EQUIVARIANT HOMOTOPY AND COHOMOLOGY THEORY

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Introduction

This volume began with Bob Piacenza's suggestion that I be the principal lecturer at an NSF/CBMS Regional Conference in Fairbanks, Alaska. That event took place in August of 1993, and the interim has seen very substantial progress in this general area of mathematics. The scope of this volume has grown accordingly.

The original focus was an introduction to equivariant algebraic topology, to stable homotopy theory, and to equivariant stable homotopy theory that was geared towards graduate students with a reasonably good understanding of nonequivariant algebraic topology. More recent material is changing the direction of the last two subjects by allowing the introduction of point-set topological algebra into stable homotopy theory, both equivariant and non-equivariant, and the last portion of the book focuses on an introduction to these new developments. There is a progression, with the later portions of the book on the whole being more difficult than the earlier portions.

Equivariant algebraic topology concerns the study of algebraic invariants of spaces with group actions. The first two chapters introduce the basic structural foundations of the subject: cellular theory, ordinary homology and cohomology theory, Eilenberg-Mac Lane G-spaces, Postnikov systems, localizations of G-spaces and completions of G-spaces. In most of this work, G can be any topological group, but we restrict attention to compact Lie groups in the rest of the book.

Chapter III, on equivariant rational homotopy theory, was written by Georgia Triantafillou. In it, she shows how to generalize Sullivan's theory of minimal models to obtain an algebraization of the homotopy category of (nilpotent) G-spaces for a finite group G. This chapter contains a first surprise: rational Hopf G-spaces need not split as products of Eilenberg-Mac Lane G-spaces. This is a hint that the calculational behavior of equivariant algebraic topology is more intricate and difficult to determine than that of the classical nonequivariant theory.

Chapter IV gives two proofs of the first main theorem of equivariant algebraic topology, which goes under the name of "Smith theory": any fixed point space of an action of a finite p-group on a mod p homology sphere is again a mod p homology sphere. One proof uses ordinary (or Bredon) equivariant cohomology and the other uses a general localization theorem in classical (or Borel) equivariant cohomology.

Parts of equivariant theory require a good deal of categorical bookkeeping, for example to keep track of fixed point data and to construct new G-spaces from diagrams of potential fixed point spaces. Some of the relevant background, such as geometric realization of simplicial spaces and the construction of homotopy colimits, is central to all of algebraic topology. These matters are dealt with in Chapter V, where Eilenberg-Mac Lane G-spaces and universal $\mathscr{F}$-spaces for families $\mathscr{F}$ of subgroups of a given group G are constructed. Special cases of such universal $\mathscr{F}$-spaces are used in Chapter VII to study the classification of equivariant bundles.

A different perspective on these matters is given in Chapter VI, which was written by Bob Piacenza. It deals with the general theory of diagrams of topological spaces, showing how to mimic classical homotopy and homology theory in categories of diagrams of topological spaces. In particular, Piacenza constructs a Quillen (closed) model category structure on any such category of diagrams and shows how these ideas lead to another way of passing from diagrams of fixed point spaces to their homotopical realization by G-spaces.

Chapter VIII combines equivariant ideas with the use of new tools in nonequivariant algebraic topology, notably Lannes' functor T in the context of unstable modules and algebras over the Steenrod algebra, to describe one of the most beautiful recent developments in algebraic topology, namely the Sullivan conjecture and its applications. While many mathematicians have contributed to this area, the main theorems are due to Haynes Miller, Gunnar Carlsson, and Jean Lannes. Although the set $[X,Y]$ of homotopy classes of based maps from a space X to a space Y is trivial to define, it is usually enormously difficult to compute. The Sullivan conjecture, in its simplest form, asserts that $[BG,X]=0$ if G is a finite group and X is a finite CW complex. It admits substantial generalizations which lead to much more interesting calculations, for example of the set of maps $[\mathrm{BG},\mathrm{BH}]$

for suitable compact Lie groups G and H. We shall see that an understanding of equivariant classifying spaces sheds light on what these calculations are really saying. There is already a large literature in this area, and we can only give an introduction. One theme is that the Sullivan conjecture can be viewed conceptually as a calculational elaboration of Smith theory. A starting point of this approach lies in work of Bill Dwyer and Clarence Wilkerson, which first exploited the study of modules over the Steenrod algebra in the context of the localization theorem in Smith theory.

We begin the study of equivariant stable homotopy theory in Chapter IX, which gives a brief introduction of some of the main ideas. The chapter culminates with a quick conceptual proof of a conjecture of Conner: if G is a compact Lie group and X is a finite dimensional G-CW complex with finitely many orbit types such that $\tilde{H}(X;\mathbb{Z}) = 0$, then $\tilde{H}(X/G;\mathbb{Z}) = 0$. This concrete statement is a direct consequence of the seemingly esoteric assertion that ordinary equivariant cohomology with coefficients in a Mackey functor extends to a cohomology theory graded on the real representation ring $RO(G)$; this means that there are suspension isomorphisms with respect to the based spheres associated to all representations, not just trivial ones. In fact, the interplay between homotopy theory and representation theory pervades equivariant stable homotopy theory.

One manifestation of this appears in Chapter X, which was written by Stefan Waner. It explains a variant theory of G-CW complexes defined in terms of representations and uses the theory to construct the required ordinary $RO(G)$-graded cohomology theories with coefficients in Mackey functors by means of appropriate cellular cochain complexes.

Another manifestation appears in Chapter XI, which was written by Gaunce Lewis and which explains equivariant versions of the Hurewicz and Freudenthal suspension theorems. The algebraic transition from unstable to stable phenomena is gradual rather than all at once. Nonequivariantly, the homotopy groups of first loop spaces are already Abelian groups, as are stable homotopy groups. Equivariantly, stable homotopy groups are modules over the Burnside ring, but the homotopy groups of Vth loop spaces for a representation V are only modules over a partial Burnside ring determined by V. The precise form of Lewis's equivariant suspension theorem reflects this algebraic fact.

