum. For general compact Lie groups G, we can only deduce that the smash product of G-CW spectra has the homotopy type of a G-CW spectrum.

8. Stability of the stable category

The observant reader will object that we have called $\bar{h}G\mathscr{S}$ the "stable category", but that we haven't given a shred of justification. As usual, we write $\Sigma^V E = E \wedge S^V$ and $\Omega^V E = F(S^V, E)$.

THEOREM 8.1. For all indexing spaces V in U, the natural maps

$$\eta : E \longrightarrow \Omega^V \Sigma^V E \quad \text{and} \quad \varepsilon : \Sigma^V \Omega^V E \longrightarrow E$$

are isomorphisms in $\bar{h}G\mathscr{S}$. Therefore Ω^V and Σ^V are inverse self-equivalences of $\bar{h}G\mathscr{S}$.

Thus we can desuspend by any representations that are in U. Once this is proven, it is convenient to write Σ^{-V} for Ω^V. There are several possible proofs, all of which depend on Theorem 6.8: that is the crux of the matter, and this means that the result is trivial in the nonequivariant context. In fact, once we have Theorem 6.8, we have that the functor Σ_V^∞ preserves G-CW homotopy types. Using (6.2), (6.4), and the unit equivalence for the smash product, we obtain

$$E \simeq E \wedge \underline{S}^0 \cong E \wedge \Sigma_V^\infty S^V \cong E \wedge (\Sigma_V^\infty S^0 \wedge S^V).$$

This proves that the functor Σ^V is an equivalence of categories. By playing with adjoints, we see that Ω^V must be its inverse. Observe that this proof is independent of the Freudenthal suspension theorem. This argument and (6.2) give the following important consistency relations, where we now drop the underline from our notation for sphere spectra:

(8.2) $$\Omega^V E \simeq E \wedge S^{-V} \quad \text{and} \quad \Sigma_V^\infty X \cong X \wedge S^{-V}, \text{ where } S^{-V} \equiv \Sigma_V^\infty S^0.$$

Since all universes contain $\mathbb{R}$, all G-spectra are equivalent to suspensions. This implies that $\bar{h}G\mathscr{S}$ is an additive category, and it is now straightforward to prove that $\bar{h}G\mathscr{S}$ is triangulated. In fact, it has two triangulations, by cofibrations and fibrations, that differ only by signs. We have already seen that it is symmetric monoidal under the smash product and that it has well-behaved function spectra. We have established a good framework in which to do equivariant stable homotopy theory, and we shall say more about how to exploit it as we go on.

9. Getting into the stable category

The stable category is an ideal world, and the obvious question that arises next is how one gets from the prespectra that occur "in nature" to objects in this category. Of course, our prespectra are all encompassing, since we assumed nothing about their constituent spaces and structure maps, and we do have the left adjoint $L : G\mathscr{P} \longrightarrow G\mathscr{S}$. However, this is a theoretical tool: its good formal properties come at the price of losing control over homotopical information. We need an alternative way of getting into the stable category, one that retains homotopical information.

We first need to say a little more about the functor L. If the adjoint structure maps $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$ of a prespectrum E are inclusions, then $(LE)(V)$ is just the union over $W \supset V$ of the spaces $\Omega^{W-V}EW$. Taking $W = V$, we obtain an inclusion $\eta : EV \longrightarrow (LE)(V)$, and these maps specify a map of prespectra. If, further, each $\tilde{\sigma}$ is a cofibration and an equivalence, then each map η is an equivalence.

Thus we seek to transform given prespectra into spacewise equivalent ones whose adjoint structural maps are cofibrations. The spacewise equivalence property will ensure that Ω-prespectra are transported to Ω-prespectra. It is more natural to consider cofibration conditions on the structure maps $\sigma : \Sigma^{W-V}EV \longrightarrow EW$, and we say that a prespectrum E is "Σ-cofibrant" if each σ is a cofibration. If E is a Σ-cofibrant prespectrum and if each EV has cofibered diagonal, in the sense that the diagonal map $EV \longrightarrow EV \times EV$ is a cofibration, then each adjoint map $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$ is a cofibration, as desired.

Observe that no non-trivial spectrum can be Σ-cofibrant as a spectrum since the structure maps σ of spectra are surjections rather than injections. We say that a spectrum is "tame" if it is homotopy equivalent to LE for some Σ-cofibrant prespectrum E. The importance of this condition was only recognized during the work of Elmendorf, Kriz, Mandell, and myself on structured ring spectra. Its use leads to key technical improvements of [EKMM] over [LMS]. For example, the sharpest versions of Theorems 3.4 and 8.1 read as follows.

Theorem 9.1. *Let $\mathscr{S}_tU$ be the full subcategory of tame spectra indexed on U. Then any two linear isometries $U \longrightarrow U'$ induce canonically and coherently equivalent functors $h\mathscr{S}_tU \longrightarrow h\mathscr{S}_tU'$. The maps $\eta : E \longrightarrow \Omega\Sigma E$ and $\varepsilon : \Sigma\Omega E \longrightarrow E$ are homotopy equivalences of spectra when E is tame.*

Moreover, analogously to (6.5), but much more usefully, if E is a Σ-cofibrant prespectrum, then

$$LE \cong \operatorname{colim} \Sigma^\infty_V EV, \tag{9.2}$$

where the maps of the colimit system are the cofibrations

$$\Sigma^\infty_W \sigma : \Sigma^\infty_V EV \cong \Sigma^\infty_W(\Sigma^{W-V}EV) \longrightarrow \Sigma^\infty_W EW.$$

Here the prespectrum level colimit is already a spectrum, so that the colimit is constructed directly, without use of the functor L. Given a G-spectrum E', there results a valuable $\lim^1$ exact sequence

$$0 \longrightarrow \lim{}^1[\Sigma EV, E'V]_G \longrightarrow [LE, E']_G \longrightarrow \lim[EV, E'V]_G \longrightarrow 0 \tag{9.3}$$

for the calculation of maps in $\bar{h}G\mathscr{S}$ in terms of maps in $\bar{h}G\mathscr{T}$.

To avoid nuisance about inverting weak equivalences here, we introduce an equivariant version of the classical CW prespectra.

Definition 9.4. A G-CW prespectrum is a Σ-cofibrant G-prespectrum E such that each EV has cofibered diagonal and is of the homotopy type of a G-CW complex.

We can insist on actual G-CW complexes, but it would not be reasonable to ask for cellular structure maps. We have the following reassuring result relating this notion to our notion of a G-CW spectrum.

PROPOSITION 9.5. If E is a G-CW prespectrum, then LE has the homotopy type of a G-CW spectrum. If E is a G-CW spectrum, then each component space EV has the homotopy type of a G-CW complex.

Now return to our original question of how to get into the stable category. The kind of maps of prespectra that we are interested in here are "weak maps" $D \longrightarrow E$, whose components $DV \longrightarrow EV$ are only required to be compatible up to homotopy with the structural maps. If D is Σ-cofibrant, then any weak map is spacewise homotopic to a genuine map. The inverse limit term of (9.3) is given by weak maps, which represent maps between cohomology theories on spaces, and its $\lim^1$ term measures the difference between weak maps and genuine maps, which represent maps between cohomology theories on spectra.

Applying G-CW approximation spacewise, using I.3.6, we can replace any G-prespectrum E by a spacewise weakly equivalent G-prespectrum $?E$ whose component spaces are G-CW complexes and therefore have cofibered diagonal maps. However, the structure maps, which come from the Whitehead theorem and are only defined up to homotopy, need not be cofibrations. The following "cylinder construction" converts a G-prespectrum E whose spaces are of the homotopy types of G-CW complexes and have cofibered diagonals into a spacewise equivalent G-CW prespectrum KE. Both constructions are functorial on weak maps.

The composite $K?$ carries an arbitrary G-prespectrum E to a spacewise equivalent G-CW prespectrum. By Proposition 9.5, $LK?E$ has the homotopy type of a G-CW spectrum. In sum, the composite $LK?$ provides a canonical passage from G-prespectra to G-CW spectra that is functorial up to weak homotopy and preserves all homotopical information in the given G-prespectra.

The version of the cylinder construction presented in [LMS] is rather clumsy. The following version is due independently to Elmendorf and Hesselholt. It enjoys much more precise properties, details of which are given in [EKMM].

