A geometric solution of the Kervaire Invariant One problem

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Let

$$f : M^{n-1} \looparrowright \mathbb{R}^n,$$

$n = 4k + 2$, $n \geq 2$ be a smooth generic immersion of a closed manifold of codimension 1. Let

$$g : N^{n-2} \looparrowright \mathbb{R}^n$$

be the immersion of the double points intersection of g.

The **Kervaire invariant** of f is defined by

$$\Theta_{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle,$$

where $\eta_N = w_2(N^{n-2})$ is the second normal Stiefel-Whitney class of N^{n-2}.

In particular, if $n = 2$, $\Theta_{sf}(f)$ is the parity of the number of self-intersection points of the curve f on the plane $\mathbb{R}^2$.

Let us denote by $Imm^{sf}(n-1, 1)$ the cobordism group of immersions in the codimension 1 of (non-oriented) closed $(n-1)$–manifold ("sf" stands for skew-framed).

Theorem

The Kervaire invariant is a well-defined homomorphism:

$$\Theta_{sf} : Imm^{sf}(4k+1,1) \longrightarrow \mathbb{Z}/2.$$

1. This homomorphism is trivial if $4k+2 \neq 2^l - 2$, $l \geq 2$.
2. For $k = 0, 1, 3, 7, 15$ the homomorphism Θ_{sf} is an epimorphism.

Part 1 is the Theorem by W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math., (2) 90 (1969) 157-186. A special case of this theorem was discovered by M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv., 34 (1960) 257-270.

Part 2 (in the case $k = 7$) was proved by M.E. Mahowald and M.C. Tangora, *Some differentials in the Adams spectral sequence*, Topology 6 (1967), 349-369. The case $k = 15$ was proved by M. G. Barratt, J. D. S. Jones and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, J.London Math.Soc. (2) 30 (1984), no. 3, 533-550.

Main Theorem

There exists an integer l_0, such that for an arbitrary $l \geq l_0$, the Kervaire invariant

$$\Theta_{sf} : Imm^{sf}(2^l - 3, 1) \longrightarrow \mathbb{Z}/2$$

is the trivial homomorphism.

The proof of this theorem is based on the approach proposed by P.J.Eccles *Codimension One Immersions and the Kervaire Invariant One Problem*, Math. Proc. Cambridge Phil. Soc., vol.90 (1981) 483-493.

A commutative diagram that we use to define dihedral structure of self-intersection manifold of skew-framed immersions:

$$\begin{array}{ccc} Imm^{sf}(n-1,1) & \stackrel{J^k_{sf}}{\longrightarrow} & Imm^{sf}(n-k,k) \\ \downarrow \delta_{\mathbb{Z}/2^{[2]}} & & \downarrow \delta^k_{\mathbb{Z}/2^{[2]}} \\ Imm^{\mathbb{Z}/2^{[2]}}(n-2,2) & \stackrel{J^k_{\mathbb{Z}/2^{[2]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k). \end{array}$$

A commutative diagram that we use to define the Kervaire invariant in the codimension k:

$$\begin{array}{ccccc}
Imm^{sf}(n-k,k) & \stackrel{\Theta^k_{sf}}{\longrightarrow} & \mathbb{Z}/2 & \frac{skew-framed}{immersions} \\
\downarrow \delta^k_{\mathbb{Z}/2^{[2]}} & & \| & \\
Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) & \stackrel{\Theta^k_{\mathbb{Z}/2^{[2]}}}{\longrightarrow} & \mathbb{Z}/2 & \frac{dihedral}{immersions}.
\end{array}$$

Structure groups of immersions

Let us consider the following collection of $(d-1)$ sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2,$$

where each set consists of proper coordinate subspaces of $\mathbb{R}^{2^{d-1}}$.

The set of the subspaces

$$\Upsilon_i, \quad 2 \le i \le d,$$

(we will use only the case $d = 6$) consists of 2^{i-1} coordinate subspaces generated by the basis vectors:

$$((\mathbf{e}_1, \dots \mathbf{e}_{2^{d-i}}), \dots, (\mathbf{e}_{2^{d-1}-2^{d-i}+1}, \dots, \mathbf{e}_{2^{d-1}})).$$

Let us denote by $\mathbb{Z}/2^{[d]}$ the subgroup

$$\mathbb{Z}/2 \wr \Sigma_{2^{d-1}} \subset O(2^{d-1})$$

under the following condition:

