Algebraicity in monochromatic homotopy theory

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Abstract

Using Patchkoria–Pstrągowski's version of Franke's algebraicity theorem, we prove that the category of $K_p(n)$-local spectra is exotically equivalent to the category of derived I_n-complete periodic comodules over the Adams Hopf algebroid (E_*, E_*E) for large primes. This gives a finite prime result analogous to the asymptotic algebraicity for $\mathrm{Sp}_{K_p(n)}$ of Barthel–Schlank–Stapleton.

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1 Introduction

The central idea in chromatic homotopy theory is to study the symmetric monoidal stable ∞-category of spectra, Sp, via its smaller building blocks. These are the categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ of $E_{n,p}$-local and $K_p(n)$-local spectra, where $E = E_{n,p}$ is Morava E-theory, and $K_p(n)$ is Morava K-theory, see for example [HS99]. These categories depend on a prime p and an integer n, called the height. For a fixed height n, increasing the prime p makes both categories behave more algebraic. This manifests itself, for example, in the E-Adams spectral sequence of signature

$$E_2^{s,t}(L_n\mathbb{S}) = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \Longrightarrow \pi_{t-s}L_n\mathbb{S}$$

computing the homotopy groups of the E-local sphere. By the smash product theorem of Ravenel, see [Rav92, 7.5.6], this spectral sequence has a horizontal vanishing line at a finite page. If $p > n+1$, this vanishing line appears already on the second page, where the information is completely described by the homological algebra of Comod_{E_*E} – the Grothendieck abelian category of comodules over the Hopf algebroid (E_*, E_*E).

Increasing the prime p correspondingly increases the distance between objects appearing in the E-Adams spectral sequence. When $2p-2$ exceeds n^2+n, there is no longer room for any differentials, and the spectral sequence in fact collapses to an isomorphism

$$\pi_*L_n\mathbb{S} \cong \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*),$$

for degree reasons. In other words, the homotopy groups are completely algebraic in this range.

A natural question to ask is whether this collapse is a feature solely of the E-Adams spectral sequence or if it is a feature of the category $\mathrm{Sp}_{n,p}$. More precisely, is the entire category of E-local spectra algebraic, in the sense that it is equivalent to a derived category of an abelian category, whenever $2p-2 > n^2+n$?

At height $n = 0$, the category $\mathrm{Sp}_{n,p}$ is the category of rational spectra $\mathrm{Sp}_{\mathbb{Q}}$, which can be seen to be equivalent to the derived ∞-category of rational vector spaces, but at positive heights $n > 0$, there can never be an equivalence of ∞-categories $\mathrm{Sp}_{n,p} \simeq D(\mathcal{A})$.

However, in [Bou85] Bousfield showed that for $p > 2$ and $n = 1$, that there is an equivalence of homotopy categories

$$h\mathrm{Sp}_{1,p} \simeq h\mathrm{Fr}_{1,p},$$

where $\mathrm{Fr}_{n,p}$ is a certain derived ∞-category of twisted comodules over the Hopf algebroid (E_*, E_*E). As this cannot be lifted to an equivalence of ∞-categories, it is sometimes referred to as an *exotic* equivalence.

Franke expanded upon this in [Fra96] by conjecturing – and attempting to prove – that for $2p-2 > n^2+n$ there should be an equivalence of homotopy categories

$$h\mathrm{Sp}_{n,p} \simeq h\mathrm{Fr}_{n,p}.$$

Unfortunately, a subtle error was discovered in the proof by Patchkoria in [Pat12], but the result was recovered in [Pst21] with a slightly worse bound: $2p-2 > 2n^2+2n$. Pstrągowski also proved that this equivalence gets "stronger" the larger the prime, where we not only get an equivalence of categories but an equivalence of k-categories

$$h_k\mathrm{Sp}_{n,p} \simeq h_k\mathrm{Fr}_{n,p},$$

for $k = 2p-2-n^2-n$. Here $h_k\mathcal{C}$ denotes taking the homotopy k-category, given by $(k-1)$-truncating the mapping spaces in $\mathcal{C}$. At $k=1$, this gives the classical situation of taking the homotopy category $h\mathcal{C}$. Using and developing a more general machinery, Pstrągowski and Patchkoria proved in [PP21] that the above equivalence holds in Franke's conjectured bound, $2p-2 > n^2+n$.

These results imply that increasing the prime p decreases how destructive the k-truncation of the mapping spaces needs to be. In the limit $p \to \infty$, we might expect that there is no need to truncate at all, giving an equivalence of ∞-categories. But, there needs to be an appropriate notion of what "going to the infinite prime" should be. In [BSS20], the authors use a notion of ultraproducts over a non-principal ultrafilter $\mathcal{F}$ of primes to formalize this limiting process. They use this to prove the existence of a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{n,p} \simeq \prod_{\mathcal{F}} \mathrm{Fr}_{n,p}.$$

Expanding on their work, Barthel, Schlank, and Stapleton proved in [BSS21] a $K_p(n)$-local version of the above result. More precisely, they show that there is a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{K_p(n)} \simeq \prod_{\mathcal{F}} \widehat{\mathrm{Fr}}_{n,p},$$

where the right-hand side consists of derived complete twisted comodules for the naturally occurring Landweber ideal $I_n \subseteq E_*$.

Statement of results

We can summarize the most general of the above algebraicity results in the following table,

	$p < \infty$	$p \to \infty$
$\mathrm{Sp}_{n,p}$	[PP21]	[BSS20]
$\mathrm{Sp}_{K_p(n)}$		[BSS21]

A natural question arises: Is there a finite prime exotic algebraicity for $\mathrm{Sp}_{K_p(n)}$? The goal of this paper is to give an affirmative answer. More precisely, we prove the following.

Theorem A (Theorem 7.10)**.** *Let p be a prime and n a natural number. If $k = 2p-2-n^2 > 0$, then there is an equivalence of k-categories*

$$h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\widehat{\mathrm{Fr}}_{n,p}.$$

In other words, $K_p(n)$-local spectra are exotically algebraic at large primes.

The available tools for proving such a statement require an abelian category with enough injective objects admitting lifts to a stable ∞-category. In lack of such a well-behaved abelian approximation for $\mathrm{Sp}_{K_p(n)}$, we take inspiration from [BSS21] and instead use the dual category $\mathcal{M}_{n,p}$ of monochromatic spectra, which we show has the needed properties. Theorem A then follows from the following result.

Theorem B (Theorem 7.7)**.** *Let p be a prime and n a natural number. If $k = 2p - 2 - n^2 > 0$, then there is an equivalence of k-categories $h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-tors}$.*

Remark. Note that whenever there is an equivalence $h_k\mathrm{Sp}_{n,p} \simeq \mathrm{Sp}_{n,p}h_k\mathrm{Fr}_{n,p}$ as in [PP21], we inherit an equivalence $h_k\mathrm{Sp}_{K_p(n)} \simeq \widehat{\mathrm{Fr}}_{n,p}$ as full subcategories. But, for a fixed height n we obtain a sharper bound for the prime p. For example, at $n = 2$, the equivalence induced from $\mathrm{Sp}_{n,p}$ only applies for $p \geqslant 7$, but our results also give an equivalence for $p = 5$. Furthermore, fixing both a height n and a prime p, we can often conclude with stronger statements. For example, if $p = 3$ and $n = 1$ [PP21] induced on the subcategory gives $h_2\mathrm{Sp}_{K_3(1)} \simeq h_2\widehat{\mathrm{Fr}}_{1,3}$, but we obtain $h_3\mathrm{Sp}_{K_3(1)} \simeq h_3\widehat{\mathrm{Fr}}_{1,3}$.

In order to prove Theorem B, we first prove the analogous statement for monochromatic E-modules.

Theorem C (Theorem 6.6)**.** *Let p be a prime and n a natural number. If $k = 2p - 2 - n > 0$, then there is an equivalence of k-categories $h_k\mathrm{Mod}_E^{I_n-tors} \simeq h_kD^{per}(\mathrm{Mod}_{E_*}^{I_n-tors})$.*

Overview of the paper

Be aware that the paper contains a significant amount of exposition – perhaps more than some would like. This is done with non-experts in mind. Readers familiar with local duality, chomatic homotopy theory, and algebraicity, can skip to the last two sections, where most of the new material is proved.

