Geometric approach to stable homotopy groups of spheres. I. The Hopf invariant

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dedicated to the memory of Professor M.M. Postnikov

Abstract

A geometric approach to the stable homotopy groups of spheres is developed in this paper, based on the Pontryagin-Thom construction. The task of this approach is to obtain an alternative proof of the Hill-Hopkins-Ravenel theorem [H-H-R] on Kervaire invariants in all dimensions, except, possibly, a finite number of dimensions. In the framework of this approach, the Adams theorem on the Hopf invariant is studied, for all dimensions with the exception of 15, 31, 63. The new approach is based on the methods of geometric topology.

Introduction

Let $\pi_{n+m}(S^m)$ be the homotopy groups of spheres. Under the condition $m \geq n+2$ this group is independent of m and is denoted by Π_n. It is called the stable homotopy group of spheres in dimension n. The problem of calculating the stable homotopy groups of spheres is one of the main unsolved problems of topology. A development of the Pontryagin-Thom construction leads to various applications having important practical significance: an approach by V.I.Arnol'd to bifurcations of critical points in multiparameter families of functions [V] chapter 3, section 2.2 and Theorem 1, section 2.4, Lemma 4, approximation of maps by embeddings [Me], and gives many unsolved geometrical problems [E2].

For the calculation of elements of the stable homotopy groups of spheres, one frequently studies algebraic invariants which are defined for all dimensions at once (or for some infinite sequence of dimensions). Nevertheless, as a rule these invariants turn out to be trivial, and are nonzero only in exceptional cases, see [M1]. As Prof. Peter Landweber noted: "This a very interesting "philosophy". Are there examples to illustrate this, apart from the Hopf invariant and the Kervaire invariant? There might be one in N. Minami's paper [M2]."

A basic invariant is the Hopf invariant, which is defined as follows in the framework of stable homotopy theory. The Hopf invariant (also called the stable Hopf invariant), or the Steenrod-Hopf invariant is a homomorphism

$$h : \Pi_{2k-1} \to \mathbb{Z}/2,$$

for details see [W],[M-T]. The stable Hopf invariant is studied in this paper.

The following theorem was proved by J.F.Adams in [A].

Theorem. *The stable Hopf invariant* $h : \Pi_n \to \mathbb{Z}/2$, $n \equiv 1 \pmod 2$ *is a trivial homomorphism if and only if* $n \neq 1, 3, 7$.

Remark. The case n = 15 was proved by Toda (cf. [M-T] Ch. 18).

Later Adams and Atiyah offered an alternative approach to the study of the Hopf invariant, based on results of K-theory and the Bott periodicity theorem, cf. [A-A]. This approach was also extended in subsequent works. A simple proof of the theorem of Adams, close to the proof of Adams and Atiyah, was given by V.M. Buchstaber in [B], section 2.

The definition of the stable Hopf invariant is reformulated in the language of the cobordism groups of immersions of manifolds [E1, K2, K-S1, K-S2, La]. Using the Pontryagin-Thom theorem in the form of Wells on the representation of the stable homotopy groups of infinite dimensional real projective space (which by the Kahn-Priddy theorem surject onto the 2-components of the stable homotopy groups of spheres), we classify the cobordism of immersions of (in general nonorientable) manifolds in codimension 1. The Hopf invariant is expressed as a characteristic number of the manifold of double points of self-intersection of an immersion of a manifold representing the given element of the stable homotopy group. This is explicitly formulated in [E1], Lemma 3.1. This lemma is reformulated in the standard way by means of the Pontryagin-Thom construction for immersions.

The Theorem of Adams admits a simple geometric proof for dimensions $n \neq 2^\ell - 1$. In the case $n \not\equiv 3 \pmod 4$ a proof, using the elements of the theory of immersions, was given by A. Szücs in [Sz]. The next case in complexity arises for $n \neq 2^\ell - 1$. The proof of Adams' Theorem under this assumption was given by Adem [Adem] using algebraic methods. In this paper the Adem relations on the multiplicative generators of the Steenrod algebra were used. The theorem of Adem was reproved using geometric methods in a joint paper of the author and A. Szücs [A-Sz].

We assume below that $n = 2^\ell - 1$. Define a positive integer $\sigma = \sigma(\ell)$ by the formula:

$$\sigma = \left[\frac{\ell - 1}{2}\right]. \qquad (1)$$

In particular, for $\ell = 7$, $\sigma = 3$. Denote $n_s = 2^s - 1$. Assume that s is a positive integer, then n_s is a positive integer.

The following is the main result of Part I.

Main Theorem

Assume that $\ell \geq 7$, therefore $n \geq 127$. Let $g : M^{\frac{3n+n_\sigma}{4}} \looparrowright \mathbb{R}^n$ be an arbitrary smooth immersion of a closed manifold M, $\dim(M) = \frac{3n+n_\sigma}{4}$, where the normal bundle $\nu(g)$ to the immersion g is isomorphic to the Whitney sum of $(\frac{n-n_\sigma}{4})$ copies of a line bundle κ over M, $\nu(g) = (\frac{n-n_\sigma}{4})$. (In particular, $w_1(M) = 0$, where w_1 is the first Stiefel-Whitney class, because the codimension of the immersion g is even and $w_1(M) = (\frac{n-n_\sigma}{4})w_1(\kappa) = 0$), in general M is nonconnected.) Then the equation $\langle w_1(\kappa)^{\dim(M)}; [M]\rangle = 0$ is valid.

The Main Theorem is deduced from Theorem 12. Theorem 12 is deduced from Propositions 28, 29; these propositions follow from Lemmas 32 and 32. The proofs of these lemmas are given in part III [A3].

The proof of the Main Theorem is based on the principle of geometric control due to M.Hirsh, see Proposition 30. This proposition permits one to find within a cobordism class of immersions an immersion with additional properties of self-intersection manifold (see Propositions 28, 29). In this case we say that the immersion admits a cyclic or quaternionic structure (see Definitions 19, 20).

We can deduce the following from the Main Theorem by standard arguments.

Main Corollary

Let $g : M^{n-1} \looparrowright \mathbb{R}^n$ be an arbitrary smooth immersion of the closed manifold M^{n-1}, which in general is not assumed to be orientable. Then under the assumption $n = 2^\ell - 1$, $n \geq 127$ (i.e., for $\ell \geq 7$), the equality $\langle w_1(M)^{n-1}; [M]\rangle = 0$ is valid.

Remark. The equivalence of the preceding assertion and the theorem of Adams (under the restriction $\ell \geq 4$) is proved in [E1,La].

We mention that in topology there are theorems which are close to the formulation of the theorem of Adams. As a rule, these theorems are corollaries of Adams' theorem. Sometimes these theorems can be given alternative proofs by simpler methods. As S.P. Novikov remarks in his survey [N], a theorem of this type is the Bott-Milnor Theorem that the tangent n-plane bundle to the standard sphere S^n is trivial if and only if $n = 1, 3$ or 7. This theorem was first proved in the paper [B-M]. An elegant modification of the known proof was recently given in [F]. One should mention the Baum-Browder Theorem [B-B] about non-immersion of the standard real projective space $\mathbb{RP}^{2^{\ell-1}+1}$ into $\mathbb{R}^{2^{\ell}-1}$ for $\ell \geq 4$. It would be interesting to discover an elementary geometrical proof of this theorem and to prove the Main Corollary for $\ell \geq 4$ as a generalization of Baum-Browder Theorem.

We turn our attention to the structure of the paper. In section 1 we recall the main definitions and constructions of the theory of immersions. The results of this section are formally new, but are easily obtained by known methods. In section 2 the Main Theorem is reformulated using the notation of section 1 (Theorem 12), which represents a basic step in its proof. The proof of Main Theorem is based on Lemmas 32 and 34. The proof of Lemma 32 is in the part III [A3] of the paper. This part of the paper also contains the Lemmas 1, for the proof of the Main Theorem in the part II of the paper.

This paper was written for repeated discussions in the seminar on algebraic topology under the direction of Professor M.M. Postnikov. The author is grateful for the discussions to Prof. V.M. Buchstaber, Prof. S.A.Bogatyi, Prof. A.V. Chernavsky, Prof. V. V. Chernov, Prof. P. J. Eccles, Prof. P. Landweber, Prof. A.S. Mishenko, Prof. O.Saeki, Prof. E.V. Scepin, Prof. A.B. Skopenkov, Prof. Yu. P. Soloviev, Prof. V.A. Vassiliev, N. Brodsky, S.A.Melikhov, R.R.Sadykov and M.B. Skopenkov. Let me mention that Prof. Peter Landweber has devoted time and care to part 1 of this paper on the Hopf invariant and part 2 of this paper on the Kervaire invariant. He pointed out (see [L]) that the approach in an earlier version of this paper is not valid. The mistake occurs in section 3 of paper [A1]. I am pleased to acknowledge his effort.

1 Preliminary information

We recall the definition of the cobordism groups of framed immersions in Euclidean space, which is a special case of a more general construction presented in the book [K1] on page 55 and in section 10. The connection with the Pontryagin-Thom construction is explained in [A-E].

Let $f : M^{n-1} \looparrowright \mathbb{R}^n$ be a smooth immersion, where the $(n-1)$-dimensional manifold M^{n-1} is closed, but in general is nonorientable and nonconnected. We introduce a relation of cobordism on the space of such immersions. We say that two immersions f_0, f_1 are connected by a cobordism, $f_0 \sim f_1$, if there exists an immersion $\Phi : (W^n, \partial W = M_0^{n-1} \cup M_1^{n-1}) \looparrowright (\mathbb{R}^n \times [0;1]; \mathbb{R}^n \times \{0,1\})$ satisfying the boundary conditions $f_i = \Phi|_{M_i^{n-1}} : M_i^{n-1} \looparrowright \mathbb{R}^n \times \{i\}$, $i = 0, 1$ and, moreover, it is required that the immersion Φ is orthogonal to $\mathbb{R} \times \{0,1\}$.

The set of cobordism classes of immersions forms an Abelian group with respect to the operation of disjoint union of immersions. For example, the trivial element of this group is represented by an empty immersion, the element that is inverse to a given element represented by an immersion f_0 is represented by the composition $S \circ f_0$, where S is a mirror symmetry of the space $\mathbb{R}^n$.

This group is denoted by $Imm^{sf}(n-1,1)$. Because $\mathbb{RP}^\infty = MO(1)$, by the Wales theorem [Wa] (see [E1],[Sz2] for references) this group maps onto the stable homotopy group of spheres $\lim_{k\to\infty} \pi_{n+k}(S^k)$.

The immersion f defines an isomorphism of the normal bundle to the manifold M^{n-1} and the orientation line bundle κ, i.e. an isomorphism $D(f) : T(M^{n-1}) \oplus \kappa \cong n\varepsilon$, where ε is the trivial line bundle over M^{n-1}. In similar constructions in surgery theory of smooth immersions one requires a stable isomorphism of the normal bundle of a manifold M^{n-1} and the orientation line bundle κ, i.e. an isomorphism $T(M^{n-1}) \oplus \kappa \oplus N\varepsilon \cong (n+N)\varepsilon$, for $N > 0$. Using Hirsch's Theorem [Hi], it is easy to verify that if two immersions of f_1, f_2 determine isomorphisms $D(f_1)$, $D(f_2)$, that belong to the same class of stable isomorphisms then the immersions f_1, f_2 are regularly cobordant and even regularly concordant (but, generally speaking, may not be regularly homotopic).

We also require groups $Imm^{sf}(n-k,k)$. An element of this group is represented by a triple (f, κ, Ξ), where $f : M^{n-k} \looparrowright \mathbb{R}^n$ is an immersion of a closed manifold, $\kappa : E(\kappa) \to M^{n-k}$ is a line bundle (to shorten notation, we shall denote the line (one-dimensional) bundle and its characteristic class in $H^1(M^{n-k};\mathbb{Z}/2)$ by the same symbol), and Ξ is a skew-framing of the normal bundle of the immersion by means of the line bundle κ, i.e., an isomorphism of the normal bundle of the immersion f and the bundle $k\kappa$. In the case of odd k, the line bundle κ turns out to be orientable over M^{n-k} and necessarily $\kappa = w_1(M^{n-k})$.

Two elements of the cobordism group, represented by triples (f_1, κ_1, Ξ_1), (f_2, κ_2, Ξ_2) are equal if the immersions f_1, f_2 are cobordant (this definition is analogous to the previous one for representatives of the group $Imm^{sf}(n-1,1)$), where in addition it is required that the immersion of the cobordism be skew-framed, and that the skew-framing of the cobordism be compatible with the given skew-framings on the components of the boundary. We remark that for $k=1$ the new definition of the group $Imm^{sf}(n-k,k)$ coincides with the original definition.

We define a homomorphism

$$J^{sf} : Imm^{sf}(n-1,1) \to Imm^{sf}(n-k,k),$$

which is called the homomorphism of transition to codimension k. Consider a manifold M'^{n-1} and an immersion $f' : M'^{n-1} \looparrowright \mathbb{R}^n$ representing an element of the first group, and consider a classifying map $\kappa' : M'^{n-1} \to \mathbb{RP}^a$ to a real projective space of large dimension ($a = n-1$ suffices), representing the first Stiefel-Whitney class $w_1(M'^{n-1})$. Consider the standard subspace $\mathbb{RP}^{a-k+1} \subset \mathbb{RP}^a$ of codimension $(k-1)$. Assume that the mapping κ' is transverse along the chosen subspace and define the submanifold $M^{n-k} \subset M'^{n-1}$ as the complete inverse image of this subspace for our mapping, $M^{n-k} = \kappa'^{-1}(\mathbb{RP}^{a-k+1})$. Define an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ as the restriction of the immersion f' to the given submanifold. Notice that the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ admits a natural skew-framing. In fact, the normal bundle to the submanifold $M^{n-k} \subset M'^{n-1}$ is naturally isomorphic to the bundle $(k-1)\kappa$, where $\kappa = \kappa'|_M$ (here and below, when a manifold is used as a subscript, the superscript indicating the dimension of the manifold is omitted). The isomorphism Ξ is defined by the standard skew-framing of the normal bundle to the submanifold $\mathbb{RP}^{a-k+1}$ of the manifold $\mathbb{RP}^a$, which is transported to the submanifold $M^{n-k} \subset M'^{n-1}$, since it is assumed that κ' is transverse regular along $\mathbb{RP}^{a-k+1}$. A further direct summand in the skew-framing of the normal bundle of the immersion f corresponds to the normal line bundle of the immersion f'. This bundle serves as orientation bundle for M'^{n-1}, hence its restriction to M^{n-k} coincides with κ. The homomorphism J^{sf} carries the element represented by the immersion f' to the element represented by the triple (f, κ, Ξ). Elementary geometrical considerations, using only the concept of transversality imply that the homomorphism J^{sf} is correctly defined.

We now define the manifold of double points of self-intersection of an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ in general position, and a canonical 2-sheeted covering over this manifold. Under the assumption that the immersion f is in general position, the subset in $\mathbb{R}^n$ of points of self-intersection of the immersion f is denoted by $\Delta = \Delta(f)$, $\dim(\Delta) = n-2k$. This subset is defined by the formula

$$\Delta = \{x \in \mathbb{R}^n : \exists x_1, x_2 \in M^{n-k}, x_1 \neq x_2, f(x_1) = f(x_2) = x\}, \qquad (2)$$

We define $\bar{\Delta} \subset M^{n-k}$ by the formula $\bar{\Delta} = f^{-1}(\Delta)$.

We recall the standard definition of the manifold of points of self-intersection and the parameterizing immersion, see e.g. [Ada] for details.

Definition of self-intersection manifold

The set N is defined by the formula

$$N = \{[(x_1, x_2)] \in (M^{n-k} \times M^{n-k})/T' : x_1 \neq x_2, f(x_1) = f(x_2)\} \qquad (3)$$

(T' is the involution permuting the coordinate factors), and its canonical covering is defined by the formula

$$\bar{N} = \{(x_1, x_2) \in M^{n-k} \times M^{n-k} : x_1 \neq x_2, f(x_1) = f(x_2)\}. \qquad (4)$$

Under the assumption that the immersion f is generic, N is a smooth manifold of dimension $\dim(N) = n - 2k$. This manifold is denoted by N^{n-2k} and is called the self-intersection manifold of f, the projection of the covering is denoted by $p : \bar{N}^{n-2k} \to N^{n-2k}$ and is called the canonical 2-sheeted covering.

The immersion $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$, parameterizing $\bar{\Delta}$, is defined by the formula $\bar{g} = p|_{\bar{N}}$. Notice that the parameterizing immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ of Δ in general, is not an immersion in general position. There is a two-sheeted covering $p : \bar{N}^{n-2k} \to N^{n-2k}$, for which $g \circ p = f \circ \bar{g}$. This covering is called the canonical covering over the manifold of points of self-intersection.

Definition 1. Let (f, κ, Ξ) represent an element in the group $Imm^{sf}(n-k, k)$. Let us define the homomorphism:

$$h_k : Imm^{sf}(n-k, k) \to \mathbb{Z}/2,$$

called the stable Hopf invariant by the following formula:

$$h_k([f, \kappa, \Xi]) = \langle \kappa^{n-k}; [M^{n-k}] \rangle.$$

The definitions of the stable Hopf invariant (in the sense of Definition 1) for distinct values of k are compatible with one another, and coincide with the definition used in the introduction. We formulate this as a separate assertion.

Proposition 2. *The homomorphism $J^{sf} : Imm^{sf}(n-1, 1) \to Imm^{sf}(n-k, k)$ preserves the Hopf invariant, i.e. the invariant $h_1 : Imm^{sf}(n-1, 1) \to \mathbb{Z}/2$ and the invariant $h_k : Imm^{sf}(n-k, k) \to \mathbb{Z}/2$ are related by the formula:*

$$h_1 = h_k \circ J^{sf}. \qquad (5)$$

Proof of Proposition 2

Let $f : M^{n-k} \looparrowright \mathbb{R}^n$ be an immersion with a skew-framing Ξ of its normal bundle and with characteristic class $\kappa \in H^1(M^{n-k};\mathbb{Z}/2)$, representing an element of $Imm^{sf}(n-k,k)$, which satisfies $J^{sf}([f']) = [f,\kappa,\Xi]$ for an element $[f'] \in Imm^{sf}(n-1,1)$, where f and f' are related as in the definition of J^{sf}. By definition, $h_k([f,\kappa,\Xi]) = \langle \kappa^{n-k}; [M^{n-k}]\rangle$.