Serious work in both equivariant and nonequivariant stable homotopy theory requires a good category of "stable spaces", called spectra, in which to work.

There is a great deal of literature on this subject. The original construction of the nonequivariant stable homotopy category was due to Mike Boardman. One must make a sharp distinction between the stable homotopy category, which is fixed and unique up to equivalence, and any particular point-set level construction of it. In fact, there are quite a few constructions in the literature. However, only one of them is known to generalize to the equivariant context, and that is also the one that is the basis for the new development of point-set topological algebra in stable homotopy theory. We give an intuitive introduction to this category in Chapter XII, beginning nonequivariantly and focusing on the construction of smash products and function spectra since that is the main technical issue. We switch to the equivariant case to explain homotopy groups, the suspension isomorphism for representation spheres, and the theory of G-CW spectra. We also explain how to transform the spectra that occur "in nature" to the idealized spectra that are the objects of the stable homotopy category.

In Chapters XIII, XIV, and XV, we introduce the most important $RO(G)$-graded cohomology theories and describe the G-spectra that represent them. We begin with an axiomatic account of exactly what $RO(G)$-graded homology and cohomology theories are and a proof that all such theories are representable by G-spectra. We also discuss ring G-spectra and products in homology and cohomology theories. We show how to construct Eilenberg-Mac Lane G-spectra by representing the zeroth term of a $\mathbb{Z}$-graded cohomology theory defined by means of G-spectrum level cochains. This implies an alternative construction of ordinary $RO(G)$-graded cohomology theories with coefficients in Mackey functors.

Chapter XIV, which was written by John Greenlees, gives an introduction to equivariant K-theory. The focus is on equivariant Bott periodicity and its use to prove the Atiyah-Segal completion theorem. That theorem states that, for any compact Lie group G, the nonequivariant K-theory of the classifying space BG is isomorphic to the completion of the representation ring $R(G)$ at its augmentation ideal I. The result is of considerable importance in the applications of K-theory, and it is the prototype for a number of analogous results to be described later.

Chapter XV, which was written by Steve Costenoble, gives an introduction to equivariant cobordism. The essential new feature is that transversality fails in general, so that geometric equivariant bordism is not same as stable (or homotopical) bordism; the latter is the theory represented by the most natural equivariant generalization of the nonequivariant Thom spectrum. Costenoble also explains the use of adjacent families of subgroups to reduce the calculation of equivariant bordism to suitably related nonequivariant calculations. The equivariant results are considerably more intricate than the nonequivariant ones. While the G-spectra that represent unoriented geometric bordism and its stable analog split as products of Eilenberg Mac Lane G-spectra for finite groups of odd order, just as in the nonequivariant case, this is false for the cyclic group of order 2.

Chapters XVI–XIX describe the basic machinery and results on which all work in equivariant stable homotopy theory depends. Chapter XVI describes fixed point and orbit spectra, shows how to relate equivariant and nonequivariant homology and cohomology theories, and, more generally, shows how to relate homology and cohomology theories defined for a group G to homology and cohomology theories defined for subgroups and quotient groups of G. These results about change of groups are closely related to duality theory, and we give basic information about equivariant Spanier-Whitehead, Atiyah, and Poincaré duality.

In Chapter XVII, we discuss the Burnside ring $A(G)$. When G is finite, $A(G)$ is the Grothendieck ring associated to the semi-ring of finite G-sets. For any compact Lie group G, $A(G)$ is isomorphic to the zeroth equivariant stable homotopy group of spheres. It therefore acts on the equivariant homotopy groups $\pi_n^G(X) = \pi_n(X^G)$ of any G-spectrum X, and this implies that it acts on all homology and cohomology groups of any G-spectrum. Information about the algebraic structure of $A(G)$ leads to information about the entire stable homotopy category of G-spectra. It turns out that $A(G)$ has Krull dimension one and an easily analyzed prime ideal spectrum, making it quite a tractable ring. Algebraic analysis of localizations of $A(G)$ leads to analysis of localizations of equivariant homology and cohomology theories. For example, for a finite group G, the localization of any theory at a prime p can be calculated in terms of subquotient p-groups of G.

In Chapter XVIII, we construct transfer maps, which are basic calculational tools in equivariant and nonequivariant bundle theory, and describe their basic properties. Special cases were vital to the earlier discussion of change of groups. The deepest property is the double coset formula, and we say a little about its applications to the study of the cohomology of classifying spaces.

In Chapter XIX, we discuss several fundamental splitting theorems in equivariant stable homotopy theory. These describe the equivariant stable homotopy groups of G-spaces in terms of nonequivariant homotopy groups of fixed point spaces. These theorems lead to an analysis of the structure of the subcategory of the stable category whose objects are the suspension spectra of orbit spaces. A Mackey functor is an additive contravariant functor from this subcategory to Abelian groups, and, when G is finite, the analysis leads to a proof that this topological definition of Mackey functors is equivalent to an earlier and simpler algebraic definition. Mackey functors describe the algebraic structure that is present on the system of homotopy groups $\pi_n^H(X) = \pi_n(X^H)$ of a G-spectrum X, where H runs over the subgroups of G. The action of the Burnside ring on $\pi_n^G(X)$ is part of this structure. It is often more natural to study such systems than to focus on the individual groups. In particular, we describe algebraic induction theorems that often allow one to calculate the value of a Mackey functor on the orbit G/G from its values on the orbits G/H for certain subgroups H. Such theorems have applications in various branches of mathematics in which finite group actions appear. Again, algebraic analysis of rational Mackey functors shows that, when G is finite, rational G-spectra split as products of Eilenberg-Mac Lane G-spectra. This is false for general compact Lie groups G.