– the transformation

$$\mathbb{Z}/2^{[d]} \times \mathbb{R}^{2^{d-1}} \to \mathbb{R}^{2^{d-1}}$$

admits the invariant collection of sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2.$$

In particular, in the case $d = 2$ we get that Υ_2 contains only one collection of subspaces and this collection is $((\mathbf{e}_1), (\mathbf{e}_2))$. Therefore $\mathbb{Z}/2^{[2]}$ is a dihedral group.

$$\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2,2) & \longrightarrow & Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) \\
\downarrow \delta_{\mathbb{Z}/2^{[3]}} & & \downarrow \delta^k_{\mathbb{Z}/2^{[3]}} \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4,4) & \stackrel{J^k_{\mathbb{Z}/2^{[3]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) \\
\downarrow \delta_{\mathbb{Z}/2^{[4]}} & & \downarrow \delta^k_{\mathbb{Z}/2^{[4]}} \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8,8) & \stackrel{J^k_{\mathbb{Z}/2^{[4]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k) \\
\downarrow \delta_{\mathbb{Z}/2^{[5]}} & & \downarrow \delta^k_{\mathbb{Z}/2^{[5]}} \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16,16) & \stackrel{J^k_{\mathbb{Z}/2^{[5]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k) \\
\downarrow \delta_{\mathbb{Z}/2^{[6]}} & & \downarrow \delta^k_{\mathbb{Z}/2^{[6]}} \\
Imm^{\mathbb{Z}/2^{[6]}}(n-32,32) & \stackrel{J^k_{\mathbb{Z}/2^{[6]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[6]}}(n-32k,32k).
\end{array}$$

$$
\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) & \longrightarrow & \mathbb{Z}/2 \\
\downarrow \delta^k_{\mathbb{Z}/2^{[3]}} & & \| \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) & \xrightarrow{\Theta^k_{\mathbb{Z}/2^{[3]}}} & \mathbb{Z}/2 \\
\downarrow \delta^k_{\mathbb{Z}/2^{[4]}} & & \| \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k) & \xrightarrow{\Theta^k_{\mathbb{Z}/2^{[4]}}} & \mathbb{Z}/2 \\
\downarrow \delta^k_{\mathbb{Z}/2^{[5]}} & & \| \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k) & \xrightarrow{\Theta^k_{\mathbb{Z}/2^{[5]}}} & \mathbb{Z}/2 \\
\downarrow \delta^k_{\mathbb{Z}/2^{[6]}} & & \| \\
Imm^{\mathbb{Z}/2^{[6]}}(n-32k,32k) & \xrightarrow{\Theta^k_{\mathbb{Z}/2^{[6]}}} & \mathbb{Z}/2.
\end{array}
$$

$$
\begin{array}{ccccc}
\mathbf{I}_d \oplus \dot{\mathbf{I}}_d & \subset & \mathbb{Z}/2^{[2]} & & \frac{Abelian}{structure} \\
\downarrow & & i_{[3]} \downarrow & & \\
\mathbf{I}_a \oplus \dot{\mathbf{I}}_d & \subset & \mathbb{Z}/2^{[3]} & & \frac{cyclic-Abelian}{structure} \\
\downarrow & & i_{[4]} \downarrow & & \\
\mathbf{I}_a \times \dot{\mathbf{I}}_a & \subset & \mathbb{Z}/2^{[4]} & & \frac{bicyclic}{structure} \\
\downarrow & & i_{[5]} \downarrow & & \\
\mathbf{Q} \times \dot{\mathbf{I}}_a & \subset & \mathbb{Z}/2^{[5]} & & \frac{quaternionic-cyclic}{structure} \\
\downarrow & & i_{[6]} \downarrow & & \\
\mathbf{Q} \times \dot{\mathbf{Q}} & \subset & \mathbb{Z}/2^{[6]} & & \frac{biquaternionic}{structure}.
\end{array}
$$

Dihedral group

The dihedral group (of the order 8) $\mathbb{Z}/2^{[2]} \subset O(2)$:

$$\{a, b \quad | \quad a^4 = b^2 = e, [a, b] = a^2\}.$$

Let $\{\mathbf{f}_1, \mathbf{f}_2\}$ be the standard base of the plane $\mathbb{R}^2$. The element a is represented by the rotation through the angle $\frac{\pi}{2}$:

$$f_1 \mapsto f_2; \quad f_2 \mapsto -f_1.$$

The element b is represented by the permutation of the base vectors

$$f_1 \mapsto f_2; \quad f_2 \mapsto f_1.$$

Elementary 2-group

The elementary subgroup $\mathbf{I}_d \times \dot{\mathbf{I}}_d \subset \mathbb{Z}/2^{[2]}$ of the rank 2:

$$\{a^2, b \quad | \quad a^4 = b^2 = e, [a^2, b] = e\}.$$

This group preserves the vectors $\mathbf{f}_1 + \mathbf{f}_2$, $\mathbf{f}_1 - \mathbf{f}_2$.

