The Arf-Kervaire invariant of framed manifolds as an obstruction to embeddability

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Abstract

We define a quadratic form which gives an obstruction to embedding $N^{4k+2} \subset \mathbb{R}^{6k+4}$ of a smooth highly connected manifold into Euclidean space, with sufficiently many nondegenerate sections of the normal bundle. As the main corollary we prove that no 14-connected (resp. 30-connected) stably parallelizable manifold N^{30} (resp. N^{62}) with Arf-Kervaire invariant one is smoothly embeddable into $\mathbb{R}^{36}$ (resp. $\mathbb{R}^{83}$).

1 A Brief Introduction on the Kervaire Problem

Let us denote the cobordism groups of immersion in codimension 1 of oriented n-manifolds by $Imm^{fr}(n,1)$. By the Pontryagin -Thom construction [19] in the form proposed in [24], this group is isomorphic to the stable n-homotopy group of spheres. The class of regular cobordism of immersion $f : N^n \looparrowright \mathbb{R}^{n+1}$ represents an element in this cobordism group. We will consider the case $n = 4k+2$.

Definition of $\mathbb{Z}/2$-quadratic form of an immersion and its Arf-invariant. Let $f : N^{4k+2} \looparrowright \mathbb{R}^{4k+3}$ be an immersion, representing an element in the group $Imm^{fr}(4k+2,1)$. The homology group $H_{2k+1}(N^{4k+2};\mathbb{Z}/2)$ will be simply denoted by H. By the Poincaré duality a bilinear non-degenerated form $b : H \times H \to \mathbb{Z}/2$ is well-defined.

Let us define the quadratic form $q : H \to \mathbb{Z}/2$ associated with b. Take a cycle $x \in H$ represented by a closed (possibly, non-oriented) embedded $(2k+1)$-manifold $i_X : X^{2k+1} \subset N^{4k+2}$. We may assume that an embedding $f \circ i_X : X^{2k+1} \subset \mathbb{R}^{4k+3}$ is generic and is equipped with a cross-section ξ_X of the normal bundle of $f \circ i_X$. This cross-section is defined by the normal vector of f along this cycle.

A self-linking number $lk(i_X, \xi_X)$ is well-defined and is independent modulo 2 of the embedding i_X in the prescribed homology class. We put $q(x) = lk(i_X, \xi_X)(mod 2)$. The Arf-invariant $Arf(H,q)$ is the equivalence class of this quadratic form in the Witt group (i. e. up to stable isomorphism). For a given immersion $f \in Imm^{fr}(4k+2,1)$ the Arf-invariant of the quadratic form (H,q) is called the Arf-Kervaire invariant of the immersion [6][18].

Proposition 1.1. *The Arf-Kervaire invariant is a well-defined homomorphism*

$$\Theta : Imm^{sf}(4k+2,1) \longrightarrow \mathbb{Z}/2$$

with the following properties:

1. *This homomorphism is trivial if* $k \neq 2^l - 1$, $l \geq 0$.

2. *For* $k = 0, 1, 3, 7, 15$ *the homomorphism* Θ *is an epimorphism.*

Part 1 is the Browder Theorem [5]. A special case of this theorem was discovered by Kervaire in [12]. Part 2 in the case $k = 7$ was proved in [16] and in the case $k = 15$ in [4]. A simplification of the last construction was proposed in [15]. An alternative approach toward this Proposition can be found in [22].

Skew-framed immersion cobordism group. Let (θ, Ψ) be a pair, where $\theta : L^{2k+1} \looparrowright \mathbb{R}^{4k+2}$ is an immersion, Ψ is a skew-framing of the normal bundle ν_θ, i.e. an isomorphism $\nu_\theta = (2k+1)\kappa$, where κ is the orientation line bundle over L^{2k+1}, $w_1(\kappa) = w_1(L^{2k+1})$. The cobordism relation of pairs is the standard one. The set of all such pairs up to cobordism forms an abelian group $Imm^{sf}(2k+1, 2k+1)$ with respect to the operation of disjoint union [3].

Browder-Eccles invariant. An alternative definition of the Arf-Kervaire invariant of framed manifolds by means of self-intersection manifolds was originally proposed by Eccles in [9] as a corollary of the Browder construction [5]. We recall this definition in the simplest form, as proposed in [2].