In Chapter XX, we turn to another of the most beautiful recent developments in algebraic topology: the Segal conjecture and its applications. The Segal conjecture can be viewed either as a stable analogue of the Sullivan conjecture or as the analogue in equivariant stable cohomotopy of the Atiyah-Segal completion theorem in equivariant K-theory. The original conjecture, which is just a fragment of the full result, asserts that, for a finite group G, the zeroth stable cohomotopy group of the classifying space BG is isomorphic to the completion of $A(G)$ at its augmentation ideal I. The key step in the proof of the Segal conjecture is due to Gunnar Carlsson. We explain the proof and also explain a number of generalizations of the result. One of these leads to a complete algebraic determination of the group of homotopy classes of stable maps between the classifying spaces of any two finite groups. This is analogous to the role of the Sullivan conjecture in the study of ordinary homotopy classes of maps between classifying spaces. Use of equivariant classifying spaces is much more essential here. In fact, the Segal conjecture is intrinsically a result about the I-adic completion of the sphere G-spectrum, and the application to maps between classifying spaces depends on a generalization in which the sphere G-spectrum is replaced by the suspension G-spectra of equivariant classifying spaces.

Chapter XXI is an exposition of joint work of John Greenlees and myself in which we generalize the classical Tate cohomology of finite groups and the periodic cyclic cohomology of the circle group to obtain a Tate cohomology theory associated to any given cohomology theory on G-spectra, for any compact Lie group G. This work has had a variety of applications, most strikingly to the computation of the topological cyclic homology and thus to the algebraic K-theory of number rings. While we shall not get into that application here, we shall describe the general Atiyah-Hirzebruch-Tate spectral sequences that are used in that work and we shall give a number of other applications and calculations. For example, we shall explain a complete calculation of the Tate theory associated to the equivariant K-theory of any finite group. This is an active area of research, and some of what we say at the end of this chapter is rather speculative. The Tate theory provides some of the most striking examples of equivariant phenomena illuminating nonequivariant phenomena, and it leads to interrelationships between the stable homotopy groups of spheres and the Tate cohomology of finite groups that have only begun to be explored.

Chapters XXII through XXV concern "brave new algebra", the study of point-set level topological algebra in stable homotopy theory. The desirability of such a theory was advertised by Waldhausen under the rubric of "brave new rings", hence the term "brave new algebra" for the new subject. Its starting point is the construction of a new category of spectra, the category of "S-modules", that has a smash product that is symmetric monoidal (associative, commutative, and unital up to coherent natural isomorphisms) on the point-set level. The construction is joint work of Tony Elmendorf, Igor Kriz, Mike Mandell, and myself, and it changes the nature of stable homotopy theory. Ever since its beginnings with Adams' use of stable homotopy theory to solve the Hopf invariant one problem some thirty-five years ago, most work in the field has been carried out working only "up to homotopy"; formally, this means that one is working in the stable homotopy category. For example, classically, the product on a ring spectrum is defined only up to homotopy and can be expected to be associative and commutative only up to homotopy. In the new theory, we have rings with well-defined point-set level products, and they can be expected to be strictly associative and commutative. In the associative case, we call these "S-algebras". The new theory permits constructions that have long been desired, but that have seemed to be out of reach technically: simple constructions of many of the most basic spectra in current use in algebraic topology; simple constructions of generalized universal coefficient, Künneth, and other spectral sequences; a conceptual and structured approach to Bousfield localizations of spectra, a generalized construction of topological Hochschild homology and of spectral sequences for its computation; a simultaneous generalization of the algebraic K-theory of rings and of spaces; etc. Working nonequivariantly, we shall describe the properties of the category of S-modules and shall sketch all but the last of the cited applications in Chapter XXII.

We return to the equivariant world in Chapter XXIII, which was written jointly with Elmendorf and Lewis, and sketch how the construction of the category of S_G-modules works. Here S_G denotes the sphere G-spectrum. The starting point of the construction is the "twisted half smash product", which is a spectrum level generalization of the half-smash product $X \ltimes Y = X_+ \wedge Y$ of an unbased G-space X and a based G-space Y and is perhaps the most basic construction in equivariant stable homotopy theory. Taking X to be a certain G-space $\mathscr{L}(j)$ of linear isometries, one obtains a fattened version $\mathscr{L}(j) \ltimes E_1 \wedge \cdots \wedge E_j$ of the j-fold smash product of G-spectra. Taking $j = 2$, insisting that the E_i have extra structure given by maps $\mathscr{L}(1) \ltimes E_i \longrightarrow E_i$, and quotienting out some of the fat, one obtains a commutative and associative smash product of G-spectra with actions by the monoid $\mathscr{L}(1)$; a little adjustment adds in the unit condition and gives the category of S_G-modules. The theory had its origins in the notion of an E_∞ ring spectrum introduced by Quinn, Ray, and myself over twenty years ago. Such rings were defined in terms of "operad actions" given by maps $\mathscr{L}(j) \ltimes E^j \longrightarrow E$, where E^j is the j-fold smash power of E, and it turns out that such rings are virtually the same as our new commutative S_G-algebras. The new theory makes the earlier notion much more algebraically tractable, while the older theory gives the basic examples to which the new theory can be applied.