Let $\tau_{[2]} \in H^2(\mathbb{Z}/2^{[2]}; \mathbb{Z}/2)$ be the universal class, $i^*_{d\times\dot{d}}(\tau_{[2]}) \in H^2(\mathbf{I}_d \times \dot{\mathbf{I}}_d; \mathbb{Z}/2)$ is the pull-back of $\tau_{[2]}$ under the inclusion $i_{d\times\dot{d}} : \mathbf{I}_d \times \dot{\mathbf{I}}_d \subset \mathbb{Z}/2^{[2]}$.

$$i^*_{d\times\dot{d}}(\tau_{[2]}) = \kappa_d \kappa_{\dot{d}},$$

$\kappa_d \in H^1(\mathbf{I}_d \times \dot{\mathbf{I}}_d; \mathbb{Z}/2)$. $p_d : \mathbf{I}_d \times \mathbf{I}_{\dot{d}} \longrightarrow \mathbf{I}_d$, $\kappa_d = p^*_d(t_d)$, $e \neq t_d \in \mathbf{I}_d \simeq \mathbb{Z}/2$, and $\kappa_{\dot{d}} \in H^1(\mathbf{I}_d \times \dot{\mathbf{I}}_d; \mathbb{Z}/2)$ is defined analogously.

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k),$$

$-f : M^{n-k} \looparrowright \mathbb{R}^n$,

$-\kappa_M$ is a line bundle over M^{n-k},

$-\Xi$ is a skew-framing of the normal bundle of f, i.e. an isomorphism $\Xi : \nu_f = k\kappa_M$.

$$y = \delta^k_{\mathbb{Z}/2^{[2]}}(x) = (g, \Psi, \eta_N) \in Imm^{\mathbb{Z}^{[2]}}(n-2k, 2k),$$

$-g : N^{n-2k} \looparrowright \mathbb{R}^n$,

$-\eta_N$ is a $\mathbb{Z}/2^{[2]}$–bundle over N^{n-2k},

$-\Psi$ is a dihedral framing of the normal bundle of f, i.e. an isomorphism $\Psi : \nu_g = k\eta_N$.

Definition of Abelian structure

A skew-framed immersion

$$(f, \Xi, \kappa_M)$$

admits an Abelian structure if there exists a map

$\eta_{d\times\dot{d},N} : N^{n-2k} \to K(\mathbf{I}_d \times \dot{\mathbf{I}}_d, 1)$ (Eilenberg-Mac Lain space),

satisfying the following condition:

$$\langle \eta_N^{\frac{n-2k}{2}}; [N^{n-2k}] \rangle = \Theta^k_{\mathbb{Z}/2^{[2]}}(y) = \langle \eta_N^{15k} \eta_{d\times\dot{d}}^{\frac{n-32k}{2}}; [N] \rangle.$$

$\eta_{d\times\dot{d}} = \eta^*_{d\times\dot{d},N}(i^*_{d\times\dot{d}}(\tau_{[2]})) \in H^2(N^{n-2k}; \mathbb{Z}/2)$, $[N]$ is the fundamental class of N^{n-2k}, $\eta_N \in H^2(N^{n-2k}; \mathbb{Z}/2)$ is the characteristic class of $\mathbb{Z}/2^{[2]}$–framing.

Definition of a Desuspension

A skew-framed cobordism class

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k)$$

admits a desuspension of order q, if this class is represented by a triple, such that

$$\kappa_M = I \circ \kappa_{(q)},$$

$\kappa_{(q)} : M^{n-k} \longrightarrow \mathbb{R}\mathrm{P}^{n-k-q-1}$, $I : \mathbb{R}\mathrm{P}^{n-k-q-1} \subset \mathbb{R}\mathrm{P}^{\infty}$.

Desuspension theorem

For an arbitrary q there exists an integer $l_0 = l_0(q)$, such that an arbitrary element $x \in Imm^{sf}(2^l - 3, 1)$, $l \geq l_0$, admits a desuspension of order q.