Recall that the stable Hopf homomorphism $h : Imm^{sf}(2k+1, 2k+1) \to \mathbb{Z}/2$ is defined as the parity of the number of self-intersection points for a given skew-framed immersion $[(\theta, \Psi_\theta)] \in Imm^{sf}(2k+1, 2k+1)$.

Definition 1.2. For an element in $Imm^{fr}(4k+2, 1)$ represented by a pair (φ, Ψ_φ) we will call the integer $\bar{\Theta}(\varphi, \Psi_\varphi) = h \circ \delta(\varphi, \Psi_\varphi)$ $(mod\ 2)$ the Browder-Eccles invariant of the framed immersion.

The following theorem is the main result of this paper.

Theorem 1.3. *Let* N^{30} *(resp.* N^{62}*) be a closed* 14*-connected (resp.* 30*-connected) framed manifold with Kervaire invariant 1. Then the product* $N^{30} \times D^{10}$ *(resp.* $N^{62} \times D^{11}$*) of this manifold with the standard 10-dimensional (resp. 11-dimensional) disk does not allow any smooth immersion into Euclidean space* $\mathbb{R}^{46}$ *(resp.* $\mathbb{R}^{94}$*). Equivalently, there is no smooth embedding of* N^{30} *(resp.* N^{62}*) into* $\mathbb{R}^{46}$ *(resp.* $\mathbb{R}^{94}$*) which is projected by the standard projection* $\mathbb{R}^{46} \to \mathbb{R}^{36}$ *(resp.* $\mathbb{R}^{94} \to \mathbb{R}^{83}$*) into an immersion. Consequently,* N^{30} *(resp.* N^{62}*) is not smoothly embeddable into* $\mathbb{R}^{36}$ *(resp.* $\mathbb{R}^{83}$*).*

Remark 1.4. By an observation of Jones and Rees [11], every 14-connected (resp. 30-connected) framed manifold N^{30} (resp. N^{62}) is PL-embeddable into $\mathbb{R}^{32}$ (resp. $\mathbb{R}^{62}$) (see the last section for an explicit definition of N^{30} (resp. N^{62}). By the Haefliger theorem N^{30} (resp. N^{62}) is smoothly embeddable into $\mathbb{R}^{47}$ (resp. $\mathbb{R}^{95}$).

Remark 1.5. Other geometric properties of the Arf-Kervaire invariant, related to the problems of immersions, were presented in [20].

2 Cobordism groups of stably skew-framed immersions

In this section we define cobordism groups of stably framed (resp. stably skew-framed) immersions, i.e. framed in the ambient space. Such a framing (resp.

skew-framing) will be called a stable framing (resp. stable skew-framing). The new cobordism groups generalize *intermediate cobordism groups*, introduced in [8]. We generalize the definition of the Arf-Kervaire invariant and the Browder-Eccles invariant for these cobordism groups.

We call the new invariants the *twisted Arf-Kervaire invariant* and the *twisted Browder-Eccles invariant*, respectively. The definition of the *twisted Arf-Kervaire invariant* is based on the concept of the *Arf-changeable* framed manifolds proposed in [11].

Definition of stably framed cobordism grups $Imm^{stfr}(4k+2, 2k+1)$**.**

Let (φ, Ξ_N) be a pair, where $\varphi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}$ be an immersion in the codimension $2k+1$, Ξ_N be the stable framing of the normal bundle of φ, i.e. the framing of the normal bundle of the composition $I \circ \varphi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3} \subset \mathbb{R}^r$, $r \geq 8k+6$.

The set of pairs described above is equipped by equivalence relation, which is given by the standard relation of concordance. Up to the standard cobordism relation the set of the pairs generates an Abelian group (the operation is determined by disjoint union of pairs). This group is denoted by $Imm^{stfr}(4k+2, 2k+1)$.

Definition of of stably skew-framed immersions cobordism groups $Imm^{stsf}(2k+1, 2k+1)$**.**

Let (ψ, Ψ) be a pair, where $\psi : L^{2k+1} \looparrowright \mathbb{R}^{4k+2}$ be an immersion, Ψ be a stable skew-framing of the normal bundle ν_ψ of the immersion ψ. A stable skew-framing is defined as an isomorphism $\Psi : \nu_\psi \oplus s_1\kappa \oplus s_2\varepsilon = (2k+1+s_2)\varepsilon + s_2\kappa$ for non-negative integers s_1, s_2.