CONSTRUCTION 9.6 (CYLINDER CONSTRUCTION). Let E be a G-prespectrum indexed on U. Define KE as follows. For an indexing space V, let $\underline{V}$ be the category of subspaces $V' \subset V$ and inclusions. Define a functor E_V from $\underline{V}$ to

G-spaces by letting $E_V(V') = \Sigma^{V-V'}EV'$. For an inclusion $V'' \longrightarrow V'$,

$$V - V'' = (V - V') \oplus (V' - V'')$$

and $\sigma : \Sigma^{V'-V''}EV'' \longrightarrow EV'$ induces $E_V(V'') \longrightarrow E_V(V')$. Define

$$(KE)(V) = \operatorname{hocolim} E_V.$$

An inclusion $i : V \longrightarrow W$ induces a functor $\underline{i} : \underline{V} \longrightarrow \underline{W}$, the functor Σ^{W-V} commutes with homotopy colimits, and we have an evident isomorphism $\Sigma^{W-V}E_V \cong E\underline{i}$ of functors $\underline{V} \longrightarrow \underline{W}$. Therefore $\underline{i}$ induces a map

$$\sigma : \Sigma^{W-V} \operatorname{hocolim} E_V \cong \operatorname{hocolim} \Sigma^{W-V} E_V \cong \operatorname{hocolim} E\underline{i} \longrightarrow \operatorname{hocolim} E_W.$$

One can check that this map is a cofibration. Thus, with these structural maps, KE is a Σ-cofibrant prespectrum. The structural maps $\sigma : E_V V' \longrightarrow EV$ specify a natural transformation to the constant functor at EV and so induce a map $r : (KE)(V) \longrightarrow EV$, and these maps r specify a map of prespectra. Regarding the object V as a trivial subcategory of $\underline{V}$, we obtain $j : EV \longrightarrow (KE)(V)$. Clearly $rj = \mathrm{Id}$, and $jr \simeq \mathrm{Id}$ via a canonical homotopy since V is a terminal object of $\underline{V}$. The maps j specify a weak map of prespectra, via canonical homotopies. Clearly K is functorial and homotopy-preserving, and r is natural. If each space EV has the homotopy type of a G-CW complex, then so does each $(KE)(V)$, and similarly for the cofibered diagonals condition.

A striking property of this construction is that it commutes with smash products: if E and E' are prespectra indexed on U and U', then $KE \wedge KE'$ is isomorphic over $E \wedge E'$ to $K(E \wedge E')$.

CHAPTER XIII

$RO(G)$-graded homology and cohomology theories

1. Axioms for RO(G)-graded cohomology theories

Switching to a homological point of view, we now consider $RO(G)$-graded homology and cohomology theories. There are several ways to be precise about this, and there are several ways to be imprecise. The latter are better represented in the literature than the former. As we have already said, no matter how things are set up, "$RO(G)$-graded" is technically a misnomer since one cannot think of representations as isomorphism classes and still keep track of signs. We give a formal axiomatic definition here and connect it up with G-spectra in the next section.

From now on, we shall usually restrict attention to reduced homology and cohomology theories and shall write them without a tilde. Of course, a $\mathbb{Z}$-graded homology or cohomology theory on G-spaces is required to satisfy the redundant axioms: homotopy invariance, suspension isomorphism, exactness on cofiber sequences, additivity on wedges, and invariance under weak equivalence. Here exactness only requires that a cofiber sequence $X \longrightarrow Y \longrightarrow Z$ be sent to a three term exact sequence in each degree. The homotopy and weak equivalence axioms say that the theory is defined on $\bar{h}G\mathscr{T}$. Such theories determine and are determined by unreduced theories that satisfy the Eilenberg-Steenrod axioms, minus the dimension axiom. Since

$$k_G^{-n}(X) \cong k_G^0(\Sigma^n X),$$

only the non-negative degree parts of a theory need be specified, and a non-negative integer n corresponds to $\mathbb{R}^n$. Indexing on $\mathbb{Z}$ amounts to either choosing a basis for $\mathbb{R}^\infty$ or, equivalently, choosing a skeleton of a suitable category of trivial representations.

Now assume given a G-universe U, say $U = \oplus(V_i)^\infty$ for some sequence of distinct irreducible representations V_i with $V_1 = \mathbb{R}$. An $RO(G;U)$-graded theory can be thought of as graded on the free Abelian group on basis elements corresponding to the V_i. It is equivalent to grade on the skeleton of a category of representations embeddable in U, or to grade on this entire category. The last approach seems to be preferable when considering change of groups, so we will adopt it.

Thus let $\mathscr{RO}(G;U)$ be the category whose objects are the representations embeddable in U and whose morphisms $V \longrightarrow W$ are the G-linear isometric isomorphisms. Say that two such maps are homotopic if their associated based G-maps $S^V \longrightarrow S^W$ are *stably* homotopic, and let $h\mathscr{RO}(G;U)$ be the resulting homotopy category. For each W, we have an evident functor

$$\Sigma^W : \mathscr{RO}(G;U) \times \bar{h}G\mathscr{T} \longrightarrow \mathscr{RO}(G;U) \times \bar{h}G\mathscr{T}$$

that sends (V, X) to $(V \oplus W, \Sigma^W X)$.

Definition 1.1. An $RO(G;U)$-graded cohomology theory is a functor

$$E_G^* : h\mathscr{RO}(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

written $(V, X) \longrightarrow E_G^V(X)$ on objects and similarly on morphisms, together with natural isomorphisms $\sigma^W : E_G^* \longrightarrow E_G^* \circ \Sigma^W$, written

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X),$$

such that the following axioms are satisfied.

(1) For each representation V, the functor E_G^V is exact on cofiber sequences and sends wedges to products.
(2) If $\alpha : W \longrightarrow W'$ is a map in $\mathscr{RO}(G,U)$, then the following diagram commutes:

$$\begin{array}{ccc}
E_G^V(X) & \xrightarrow{\sigma^W} & E_G^{V\oplus W}(\Sigma^W X) \\
{\scriptstyle \sigma^{W'}} \downarrow & & \downarrow {\scriptstyle E_G^{\mathrm{id}\,\oplus\alpha}(\mathrm{id})} \\
E_G^{V\oplus W'}(\Sigma^{W'} X) & \xrightarrow[(\Sigma^\alpha\,\mathrm{id})^*]{} & E_G^{V\oplus W'}(\Sigma^W X).
\end{array}$$

(3) $\sigma^0 = \mathrm{id}$ and the σ are transitive in the sense that the following diagram commutes for each pair of representations (W, Z):

$$\begin{array}{ccc} E_G^V(X) & \xrightarrow{\sigma^W} & E_G^{V\oplus W}(\Sigma^W X) \\ & \searrow^{\sigma^{W\oplus Z}} \quad \swarrow_{\sigma^Z} & \\ & E_G^{V\oplus W\oplus Z}(\Sigma^{W\oplus Z} X). & \end{array}$$

We extend a theory so defined to "formal differences $V \ominus W$" for any pair of representations (V, W) by setting

$$E_G^{V\ominus W}(X) = E_G^V(\Sigma^W X).$$

We use the symbol $\ominus$ to avoid confusion with either orthogonal complement or difference in the representation ring. Rigorously, we are thinking of $V \ominus W$ as an object of the category $h\mathscr{R}O(G;U) \times h\mathscr{R}O(G;U)^{op}$, and, for each X, we have defined a functor from this category to the category of Abelian groups.

The representation group $RO(G;U)$ relative to the given universe U is obtained by passage to equivalence classes from the set of formal differences $V \ominus W$, where $V \ominus W$ is equivalent to $V' \ominus W'$ if there is a G-linear isometric isomorphism

$$\alpha : V \oplus W' \longrightarrow V' \oplus W;$$

$RO(G;U)$ is a ring if tensor products of representations embeddable in U are embeddable in U.

When interpreting $RO(G;U)$-graded cohomology theories, we must keep track of the choice of α, and we see that a given α determines the explicit isomorphism displayed as the unlabelled arrow in the diagram of isomorphisms

$$\begin{array}{ccc} E_G^V(\Sigma^W X) & \xrightarrow{\sigma^{W'}} & E_G^{V\oplus W'}(\Sigma^{W\oplus W'} X) \\ \Big\downarrow & & \Big\downarrow {\scriptstyle E_G^{\alpha}(\Sigma^{\tau}\,\mathrm{id})} \\ E_G^{V'}(\Sigma^{W'} X) & \xrightarrow[\sigma^W]{} & E_G^{V'\oplus W}(\Sigma^{W'\oplus W} X), \end{array}$$

where $\tau : W \oplus W' \longrightarrow W' \oplus W$ is the transposition isomorphism.