Abelian structure immersion theorem

Let q be an arbitrary integer divisible by 16, and let $n = 2^l - 2$ with l is sufficiently large. Put

$$k = k(q) = \frac{n+2}{32} - \frac{q}{16}.$$

Let us assume that $x \in Imm^{sf}(n-k, k)$ admits a desuspension of order q. Then the class x is represented by a triple (f, Ξ, κ_M), such that this skew-framed immersion admits an Abelian structure.

Cyclic group

The cyclic index 2 subgroup of the order 4:

$$\mathbf{I}_a = \{a \quad | \quad a^4 = e\} \subset \mathbb{Z}/2^{[2]}.$$

Bicyclic group

The bicyclic index 2^{11} subgroup of the order 16:

$$\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[2]} \times \mathbb{Z}/2^{[2]} \subset \mathbb{Z}/2^{[4]}.$$

The universal cohomology class of the bicyclic group

There exists $\tau_{[4]} \in H^8(\mathbb{Z}/2^{[4]};\mathbb{Z}/2)$ (the universal class), $i^*_{a\times\dot{a}}(\tau_{[4]}) \in H^8(\mathbf{I}_a \times \dot{\mathbf{I}}_a;\mathbb{Z}/2)$ is the pull-back of $\tau_{[4]}$ under the inclusion $i_{a\times\dot{a}} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[4]}$.

$$i^*_{a\times\dot{a}}(\tau_{[4]}) = \eta_a^2\eta_{\dot{a}}^2,$$

$\eta_a, \eta_{\dot{a}} \in H^2(\mathbf{I}_a \times \dot{\mathbf{I}}_a;\mathbb{Z}/2)$. Here η_a, $\eta_{\dot{a}}$ are defined similar to κ_a, $\kappa_{\dot{a}}$.

$$x \in Imm^{sf}(n-k,k),$$

$$y = \delta_{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k),$$

$$z = \delta_{\mathbb{Z}/2^{[3]}} \circ \delta_{\mathbb{Z}/2^{[2]}} \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k),$$

$$u = \delta_{\mathbb{Z}^{[4]}} \circ \delta_{\mathbb{Z}^{[3]}} \circ \delta_{\mathbb{Z}^{[2]}}(x)$$

$$u = [(h,\Lambda,\zeta_L)] \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k),$$

–$h : L^{n-8k} \looparrowright \mathbb{R}^n$,

–ζ_L is a $\mathbb{Z}/2^{[4]}$–bundle over L^{n-8k},

–Λ is a an 8–dimensional $\mathbb{Z}/4^{[4]}$-framing of the normal bundle of h, i.e. an isomorphism $\Lambda : \nu_h \simeq k\zeta_L$.

Definition of bicyclic structure

A $\mathbb{Z}/2^{[3]}$–immersion

$$[(g', \Psi', \eta_{N'})] = \delta_{\mathbb{Z}/2^{[3]}} \circ \delta_{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$$

admits a bicyclic structure if there exists a map

$\zeta_{a \times \dot{a}, L} : L^{n-8k} \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$ (Eilenberg-Mac Lain space),

satisfying the following equation:

$$\Theta^k_{\mathbb{Z}/2^{[3]}}(z) = \langle \pi^*_{d \times \dot{d}, a \times \dot{a}, L}(\zeta_L)^{3k} \bar{\zeta}_{d \times \dot{d}, L}^{\frac{n-32k}{2}}; [\bar{L}_{d \times \dot{d}}] \rangle .$$

Here the cohomology class $\bar{\zeta}_{d \times \dot{d}, L}$ is defined by means of $\zeta_{a \times \dot{a}, L}$.

In the previous formula:

– L^{n-8k} is the double-point $\mathbb{Z}/2^{[4]}$–manifold of g'

– $[\bar{L}_{d\times\dot{d}}]$ is the fundamental class of the corresponding 4-sheeted cover

$$\pi_{d\times\dot{d},a\times\dot{a}} : \bar{L}^{n-8k}_{d\times\dot{d}} \longrightarrow L^{n-8k},$$

induced from the 4-sheeted cover of Eilenberg-Maclain spaces $K(\mathbf{I}_d \times \dot{\mathbf{I}}_d, 1) \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$ by the map $\zeta_{a\times\dot{a},L} : L^{n-8k} \longrightarrow K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$