On the other hand, $M^{n-k} \subset M'^{n-1}$ is a cycle dual in the sense of Poincaré to the cohomology class $\kappa'^{k-1} \in H^{k-1}(M'^{n-1};\mathbb{Z}/2)$. The formula (5) is valid, since $\langle \kappa'^{n-1}; [M'^{n-1}]\rangle = \langle \kappa^{n-k}; [M^{n-k}]\rangle$. Proposition 2 is proved.

Let us formulate another (equivalent) definition of the stable Hopf invariant (assuming $n-2k>0$).

Definition 3. Let (f,κ,Ξ) represent an element in the group $Imm^{sf}(n-k,k)$, $n-2k>0$. Let N^{n-2k} be the manifold of the double points of the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$, $\bar{N}$ be the canonical 2-sheeted cover over N, $\kappa_{\bar{N}} \in H^1(\bar{N};\mathbb{Z}/2)$ be induced from $\kappa \in H^1(M^{n-k};\mathbb{Z}/2)$ by the immersion $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$.

Let us define the homomorphism $h_k^{sf} : Imm^{sf}(n-k,k) \to \mathbb{Z}/2$ by the formula:

$$h_k^{sf}([f,\kappa,\Xi]) = \langle \kappa_{\bar{N}}^{n-2k}; [\bar{N}^{n-2k}]\rangle.$$

The following proposition establishes the equivalence of Definitions 1 and 3.

Proposition 4. *Let us assume that the conditions of Definition 3 are satisfied. Then we have:*

$$\langle \kappa_{\bar{N}}^{n-2k}; [\bar{N}^{n-2k}]\rangle = \langle \kappa^{n-k}; [M^{n-k}]\rangle. \qquad (6)$$

Proof of Proposition 4

Let $f : M^{n-k} \looparrowright \mathbb{R}^n$ be an immersion with a skew framing Ξ and with the characteristic class $\kappa \in H^1(M^{n-k};\mathbb{Z}/2)$; then the triple (f,Ξ,κ) represents an element in the group $Imm^{sf}(n-k,k)$. Let N^{n-2k} be the manifold of self-intersection points of the immersion f, $g : N^{n-2k} \looparrowright \mathbb{R}^n$ be the parameterizing immersion, and $\bar{N}^{n-2k} \to N^{n-2k}$ be the canonical double covering. Consider the image of the fundamental class $\bar{g}_*([\bar{N}^{n-2k}]) \in H_{n-2k}(M^{n-k}; \mathbb{Z}/2)$ by the immersion $\bar{g} : \bar{N}^{n-2k} \looparrowright M^{n-k}$ and denote by $m \in H^k(M^{n-k}; \mathbb{Z}/2)$ the cohomology class Poincaré dual to the homology class $\bar{g}_*([\bar{N}^{n-2k}])$. Consider also the cohomology Euler class of the normal bundle immersion f, which is denoted by $e \in H^k(M^{n-k}; \mathbb{Z}/2)$.

By the Herbert Theorem for the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ with self-intersection manifold N^{n-2k} (see [E-G], Theorem 1.1 the case $r = 1$ coefficients is $\mathbb{Z}/2$; see also this theorem in the original papers [He], [L-S]) the following formula is valid:

$$e + m = 0. \tag{7}$$

Since the Euler class e of the normal bundle $k\kappa$ of the immersion f is equal to κ^k (line bundles and their corresponding characteristic cohomology classes are denoted by the same symbols), then the cycle $\bar{g}_*([\bar{N}]) \in H_{n-2k}(M^{n-k}; \mathbb{Z}/2)$ is Poincaré dual to the cohomology class $\kappa^k \in H^k(M^{n-k}; \mathbb{Z}/2)$. Therefore, the formula (6) and Proposition 4 are proved.

It is more convenient to reformulate Proposition 4 (in a more general form) by means of the language of commutative diagrams. The desired reformulation is given in Lemma 7 below. We turn to the relevant definitions.

Let $g : N^{n-2} \looparrowright \mathbb{R}^n$ be the immersion of the double self-intersection points of the immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$ of codimension 1. We denote by $\nu_N : E(\nu_N) \to N^{n-2}$ the normal 2-dimensional bundle of the immersion g. (Note that the disk bundle associated with the vector bundle ν_N is diffeomorphic to a regular closed tubular neighborhood of the immersion g.)

In comparison with an arbitrary vector bundle, this bundle carries an additional structure, namely its structure group as an $O(2)$-bundle admits a reduction to a discrete dihedral group which we denote by $\mathbf{D}$. This group has order 8, and is defined as the group of orthogonal transformations of the plane which carry the standard pair of coordinate axes into themselves (with possible change of orientation and order).

In the standard presentation of the group $\mathbf{D}$ there are two generators a, b which are connected by the relations $\{a^4 = b^2 = 1, [a, b] = a^2\}$. The generator a is represented by the rotation of the plane through an angle $\frac{\pi}{2}$, and the generator b is represented by a reflection with respect to the bisector of the first and second coordinate axes. Notice that the element ba (the product means the rule of composition $b \circ a$ of transformations in $O(2)$) is represented by the reflection with respect to the first coordinate axis.

The structure group of the normal bundle of the manifold of self intersection points for an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$, in case $k = 1$

Let us use the transversality condition for the immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$. Let N^{n-2} be the self-intersection manifold of the immersion f, $g : N^{n-2} \looparrowright \mathbb{R}^n$ be the parameterizing immersion. In the fiber $E(\nu_N)_x$ of the normal bundle ν_N over the point $x \in N$ an unordered pair of axes is fixed. These axes are formed by the tangents to the curves of intersection of the fiber $E(\nu_N)_x$ with two sheets of immersed manifolds intersecting transversely in the neighborhoods of this point. By construction, the bundle ν_N has the structure group $\mathbf{D} \subset O(2)$.

Over the space $K(\mathbf{D}, 1)$ the universal 2-dimensional $\mathbf{D}$–bundle is defined. This bundle will be denoted by $\psi : E(\psi) \to K(\mathbf{D}, 1)$. We say that the mapping $\eta : N \to K(\mathbf{D}, 1)$ is classifying for the bundle ν_N, if an isomorphism $\Xi : \eta^*(\psi) \cong \nu_N$ is well defined, where $\eta^*(\psi)$ is the inverse image of the bundle ψ and ν_N is the normal bundle of the immersion g. Further a bundle itself and its classifying map will be denoted the same; in the considered case we have $\eta \cong \nu_N$. The isomorphism Ξ will be called a $\mathbf{D}$–framing of the immersion g, and the mapping η will be called the characteristic mapping of the $\mathbf{D}$–framing Ξ.

Remark. In fact, we have described only part of a more general construction. The structure group of the s-dimensional normal bundle to the submanifold N_s of points of self-intersection of multiplicity s of an immersion f admits a reduction to the structure group $\mathbb{Z}/2 \int \Sigma(s)$, the wreath product of the cyclic group $\mathbb{Z}/2$ with the group of permutations of a set of s elements (cf., for example, [E1]).

Let a triple (f, κ, Ξ), where $f : M^{n-k} \looparrowright \mathbb{R}^n$ is an immersion and Ξ is a skew–framing of f with the characteristic class $\kappa \in H^1(M; \mathbb{Z}/2)$, represent an element of the group $Imm^{sf}(n-k, k)$. We need to generalize the previous construction for $k = 1$ and to describe the structure group of the normal bundle ν_N to the manifold N^{n-2k} of self–intersection points of a generic immersion of an arbitrary codimension k, $k \le [\frac{n}{2}]$.

Proposition 5. *The normal bundle ν_N, $\dim(\nu_N) = 2k$, of the immersion g is a direct sum of k copies of a two-plane bundle η over N^{n-2k}, where each two-plane bundle has structure group $\mathbf{D}$ and is classified by a classifying mapping $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ (an analogous proposition is proved in* [Sz2]).

Proof of Proposition 5

Let $x \in N^{n-2k}$ be a point in the manifold of double points. Denote by $\bar{x}_1, \bar{x}_2 \in \bar{N}^{n-2k} \looparrowright M^{n-k}$ the two preimages of this point under the canonical covering map by the double covering. The orthogonal complement in the space $T_{g(x)}(\mathbb{R}^n)$ to the subspace $g_*(T_x(N^{n-2k}))$ is the fiber of the normal bundle $E(\nu_N)$ of the immersion g over a point $x \in N^{n-2k}$. This fiber is represented as a direct sum of two linear spaces, $E(\nu_N)_x = \bar{E}_{x,1} \oplus \bar{E}_{x,2}$, where each subspace $\bar{E}_{x,i} \subset E(\nu_N)_x$ is a fiber of the normal bundle of the immersion f at the point $\bar{x}_i$.

Each subspace $\bar{E}_{x,i}$ of the fiber is canonically a direct sum of k ordered copies of the fiber of a line bundle, since the normal bundle to the immersion f is equipped with a skew-framing. We group the fibers $\bar{E}(\kappa_{x,j,i})$, $j = 1, \dots, k$, $i = 1, 2$ with a corresponding index into a two-dimensional subfiber of the fiber of the normal bundle $E(\nu_N)_x$. As a result, we obtain a decomposition of the fiber $E(\nu_N)_x$ over each point $x \in N^{n-2k}$ into a direct sum of k copies of a two-dimensional subspace. This construction depends continuously on the choice of the point x, and can be carried out simultaneously for each point of the base N^{n-2k}. As a result, we obtain the required decomposition of the bundle ν_N into a direct sum of a number of canonically isomorphic two-plane bundles. Each two-dimensional summand is classified by a structure map $\eta : N \to K(\mathbf{D}, 1)$, which proves Proposition 5.

Definition 6. We define the cobordism group of immersions $Imm^{\mathbf{D}}(n - 2k, 2k)$, assuming $n > 2k$. Let (g, η, Ψ) be a triple, which determines a $\mathbf{D}$–framed immersion of codimension $2k$. Here $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion and $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ is the classifying map of the $\mathbf{D}$–framing Ψ. The cobordism relation of triples is standard.

Lemma 7. *Under the assumption $k_1 < k$, $2k < n$, the following commutative diagram of groups is well defined:*

$$\begin{array}{ccccc} Imm^{sf}(n-k_1, k_1) & \stackrel{J^{sf}}{\longrightarrow} & Imm^{sf}(n-k, k) & \stackrel{h_k^{sf}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow \delta_{k_1} & & \downarrow \delta_k & & \| \\ Imm^{\mathbf{D}}(n-2k_1, 2k_1) & \stackrel{J^{\mathbf{D}}}{\longrightarrow} & Imm^{\mathbf{D}}(n-2k, 2k) & \stackrel{h_k^{\mathbf{D}}}{\longrightarrow} & \mathbb{Z}/2. \end{array} \qquad (8)$$

Proof of Lemma 7

We define the homomorphisms in the diagram (8). The homomorphism

$$Imm^{sf}(n-k_1,k_1) \stackrel{J_{sf}}{\longrightarrow} Imm^{sf}(n-k,k)$$

is defined exactly was the homomorphism J_{sf} for the case $k=1$.

The homomorphism

$$Imm^{\mathbf{D}}(n-2k_1,2k_1) \stackrel{J_{\mathbf{D}}}{\longrightarrow} Imm^{\mathbf{D}}(n-2k,2k)$$

is defined analogously to the homomorphism J_{sf}. Let a triple (g,η,Ψ) represent an element in the cobordism group $Imm^{\mathbf{D}}(n-2k,2k)$, where $g: N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion with a dihedral framing. Take the universal $\mathbf{D}$–bundle ψ over $K(\mathbf{D},1)$ and take the pull-back of this bundle by means of the classifying map η, $\eta^*(\psi)$. Take a submanifold $N'^{n-2k_1} \subset N^{n-2k}$ which represents the Euler class of the bundle $(k-k_1)\eta^*(\psi)$. The triple (g',η',Ψ') is well defined, where $g'=g|_{N'}$ and $\eta'=\eta|_{N'}$. Let us define the $\mathbf{D}$-framing Ψ'.

Let us consider the normal bundle $\nu_{g'}$ of the immersion g'. This bundle is decomposed into the Whitney sum of the two bundles: $\nu_{g'} = \nu_g|_N \oplus \nu_{N'\subset N}$, where $\nu_{N'\subset N}$ is the normal bundle of the submanifold $N'^{n-2k_1} \subset N^{n-2k}$. The bundles $\nu_{N'\subset N}$ and $(k-k_1)\eta^*(\psi)$ are isomorphic and this bundle is equipped with the standard $\mathbf{D}$–framing. The bundle $\nu_g|_N$ is also equipped with the $\mathbf{D}$–framing. Therefore the bundle $\nu_{g'}$ is equipped with the dihedral framing $\Psi' : \nu_{g'} \cong k_1\eta^*(\psi)$. The triple (g',η',Ψ') represent the element $J^{\mathbf{D}}(g,\eta,\Psi) \in Imm^{\mathbf{D}}(n-2k_1,2k_1)$.

The homomorphism

$$Imm^{sf}(n-k,k) \stackrel{\delta_k}{\longrightarrow} Imm^{\mathbf{D}}(n-2k,2k)$$

transforms the cobordism class of a triple (f,κ,Ξ) to the cobordism class of the triple (g,η,Ψ), where $g: N^{n-2k} \looparrowright \mathbb{R}^n$ is the immersion parameterizing the self-intersection points manifold of the immersion f (it is assumed that the immersion f intersects itself transversally), Ψ is the $\mathbf{D}$–framing of the normal bundle of the immersion g, and η is the classifying map of the $\mathbf{D}$–framing Ψ.

We turn to the definition of the homomorphism

$$Imm^{\mathbf{D}}(n-2k,2k) \stackrel{h_k^{\mathbf{D}}}{\longrightarrow} \mathbb{Z}/2,$$

which will be called the dihedral Hopf invariant. Define the subgroup

$$\mathbf{I}_c \subset \mathbf{D}, \qquad (9)$$

generated by the transformations of the plane that preserve the subspace spanned by each basis vector. The subgroup $\mathbf{I}_c$ is an elementary abelian 2-group of rank 2. Define the homomorphism

$$l : \mathbf{I}_c \to \mathbb{Z}/2, \tag{10}$$

by sending an element x of $\mathbf{I}_c$ to 0 if x fixes the first basis vector, and to 1 if x sends the first basis vector to its negative.

The subgroup (9) has index 2 and the following 2-sheeted covering:

$$K(\mathbf{I}_c, 1) \to K(\mathbf{D}, 1), \tag{11}$$

induced by this subgroup is well defined.

Denote by

$$\bar{N}^{n-2k} \to N^{n-2k} \tag{12}$$

the 2-sheeted covering induced by the classifying map $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ from the covering (11). The following characteristic class is well defined:

$$\bar{\eta}_{sf} = l \circ \bar{\eta}_{\mathbf{I}_c} : \bar{N}^{n-2k} \to K(\mathbb{Z}/2, 1),$$

where $\bar{\eta}_{\mathbf{I}_c} : \bar{N}^{n-2k} \to K(\mathbf{I}_c, 1)$ is the double covering over the classifying map $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ induced by the coverings (11), (12) over the target and the source of the map η respectively.

Let us define the homomorphism $Imm^{\mathbf{D}}(n-2k, 2k) \stackrel{h_k^{\mathbf{D}}}{\longrightarrow} \mathbb{Z}/2$ by the formula:

$$h_k^{\mathbf{D}}([g, \eta, \Psi]) = \left\langle (\bar{\eta}_{sf})^{n-2k}; [\bar{N}^{n-2k}] \right\rangle. \tag{13}$$

The diagram (8) is now well defined. Commutativity of the right square of the diagram follows for $k_1 = 1$ by Proposition 4, and for an arbitrary k_1 the proof is similar. Let us prove the commutativity of the left square of the diagram.

Let us consider the canonical double covering $p : \bar{N}^{n-2k} \to N^{n-2k}$ over the self-intersection points manifold of g. The manifold $\bar{N}^{n-2k}$ is naturally immersed into M^{n-k}: $\bar{N}^{n-2k} \looparrowright M^{n-k}$. Let us consider the submanifold $M'^{n-k_1} \subset M^{n-k}$, dual to κ^{k-k_1}. Let us consider the submanifold $M'^{n-k_1} \cap \bar{N}^{n-2k} \subset \bar{N}^{n-2k}$, assuming that M'^{n-k_1} intersects the immersion $\bar{N}^{n-2k} \looparrowright M^{n-k}$ in a general position. We denote this submanifold of the intersection by $\tilde{N}'$. Obviously, $\dim(\tilde{N}') = n-k-k_1$ and the codimension of the submanifold $\tilde{N}'^{n-k-k_1} \subset \bar{N}^{n-2k}$ is equal to $k - k_1$.

Let us consider the involution $T : \bar{N}^{n-2k} \to \bar{N}^{n-2k}$ in the covering p. Let us consider the manifold $T(\tilde{N}'^{n-k-k_1})$ and the intersection $T(\tilde{N}'^{n-k-k_1}) \cap \tilde{N}'^{n-k-k_1}$ inside $\bar{N}^{n-2k}$. Assuming that this intersection is generic, then $T(\tilde{N}'^{n-k-k_1}) \cap \tilde{N}'^{n-k-k_1}$ is a smooth closed manifold, let us denote this manifold by $\bar{N}'$, $\dim(\bar{N}') = n-2k_1$. Moreover, the manifold $\bar{N}'^{n-2k_1}$ is equivariant with respect to the involution $T' = T|_{\bar{N}'^{n-2k_1}}$. The factor-space $\bar{N}'^{n-2k_1}/T$ is well defined, this is smooth closed manifold. Let us denote this manifold by N'^{n-2k_1} and the restriction of the canonical double cover over this manifold by $p' : \bar{N}'^{n-2k_1} \to N'^{n-2k_1}$.