In Chapter XXIV, which was written jointly with Greenlees, we give a series of algebraic definitions, together with their brave new algebra counterparts, and we show how these notions lead to a general approach to localization and completion theorems in equivariant stable homotopy theory. We shall see that Grothendieck's local cohomology groups are relevant to the study of localization theorems in equivariant homology and that analogs called local homology groups are relevant to the study of completion theorems in equivariant cohomology. We use these constructions to prove a general localization theorem for suitable commutative S_G-algebras R_G. Taking R to be the underlying S-algebra of R_G and taking M to be the underlying R-module of an R_G-module M_G, the theorem implies both a localization theorem for the computation of $M_*(BG)$ in terms of $M^G_*(pt)$

and a completion theorem for the computation of $M^*(BG)$ in terms of $M^*_G(pt)$. Of course, this is reminiscent of the Atiyah-Segal completion for equivariant K-theory and the Segal conjecture for equivariant cobordism. The general theorem does apply to K-theory, giving a very clean description of $K_*(BG)$, but it does not apply to cohomotopy: there the completion theorem for cohomology is true but the localization theorem for homology is false.

We are particularly interested in stable equivariant complex bordism, represented by MU_G, and modules over it. We explain in Chapter XXII how simple it is to construct all of the usual examples of MU-module spectra in the homotopical sense, such as Morava K-theory and Brown-Peterson spectra, as brave new point-set level MU-modules. We show in Chapter XXIII how to construct equivariant versions M_G as brave new MU_G-modules of all such MU-modules M, where G is any compact Lie group. We would like to apply the localization theorem of Chapter XXIV to MU_G and its module spectra, but its algebraic hypotheses are not satisfied. Nevertheless, as Greenlees and I explain in chapter XXV, the localization theorem is in fact true for MU_G when G is finite or a finite extension of a torus. The proof involves the construction of a multiplicative norm map in MU^G_*, together with a double coset formula for its computation. This depends on the fact that MU_G can be constructed in a particularly nice way, codified in the notion of a "global $\mathscr{I}_*$-functor with smash product", as a functor of G.

These results refocus attention on stable equivariant complex bordism, whose study lapsed in the early 1970's. In fact, some of the most significant calculational results obtained then were never fully documented in the literature. In Chapter XXVI, which was written by Gustavo Comezana, new and complete proofs of these results are presented, along with results on the relationship between geometric and stable equivariant complex cobordism. In particular, when G is a compact Abelian Lie group, Comezana proves that MU^G_* is a free MU_*-module on even degree generators.

In Chapter XXVI, and in a few places earlier on, complete proofs are given either because we feel that the material is inadequately treated in the published literature or because we have added new material. However, most of the material in the book is known and has been treated in full detail elsewhere. Our goal has been to present what is known in a form that is more readily accessible and assimilable, with emphasis on the main ideas and the structure of the theory and with pointers to where full details and further developments can be found.

Most sections have their own brief bibliographies at the end; thus, if an author's work is referred to in a section, the appropriate reference is given at the end of that section. There is also a general bibliography but, since it has over 200 items, I felt that easily found local references would be more helpful. With a few exceptions, the general bibliography is restricted to items actually referred to in the text, and it makes no claim to completeness. A full list of relevant and interesting papers would easily double the number of entries. I offer my apologies to authors not cited who should have been. Inevitably, the choice of topics and of material within topics has had to be very selective and idiosyncratic.

There are some general references that should be cited here (reminders of their abbreviated names will be given where they are first used). Starting with Chapter XII, references to [LMS] (= [133]) are to

L.G. Lewis, J.P. May, and M. Steinberger (with contributions by J.E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Vol. 1213. 1986.

Most of the material in Chapter XII and in the five chapters XV–XIX is based on joint work of Gaunce Lewis and myself that is presented in perhaps excruciating detail in that rather encyclopedic volume. There are also abbreviated references in force in particular chapters: [L1]–[L3] = [128, 129, 130] in Chapter XI and [tD] = [55] in Chapter XVII.

The basic reference for the proofs of the claims in Chapters XXII and XXIII is

[EKMM] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Preprint, 1995.

We shall also refer to the connected sequence of expository papers [73, 88, 89]

[EKMM′] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory.

[GM1] J. P. C. Greenlees and J. P. May. Completions in algebra and topology.

[GM2] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory.

These are all in the "Handbook of Algebraic Topology", edited by Ioan James, that came out in 1995. While these have considerable overlap with Chapters XXII through XXIV, we have varied the perspective and emphasis, and each exposition includes a good deal of material that is not discussed in the other. In particular, we point to the application of brave new algebra to chromatic periodicity in [GM1]: the ideas there have yet to be fully exploited and are not discussed here.

In view of the broad and disparate range of topics, we have tried very hard to make the chapters, and often even the sections, independent of one another. We have also broken the material into short and hopefully manageable chunks; only a few sections are as long as five pages, and all chapters are less than twenty-five pages long. Very few readers are likely to wish to read straight through, and the reader should be unafraid to jump directly to what he or she finds of interest. The reader should also be unintimidated by finding that he or she has insufficient background to feel comfortable with particular sections or chapters. Unfortunately, the subject of algebraic topology is particularly badly served by its textbooks. For example, none of them even mentions localizations and completions of spaces, although those have been standard tools since the early 1970's. We have tried to include enough background to give the basic ideas. Modern algebraic topology is a thriving subject, and perhaps jumping right in and having a look at some of its more recent directions may give a better perspective than trying to start at the beginning and work one's way up.

As the reader will have gathered, this book is a cooperative enterprise. Perhaps this is the right place to try to express just how enormously grateful I am to all of my friends, collaborators, and students. This book owes everything to our joint efforts over many years. When planning the Alaska conference, I invited some of my friends and collaborators to give talks that would mesh with mine and help give a reasonably coherent overview of the subject. Most of the speakers wrote up their talks and gave me license to edit them to fit into the framework of the book. Since TeX is refractory about listing authors inside a Table of Contents, I will here list those chapters that are written either solely by other authors or jointly with me.