– $\bar{\zeta}_{d\times\dot{d},L} \in H^2(\bar{L}^{n-8k}_{d\times\dot{d}}; \mathbb{Z}/2)$ is the universal cohomological $\mathbf{I}_d \times \dot{\mathbf{I}}_d$-class, constructed by means of the map $\zeta_{\mathbf{Q}\times\dot{\mathbf{Q}},L}$

– $\zeta_L \in H^8(L^{n-8k};\mathbb{Z}/2)$ is the top characteristic class of the $\mathbb{Z}/2^{[4]}$– framing and it is the pull-back of the universal class $\tau_{[4]} \in H^8(\mathbb{Z}/2^{[4]};\mathbb{Z}/2)$ under the classifying map $L^{n-8k} \to K(\mathbb{Z}/2^{[4]},1)$ of the corresponded $\mathbb{Z}/2^{[4]}$–bundle previously denoted by ζ_L.

– $\pi^*_{d\times\dot{d},a\times\dot{a},L}(\zeta_L) \in H^8(\bar{L}^{n-8k}_{d\times\dot{d}};\mathbb{Z}/2)$ is the pull-back of the class ζ_L under the 4-sheeted cover

$$\pi_{d\times\dot{d},a\times\dot{a}} : \bar{L}^{n-8k}_{d\times\dot{d}} \to L^{n-8k}.$$

Bicyclic structure immersion Theorem

Let us assume that $x \in Imm^{sf}(n-k,k)$, $k = \frac{n-2^s+2}{32}$, $s \geq 6$, admits a desuspension of the order $q = \frac{2^s-2}{2}$. Then the class

$$z = \delta^k_{\mathbb{Z}[3]} \circ \delta^k_{[2]}(x) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$$

is represented by a triple (h, Λ, ζ_L), such that this skew-framed immersion admits an Abelian structure.

Quaternionic group

The quaternionic group of the order 8:

$$\mathbf{Q} = \{\mathbf{i}, \mathbf{j}, \mathbf{k} \quad | \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik},$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}.$$

This is an index 16 subgroup $\mathbf{Q} \subset \mathbb{Z}/2^{[3]}$. The standard representation $\chi_+ : \mathbf{Q} \to \mathbb{Z}/2^{[3]}$ transforms the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the following matrices:

$$\mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Biquaternionic group

The biquaternionic index 2^{57}–subgroup of the order 64:

$$\mathbf{Q} \times \mathbf{Q} \subset \mathbb{Z}/2^{[3]} \times \mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[6]}.$$

The universal cohomology class of the biquaternionic group

There exists $\tau_{[6]} \in H^{32}(\mathbb{Z}/2^{[6]};\mathbb{Z}/2)$ (the universal class), $i^*_{\mathbf{Q}\times\dot{\mathbf{Q}}}(\tau_{[6]}) \in H^{32}(\mathbf{Q}\times\dot{\mathbf{Q}};\mathbb{Z}/2)$ is the pull-back of $\tau_{[6]}$ under the inclusion $i_{\mathbf{Q}\times\dot{\mathbf{Q}}} : \mathbf{Q}\times\dot{\mathbf{Q}} \subset \mathbb{Z}/2^{[6]}$.

$$i^*_{\mathbf{Q}\times\dot{\mathbf{Q}}}(\tau_{[6]}) = \zeta^4_{\mathbf{Q}}\zeta^4_{\dot{\mathbf{Q}}},$$

$\zeta_{\mathbf{Q}}, \zeta_{\dot{\mathbf{Q}}} \in H^4(\mathbf{Q}\times\dot{\mathbf{Q}};\mathbb{Z}/2)$. Here $\zeta_{\mathbf{Q}}$, $\zeta_{\dot{\mathbf{Q}}}$ are defined similar to η_a, $\eta_{\dot{a}}$.

$$x \in Imm^{sf}(n-k,k),$$

$$y = \delta_{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k),$$

$$z = \delta_{\mathbb{Z}/2^{[3]}} \circ \delta_{\mathbb{Z}/2^{[2]}} \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k),$$

$$u = \delta_{\mathbb{Z}^{[4]}} \circ \delta_{\mathbb{Z}^{[3]}} \circ \delta_{\mathbb{Z}^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k),$$

$$v = \delta_{\mathbb{Z}^{[5]}} \circ \delta_{\mathbb{Z}^{[4]}} \circ \delta_{\mathbb{Z}^{[3]}} \circ \delta_{\mathbb{Z}^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k),$$

$$w = \delta_{\mathbb{Z}^{[6]}} \circ \delta_{\mathbb{Z}^{[5]}} \circ \delta_{\mathbb{Z}^{[4]}} \circ \delta_{\mathbb{Z}^{[3]}} \circ \delta_{\mathbb{Z}^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k).$$