Section 2 introduces the theory of local duality, as well as some Barr-Beck type statements related to it. In Section 3, we introduce Hopf algebroids and the theory of torsion comodules, which we at the end of the section relate back to local duality. In Section 4, we look at chromatic homotopy theory, with a particular focus on monochromatic homotopy theory and its relationship to local duality. Section 5 focuses on the relationship between the two sections prior, where we introduce exotic adapted homology theories and Franke's algebraicity theorem. Most of the new results are presented in Section 6 and Section 7, where we prove Theorem A, Theorem B and Theorem C.

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2 Torsion and completion

The goal of this section is to set up some necessary results that will be used throughout the paper. More precisely, we present the theory of local duality, proved in [HPS97] and generalized to the ∞-categorical setting in [BHV18]. This theory will be used in Section 3 and Section 4 to describe the "irreducible pieces" of certain categories, as well as making arguments in Section 6 and Section 7 simpler by dualizing to a category that behaves better in respect to existing tools and techniques.

2.1 Local duality

We freely use the language of ∞-categories, as developed by Joyal [Joy02] and Lurie [Lur09; Lur17]. Even though we are dealing with both classical 1-categories and ∞-categories in this paper, we will mostly refer to them both as *categories*, hoping that the prefix is clear from the context.

Notation 2.1. We denote by Pr^L_{st} the ∞-category of presentable stable ∞-categories and colimit preserving functors. Together with the Lurie tensor product, it is a symmetric monoidal ∞-category.

Construction 2.2. The category of commutative algebra objects $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is the ∞-category of presentable stable symmetric monoidal ∞-categories, where the monoidal product commutes with colimits separately in each variable. Any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$, then, has an internal hom-functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -)$ that is a right adjoint to $(-) \otimes_{\mathcal{C}} X$, making the symmetric monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ into a closed symmetric monoidal category. This means we have an internal duality functor $\mathbb{D}_{\mathcal{C}}(-)\colon \mathcal{C}^{op} \longrightarrow \mathcal{C}$, sending and object $X \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbb{1}_{\mathcal{C}})$.

Remark 2.3. If it is clear from the context we will sometimes omit the subscript from the notation, simply using $(-) \otimes (-)$, $\underline{\mathrm{Hom}}(-,-)$ and $\mathbb{D}(-)$.

Definition 2.4. *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. An object $X \in \mathcal{C}$ is said to be compact if the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ corepresented by X preserves filtered colimits. It is said to be dualizable if the natural map $\mathbb{D}X \otimes Y \longrightarrow \underline{\mathrm{Hom}}(X, Y)$is an equivalence for all $Y \in \mathcal{C}$.*

Definition 2.5. *If there is a set of compact objects $\mathcal{K}$ in $\mathcal{C}$ such that $\mathcal{K}$ generates $\mathcal{C}$ under filtered colimits, then we say $\mathcal{C}$ is compactly generated. If all objects in $\mathcal{K}$ are in addition dualizable, then we say $\mathcal{C}$ is compactly generated by dualizables.*

Remark 2.6. Being compact or dualizable can be thought of as smallness conditions. By [Lur17, 1.4.4.2] any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is generated by κ-compact generators. Being compactly generated means κ can be chosen to be ω.

Remark 2.7. If $\mathcal{C}$ is compactly generated by a set of dualizable objects $\mathcal{K}$ which contains the monoidal unit $\mathbb{1}_{\mathcal{C}}$, then any object X is compact if and only if it is dualizable. This will hold for many of the categories we meet, but not all of them. If $\mathbb{1}_{\mathcal{C}}$ is not compact, then compact objects are still dualizable, but the converse fails in general.

We will, throughout the paper, be interested in certain functors called localizations. In spirit, these are functors that invert a certain class of maps.

Definition 2.8. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ a functor. A map f in $\mathcal{C}$ is called an L-equivalence if Lf is an equivalence. The functor L is said to be tensor-compatible if being an L-equivalence is stable under tensor product: in the sense that for any L-equivalence $X \longrightarrow Y$ and object $Z \in \mathcal{C}$, the induced map $X \otimes Z \longrightarrow Y \otimes Z$ is again an L-equivalence.*

Definition 2.9. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. A (monoidal) localization is a tensor-compatible functor $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ with a fully faithful right adjoint i.*

Remark 2.10. Note that in the litterature localizations are not always assumed to be tensor-compatible. We will, however, assume that all of our localizations satisfy this, and omit the prefix monoidal. This is not a very restrictive assumption, and is, for example, satisfied by all Bousfield localizations of spectra.

Remark 2.11. Let $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a localization. The composition of f with The fully faithful right adjoint i is denoted L. The functor i gives an equivalence between $\mathcal{D}$ and a full subcategory of $\mathcal{C}$, denoted $\mathcal{C}_L$. By [Lur09, 5.2.7.4] there is an equivalence between localizations $f\colon \mathcal{C} \longrightarrow \mathcal{D}$ and functors $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ (L viewed as a functor to its essential image) that are left adjoint to the inclusion. Hence, by abuse of notation, we will also call $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ a localization.

Definition 2.12. *Given a localization $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$, any object $C \in \mathcal{C}$ admits a map $C \longrightarrow LC$ coming from the unit of the adjunction, called its L-localization. The object C is said to be L-local if this is an L-equivalence.*

Proposition 2.13 ([Lur17, 1.3.4.3])**.** *Let $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ be a localization. Then $\mathcal{C}_L$ is equivalent to the full subcategory of $\mathcal{C}$ obtained by inverting the collection of L-equivalences W_L. In other words, $\mathcal{C}_L \simeq \mathcal{C}[W_L^{-1}]$.*

Remark 2.14. Let $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ be a localization. The symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal structure on $\mathcal{C}_L$, defined by $L(- \otimes_{\mathcal{C}} -)$, making L into a symmetric monoidal functor. This follows from [Lur17, 2.2.1.9] by our standing assumption that all localizations are tensor-compatible, see Remark 2.10.

Remark 2.15. Similarly to localizations, we can define colocalizations as functors $g\colon \mathcal{C} \longrightarrow \mathcal{D}$ admitting a fully faithful left adjoint i. The composition $i \circ g$ is denoted Γ. The adjoint gives an equivalence between $\mathcal{D}$ and a full subcategory $\mathcal{C}^\Gamma$ of $\mathcal{C}$, and the datum of a colocalization is equivalent to the datum of a functor $\Gamma\colon \mathcal{C} \longrightarrow \mathcal{C}^\Gamma$ that is right adjoint to the inclusion. Dually to localizations, we get for any $C \in \mathcal{C}$ a colocalization map $\Gamma C \to C$, and we say C is Γ-colocal if this is an equivalence.

For any localization $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$, the image of the unit $L\mathbb{1}_{\mathcal{C}}$ is a ring object, and any L-local object X admits the structure of an $L\mathbb{1}_{\mathcal{C}}$ module via a map $L\mathbb{1}_{\mathcal{C}} \otimes X \longrightarrow X$. Equivalently, there is a map of functors $L\mathbb{1}_{\mathcal{C}} \otimes L(-) \longrightarrow L(-)$. Via the tensor-internal hom adjunction this is equivalently a map $L(-) \longrightarrow \underline{\mathrm{Hom}}(L\mathbb{1}_{\mathcal{C}}, -)$.

Definition 2.16. *We say a localization L is smashing if the $L\mathbb{1}_{\mathcal{C}}$-module map above is an equivalence. This is equivalent to the dual map $L(-) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(L\mathbb{1}, -)$ being an equivalence.*

Remark 2.17. Similarly, for a colocalization Γ there are maps $\Gamma\mathbb{1}_{\mathcal{C}} \otimes \Gamma(-) \longrightarrow \Gamma(-)$ and $\Gamma(-) \longrightarrow \underline{\mathrm{Hom}}(\Gamma\mathbb{1}_{\mathcal{C}}, -)$. The colocalization Γ is said to be smashing if the former is an equivalence, and cosmashing if the latter is. Note that being smashing and cosmashing is not equivalent for colocalizations, as was the case for localizations. We plan to investigate this distinction, and its relationship with comodules, contramodules and Positselski duality, in future work.