The standard cobordism relation of the pairs allows us to define the Abelian group $Imm^{stsf}(2k+1, 2k+1)$.

Homomorphisms $A : Imm^{fr}(4k+2, 1) \longrightarrow Imm^{stfr}(4k+2, 2k+1)$**,** $B : Imm^{stfr}(4k+2, 2k+1) \to Imm^{fr}(4k+2, 1)$**.**

An arbitrary framed immersion $\varphi : N^{4k+2} \looparrowright \mathbb{R}^{4k+3}$ determines a stably-framed immersion (φ, Ξ_φ) in the codimension $2k+1$. The homomorphism A is well-defined.

An arbitrary stably-framed immersion $(\xi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}, \Psi_\xi)$ is, in particular, a framed immersion into $\mathbb{R}^r$, $r > 4k+2$, with the framing of the normal bundle. After the Smale-Hirsch compression to codimension 1 we obtain an immersion $\varphi : N^{4k+2} \looparrowright \mathbb{R}^{4n+3}$ in the prescribed regular cobordism class. The homomorphism B is well-defined. Obviously, $B \circ A = Id : Imm^{fr}(4k+2, 1) \to Imm^{fr}(4k+2, 1)$.

Twisted Arf-Kervaire invariant $\Theta : Imm^{stfr}(4k+2, 2k+1) \longrightarrow \mathbb{Z}/2$. We generalize the Arf-Kervaire homomorphism $\Theta : Imm^{fr}(4k+1, 1) \to \mathbb{Z}/2$ and define a homomorphism $\Theta^{st} : Imm^{stfr}(4k+2, 2k+1) \to \mathbb{Z}/2$, called the *twisted Arf-Kervaire homomorphism* such that the following diagram is commutative:

$$\begin{array}{ccc} Imm^{fr}(4k+2,1) & \stackrel{A}{\longrightarrow} & Imm^{stfr}(4k+2,2k+1) \\ \searrow \Theta & & \swarrow \Theta^{st} \\ & \mathbb{Z}/2 & . \end{array} \qquad (1)$$

□

The homomorphism $\pi : H_{2k+1}(N^{4k+2}; \mathbb{Z}/2) \to \mathbb{Z}/2$.

Let us describe the homomorphism

$$\pi : H_{2k+1}(N^{4k+2}; \mathbb{Z}/2) \to \mathbb{Z}/2.$$

Take a cycle $x \in H$. This cycle is represented by a mapping $l_x : K^{2k+1} \to N^{4k+2}$. Let Ξ_x be the restriction of the stable framing of $\nu = \nu_\varphi$ on $l_x(K^{2k+1}, \Xi_x : \nu \oplus s_1\varepsilon = (2k+1+s_1)\varepsilon)$.

By general position argument we may assume that $(s_1 - 1)$-vectors of the framing $\nu \oplus s_1\varepsilon$ coincide with the first $(s_1 - 1)$-vectors of the trivialization. Then the last vector of the framing is defined by a mapping $\xi_x : K^{2k+1} \to S(2k+2\varepsilon) = S^{2k+1} \times K^{2k+1} \to S^{2k+1}$. We define that $\pi(x)$ is equal to the degree of this composition modulo 2. The correctness of the definition can be easily established by straightforward considerations.

Definition 2.1. Let $q : H \to \mathbb{Z}/2$, $H = H_{2k+1}(N^{4k+2};\mathbb{Z}/2)$, be the quadratic form of the frame manifold. The twisted quadratic form is defined by $q^{tw} = q + \pi : H \to \mathbb{Z}/2$. The Arf-invariant of this twisted quadratic form determines the twisted Arf-Kervaire homomorphism $\Theta^{st} : Imm^{stfr}(4k+2, 2k+1) \longrightarrow \mathbb{Z}/2$.