Chapter III. Equivariant rational homotopy theory

by Georgia Triantafillou

Chapter VI. The homotopy theory of diagrams

by Robert Piacenza

Chapter X. G-CW(V) complexes and $RO(G)$-graded cohomology

by Stefan Waner

Chapter XI. The equivariant Hurewicz and suspension theorems

by L. G. Lewis Jr.

Chapter XIV. An introduction to equivariant K-theory

by J. P. C. Greenlees

Chapter XV. An introduction to equivariant cobordism by Steven Costenoble

Chapter XXI. Generalized Tate cohomology

by J. P. C. Greenlees and J. P. May

Chapter XXIII. Brave new equivariant foundations

by A. D. Elmendorf, L. G. Lewis Jr., and J. P. May

Chapter XXIV. Brave new equivariant algebra

by J. P. C. Greenlees and J. P. May

Chapter XXV. Localization and completion in complex cobordism

by J. P. C. Greenlees and J. P. May

Chapter XXVI. Some calculations in complex equivariant bordism

by Gustavo Costenoble

My deepest thanks to these people and to Stefan Jackowski and Chun-Nip Lee, who also gave talks; their topics were the subjects of their recent excellent survey papers [107] and [125] and were therefore not written up for inclusion here. I would also like to thank Jim McClure, whose many insights in this area are reflected throughout the book, and Igor Kriz, whose collaboration over the last six years has greatly influenced the more recent material. I would also like to thank my current students at Chicago — Maria Basterra, Mike Cole, Dan Isaksen, Mike Mandell, Adam Przezdziecki, Laura Scull, and Jerome Wolbert — who have helped catch many soft spots of exposition and have already made significant contributions to this general area of mathematics.

It is an especial pleasure to thank Bob Piacenza and his wife Lyric Ozburn for organizing the Alaska conference and making it a memorably pleasant occasion for all concerned. Thanks to their thoughtful arrangements, the intense all day mathematical activity took place in a wonderfully convivial and congenial atmosphere. Finally, my thanks to all of those who attended the conference and helped make the week such a pleasant mathematical occasion: thanks for bearing with me.

J. Peter May

December 31, 1995

CHAPTER I

Equivariant Cellular and Homology Theory

1. Some basic definitions and adjunctions

The objects of study in equivariant algebraic topology are spaces equipped with an action by a topological group G. That is, the subject concerns spaces X together with continuous actions $G \times X \longrightarrow X$ such that $ex = x$ and $g(g'x) = (gg')x$. Maps $f : X \longrightarrow Y$ are equivariant if $f(gx) = gf(x)$. We then say that f is a G-map. The usual constructions on spaces apply equally well in the category $G\mathscr{U}$ of G-spaces and G-maps. In particular G acts diagonally on Cartesian products of spaces and acts by conjugation on the space $\mathrm{Map}(X, Y)$ of maps from X to Y. That is, we define $g \cdot f$ by $(g \cdot f)(x) = gf(g^{-1}x)$.

As usual, we take all spaces to be compactly generated (which means that a subspace is closed if its intersection with each compact Hausdorff subspace is closed) and weak Hausdorff (which means that the diagonal $X \subset X \times X$ is a closed subset, where the product is given the compactly generated topology). Among other things, this ensures that we have a G-homeomorphism

$$(1.1) \qquad \mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

for any G-spaces X, Y, and Z.

For us, subgroups of G are assumed to be closed. For $H \subset G$, we write $X^H = \{x | hx = x \text{ for } h \in H\}$. For $x \in X$, $G_x = \{h | hx = x\}$ is called the isotropy group of x. Thus $x \in X^H$ if H is contained in G_x. A good deal of the formal homotopy theory of G-spaces reduces to the ordinary homotopy theory of fixed point spaces. We let NH be the normalizer of H in G and let $WH = NH/H$. (We sometimes write N_GH and W_GH.) These "Weyl groups" appear ubiquitously in the theory. Note that X^H is a WH-space. In equivariant theory, orbits G/H play the role of points, and the set of G-maps $G/H \longrightarrow G/H$ can be identified with the group WH. We also have the orbit spaces X/H obtained by identifying points of X in the same orbit, and these too are WH-spaces. For a space K regarded as a G-space with trivial G-action, we have

$$G\mathscr{U}(K,X) \cong \mathscr{U}(K,X^G) \tag{1.2}$$

and

$$G\mathscr{U}(X,K) \cong \mathscr{U}(X/G,K). \tag{1.3}$$

If Y is an H-space, there is an induced G-space $G \times_H Y$. It is obtained from $G \times Y$ by identifying (gh, y) with (g, hy) for $g \in G$, $h \in H$, and $y \in Y$. A bit less obviously, we also have the "coinduced" G-space $\mathrm{M}ap_H(G,Y)$, which is the space of H-maps $G \longrightarrow Y$ with left action by G induced by the right action of G on itself, $(g \cdot f)(g') = f(g'g)$. For G-spaces X and H-spaces Y, we have the adjunctions

$$G\mathscr{U}(G \times_H Y, X) \cong H\mathscr{U}(Y,X) \tag{1.4}$$

and

$$H\mathscr{U}(X,Y) \cong G\mathscr{U}(X, \mathrm{Map}_H(G,Y)). \tag{1.5}$$

Moreover, for G-spaces X, we have G-homeomorphisms

$$G \times_H X \cong (G/H) \times X \tag{1.6}$$

and

$$\mathrm{Map}_H(G,X) \cong \mathrm{Map}(G/H,X). \tag{1.7}$$

For the first, the unique G-map $G \times_H X \longrightarrow (G/H) \times X$ that sends $x \in X$ to (eH, x) has inverse that sends (gH, x) to the equivalence class of $(g, g^{-1}x)$.