Note that the manifold N'^{n-2k_1} is a submanifold in N^{n-2k} and this submanifold coincides with the self-intersection manifold of the immersion g'. Let us consider the manifold $N_2^{n-2k_1} \subset N^{n-2k}$, this manifold represents the Euler class of the bundle $\eta^*(\psi)$. Let us prove that the submanifold $N'^{n-2k} \subset N^{n-2k}$ also represents the Euler class of the bundle $\eta^*(\psi)$. Therefore we may put $N_2^{n-2k_1} = N'^{n-2k_1}$.

Let us denote the bundle $\eta^*(\psi)$ by ξ for short. Take the bundle $p'^*(\eta^*(\psi))$, let us denote this bundle by $\bar{\xi}$, and take the bundle $T^*(p'^*(\eta^*(\psi))$. Obviously, the bundle $\bar{\xi}$ decomposes into the Whitney sum of the two $k-k_1$–dimensional bundles, denoted by $\bar{\xi} = \bar{\xi}_+ \oplus \bar{\xi}_-$. For the same reason we get $T^*(\bar{\xi}) = T^*(\bar{\xi}_+) \oplus T^*(\bar{\xi}_-)$. Moreover, $T^*(\bar{\xi}_+) = \bar{\xi}_-$, $T^*(\bar{\xi}_-) = \bar{\xi}_+$. The bundle $\bar{\xi}_+$ is isomorphic to the Whitney sum of $k-k_1$ copies of the line bundle $\kappa|_{\bar{N}'}$.

The submanifold $\bar{N}_2^{n-2k_1} \subset \bar{N}^{n-2k}$ is well defined as the covering space of p', restricted to the submanifold $N_2^{n-2k_1} \subset N^{n-2k}$ in the base of p. This manifold represents the equivariant Euler class of the bundle $\bar{\xi}$. This submanifold is well defined as the intersection of the two submanifolds in $\bar{N}^{n-2k}$, denoted by $\bar{N}_{2,+}^{n-k-k_1}$ and $\bar{N}_{2,-}^{n-k-k_1}$. The submanifold $\bar{N}_{2,+}^{n-k-k_1} \subset \bar{N}^{n-2k}$ represents the Euler class of the bundle $\bar{\xi}_+$. The submanifold $\bar{N}_{2,-}^{n-k-k_1} \subset \bar{N}^{n-2k}$ represents the Euler class of the bundle $\bar{\xi}_-$. Note that the submanifold $\bar{N}_{2,-}^{n-k-k_1} \subset \bar{N}^{n-2k}$ coincides by definition with $\tilde{N}'^{n-k-k_1}$. The submanifold $\bar{N}_{2,-}^{n-k-k_1} \subset \bar{N}^{n-2k}$ coincides by definition with $T(\tilde{N}'^{n-k-k_1})$. Therefore $\bar{N}_2^{n-2k_1}$ coincides with $\bar{N}'^{n-2k_1}$ and $N_2^{n-2k_1}$ coincides with N'^{n-2k_1}. The commutativity of the left square of the diagram is proved. Lemma 7 is proved.

We need an equivalent definition of the dihedral Hopf invariant in the case of **D**–framed immersions in the codimension $2k$, $n-4k > 0$. Consider the subgroup of the orthogonal group $O(4)$ that transforms the set of vectors $(\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_4)$ of the standard basis into itself, perhaps by changing the direction of some vectors and, moreover, preserving the non-ordered pair of 2-dimensional subspaces $Lin(\mathbf{e}_1, \mathbf{e}_2)$, $Lin(\mathbf{e}_3, \mathbf{e}_4)$ generated by basis vectors

$(\mathbf{e}_1,\mathbf{e}_2)$, $(\mathbf{e}_3,\mathbf{e}_4)$. Thus these 2-dimensional subspaces may be preserved or interchanged. Denote the subgroup of these transformations by $\mathbb{Z}/2^{[3]}$. This group has order 2^7. Define the chain of subgroups of index 2:

$$\mathbf{I}_c \times \mathbf{D} \subset \mathbf{D} \times \mathbf{D} \subset \mathbb{Z}/2^{[3]}. \tag{14}$$

The subgroup $\mathbf{D} \times \mathbf{D} \subset \mathbb{Z}/2^{[3]}$ is defined as the subgroup of transformations leaving invariant each 2-dimensional subspace $Lin(\mathbf{e}_1,\mathbf{e}_2)$, $Lin(\mathbf{e}_3,\mathbf{e}_4)$ spanned by pairs of vectors $(\mathbf{e}_1,\mathbf{e}_2)$, $(\mathbf{e}_3,\mathbf{e}_4)$. This subgroup is isomorphic to a direct product of two copies of $\mathbf{D}$, each factor leaving invariant the corresponding 2-dimensional subspace. The subgroup $\mathbf{I}_c \times \mathbf{D} \subset \mathbf{D} \times \mathbf{D}$ is defined as the subgroup of transformations that leave invariant each linear subspace $Lin(\mathbf{e}_1)$, $Lin(\mathbf{e}_2)$ generated by vectors $\mathbf{e}_1$, $\mathbf{e}_2$.

Let the triple (g,η,Ψ) represent an element of $Imm^{\mathbf{D}}(n-2k,2k)$, assuming that $n-4k>0$ and that g is an immersion in general position. Let L^{n-4k} be the manifold of double self-intersection points of the immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$. The following tower of 2-sheeted coverings

$$\bar{L}^{n-4k}_{\mathbf{I}_c\times\mathbf{D}} \to \bar{L}^{n-4k}_{\mathbf{D}\times\mathbf{D}} \to L^{n-4k} \tag{15}$$

is well defined by the following construction. (The covering $\bar{L}^{n-4k}_{\mathbf{D}\times\mathbf{D}} \to L^{n-4k}$ was considered above as the canonical covering over self-intersection manifold of the immersion g and was denoted by $\bar{L}^{n-4k} \to L^{n-4k}$.) Let us consider the parameterizing immersion $h : L^{n-4k} \looparrowright \mathbb{R}^n$. The normal bundle of the immersion h will be denoted by ν_L. This bundle is classified by a mapping $\zeta : L^{n-4k} \to K(\mathbb{Z}/2^{[3]},1)$.

The chain of subgroups (14) induces a tower of 2-sheeted coverings of classifying spaces:

$$K(\mathbf{I}_c \times \mathbf{D},1) \subset K(\mathbf{D}\times\mathbf{D},1) \subset K(\mathbb{Z}/2^{[3]},1) \tag{16}$$

over the target of the classifying map ζ and the tower of 2-sheeted coverings (15) over the domain of the mapping ζ. The covering $\bar{L}_{\mathbf{I}_c\times\mathbf{D}} \to L^{n-4k}$, defined by the formula (15), will be called the canonical 4-sheeted covering over the manifold of points of self-intersections of the immersion g.

Define the epimorphism

$$l^{[3]} : \mathbf{I}_c \times \mathbf{D} \to \mathbf{I}_d, \tag{17}$$

by sending an element x of $\mathbf{I}_c \times \mathbf{D}$ to 0 if x fixes the first basis vector, and to 1 if x sends the first basis vector to its negative. This map induces the map of the classifying spaces:

$$K(\mathbf{I}_c \times \mathbf{D},1) \to K(\mathbf{I}_d,1). \tag{18}$$

The classifying mapping $\bar{\zeta}_{\mathbf{I}_c\times\mathbf{D}} : \bar{L}^{n-4k}_{\mathbf{I}_c\times\mathbf{D}} \to K(\mathbf{I}_c \times \mathbf{D}, 1)$ is well defined as a result of the transition to a 4-sheeted covering over the mapping ζ and the classifying map $\bar{\zeta}_{sf} : \bar{L}^{n-4k}_{\mathbf{I}_c\times\mathbf{D}} \to K(\mathbf{I}_d, 1)$ is well defined as a result of the composition of the classifying map $\bar{\zeta}_{\mathbf{I}_c\times\mathbf{D}}$ with the map (18).

Proposition 8. *Suppose that a* $\mathbf{D}$*–framed immersion* (g, η, Ψ) *represents an element of* $Imm^{\mathbf{D}}(n-2k, 2k)$*, the following formula is satisfied:*

$$\langle (\bar{\zeta}_{sf})^{n-4k}; [\bar{L}^{n-4k}_{\mathbf{I}_c\times\mathbf{D}}]\rangle = \langle (\bar{\eta}_{sf})^{n-2k}; [\bar{N}^{n-2k}]\rangle. \tag{19}$$

Proof of Proposition 8

Let $(g : N^{n-2k} \looparrowright \mathbb{R}^n, \Psi, \eta)$ be a $\mathbf{D}$–framed immersion, representing an element of $Imm^{\mathbf{D}}(n-2k, 2k)$ in the image of the homomorphism δ^k. Let L^{n-4k} be the double-points manifold of the immersion g, $h : L^{n-4k} \looparrowright \mathbb{R}^n$ be the parameterizing immersion, and $\bar{L}^{n-4k} \to L^{n-4k}$ be the canonical double covering. Consider the image of the fundamental class $\bar{h}_*([\bar{L}^{n-4k}]) \in H_{n-4k}(N^{n-2k}; \mathbb{Z}/2)$ by means of the immersion $\bar{h} : \bar{L}^{n-4k} \looparrowright N^{n-2k}$ and let us denote by $m \in H^{2k}(N^{n-2k}; \mathbb{Z}/2)$ the cohomology class that is Poincaré-dual to the homology class $\bar{h}_*([\bar{L}^{n-4k}])$. Consider also the cohomology Euler class of the normal bundle immersion g, which is denoted by $e \in H^{2k}(N^{n-2k}; \mathbb{Z}/2)$.

By the Herbert Theorem (see [E-G], Theorem 1.1, coefficients $\mathbb{Z}/2$) for immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ with the self-intersection manifold L^{n-4k} the formula $e = m$ given in (7) is valid. Let us consider the classifying map $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$. Let us consider the 2–sheeted cover $K(\mathbf{I}_c, 1) \to K(\mathbf{D}, 1)$ over the classifying space. Let us induced η the 2-sheeted covering map over the map η, denoted by $\bar{\eta}_{sf} : \bar{N}^{n-2k}_{sf} \to K(\mathbf{I}_c, 1)$. (Note that in the case N^{n-2k} is a self-intersection points manifold of a skew-framed immersion, the manifold $\bar{N}^{n-2k}_{sf}$ was considered above and this manifold was called the canonical covering manifold over N^{n-2k}, this manifold was denoted by $\bar{N}^{n-2k}$.)

Let us denote by $\bar{e} \in H^{2k}(\bar{N}^{n-2k}_{sf}; \mathbb{Z}/2)$, $\bar{m} \in H^{2k}(\bar{N}^{n-2k}_{sf}; \mathbb{Z}/2)$ the images of the cohomology classes e, m, respectively, under the canonical double cover $\bar{N}^{n-2k}_{sf} \to N^{n-2k}$. The Herbert Theorem implies that:

$$\bar{e} = \bar{m},$$

in particular, the following formula holds:

$$\langle (\bar{\eta}_{sf})^{n-4k}\bar{m}; [\bar{N}^{n-2k}_{sf}]\rangle = \langle (\bar{\eta}^{sf})^{n-4k}\bar{e}; [\bar{N}^{n-2k}_{sf}]\rangle. \tag{20}$$

Because $\bar{\eta}_{sf}$ coincides with $\bar{e}$, the right side of the formula is equal to $\langle (\bar{\eta}_{sf})^{n-2k}; [\bar{N}^{n-2k}]\rangle$. Because $\bar{m}$ is dual to the cohomology class $\bar{h}_*[\bar{L}^{n-4k}_{\mathbf{H}_{\bar{c}}}]$, the left side of the formula (20) can be rewritten in the form: $\langle \bar{\eta}_{sf})^{n-4k}; [\bar{L}^{n-2k}_{H_{\bar{c}}}]\rangle$. Because the classifying mappings $\bar{\zeta}_{sf} : \bar{L}^{n-4k}_{\mathbf{H}_{\bar{c}}} \to K(\mathbb{Z}/2,1)$ and $\bar{\eta}_{sf}|_{\bar{L}_{\mathbf{H}_{\bar{c}}}}$ coincide, the left side of the formula (20) is equal to the characteristic number $\langle (\bar{\zeta}_{sf})^{n-4k}; [\bar{L}^{n-4k}_{\mathbf{H}_{\bar{c}}}]\rangle$. Proposition 8 is proved.

Let us generalize Proposition 5, Definition 6, and Lemma 7.

Let us assume that the triple (g,η,Ψ), where $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion, Ψ is a $\mathbf{D}$–framing of the normal bundle of the immersion g with the characteristic class $\eta : N^{n-2k} \to K(\mathbf{D},1)$, represents an element in $Imm^{\mathbf{D}}(n-2k,2k)$. Let $h : L^{n-4k} \looparrowright \mathbb{R}^n$ be an immersion, that gives a parametrization of the self-intersection manifold of the immersion g.

Proposition 9. *The normal $4k$–dimensional bundle ν_L of the immersion h is isomorphic to the Whitney sum of k copies of a 4-dimensional bundle ζ over L^{n-4k}, each 4-dimensional direct summand has the structure group $\mathbb{Z}/2^{[3]}$ and is classified by a classifying mapping $\zeta : L^{n-4k} \to K(\mathbb{Z}/2^{[3]},1)$.*

Proof of Proposition 9

The proof is omitted, this proof is analogous to the proof of Proposition 5.

Definition 10. We define the cobordism group of immersions $Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k)$, assuming $n>4k$. Let (h,ζ,Λ) be a triple, which determines a $\mathbb{Z}/2^{[3]}$–framed immersion of codimension $4k$. Here $h : L^{n-4k} \looparrowright \mathbb{R}^n$ is an immersion and $\zeta : L^{n-4k} \to K(\mathbb{Z}/2^{[3]},1)$ is the characteristic map of the $\mathbb{Z}/2^{[3]}$–framing Λ. The cobordism relation of triples is standard.

Lemma 11. *Under the assumption $k_1 < k$, $4k < n$, the following commutative diagram of groups is well defined:*

$$\begin{array}{ccccc} Imm^{\mathbf{D}}(n-2k_1,2k_1) & \stackrel{J^{\mathbf{D}}}{\longrightarrow} & Imm^{\mathbf{D}}(n-2k,2k) & \stackrel{h_k^{\mathbf{D}}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow \delta^{\mathbf{D}}_{k_1} & & \downarrow \delta^{\mathbf{D}}_{k} & & \| \\ Imm^{\mathbb{Z}/2^{[3]}}(n-4k_1,4k_1) & \stackrel{J^{\mathbb{Z}/2^{[3]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) & \stackrel{h_k^{\mathbb{Z}/2^{[3]}}}{\longrightarrow} & \mathbb{Z}/2. \end{array} \qquad (21)$$

Proof of Lemma 11

Let us define the homomorphisms in the diagram (21). The homomorphism $h^{\mathbf{D}}_k$ is given by the characteristic number in the right side of the formula (13). The homomorphism $h_k^{\mathbb{Z}/2^{[3]}}$ is defined by means of the characteristic number in the left side of the formula (19). The commutativity of the right square of the diagram is proved in Proposition 8.

We define the further homomorphisms in the diagram (21). The homomorphism

$$Imm^{\mathbb{Z}/2^{[3]}}(n-4k_1,4k_1) \stackrel{J^{\mathbb{Z}/2^{[3]}}}{\longrightarrow} Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k)$$

is defined exactly was the homomorphism $J^{\mathbf{D}}$ in the bottom row of the diagram (8). Namely, let a triple (h,ζ,Λ) represent an element in the cobordism group $Imm^{\mathbb{Z}/3^{[3]}}(n-4k,4k)$, where $h: L^{n-4k} \looparrowright \mathbb{R}^n$ is an immersion with $\mathbb{Z}/2^{[3]}$–framing Λ. Take the universal $\mathbb{Z}/2^{[3]}$–bundle $\psi_{[3]}$ over $K(\mathbb{Z}/2^{[3]},1)$ and take the pull-back of this bundle by means of the classifying map ζ, $\zeta^*(\psi_{[3]})$. Take a submanifold $L'^{n-4k} \subset L^{n-4k_1}$ which represents the Euler class of the bundle $(k-k_1)\eta^*(\psi_{[3]})$. The triple (h',ζ',Λ') is well defined, where $h'=h|_{L'}$ и $\zeta'=\zeta|_{L'}$. Let us define the $\mathbb{Z}/2^{[3]}$-framing Λ'.

Let us consider the normal bundle $\nu_{h'}$ of the immersion h'. This bundle decomposes into the Whitney sum of the two bundles: $\nu_{h'}=\nu_h|_L \oplus \nu_{L'\subset L}$, where $\nu_{L'\subset L}$ is the normal bundle of the submanifold $L'^{n-2k_1} \subset L^{n-2k}$. The bundles $\nu_{L'\subset L}$ and $(k-k_1)\zeta^*(\psi_{[3]})$ are isomorphic and this bundle is equipped with the standard $\mathbb{Z}/2^{[3]}$–framing. The bundle $\nu_h|_L$ is also equipped with the $\mathbb{Z}/2^{[3]}$–framing. Therefore the bundle $\nu_h|_L$ is equipped with the dihedral framing $\Lambda' : \nu_{h'} \cong k_1\zeta^*(\psi_{[3]})$. The triple (h',ζ',Λ') represent the element $J^{\mathbb{Z}/2^{[3]}}(h,\zeta,\Lambda) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k_1,4k_1)$.

The homomorphism

$$Imm^{\mathbf{D}}(n-2k,2k) \stackrel{\delta_k^{\mathbf{D}}}{\longrightarrow} Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k)$$

transforms the cobordism class of a triple (g,η,Ψ) to the cobordism class of the triple (h,ζ,Λ), where $h: L^{n-4k} \looparrowright \mathbb{R}^n$ is the immersion parameterizing the self-intersection points manifold of the immersion g (it is assumed that the immersion g intersects itself transversely), Λ is the $\mathbb{Z}/2^{[3]}$–framing of the normal bundle of the immersion h, and ζ is the classifying mapping of the $\mathbb{Z}/2^{[3]}$–framing Λ.

The commutativity of the left square in the diagram (21) is proved analogously with the commutativity of the left square in the diagram (8). Lemma 11 is proved.

2 Proof of the main theorem

We reformulate the Main Theorem ($\sigma(n)$ is defined by the formula (1)), taking into account the notation of the previous section.