Definition of biquaternionic structure

A $\mathbb{Z}/2^{[5]}$–immersion $[(h', \Lambda', \zeta_{L'})] =$

$\delta_{\mathbb{Z}/2^{[5]}} \circ \delta_{\mathbb{Z}/2^{[4]}} \circ \delta_{\mathbb{Z}/2^{[3]}} \circ \delta_{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$

admits a biquaternionic structure if there exists a map

$\omega_{\mathbf{Q}\times\dot{\mathbf{Q}},K} : K^{n-32k} \to K(\mathbf{Q}\times\dot{\mathbf{Q}}, 1)$ (Eilenberg-Mac Lain space

satisfying the following equation:

$$\Theta^k_{\mathbb{Z}/2^{[5]}}(w) = \langle \bar{\omega}^{\frac{n-32k}{2}}_{d\times\dot{d},K}; [\bar{K}_{d\times\dot{d}}] \rangle.$$

Here the cohomology class $\bar{\omega}_{d\times\dot{d},K}$ is defined by means of

$\omega_{\mathbf{Q}\times\dot{\mathbf{Q}},K}$.

In the previous formula:

– K^{n-32k} is the double-point $\mathbb{Z}/2^{[6]}$–manifold of h'

– $[\bar{K}_{d\times\dot{d}}]$ is the fundamental class of the 16-sheeted cover

$$\pi_{d\times\dot{d},\mathbf{Q}\times\dot{\mathbf{Q}}}:\bar{K}^{n-32k}_{d\times\dot{d}}\longrightarrow K^{n-32k},$$

induced from the 16-sheeted cover of Eilenberg-Mac Lain spaces $K(\mathbf{I}_d\times\dot{\mathbf{I}}_d,1)\to K(\mathbf{Q}\times\dot{\mathbf{Q}},1)$ by the map $\omega_{\mathbf{Q}\times\dot{\mathbf{Q}},K}:K^{n-32k}\longrightarrow K(\mathbf{Q}\times\dot{\mathbf{Q}},1)$

– $\bar{\omega}_{d\times\dot{d},K}\in H^2(\bar{K}^{n-32k}_{d\times\dot{d}};\mathbb{Z}/2)$ is the universal cohomology $\mathbf{I}_d\times\dot{\mathbf{I}}_d$-class, constructed by means of the map $\omega_{\mathbf{Q}\times\dot{\mathbf{Q}},K}$.

Biquaternionic structure immersion theorem

Let $k = \frac{n-2^s+2}{32}$, $s \geq 6$, ($q = \frac{2^s-2}{2}$), q be an integer divisible by 16, and let $n = 2^l - 2$ with l sufficiently large. Put

$$k = k(q) = \frac{n+2}{32} - \frac{q}{16}.$$

Let us assume that $x \in Imm^{sf}(n-k, k)$ admits a desuspension of the order $q = \frac{2^s-2}{2}$. Then the class $w = \delta^k_{\mathbb{Z}[5]} \circ \delta^k_{\mathbb{Z}[4]} \circ \delta^k_{\mathbb{Z}[3]} \circ \delta^k_{\mathbb{Z}[2]}(x) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$ is represented by a triple $(h', \Lambda', \zeta_{L'})$ such that this triple admits a biquaternionic structure.

Biquaternionic Kervaire Invariant Theorem

Assume that $w \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$, $n = 2^l - 2$, $k \cong 0 \pmod{64}$, $k > 0$, $n - 32k > 0$ admits a biquaternionic structure. Then $\Theta_{\mathbb{Z}/2^{[5]}}(w) = 0$.

As a corollary we get

Main Theorem

There exists an integer l_0, such that for an arbitrary $l \geq l_0$, the Kervaire invariant

$$\Theta_{sf} : Imm^{sf}(2^l - 3, 1) \longrightarrow \mathbb{Z}/2$$

is the trivial homomorphism.

Proof of Biquaternionic Theorem

Let $w \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$,

$$w = [(e, \Omega, \omega_K)],$$

S^{n-32k} be the double point manifold of the immersion e,

$$\omega_{\mathbf{Q}} \times \omega_{\dot{\mathbf{Q}}} : S^{n-32k} \to K(\mathbf{Q}, 1) \times K(\dot{\mathbf{Q}}, 1)$$

be the biquaternionic map.