A homotopy between G-maps $X \longrightarrow Y$ is a homotopy $h : X \times I \longrightarrow Y$ that is a G-map, where G acts trivially on I. There results a homotopy category $hG\mathscr{U}$. Recall that a map of spaces is a weak equivalence if it induces an isomorphism of all homotopy groups. A G-map $f : X \longrightarrow Y$ is said to be a weak equivalence if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all $H \subset G$. We let $\bar{h}G\mathscr{U}$ denote the category constructed from $hG\mathscr{U}$ by adjoining formal inverses to the weak equivalences. We shall be more precise shortly. The algebraic invariants of G-spaces that we shall be interested in will be defined on the category $\bar{h}G\mathscr{U}$.

General References

G. E. Bredon. Introduction to compact transformation groups. Academic Press. 1972.

T. tom Dieck. Transformation groups. Walter de Gruyter. 1987.

(This reference contains an extensive Bibliography.)

2. Analogs for based G-spaces

It will often be more convenient to work with based G-spaces. Basepoints are G-fixed and are generically denoted by $*$. We write X_+ for the union of a G-space X and a disjoint basepoint. The wedge, or 1-point union, of based G-spaces is denoted by $X \vee Y$. The smash product is defined by $X \wedge Y = X \times Y / X \vee Y$. We write $F(X, Y)$ for the based G-space of based maps $X \longrightarrow Y$. Then

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)). \tag{2.1}$$

We write $G\mathscr{T}$ for the category of based G-spaces, and we have

$$G\mathscr{T}(K, X) \cong \mathscr{T}(K, X^G) \tag{2.2}$$

and

$$G\mathscr{T}(X, K) \cong \mathscr{T}(X/G, K) \tag{2.3}$$

for a based space K and a based G-space X. Similarly, for a based G-space X and a based H-space Y, we have

$$G\mathscr{T}(G_+ \wedge_H Y, X) \cong H\mathscr{T}(Y, X) \tag{2.4}$$

and

$$H\mathscr{T}(X, Y) \cong G\mathscr{T}(X, F_H(G_+, Y)), \tag{2.5}$$

where $F_H(G_+, Y) = \mathrm{Map}_H(G, X)$ with the trivial map as basepoint, and we have G-homeomorphisms

$$G_+ \wedge_H X \cong (G/H)_+ \wedge X \tag{2.6}$$

and

$$F_H(G_+, X) \cong F(G/H_+, X). \tag{2.7}$$

A based homotopy between based G-maps $X \longrightarrow Y$ is given by a based G-map $X \wedge I_+ \longrightarrow Y$. Here the based cylinder $X \wedge I_+$ is obtained from $X \times I$ by collapsing the line through the basepoint of X to the basepoint. There results a homotopy category $hG\mathscr{T}$, and we construct $\bar{h}G\mathscr{T}$ by formally inverting the weak equivalences. Of course, we have analogous categories $hG\mathscr{U}$, and $\bar{h}G\mathscr{U}$ in the unbased context.

In both the based and unbased context, cofibrations and fibrations are defined exactly as in the nonequivariant context, except that all maps in sight are G-maps. Their theory goes through unchanged. A based G-space X is nondegenerately based if the inclusion $\{*\} \longrightarrow X$ is a cofibration.

3. G-CW complexes

A G-CW complex X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along attaching G-maps $G/H \times S^n \longrightarrow X^n$. Such an attaching map is determined by its restriction $S^n \longrightarrow (X^n)^H$, and this allows the inductive analysis of G-CW complexes by reduction to nonequivariant homotopy theory. Subcomplexes and relative G-CW complexes are defined in the obvious way. I will review my preferred way of developing the theory of G-CW complexes since this will serve as a model for other versions of cellular theory that we shall encounter.

We begin with the Homotopy Extension and Lifting Property. Recall that a map $f : X \longrightarrow Y$ is an n-equivalence if $\pi_q(f)$ is a bijection for $q < n$ and a surjection for $q = n$ (for any choice of basepoint). Let ν be a function from conjugacy classes of subgroups of G to the integers ≥ -1. We say that a map $e : Y \longrightarrow Z$ is a ν-equivalence if $e^H : Y^H \longrightarrow Z^H$ is a $\nu(H)$-equivalence for all H. (We allow $\nu(H) = -1$ to allow for empty fixed point spaces.) We say that a G-CW complex X has dimension ν if its cells of orbit type G/H all have dimension $\leq \nu(H)$.

Theorem 3.1 (HELP). *Let A be a subcomplex of a G-CW complex X of dimension ν and let $e : Y \longrightarrow Z$ be a ν-equivalence. Suppose given maps $g : A \longrightarrow Y$, $h : A \times I \longrightarrow Z$, and $f : X \longrightarrow Z$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"'] & & X. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

PROOF. We construct $\tilde{g}$ and $\tilde{h}$ on $A \cup X^n$ by induction on n. When we pass from the n-skeleton to the $(n+1)$-skeleton, we may work one cell at a time, dealing with the cells of X not in A. By considering attaching maps, we quickly reduce the proof to the case when $(X, A) = (G/H \times D^{n+1}, G/H \times S^n)$. But this case reduces directly to the nonequivariant case of (D^{n+1}, S^n). $\square$

THEOREM 3.2 (WHITEHEAD). Let $e : Y \longrightarrow Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : hG\mathscr{U}(X, Y) \longrightarrow hG\mathscr{U}(X, Z)$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.

PROOF. Apply HELP to the pair $(X, \emptyset)$ for the surjectivity. Apply HELP to the pair $(X \times I, X \times \partial I)$ for the injectivity. $\square$

COROLLARY 3.3. If $e : Y \longrightarrow Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.