Theorem 12. *For $\ell \geq 7$ the homomorphism $h^{sf}_{2^{\ell-2}-n_\sigma} : Imm^{sf}(3 \cdot 2^{\ell-2} + n_{\sigma-4}, 2^{\ell-2} - 1 - n_{\sigma-4}) \to \mathbb{Z}/2$, given by the equivalent Definitions 1 and 3 is trivial.*

Remark 13. In the cases $\ell = 4$ and $\ell = 5$ the proof by means of the considered approach is unknown, because of the dimensional restriction in Lemmas 32, 34, in the case $\ell = 6$ because of the dimensional restrictions in Lemma 34. For a possible approach in the cases $n = 15$, $n = 31$ and $n = 63$ see a remark in the Introduction of the part III [A3].

Consider the homomorphism $J^{sf} : Imm^{sf}(n-1, 1) \to Imm^{sf}(3 \cdot 2^{\ell-2} + n_{\sigma-4}, 2^{\ell-2} - 1 - n_{\sigma-4})$. According to Proposition 2, $h^{sf}_1 = h^{sf}_{2^{\ell-2}-n_\sigma-1} \circ J^{sf}$. Let an element of the group $Imm^{sf}(n-1,1)$ be represented by an immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$, $w_1(M) = \kappa$. The value $h^{sf}_k(J^{sf}(f))$, where $k = 2^{\ell-2} - n_\sigma - 1$, coincides with the characteristic number $\langle \kappa^{n-1}; [M^{n-1}] \rangle$. Applying Theorem 12, we conclude the proof of the Main Theorem.

For the proof of Theorem 12 we shall need the fundamental Definitions 18, 20, whose formulation will require some preparation.

Definition of the subgroups $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{D}$, $\mathbf{I}_b \subset \mathbf{D}$

We denote by $\mathbf{I}_a \subset \mathbf{D}$ the cyclic subgroup of order 4 and index 2, containing the nontrivial elements $a, a^2, a^3 \in \mathbf{D}$ (i.e., generated by the plane rotation which exchanges the coordinate axes). We denote by $\mathbf{I}_d \subset \mathbf{I}_a$ the subgroup of index 2 with nontrivial elements a^2. We denote by $\mathbf{I}_b \subset \mathbf{D}$ the subgroup of index 2 with nontrivial elements a^2, ab, a^3b (i.e., generated by the reflections with respect to the bisectors of the coordinate axes).

The following inclusion homomorphisms of subgroups are well defined: $i_{d,a} : \mathbf{I}_d \subset \mathbf{I}_a$, $i_{d,b} : \mathbf{I}_d \subset \mathbf{I}_b$. When the image coincides with the entire group $\mathbf{D}$ the corresponding index for the inclusion homomorphism will be omitted: $i_d : \mathbf{I}_d \subset \mathbf{D}$, $i_a : \mathbf{I}_a \subset \mathbf{D}$, $i_b : \mathbf{I}_b \subset \mathbf{D}$.

Definition of the subgroup $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$

Let $\mathbf{Q}$ be the quaternion group of order 8. This group has presentation $\{\mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}$. There is a standard representation $\chi : \mathbf{Q} \to O(4)$. The representation χ (a matrix acts to the left on a vector) carries the unit quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{22}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

These matrices give the action of **left** multiplication by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ on the standard basis for the quaternions. This representation χ defines the subgroup $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]} \subset O(4)$.

Definition of the subgroups $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{Q}$

Denote by $i_{\mathbf{I}_d,\mathbf{Q}} : \mathbf{I}_d \subset \mathbf{Q}$ the central subgroup of the quaternion group, which is also the center of the whole group $\mathbb{Z}/2^{[3]}$.

Denote by $i_{\mathbf{I}_a,\mathbf{Q}} : \mathbf{I}_a \subset \mathbf{Q}$ the subgroup of the quaternion group generated by the quaternion $\mathbf{i}$.

The following inclusions are well defined: $i_{\mathbf{I}_d} : \mathbf{I}_d \subset \mathbb{Z}/2^{[3]}$, $i_{\mathbf{Q}} : \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$.

Definition 14. We say that a classifying map $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ is cyclic if it can be factored as a composition of a map $\mu_a : N^{n-2k} \to K(\mathbf{I}_a, 1)$ and the inclusion $i_a : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$. We say that the mapping μ_a determines a reduction of the classifying mapping η.

Definition 15. We say that a classifying map $\zeta : L^{n-4k} \to K(\mathbb{Z}/2^{[3]}, 1)$ is quaternionic if it can be factored as a composition of a map $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$ and the inclusion $i_{\mathbf{Q}} : K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$. We also say that the mapping λ determines a reduction of the classifying mapping ζ.

We shall later require the construction of the Eilenberg-Mac Lane spaces $K(\mathbf{I}_a,1)$, $K(\mathbf{Q},1)$ and a description of the finite dimensional skeleta of these spaces, which we now recall.

Consider the infinite dimensional sphere S^∞ (a contractible space), which it is convenient to define as a direct limit of an infinite sequence of inclusions of standard spheres of odd dimension,

$$S^\infty = \lim_{\longrightarrow} \quad (S^1 \subset S^3 \subset \dots \subset S^{2j-1} \subset S^{2j+1} \subset \cdots \quad).$$

Here S^{2n-1} is defined by the formula $S^{2j-1} = \{(z_1,\dots,z_j) \in \mathbb{C}^j, \quad |z_1|^2 + \dots + |z_j|^2 = 1\}$. Let $\mathbf{i}(z_1,\dots,z_j) = (\mathbf{i}z_1,\dots,\mathbf{i}z_j)$.

Then the space $S^{2j-1}/\mathbf{i}$, which is called the $(2j-1)$-dimensional lens space over $\mathbb{Z}/4$, is the $(2j-1)$–dimensional skeleton of the space $K(\mathbf{I}_a,1)$. The space $S^\infty/\mathbf{i}$ itself is the Eilenberg-Mac Lane space $K(\mathbf{I}_a,1)$. The cohomology ring of this space is well-known, see e.g. [A-M].

Let us define the Eilenberg-Mac Lane space $K(\mathbf{Q},1)$. Consider the infinite dimensional sphere S^∞, which now it is convenient to define as a direct limit of an infinite sequence of inclusions of standard spheres of dimensions $4j+3$:

$$S^\infty = \lim_{\longrightarrow} \quad (S^3 \subset S^7 \subset \dots \subset S^{4j-1} \subset S^{4j+3} \subset \cdots \quad).$$

A coordinate action $\mathbf{Q} \times (\mathbb{C}^2)^j \to (\mathbb{C}^2)^j$ is defined on each direct summand $\mathbb{H} = \mathbb{C}^2$ in accordance with the formulas (22), (23), (24). Thus, the space $S^{4j-1}/\mathbf{Q}$ is a $(4j-1)$-dimensional skeleton of the space $S^\infty/\mathbf{Q}$ and this space is called the $(4j-1)$-dimensional lens space over $\mathbf{Q}$. The space $S^\infty/\mathbf{Q}$ itself is the Eilenberg-Mac Lane space $K(\mathbf{Q},1)$. The cohomology ring of this space is well known, see [At] section 13.

Definition of the characteristic number $h_{\mu_a,k}$

Let us assume $n > 4k$ and let us assume that on the manifold N^{n-2k} of self-intersection points of a skew-framed immersion there is defined a map $\mu_a : N^{n-2k} \to K(\mathbf{I}_a,1)$. Define the characteristic value $h_{\mu_a,k}$ by the formula:

$$h_{\mu_a,k} = \langle \bar{e}_g \bar{\mu}_a^* x; [\bar{N}_a^{n-2k}] \rangle, \qquad (25)$$

where $\bar{\mu}_a : \bar{N}_a^{n-2k} \to K(\mathbf{I}_d,1)$ is a double cover over the map $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a,1)$, induced by the cover $K(\mathbf{I}_d,1) \to K(\mathbf{I}_a,1)$, $x \in H^{n-4k}(K(\mathbf{I}_d,1);\mathbb{Z}/2)$ is the generator, $\bar{e}_g \in H^k(\bar{N}_a^{n-2k};\mathbb{Z}/2)$ is the image of the Euler class $e_g \in H^k(N^{n-2k};\mathbb{Z}/2)$ of the immersion g by means of the covering $p_a : \bar{N}_a^{n-2k} \to N^{n-2k}$, $\bar{e}_g = p_a^*(e_g)$, and $[\bar{N}_a^{n-2k}]$ is the fundamental class of the manifold $\bar{N}_a^{n-2k}$. (The manifold $\bar{N}_a^{n-2k}$ coincides with the canonical 2-sheeted covering

$\bar{N}^{n-2k}$, if and only if the classifying mapping $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$ is cyclic, see Definition 14.)

Let us assume that $k \equiv 0 \pmod 2$. Then the characteristic number (25) is the reduction modulo 2 of the following characteristic number, denoted the same, determined modulo 4:

$$\langle e_g \mu_a^* x; [N^{n-2k}] \rangle, \tag{26}$$

where $x \in H^{n-4k}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$ is the generator, $e_g \in H^k(N^{n-2k}; \mathbb{Z}/4)$ is the Euler class of the co-oriented immersion g with coefficients modulo 4, $[N^{n-2k}]$ is the fundamental class of the oriented manifold N^{n-2k} with coefficients modulo 4.

Definition of the characteristic number $h_{\lambda,k}$

Let us assume $n > 4k$ and let us assume that on the manifold L^{n-4k} of self-intersection points of a $\mathbf{D}$-framed immersion there is defined a map $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$. Define the characteristic value h_λ^k by the formula:

$$h_{\lambda,k} = \langle \bar{\lambda}^* y; [\bar{L}_{\mathbf{I}_d}] \rangle, \tag{27}$$

where $y \in H^{n-4k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ is a generator,

$$\bar{\lambda}_{\mathbf{I}_d} : \bar{L}_{\mathbf{I}_d}^{n-4k} \to K(\mathbf{I}_d, 1) \tag{28}$$

is a 4-sheeted cover over the map $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$, induced by the cover $K(\mathbf{I}_d, 1) \to K(\mathbf{Q}, 1)$, and $[\bar{L}_{\mathbf{I}_d}]$ is the fundamental class of the manifold $\bar{L}_{\mathbf{I}_d}^{n-4k}$. (The manifold $\bar{L}_{\mathbf{I}_d}^{n-4k}$ coincides with the canonical 4-sheeted covering $\bar{L}_{\mathbf{I}_c \times \mathbf{D}}^{n-4k}$, if and only if the classifying mapping $\zeta : L^{n-4k} \to K(\mathbb{Z}/2^{[3]}, 1)$ is quaternionic, see Definition 15.)

Let us assume that $k \equiv 0 \pmod 2$. Then the characteristic number (27) is the reduction modulo 2 of the following characteristic number, denoted the same, determined modulo 4:

$$\langle \bar{\lambda}^* y; [\bar{L}] \rangle, \tag{29}$$

where $y \in H^{n-4k}(K(\mathbf{I}_a, 1); \mathbb{Z}/4)$ is the generator, $[\bar{L}]$ is the fundamental class of the oriented manifold $\bar{L}^{n-4k}$ (the manifold $\bar{L}^{n-4k}$ is the canonical 2-sheeted covering over the manifold L^{n-4k}) with coefficients modulo 4.

Lemma 16. *For an arbitrary skew-framed immersion $(f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa, \Xi)$ with self-intersection manifold N^{n-2k} for which the classifying mapping η of the normal bundle is cyclic, the following equality is satisfied:*

$$h_k^{sf}(f, \kappa, \Xi) = h_{\mu_a,k},$$

where the characteristic value on the right side is calculated for a mapping μ_a, satisfying the condition $\eta = i_a \circ \mu_a$, $i_a : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$.

Proof of Lemma 16

Consider the double cover $\bar{\mu}_a : \bar{N}_a^{n-2k} \to K(\mathbf{I}_d, 1)$ over the mapping $\mu_a : N^{n-2k} \to K(\mathbf{I}_a, 1)$, induced by the double cover $K(\mathbf{I}_d, 1) \to K(\mathbf{I}_a, 1)$ over the target space of the map. Since the structure mapping η is cyclic, the manifold $\bar{N}_a^{n-2k}$ coincides with the canonical 2-sheeted cover $\bar{N}^{n-2k}$ over the self-intersections manifold N^{n-2k} of the immersion f, the class $\bar{e}_g \in H^{2k}(N^{n-2k};\mathbb{Z}/2)$ coincides with the class $\bar{\eta}_{sf}^{2k}$, $\bar{\eta}_{sf} \in H^1(\bar{N}^{n-2k};\mathbb{Z}/2)$. The proof of the lemma follows from Lemma 7 since the mappings $\bar{\mu}_a$ and $\bar{\eta}$ coincide and the characteristic number $h_{\mu_a,k}$ is computed as in the right side of the equation (13).

Lemma 17. *For an arbitrary* $\mathbf{D}$*-framed immersion* $(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)$ *with a self-intersection manifold* L^{n-4k}*, for which the classifying mapping* ζ *of the normal bundle is quaternionic, the following equality is satisfied:*

$$h_k^{sf}(g, \eta, \Psi) = h_{\lambda,k},$$

where the characteristic value on the right side is calculated by the formula (27) for a mapping λ*, satisfying the condition* $\zeta = i_a \circ \lambda$*,* $i_a : K(\mathbf{Q}, 1) \subset K(\mathbb{Z}/2^{[3]}, 1)$*.*

Proof of Lemma 17

Define the 2-sheeted covering

$$\bar{\lambda}_{\mathbf{I}_a} : \bar{L}_{\mathbf{I}_a}^{n-4k} \to K(\mathbf{I}_a, 1) \qquad (30)$$

over the mapping $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$, induced by the 2-sheeted cover $K(\mathbf{I}_a, 1) \to K(\mathbf{Q}, 1)$ over the target space of the map. Let us consider the 4-sheeted covering $\bar{\lambda}_{\mathbf{I}_d} : \bar{L}_{\mathbf{I}_d}^{n-4k} \to K(\mathbf{I}_d, 1)$ over the mapping $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$, defined by the formula (28), over the target space of the map.

Since the structure mapping ζ is quaternionic, the manifold $\bar{L}_{\mathbf{I}_d}^{n-4k}$ coincides with the canonical 4-sheeted covering $\bar{L}^{n-4k}$ over the self-intersections manifold L^{n-4k} of the immersion g. The proof of the lemma follows from Proposition 8, since the mappings $\bar{\lambda}$ and $\bar{\zeta}$ coincide and the characteristic number h_λ^k is computed as in the left side of the equation (19).

Definition 18. Let N^{n-2k} be the manifold of double self-intersection points of a skew-framed immersion (f, κ, Ξ), where the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ is assumed to be in general position. We say that this skew-framed immersion admits a cyclic structure, if a marked component (possibly, non-connected) component $N_a^{n-2k} \subset N^{n-2k}$ is fixed, and a mapping mapping $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a, 1)$ is defined. The following conditions are satisfied:

–1. The mapping μ_a determines a reduction of the restriction of the classifying mapping η to the component N_a^{n-2k}, to a mapping into the subspace $K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$ (see Definition 14 about the notion "‘reduction"’).

–2 The following equation is satisfied:

$$h_{\mu_a,k} = h_k^{sf}(f, \Xi, \kappa), \tag{31}$$

where the characteristic number in the left side of the formula is given by (25) and on the right side is given by Definition 3.

Example 19. Lemma 16 implies that if a classifying map η is cyclic, the cyclic structure can be defined by the mapping $\mu_a : N^{n-2k} \to K(\mathbf{I}_a, 1)$, where $i_a \circ \mu_a = \eta$, $i_a : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$.

Let us define the subgroup $\mathbf{H}_b \subset \mathbb{Z}/2^{[3]}$ as a product of the subgroup $\mathbf{I}_a \subset \mathbf{Q} \subset \mathbb{Z}/2^{[3]}$ and an elementary subgroup, with the only non-trivial element t given by the transformation transposing each pair of the corresponding basis vectors $\mathbf{e}_1 = \mathbf{1}$ and $\mathbf{e}_3 = \mathbf{j}$ and the pair of the basis vectors $\mathbf{e}_2 = \mathbf{i}$ and $\mathbf{e}_4 = \mathbf{k}$, preserving their direction. It is easy to verify that the group $\mathbf{H}_b$ has the order 8 and this group is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/2$. The groups $\mathbf{H}_b$ and $\mathbf{Q}$ contains the common index 2 subgroup: $\mathbf{I}_a \subset \mathbf{H}_b$, $\mathbf{I}_a \subset \mathbf{Q}$.

Definition 20. Let (g, Ψ, η) be a $\mathbf{D}$-framed immersion, where the immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is assumed to be in a general position with self-intersection manifold denoted by L^{n-4k}. Assume that the manifold N^{n-2k} contains a marked component $N_a^{n-2k} \subset N^{n-2k}$, with self-intersection manifold $L_a^{n-4k} \subset L^{n-4k}$.

Let the component N_a^{n-2k} be equipped with a mapping $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a, 1)$, which is determined a reduction of the restriction of the classifying mapping η to the component N_a^{n-2k} (see property 1 in Definition (18)).