Recall that $n - 32k = \dim(S) \geq 14$. Let $i_T : T^{14} \subset S^{n-32k}$ be a closed submanifold dual to the cohomology class

$$(\omega_{S;\mathbf{Q}}\omega_{S;\dot{\mathbf{Q}}})^{\frac{n-32k-14}{8}} \in H^{n-32k-14}(S^{n-32k}; \mathbb{Z}/2),$$

where $\omega_{S;\mathbf{Q}} = \omega_{\mathbf{Q}}^*(\zeta_{\mathbf{Q}}); \omega_{S;\dot{\mathbf{Q}}} = \omega_{\dot{\mathbf{Q}}}^*(\zeta_{\dot{\mathbf{Q}}}) \in H^4(S^{n-32k}; \mathbb{Z}/2)$.

The following (non-standard) representation $\chi_- : \mathbf{Q} \to \mathbb{Z}/2^{[3]}$ transforms the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the following matrices:

$$\mathbf{i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let us define the vector bundles ζ_+, ζ_- be over $S^{15}/\mathbf{Q}_a$. The bundle ζ_+ is defined by means of the representation χ_+. The bundle ζ_- is defined by means of the representation χ_-.

The bundle ζ_+ admits a complex structure. Note that $c_1(\zeta_+) = 0$, because the restriction of the bundle ζ_+ over $S^3/\mathbf{Q} \subset S^{15}/\mathbf{Q}$ is the trivial complex bundle and $H^2(S^{15}/\mathbf{Q};\mathbb{Z}) \to H^2(S^3/\mathbf{Q};\mathbb{Z})$ is an isomorphism.

Therefore,

$$p_1(2\zeta_+) = c_1^2(2\zeta_+) - 2c_2(2\zeta_+) =$$

$$4c_1^2(\zeta_+) - 4c_2(\zeta_+) = 4\zeta_{\mathbf{Q}} \in H^4(K(\mathbf{Q}, 1); \mathbb{Z}).$$

By the analogical computation:

$$p_1(2\dot{\zeta}_+) = 4\zeta_{\dot{\mathbf{Q}}} \in H^4(K(\dot{\mathbf{Q}}, 1); \mathbb{Z}).$$

The bundle ζ_- admits a complex structure. Note that $c_1(\zeta_-) = 0$ by analogical calculations. Therefore,

$$p_1(\zeta_+ \oplus \zeta_-) = c_1^2(\zeta_+ \oplus \zeta_-) - 2c_2(\zeta_+ \oplus \zeta_-) =$$

$$c_1^2(\zeta_+) + c_1^2(\zeta_-) + 2c_1(\zeta_+)c_1(\zeta_-) - 2c_2(\zeta_+) - 2c_2(\zeta_-) = 0,$$

because the Euler classes $e(\zeta_+) \in H^4(S^{15}/\mathbf{Q};\mathbb{Z})$ $e(\zeta_-) \in H^4(S^{15}/\mathbf{Q};\mathbb{Z})$ are opposite: $e(\zeta_+) = -e(\zeta_-)$.

The normal bundle ν_T is stably isomorphic to the bundle $l\zeta_{T,+} \oplus l\dot{\zeta}_{T,+}$, where l is an integer, $l = 2 \pmod 4$.

The bundle $\zeta_{T,+}$ is the 4-dimensional vector bundle over T defined as

$$\zeta_{T,+} = \omega^*_{T,\mathbf{Q}}(\zeta_+),$$

$$\omega_{T,\mathbf{Q}} = \omega_{\mathbf{Q}}|_T : T^{14} \to K(\mathbf{Q}, 1).$$

The bundle $\dot{\zeta}_{T,+}$ is the 4-dimensional vector bundle over T defined as

$$\dot{\zeta}_{T,+} = \omega^*_{T,\dot{\mathbf{Q}}}(\zeta_+),$$

$$\omega_{T,\dot{\mathbf{Q}}} = \omega_{\dot{\mathbf{Q}}}|_T : T^{14} \to K(\dot{\mathbf{Q}}, 1).$$

Put $-T^{14}$ to be T^{14} with the opposite orientation. The normal bundle ν_{-T} is stably isomorphic to the bundle $(l-1)\zeta_{-T,+} \oplus \zeta_{-T,-} \oplus l\dot{\zeta}_{-T,+}$ (we will put after $l=2$ for the shortness).