Twisted Browder-Eccles invariant. We will define the homomorphism $\bar{\Theta}^{st} : Imm^{stfr}(4k+2, 2k+1) \to \mathbb{Z}/2$ called the twisted Browder-Eccles invariant. Let us describe the homomorphism $\delta^{st} : Imm^{stsf}(4k+2, 2k+1) \to Imm^{stsf}(2k+1, 2k+1)$. Let an element in $Imm^{stfr}(4k+2, 2k+1)$ be represented by $(\xi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}, \Psi_\xi)$. Let us consider the double point manifold L^{2k+1} for ξ. This manifold is embedded into $\mathbb{R}^{6k+3}$ and the normal $(4k+2)$-bundle ν_L admits (for a sufficiently great positive integer l) a stable isomorphism $\Xi' : \nu_L \oplus l\varepsilon \oplus l\kappa = (l+2k+1)\varepsilon \oplus (l+2k+1)\kappa$.

This stable isomorphism uniquely determines the stable isomorphism $\Xi : \nu_L \oplus l\varepsilon = (2k+1)\kappa \oplus l\varepsilon$. This determines the element $\delta^{st}([(\xi, \Psi_\xi)])$ a regular cobordism class of the skew-framed immersion $(\varphi' : L^{2k+1} \looparrowright \mathbb{R}^{4k+2}, \Psi)$.

Let $Imm^{stsf}(2k+1, 2k+1) \overset{h^{\mathbf{D}_4}}{\longrightarrow} Imm^{\mathbf{D}_4}(0, 4k+2) = \mathbb{Z}/2$ be the stable Hopf invariant for a skew-framed immersion. The Browder-Eccles invariant $\bar{\Theta}^{st}$ is defined as a composition $Imm^{stfr}(4k+2, 2k+1) \overset{h^{\mathbf{D}_4} \circ \bar{\delta}^{st}}{\longrightarrow} Imm^{\mathbf{D}_4}(0, 4k+2) = \mathbb{Z}/2$.

A subgroup $Imm^{stfr}(4k+2, 2k+1)^* \subset Imm^{stfr}(4k+2, 2k+1)$.

The subgroup $Imm^{stfr}(4k+2, 2k+1)^* \subset Imm^{stfr}(4k+2, 2k+1)$ is defined as the preimage $A^{-1}(Imm^{sf}(2k+1, 2k+1) \subset Imm^{stsf}(2k+1, 2k+1))$. An explicit definition is following. Let a pair $(\xi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}, \Psi_\xi)$ represents an element in $x \in Imm^{stsf}(4k+2, 2k+1)$, and a pair $(\varphi : L^{2k+1} \looparrowright \mathbb{R}^{4k+2}, \Psi)$ represents an element $\delta^{st}(x)$, where L^{2k+1} is the double point manifold of the immersion ξ. Let us consider the canonical covering $\bar{L}^{2k+1} \looparrowright N^{4k+2}$ over the double point manifold of ξ as the cycle $[\bar{L}^{2k+1}]$ in $H_{2k+1}(N^{4k+2};\mathbb{Z}/2)$ and take an integer $\pi([\bar{L}^{2k+1}])$, where $\pi : H \to \mathbb{Z}/2$ (see the definition of the twisted Arf-Kervaire invariant).

Let us assume that $\bar{L}^{2k+1}$ is connected and $\pi_([\bar{L}^{2k+1}]) = 0$. Then $x \in Imm^{stsf}(4k+2, 2k+1)$ iff the restriction of the normal bundle ν_ξ over $\bar{L}^{2k+1}$ is trivial (not only stably trivial).

The homomorphisms constructed above are included into the following commutative diagramm

$$\begin{array}{ccccc} Imm^{fr}(4k+2,1) & \stackrel{A}{\longrightarrow}\stackrel{B}{\longleftarrow} & Imm^{stfr}(4k+2,2k+1)^{*} & \subset & Imm^{stsf}(4k+2,2k+1) \\ \downarrow \delta^{\mathbf{D}_4} & & \downarrow \delta^{st,\mathbf{D}_4} & & \downarrow \delta^{st,\mathbf{D}_4} \\ Imm^{sf}(2k+1,2k+1) & = & Imm^{sf}(2k+1,2k+1) & \subset & Imm^{stsf}(2k+1,2k+1) \\ \downarrow h^{\mathbf{D}_4} & & & & h^{st,\mathbf{D}_4} \\ Z/2 = Imm^{\mathbf{D}_4}(0,4k+2) & & = & & Imm^{\mathbf{D}_4}(0,4k+2) \end{array}$$

3 Arf-Kervaire and Browder-Eccles (twisted) homomorphisms coincide

The following lemma is necessary for our proof of Theorem 1.4.