Remark 2.18. Any localization L equips $\mathcal{C}_L$ with a symmetric monoidal structure, as seen in Remark 2.14. If L is a smashing localization, then the induced tensor product is the same as in the category $\mathcal{C}$. The same applies to smashing colocalizations.

There are several ways to construct localizations, but one method particularly important for us will be via localizing subcategories.

Definition 2.19. *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. A subcategory $\mathcal{L} \subseteq \mathcal{C}$ is said to be a localizing subcategory if it is closed under retracts, suspensions, and filtered colimits. If in addition $L \otimes C \in \mathcal{L}$ for all $L \in \mathcal{L}$ and $C \in \mathcal{C}$, then $\mathcal{L}$ is called a localizing ideal.*

Notation 2.20. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $\mathcal{K} \subseteq \mathcal{C}$ a set of objects. We denote by $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ the smallest localizing ideal in $\mathcal{C}$ containing $\mathcal{K}$. It will sometimes be referred to as the localizing ideal generated by $\mathcal{K}$.

Definition 2.21. *Let $\mathcal{L} \subseteq \mathcal{C}$ be a full subcategory. The left orthogonal complement of $\mathcal{L}$, is the full subcategory $\mathcal{L}^{\perp}$ consisting of objects $C \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(L, C) \simeq 0$ for all $L \in \mathcal{L}$.*

Example 2.22. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $\mathcal{L}$ a localizing ideal. The inclusion of the complement $\mathcal{L}^{\perp} \hookrightarrow \mathcal{C}$ is fully faithful and has a left adjoint $L\colon \mathcal{C} \longrightarrow \mathcal{L}^{\perp}$. Viewed as an endofunctor on $\mathcal{C}$, this is a localization, and its kernel is precisely $\mathcal{L}$.

We are now ready to present the setup for local duality. In essence, it can be viewed as a natural duality theory occurring whenever the localizing ideal $\mathcal{L}$ is generated by a set of compact objects.

Definition 2.23. *A pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is compactly generated by dualizables, and $\mathcal{K}$ is a subset of compact objects, is called a local duality context.*

Definition 2.24. *Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We define $\mathcal{C}^{\mathcal{K}-tors}$ to be the localizing tensor ideal $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$. In addition, $\mathcal{C}^{\mathcal{K}-loc}$ denotes the left orthogonal complement $(\mathcal{C}^{\mathcal{K}-tors})^{\perp}$, and $\mathcal{C}^{\mathcal{K}-comp}$ denotes the double left-orthogonal complement $(\mathcal{C}^{\mathcal{K}-loc})^{\perp}$. These full subcategories are respectively called the $\mathcal{K}$-torsion, $\mathcal{K}$-local and $\mathcal{K}$-complete objects in $\mathcal{C}$.*

Remark 2.25. By definition $\mathcal{C}^{\mathcal{K}-tors}$ is compactly generated, and by [BHV18, 2.17] both $\mathcal{C}^{\mathcal{K}-loc}$ and $\mathcal{C}^{\mathcal{K}-comp}$ are as well.

Notation 2.26. The categories $\mathcal{C}^{\mathcal{K}-tors}$, $\mathcal{C}^{\mathcal{K}-loc}$ and $\mathcal{C}^{\mathcal{K}-comp}$ are full subcategories of $\mathcal{C}$, and have full inclusions into $\mathcal{C}$, denoted $i_{\mathcal{K}-tors}$, $i_{\mathcal{K}-loc}$ and $i_{\mathcal{K}-comp}$ respectively. When the $\mathcal{K}$ is understood, we sometimes omit it from the notation. By the adjoint functor theorem, [Lur09, 5.5.2.9], the inclusions $i_{\mathcal{K}-loc}$ and $i_{\mathcal{K}-comp}$ have left adjoints $L_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ respectively, while $i_{\mathcal{K}-tors}$ and $i_{\mathcal{K}-loc}$ have right adjoints $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ respectively.

For any $X \in \mathcal{C}$, these functors assemble into two cofiber sequences:

$$\Gamma_{\mathcal{K}} X \longrightarrow X \longrightarrow L_{\mathcal{K}} X \quad \text{and} \quad V_{\mathcal{K}} X \longrightarrow X \longrightarrow \Lambda_{\mathcal{K}} X.$$

Note also that these functors only depend on the localizing subcategory $\mathcal{C}^{\mathcal{K}-tors}$, not on the particular choice of generators $\mathcal{K}$. Thus, when the set $\mathcal{K}$ is clear from the context, we often omit it as a subscript when writing the functors.

Construction 2.27. As $\mathcal{C}^{\mathcal{K}-tors}$ is a localizing ideal, it automatically inherits a non-unitally symmetric monoidal structure from $\mathcal{C}$. As $\mathcal{C}^{\mathcal{K}-loc}$ is the left complement of a localizing ideal, the functor L is a localization, as mentioned in Example 2.22. By Remark 2.14, we have a symmetric monoidal structure on its local objects – in this case, $\mathcal{C}^{\mathcal{K}-loc}$ – that is compatible with the localization L. As L is a localization, Γ is a colocalization, and is in particular idempotent. Hence, the non-unitally symmetric monoidal structure on $\mathcal{C}^{\mathcal{K}-tors}$ is quasi-unital, which by [Lur17, 5.4.4.7] can be made into a symmetric monoidal structure. This makes Γ into a symmetric monoidal functor by the same argument as for localizations. Similarly, as $\mathcal{C}^{\mathcal{K}-tors}$ is generated by a set of compact objects [BHV18, 2.17] makes sure that also $\mathcal{C}^{\mathcal{K}-loc}$ is a localizing ideal, making Λ into a localization as well. In particular, $\mathcal{C}^{\mathcal{K}-comp}$ inherits a symmetric monoidal structure compatible with Λ.

We are now ready to state the abstract local duality theorem. We have chosen a slightly restricted version compared to that of [HPS97, 3.3.5] and [BHV18, 2.21], where we focus only on the parts we will need.

Theorem 2.28. *Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. Then*

1. *the functors Γ and L are smashing, i.e. $\Gamma X \simeq X \otimes \Gamma \mathbb{1}$ and $LX \simeq X \otimes L\mathbb{1}$,*
2. *the functors Λ and V are cosmashing, i.e. $\Lambda X \simeq \underline{\mathrm{Hom}}(\Gamma \mathbb{1}, X)$ and $VX \simeq \underline{\mathrm{Hom}}(L\mathbb{1}, x)$, and*
3. *the functors $\Gamma\colon \mathcal{C}^{\mathcal{K}-comp} \longrightarrow \mathcal{C}^{\mathcal{K}-tors}$ and $\Lambda\colon \mathcal{C}^{\mathcal{K}-tors} \longrightarrow \mathcal{C}^{\mathcal{K}-comp}$ are mutually inverse equivalences of categories,*

This can be summarized by the following diagram of adjoints

$$\begin{tikzcd}
 & \mathcal{C}^{\mathcal{K}-loc} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathcal{C} \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathcal{C}^{\mathcal{K}-tors} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathcal{C}^{\mathcal{K}-comp} \arrow[ul, shift left]
\end{tikzcd}$$

Remark 2.29. Although not stated in [HPS97] or [BHV18] the equivalence $\mathcal{C}^{\mathcal{K}-tors} \simeq \mathcal{C}^{\mathcal{K}-comp}$ is in fact monoidal. This is because the equivalence is a composition of a lax-monoidal functor i_{comp} (it is lax as it is adjoint to the symmetric monoidal functor $\Lambda_{\mathcal{K}}$) and a symmetric monoidal functor $\Gamma_{\mathcal{K}}$.

The following example will be useful later, and was also used as an instructive example of stable recollments in [BRW23, 3.4].