If $V^G = 0$, write $V \oplus \mathbb{R}^n = V + n$. Axiom (1) ensures that, for each such V, the E_G^{V+n} and σ^1 define a $\mathbb{Z}$-graded cohomology theory. Axiom (2), together with some easy category theory, ensures that we obtain complete information if we restrict attention to one object in each isomorphism class of representations, that is, if we restrict to any skeleton of the category $\mathscr{R}O(G;U)$. One can even restrict further to a skeleton of its homotopy category. We shall say more about this in the next section.

We can replace the category $\bar{h}G\mathscr{T}$ of based G-spaces by the category $\bar{h}G\mathscr{S}U$ of G-spectra in the definition just given and so define an $RO(G;U)$-graded cohomology theory on G-spectra. Observe that, by our definition of the category $\mathscr{R}O(G;U)$, the isomorphism type of the functor E^V_G depends only on the stable homotopy type of the G-sphere S^V. Such stable homotopy types have been classified by tom Dieck.

We have the evident dual axioms for $RO(G;U)$-graded homology theories on G-spaces or G-spectra. The only point that needs to be mentioned is that homology theories must be given by contravariant functors on $\mathscr{R}O(G;U)$ in order to make sense of the homological counterpart of Axiom (2).

T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics. Vol. 766. 1979.

2. Representing $RO(G)$-graded theories by G-spectra

With our categorical definition of $RO(G;U)$-graded cohomology theories, it is not obvious that they are represented by G-spectra. We show that they are in this and the following section, first showing how to obtain an $RO(G;U)$-graded theory from a G-spectrum and then showing how to obtain a G-spectrum from an $RO(G;U)$-graded theory. Since I find the equivariant forms of these results in the literature to be unsatisfactory, I shall go into some detail. The problem is to pass from indexing spaces to general representations embeddable in our given universe U, and the idea is to make explicit structure that is implicit in the notion of a G-spectrum and then exploit standard categorical techniques. We begin with some of the latter.

Let $\mathscr{I}O(G;U)$ and $h\mathscr{I}O(G;U)$ be the full subcategories of $\mathscr{R}O(G;U)$ and $h\mathscr{R}O(G;U)$ whose objects are the indexing spaces in U, let

$$\Psi : \mathscr{I}O(G;U) \longrightarrow \mathscr{R}O(G;U)$$

be the inclusion, and also write Ψ for the inclusion $h\mathscr{I}O(G;U) \longrightarrow h\mathscr{R}O(G;U)$. For each representation V that is embeddable in U, choose an indexing space ΦV in U and a G-linear isomorphism $\phi_V : V \longrightarrow \Phi V$. If V is itself an indexing space in U, choose $\Phi V = V$ and let ϕ_V be the identity map. Extend Φ to a functor

$$\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$$

by letting $\Phi\alpha$, $\alpha : V \longrightarrow V'$, be the composite

$$\Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\alpha} V' \xrightarrow{\phi_{V'}} \Phi V'.$$

Then $\Phi \circ \Psi = \mathrm{Id}$ and the ϕ_V define a natural isomorphism $\mathrm{Id} \longrightarrow \Psi \circ \Phi$. This equivalence of categories induces an equivalence of categories between $h\mathscr{I}O(G;U)$ and $h\mathscr{R}O(G;U)$. A functor F from $h\mathscr{I}O(G;U)$ to any category $\mathscr{C}$ extends to the functor $F\Phi$ from $h\mathscr{R}O(G;U)$ to $\mathscr{C}$, and we agree to write F instead of $F\Phi$ for such an extended functor.

LEMMA 2.1. Let E be an ΩG-prespectrum. Then E gives the object function of a functor $E : h\mathscr{R}O(G;U) \longrightarrow \bar{h}G\mathscr{T}$.

PROOF. By the observations above, it suffices to define E as a functor on $h\mathscr{I}O(G;U)$. Suppose given indexing spaces V and V' in U and a G-linear isomorphism $\alpha : V \longrightarrow V'$. Choose an indexing space W large enough that it contains both V and V' and that $W - V$ and $W - V'$ both contain copies of representations isomorphic to V and thus to V'. Then there is an isomorphism $\beta : W - V \longrightarrow W' - V'$ such that

$$\beta \wedge \alpha : S^W \cong S^{W-V} \wedge S^V \longrightarrow S^{W-V'} \wedge S^{V'} \cong S^W$$

is stably homotopic to the identity. (For the verification, one relates smash product to composition product in the zero stem $\pi_0^G(S^0)$, exactly as in nonequivariant stable homotopy theory.) Then define $E\alpha : EV \longrightarrow EV'$ to be the composite

$$EV \xrightarrow{\tilde{\sigma}} \Omega^{W-V} EW \xrightarrow{\Omega^{\beta^{-1}}} \Omega^{W'-V'} EW \xrightarrow{\sigma^{-1}} EV'.$$

It is not hard to check that this construction takes stably homotopic maps α and α' to homotopic maps $E\alpha$ and $E\alpha'$ and that the construction is functorial on $\mathscr{I}O(G;U)$. $\square$

PROPOSITION 2.2. An Ω-G-prespectrum E indexed on a universe U represents an $RO(G;U)$-graded cohomology theory E_G^* on based G-spaces.

PROOF. For a representation V that embeds in U, define

$$E_G^V(X) = [X, E\Phi V]_G.$$

For each $\alpha : V \longrightarrow V'$, define

$$E_G^\alpha(X) = [X, E\Phi\alpha]_G.$$

This gives us the required functor

$$E_G^* : h\mathscr{R}O(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

and it is obvious that Axiom (1) of Definition 1.1 is satisfied.

Next, suppose given representations V and W that embed in U. We may write

$$\Phi(V \oplus W) = V' + W',$$

where $V' = \phi_{V\oplus W}(V)$ and $W' = \phi_{V\oplus W}(W)$. There result isomorphisms

$$\iota_V : \Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\phi'_V} V' \quad \text{and} \quad \iota_W : \Phi W \xrightarrow{\phi_W^{-1}} W \xrightarrow{\phi'_W} W',$$

where $\phi'_V = \phi_{V\oplus W}|_V$ and $\phi'_W = \phi_{V\oplus W}|_W$. Define

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X)$$

by the commutativity of the following diagram:

$$\begin{array}{ccc}
E_G^V X = [X, E\Phi V]_G & \xrightarrow{[\mathrm{id}, E\iota_V]} & [X, EV']_G \\
 & & \downarrow{\scriptstyle [\mathrm{id},\tilde{\sigma}]} \\
{\scriptstyle \sigma^W}\downarrow & & [X, \Omega^{W'}E(V'+W')]_G \\
 & & \downarrow{\scriptstyle \cong} \\
E_G^{V\oplus W}(\Sigma^W X) = [\Sigma^W X, E\Phi(V\oplus W)]_G & \xleftarrow{[\Sigma^{\phi'_W}\mathrm{id},\mathrm{id}]} & [\Sigma^{W'}X, E(V'\oplus W')]_G.
\end{array}$$

Diagram chases from the definitions demonstrate that Σ^W is natural, that the diagram of Axiom (2) of Definition 1.1 commutes, and that the transitivity diagram of Axiom 3 commutes because of the transitivity condition that we gave as part of the definition of a G-prespectrum. □

There is an analog for homology theories.

A slight variant of the proof above could be obtained by first replacing the given Ω-G-prespectrum by a spacewise equivalent G-spectrum indexed on U and then specializing the following result to suspension G-spectra. Recall that, for an indexing space V, we have the shift desuspension functor Σ_V^∞ from based G-spaces to G-spectra. It is left adjoint to the Vth space functor:

$$[\Sigma_V^\infty X, E]_G \cong [X, EV]_G. \tag{2.3}$$

DEFINITION 2.4. For a formal difference $V \ominus W$ of representations of G that embed in U, define the sphere G-spectrum $S^{V\ominus W}$ by

$$S^{V\ominus W} = \Sigma_{\Phi W}^\infty S^V, \tag{2.5}$$

where $\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$ is the equivalence of categories constructed above.