Assume that the manifold L_a^{n-4k} is the disjoint union of the two closed submanifolds: $L_a^{n-4k} = L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_b}^{n-4k}$. Moreover, there exists a pair of mappings (μ_a, λ), where $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a, 1)$, $\lambda = \lambda_{\mathbf{Q}} \cup \lambda_{\mathbf{H}_b} : L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_b}^{n-4k} \to K(\mathbf{Q}, 1) \cup K(\mathbf{H}_b, 1)$. Define the manifold $\bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k}$ and its mapping

$$\bar{\lambda} = \bar{\lambda}_{\mathbf{Q}} \cup \bar{\lambda}_{\mathbf{H}_b} : \bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k} \to K(\mathbf{I}_a, 1) \cup K(\mathbf{I}_a, 1), \tag{32}$$

as the 2-sheeted covering mapping over the disjoint union of the mappings $\lambda_{\mathbf{Q}} : L_{\mathbf{Q}}^{n-4k} \to K(\mathbf{Q},1)$, $\lambda_{\mathbf{H}_b} : L_{\mathbf{H}_b}^{n-4k} \to K(\mathbf{H}_b,1)$ which is induced from 2-sheeted coverings $K(\mathbf{I}_a,1) \to K(\mathbf{Q},1)$, $K(\mathbf{I}_a,1) \to K(\mathbf{H}_b,1)$ over the target space of the mapping λ. We say that this $\mathbf{D}$-framed immersion (g,Ψ,η) admits a quaternionic structure if the following two conditions are satisfied:

–1. The manifold $\bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k}$ is diffeomorphic to the canonical 2-sheeted covering manifold $\bar{L}^{n-4k}$ over the self-intersection manifold of the immersion g, and the mapping

$$(Id \cup Id) \circ \bar{\lambda} : \bar{L}_{\mathbf{Q}}^{n-4k} \cup \bar{L}_{\mathbf{H}_b}^{n-4k} \to K(\mathbf{I}_a,1) \cup K(\mathbf{I}_a,1) \to K(\mathbf{I}_a,1) \to K(\mathbf{I}_a,1), \quad (33)$$

where $Id \cup Id : K(\mathbf{I}_a,1) \cup K(\mathbf{I}_a,1) \to K(\mathbf{I}_a,1)$ is the identity map over each component, coincides with the restriction of the mapping μ_a to the submanifold $\bar{L}^{n-4k} \looparrowright N^{n-2k}$.

–2. The following equation is satisfied:

$$h_{\mu_a,k} = h_k^{\mathbf{D}}(g,\eta,\Xi), \quad (34)$$

where the characteristic number in the left side of the formula is given by (25) and on the right side is given by the formula (13). (comp. with condition 2 from Definition 18).

Example 21. Let us assume that the classifying map ζ is quaternionic. The quaternionic structure is defined by the mappings $\mu_a : N^{n-2k} \to K(\mathbf{I}_a,1)$, $\lambda : L^{n-4k} \to K(\mathbf{Q},1) \subset K(\mathbb{Z}/2^{[3]},1)$, where $i_{\mathbf{I}_a} \circ \mu_a = \eta$, $i_{\mathbf{I}_a} : K(\mathbf{I}_a,1) \subset K(\mathbf{D},1)$, $i_{\mathbf{Q}} \circ \lambda = \zeta$, $i_{\mathbf{Q}} : K(\mathbf{Q},1) \subset K(\mathbb{Z}/2^{[3]},1)$.

Lemma 22. *Assume that a pair (μ_a,λ) determines a quaternionic structure for a $\mathbf{D}$–framed immersion (g,η,Ψ). Then the characteristic number $h_{\lambda,k}$, determined by the formula (27), coincides with the characteristic number $h_{\mu_a}^k$, given by the formula (25).*

Proof of Lemma 22

The proof is analogous to the proof of Proposition 8.

Corollary 23. *Assume that a pair (μ_a,λ) determines a quaternionic structure for a $\mathbf{D}$–framed immersion (g,η,Ψ). Then*

$$h_{\mu_a,k} = \langle \bar{e}_g \bar{\mu}_a^* x; [\bar{N}_a^{n-2k}] \rangle = h_{\lambda,k}(L_{\mathbf{Q}}) + h_{\lambda,k}(L_{\mathbf{H}_b}), \quad (35)$$

where the terms in the right side are defined by the formula (27) for each corresponding component of the manifold L^{n-4k}.

We need to reformulate the notion of a cyclic structure and of a quaternionic structure without the assumption that the corresponding maps $f : M^{n-k} \to \mathbb{R}^n$ and $g : N^{n-2k} \to \mathbb{R}^n$ are immersions. We formulate the necessary definition in minimal generality, under the assumption that $M^{n-k} = \mathbb{RP}^{n-k}$, $N^{n-2k} = S^{n-2k}/\mathbf{i}$.

Let

$$d : \mathbb{RP}^{n-k} \to \mathbb{R}^n \tag{36}$$

be an arbitrary PL-mapping. Consider the two-point configuration space

$$(\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Delta_{\mathbb{RP}^{n-k}})/T', \tag{37}$$

which is called the "deleted square" of the space $\mathbb{RP}^{n-k}$. This space is obtained as the quotient of the direct product without the diagonal by the involution $T' : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$, exchanging the coordinates. This space is an open manifold. It is convenient to define an analogous space, which is a manifold with boundary.

Define the space $\bar{\Gamma}$ as a spherical blow-up of the space $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag}$ in the neighborhood of the diagonal. The spherical blow-up is a manifold with boundary, which is defined as a result of compactification of the open manifold $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag}$ by the fiberwise glue-in of the fibers of the unit sphere bundle $ST\Sigma_{diag}$ of the tangent bundle $T\Sigma_{diag}$ in the neighborhood of zero–sections of the normal bundle of the diagonal $\Sigma_{diag} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$. The following natural inclusions are well defined:

$$\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Sigma_{diag} \subset \bar{\Gamma},$$

$$ST\Sigma_{diag} \subset \bar{\Gamma}.$$

On the space $\bar{\Gamma}$ the free involution $\bar{T}' : \bar{\Gamma}_0 \to \bar{\Gamma}$, which is an extension of the involution T' is well defined.

The quotient $\bar{\Gamma}/\bar{T}'$ is denoted by Γ, and the corresponding double covering by

$$p_{\Gamma} : \bar{\Gamma}/\bar{T}' \to \Gamma.$$

The space Γ is a manifold with boundary and it is called the resolution space of the configuration space (37). The projection $p_{\partial\Gamma} : \partial\Gamma \to \mathbb{RP}^{n-k}$ is well defined, and is called a resolution of the diagonal.

For an arbitrary mapping (36) the space of self-intersection points of the mapping d is defined by the formula:

$$\mathbf{N}(d) = Cl\{([x, y]) \in int(\Gamma) : y \neq x, d(y) = d(x)\}. \tag{38}$$

By Porteous' Theorem [Por] under the assumption that the map d is generic smooth, the space $\mathbf{N}$ is a manifold with boundary of dimension $n-2k$. In the case when d is a generic PL-mapping, the space N is a polyhedron, the interior of this polyhedron

$$\mathbf{N}_\circ = \{([x,y]) \in int(\Gamma) : y \ne x, d(y) = d(x)\}. \qquad (39)$$

is an open PL–manifold.

This polyhedron is called the polyhedron of self-intersection points of the mapping d. The formula (38) defines an embedding:

$$i_{\mathbf{N}} : (\mathbf{N}, \partial\mathbf{N}) \subset (\Gamma, \partial\Gamma).$$

The boundary $\partial\mathbf{N}$ of this polyhedron $\mathbf{N}$ is called the resolution manifold of critical points of the map d. The map $p_{\partial\Gamma} \circ i_{\partial\mathbf{N}}|_{\partial\mathbf{N}} : \partial\mathbf{N} \subset \partial\bar{\Gamma} \to \mathbb{RP}^{n-k}$ is called the resolution map of singularities of the map d, we denote this mapping by $res_d : \partial\mathbf{N} \to \mathbb{RP}^{n-k}$. There is a canonical double covering

$$p_{\mathbf{N}} : \bar{\mathbf{N}} \to \mathbf{N}, \qquad (40)$$

ramified over the boundary $\partial\mathbf{N}$ (above this boundary the cover is a diffeomorphism). The following diagram is commutative:

$$\begin{array}{ccc} i_{\bar{\mathbf{N}}} : (\bar{\mathbf{N}}, \partial\mathbf{N}) & \subset & (\bar{\Gamma}, \partial\Gamma) \\ \downarrow p_{\mathbf{N}} & & \downarrow p_{\Gamma} \\ i_{\mathbf{N}} : (\mathbf{N}, \partial\mathbf{N}) & \subset & (\Gamma, \partial\Gamma). \end{array}$$

Formal (equivariant) mapping with holonomic self-intersection

Denote by $T_{\mathbb{RP}^{n-k}}$, $T_{\mathbb{R}^n}$ the standard involutions on the spaces $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$, $\mathbb{R}^n \times \mathbb{R}^n$, which permutes the coordinates. Let

$$d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n \qquad (41)$$

be an arbitrary $T_{\mathbb{RP}^{n-k}}$, $T_{\mathbb{R}^n}$–equivariant mapping, which is transversal along the diagonal of the source space. Denote $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{RP}^{n-k}}$ by $\mathbf{N} = \mathbf{N}(d^{(2)})$, let us call this polyhedron a self-intersection (formal) polyhedron of the mapping $d^{(2)}$. In the case the formal mapping $d^{(2)}$ is the extension of a mapping (36), the polyhedron $\mathbf{N}(d^{(2)})$ coincides with the polyhedron, denoted by the formula (38). The formula (39), (40) are applied for polyhedral of formal self-intersection. We need to define an intermediate notion between the formal mapping (41) and the extension of a mapping (36).

Definition 24. Let us assume that the polyhedron $\mathbf{N}_\circ$ contains a closed subpolyhedron, which is denoted by $\mathbf{N}_a \subset \mathbf{N}_\circ$, the complement is a polyhedron, possibly, with open components, which is denoted by $\mathbf{N}_{\mathbf{b}\circ} \subset \mathbf{N}_\circ$. Therefore we have:

$$\mathbf{N}_a \cup \mathbf{N}_{\mathbf{b}\circ} = \mathbf{N}_\circ. \tag{42}$$

For an arbitrary point $x = (x_1, x_2) \in \mathbf{N}_a$, $x_1 \neq x_2$, let us denote by $U(x)$ a neighborhood of x, which is a Cartesian product of two neighborhoods $x_1 \in V(x_1) \subset \mathbb{R}\mathrm{P}^{n-k}$ and $x_2 \in V(x_2) \subset \mathbb{R}\mathrm{P}^{n-k}$.

Let us call that an equivariant mapping $d^{(2)}$, which is determined by the formula (41), has a holonomic (formal) self-intersection along $\mathbf{N}_a$, if for an arbitrary point $x = (x_1, x_2) \in \bar{\mathbf{N}}_a$ the mapping $d^{(2)}$ in a small neighborhood $U(x) = V(x_1) \times V(x_2)$ of x is a Cartesian product of the two mappings $f_1 : V(x_1) \to \mathbb{R}^n$ and $f_2 : V(x_2) \to \mathbb{R}^n$, $d^{(2)} = f_1 \times f_2$, $d^{(2)} = f_1 \times f_2$. Equivalently, if for a regular neighborhood $\bar{\mathbf{N}}_a \subset V(\bar{\mathbf{N}}_a) \subset (\mathbb{R}\mathrm{P}^{n-k})^2$ there exists an immersion $d : V(\bar{\mathbf{N}}_a) \looparrowright \mathbb{R}^n$, for which the formal extension coincides with the restriction of the equivariant mapping $d^{(2)}$ on $V(\bar{\mathbf{N}}_a)/T_{\mathbb{R}\mathrm{P}^{n-k}\times\mathbb{R}\mathrm{P}^{n-k}}$.

Remark

Definition (24) is generalized for non-closed $\mathbf{N}_a$, even if the closure of this polyhedron contains critical points of the formal mapping $d^{(2)}$, see [Definition 9, A2].

Example

Let us assume that the equivariant mapping (41) is the formal extension of an immersion (36). Then the formal mapping $(d^{(2)}, d)$ has a holonomic self-intersection along the self-intersection manifold.

Structural map $\eta_{\mathbf{N}\circ} : \mathbf{N}(d)_\circ \to K(\mathbf{D}, 1)$

Define the mapping

$$\eta_\Gamma : \Gamma \to K(\mathbf{D}, 1), \tag{43}$$

which we shall call the structure mapping of the "deleted square". Note that the inclusion $\bar{\Gamma} \subset \mathbb{R}\mathrm{P}^{n-k} \times \mathbb{R}\mathrm{P}^{n-k}$ induces an isomorphism of fundamental groups, since the codimension of the diagonal $\Delta_{\mathbb{R}\mathrm{P}^{n-k}} \subset \mathbb{R}\mathrm{P}^{n-k} \times \mathbb{R}\mathrm{P}^{n-k}$ is equal to $n-k$ and satisfies the inequality $n-k \geq 3$. Therefore, the equality is satisfied:

$$\pi_1(\bar{\Gamma}) = H_1(\bar{\Gamma};\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{44}$$

Consider the induced automorphism $T'_{ast} : H_1(\bar{\Gamma};\mathbb{Z}/2) \to H_1(\bar{\Gamma};\mathbb{Z}/2)$. Note that this automorphism is not the identity. Fix an isomorphism of the groups $H_1(\bar{\Gamma};\mathbb{Z}/2)$ and $\mathbf{I}_c$, which maps the generator of the first (respectively second) factor of $H_1(\bar{\Gamma};\ Z/2)$, see (44), into the generator $ab \in \mathbf{I}_c \subset \mathbf{D}$ (respectively, $ba \in \mathbf{I}_c \subset \mathbf{D}$), which in the standard representation of the group $\mathbf{D}$ is defined by the reflection with respect to the second (respectively, the first) coordinate axis.

It is easy to verify that the automorphism of the conjugation with respect to the subgroup $\mathbf{I}_c \subset \mathbf{D}$ by means of the element $b \in \mathbf{D} \setminus \mathbf{I}_c$ (in this formula the element b can be chosen arbitrarily), defined by the formula $x \mapsto bxb^{-1}$, corresponds to the automorphism T'_*. The fundamental group $\pi_1(\Gamma)$ is a quadratic extension of $\pi_1(\bar{\Gamma})$ by means of the element b, and this extension is uniquely defined up to isomorphism by the automorphism T'_*. Therefore $\pi_1(\Gamma) \simeq \mathbf{D}$, and hence the mapping $\eta_\Gamma : \Gamma \to K(\mathbf{D},1)$ is well defined.

It is easy to verify that the mapping $\eta_\Gamma|_{\partial\Gamma}$ takes values in the subspace $K(\mathbf{I}_{b\times\dot{b}},1) \subset K(\mathbf{D},1)$. The mapping η_Γ, which is defined by the formula (43), induces the mapping

$$\eta_{\mathbf{N}_\circ} : (\mathbf{N}_\circ, U(\partial\mathbf{N})_\circ) \to (K(\mathbf{D},1), K(\mathbf{I}_{b\times\dot{b}},1)), \tag{45}$$

which we call the structure mapping. (The notion of the structure mapping is analogous to the notion of the classifying mapping for $\mathbf{D}$–framed immersion.)

Also, it is easy to verify that the homotopy class of the composition $U(\partial\mathbf{N})_\circ \stackrel{\eta_{\mathbf{N}_\circ}}{\longrightarrow} K(\mathbf{I}_{b\times\dot{b}},1) \stackrel{p_b}{\longrightarrow} K(\mathbf{I}_d,1)$, where the map $K(\mathbf{I}_{b\times\dot{b}},1) \stackrel{p_b}{\longrightarrow} K(\mathbf{I}_d,1)$ is induced by the homomorphism $\mathbf{I}_{b\times\dot{b}} \to \mathbf{I}_d$ with the kernel $\mathbf{I}_b$, $\partial\mathbf{N}(d) \stackrel{\eta}{\longrightarrow} K(\mathbf{I}_b,1) \stackrel{p_b}{\longrightarrow} K(\mathbf{I}_d,1)$ is extended to a map on $\partial\mathbf{N}$ and this extension coincides to the map $\kappa \circ res_d : \partial\mathbf{N}(d) \to \mathbb{RP}^{n-k} \to K(\mathbf{I}_d,1)$, which is the composition of the resolution map $res_d : \partial\mathbf{N}(d) \to \mathbb{RP}^{n-k}$ and the embedding of the skeleton $\mathbb{RP}^{n-k} \subset K(\mathbf{I}_d,1)$ in the classifying space.

Assume that the following mapping

$$\mu_a : \mathbf{N}_a \to K(\mathbf{I}_a,1) \tag{46}$$

determines the reduction of the restriction of the structure mapping (45) to the marked component in the formula (42).

The following characteristic number

$$\langle \mu_a^*(t); [\mathbf{N}_a] \rangle, \tag{47}$$

is well defined, where $t \in H^{n-2k}(K(\mathbf{I}_a,1);\mathbb{Z}/2)$ is the generic cohomology class $[\mathbf{N}_a]$ is the fundamental class of the polyhedron $\mathbf{N}_a$ (this polyhedron is a PL-manifold).

Definition 25. Cyclic structure of a formal mapping $(d^{(2)},d)$ **with holonomic self-intersection along** $\mathbf{N}_a$ The mapping (46) is called the cyclic structure of the equivariant mapping with holonomic self-intersection along the polyhedron $\mathbf{N}_a$, if the characteristic number (47) satisfies the following equation:

$$\langle \mu_a^*(t);[\mathbf{N}_a]\rangle = 1. \qquad (48)$$

Let $(\mathbf{N},\partial\mathbf{N})$ be the polyhedron (with boundary) of a formal self-intersection of the formal (equivariant) mapping (41), which contains a closed component $\mathbf{N}_a \subset \mathbf{N}$, and assume that along this component the self-intersection is holonomic. Suppose given a map $\mu_a : \mathbf{N}_a \to K(\mathbf{I}_a,1)$ like in Definition . We need a criterion to verify that the mapping μ_a satisfies the equation (47).

Consider the double covering

$$p_a : \bar{\mathbf{N}}_a \to \mathbf{N}_a, \qquad (49)$$

induced from the universal double cover

$$K(\mathbf{I}_d,1) \to K(\mathbf{I}_a,1) \qquad (50)$$

by the mapping $\mu_a : \mathbf{N}_a \to K(\mathbf{I}_a,1)$. Denote by $\bar{\mu}_a : \bar{\mathbf{N}}_a \to K(\mathbf{I}_d,1)$ the mapping double covering over the mapping μ_a. Obviously, because the structure mapping $\eta : \mathbf{N}_a \to K(\mathbf{D},1)$ is cyclic, the covering (52) coincides with the canonical 2-sheeted covering, which is determined by the formula (40).

The following homology class

$$\bar{\mu}_{a*}([\bar{\mathbf{N}}_a]) \in H_{n-2k}(K(\mathbf{I}_d,1)) \qquad (51)$$

is well defined as the image of the fundamental homology class $[\mathbf{N}_a]$ by the mapping $\bar{\mu}_a$.