PROOF. A map $f : Z \longrightarrow Y$ such that $e_*[f] = \mathrm{id}$ is a homotopy inverse to e. $\square$

The cellular approximation theorem works equally simply. A map $f : X \longrightarrow Y$ between G-CW complexes is said to be cellular if $f(X^n) \subset Y^n$ for all n, and similarly in the relative case.

THEOREM 3.4 (CELLULAR APPROXIMATION). Let (X, A) and (Y, B) be relative G-CW complexes, (X', A') be a subcomplex of (X, A), and $f : (X, A) \longrightarrow (Y, B)$ be a G-map whose restriction to (X', A') is cellular. Then f is homotopic rel $X' \cup A$ to a cellular map $g : (X, A) \longrightarrow (Y, B)$.

PROOF. This again reduces to the case of a single nonequivariant cell. $\square$

COROLLARY 3.5. Let X and Y be G-CW complexes. Then any G-map $f : X \longrightarrow Y$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.

PROOF. Apply the theorem in the cases $(X, \emptyset)$ and $(X \times I, X \times \partial I)$. $\square$

THEOREM 3.6. For any G-space X, there is a G-CW complex ΓX and a weak equivalence $\gamma : \Gamma X \longrightarrow X$.

PROOF. We construct an expanding sequence of G-CW complexes $\{Y_i | i \geq 0\}$ together with maps $\gamma_i : Y_i \longrightarrow X$ such that $\gamma_{i+1}|Y_i = \gamma_i$. Choose a representative map $f : S^q \longrightarrow X^H$ for each element of $\pi_q(X^H, x)$. Here q runs over the non-negative integers, H runs over the conjugacy classes of subgroups of G, and x runs over the components of X^H. Let Y_0 be the disjoint union of spaces $G/H \times S^q$, one for each chosen map f, and let γ_0 be the G-map induced by the maps f. Inductively, assume that $\gamma_i : Y_i \longrightarrow X$ has been constructed. Choose representative maps (f, g) for each pair of elements of $\pi_q((Y_i)^H, y)$ that are equalized by $\pi_q(\gamma_i)$; here again, q runs over the non-negative integers, H runs over the conjugacy classes of subgroups of G, and y runs over the components of $(Y_i)^H$. We may arrange that f and g have image in the q-skeleton of Y_i. Let Y_{i+1} be the homotopy coequalizer of the disjoint union of these pairs of maps; that is Y_{i+1} is obtained by attaching a tube $(G/H_+ \wedge S^q \times I_+$ via each chosen pair (f, g). Define γ_{i+1} by use of homotopies $h : \gamma_i f \simeq \gamma_i g$ based at $\gamma_i(y)$. It is easy to triangulate Y_{i+1} as a G-CW complex that contains Y_i as a subcomplex. Taking ΓX to be the union of the Y_i and γ to be the map induced by the γ_i, we obtain the desired weak equivalence. □

The Whitehead theorem implies that the G-CW approximation ΓX is unique up to G-homotopy equivalence. If $f : X \longrightarrow X'$ is a G-map, there is a unique homotopy class of G-maps $\Gamma f : \Gamma X \longrightarrow \Gamma X'$ such that $\gamma' \circ \Gamma f \simeq f \circ \gamma$. That is, Γ becomes a functor $hG\mathscr{U} \longrightarrow hG\mathscr{U}$ such that γ is natural. A construction of Γ that is functorial even before passage to homotopy is possible (Seymour). It follows that the morphisms of the category $\bar{h}G\mathscr{U}$ can be specified by

$$\bar{h}G\mathscr{U}(X, X') = hG\mathscr{U}(\Gamma X, \Gamma X') = hG\mathscr{C}(\Gamma X, \Gamma X'), \tag{3.7}$$

where $G\mathscr{C}$ is the category of G-CW complexes and cellular maps. From now on, we shall write $[X, X']_G$ for this set, or for its based variant, depending on the context.

Almost all of this works just as well in the based context, giving a theory of "G-CW based complexes", which are required to have based attaching maps. This notion is to be distinquished from that of a based G-CW complex, which is just a G-CW complex with a G-fixed base vertex. In detail, a G-CW based complex X is the union of based sub G-spaces X^n such that X^0 is a point and X^{n+1} is obtained from X^n by attaching G-cells $G/H_+ \wedge D^{n+1}$ along based attaching G-maps $G/H_+ \wedge S^n \longrightarrow X^n$. Observe that such G-CW based complexes are G-connected in the sense that all of their fixed point spaces are non-empty and connected.

Nonequivariantly, one often starts proofs with the simple remark that it suffices to consider connected spaces. Equivariantly, this won't do; many important foundational parts of homotopy theory have only been worked out for G-connected G-spaces.

I should emphasize that G has been an arbitrary topological group in this discussion. When G is a compact Lie group — and we shall later restrict attention to such groups — there are important results saying that reasonable spaces are triangulable as G-CW complexes or have the homotopy types of G-CW complexes. It is fundamental for our later work that smooth compact G-manifolds are triangulable as finite G-CW complexes (Verona, Illman). In contrast to the nonequivariant situation, this is false for topological G-manifolds, which have the homotopy types of G-CW complexes but not necessarily finite ones. Metric G-ANR's have the homotopy types of G-CW complexes (Kwasik). Milnor's results on spaces of the homotopy type of CW complexes generalize to G-spaces (Waner). In particular, $\mathrm{Map}(X, Y)$ has the homotopy type of a G-CW complex if X is a compact G-space and Y has the homotopy type of a G-CW complex, and similarly for based function spaces.

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4. Ordinary homology and cohomology theories

Let $\mathscr{G}$ denote the category of orbit G-spaces G/H; the standard notation is $\mathscr{O}_G$. Observe that there is a G-map $f : G/H \longrightarrow G/K$ if and only if H is subconjugate to K since, if $f(eH) = gK$, then $g^{-1}Hg \subset K$. Let $h\mathscr{G}$ be the homotopy category of $\mathscr{G}$. Both $\mathscr{G}$ and $h\mathscr{G}$ play important roles and it is essential to keep the distinction in mind.