PROPOSITION 2.6. *A G-spectrum E indexed on U determines an $RO(G;U)$-graded homology theory E_*^G and an $RO(G;U)$-graded cohomology theory E_G^* on G-spectra.*

PROOF. For G-spectra X and formal differences $V \ominus W$ of representations that embed in U, we define

$$E_{V\ominus W}^G(X) = [S^{V\ominus W}, E \wedge X]_G \tag{2.7}$$

and

$$E_G^{V\ominus W}(X) = [S^{W\ominus V} \wedge X, E]_G = [S^{W\ominus V}, F(X,E)]_G. \tag{2.8}$$

Of course, in cohomology, to verify the axioms, we may as well restrict attention to the case $W = 0$, and similarly in homology. Obviously, the verification reduces to the study of the properties of the G-spheres $\Sigma_V S^0$, or of the functors Σ_V. First, we need functoriality on $\mathscr{R}O(G;U)$, but this is immediate from (2.3) and the functoriality of the EV given by Lemma 2.1. With the notations of the previous proof, we obtain the σ^W from the composite isomorphism of functors

$$\Sigma_{\Phi V}^\infty \cong \Sigma_{V'}^\infty \cong \Sigma^{W'}\Sigma_{V'+W'}^\infty \cong \Sigma^W \Sigma_{\Phi(V\oplus W)}^\infty,$$

where the three isomorphisms are given by use of ι_V, passage to adjoints from the homeomorphism $\tilde{\sigma} : EV' \longrightarrow \Omega^{W'}E(V'+W')$, and use of ϕ'_W. From here, the verification of the axioms is straightforward. □

3. Brown's theorem and $RO(G)$-graded cohomology

We next show that, conversely, all $RO(G)$-graded cohomology theories on based G-spaces are represented by Ω-G-prespectra and all theories on G-spectra are represented by G-spectra. We then discuss the situation in homology, which is considerably more subtle equivariantly than nonequivariantly.

We first record Brown's representability theorem. Brown's categorical proof applies just as well equivariantly as nonequivariantly, on both the space and the spectrum level. Recall that homotopy pushouts are double mapping cylinders and that weak pullbacks satisfy the existence but not the uniqueness property of pullbacks. Recall that a G-space X is said to be G-connected if each of its fixed point spaces X^H is non-empty and connected.

THEOREM 3.1 (BROWN). A contravariant set-valued functor k on the homotopy category of G-connected based G-CW complexes is representable in the form $kX \cong [X, K]_G$ for a based G-CW complex K if and only if k satisfies the wedge and Mayer-Vietoris axioms: k takes wedges to products and takes homotopy pushouts to weak pullbacks. The same statement holds for the homotopy category of G-CW spectra indexed on U for any G-universe U.

COROLLARY 3.2. An $RO(G;U)$-graded cohomology theory E^*_G on based G-spaces is represented by an Ω-G-prespectrum indexed on U.

PROOF. Restricting attention to G-connected based G-spaces, which is harmless in view of the suspension axiom for trivial representations, we see that (1) of Definition 1.1 implies the Mayer-Vietoris and wedge axioms that are needed to apply Brown's representability theorem. This gives that E^V_G is represented by a G-CW complex EV for each indexing space V in U. If $V \subset W$, then the suspension isomorphism

$$\sigma^{W-V} : E^V_G(X) \cong E^W_G(\Sigma^{W-V} X)$$

is represented by a homotopy equivalence $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$. The transitivity of the given system of suspension isomorphisms only gives that the structural maps are transitive up to homotopy, whereas the definition of a G-prespectrum requires that the structural maps be transitive on the point-set level. If we restrict to a cofinal sequence of indexing spaces, then we can use transitivity to define the structural weak equivalences for non-consecutive terms of the sequence. We can then interpolate using loop spaces to construct a representing Ω-G-prespectrum indexed on all indexing spaces. □

We emphasize a different point of view of the spectrum level analog. In fact, we shall exploit the following result to construct ordinary $RO(G)$-graded cohomology theories in the next section.

COROLLARY 3.3. A $\mathbb{Z}$-graded cohomology theory on G-spectra indexed on U is represented by a G-spectrum indexed on U and therefore extends to an $RO(G;U)$-graded cohomology theory on G-spectra indexed on U.

PROOF. Since the loop and suspension functors are inverse equivalences on the stable category $\bar{h}G\mathscr{S}U$, we can reconstruct the given theory from its zeroth term, and Brown's theorem applies to represent the zeroth term. □

We showed in the previous chapter that an Ω-G-prespectrum determines a spacewise equivalent G-spectrum, so that a cohomology theory on based G-spaces extends to a cohomology theory on G-spectra. The extension is unique up to non-unique isomorphism, where the non-uniqueness is measured by the $\lim^1$ term in (XII.9.3).

Adams proved a variant of Brown's representability theorem for functors defined only on connected finite CW complexes, removing a countability hypothesis that was present in an earlier version due to Brown. This result also generalizes to the equivariant context, with the same proof as Adams' original one.

THEOREM 3.4 (ADAMS). A contravariant group-valued functor k defined on the homotopy category of G-connected finite based G-CW complexes is representable in the form $kX \cong [X, K]_G$ for some G-CW spectrum K if and only if k converts finite wedges to direct products and converts homotopy pushouts to weak pullbacks of underlying sets. The same statement holds for the homotopy category of finite G-CW spectra.

Here the representing G-CW spectrum K is usually infinite and is unique only up to non-canonical equivalence. More precisely, maps $g, g' : Y \longrightarrow Y'$ are said to be weakly homotopic if gf is homotopic to $g'f$ for any map $f : X \to Y$ defined on a finite G-CW spectrum X, and K is unique up to isomorphism in the resulting weak homotopy category of G-CW spectra.

Nonequivariantly, we pass from here to the representation of homology theories by use of Spanier-Whitehead duality. A finite CW spectrum X has a dual DX

that is also a finite CW spectrum. Given a homology theory E_* on based spaces or on spectra, we obtain a dual cohomology theory on finite X by setting

$$E^n(X) = E_{-n}(DX).$$

We then argue as above that this cohomology theory on finite X is representable by a spectrum E, and we deduce by duality that E also represents the originally given homology theory.

Equivariantly, this argument works for a complete G-universe U, but it does not work for a general universe. The problem is that, as we shall see later, only those orbit spectra $\Sigma^\infty G/H_+$ such that G/H embeds equivariantly in U have well-behaved duals. For example, if the universe U is trivial, then inspection of definitions shows that $F(G/H_+, S) = S$ for all $H \subseteq G$, where S is the sphere spectrum with trivial G-action. Thus X is not equivalent to DDX in general and we cannot hope to recover $E_*(X)$ as $E^*(DX)$.

Corollary 3.5. If U is a complete G-universe, then an $RO(G;U)$-graded homology theory on based G-spaces or on G-spectra is representable.

From now on, unless explicitly stated otherwise, we take our given universe U to be complete, and we write $RO(G) = RO(G;U)$. As shown by long experience in nonequivariant homotopy theory, even if one's primary interest is in spaces, the best way to study homology and cohomology theories is to work on the spectrum level, exploiting the virtues of the stable homotopy category.

J. F. Adams. A variant of E. H. Brown's representability theorem. Topology, 10(1971), 185–198.
E. H. Brown, Jr. Cohomology theories. Annals of Math. 75(1962), 467–484.
E. H. Brown, Jr. Abstract homotopy theory. Trans. Amer. Math. Soc. 119(1965), 79–85.

4. Equivariant Eilenberg-MacLane spectra

From the topological point of view, a coefficient system is a contravariant additive functor from the stable category of naive orbit spectra to Abelian groups. In fact, it is easy to see that the group of stable maps $G/H_+ \longrightarrow G/K_+$ in the naive sense is the free Abelian group on the set of G-maps $G/H \longrightarrow G/K$.

Recall from IX§4 that a Mackey functor is defined to be an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$. Clearly the Burnside category $\mathscr{B} = \mathscr{B}_G$ introduced there is just the full subcategory of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H_+$. The only difference is that, when defining $\mathscr{B}_G$, we abbreviated the names of objects to G/H.

From this point of view, the forgetful functor that takes a Mackey functor to a coefficient system is obtained by pullback along the functor i^* from the stable category of genuine orbit spectra to the stable category of naive orbit spectra. In X§4, Waner described a space level construction of an $RO(G)$-graded cohomology theory with coefficients in a Mackey functor M that extends the ordinary $\mathbb{Z}$-graded cohomology theory determined by its underlying coefficient system i^*M. We shall here give a more sophisticated, and I think more elegant and conceptual, spectrum level construction of such "ordinary" $RO(G)$-graded cohomology theories, and similarly for homology.

Our strategy is to construct a genuine Eilenberg-MacLane G-spectrum $HM = K(M, 0)$ to represent our theory. Just as nonequivariantly, an Eilenberg-Mac Lane G-spectrum HM is one such that $\underline{\pi_n}(HM) = 0$ for $n \neq 0$. Of course, $\underline{\pi}_0(HM) = M$ must be a Mackey functor since that is true of $\underline{\pi}_n(E)$ for any n and any G-spectrum E. We shall explain the following result.