Let us consider the canonical 2-sheeted covering over the polyhedron $\mathbf{N}$, which is, probably, branched over the boundary:

$$p : \bar{\mathbf{N}} \to \mathbf{N}. \qquad (52)$$

The total space of this covering is a closed polyhedron $\bar{\mathbf{N}}$ of the dimension $n-2k$. This polyhedron is decomposed into the union of the two subpolyhedra: $\bar{\mathbf{N}} = \bar{\mathbf{N}}_a \cup \bar{\mathbf{N}}_{\mathbf{b}}$, this decomposition corresponds with the decomposition in the formula (42).

Consider the mapping $p_{\mathbf{I}_c,\mathbf{I}_d} \circ \bar{\eta} : \bar{\mathbf{N}}_{\mathbf{b}} \to K(\mathbf{I}_c, 1) \to K(\mathbf{I}_d, 1)$, where the mapping $p_{\mathbf{I}_c,\mathbf{I}_d} : K(\mathbf{I}_c, 1) \to K(\mathbf{I}_d, 1)$ is induced by the epimorphism $\mathbf{I}_c \to \mathbf{I}_d$, with the kernel, generated by the element $ab \in \mathbf{I}_c$. Denote the restriction of this mapping to the component $\bar{\mathbf{N}}_{\mathbf{b}}$ by $\bar{\eta}_{\mathbf{b}} : \bar{\mathbf{N}}_{\mathbf{b}} \to K(\mathbf{I}_d, 1)$.

Lemma 26. *Let a mapping $\mu_a : \mathbf{N}_a \to K(\mathbf{I}_a, 1)$ be given. The condition (47) is a corollary of the following two conditions.*

–1. The restriction of the structure mapping $\eta_\circ$ to the component N_a is cyclic and the mapping μ_a determines the reduction of the structure mapping to a mapping into the subspace $K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$.

–2. The homology class

$$\bar{\eta}_{\mathbf{b}*}([\bar{\mathbf{N}}_{\mathbf{b}}]) \in H_{n-2k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$$

is trivial.

Proof of Lemma 26

Let us consider a sketch of the proof. The homology class $\bar{\eta}_*([\bar{\mathbf{N}}]) \in H_{n-2k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ is the generator, because the fundamental class of the subpolyhedron $\bar{\mathbf{N}} \subset \mathbb{R}\mathrm{P}^{n-k}$ is dual to the (normal) characteristic class of the dimension $2k$, which is the generator in $H_{n-2k}(\mathbb{R}\mathrm{P}^{n-k})$, because $n = 2^\ell - 1$. Lemma 26 is proved.

Let $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ be a PL–mapping in a general position.

Consider the configuration space

$$((S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Delta_{S^{n-2k}})/T', \qquad (53)$$

which is called the "deleted square" of the lens space $S^{n-2k}/\mathbf{i}$. This space is obtained as the quotient of the direct product without the diagonal by the involution $T' : S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i} \to S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$, exchanging the coordinates. This space is an open manifold. It is convenient to define an analogous space, which is a manifold with boundary.

Define the space $\bar{\Gamma}_1$ as a spherical blow-up of the space $(S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Delta_{S^{n-2k}}$ in the neighborhood of the diagonal. The spherical blow-up is a manifold with boundary, which is defined as a result of compactification of the open manifold $(S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}) \setminus \Sigma_{diag}$ by the fiberwise glue-in of the fibers of the spherization $ST\Sigma_{diag}$ of the tangent bundle $T\Sigma_{diag}$ in the neighborhood of zero–sections of the normal bundle of the diagonal $\Sigma_{diag} \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$. The following natural inclusions are well defined:

$$S^{n-2k} \times S^{n-2k} \setminus \Sigma_{diag} \subset \bar{\Gamma}_1,$$

$$ST\Sigma_{diag} \subset \bar{\Gamma}_1.$$

On the space $\bar{\Gamma}_1$ the free involution

$$\bar{T}' : \bar{\Gamma}_1 \to \bar{\Gamma}_1 \tag{54}$$

which is an extension of an involution T' is well defined.

The quotient $\bar{\Gamma}_1/\bar{T}'$ is denoted by Γ_1, and the corresponding double covering by

$$p_{\Gamma_1} : \bar{\Gamma}_1/\bar{T}' \to \Gamma_1.$$

The space Γ_1 is a manifold with boundary and it is called the resolution space of the configuration space (53). The projection $p_{\partial\Gamma_1} : \partial\Gamma_1 \to S^{n-2k}/\mathbf{i}$ is well defined, this map is called a resolution of the diagonal.

For an arbitrary mapping $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ the polyhedron $\mathbf{L}(c)$ of self-intersection points of the mapping c is defined by the formula:

$$\mathbf{L}(c) = Cl\{([x,y]) \in int(\Gamma_1) : y \neq x, c(y) = c(x)\}. \tag{55}$$

By Porteous' Theorem [Por] under the assumption that the map c is smooth and generic, the polyhedron $\mathbf{L}(c)$ is a manifold with boundary of dimension $n - 4k$. This polyhedron is denoted by $\mathbf{L}^{n-4k}(c)$ and called the polyhedron of self-intersection of the map c. This formula (55) defines an embedding of polyhedra into manifold:

$$i_{\mathbf{L}(c)} : (\mathbf{L}^{n-4k}(c), \partial\mathbf{L}^{n-4k}(c)) \subset (\Gamma_1, \partial\Gamma_1).$$

The boundary $\partial\mathbf{L}^{n-4k}(c)$ of the manifold $\mathbf{L}^{n-4k}(c)$ is called the resolution manifold of critical points of the map c. The map $p_{\partial\Gamma_1} \circ i_{\partial\mathbf{L}(c)}|_{\partial\mathbf{L}(c)} : \partial\mathbf{L}^{n-4k}(c) \subset \partial\bar{\Gamma}_1 \to S^{n-2k}$ is called the resolution map of singularities of the map c, we denote this mapping by $res_c : \partial\mathbf{L}(c) \to S^{n-2k}/\mathbf{i}$.

Consider the canonical double covering

$$p_{\mathbf{L}(c)} : \bar{\mathbf{L}}(c)^{n-4k} \to \mathbf{L}(c)^{n-4k}, \tag{56}$$

ramified over the boundary $\partial\mathbf{L}(c)^{n-4k}$ (over this boundary the cover is a diffeomorphism). The next diagram is commutative:

$$\begin{array}{ccc} i_{\bar{\mathbf{L}}(c)} : (\bar{\mathbf{L}}^{n-4k}(c), \partial\mathbf{L}^{n-4k}(c)) & \subset & (\bar{\Gamma}_1, \partial\Gamma_1) \\ \downarrow p_{\mathbf{L}(c)} & & \downarrow p_{\Gamma_1} \\ i_{\mathbf{L}(c)} : (\mathbf{L}^{n-4k}(c), \partial\mathbf{L}^{n-4k}(c)) & \subset & (\Gamma_1, \partial\Gamma_1). \end{array}$$

Definition of the subgroup $\mathbf{H} \subset \mathbb{Z}/2^{[3]}$

Consider the space $\mathbb{R}^4$ with the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$. The basis vectors are conveniently identified with the basic unit quaternions $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$, which sometimes will be used to simplify the formulas for some transformations. Define the subgroup

$$\mathbf{H} \subset \mathbb{Z}/2^{[3]} \tag{57}$$

as a subgroup of transformations, of the following two types:

– in each plane $Lin(\mathbf{e}_1 = \mathbf{1}, \mathbf{e}_2 = \mathbf{i})$, $Lin(\mathbf{e}_3 = \mathbf{j}, \mathbf{e}_4 = \mathbf{k})$ may be (mutually independent) transformations by the multiplication with the quaternion $\mathbf{i}$. The subgroup of all such transformations is denoted by $\mathbf{H}_c$, this subgroup is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$.

- the transformation exchanging each pair of the corresponding basis vectors $\mathbf{e}_1 = \mathbf{1}$ and $\mathbf{e}_3 = \mathbf{j}$ and the pair of the basis vectors $\mathbf{e}_2 = \mathbf{i}$ and $\mathbf{e}_4 = \mathbf{k}$, preserving their direction. Denote this transformation by

$$t \in \mathbf{H} \setminus \mathbf{H}_c. \tag{58}$$

It is easy to verify that the group itself is $\mathbf{H}$, has order 32, and is a subgroup $\mathbf{H} \subset \mathbb{Z}/2^{[3]}$ of index 4.

Definition of the subgroup $\mathbf{H}_{b \times \dot{b}} \subset \mathbf{H}$ and the monomorphism $i_{\mathbf{I}_a,\mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$

Define the inclusion $i_{\mathbf{I}_a,\mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$, which translates the generator of the group $\mathbf{I}_a$ into the operator of multiplication by the quaternion $\mathbf{i}$, acting simultaneously in each plane $Lin(\mathbf{e}_1 = \mathbf{1}, \mathbf{e}_2 = \mathbf{i})$, $Lin(\mathbf{e}_3 = \mathbf{j}, \mathbf{e}_4 = \mathbf{k})$. Define the subgroup $\mathbf{H}_{b \times \dot{b}} \subset \mathbf{H}$ as the product of the subgroup $i_{\mathbf{I}_a,\mathbf{H}} : \mathbf{I}_a \subset \mathbf{H}$ and the subgroup generated by the generator $t \in \mathbf{H}$. It is easy to verify that the group $\mathbf{H}_{b \times \dot{b}}$ is of the order 8 and this group is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/2$. The subgroup $\mathbf{I}_a \subset \mathbf{H}_{b \times \dot{b}}$ is of the index 2. The inclusion homomorphism $i_{\mathbf{I}_a,\mathbf{H}_{b \times \dot{b}}} : \mathbf{I}_a \subset \mathbf{H}_{b \times \dot{b}}$ of the subgroup and the projection $p_{\mathbf{H}_{b \times \dot{b}},\mathbf{I}_a} : \mathbf{H}_{b \times \dot{b}} \to \mathbf{I}_a$ are defined, such that the composition $\mathbf{I}_a \stackrel{i_{\mathbf{I}_a,\mathbf{H}_{b \times \dot{b}}}}{\longrightarrow} \mathbf{H}_{b \times \dot{b}} \stackrel{p_{\mathbf{H}_a,\mathbf{I}_a}}{\longrightarrow} \mathbf{I}_a$ is the identity.

Structure map $\zeta_\circ : \mathbf{L}_\circ(c) \to K(\mathbf{H}, 1)$

Define the map $\zeta_{\Gamma_1} : \Gamma_1 \to K(\mathbf{H}, 1)$, which we call the structure mapping of the “deleted square”. Note that the inclusion $\bar{\Gamma}_1 \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$ induces an isomorphism of fundamental groups, since the codimension of the diagonal $\Delta_{S^{n-2k}/\mathbf{i}} \subset S^{n-2k}/\mathbf{i} \times S^{n-2k}/\mathbf{i}$ satisfies the inequality $n-2k \geq 3$. Therefore, the following equality is satisfied:

$$\pi_1(\bar{\Gamma}_1) = H_1(\bar{\Gamma}_1; \mathbb{Z}/4) = \mathbb{Z}/4 \times \mathbb{Z}/4. \tag{59}$$

Consider the induced automorphism $\bar{T}'_* : H_1(\bar{\Gamma}_1; \mathbb{Z}/2) \to H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$, induced by the involution (54). Note that this automorphism is not the identity and permutes the factors. Fix an isomorphism of the groups $H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$ and $\mathbf{H}_c$, which maps the generator of the first (respectively second) factor of $H_1(\bar{\Gamma}_1; \mathbb{Z}/2)$, see (59), into the generator, defined by the multiplication by the quaternion $\mathbf{i}$ in the plane $Lin(\mathbf{1}, \mathbf{i})$ (respectively, in the plane $Lin(\mathbf{j}, \mathbf{k})$) and is the identity on the complement.

It is easy to verify that the automorphism of the conjugation with respect to the subgroup $\mathbf{H}_c \subset \mathbf{H}$ by means of the element $t \in \mathbf{H} \setminus \mathbf{H}_c$, (in this formula the element t is given by the equation (58)), defined by the formula $x \mapsto txt^{-1}$, corresponds to conjugation by means of the automorphism $\bar{T}'_*$. The fundamental group $\pi_1(\Gamma_1)$ is a quadratic extension of $\pi_1(\bar{\Gamma}_1)$ by means of the element t, and this extension is uniquely defined up to isomorphism by the automorphism T'_*. Therefore $\pi_1(\Gamma_1) \simeq \mathbf{H}$, and hence the mapping $\zeta_{\Gamma_1} : \Gamma_1 \to K(\mathbf{H}, 1)$ is well defined.

It is easy to verify that the mapping $\zeta_{\Gamma_1}|_{\partial\Gamma_1}$ takes values in the subspace $K(\mathbf{H}_{b\times\dot{b}}, 1) \subset K(\mathbf{H}, 1)$. The mapping ζ_{Γ_1} induces the map $\zeta_\circ : (\mathbf{L}_\circ(c), U(\partial\mathbf{L}(c))_\circ) \to (K(\mathbf{H}, 1), K(\mathbf{H}_{b\times\dot{b}}, 1))$, which we call the structure map. (In the considered case the notion of the structure mapping is analogous to the notion of the classifying mapping for $\mathbb{Z}/2^{[3]}$–framed immersion.) Also, it is easy to verify that the homotopy class of the composition $U(\partial\mathbf{L}(c)_\circ) \stackrel{\zeta_\circ}{\longrightarrow} K(\mathbf{H}_{b\times\dot{b}}, 1) \stackrel{p_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a}}{\longrightarrow} K(\mathbf{I}_a, 1)$ is extended to $\partial\mathbf{L}(c)$ and coincides with the characteristic map $\eta \circ res_c : \partial\mathbf{L}(c) \to S^{n-2k}/\mathbf{i} \to K(\mathbf{I}_a, 1)$, which is the composition of the resolution map $res_c : \partial\mathbf{L}(c) \to S^{n-2k}/\mathbf{i}$ and the embedding of the skeleton $S^{n-2k}/\mathbf{i} \subset K(\mathbf{I}_a, 1)$ in the classifying space.

Definition 27. Quaternionic structure for a mapping $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ with singularities

Let $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ be a map in general position, having critical points, where $k \equiv 0 \pmod 2$. Let $\mathbf{L}(c)_\circ$ be the polyhedron of double self-intersection points of the map c with the boundary $\partial\mathbf{L}(c)$.

Let us assume that the polyhedron $\mathbf{L}(c)_\circ$ is the disjoint union of the two components

$$\mathbf{L}(c)_\circ = \mathbf{L}_{\mathbf{Q}} \cup \mathbf{L}_{\mathbf{H}_{b\times\dot{b}}\circ}, \tag{60}$$

where the polyhedron $\mathbf{L_Q}$ is closed, and the polyhedron $\mathbf{L}_{\mathbf{H}_{b\times\dot{b}}\circ}$, generally speaking, contains a regular neighborhood of the boundary $U(\partial \mathbf{L}_{\mathbf{H}_{b\times\dot{b}}})_\circ = U(\partial \mathbf{L}(c))_\circ$.

We say that this map c admits a (relative) quaternionic structure, if the structure map $\zeta_\circ : (\mathbf{L}(c)_\circ, U(\partial \mathbf{L}(c))_\circ) \to (K(\mathbf{H},1), K(\mathbf{H}_{b\times\dot{b}},1))$ is given by the composition:

$$\lambda_{\mathbf{L}(c)} : \mathbf{L_Q} \cup \mathbf{L}_{\mathbf{H}_{b\times\dot{b}}\circ} \stackrel{\lambda_{\mathbf{L_Q}}\cup\lambda_{H_{b\times\dot{b}}\circ}}{\longrightarrow} K(\mathbf{Q},1) \cup K(\mathbf{H}_{b\times\dot{b}},1) \qquad (61)$$

$$\stackrel{i_{\mathbf{Q},\mathbf{H}}\cup i_{\mathbf{H}_{b\times\dot{b}},\mathbf{H}}}{\longrightarrow} K(\mathbf{H},1) \cup K(\mathbf{H},1) \stackrel{Id\cup Id}{\longrightarrow} K(\mathbf{H},1).$$

Proposition 28. *If $k \equiv 0 \pmod 4$, $k \geq 10$, an arbitrary element of $Imm^{sf}(n-k,k)$ is represented by a skew-framed immersion (f,κ,Ξ), admitting a cyclic structure in the sense of* Definition *18.*

Proposition 29. *If $n = 4k + n_\sigma$, $n \geq 127$, an arbitrary element of the group $Imm^{\mathbf{D}}(n-2k,2k)$, in the image of the homomorphism $\delta_k : Imm^{sf}(n-k,k) \to Imm^{\mathbf{D}}(n-2k,2k)$, is represented by a $\mathbf{D}$–framed immersion (g,η,Ψ), admitting a quaternionic structure in the sense of* Definition *20.*

Propositions 28, 29 are based on the application of the following principle of density of the subspace of immersions in the space of continuous maps equipped with the compact-open topology, see [Hi, Theorem 5.10].

Proposition 30. *Let $f_0 : M \looparrowright R$ (we will use the case $R = \mathbb{R}^n$) be a smooth immersion between manifolds, where the manifold M is compact, the manifold R is equipped with the metric* dist *and* $\dim(M) < \dim(R)$. *Let $g : M \to R$ be a continuous mapping homotopic to the immersion f_0. Then $\forall \varepsilon > 0$ there exists an immersion $f : M \looparrowright R$, regularly homotopic to the immersion f_0, for which* $\mathrm{dist}(g;f)_{C^0} < \varepsilon$ *in the space of maps with the induced metric.*

We need the following corollary of Proposition 30.