Lemma 3.1. *The twisted Arf-Kervaire homomorphism* $\Theta^{st} : Imm^{stfr}(4k+2,2k+1)^{*} \to \mathbb{Z}/2$ *coincides with the twisted Browder-Eccles homomorphism* $\bar{\Theta}^{st} : Imm^{stfr}(4k+2,2k+1)^{*} \to \mathbb{Z}/2$.

We shall derive Lemma 3.1 from the following lemma.

Lemma 3.2. *The homomorphisms* Θ^{st} *and* $\bar{\Theta}^{st}$ *coincide on the subgroup* $ImA = KerB \cap Imm^{stfr}(4k+2,2k+1)^{*} \subset Imm^{stfr}(4k+2,2k+1)$.

Proof of Lemma 3.2. From the main result of [9] reformulated in [2], in the required form, it follows that the homomorphisms Θ^{st} and $\bar{\Theta}^{st}$ coincide on the subgroup $ImA \subset Imm^{stfr}(4k+2,2k+1)^{*}$. □

Proof of Lemma 3.1. Let $(\xi : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}, \Psi_{\xi})$ be a stably framed immersion. The double point manifold of this immersion equipped with the (stable) skew-framing will be denoted by (L^{2k+1}, Ξ_L). Let $\eta : W^{4k+3} \to \mathbb{R}^{6k+3} \times \mathbb{R}^1_+$ be a generic mapping of a framed boundary (W^{4k+3}, Ψ_W) of (N^{4k+2}, Ψ). One may assume that the critical point manifold Σ of η satisfies the following properties.

(1) Σ is connected with connected canonical double covering $\bar{\Sigma}$.

(2) $\eta(\Sigma)$ belongs to the hyperspace $\mathbb{R}^{6k+3} \times \{1\}$ and the double point manifold K^{2k+2} of η with $\partial K^{2k+2} = L^{2k+1} \cup \Sigma^{2k+1}$ is regular in a small neighborhood of the boundary with respect to the projection onto $\mathbb{R}^1_+$ (i.e. has a non-singular projection with the image in the intervals $[0; 0+\varepsilon)$,$[1; 1-\varepsilon)$).

Let $\eta_{1-\varepsilon} : N^{4k+2}_{1-\varepsilon} \looparrowright \mathbb{R}^{6k+3} \times \{1-\varepsilon\}$ be the immersion defined as the intersection of η with the hyperspace $\mathbb{R}^{6k+3} \times \{1-\varepsilon\}$. Let $L^{2k+1}_{1-\varepsilon}$ be the component of the double point manifold $\eta_{1-\varepsilon}(N^{4k+2}_{1-\varepsilon})$ in the neighborhood of the critical point boundary Σ^{2k+1} of K^{2k+2}. From the assumption $\pi(\xi) = 0$ we may deduce (see Remark 2.2) that the normal bundle $\nu_{1-\varepsilon}$ of the manifold $L^{2k+1}_{1-\varepsilon}$ is decomposed into the direct sum of the trivial bundle $\nu_{\varepsilon} = (2k+1)\varepsilon$ and a nontrivial bundle $\nu_{\kappa} = \nu_{\varepsilon} \otimes \kappa$, where κ is the orientation line bundle over $L^{2k+1}_{1-\varepsilon}$.

The proof of Lemma 3.1 will be completed in Section 6.

4 The construction of the standard stably-framed immersion

This section is devoted to the construction of the standard stably-framed immersion. Let us describe the simplest possible stably framed immersion ($\xi_0 : N_0^{4k+2} \looparrowright \mathbb{R}^{6k+3}, \Psi_0$) such that the double point manifold L_0^{2k+1} (equipped with the normal bundle structure Ξ_0) is diffeomorphic to the manifold $L_{1-\varepsilon}^{2k+1}$ (see the prove of the Lemma 3.1).