The bundle $\zeta_{-T,+}$ is the 4-dimensional vector bundle defined as

$$\zeta_{-T,+} = \omega^*_{-T,\mathbf{Q}}(\zeta_+),$$

$$\omega_{-T,\mathbf{Q}} = \omega_{\mathbf{Q}}|_{-T} : -T^{14} \to K(\mathbf{Q},1).$$

The bundle $\zeta_{-T,-}$ is the 4-dimensional vector bundle defined as

$$\zeta_{-T,-} = \omega^*_{-T,\mathbf{Q}}(\zeta_-).$$

The bundle $\zeta_{-T,+}$ is the 4-dimensional vector bundle defined as

$$\dot{\zeta}_{-T,+} = \omega^*_{-T,\dot{\mathbf{Q}}}(\zeta_+),$$

$$\omega_{-T,\dot{\mathbf{Q}}} = \omega_{-T,\dot{\mathbf{Q}}}|_{-T} : -T^{14} \to K(\dot{\mathbf{Q}}, 1).$$

Let us assume that $\Theta_{\mathbb{Z}/5}(w) = 1.$. Then the decomposition of the cycle $\omega_{\mathbf{Q}\oplus\dot{\mathbf{Q}},*}([T])$ in the standard base of $H_{14}(\mathbf{Q} \oplus \dot{\mathbf{Q}}; \mathbb{Z})$ involves the element $u_7 \otimes v_7$, where $u_7 \in H_7(K(\mathbf{Q}, 1); \mathbb{Z}) = \mathbb{Z}/8$,
$v_7 \in H_7(K(\dot{\mathbf{Q}}, 1); \mathbb{Z}) = \mathbb{Z}/8$ are the generators,
$H_7(K(\mathbf{Q}, 1); \mathbb{Z}) \otimes H_7(K(\dot{\mathbf{Q}}, 1); \mathbb{Z}) \subset H_{14}(K(\mathbf{Q} \times \dot{\mathbf{Q}}, 1); \mathbb{Z})$.

Let

$$F = id \cup -id : T^{14} \cup -T^{14} \to T^{14}$$

be the standard degree 0 map. Let us consider the following homology class:

$$\aleph = (\omega_{\mathbf{Q}\times\dot{\mathbf{Q}}} \circ F)_*([p_1(\nu_T)]^{op} + [p_1(\nu_{-T})]^{op}]) \in$$

$$H_{10}(K(\mathbf{Q} \times \dot{\mathbf{Q}}, 1); \mathbb{Z}),$$

where the upper index "op" stands for Poincaré dual.

Let us prove that $\aleph$ involves the element $4u_3 \otimes v_7 \in H_3(K(\mathbf{Q},1);\mathbb{Z}) \otimes H_7(K(\dot{\mathbf{Q}},1);\mathbb{Z}) \subset H_{10}(K(\mathbf{Q} \times \dot{\mathbf{Q}},1);\mathbb{Z})$. Without loss of the generality we may assume that $\omega_{\mathbf{Q}\oplus\dot{\mathbf{Q}},*}([T]) = u_7 \otimes v_7 + xu_3 \otimes v_{11} + \dots$, where x is an arbitrary integer. (For all last terms in this formula the characteristic class $\aleph$ does not involve the element $u_3 \otimes v_7$ by the dimension reason). Under this assumption by the computation above we get:

$$F_*([p_1(\nu_T)]^{op}) = 4u_3 \otimes v_7 + 4xu_3 \otimes v_7 + \dots \in$$

$$H_3(K(\mathbf{Q},1);\mathbb{Z}) \otimes H_7(K(\dot{\mathbf{Q}},1);\mathbb{Z}) \subset H_{10}(K(\mathbf{Q} \times \dot{\mathbf{Q}},1);\mathbb{Z}),$$

$$F_*([p_1(\nu_{-T})]^{op}) = 4xu_3 \otimes v_7 + \dots .$$

Therefore the first (normal) Pontrjagin class satisfy the equation:

$$0 \neq 4u_3 \otimes v_7 + \dots = (\omega_{\mathbf{Q} \times \dot{\mathbf{Q}}} \circ F)_*([p_1(\nu_T)]^{op} + [p_1(\nu_{-T})]^{op}]).$$

In particular, F is not cobordant to zero. But the mapping F is cobordant to zero by definition. Contradiction. Therefore $\Theta_{\mathbb{Z}/5}(w) = 0$.