Corollary 31. *Let* (f', κ, Ξ') *be an arbitrary skew-framed immersion,* $f' : M^{n-k} \looparrowright \mathbb{R}^n$*, representing an element* $[(f', \kappa, \Xi')] \in Imm^{sf}(n-k, k)$ *and let* $f_1 : M^{n-k} \to \mathbb{R}^n$ *be an arbitrary continuous mapping. Then for an arbitrary (arbitrarily small)* $\varepsilon > 0$ *there is a skew-framed immersion* (f, κ, Ξ)*, where* $f : M^{n-k} \looparrowright \mathbb{R}^n$ *is an immersion of the same manifold, which is regular homotopic to the immersion* f_1 *(in particular,* (f, κ, Ξ) *represents the same element* $[(f', \kappa, \Xi')] \in Imm^{sf}(n-k, k)$*) and satisfying*

$$\mathrm{dist}(f_1; f) < \varepsilon. \tag{62}$$

Proof of Corollary 31

By Proposition 30 there exists an immersion f, regularly homotopic to f', for which the condition (62) is satisfied. A regular homotopy of a skew-framed immersion continues by a regular homotopy in the class of skew-framed immersions. Therefore f is a skew-framed and the elements $[(f', \kappa, \Xi')]$, $[(f, \kappa, \Xi)]$ are equal. Corollary 31 is proved.

Lemma 32.

Assuming the dimensional restriction

$$k \geq 10, \quad n-k \equiv -1 \pmod 8, \tag{63}$$

in particular for

$$n = 2^\ell - 1, \quad \ell \geq 7, \quad n - 4k = 7, \tag{64}$$

there exists an equivariant generic mapping $d^{(2)}$*, which admits a cyclic structure in the sense of Definition 2.*

The proof of Proposition 28 by means of Lemma 32

Let us consider the equivariant mapping $d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n$, constructed in Lemma 32. Let us consider the closed marked component $\mathbf{N}_a$ of the polyhedron of (formal) self-intersection of the formal mapping $d^{(2)}$, along which the mapping $d^{(2)}$ is holonomic. Consider the canonical 2-sheeted covering $\bar{\mathbf{N}}_a \to \mathbf{N}_a$. The holonomic condition determines an immersion $\varphi_{\bar{\mathbf{N}}_a} : \bar{\mathbf{N}}_a \looparrowright \mathbb{RP}^{n-k}$.

Denote by $\varphi_{\overline{U\mathbf{N}}_a} : \overline{U\mathbf{N}}_a \looparrowright \mathbb{RP}^{n-k}$ an immersed regular neighborhood, for which the immersion $\varphi_{\bar{\mathbf{N}}_a}$ is an immersion of the central PL-submanifold.

Denote by $U(\bar{\mathbf{N}}_a) \subset \mathbb{RP}^{n-k}$ the image of the neighborhood $\overline{U\mathbf{N}}_a$ by the immersion $\varphi_{\overline{U\mathbf{N}}_a}$. By the construction the following immersion $\overline{U\mathbf{N}}_a \looparrowright U(\bar{\mathbf{N}}_a)$ is well-defined, denote this immersion by $\varphi_{\overline{U\mathbf{N}}_a}$. Because of the holonomic condition, the following mapping $\theta : \overline{U\mathbf{N}}_a \to \mathbb{R}^n$ is well-defined. The self-intersection polyhedron of the mapping θ contains a closed component, which is PL-homeomorphic to the polyhedron $\mathbf{N}_a$.

Let us consider an arbitrary element of the group $Imm^{sf}(n-k, k)$, this element is represented by an immersion $f_0 : M_0^{n-k} \looparrowright \mathbb{R}^n$, equipped with a skew-framing (κ_0, Ξ_0). Let us consider the characteristic class $\kappa_0 : M_0^{n-k} \to \mathbb{RP}^{n-k}$ of this skew-framing. By dimensional reason, without loss of a generality, the image of κ_0 is inside the standard skeleton $\mathbb{RP}^{n-k} \subset K(\mathbf{I}_d, 1)$.

Consider an open domain $U(\bar{\mathbf{N}}_a) \subset \mathbb{RP}^{n-k}$ and denote the inverse image $\kappa_0^{-1}(U(\bar{\mathbf{N}}_a))$ of this domain by $VM^{n-k} \subset M_0^{n-k}$. The restriction of the mapping κ_0 $VM^{n-k} \to U(\bar{\mathbf{N}}_a)$ is well-defined, let us call this restriction a projection.

Define an open manifold WM^{n-k}, an immersion $WM^{n-k} \looparrowright VM^{n-k}$ and a projection $\pi_{WM} : WM^{n-k} \to \overline{U\mathbf{N}}_a$, such that the following commutative diagram is well defined.

$$\begin{array}{ccc} WM^{n-k} & \looparrowright & VM^{n-k} \\ \downarrow & & \downarrow \\ \overline{U\mathbf{N}}_a & \looparrowright & U(\bar{\mathbf{N}}_a). \end{array} \qquad (65)$$

In this diagram the horizontal rows are immersions and vertical rows are projections.

Take $\varepsilon > 0$, which is much smaller then the radius of the regular neighborhood $\overline{U\mathbf{N}}_a$ of the polyhedron $\bar{\mathbf{N}}_a$. By Proposition 30 there exists a self-transversal immersion $\alpha_1 : WM^{n-k} \looparrowright \mathbb{R}^n$, which is ε–closed to the mapping $\theta \circ \kappa_0 \circ \pi_{WM} : WM^{n-k} \to \mathbb{R}^n$. Let us consider the self-intersection manifold of the immersion α_1. This manifold contains a closed component, which is inside ε–regular neighborhood of the immersed polyhedron $\mathbf{N}_a$. Denote this component by WN_a^{n-2k}. The manifold WN_a^{n-2k} is a $\mathbf{D}$–framed immersed manifold in a codimension k, because the considered manifold is defined as regular self-intersection of a skew-framed immersion in the codimension k. Denote the constructed $\mathbf{D}$–framed immersion by $(g_{WN_a}, \eta_{WN_a}, \Psi_{WN_a})$.

Define a projection

$$p_{WN_a} : WN_a^{n-2k} \to \mathbf{N}_a. \qquad (66)$$

The following mapping

$$\mu_a : WN_a^{n-2k} \to K(\mathbf{I}_a, 1) \qquad (67)$$

is well-defined as the induced mapping from $\mathbf{N}_a$ to WN_a^{n-2k} by the projection (66), the new mapping is denoted the same.

Lemma 33. *Let $x \in Imm^{sf}(n-k,k)$ be an arbitrary element, $y \in Imm^{\mathbf{D}}(n-2k,2k)$ be an arbitrary element, represented by the triple (g,η,Ξ). Then there exists a triple (f,κ,Ψ), $f : M^{n-k} \looparrowright \mathbb{R}^n$, which represents the element x, for which the immersion f is self-transversal and the immersion of the self-intersection maniflod of f, which is represented an element $\delta_k^{\mathbf{D}}(f,\kappa,\Psi)$, contains a closed component, such that the immersion of this component coincides with the $\mathbf{D}$–framed immersion (g,η,Ξ) up to regular homotopy. Moreover, if the characteristic mapping η is cyclic in the sense of Definition 14, the canonical covering mapping $\bar{\eta} : \bar{N}^{n-2k} \to K(\mathbf{I}_d,1)$ is induced from the mapping κ by means of the immersion $\bar{N}^{n-2k} \looparrowright M^{n-k}$.*

Proof of Lemma 33

Let us re-denote the $\mathbf{D}$-framed immersion (g,η,Ξ) by $(g,\eta,\Xi)_+$. Define the manifold with boundary M_+^{n-k} as the total space of the 1-disk bundle, which is associated with the line bundle $\kappa|_{\bar{N}_+^{n-2k}}$ over $\bar{N}_+^{n-2k}$. Define an immersion $f_+ : M_+^{n-k} \looparrowright \mathbb{R}^n$, which is self-intersects along $\bar{N}_+^{n-2k}$, and for which $\bar{N}_+^{n-2k}$ is a closed component of the self-intersection manifold.

Denote the mirror copy of the $\mathbf{D}$-framed immersion $(g,\eta,\Xi)_+$ by $(g,\eta,\Xi)_-$. The self-intersection manifold of the mirror immersion $f_- : M_-^{n-k} \looparrowright \mathbb{R}^n$, contains the closed component $\bar{N}_-^{n-2k}$.

Let us proved that the immersion $f_+ \cup f_-$ is extended to an immersion f_0 of a closed manifold M^{n-k}. Define M^{n-k} as the result of the gluing of the cylinder $\partial\bar{M}^{n-k} \times [+1,-1]$ to the boundaries $\partial(M_+^{n-k})$ and $\partial(M_-^{n-k})$ along the upper and the lower components. Evidently, there exists an immersion f_0, for which the restriction over the component M_+^{n-k} coincides to f_+. (For example, we may first construct a fiberwise monomorphism of the tangent bundle $T(M^{n-k})$ into the trivial bundle $T(\mathbb{R}^n)$) such that the restriction of this fiberwise monomorphism on M_+^{n-k} coincides to f_+, then by the main theorem of [Hi] the required self-transversal immersion f_0 is well-defined).

The immersion f_0 is a skew-framed immersion and this immersion self-intersects along a $\mathbf{D}$–framed immersion, which contains a closed component $(g,\eta,\Xi)_+$. Define $(f,\kappa,\Psi) = (f_0,\kappa_0,\Psi_0) \cup (f_1,\Psi_1,\kappa_1)$, such that (f,κ,Ψ) represents the given element $x \in Imm^{sf}(n-k,k)$. The skew-framed immersion (f,Ψ,κ) is required. Lemma 33 is proved.

In the statement of Lemma 33 take the triple $(g,\eta,\Xi) = (g_{WN_a},\eta_{WN_a},\Psi_{WN_a})$. In the skew-framed regular cobordism class of

$[(f_0, \kappa_0, \Psi_0)]$ there exists a skew-framed immersion (f_1, κ_1, Ψ_1), for which the self-intersection manifold (g_1, η_1, Ξ_1), $g_1 : N_1^{n-2k} \looparrowright \mathbb{R}^n$ contains the component $(g_{WN_a}, \eta_{WN_a}, \Psi_{WN_a})$, $WN_a \subset N_1^{n-2k}$. Define the mapping $\mu_a : WN_a^{n-2k} \to K(\mathbf{I}_a, 1)$ on the marked component $WN_a^{n-2k} \subset N_1^{n-2k}$ as the mapping (67). Let us prove that the mapping μ_a determines a cyclic structure for the skew-framed immersion (f_1, κ_1, Ψ_1). For this it is sufficient to check the equation (34).

The Hopf invariant $h(f_0, \kappa_0, \Xi_0)$ coincides with the degree modulo 2 of the mapping κ_0. By a geometrical argument the degree of the mapping κ_0 coincides with the degree of the mapping (66). Therefore the right side of the equation (34) coincides the degree of the mapping (66). Because the mapping μ_a is cyclic, the characteristic class in the left side of the formula (34), which is defined by the formula (25), using the formula (48), also coincides with the degree of the mapping (66). Proposition 28 is proved.

Lemma 34. *Let us assume that $n = 4k + n_\sigma$, $n \geq 127$. Then there exists a generic mapping $: S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ with singularities that admits a quaternionic structure in the sense of* Definition *27.*

The proof of Lemma 34 is simpler than the proof of Lemma 32 because of dimensional restriction. The proof is analogous to the proof of Lemma 4 in [Akh].

The proof of Proposition 29 from Proposition 28 and Lemma 34

Consider a skew-framed immersion (f, κ, Ξ), so that $\delta_k([(f, \kappa, \Xi)]) = [(g_1 : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$ is an element of the group $Imm^{\mathbf{D}}(n-2k, 2k)$. By Proposition 28, without loss of generality we can assume that the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ admits a cyclic structure in the sense of Definition 18. Denote by $g' : N_a^{n-2k} \looparrowright \mathbb{R}^n$ the restriction of the immersion g to the marked component of the self-intersection manifold N^{n-2k} of the immersion f, and by $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a, 1)$ the mapping on the marked component, which determines a cyclic structure for the immersion f.

Consider the map $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$, constructed in Lemma 34. Consider an immersion $g_a : N_a^{n-2k} \looparrowright \mathbb{R}^n$, defined by Proposition 30 as a C^0–approximation of the composition $c \circ \mu_a : N_a^{n-2k} \to S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$ in the regular homotopy class of g'.

Analogously to the construction of the map μ_a in the Proposition 28, we conclude that a quaternionic structure of the $\mathbf{D}$-framed immersion $[(g, \eta, \Psi)]$ is well defined. Proposition 29 is proved.

Proof of Theorem 12

Let us take a positive integer k under the condition $n-4k = n_\sigma$, $k \geq 8$, this is possible if $n \geq 127$. Let the triple $[(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$ represent the given element in the cobordism group $Imm^{\mathbf{D}}(n-2k, 2k)$. Let us denote by L_a^{n-4k} the self-intersection manifold of the immersion g_a, which is the restriction of g on the marked component $N_a^{n-2k} \subset N^{n-2k}$. Let us consider a skew-framed immersion (f, κ, Ξ), such that $\delta_k^{sf}([(f, \kappa, \Xi)]) = [(g : N^{n-2k} \looparrowright \mathbb{R}^n, \eta, \Psi)]$. By Proposition 29 we may assume that the triple $[(g, \eta, \Psi)]$ admits a quaternionic structure in the sense of Definition 20.

In the first step let us assume that the classifying map η of the $\mathbf{D}$–framed immersion is cyclic in the sense of Definition 14. This means that for the marked component the following equation is satisfied:

$$\eta = i_a \circ \mu_a,$$

where $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a, 1)$, $N_a^{n-2k} = N^{n-2k}$ and $i_a : K(\mathbf{I}_a, 1) \to K(\mathbf{D}, 1)$ is the natural map induced by the inclusion of the subgroup.

Let us also assume that the classifying map ζ of the $\mathbb{Z}/2^{[3]}$–framed immersion $\delta_k^{\mathbf{D}}(g, \eta, \Psi) = (h, \zeta, \Lambda)$ is quaternionic in the sense of Definition 15. This means that $L^{n-4k} = L_{\mathbf{Q}}^{n-4k}$ and the following equation is satisfied:

$$\zeta = i_{\mathbf{Q},\mathbb{Z}/2^{[3]}} \circ \lambda_{\mathbf{Q}}, \qquad (68)$$

where $\lambda_{\mathbf{Q}} : L_{\mathbf{Q}}^{n-4k} \to K(\mathbf{Q}, 1)$, $L_{\mathbf{Q}}^{n-4k} = L^{n-4k}$, and $i_{\mathbf{Q},\mathbb{Z}/2^{[3]}} : K(\mathbf{Q}, 1) \to K(\mathbb{Z}/2^{[3]}, 1)$ is the natural map, induced by the inclusion of the subgroup (see Example 21). Let us prove the theorem in this case.

Let us consider the classifying mapping $\eta : N^{n-2k} \to K(\mathbf{D}, 1)$. Let us denote by $\tilde{N}^{n-2k-2} \subset N^{n-2k}$ the submanifold, representing the Euler class of the vector bundle $\eta^*(\psi_{\mathbf{D}})$, where by $\psi_{\mathbf{D}}$ is denoted the universal 2-dimensional vector bundle over the classifying space $K(\mathbf{D}, 1)$. Because the classifying map η is cyclic, the submanifold $\tilde{N}^{n-2k-2} \subset N^{n-2k}$ is co-oriented, moreover we have

$$\eta^*(\psi_{\mathbf{D}}) = \mu_a^*(\psi_+),$$

where by ψ_+ we denote the 2-dimensional universal $SO(2)$–bundle over $K(\mathbf{I}_a, 1)$.

Let us denote by $\tilde{g} : \tilde{N}^{n-2k-2} \looparrowright \mathbb{R}^n$ the restriction of the immersion g on the submanifold $\tilde{N}^{n-2k-2} \subset N^{n-2k}$, assuming that the immersion $\tilde{g}$ is generic. The immersion $\tilde{g}$ is a $\mathbf{D}$–framed immersion by $\tilde{\Psi}$, the classifying map $\tilde{\eta}$ of this $\mathbf{D}$–framed immersion is the restriction of η to the submanifold, this map is cyclic. The triple $(\tilde{g}, \tilde{\eta}, \tilde{\Psi})$ is constructed from the triple (g, η, Ψ) by means of the transfer homomorphism $J^{\mathbf{D}}$ in the bottom row of the diagram (8) (in this diagram k_1 is changed to k, k is changed to $k+1$).

Let us denote by $\tilde{L}^{n-4k-4}$ the self-intersection manifold of the immersion $\tilde{g}$. The manifold $\tilde{L}^{n-4k-4}$ is a submanifold of the manifold L^{n-4k}, $\tilde{L}^{n-4k-4} \subset L^{n-4k}$. The parameterized immersion $\tilde{h} : \tilde{L}^{n-4k-4} \looparrowright \mathbb{R}^n$ is well defined, this immersion is a $\mathbb{Z}/2^{[3]}$–framed immersion by means of $\tilde{\Lambda}$, the classifying map $\tilde{\zeta}$ of this $\mathbb{Z}/2^{[3]}$–framed immersion is quaternionic. The triple $(\tilde{h}, \tilde{\zeta}, \tilde{\Lambda})$ is defined from the triple (h, ζ, Λ) by means of the homomorphism $J^{\mathbb{Z}/2^{[3]}}$ in the bottom row of the diagram (21) (in this diagram k_1 is changed to k, k is changed to $k+1$).

By Lemma 11 the submanifold $\tilde{L}^{n-4k-4} \subset L^{n-4k}$ represents the Euler class of the bundle $\zeta^*(\psi_{[3]})$. This submanifold is the source manifold of a $\mathbb{Z}/2^{[3]}$-immersion, representing the image of the left bottom horizontal homomorphism in the diagram (21) (in the diagram $k_1 = k$, $k = k+1$).

Let us consider the canonical 2-sheeted covering $\tilde{p} : \bar{\tilde{L}}^{4k-4} \to \tilde{L}^{n-4k}$. The submanifold $\bar{\tilde{L}}^{4k-4} \subset \bar{L}^{n-4k}$ represents the Euler class of the bundle $\tilde{p}^*(\zeta^*(\psi_{[3]}))$. This vector bundle is naturally isomorphic to the vector bundle $\bar{\zeta}^*(\psi^!_{[3]})$, where $\bar{\zeta} : \bar{L}^{n-4k} \to K(\mathbf{H}_c, 1)$ is the canonical 2-sheeted covering over the classifying map ζ ($\mathbf{H}_c \cong \mathbf{D} \times \mathbf{D}$), $\psi^!_{[3]}$ is the pull-back of the universal vector bundle $\psi_{[3]}$ over $K(\mathbb{Z}/2^{[3]}, 1)$ by means of the covering $K(\mathbf{H}_c, 1) \to K(\mathbb{Z}/2^{[3]}, 1)$.