Let us start the construction by the description of the standard immersion $f : S^{2k+1} \looparrowright \mathbb{R}^{4k+2}$ with the self-intersection points at the origin. Take the standard coordinate subspaces $\mathbb{R}_1^{2k+1} \oplus \mathbb{R}_2^{2k+1} = \mathbb{R}^{4k+2}$ and let $\mathbb{R}_{diag}^{2k+1}$, $\mathbb{R}_{antidiag}^{2k+1}$ be two other coordinate subspaces defined by means of the sum and the difference of the bases of the planes $\mathbb{R}_1^{2k+1}, \mathbb{R}_2^{2k+1}$.

Take the two standard unit disks $D_1^{2k+1} \subset \mathbb{R}_1^{2k+1}$, $D_2^{2k+1} \subset \mathbb{R}_2^{2k+1}$. Take a manifold C diffeomorphic to $S^{2k} \times I$ defined as the collection of the segments, each starts in a point in ∂D_1^{2k+1} and ends at the corresponding point (i.e. at the point with the same coordinates) in ∂D_2^{2k+1}. The union of the two disks $D_1^{2k+1} \cup D_2^{2k+1}$ with C (after the identification of the corresponding components of the boundary) is the sphere $f(S^{2k+1})$ PL-immersed into $\mathbb{R}^{4k+2}$ with one self-intersection point (in the origin). After the smoothing of corners along ∂C we obtain the smoothly immersed sphere.

Let us describe the manifold N_0^{4k+2}, the stable framing Ξ_0 over this manifold and the immersion $\xi_0 : N_0^{4k+2} \looparrowright \mathbb{R}^{6k+3}$, $[(\eta_0, L^{2k+1}, \Xi_0)] \in Imm^{sf}(2k+1, 2k+1)$. Take the embedding $\eta_0 : L_0^{2k+1} \subset \mathbb{R}^{6k+3}$ with the normal bundle $\nu_{L_0} = \nu_1 \oplus \nu_1 \otimes \kappa$, where ν_1 is the trivial $(2k+1)$-dimensional bundle (with the prescribed trivialization), κ is the orientation line bundle over L_0^{2k+1}. Take a bundle $\nu_1 \otimes \kappa \oplus \varepsilon$ over L_0^{2k+1} and take N_0 as the boundary $S(\nu_1 \otimes \kappa \oplus \varepsilon)$ of the disk bundle of this vector bundle.

The fibration $p : N_0^{4k+2} \to L_0^{2k+1}$ is well-defined. Because $\nu_1 \otimes \kappa \oplus \varepsilon$ is the normal bundle of L_0^{2k+1}, the manifold N_0^{4k+2} addmits an immersion in codimension 1. The framing over N_0^{4k+2} is well-defined. The reflection of the fiber corresponding to the structure group of the fibration will be denoted by $\lambda : S^{2k+1} \to S^{2k+1}$. This is the standard reflection of the sphere by means of the hyperplane.

Let us define the immersion $\xi_0 : N_0^{4k+2} \looparrowright \mathbb{R}^{6k+3}$. Take the normal bundle $\nu_1 \otimes \kappa \oplus \nu_1$ of the immersion η_0 and take a collection of the standard immersed spheres $f(S^{2k+1})$ constructed above in each fiber of η_0. The immersion ξ_0 is well-defined. The pair $(\xi_0, \Psi_0 = \Psi)$ is the stably framed immersion under construction.

5 Calculation of twisted invariants of the standard skew-framed immersion

This section is devoted to the calculation of the twisted Browder-Eccles and the twisted Arf-Kervaire invariants for the standard stably framed immersion constructed in the section 4.

The group $H = H_{2k+1}(N_0^{4k+2}; \mathbb{Z}/2)$ is generated by two elements. The first generator $x \in H$ is represented by a spherical fiber of the fibration p :

$N_0^{4k+2} \to L_0^{2k+1}$. The fibration p has a standard section p^{-1} and the image of the fundamental class of the base L_0^{2k+1} represents the second generator $y \in H$. Let us prove under the assumption $2k+1 > 7$ that the homomorphism $\pi : H \to \mathbb{Z}/2$ is defined by $\pi(x) = 1, \pi(y) = 0$.