Example 2.30. We denote by $D(\mathbb{Z})$ the derived category of abelian groups. An abelian group A is said to be p-local for a prime p if all other primes act invertably on A. Equivalently, it is a module over the p-local integers $\mathbb{Z}_{(p)}$. The derived category of p-local abelian groups is denoted $D(\mathbb{Z})_{(p)}$ and can equivalently be described by modules over $\mathbb{Z}_{(p)}$, treated as a chain complex in degree zero, in $D(\mathbb{Z})$. The object $\mathbb{Z}_{(p)}/p$ is compact in $D(\mathbb{Z})_{(p)}$, hence the pair $(D(\mathbb{Z})_{(p)}, \{\mathbb{Z}_{(p)}/p\})$ is a local duality context. By Theorem 2.28, we have a local duality diagram

$$\begin{tikzcd}
 & D(\mathbb{Q}) \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & D(\mathbb{Z})_{(p)} \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left] & \\
D(\mathbb{Z})^{p-tors} \arrow[ur, shift left] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & D(\mathbb{Z})^{\wedge}_{p} \arrow[ul, shift left, "\Lambda"]
\end{tikzcd}$$

We have identified $D(\mathbb{Z})^{p-loc}_{(p)}$ with $D(\mathbb{Q})$ as every prime is invertible, giving us the rationals $\mathbb{Q}$. We have also identified $D(\mathbb{Z})^{p-comp}_{(p)}$ with $D(\mathbb{Z})^{\wedge}_{p}$, the category of derived p-complete abelian groups. The category $D(\mathbb{Z})^{p-tors}$ is the full subcategory of objects with p-torsion homology. For more on derived completion, see Remark 3.18. The functor Λ is then identified with the total left derived functor of p-adic completion,

$$\Lambda(X) \simeq \mathbb{L}C_p(X) \simeq \lim_k \Sigma^{-1}\mathbb{Z}/p^k \otimes X,$$

while the functor Γ is identified with the total right derived functor of p-torsion

$$\Gamma(X) \simeq \mathbb{R}T_p(X) \simeq \mathrm{colim}_k \, \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^k, X),$$

see [BHV18, 3.16, 3.18] for details. This, then, reduces the local duality for $(D(\mathbb{Z})_{(p)}, \mathbb{Z}/p)$ to the classical duality theory between derived p-torsion and derived p-completion in abelian groups.

Remark 2.31. We will see more examples of this duality in later sections. In Section 3, we will study a local duality of a particular class of derived categories, while in Section 4 we will study a local duality in chromatic homotopy theory. Hence, we give no more examples here, except to state that the above result puts several known duality results on a common framework, for example, Grothendieck duality and Greenlees-May duality, see [BHV18].

2.2 Barr-Beck for localizing ideals

As stated earlier, we will later be particularly interested in smashing localizations on categories. The goal of this section is essentially to prove that such smashing localizations interact nicely with local duality as in Theorem 2.28.

Remark 2.32. Let L be localization on $\mathcal{C}$. As noted before, the image $L\mathbb{1}$ is a ring object, and every L-local object admits the structure of an $L\mathbb{1}$-module. If L is a smashing localization, then $L\mathbb{1}$ is an idempotent algebra in $\mathcal{C}$. The image of the inclusion $\mathcal{C}_L \hookrightarrow \mathcal{C}$ is then identified with the category of modules $\mathrm{Mod}_{L\mathbb{1}}(\mathcal{C})$. In particular, smashing localizations are in bijection with idempotent algebras.

There is a more general way to identify which categories are equivalent to modules over an algebra. This is the contents of the Barr-Beck-Lurie theorem, which we now cover.

Definition 2.33. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$. An adjoint pair $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called a monoidal adjunction if F admits the structure of a symmetric monoidal functor.*

Remark 2.34. If $(F \dashv G)$ is a monoidal adjunction, then the right adjoint G is a lax monoidal functor. In particular, it sends algebras to algebras.

Definition 2.35. *Let $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ be an adjoint pair. The composition GF is a monad on $\mathcal{C}$, i.e. an object in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{End}(\mathcal{C}))$. The functor G factors as*

$$\mathcal{D} \xrightarrow{\bar{G}} \mathrm{Mod}_{GF}(\mathcal{C}) \xrightarrow{f} \mathcal{C}$$

where $\mathrm{Mod}_{GF}(\mathcal{C})$ is the Eilenberg-Moore category of GF and f is the forgetful functor. We say the adjunction $(F \dashv G)$ is monadic if $\bar{G}$ is an equivalence. If the adjunction is monoidal and $\bar{G}$ is a monoidal equivalence, we say $(F \dashv G)$ is monoidally monadic.

Definition 2.36. *Let $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a monoidal adjunction. For any $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ there is a natural map $\phi_{X,Y}\colon G(X) \otimes_{\mathcal{C}} Y \longrightarrow G(X \otimes_{\mathcal{D}} F(Y))$, called the projection formula, adjoint to the composition*

$$G(X \otimes_{\mathcal{D}} F(Y)) \longrightarrow G(X) \otimes_{\mathcal{C}} GF(Y) \longrightarrow G(X) \otimes_{\mathcal{C}} Y.$$

If $\phi_{X,Y}$ is an isomorphism for all X and Y, we say the projection formula holds for $(F \dashv G)$.

The following theorem is the monoidal Barr-Beck-Lurie theorem. It is a monoidal version of Lurie's ∞-categorical version of the classical Barr-Beck monadicity theorem, see [Lur17, Section 4.7].

Theorem 2.37 ([MNN17, 5.29])**.** *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a monoidal adjunction. If in addition*

1. *G is conservative*
2. *G preserves colimits*
3. *The projection formula holds*

then (F, G) is a monoidally monadic adjunction and the monad GF is equivalent to the monad $G(\mathbb{1}_{\mathcal{D}}) \otimes (-)$. In particular this gives a symmetric monoidal equivalence $\mathcal{D} \simeq \mathrm{Mod}_{G(\mathbb{1}_{\mathcal{D}})}(\mathcal{C})$.

Proof. By [Lur17, 4.7.0.3] the adjunction is monadic by the first two criteria, giving an equivalence $\mathcal{D} \simeq \mathrm{Mod}_{GF}(\mathcal{C})$. The projection formula applied to the unit $\mathbb{1}_{\mathcal{D}}$ gives an equivalence of monads $GF \simeq G(\mathbb{1}_{\mathcal{D}}) \otimes \mathcal{C}$. ∎

Definition 2.38. *When the three criteria above hold for a given monoidal adjunction $(F \dashv G)$, we will say that the adjunction satisfies the monoidal Barr-Beck criteria or that it is a monoidal Barr-Beck adjunction. We will sometimes omit the prefix monoidal when it is clear from context.*

Example 2.39. This gives an immediate proof of the claim in Remark 2.32. Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $L\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ any smashing localization. Then $L\mathbb{1}_{\mathcal{C}}$ is an idempotent algebra and the monoidal adjunction $(L \dashv i)\colon \mathcal{C} \longrightarrow \mathcal{C}_L$ satisfies the Barr-Beck criteria. Hence Theorem 2.37 gives $\mathcal{C}_L \xrightarrow{\simeq} \mathrm{Mod}_{L\mathbb{1}_{\mathcal{C}}}(\mathcal{C})$, which is exactly the claimed equivalence.

Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. The goal for this section was to prove that local duality interacts nicely with smashing localizations. Since smashing localizations are examples of the Barr-Beck adjunction, we take a more general approach and prove that local duality is compatible with Theorem 2.37. By modifying [BS20, 3.7] slightly, we know that any Barr-Beck adjunction induces a Barr-Beck adjunction on $\mathcal{K}$-local and $\mathcal{K}$-complete objects. Hence, it remains only to prove a similar statement for the $\mathcal{K}$-torsion objects.

Definition 2.40. *Let $(\mathcal{C}, \mathcal{K})$ and $(\mathcal{D}, \mathcal{L})$ be local duality contexts. A map of local duality contexts is a symmetric monoidal colimit-preserving functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(\mathcal{K}) \subseteq \mathcal{L}$. If, in addition $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) \simeq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, then we say F is a strict map of local duality contexts. A monoidal adjunction $(F \dashv G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that F is a strict map of local duality contexts is called a local duality adjunction, sometimes denoted*

$$(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \longrightarrow (\mathcal{D}, \mathcal{L}).$$

Given a local duality context and an appropriate functor, one can always extend the functor to a strict map of local duality context in the following way.