Theorem 4.1. For a Mackey functor M, there is an Eilenberg-MacLane G-spectrum HM such that $\underline{\pi}_0(HM) = M$. It is unique up to isomorphism in $\bar{h}G\mathscr{S}$. For Mackey functors M and M', $[HM, HM']_G$ is the group of maps of Mackey functors $M \longrightarrow M'$.

There are several possible proofs. For example, one can exploit projective resolutions of Mackey functors. The proof that we shall give is the original one of Lewis, McClure, and myself, which I find rather amusing.

What is amusing is that, motivated by the desire to construct an $RO(G)$-graded cohomology theory, we instead construct a $\mathbb{Z}$-graded theory. However, this is a $\mathbb{Z}$-graded theory defined on G-spectra. As observed in Corollary 4.3, it can be represented and therefore extends to an $RO(G)$-graded theory. The representing G-spectrum is the desired Eilenberg-MacLane G-spectrum HM. What is also amusing is that the details that we shall use to construct the desired cohomology theories are virtually identical to those that we used to construct ordinary theories in the first place.

We start with G-CW spectra X. They have skeletal filtrations, and we define Mackey-functor valued cellular chains by setting

$$\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1}). \tag{4.2}$$

We used homology groups in I§4, but, aside from nuisance with the cases $n = 0$

and $n = 1$, we could equally well have used homotopy groups. Of course, X^n/X^{n-1} is a wedge of n-sphere G-spectra $S^n_H \simeq G/H_+ \wedge S^n$. We see that the $\underline{C}_n(X)$ are projective objects of the Abelian category of Mackey functors by essentially the same argument that we used in I§4. As there, the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}) specifies a map of Mackey functors

$$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X),$$

and $d^2 = 0$. Write $\mathrm{Hom}_{\mathscr{B}}(M, M')$ for the Abelian group of maps of Mackey functors $M \longrightarrow M'$. For a Mackey functor M, define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathscr{B}}(\underline{C}_n(X), M), \quad \text{with} \quad \delta = \mathrm{Hom}_{\mathscr{B}}(d, \mathrm{Id}). \tag{4.3}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. We denote its homology by $H^*_G(X; M)$.

The evident cellular versions of the homotopy, exactness, wedge, and excision axioms admit exactly the same quick derivations as on the space level, and we use G-CW approximation to extend from G-CW spectra to general G-spectra: we have a $\mathbb{Z}$-graded cohomology theory on $\bar{h}\mathscr{G}\mathscr{S}$. It satisfies the dimension axiom

$$H^*_G(S^0_H; M) = H^0_G(S^0_H; M) = M(G/H), \tag{4.4}$$

these giving isomorphisms of Mackey functors. The zeroth term is represented by a G-spectrum HM, and we read off its homotopy group Mackey functors directly from (4.4):

$$\underline{\pi}_0(HM) = M \quad \text{and} \quad \underline{\pi}_n(HM) = 0 \text{ if } \ n \neq 0.$$

The uniqueness of HM is evident, and the calculation of $[HM, HM']_G$ follows easily from the functoriality in M of the theories $H^*_G(X; M)$.

We should observe that spectrum level obstruction theory works exactly as on the space level, modulo connectivity assumptions to ensure that one has a dimension in which to start inductions.

For G-spaces X, we now have two meanings in sight for the notation $H^n_G(X; M)$: we can regard our Mackey functor as a coefficient system and take ordinary cohomology as in I§4, or we can take our newly constructed cohomology. We know by the axiomatic characterization of ordinary cohomology that these must in fact be isomorphic, but it is instructive to check this directly. At least after a single suspension, we can approximate any G-space by a weakly equivalent G-CW based complex, with based attaching maps. The functor Σ^∞ takes G-CW based complexes to G-CW spectra, and we find that the two chain complexes in sight are isomorphic. Alternatively, we can check on the represented level:

$$[\Sigma^\infty X, \Sigma^n HM]_G \cong [X, \Omega^\infty \Sigma^n HM]_G \cong [X, K(M,n)]_G.$$

What about homology? Recall that a coMackey functor is a covariant functor $N : \mathscr{B} \longrightarrow \mathscr{A}b$. Using the usual coend construction, we define

$$C_n^G(X;N) = \underline{C}_*(X) \otimes_{\mathscr{B}} N, \quad \text{with} \quad \partial = d \otimes \mathrm{Id}. \tag{4.5}$$

Then $C_*^G(X;N)$ is a chain complex of Abelian groups. We denote its homology by $H_*^G(X;N)$. Again, the verification of the axioms for a $\mathbb{Z}$-graded homology theory on $\bar{h}\mathscr{G}\mathscr{S}$ is immediate. The dimension axiom now reads

$$H_*^G(S_H^0;N) = H_0^G(S_H^0;N) = N(G/H). \tag{4.6}$$

We define a cohomology theory on finite G-spectra X by

$$H_G^*(X;N) = H_{-*}^G(DX;N). \tag{4.7}$$

Applying Adams' variant of the Brown representability theorem, we obtain a G-spectrum JN that represents this cohomology theory. For finite X, we obtain

$$H_*^G(X;N) = H_G^{-*}(DX;N) \cong [DX, JN]_G^{-*} \cong [S, JN \wedge X]_*^G = JN_*^G(X).$$

Thus JN represents the $\mathbb{Z}$-graded homology theory that we started with and extends it to an $RO(G)$-graded theory. We again see that, on G-spaces X, $H_*^G(X;N)$ agrees with the homology of X with coefficients in the underlying covariant coefficient system of N, as defined in I§4.

What are the homotopy groups of JN? The answer must be

$$\pi_n^H(JN) = H_n^G(D(G/H_+);N).$$

For finite G, orbits are self-dual and the resulting isomorphism of the stable orbit category with its opposite category induces the evident self-duality of the algebraically defined category of Mackey functors to be discussed in XIX§3. This allows us to conclude that

$$JN = H(N^*),$$

where N^* is the Mackey functor dual to the coMackey functor N.

For general compact Lie groups, however, the dual of G/H_+ is $G \ltimes_H S^{-L(H)}$, and it is not easy to calculate the homotopy groups of JN. This G-spectrum is bounded below, but it is not connective. We must learn to live with the fact that we have two quite different kinds of Eilenberg-MacLane G-spectra, one that is suitable for representing "ordinary" cohomology and the other that is suitable for representing "ordinary" homology.

G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. Bulletin Amer. Math. Soc. 4(1981), 208-212.

5. Ring G-spectra and products

Given our precise definition of $RO(G)$-graded theories and our understanding of their representation by G-spectra, the formal apparatus of products in homology and cohomology theories can be developed in a straightforward manner and is little different from the nonequivariant case in classical lectures of Adams. However, in that early work, Adams did not take full advantage of the stable homotopy category. We here recall briefly the basic definitions from the equivariant treatment in [LMS, III§3].

There are four basic products to consider, two external products and two slant products. The reader should be warned that the treatment of slant products in the literature is inconsistent, at best, and often just plain wrong. These four products come from the following four natural maps in $\bar{h}G\mathscr{S}$; all variables are G-spectra.

(5.1)
$$X \wedge E \wedge X' \wedge E' \xrightarrow{\mathrm{id}\wedge\tau\wedge\mathrm{id}} X \wedge X' \wedge E \wedge E'$$

(5.2)
$$F(X,E) \wedge F(X',E') \xrightarrow{\ \wedge\ } F(X \wedge X', E \wedge E')$$

(5.3)
$$\begin{array}{ccc} F(X \wedge X', E) \wedge X \wedge E' & \xrightarrow{\ /\ } & F(X', E \wedge E') \\ \cong \big\downarrow & & \big\uparrow \nu \\ F(X, F(X',E)) \wedge X \wedge E' & \xrightarrow[\varepsilon\wedge\mathrm{id}]{} & F(X',E) \wedge E' \end{array}$$

(5.4)
$$\begin{array}{ccc} X \wedge X' \wedge E \wedge F(X,E') & \xrightarrow{\qquad\quad \backslash \qquad\quad} & X' \wedge E \wedge E' \\ & \searrow^{\tau} \qquad\qquad \nearrow_{\mathrm{id}\wedge\mathrm{id}\wedge\varepsilon} & \\ & X' \wedge E \wedge F(X,E') \wedge X & \end{array}$$

The τ are transposition maps and the ε are evaluation maps. The map ν can be described formally, but it is perhaps best understood by pretending that F means Hom and $\wedge$ means $\otimes$ over a commutative ring and writing down the obvious analog. Categorically, such coherence maps are present in any symmetric monoidal category with an internal hom functor. A categorical coherence theorem asserts that any suitably well formulated diagram involving these transformations will commute.