Because the classifying map ζ is quaternionic, the submanifold $\tilde{L}^{n-4k-4} \subset L^{n-4k}$ is co-oriented and represents the homological Euler class of the $SO(4)$–bundle $\lambda^*(\psi_{\mathbf{Q}})$, and moreover for the corresponding $O(4)$–bundles $\bar{\zeta}^*(\psi_{\mathbf{H}_c}) = \bar{\lambda}^*(\psi^!_{\mathbf{Q}})$, where:

–$\psi_{\mathbf{Q}}$ is the universal $SO(4)$–vector bundle over the classifying space $K(\mathbf{Q}, 1)$. This bundle is given by the quaternionic-conjugated representation with respect to the representation (22) -(24). The bundle $\psi_{\mathbf{Q}}$, as a $O(4)$–bundle, is defined by the formula: $\psi_{\mathbf{Q}} = i_a^*(\psi_{[3]})$, $\psi^!_{\mathbf{Q}} = i^*_{\mathbf{I}_a,\mathbf{Q}}(\psi_{\mathbf{Q}})$.

– $\psi_{\mathbf{H}_c}$ is the universal $O(4)$–bundle over $K(\mathbf{H}_c, 1)$ ($\mathbf{H}_c \cong \mathbf{D} \times \mathbf{D}$).

–$\bar{\lambda} : \bar{L}^{n-4k} \to K(\mathbf{I}_a, 1)$ is the 2-sheeted covering over the classifying mapping $\lambda : L^{n-4k} \to K(\mathbf{Q}, 1)$, induced by the 2-sheeted covering $K(\mathbf{I}_a, 1) \to K(\mathbf{Q}, 1)$ over the target space of the map λ.

For the universal $SO(4)$–bundle $\psi^!_{\mathbf{Q}}$ the following formula is satisfied:

$$\psi^!_{\mathbf{Q}} = \psi_+ \oplus \psi_-,$$

where the bundle ψ_+ admits a lift ψ_+^U to a complex $U(1)$–bundle, the bundle ψ_- is a $SO(2)$–bundle, obtained from ψ_+^U by means of the complex conjugation and forgetting the complex structure. The bundles ψ_+, ψ_- satisfy the equation: $e(\psi_+) = -e(\psi_-)$, and the Euler class $e(\psi_+)$ of the bundle ψ_+ is equal to the generator $t \in H^2(K(\mathbf{I}_a,1);\mathbb{Z})$ in the standard basis, the Euler class $e(\psi_-)$ of the bundle ψ_- is equal to $-t$ and is opposite to the generator t of the standard basis.

Let us denote by $m \in H^{4k}(N^{n-2k};\mathbb{Z})$ the cohomology class, dual to the fundamental class of the oriented submanifold $\bar{L}^{n-4k} \subset N^{n-2k}$ in the oriented manifold N^{n-2k}. Let us denote by $e_g \in H^{4k}(N^{n-2k};\mathbb{Z})$ the Euler class of the immersion g (this is the top class of the normal bundle ν_g). By the Herbert theorem for the immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ with the self-intersection manifold L^{n-4k} (see [E-G], Theorem 1.1 the case $r=1$, the coefficients is $\mathbb{Z}$) the following formula are satisfied:

$$e_g + m = 0. \tag{69}$$

Let us denote by $\tilde{m} \in H^{4k-4}(N^{n-2k};\mathbb{Z})$ the cohomology class, dual to the fundamental class of the oriented submanifold $\bar{\tilde{L}}^{n-4k-4} \subset \tilde{N}^{n-2k-2} \subset N^{n-2k}$ in the oriented manifold N^{n-2k}. Let us denote by $e_{\tilde{g}} \in H^{4k-4}(N^{n-2k};\mathbb{Z})$ the cohomology class, dual to the image of the homology Euler class of the immersion $\tilde{g}$ by the inclusion $\tilde{N}^{n-2k-2} \subset N^{n-2k}$. By the Herbert theorem for the immersion $\tilde{g} : N^{n-2k} \looparrowright \mathbb{R}^n$ with the self-intersection manifold $\tilde{L}^{n-4k}$ (see. [E-G], Theorem 1.1 the case $r=1$, the coefficients is $\mathbb{Z}$) the following formula are satisfied:

$$e_{\tilde{g}} + \tilde{m} = 0. \tag{70}$$

Because $\bar{\lambda} = \mu_a$, we may use the equation: $\bar{\lambda}^*(\psi^!_{\mathbf{Q}}) = \mu_a^*(\psi_+) \oplus \mu_a^*(\psi_-)$. The following equation are satisfied: $\tilde{m} = me(\mu_a^*(\psi_+))e(\mu_a^*(\psi_-))$, where the right side is the product of the three cohomology classes: m and the two Euler classes of the corresponding bundles. The following equation are satisfied: $e_{\tilde{g}} = e_g e^2(\mu_a^*(\psi_+))$. The equation (70) can be rewritten in the following form:

$$e_g e^2(\mu_a^*(\psi_+)) + me(\mu_a^*(\psi_+))e(\mu_a^*(\psi_-)) = 0. \tag{71}$$

Then we may take into account (69) and the equation $e(\mu_a^*(\psi_-)) = -e(\mu_a^*(\psi_+))$. Let us rewrite the previous formula as follows:

$$2e_g e^2(\mu_a^*(\psi_+)) = 0. \tag{72}$$

Because of the equation $e_g = e(\mu_a^*(\psi_+))^k$, we obtain:

$$2e^{k+2}(\mu_a^*(\psi_+)) = 0. \tag{73}$$

Let us recall that $\dim(L) = n - 4k = n_\sigma \geq 7$ and $\dim(\tilde{L}) = n_\sigma - 4 \geq 3$. The formula for the Hopf invariant for $\mathbf{D}$-framed immersion (g,η,Ψ) using (26) is the following:

$$h_k^{\mathbf{D}}((g,\eta,\Psi)) = \langle e^{k+2}(\mu_a^*(\psi_+))\mu_a^*(\tau_{n-4k-4}); [N^{n-2k}]\rangle \pmod 2, \tag{74}$$

where $\tau_{n-4k-4} \in H^{n-4k-4}(K(\mathbf{I}_a,1);\mathbb{Z}/4)$ is the generic class modulo 4, the cohomology class $e(\mu_a^*(\psi_+))$ is modulo 4, and the fundamental class $[N^{n-2k}]$ of the oriented manifold N^{n-2k} is modulo 4. The condition $h_k^{\mathbf{D}}((g,\eta,\Psi))=1$ implies the following condition: the cohomology class $e^{k+2}(\mu_a^*(\psi_+))$ is of order 4. This contradicts the formula (73). Therefore, $h_k^{sf}(f,\kappa,\Xi)=h_k^{\mathbf{D}}((g,\eta,\Psi))=0$ and the theorem in the particular case is proved.

Let us prove the theorem in the general case. Let us consider the pair of mappings (μ_a,λ), where $\mu_a : N_a^{n-2k} \to K(\mathbf{I}_a,1)$, $N_a^{n-2k} \subset N^{n-2k}$, $\lambda = \lambda_{\mathbf{Q}} \cup \lambda_{\mathbf{H}_{b\times\dot b}} : L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_{b\times\dot b}}^{n-4k} \to K(\mathbf{Q},1) \cup K(\mathbf{H}_{b\times\dot b},1)$, where $L_a^{n-4k} = L_{\mathbf{Q}}^{n-4k} \cup L_{\mathbf{H}_{b\times\dot b}}^{n-4k}$, $L_a^{n-4k} \subset L^{n-4k}$, these two mappings determine the quaternionic structure of the $\mathbf{D}$–framed immersion (g,η,Ψ) in the sense of Definition 20.

Let us consider the manifold $\bar L_a^{n-4k} = \bar L_{\mathbf{Q}}^{n-4k} \cup \bar L_{\mathbf{H}_{b\times\dot b}}^{n-4k}$, defined by the formula (32). The manifold $\bar L_a^{n-4k}$ is the canonical 2-sheeted covering over the manifold L_a^{n-4k}.

The formula (69) is valid, and additionally the cohomology class m (this class is dual to the fundamental class $[\bar L_a]$ of the submanifold $\bar L_a^{n-4k} \subset N_a^{n-2k}$) decomposes into the following sum:

$$m = m_{\mathbf{Q}} + m_{\mathbf{H}_{b\times\dot b}}, \tag{75}$$

corresponding to the type of the components $L_{\mathbf{Q}}^{n-4k}$, $L_{\mathbf{H}_{b\times\dot b}}^{n-4k}$ of the self-intersection manifold (see the formula (35)).

Let us consider the submanifold $\tilde N_a^{n-2k-2} \subset N_a^{n-2k}$, representing the Euler class of the bundle $\mu_a^*(\psi^+)$. The following immersion $\tilde g_a : \tilde N_a^{n-2k-2} \looparrowright \mathbb{R}^n$ is well defined by the restriction of the immersion g_a to the submanifold $\tilde N_a^{n-2k-2} \subset N_a^{n-2k}$. Let us denote by $\tilde L_a^{n-4k-4}$ the self-intersection manifold of the immersion $\tilde g_a$ (compare with the corresponding definition of the previous step).

The inclusion $\tilde L_a^{n-2k-4} \subset L_a^{n-2k}$ is well defined. In particular, the manifold $\tilde L_a^{n-4k-4}$ is represented by the union of the following two components: $\tilde L_a^{n-4k-4} = \tilde L_{\mathbf{Q}}^{n-4k-4} \cup \tilde L_{\mathbf{H}_{b\times\dot b}}^{n-4k-4}$.

Lemma 35. *The co-oriented submanifold $\tilde L_{\mathbf{Q}}^{n-2k-4} \subset L_{\mathbf{Q}}^{n-2k}$ represents the Euler class of the $SO(4)$–bundle $\lambda_{\mathbf{Q}}^*(\psi_{\mathbf{Q}})$.*

The submanifold $\tilde L_{\mathbf{H}_{b\times\dot b}}^{n-2k-4} \subset L_{\mathbf{H}_{b\times\dot b}}^{n-2k}$ represents the Euler class of the $SO(4)$–bundle $\lambda_{\mathbf{H}_{b\times\dot b}}^*(\psi_{\mathbf{H}_{b\times\dot b}})$, where $\psi_{\mathbf{H}_{b\times\dot b}}$ is the universal $SO(4)$–bundle over the space $K(\mathbf{H}_{b\times\dot b},1)$. The corresponding $O(4)$–bundle is standardly defined as the inverse image of the bundle $\psi_{\mathbb{Z}/2^{[3]}}$ over $K(\mathbb{Z}/2^{[3]},1)$ by means of the inclusion $K(\mathbf{H}_{b\times\dot b},1) \subset K(\mathbb{Z}/2^{[3]},1)$.

Proof of Lemma 35

The proof follows from the arguments above in the proof of commutativity of the left squares of the diagrams (8) and (21).

The bundle $\psi_{\mathbf{H}_{b\times\dot{b}}}$ is isomorphic to the Whitney sum of the two 2-bundles: $\psi_{\mathbf{H}_{b\times\dot{b}}} = p^*_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a}(\psi_{\mathbf{I}_a}) \oplus p^*_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a}(\psi_{\mathbf{I}_a}) \otimes l_{\mathbb{Z}/2}$, where $p^*_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a}(\psi_{\mathbf{I}_a})$ is the 2-dimensional bundle, defined as the pull-back of the canonical 2-dimensional bundle ψ^+ over $K(\mathbf{I}_a,1)$ by means of the natural mapping $p_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a} : K(\mathbf{H}_{b\times\dot{b}},1) \to K(\mathbf{I}_a,1)$, induced by the homomorphism $p_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a} : \mathbf{H}_{b\times\dot{b}} \to \mathbf{I}_a$, $l_{\mathbb{Z}/2}$ is a line bundle, defined as the inverse image of the canonical line bundle over $K(\mathbb{Z}/2,1)$ by means of the projection $K(\mathbf{H}_{b\times\dot{b}},1) \to K(\mathbb{Z}/2,1)$, this projection corresponds to the epimorphism $\mathbf{H}_{b\times\dot{b}} \to \mathbb{Z}/2$ with the kernel $\mathbf{I}_a \subset H_{b\times\dot{b}}$.

By analogous arguments the class $\tilde{m}$ is well defined as in the formula (70), moreover, the following formula is satisfied:

$$\tilde{m} = \tilde{m}_{\mathbf{Q}} + \tilde{m}_{\mathbf{H}_{b\times\dot{b}}}, \tag{76}$$

where the terms in the right side of the formula are defined as the cohomology classes, dual to the fundamental classes $[\bar{\bar{L}}_{\mathbf{Q}}]$, $[\bar{\bar{L}}_{\mathbf{H}_{b\times\dot{b}}}]$ of the canonical coverings over the corresponding component.

The formula relating $m_{\mathbf{Q}}$ and $\tilde{m}_{\mathbf{Q}}$ is the following: $\tilde{m}_{\mathbf{Q}} = m_{\mathbf{Q}} e(\mu_a^*(\psi_+)) e(\mu_a^*(\psi_-))$. The formula relating $m_{\mathbf{H}_{b\times\dot{b}}}$ and $\tilde{m}_{\mathbf{H}_{b\times\dot{b}}}$ is the following: $\tilde{m}_{\mathbf{H}_{b\times\dot{b}}} = m_{\mathbf{H}_{b\times\dot{b}}} e^2(\mu_a^*(\psi_+))$. To prove the last equation we use the following fact: the bundle $i^*_{\mathbf{I}_a,\mathbf{H}_{b\times\dot{b}}}(\psi_{\mathbf{H}_{b\times\dot{b}}})$, where the mapping $i_{\mathbf{I}_a,\mathbf{H}_{b\times\dot{b}}} : K(\mathbf{I}_a,1) \to K(\mathbf{H}_{b\times\dot{b}},1)$ corresponds to the index 2 subgroup $i_{\mathbf{I}_a,\mathbf{H}_{b\times\dot{b}}} : \mathbf{I}_a \subset \mathbf{H}_{b\times\dot{b}}$, is isomorphic to the bundle $\psi^+ \oplus \psi^+$.

The analog of the formula (71) is the following:

$$e_g e^2(\mu_a^*(\psi_+)) - m_{\mathbf{Q}} e^2(\mu_a^*(\psi_+)) + m_{\mathbf{H}_{b\times\dot{b}}} e^2(\mu_a^*(\psi_+)) = 0. \tag{77}$$

Let us multiply both sides of the formula (75) by the cohomology class $e^2(\mu_a^*(\psi_+))$ and take the sum with the opposite sign with (77), we get:

$$2m_{\mathbf{Q}} e^2(\mu_a^*(\psi_+)) = 0. \tag{78}$$

This is an analog of the formula (72).

Let us prove that the Hopf invariant of the $\mathbf{D}$–framed immersion (g,η,Ψ) is trivial. By Corollary 23 the Hopf invariant is given by the formula (35).

Let us prove that the each term in this formula is equal to zero. The first term $h_\lambda(L_{\mathbf{Q}})$, according to (29), is calculated as the reduction modulo 2 of the following characteristic number modulo 4:

$$h_\lambda(L_{\mathbf{Q}}) = \langle m_{\mathbf{Q}}\mu_a^*(x); [N_a^{n-2k}]\rangle,$$

where $x \in H^{n-4k}(\mathbf{I}_a;\mathbb{Z}/4)$ is the generator. Analogously, the second term $h_\lambda(L_{\mathbf{H}_{b\times\dot b}})$ is the reduction modulo 2 of the following number modulo 4:

$$h_\lambda(L_{\mathbf{H}_{b\times\dot b}}) = \langle m_{\mathbf{H}_{b\times\dot b}}\mu_a^*(x); [N_a^{n-2k}]\rangle.$$

Note that $x = \tau^2 y$, where $\tau \in H^2(K(\mathbf{I}_a,1);\mathbb{Z}/4)$, $y \in H^{n-4k-4}(K(\mathbf{I}_a,1);\mathbb{Z}/4)$ are the generators. We have $\mu_a(\tau) = e(\mu_a(\psi^+))$, because τ is the Euler class of the bundle ψ_+. Therefore, from (78) we get

$$h_\lambda(L_{\mathbf{Q}}) = 0,$$

because $m_{\mathbf{Q}}e^2(\mu_a^*(\psi_+)) = m_{\mathbf{Q}}(\mu_a^*(\tau))^2 = m_{\mathbf{Q}}\mu_a^*(x)$.

To calculate the second term $h_\lambda(L_{\mathbf{H}_{b\times\dot b}})$ it is sufficient to note that

$$\langle m_{\mathbf{H}_{b\times\dot b}}\mu_a^*(x); [N_a^{n-2k}]\rangle = \langle \mu_a^*(x); [\bar L_{\mathbf{H}_{b\times\dot b}}^{n-4k}]\rangle = \langle p^*((\lambda_{\mathbf{H}_{b\times\dot b}})^*(x')); [\bar L_{\mathbf{H}_{b\times\dot b}}^{n-4k}]\rangle = 0,$$

where $p : \bar L_{\mathbf{H}_{b\times\dot b}}^{n-4k} \to L_{\mathbf{H}_{b\times\dot b}}^{n-4k}$ – is the 2-sheeted covering, corresponding to the subgroup $i_{\mathbf{I}_a,\mathbf{H}_{b\times\dot b}}$, $x' \in H^{n-4k}(K(\mathbf{H}_{b\times\dot b},1);\mathbb{Z}/4)$ is a cohomology class, such that $i^*_{\mathbf{I}_a,\mathbf{H}_{b\times\dot b}}(x') = x$, $[\bar L_{\mathbf{H}_{b\times\dot b}}^{n-4k}]$ is the fundamental class of the total manifold of the canonical 2-sheeted covering p.

Theorem 12 is proved.

Remark 36. A straightforward generalization of Theorem 12 for mappings with singularities $c : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$, which admits a (relative) quaternionic structure in the sense of Definition 27 is not possible.

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