Construction 2.41. Let $(\mathcal{C}, \mathcal{K})$ be a local duality context, $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$ and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a symmetric monoidal colimit-preserving functor. The image of $\mathcal{K}$ under F generates a localizing ideal $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K}))$ in $\mathcal{D}$, which makes F a map of local duality contexts. We call this the local duality context on $\mathcal{D}$ induced by $\mathcal{C}$ via F.

The following lemma is essentially the "non-geometric" version of [BS17, 5.11]. The proof is also similar, but as we have phrased it in a different and slightly more general language, we present a full proof.

Lemma 2.42. *Let $(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \longrightarrow (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. Then, the adjunction induces a monoidal adjunction on localizing ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}).$$

Proof. Recall first that localizing ideals inherit symmetric monoidal structures by Construction 2.27. Since the functors $\Gamma_{\mathcal{K}}\colon \mathcal{C} \longrightarrow \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\Gamma_{\mathcal{L}}\colon \mathcal{D} \longrightarrow \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ are both smashing, these symmetric monoidal structures are the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, restricted to $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{D})$ respectively.

Since F is a map of local duality contexts, we have an inclusion $F(\mathcal{K}) \subseteq \mathcal{L}$, which gives inclusions

$$F(\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}),$$

meaning that the functor F restricts to the torsion objects. In particular we have for any object $X \in \mathcal{C}^{\mathcal{K}-tors}$ an equivalence $\Gamma_{\mathcal{L}}F(X) \simeq F(X)$. We let $F' = F_{|\,\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})}$ and define U' to be the composition

$$Loc^{\otimes}_{\mathcal{D}}(\mathcal{L}) \xrightarrow{i_{\mathcal{L}-tors}} \mathcal{D} \xrightarrow{U} \mathcal{C} \xrightarrow{\Gamma_{\mathcal{K}}} \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}),$$

which is an adjoint to F'. We need to show that F is a symmetric monoidal functor, but, as the inclusions $i_{\mathcal{K}-loc}$ and $i_{\mathcal{L}-loc}$ are non-unitally monoidal all that remains to be proven is that F' sends the monoidal unit $\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}$ to the monoidal unit $\Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$.

The localizing ideals $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ are equivalent to the localizing ideals generated by the respective units, i.e.

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \simeq \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \quad \text{and} \quad \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}) \simeq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}).$$

Since $(F \dashv U)$ is a local duality adjunction we also know that $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) \simeq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, which also means $\mathrm{Loc}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) \simeq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$. Let $\mathcal{G}$ be the full subcategory of $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ where $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ acts as a unit, in other words objects $M \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ such that $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M \simeq M$. In particular, $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ is in $\mathcal{G}$. The category $\mathcal{G}$ is closed under retracts, suspension, and colimits, as well as tensoring with objects in $\mathcal{D}$, as we have

$$F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (M \otimes_{\mathcal{D}} D) \simeq (F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M) \otimes_{\mathcal{D}} D \simeq M \otimes_{\mathcal{D}} D$$

for any $M \in \mathcal{G}$ and $D \in \mathcal{D}$. Hence, it is a localizing tensor ideal of $\mathcal{D}$, with a symmetric monoidal structure where the unit is $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$. In particular, $\mathcal{G} \simeq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}))$, which we already know is equivalent to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Since the ideals are equivalent, and the unit is unique, we must have $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{D}$, which finishes the proof. $\square$

The key feature for us is that such an induced adjunction inherits the property of being a Barr-Beck adjunction, i.e., that the right adjoint is conservative, preserves colimits, and has a projection formula. An analogous, but not equivalent, statement was proven in [BS20, 4.5]. Another related, but not equivalent statement, is Greenlees and Shipleys Cellularization principle, [GS13].

Theorem 2.43. *Let $(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \longrightarrow (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. If $(F \dashv G)$ satisfies the Barr-Beck criteria, then the induced monoidal adjunction on localizing ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$$

constructed in Lemma 2.42, also satisfies the Barr-Beck criteria.

Proof. We need to prove that G' is conservative and colimit-preserving and that the projection formula holds. The first two will both follow from the following computation, showing that also G' is just the restriction of G to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Let $X \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$. By definition we have $G'(X) = \Gamma_{\mathcal{K}} G(X)$, where we have omitted the inclusions from the notation for simplicity. Since $\Gamma_{\mathcal{K}}$ is smashing and $(F \dashv G)$ by assumption has a projection formula we have

$$\Gamma_{\mathcal{K}} G(X) \simeq G(X) \otimes_{\mathcal{C}} \Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}} \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}})).$$

By Lemma 2.42 F' is symmetric monoidal, hence $F(\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}$, which acts on X as the monoidal unit. Thus, we can summarize with

$$G'(X) \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}})) \simeq G(X \otimes_{\mathcal{D}} \Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}) \simeq G(X),$$

which shows that also G' is the restriction of G.

Now, as U is both conservative and preserves colimits, and colimits in the localizing ideals are computed in $\mathcal{C}$ and $\mathcal{D}$ respectively, then also U' is conservative and colimit-preserving. The projection formula for $(F' \dashv U')$ also automatically follows from the projection formula for $(F \dashv U)$. ☐

3 Hopf algebroids and comodules

Before we pass to the world of stable homotopy theory in Section 4 we study a related but simpler theory arising in algebra via comodules over Hopf algebroids. The contents of this section can be thought of as an algebraic analog to Section 4, and the goal of Section 6 and Section 7 is to study how precise this analogy is.

Definition 3.1. *A (graded) Hopf algebroid is a cogroupoid object (A, Ψ) in the category of graded commutative rings.*

The use of Hopf algebroids in situations related to homotopy theory was studied by Ravenel in [Rav86, A.1] and later in more detail by Hovey in [Hov04].

Remark 3.2. In the literature outside of topology, the assumptions of being commutative and graded are usually not present. But, as all our examples will be of this kind, we keep in line with the topological tradition.

Definition 3.3. *Let (A, Ψ) be a Hopf algebroid. A Ψ-comodule is an A-module M together with a coassociative and counital map $\psi\colon M \longrightarrow M \otimes_A \Psi$. The category of comodules over (A, Ψ) is denoted Comod_{Ψ}.*

Example 3.4. For any commutative graded ring A, the pair (A, A) is a called a discrete Hopf algebroid. The category of comodules over this Hopf algebroid is the normal abelian category Mod_A of modules over A.

Remark 3.5. In algebraic geometry, Hopf algebroids are usually formulated dually as groupoid objects in affine schemes. The left and right unit maps $A \rightrightarrows \Psi$ induces a presentation of stacks $\mathrm{Spec}(\Psi) \rightrightarrows \mathrm{Spec}(A)$, and the category Comod_Ψ is equivalent to the category of quasi-coherent sheaves on the presented stack, see [Nau07, Thm 8].

Construction 3.6. Given an Adams Hopf algebroid (A, Ψ), we can define a discretization map $\varepsilon\colon (A, \Psi) \longrightarrow (A, A)$, which is given by the identity on A and the counit on Ψ. By [Rav86, A1.2.1] and [BHV18, 4.6] it induces a faithful exact forgetful functor $\varepsilon_*\colon \mathrm{Comod}_\Psi \longrightarrow \mathrm{Mod}_A$ with a right adjoint ε^* given by $\varepsilon^*(M) \simeq \Psi \otimes_A M$. A comodule in the essential image of ε^* is called an extended comodule.