On passage to homotopy groups, these maps give rise to four products in $RO(G)$-graded homology and cohomology. With our details on $RO(G)$-grading, we leave it as an exercise for the reader to check exactly how the grading behaves.

$$E_*^G(X) \otimes E'^G_*(X') \longrightarrow (E \wedge E')_*^G(X \wedge X') \tag{5.5}$$

$$E^*_G(X) \otimes E'^*_G(X') \longrightarrow (E \wedge E')^*_G(X \wedge X') \tag{5.6}$$

$$/ : E^*_G(X \wedge X') \otimes E'^G_*(X) \longrightarrow (E \wedge E')^*_G(X') \tag{5.7}$$

$$\backslash : E_*^G(X \wedge X') \otimes E'^*_G(X) \longrightarrow (E \wedge E')_*^G(X') \tag{5.8}$$

A ring G-spectrum E is one with a product $\phi : E \wedge E \longrightarrow E$ and a unit map $\eta : S \longrightarrow E$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{array}{ccccc} S \wedge E & \xrightarrow{\eta \wedge 1} & E \wedge E & \xleftarrow{1 \wedge \eta} & E \wedge S \\ & {\scriptstyle\simeq}\searrow & \downarrow{\scriptstyle\phi} & \swarrow{\scriptstyle\simeq} & \\ & & E & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{1 \wedge \phi} & E \wedge E \\ {\scriptstyle\phi \wedge 1}\downarrow & & \downarrow{\scriptstyle\phi} \\ E \wedge E & \xrightarrow{\phi} & E. \end{array}$$

The unlabelled equivalences are canonical isomorphisms in $\bar{h}G\mathscr{S}$ that give the unital property, and we have suppressed such an associativity isomorphism in the second diagram. Of course, there is a weaker notion in which associativity is not required; E is commutative if the following diagram commutes in $\bar{h}G\mathscr{S}$:

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\tau} & E \wedge E \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & E. & \end{array}$$

An E-module is a spectrum M together with a map $\mu : E \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{array}{ccc} S \wedge M & \xrightarrow{\eta\wedge 1} & E \wedge M \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} \\ & & M \end{array} \quad \text{and} \quad \begin{array}{ccc} E \wedge E \wedge M & \xrightarrow{1\wedge\mu} & E \wedge M \\ {\scriptstyle \phi\wedge 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ E \wedge M & \xrightarrow{\mu} & M. \end{array}$$

We obtain various further products by composing the four external products displayed above with the multiplication of a ring spectrum or with its action on a module spectrum. If $X = X'$ is a based G-space (or rather its suspension spectrum), we obtain internal products by composing with the reduced diagonal $\Delta : X \longrightarrow X \wedge X$. Of course, it is more usual to think in terms of unbased spaces, but then we adjoin a disjoint basepoint. In particular, for a ring G-spectrum E and a based G-space X, we obtain the cup and cap products

$$\cup : E^*_G(X) \otimes E^*_G(X) \longrightarrow E^*_G(X) \tag{5.9}$$

and

$$\cap : E_*^G(X) \otimes E^*_G(X) \longrightarrow E_*^G(X) \tag{5.10}$$

from the external products $\wedge$ and $\backslash$.

It is natural to ask when HM is a ring G-spectrum. In fact, in common with all such categories of additive functors, the category of Mackey functors has an internal tensor product (see Mitchell). In the present topological context, we can define it simply by setting

$$M \otimes M' = \underline{\pi}_0(HM \wedge HM').$$

There results a notion of a pairing $M \otimes M' \longrightarrow M''$ of Mackey functors. By killing the higher homotopy groups of $HM \wedge HM'$, we obtain a canonical map

$$\iota : HM \wedge HM' \longrightarrow H(M \otimes M'),$$

and ι induces an isomorphism on $H^0_G(\cdot; M'') = [\cdot, HM'']_G$. It follows that pairings of G-spectra $HM \wedge HM' \longrightarrow HM''$ are in bijective correspondence with pairings $M \otimes M' \longrightarrow M''$. From here, it is clear how to define the notion of a ring in the category of Mackey functors — such objects are called Green functors — and to conclude that a ring structure on the G-spectrum HM determines and is determined by a structure of Green functor on the Mackey functor M. These observations come from work of Greenlees and myself on Tate cohomology.

There is a notion of a ring G-prespectrum; modulo $\lim^1$ problems, its associated G-spectrum (here constructed using the cylinder construction since one wishes to retain homotopical information) inherits a structure of ring G-spectrum. A good nonequivariant exposition that carries over to the equivariant context has been given by McClure.

J. F. Adams. Lectures on generalized cohomology. in Springer Lecture Notes in Mathematics, Vol. 99, 1-138.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology (§8). Memoirs Amer. Math. Soc. Number 543. 1995.

J. E. McClure. H_∞-ring spectra via space-level homotopy theory (§§1-2). In R. Bruner, et al, H_∞-ring spectra and their applications. Springer Lecture Notes in Mathematics, Vol. 1176. 1986.

B. Mitchell. Rings with several objects. Advances in Math 8(1972), 1-16.

CHAPTER XIV

An introduction to equivariant K-theory

by J. P. C. Greenlees

1. The definition and basic properties of K_G-theory

The aim of this chapter is to explain the basic facts about equivariant K-theory through the Atiyah-Segal completion theorem. Throughout, G is a compact Lie group and we focus on complex K-theory. Real K-theory works similarly.

We briefly outline the geometric roots of equivariant K-theory. A G-vector bundle over a G-space X is a G-map $\xi : E \longrightarrow X$ which is a vector bundle such that G acts linearly on the fibers, in the sense that $g : E_x \longrightarrow E_{gx}$ is a linear map. Since G is compact, all short exact sequences of G-vector bundles split. If X is a compact space, then $K_G(X)$ is defined to be the Grothendieck group of finite dimensional G-vector bundles over X. Tensor product of bundles makes $K_G(X)$ into a ring.

Many applications arise; for example, the equivariant K-groups are the homes for indices of G-manifolds and families of elliptic operators.

Any complex representation V of G defines a trivial bundle over X and, by the Peter-Weyl theorem, any G-vector bundle over a compact base space is a summand of such a trivial bundle. The cokernel of $K_G(*) \longrightarrow K_G(X)$ can therefore be described as the group of stable isomorphism classes of bundles over X, where two bundles are stably isomorphic if they become isomorphic upon adding an appropriate trivial bundle to each. When X has a G-fixed basepoint $*$, we write $\tilde{K}_G(X)$ for the isomorphic group $\ker(K_G(X) \longrightarrow K_G(*))$.

The definition of a G-vector bundle makes it clear that G-bundles over a free G-space correspond to vector bundles over the quotient under pullback. We deduce the basic reduction theorem:

$$K_G(X) = K(X/G) \text{ if } X \text{ is } G-\text{free}. \tag{1.1}$$

This is essentially the statement that K-theory is split in the sense to be discussed in XVI§2. It provides the fundamental link between equivariant and nonequivariant K-theory.

Restriction and induction are the basic pieces of structure that link different ambient groups of equivariance.

If $i : H \longrightarrow G$ is the inclusion of a subgroup it is clear that a G-space or bundle can be viewed as an H-space or bundle; we thereby obtain a restriction map

$$i^* : K_G(X) \longrightarrow K_H(X).$$

There is another way of thinking about this map. For an H-space Y,

$$K_G(G \times_H Y) \cong K_H(Y) \tag{1.2}$$

since a G-bundle over $G \times_H Y$ is determined by its underlying H-bundle over Y. For a G-space X, $G \times_H X \cong G/H \times X$, and the restriction map coincides with the map

$$K_G(X) \longrightarrow K_G(G/H \times X) \cong K_H(X)$$

induced by the projection $G/H \longrightarrow *$.

If H is of finite index in G, an H-bundle over a G-space may be made into a G-bundle by applying the functor $\mathrm{Hom}_H(G, \bullet)$. We thus obtain an induction map $i_* : K_H(X) \longrightarrow K_G(X)$. However if H is of infinite index this construction gives an infinite dimensional bundle. There are three other constructions one may hope to use. First, there is smooth induction, which Segal describes for the representation ring and which should apply to more general base manifolds than a point.