Define a coefficient system to be a contravariant functor $h\mathscr{G} \longrightarrow \mathscr{A}b$. One example to keep in mind is the system $\underline{\pi}_n(X)$ of homotopy groups of a based

G-space X : $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$. Formally, we have an evident fixed point functor $X^* : \mathscr{G} \longrightarrow \mathscr{T}$. The map $X^K \longrightarrow X^H$ induced by a G-map $f : G/H \longrightarrow G/K$ such that $f(eH) = gK$ sends x to gx. Any covariant functor $h\mathscr{T} \longrightarrow \mathscr{A}b$, such as π_n, can be composed with this functor to give a coefficient system. It should be intuitively clear that obstruction theory must be developed in terms of ordinary cohomology theories with coefficients in such coefficient systems. The appropriate theories were introduced by Bredon.

Since the category of coefficient systems is Abelian, with kernels and cokernels defined termwise, we can do homological algebra in it. Let X be a G-CW complex. We have a coefficient system

$$\underline{C}_n(X) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}). \tag{4.1}$$

That is, the value on G/H is $H_n((X^n)^H, (X^{n-1})^H)$. The connecting homomorphisms of the triples $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$ specify a map

$$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X)$$

of coefficient systems, and $d^2 = 0$. That is, we have a chain complex of coefficient systems $\underline{C}_*(X)$. For based G-CW complexes, we define $\tilde{\underline{C}}_*(X)$ similarly. Write $\mathrm{Hom}_{\mathscr{G}}(M, M')$ for the Abelian group of maps of coefficient systems $M \longrightarrow M'$ and define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathscr{G}}(\underline{C}_n(X), M), \text{ with } \delta = \mathrm{Hom}_{\mathscr{G}}(d, \mathrm{id}). \tag{4.2}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. Its homology is the Bredon cohomology of X, denoted $H^*_G(X; M)$.

To define Bredon homology, we must use covariant functors $N : h\mathscr{G} \longrightarrow \mathscr{A}b$ as coefficient systems. If $M : h\mathscr{G} \longrightarrow \mathscr{A}b$ is contravariant, we define an Abelian group

$$M \otimes_{\mathscr{G}} N = \sum M(G/H) \otimes N(G/H)/(\approx),$$

where the equivalence relation is specified by $(mf^*, n) \approx (m, f_*n)$ for a map $f : G/H \longrightarrow G/K$ and elements $m \in M(G/K)$ and $n \in N(G/H)$. Here we write contravariant actions from the right to emphasize the analogy with tensor products. Such "coends", or categorical tensor products of functors, occur very often in equivariant theory and will be formalized later. We define cellular chains by

$$C^G_n(X; N) = \underline{C}_n(X) \otimes_{\mathscr{G}} N, \text{ with } \partial = d \otimes 1. \tag{4.3}$$

Then $C^G_*(X;N)$ is a chain complex of Abelian groups. Its homology is the Bredon homology of X, denoted $H^G_*(X;N)$.

Clearly Bredon homology and cohomology are functors on the category $G\mathscr{C}$ of G-CW complexes and cellular maps. A cellular homotopy is easily seen to induce a chain homotopy of cellular chain complexes in our Abelian category of coefficient systems, so homotopic maps induce the same homomorphism on homology and cohomology with any coefficients.

The development of the properties of these theories is little different from the nonequivariant case. A key point is that $\underline{C}_*(X)$ is a projective object in the category of coefficient systems. To see this, observe that $\underline{C}_*(X)$ is a direct sum of coefficient systems of the form

$$\tilde{\underline{H}}_n(G/K_+ \wedge S^n) \cong \tilde{\underline{H}}_0(G/K_+) \cong \underline{H}_0(G/K). \tag{4.4}$$

If F denotes the free Abelian group functor on sets, then

$$\underline{H}_0(G/K)(G/H) = H_0((G/K)^H) = F\pi_0((G/K)^H) = F[G/H, G/K]_G. \tag{4.5}$$

Therefore $\mathrm{Hom}_{\mathscr{G}}(\underline{H}_0(G/K), M) \cong M(G/K)$ via $\phi \longrightarrow \phi(1_{G/K}) \in M(G/K)$. In detail, for a G-map $f : G/H \longrightarrow G/K$, we have $f = f^*(1_{G/K})$,

$$f^* : F[G/K, G/K]_G \longrightarrow F[G/H, G/K]_G,$$

so that $\phi(1_{G/K})$ determines ϕ via $\phi(f) = f^*\phi(1_{G/K})$. This calculation implies the claimed projectivity. It also implies the dimension axiom:

$$H^*_G(G/K; M) = H^0_G(G/K; M) \cong M(G/K) \tag{4.6}$$

and

$$H^G_*(G/K; N) = H^G_0(G/K; N) \cong N(G/K), \tag{4.7}$$

these giving isomorphisms of coefficient systems, of the appropriate variance, as K varies.

If A is a subcomplex of X, we obtain the relative chain complex $\underline{C}_*(X, A) = \tilde{\underline{C}}_*(X/A)$. The projectivity just proven implies the expected long exact sequences of pairs. For additivity, just note that the disjoint union of G-CW complexes is a G-CW complex. For excision, if X is the union of subcomplexes A and B, then $B/A \cap B \cong X/A$ as G-CW complexes. We take the "weak equivalence axiom" as a definition. That is, for general G-spaces X, we define

$$H^*_G(X; M) \equiv H^*_G(\Gamma X; M) \quad \text{and} \quad H^G_*(X; N) \equiv H^G_*(\Gamma X; N).$$