Definition 3.7. *We say a Hopf algebroid (A, Ψ) is of Adams type if Ψ is a filtered colimit $\mathrm{colim}_k \Psi_k \simeq \Psi$ of dualizable comodules Ψ_k.*

Proposition 3.8 ([Hov04, 1.3.1, 1.4.1])**.** *Let (A, Ψ) be an Adams Hopf algebroid. Then, the category Comod_Ψ is a Grothendieck abelian category generated by the dualizable comodules. There is a symmetric monoidal product $-\otimes_\Psi -$, which on the underlying modules is the normal tensor product of A-modules. It has a right adjoint $\underline{\mathrm{Hom}}_\Psi(-,-)$, making Comod_Ψ a closed symmetric monoidal category.*

As in Section 2.1, we have certain objects that are especially important – the compact objects and the dualizable objects. In Grothendieck abelian categories it is, in addition, important to understand the injective objects. This will also become important later in Section 5, as we will use injective objects to approximate other objects and to build certain spectral sequences.

Proposition 3.9. *Let (A, Ψ) be an Adams Hopf algebroid. A Ψ-comodule M is dualizable if and only if its underlying A-module $\varepsilon_* M$ is dualizable, i.e., it is finitely generated and projective. Similarly, a Ψ-comodule is compact if and only if its underlying A-module is compact, which coincides with being finitely presented.*

Proof. The first claim is [Hov04, 1.3.4] and the second is [Hov04, 1.4.2]. □

Remark 3.10. As colimits in Comod_Ψ are exact and are computed in Mod_A, all the dualizable comodules are compact. Hence, the full subcategory of dualizable comodules is a set of compact generators for $Comod_\Psi$.

Proposition 3.11 ([HS05b, 2.1])**.** *Let (A, Ψ) be an Adams Hopf alebroid. If I is an injective object in Comod_Ψ, then there is an injective A-module Q, such that I is a retract of the extended comodule $\Psi \otimes_A Q$.*

Remark 3.12. Note that as Comod_Ψ is Grothendieck abelian, it has enough injective objects. This allows us to construct injective resolutions and thus Ext-groups, which we will see later, greatly help in computing information in stable homotopy theory. For example, the pair $(\mathbb{F}_2, \mathcal{A}_*)$ where $\mathcal{A}_*$ is the dual Steenrod algebra is a Hopf algebroid, and the groups $\mathrm{Ext}^s_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)$ are used in the Adams spectral sequence to approximate homotopy groups of spheres, see [Ada58].

Given an Adams Hopf algebroid (A, Ψ), we also have an associated derived category. By [Hov04, 2.1.2, 2.1.3] the category of chain complexes of Ψ-comodules, Ch_Ψ, has a cofibrantly generated stable symmetric monoidal model structure. In [BR11] this model structure was modified slightly to more easily compare it to the periodic derived category, which we will construct in Section 3.2. The homotopy category associated to this model structure is the usual unbounded derived category $D(\mathrm{Comod}_\Psi)$ associated to the Grothendieck abelian category Comod_Ψ.

Notation 3.13. We will use $D(\Psi)$ as our notation for the underlying symmetric monoidal stable ∞-category associated with the above model structure. The monoidal unit is A, treated as a chain complex centered in degree 0.

Remark 3.14. We warn the reader that some authors use the notation $D(\Psi)$ to reffer to the above-mentioned periodic derived category of (A, Ψ). This is the case, for example, in [Pst21].

We also get an induced discretization adjunction on the level of derived categories.

Proposition 3.15. *Let (A, ψ) be an Adams Hopf algebroid. Then the discretization adjunction $(\varepsilon_* \dashv \varepsilon^*)\colon \mathrm{Comod}_\Psi \longrightarrow \mathrm{Mod}_A$ induces an adjunction $(\varepsilon_* \dashv \varepsilon^*)\colon D(\Psi) \longrightarrow D(A)$.*

Proof. This follows from the fact that Ψ is flat over A, which implies that both ε_* and ε^* on the abelian categories are exact. □

3.1 Torsion and completion for comodules

There are two approaches to studying torsion and completion in $D(\Psi)$ – one "internal" and one "external". The internal approach uses the classical theory of torsion objects in abelian categories, while the external uses local duality, as in Theorem 2.28. These two approaches are luckily equivalent in the situations we are interested in.

We first review the abelian situation: the internal approach. We follow [BHV18] and [BHV20] in notation and results.

Definition 3.16. *Let A be a commutative ring and $I \subseteq R$ a finitely generated ideal. The I-power torsion of an A-module M is defined as*

$$T_I^A M = \{x \in M \mid I^k x = 0 \textit{ for some } k \in \mathbb{N}\}.$$

We say a module M is I-torsion if the natural map $T_I^A M \longrightarrow M$ is an equivalence.

Definition 3.17. *Let A be a commutative ring and $I \subseteq R$ a finitely generated ideal. The I-adic completion of an A-module M is defined as*

$$C_I^A M = \lim_k A/I^k \otimes_A M.$$

We say a module M is I-adically complete if the natural map $M \longrightarrow C_I^A M$.

Remark 3.18. The resulting category of I-adically complete modules is not very well-behaved. The I-adic completion functor is often neither left nor right exact, and the category is often not abelian. To fix these issues, Greenlees and May introduced the notion of L-complete modules in [GM92], using instead the zeroth left derived functor $L = \mathbb{L}_0 C_I^A$. Thus, it is also sometimes referred to as derived completion. One then defines I-complete modules, also called

L-complete or derived complete, to be those R-modules such that the natural map $M \longrightarrow LM$ is an equivalence.

Notation 3.19. We denote the full subcategory consisting of I-power torsion A-modules by Mod_A^{I-tors} and the full subcategory of I-complete A-modules by Mod_A^{I-comp}.

Remark 3.20. The category Mod_A^{I-tors} is a Grothendieck abelian category. On the other hand, Mod_A^{I-comp} is abelian, but not Grothendieck in general.

The inclusion of the full subcategory $\mathrm{Mod}_A^{I-tors} \hookrightarrow \mathrm{Mod}_A$ has a right adjoint, which coincides with the I-power torsion $T_I^A(-)$. This gives the I-power torsion another description as the colimit

$$T_I^A M \cong \mathrm{colim}_k \, \underline{\mathrm{Hom}}_A(A/I^k, M).$$

We want to extend the construction of I-torsion and L-complete modules to general Adams Hopf algebroids (A, Ψ). For this, we need to choose sufficiently nice ideals that interact nicely with the additional comodule structure.

Definition 3.21. *Let (A, Ψ) be an Adams Hopf algebroid, and I an ideal in A. We say I is an invariant ideal if, for any comodule M, the comodule IM is a subcomodule of M. If I is finitely generated by $(x_1, \ldots, x_r)$ and x_i is non-zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for each $i = 1, \ldots, r$, then we say I is regular.*

Definition 3.22. *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. The I-power torsion of a comodule M is defined as*

$$T_I^\Psi M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a comodule M is I-torsion if the natural map $T_I^\Psi M \longrightarrow M$ is an equivalence.

Remark 3.23. By [BHV18, 5.10] the full subcategory of I-torsion comodules, which we denote $\mathrm{Comod}_\Psi^{I-tors}$, is a Grothendieck abelian category. It also inherits a symmetric monoidal structure from Comod_Ψ. This also makes Mod_A^{I-tors} Grothendieck abelian and symmetric monoidal by Example 3.4.

Remark 3.24. Unfortunately, the corresponding versions of I-adically complete and L-complete comodules do not form abelian categories in general, as we can have problems with the comodule structure on certain cokernels.

As for the case of modules, the inclusion $\mathrm{Comod}_\Psi^{I-tors} \hookrightarrow \mathrm{Comod}_\Psi$ has a right adjoint that corresponds to the I-power torsion construction T_I^Ψ. This, by [BHV18, 5.5] also has the alternative description

$$T_I^\Psi M \cong \mathrm{colim}_k \, \underline{\mathrm{Hom}}_\Psi(A/I^k, M).$$

The construction of I-power torsion in Mod_A and Comod_Ψ are completely analogous, so one can wonder whether they agree on the underlying modules. This turns out to be the case.

Lemma 3.25 ([BHV18, 5.7])**.** *For any Ψ-comodule M there is an isomorphism of A-modules $\varepsilon_* T_I^\Psi M \cong T_I^A \varepsilon_* M$. Furthermore, if an A-module N is I-power torsion, then the extended* *comodule $\Psi \otimes_A N$ is I-power torsion. In particular, a Ψ-comodule M is I-power torsion if and only if the underlying A-module is I-power torsion.*

As mentioned above, we will later make use of injectives in $\mathrm{Comod}_\Psi^{I-tors}$. Hence, we relate some facts about these.