Second, there is the holomorphic transfer, which one only expects to exist when G/H admits the structure of a projective variety. The most important case is when H is the maximal torus in the unitary group $U(n)$, in which case a construction using elliptic operators is described by Atiyah. Its essential property is that it satisfies $i_*i^* = 1$. It is used in the proof of Bott periodicity.

Third, there is a transfer map

$$tr : \tilde{K}_H(\Sigma^W X) \cong \tilde{K}_G(G_+ \wedge_H \Sigma^W X) \longrightarrow \tilde{K}_G(\Sigma^V X)$$

induced by the Pontrjagin-Thom construction $t : S^V \longrightarrow G_+ \wedge_H S^W$ associated to an embedding of G/H in a representation V, where W is the complement of the image in V of the tangent H-representation $L = L(H)$ at the identity coset of G/H. Once we use Bott periodicity to set up $RO(G)$-graded K-theory, this may be interpreted as a dimension-shifting transfer $\tilde{K}_H^{q+L}(X) \longrightarrow \tilde{K}_G^q(X)$. Clearly this transfer is not special to K-theory: it is present in any $RO(G)$-graded theory.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.
G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

2. Bundles over a point: the representation ring

Bundles over a point are representations and hence equivariant K-theory is module-valued over the complex representation ring $R(G)$. More generally, any G-vector bundle over a transitive G-space G/H is of the form $G \times_H V \longrightarrow G \times_H * = G/H$ for some representation V of H. Hence $K_G(G/H) = R(H)$. It follows that $K_G(X)$ takes values in the category of $R(G)$-modules, and thus it is important to understand the algebraic nature of $R(G)$.

Before turning to this, we observe that if G acts trivially on X, then

$$K_G(X) \cong R(G) \otimes K(X).$$

Indeed, the map $K(X) \longrightarrow K_G(X)$ obtained by regarding a vector bundle as a G-trivial G-vector bundle extends to a map $\mu : R(G) \otimes K(X)$ of $R(G)$-modules, and this map is the required isomorphism. An explicit inverse can be constructed as follows. For a representation V, let V denote the trivial G-vector bundle $X \times V \longrightarrow X$. The functor that sends a G-vector bundle ξ to the vector bundle $\mathrm{Hom}_G(V, \xi)$ induces a homomorphism $\varepsilon_V : K_G(X) \longrightarrow K(X)$. Let $\{V_i\}$ run through a set consisting of one representation V_i from each isomorphism class $[V_i]$ of irreducible representations. Then a G-vector bundle ξ over X breaks up as the Whitney sum of its subbundles $\boldsymbol{V_i} \otimes \mathrm{Hom}_G(\boldsymbol{V_i}, \xi)$. Define $\nu : K_G(X) \longrightarrow R(G) \otimes K(X)$ by $\nu(\alpha) = \sum_i [V_i] \otimes \varepsilon_{V_i}(\alpha)$. It is then easy to check that μ and ν are inverse isomorphisms.

To understand the algebra of $R(G)$, one should concentrate on the so called "Cartan subgroups" of G. These are topologically cyclic subgroups H with finite Weyl groups $W_G(H) = N_G(H)/H$. Conjugacy classes of Cartan subgroups are in one-to-one correspondence with conjugacy classes of cyclic subgroups of the component group $\pi_0(G)$. Every element of G lies in some Cartan subgroup, and therefore the restriction maps give an injective ring homomorphism

$$R(G) \longrightarrow \prod_{(C)} R(C) \tag{2.1}$$

where the product is over conjugacy classes of Cartan subgroups.

The ring $R(G)$ is Noetherian. Indeed, by explicit calculation, $R(U(n))$ is Noetherian and the representation ring of a maximal torus T is finite over it. Any group G may be embedded in some $U(n)$, and it is enough to show that $R(G)$ is finitely generated as an $R(U(n))$-module. Now $R(G)$ is detected on finitely many topologically cyclic subgroups C, so it is enough to show each $R(C)$ is finitely generated over $R(U(n))$. But each such C is conjugate to a subgroup of T, and $R(C)$ is finite over $R(T)$.

The map (2.1) makes the codomain a finitely generated module over the domain and consequently the induced map of prime spectra is surjective and has finite fibers. By identifying the fibers it can then be shown that for any prime $\wp$ of $R(G)$ the set of minimal elements of

$$\{H \subseteq G \mid \wp \text{ is the restriction of a prime of } R(H)\}$$

constitutes a single conjugacy class (H) of subgroups, with H topologically cyclic. We say that (H) is the *support* of $\wp$. If $R(G)/\wp$ is of characteristic $p > 0$ then the component group of H has order prime to p.

The first easy consequence is that the Krull dimension of $R(G)$ is one more than the rank of G.

A more technical consequence which will become important to us later is that completion is compatible with restriction. Indeed restriction gives a ring homomorphism $\text{res} : R(G) \longrightarrow R(H)$ by which we may regard an $R(H)$-module as an $R(G)$-module. Using supports, we see that if $I(G) = \ker\{\dim : R(G) \longrightarrow \mathbb{Z}\}$ is the augmentation ideal, the ideals $I(H)$ and $\text{res}(I(G)).R(H)$ have the same radical. Consequently the $I(H)$-adic and $I(G)$-adic completions of an $R(H)$-module coincide.

Finally, using supports it is straightforward to understand localizations of equivariant K-theory at primes of $R(G)$. In fact if (H) is the support of $\wp$ the inclusion $X^{(H)} \longrightarrow X$ induces an isomorphism of $K_G(\ )_\wp$, where $X^{(H)}$ is the union of the fixed point spaces $X^{H'}$ with H' conjugate to H.

G. B.Segal. The representation ring of a compact Lie group. Pub. IHES 34(1968), 113-128.

3. Equivariant Bott periodicity

Equivariant Bott periodicity is the most important theorem in equivariant K-theory and is even more extraordinary than its nonequivariant counterpart. It underlies all of the amazing properties of equivariant K-theory. For a locally compact G-space X, define $K_G(X)$ to be the reduced K-theory of the one-point compactification $X_\#$ of X. That is, writing $*$ for the point at infinity,

$$K_G(X) = \ker(K_G(X_\#) \longrightarrow K_G(*).$$

When X is compact, $X_\#$ is the union X_+ of X and a disjoint G-fixed basepoint. We issue a warning: in general, for infinite G-CW complexes, $K_G(X)$ as just defined will not agree with the represented K_G-theory of X that will become available when we construct the K-theory G-spectrum in the next section.

Theorem 3.1 (Thom isomorphism). For vector bundles E over locally compact base spaces X, there is a natural Thom isomorphism

$$\phi : K_G(X) \xrightarrow{\cong} K_G(E).$$

There is a quick reduction to the case when X is compact, and in this case we can use that any G-bundle is a summand of the trivial bundle of some representation V to reduce to the case when $E = V \times X$. Here, with an appropriate description of the Thom isomorphism, one can reinterpret the statement as a convenient and explicit version of Bott periodicity. To see this, let $\lambda(V) \in R(G)$ denote the alternating sum of exterior powers

$$\lambda(V) = 1 - V + \lambda^2 V - \cdots + (-1)^{dim V}\lambda^{dim V} V,$$

let $e_V : S^0 \longrightarrow S^V$ be the based map that sends the non-basepoint to 0, and, taking X to be a point, let $b_V = \phi(1) \in \tilde{K}(S^V)$. Observe that e_V induces

$$e_V^* : \tilde{K}(S^V) \longrightarrow \tilde{K}(S^0) = R(G).$$

Theorem 3.2 (Bott periodicity). For a compact G-space X and a complex representation V of G, multiplication by b_V specifies an isomorphism

$$\phi : \tilde{K}_G(X_+) = K_G(X) \xrightarrow{\cong} K_G(V \times X) = \tilde{K}(S^V \wedge X_+).$$

Moreover, $e(V)^*(b_V) = \lambda(V)$.

The Thom isomorphism can be proven for line bundles, trivial or not, by arguing with clutching functions, as in the nonequivariant case. The essential point is to show that the K-theory of the projective bundle $P(E \oplus \mathbb{C})$ is the free $K_G(X)$-module generated by the unit element $\{1\}$ and the Hopf bundle H. This implies the case when E is a sum of trivial line bundles. If G is abelian, every V is a sum of one dimensional representations so the theorem is proved. This deals with the case of a torus T. The significantly new feature of the equivariant case is the use of holomorphic transfer to deduce the case of $U(n)$. Finally, by change of groups, the result follows for any subgroup of $U(n)$.

For real equivariant K-theory KO_G, the Bott periodicity theorem is true as stated provided that we restrict V to be a Spin representation of dimension divisible by eight. However, the proof is significantly more difficult, requiring the use of pseudo-differential operators.