The equation $\pi(x) = 1$ holds since for an arbitrary (in particular, for the standard) framing of the normal $(2k+2)$-dimensional bundle of an immersed sphere the Hopf invariant has to be even, but for the section orthogonal to the collection of prescribed $(2k+1)$-subfibers of the bundle ν_{L_0}, the Hopf invariant is odd. Equivalently, the normal bundle of the fiber immersed into $\mathbb{R}^{4k+2}$ is non-trivial.

Let us prove the equation

$$\pi(y) = 0. \tag{4}$$

The cycle $y \in H$ is represented by a section $p^{-1}(L_0^{2k+1}) \subset N_0^{4k+2}$. The collection of the bases in the fibers of the subbundle $\nu_1 \subset \nu_{L_0}$ determines the trivialisation of the normal bundle of the immersion ξ_0 over this submanifold $p^{-1}(L_0^{2k+1}) \subset N_0^{4k+2}$. This proves (4).

The twisted Arf-Kervaire invariant for the stably-framed immersion (ξ_0, Ψ_0) is equal to $q(y)$ i.e. coincides with the twisted Browder-Eccles invariant of the stable framed immersion given by the cobordism class of the skew-framing manifold L_0^{2k+1}. This gives the required computations. $\square$

6 Proof of Theorem 1.3

Let us finish the proof of Lemma 3.1. Take a stably framed immersion $(\xi_{1-\varepsilon}, \Xi_{1-\varepsilon})$ with the double point skew-framed manifold $(L_{1-\varepsilon}^{2k+1}, \Psi_{1-\varepsilon})$. By the construction from the section 4 take the standard stably-framed immersion (ξ_0, Ξ_o) with the double point manifold, in the opposite skew-framed cobordism class, with respect to $(L_{1-\varepsilon}^{2k+1}, \Psi_{1-\varepsilon})$.

The disjoint union $(\xi_{1-\varepsilon}, \Xi_{1-\varepsilon}) \cup (\xi_0, \Xi_0)$ is a stably framed boundary. The twisted Arf-Kervaire and the twisted Browder-Eccles invariants for (ξ_0, Ξ_0) coincide. For $(\xi_{1-\varepsilon}, \Xi_{1-\varepsilon}) \cup (\xi_0, \Xi_0)$ both invariants are trivial. Therefore for the stably framed immersion $(\xi_{1-\varepsilon}, \Xi_{1-\varepsilon})$ the invariants coincide. This proves that for the stably framed immersion $(\xi, \Xi) \in Imm^{stfr}(4k+2, 2k+1)^{\cdot}$ the invariants coincide. This completes the proof of Lemma 3.1. $\square$

Proof of Theorem 1.3. Let $4k+2 = 30$, or 62, (N^{4k+2}, Ξ) be a framed $2k$-connected manifold with the Arf-Kervaire invariant 1. By the Browder-Novikov classification of simply-connected manifolds the manifold N^{4k+2} is decomposed into the connected sum $N^{4k+2} = N_0^{4k+2} \sharp \cup_i (S^{2k+1} \times S^{2k+1})_i \sharp \Sigma$, where N_0^{4k+2} is the standard framed manifold with the Kervaire invariant 1 (we recall the definition below), $S^{2k+1} \times S^{2k+1}$ is the standard torus framed as a hypersurface in $\mathbb{R}^{4k+3}$, Σ^{4k+2} is a homotopy sphere (see e.g. [5][6][14]).

Let as recall that N_0^{4k+2} is defined by means of the gluing the boundary of the plumbing M_0^{4k+2}, $\partial M_0^{4k+2} = S^{4k+1}$ of the two disk bundles of the tangent bundle $T(S^{2k+1})$ by the standard disk. The boundary of the plumbing is diffeomorphic to the standard $(4k+1)$-sphere. The space of the plumbing is realized as a Seifert surface for a trefoil knot $S^{4k+1} \subset \mathbb{R}^{4k+3}$.

The connected sum $M^{4k+2} = M_0^{4k+2} \sharp \cup_i S^{2k+1} \times S_i^{2k+1}$ is realized as a Seifert surface with handles and the boundary of this manifold is also diffeomorphic to the standard sphere S^{4k+1}. The manifold N^{4k+2} is obtained from M^{4k+2}

by the gluing the boundary sphere by the standard disk corresponding to an appropriate diffeomorphism of the two boundary spheres.