Lemma 3.26. *Let (A, Ψ) be an Adams Hopf algebroid and I a regular invariant ideal.*

1. *If J is an injective in Comod_Ψ then $T_I^\Psi J$ is an injective in $\mathrm{Comod}_\Psi^{I-tors}$.*
2. *There are enough injectives in $\mathrm{Comod}_\Psi^{I-tors}$.*
3. *Any injective J' in $\mathrm{Comod}_\Psi^{I-tors}$ is a retract of an object of the form $T_I^\Psi J$ for an injective Ψ-comodule J.*

Proof. The first point is [BS12, 2.1.4], while the second is a consequence of $\mathrm{Comod}_\Psi^{I-tors}$ being Grothendieck abelian, as mentioned in Remark 3.23. The third point is stated in the proof of [BHV20, 3.16]. ☐

Remark 3.27. Choosing a discrete Hopf algebroid (A, A), Lemma 3.26 implies that injectives in Mod_A^{I-tors} are retracts of $T_I^A(Q)$ for some injective A-module Q and that T_I^A preserves injectives. As noted in Proposition 3.11, an injective object in Comod_Ψ is a retract of an extended comodule of the form $\Psi \otimes_A Q$ for an injective A-module Q. This means that all injectives J in $\mathrm{Comod}_\Psi^{I-tors}$ are retracts of $T_I^\Psi(\Psi \otimes_A Q)$ where Q is an injective A-module.

Remark 3.28. As colimits in $\mathrm{Comod}_\Psi^{I-tors}$ are computed in Comod_Ψ, we have, similar to Proposition 3.9, that an I-power torsion Ψ-comodule M is dualizable (resp. compact) if and only if its underlying A-module is finitely generated and projective (resp. finitely presented).

Lemma 3.29. *Let (A, Ψ) be an Adams Hopf algebroid, where A is noetherian and $I \subseteq A$ a regular invariant ideal. Then $\mathrm{Comod}_\Psi^{I-tors}$ is generated under filtered colimits by the compact I-power torsion comodules.*

Proof. By [BHV20, 3.4] $\mathrm{Comod}_\Psi^{I-tors}$ is generated by the set

$$\mathrm{Tors}_\Psi^{fp} := \{G \otimes A/I^k \mid G \in \mathrm{Comod}_\Psi^{fp}, k \geqslant 1\},$$

where Comod_Ψ^{fp} is the full subcategory of dualizable Ψ-comodules. Since I is finitely generated and regular, A/I^k is finitely presented as an A-module, hence it is compact in $\mathrm{Comod}_\Psi^{I-tors}$ by Proposition 3.9 and Remark 3.28. As A is noetherian, being finitely generated and finitely presented coincide. The tensor product of finitely generated modules is finitely generated, hence any element in Tors_Ψ^{fp} is compact. ☐

Remark 3.30. The assumption that the ring A is noetherian can most likely be removed, but it makes no difference to the results in this paper.

Notation 3.31. Since $\mathrm{Comod}_\Psi^{I-tors}$ is Grothendieck abelian we have an associated derived stable ∞-category $D(\mathrm{Comod}_\Psi^{I-tors})$ which we denote simply by $D(\Psi^{I-tors})$.

We now move to the external approach, using local duality.

Construction 3.32. Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. Then A/I, treated as a complex concentrated in degree zero, is by [BHV18, 5.13] a compact object in $D(\Psi)$. Thus, $(D(\Psi), A/I)$ is a local duality context, and we can consider the corresponding local duality diagram

$$\begin{tikzcd}
 & D(\Psi)^{I-loc} \arrow[dd, dashed, bend left] \arrow[d, shift left] & \\
 & D(\Psi) \arrow[u, shift left, "L_I^{\Psi}"] \arrow[dl, shift left, "\Gamma_I^{\Psi}"] \arrow[dr, shift left, "\Delta_I^{\Psi}"] & \\
D(\Psi)^{I-tors} \arrow[rr, "\simeq"] \arrow[ur, shift left] \arrow[uur, dashed, bend left] & & D(\Psi)^{I-comp} \arrow[ul, shift left]
\end{tikzcd}$$

where we have used the superscript I instead of A/I for simplicity. This gives, in particular, a definition of I-torsion objects in $D(\Psi)$ as $D(\Psi)^{I-tors}$.

Our goal was to give two constructions and prove that they were equal in the cases we were interested in.

Lemma 3.33 ([BHV20, 3.7(2)]). *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. There is an equivalence of categories*

$$D(\Psi)^{I-tors} \simeq D(\Psi^{I-tors}).$$

Furthermore, an object $M \in D(\Psi)$ is I-torsion if and only if the homology groups H_*M are I-power torsion Ψ-comodules.

Remark 3.34. One can wonder whether the same is true for the I-complete derived category, but this is unfortunately not true as $\mathrm{Comod}_{\Psi}^{I-comp}$ is not abelian. A partial result can, however, be recovered for discrete Hopf algebroids (A, A).

We follow [BHV20] in the following construction.

Construction 3.35. Recall that Mod_A^{I-comp} denotes the category of L-complete A-modules for $I \subseteq A$ a regular ideal. By [BHV20, 2.11] the category has enough projectives, hence by [Lur17, 1.3.2] we can associate to it the right bounded category $D^-(\mathrm{Mod}_A^{I-comp})$. This has a by [Lur17, 1.3.2.19, 1.3.3.16] a left complete t-structure with heart equivalent to Mod_A^{I-comp}. We can then form its right completion, which we denote $\overline{D}(\mathrm{Mod}_A^{I-comp})$, and call the completed derived category of Mod_A^{I-comp}.

This is what allows us the partial version of Lemma 3.33 in the case of I-completion.

Proposition 3.36 ([BHV20, 3.7(1)]). *Let A be a commutative ring and $I \subseteq A$ a regular ideal. Then, there is an equivalence*

$$D(\mathrm{Mod}_A)^{I-comp} \simeq \overline{D}(\mathrm{Mod}_A^{I-comp}),$$

where the former category is the full subcategory of A/I-complete objects in $D(\mathrm{Mod}_A)$ while the latter is the completed derived category of Mod_A^{I-comp}.

3.2 The periodic derived category

The standard derived ∞-category will, for our purposes, not be the correct category to work with. The periodic derived category was constructed by Franke to serve as an algebraic version of a certain category of spectra, which we will cover in the next section.

The spirit of the periodic derived category can be captured as follows. If we have an abelian category with a local grading, for example, graded modules over a graded ring R_*, then the derived category $D(R_*)$ has two gradings, one coming from R_* and one from forming chain complexes. If we want to compare this category to a category with only one grading, then we need to remove a grading in order for the comparison to be compatible. The periodic derived category is a way to collapse the two gradings on $D(R_*)$ into a single grading.

Definition 3.37. *Let $\mathcal{D}$ be a category. A local grading on $\mathcal{D}$ is an autoequivalence $T\colon \mathcal{D} \longrightarrow \mathcal{D}$. A category together with a choice of a local grading is called a locally graded category.*

Example 3.38. Any stable ∞-category $\mathcal{C}$, together with its suspension functor $\Sigma\colon \mathcal{C} \longrightarrow \mathcal{C}$, makes $\mathcal{C}$ a locally graded category.

Example 3.39. Let R be a graded ring and Mod_R its category of graded modules. Then the grading shift functor $[1]\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ defined by $(TM)_k = M_{k-1}$ is a local grading on Mod_R. Similarly, for a (graded) Hopf algebroid (A, Ψ), the same grading shift functor T makes Comod_Ψ a locally graded abelian category.

There are several ways of constructing the periodic derived category, but we follow [Fra96] in spirit, using periodic chain complexes.