Now we may extend $K_G(\bullet)$ to a cohomology theory. Following our usual conventions, we shall write K_G^* for the reduced theory on based G-spaces X. Since we need compactness, we consider based finite G-CW complexes, and we then have the notational conventions that in degree zero

$$K_G^0(X_+) = K_G(X) \text{ for finite } G\text{-CW complexes } X$$

and

$$K_G^0(X) = \tilde{K}_G(X) \text{ for based finite } G\text{-CW complexes } X.$$

Of course we could already have made the definition $K_G^{-q}(X) = K_G^0(\Sigma^q X)$ for positive q, but we now know that these are periodic with period 2 since $\mathbb{R}^2 = \mathbb{C}$. Thus we may take

$$K_G^{2n}(X) = K_G^0(X) \quad \text{and} \quad K_G^{2n+1}(X) = K_G^0(\Sigma^1 X) \text{ for all } n.$$

Note in particular that the coefficient ring is $R(G)$ in even degrees. It is zero in odd degrees because all bundles over S^1 are pullbacks of bundles over a point, $GL_n(\mathbb{C})$ being connected. We can extend this to an $RO(G)$-graded theory that is $R(G)$-periodic, but we let the construction of a representing G-spectrum in the next section take care of this for us.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.

M. F.Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. Acta math. 112(1964), 229-247.

G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

4. Equivariant K-theory spectra

Following the procedures indicated in XII§9, we run through the construction of a G-spectrum that represents equivariant K-theory. Recall from VII.3.1 that the Grassmannian G-space $BU(n, V)$ of complex n-planes in a complex inner product G-space V classifies complex n-dimensional G-vector bundles if V is sufficiently large, for example if V contains a complete complex G-universe.

Diverging slightly from our usual notation, fix a complete G-universe $\mathscr{U}$. For each indexing space $V \subset \mathscr{U}$ and each $q \geq 0$, we have a classifying space

$$BU(q, V \oplus \mathscr{U})$$

for q-plane bundles. For $V \subseteq W$, we have an inclusion

$$BU(q, V \oplus \mathscr{U}) \longrightarrow BU(q + |W - V|, W \oplus \mathscr{U})$$

that sends a plane A to the plane $A + (W - V)$. Define

$$BU_G(V) = \coprod_{q \geq 0} BU(q, V \oplus \mathscr{U}).$$

We take the plane V in $BU(|V|, V \oplus \mathscr{U})$ as the canonical G-fixed basepoint of $BU_G(V)$. For $V \subset W$, we then have an inclusion $BU_G(V)$ in $BU_G(W)$ of based G-spaces. Define BU_G to be the colimit of the $BU_G(V)$.

For finite (unbased) G-CW complexes X, the definition of $K_G(X)$ as a Grothendieck group and the classification theorem for complex G-vector bundles lead to an isomorphism

$$[X_+, BU_G]_G \cong K_G(X) = K_G^0(X_+).$$

The finiteness ensures that our bundles embed in trivial bundles and thus have complements. In turn, this ensures that every element of the Grothendieck group is the difference of a bundle and a trivial bundle. For the proof, we may as well assume that X/G is connected. In this case, a G-map $\phi : X \longrightarrow BU_G$ factors through a map $f : BU_G(q, V \oplus \mathscr{U})$ for some q and V. If f classifies the G-bundle ξ, then the isomorphism sends ϕ to $\xi - V$.

The spaces $BU_G(V)$ and BU_G have the homotopy types of G-CW complexes. If we wish, we can replace them by actual G-CW complexes by use of the functor ? from G-spaces to G-CW complexes. For a complex representation V and based finite G-CW complexes X, Bott periodicity implies a natural isomorphism

$$[X, BU_G]_G \cong K_G^0(X) \cong K_G^0(\Sigma^V X) \cong [X, \Omega^V BU_G]_G.$$

By Adams' variant XIII.3.4 of Brown's representability theorem, this isomorphism is represented by a G-map $\tilde{\sigma} : BU_G \longrightarrow \Omega^V BU_G$, which must be an equivalence. However, we must check the vanishing of the appropriate $\lim^1$-term to see that the homotopy class of $\tilde{\sigma}$ is well-defined. Restricting to a cofinal sequence of representations so as to arrange transitivity (as in XIII.3.2), we have an Ω-G-prespectrum. It need not be Σ-cofibrant, but we can apply the cylinder construction K to make it so. Applying L, we then obtain a G-spectrum K_G. It is related to the Ω-G-prespectrum that we started with by a spacewise equivalence. Of course, the restriction to complex indexing spaces is no problem since we can extend to all real indexing spaces, as explained in XII§2.

Using real inner product spaces, we obtain an analogous G-space BO_G and an analogous isomorphism

$$[X, BO_G]_G \cong KO_G(X).$$

If we start with *Spin* representations of dimension $8n$, those being the ones for which we have real Bott periodicity, the same argument works to construct a G-spectrum KO_G that represents real K-theory.

5. The Atiyah-Segal completion theorem

It is especially important to understand bundles over the universal space EG, because of their role in the theory of characteristic classes. We have already mentioned one very simple construction of bundles. In fact for any representation V we may form the bundle $EG \times V \longrightarrow EG \times *$ and hence we obtain the homomorphism

$$\alpha : R(G) \longrightarrow K_G(EG).$$

Evidently α is induced by the projection map $\pi : EG \longrightarrow *$. The Atiyah-Segal completion theorem measures how near α is to being an isomorphism.

Of course, EG is a free G-CW complex. Any free G-CW complex is constructed from the G-spaces $G_+ \wedge S^n$ by means of wedges, cofibers, and passage to colimits. From the change of groups isomorphism $K^*_G(G_+ \wedge X) \cong K^*(X)$ we see that the augmentation ideal $I = I(G)$ acts as zero on the K-theory of any space $G_+ \wedge X$.

In particular the K-theory of a free sphere is complete as an $R(G)$-module for the topology defined by powers of I. Completeness is preserved by extensions of finitely generated modules, so we that $K^*_G(X)$ is I-complete for any finite free G-CW complex X. Completeness is also preserved by inverse limits so, *provided* $\lim^1$ *error terms vanish*, the K-theory of EG is I-complete.

Remarkably the K-theory of EG is fully accounted for by the representation ring, in the simplest way allowed for by completeness. The Atiyah-Segal theorem can be seen as a comparison between the algebraic process of I-adic completion and the geometric process of "completion" by making a space free.

The map α has a counterpart in all degrees, and it is useful to allow a parameter space, which will be a based G-space X. Thus we consider the map

$$\pi^* : K_G^*(X) \longrightarrow K_G^*(EG_+ \wedge X).$$

Note that the target is isomorphic to the non-equivariant K-theory $K^*(EG_+ \wedge_G X)$, and the following theorem may be regarded as a calculation of this in terms of the more approachable group $K_G^*(X)$.

THEOREM 5.1 (ATIYAH-SEGAL). Provided that X is a finite G-CW-complex, the map π^* above is completion at the augmentation ideal, so that

$$K_G^*(EG_+ \wedge X) \cong K_G^*(X)_I^\wedge.$$

In particular,

$$K_G^0(EG_+) = R(G)_I^\wedge \quad \text{and} \quad K_G^1(EG_+) = 0.$$

We sketch the simplest proof, which is that of Adams, Haeberly, Jackowski, and May. We skate over two technical points and return to them at the end. For simplicity of notation, we omit the parameter space X. We do not yet know that $K_G^*(EG_+)$ is complete since we do not yet know that the relevant $\lim^1$-term vanishes. If we did know this, we would be reduced to proving that $\pi : EG_+ \longrightarrow S^0$ induces an isomorphism of I-completed K-theory.

If we also knew that "completed K-theory" was a cohomology theory it would then be enough to show that the cofiber of π was acyclic. It is standard to let $\tilde{E}G$ denote this cofiber, which is easily seen to be the unreduced suspension of EG with one of the cone points as base point. That is, it would be enough to prove that $K_G^*(\tilde{E}G) = 0$ after completion.

The next simplification is adapted from a step in Carlsson's proof of the Segal conjecture. If we argue by induction on the size of the group (which is possible since chains of subgroups of compact Lie groups satisfy the descending chain condition), we may suppose the result proved for all proper subgroups H of G. Accordingly, by change of groups, $K_G^*(G/H_+ \wedge Y) = 0$ after completion for any nonequivariantly contractible space Y and hence by wedges, cofibers, and colimits $K_G^*(E \wedge Y) = 0$