Let as assume that there exists an embedding $\bar{I} : N^{4k+2} \subset \mathbb{R}^{6k+4}$, $k = 7$ or $k = 15$ and the normal bundle of this embedding is equipped with 10 (resp. 11) independent sections of the normal $(2k+2)$-bundle $\nu_{\bar{I}}$ of this embedding. The restriction of the normal bundle $\bar{\nu}(\bar{I})_i$ over an arbitrary embedded sphere $i : S^{2k+1} \subset N^{4k+2}$ is the trivial bundle $\mathbb{R}^{2k+2} \times S^{2k+1} \to S^{2k+1}$, equipped with 10 (resp. 11) sections. The first section l of the family determines a 1-framing of the immersion $\bar{I}$. Let us prove that the function $\pi : H_{2k+1}(N^{4k+1}; \mathbb{Z}/2) \to \mathbb{Z}/2$ constructed for this stably framed immersion is trivial.

Let us consider the $(2k+1)$-dimensional bundle $\nu(\bar{I})$ over N^{4k+2} that is defined as the complement to the section l inside $\bar{\nu}(\bar{I})$, $\nu(\bar{I}) \oplus \varepsilon = \bar{\nu}(\bar{I})$. The bundle $\nu(\bar{I})|_{i(S^{2k+1})}$ is a stably trivial bundle. There is only one stably-trivial but non-trivial bundle over S^{15} (resp. S^{31}); this bundle is the tangent bundle $T(S^{2k+1})$ (see [13]). The case of the non-trivial (resp. trivial) bundle corresponds to $\pi(i) = 1$ (resp. $\pi(i) = 0$). By the Adams theorem [1] the tangent bundle $T(S^{15})$ (resp. $T(S^{31})$) admits no more then 8 (resp. 9) independent sections. By our assumption the bundle $\nu(\bar{I})|_{i(S^{2k+1}}$ admits 9 (resp. 10) independent sections. Therefore $\nu(\bar{I})|_{i(S^{2k+1})}$ is the trivial bundle. This proves that the homomorphism π is trivial. Therefore the stable Arf-Kervaire invariant for $(\bar{I}, l)$ is equal to the Arf-Kervaire invariant for the the fraimed manifold (N^{4k+2}, Ξ) and is non-trivial by the assumption.

By the Rourke-Sanderson compression theorem [21] we may assume, after an isotopy of $\mathbb{R}^{6k+4}$, that the cross-section l, defined by the first vector of the framing, is vertically up. After the projection we obtain a (partial) framed immersion $I : N^{4k+2} \looparrowright \mathbb{R}^{6k+3}$. This immersion is framed outside a point. The double point manifold L^{2k+1} of the immersion I is a stably skew-framed manifold and the class of the skew-framing is trivial. Therefore this manifold is stably framed (the canonical double cover over L^{2k+1} is trivial). The stably-framed immersion (I, Ξ) determines an element in the group $Imm^{fr}(4k+2, 1)$.

The stable Browder-Eccles invariant for (I, Ξ) coincides with the Steenrod-Hopf invariant on the cobordism class of the framed manifold $(L^{2k+1}, \Psi) \in \Pi_{2k+1}$. From the Toda theorem for $2k+1 = 15$ and by the Adams theorem for $2k+1 = 31$ (see [17][23]) the Browder-Eccles invariant is trivial. By Lemma 3.1 the stable Arf-Kervaire invariant and the stable Browder-Eccles invariants for a framed immersion (I, Ξ) coincide and therefore the Arf-Kervaire invariant is trivial. This contradiction shows that N^{4k+2} is not embeddable into $\mathbb{R}^{6k+4}$ with the family of independent sections. Theorem 1.4. is proved. ☐

Remark 6.1. Eccles has constructed in [7][8] a framed 30- (resp. 62-) manifold N^{30} (resp. N^{62}) with Kervaire invariant 1, which is embeddable into $\mathbb{R}^{46}$ (resp. $\mathbb{R}^{96}$). It is easy to prove that such an embedding cannot be framed.

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