Definition 3.40. *Let $\mathcal{A}$ be an abelian category with a local grading T and denote $[1]$ the shift functor on the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ in $\mathcal{A}$. A chain complex $C \in \mathrm{Ch}(\mathcal{A})$ is called periodic if there is an isomorphism $\phi\colon C[1] \longrightarrow TC$.*

Notation 3.41. It is more common to write the chain complex grading as $C_\bullet$. We can then incorporate the isomorphism into the structure, defining a periodic chain complex to be a pair $(C_\bullet, \phi_\bullet)$. Together with chain maps that commute with the $\psi_\bullet$ isomorphisms, these objects form a category of periodic chain complexes, denoted $\mathrm{Ch}^{per}(\mathcal{A})$.

Definition 3.42. *The forgetful functor $\mathrm{Ch}^{per}(\mathcal{A}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ has a left adjoint P, called the periodization.*

Remark 3.43. By [BR11], there is an explicit formula for this periodization of a chain complex C, given by

$$P(C) = \bigoplus_{k\in\mathbb{Z}} T^k C[-k],$$

which in particular means that $P(C)_n = \bigoplus_{k\in\mathbb{Z}} T^k C_{n+k}$. The proof of the fact that this functor is left adjoint to the forgetful functor is given in [BR11, Lemma 1.2].

Definition 3.44. *Let $\mathcal{A}$ be a locally graded abelian category. Then the periodic derived category of $\mathcal{A}$, denoted $D^{per}(\mathcal{A})$ is the ∞-category obtained by localizing $\mathrm{Ch}^{per}(\mathcal{A})$ at the quasi-isomorphism. It is in fact stable by [PP21, 7.8].*

Remark 3.45. For more details on the model structure and finer details of this definition, the interested reader is referred to [BR11] and [PP21]. Other ources covering the more specific case of comodules over an Adams Hopf algebroid, are [BSS20] and [Pst21].

Remark 3.46. If $\mathcal{A}$ is a symmetric monoidal category, for example $\mathcal{A} = \mathrm{Comod}_\Psi$, then $P\mathbb{1}$ is a commutative ring object called the periodic unit. The category of periodic chain complexes $\mathrm{Ch}^{per}(\mathcal{A})$ is equivalent to $\mathrm{Mod}_{P\mathbb{1}}(\mathrm{Ch}(\mathcal{A}))$. This descends also to the derived categories, giving an equivalence

$$D^{per}(\mathcal{A}) \simeq \mathrm{Mod}_{P\mathbb{1}}(D(\mathcal{A})),$$

see for example [Pst21, 3.7].

Construction 3.47. The pair $(D^{per}(\mathcal{A}), P(A/I))$ is a local duality context with associated local duality diagram

$$\begin{tikzcd}
 & D^{per}(\Psi)^{I-loc} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & D^{per}(\Psi) \arrow[u, shift left, "L_I^\Psi"] \arrow[dl, shift left, "\Gamma_I^\Psi"] \arrow[dr, shift right] & \\
D^{per}(\Psi)^{I-tors} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & D^{per}(\Psi)^{I-comp} \arrow[ul, shift right, "\Lambda_I^\Psi"']
\end{tikzcd}$$

The functors in the diagram are induced by the functors from Construction 3.32. In fact, there is a diagram

$$\begin{tikzcd}
D(\Psi)^{I-tors} \arrow[r, shift left] \arrow[d, shift right, "P"'] & D(\Psi) \arrow[l, shift left, "\Gamma_I^\Psi"] \arrow[r, shift left, "L_I^\Psi"] \arrow[d, shift right, "P"'] & D(\Psi)^{I-loc} \arrow[l, shift left] \arrow[d, shift right, "P"'] \\
D^{per}(\Psi)^{I-tors} \arrow[r, shift left] \arrow[u, shift right] & D^{per}(\Psi) \arrow[l, shift left, "\Gamma_I^\Psi"] \arrow[r, shift left, "L_I^\Psi"] \arrow[u, shift right] & D^{per}(\Psi)^{I-loc} \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

that is commutative in all possible directions. Here where the unmarked horizontal arrows are the respective fully faithful inclusions.

Remark 3.48. In the specific case of $(A, \Psi) = (E_0, E_0E)$ and $I \subseteq E_0$ the Landweber ideal I_n, then the above construction is [BSS21, 3.12]. For more on this example, see Section 4.2.

There is now some ambiguity to take care of for our category of interest $D^{per}(\Psi)^{I-tors}$. In the picture above, we do mean that we take I-torsion objects in $D^{per}(\Psi)$, i.e., $[D^{per}(\Psi)]^{I-tors}$, but we could also take the periodization of the category $D(\Psi)^{I-tors}$ as our model. Luckily, there is no choice, as they are equivalent. This can be thought of as the periodic version of Lemma 3.33, where we had an equivalence $D(\Psi^{I-tors}) \simeq D(\Psi)^{I-tors}$.

Theorem 3.49. *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a finitely generated invariant regular ideal. Then there is an equivalence of stable ∞-categories*

$$[D^{per}(\Psi)]^{I-tors} \simeq D^{per}(\Psi^{I-tors}).$$

Proof. By Remark 3.46 we have equivalences

$$D^{per}(\Psi) \simeq \mathrm{Mod}_{P\mathbb{1}}(D(\Psi)) \quad \text{and} \quad D^{per}(\Psi^{I-tors}) \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi \mathbb{1})}(D(\Psi^{I-tors})). \tag{1}$$

By Theorem 2.43 the former induces an equivalence

$$[D^{per}(\Psi)]^{I-tors} \simeq \mathrm{Mod}_{\Gamma_I^\Psi(P\mathbb{1})}(D(\Psi)^{I-tors}).$$

Since Γ_I^Ψ is a smashing colocalization and P is given by tensoring with $P(\mathbb{1})$, they do in fact commute. By again identifying $D(\Psi)^{I-tors} \simeq D(\Psi^{I-tors})$ the above equivalence can be rewritten as

$$[D^{per}(\Psi)]^{I-tors} \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi \mathbb{1})}(D(\Psi^{I-tors})),$$

which by the second equivalence in Eq. (1) is equivalent to $D^{per}(\Psi^{I-tors})$, finishing the argument. $\square$

4 Chromatic homotopy theory

There are by now countless well-written introductions to the chromatic viewpoint of stable homotopy theory – from multiple different viewpoints. But, we still decided to include a short version of the story, as well as the key ideas and the definitions we need to state our results. In light of our overarching focus on local duality (Theorem 2.28), we have chosen a viewpoint that exemplifies the relationship between such dualities and the chromatic viewpoint. A reader interested in a more comprehensive background treatment is referred to [BB19], of which some of the below approach is inspired.

4.1 Fracture squares and field objects

In light of Waldhausen's viewpoint of stable homotopy theory as an enhancement of algebra, usually called brave new algebra, one should view the category of spectra Sp as a homotopical enrichment of the derived category of abelian groups $D(\mathbb{Z})$. We know that abelian groups can be studied one prime at the time, which corresponds to studying $D(\mathbb{Z})_{(p)}$, the p-local derived category. In [Bou79b], Bousfield developed a general machinery for studying localizations on Sp, by inverting maps that are equivalences with respect to some spectrum F. The corresponding localization dunctor is denoted L_F. We can then create p-localization on Sp, by Bousfield localizing at the p-local Moore spectrum $M\mathbb{Z}_{(p)}$. On homotopy groups this has the effect of p-localizing, i.e., inverting all primes except for p. The category of p-local spectra, denoted $\mathrm{Sp}_{(p)}$, should then be thought of as a homotopical enrichment of $D(\mathbb{Z})_{(p)}$.

Remark 4.1. Both $L_{(p)} \colon D(\mathbb{Z}) \longrightarrow D(\mathbb{Z})_{(p)}$ and $L_{(p)} \colon \mathrm{Sp} \longrightarrow \mathrm{Sp}_{(p)}$ are smashing localizations.

The study of $D(\mathbb{Z})_{(p)}$ can be further reduced to the study of its "atomic pieces", which are the minimal localizing subcategories.

Definition 4.2. *A localizing subcategory $\mathcal{L} \subseteq \mathcal{C}$ (see Definition 2.19) is said to be minimal if any proper localizing subcategory $\mathcal{L}' \subset \mathcal{L}$ is (0).*

Remark 4.3. If $\mathcal{L}$ is a minimal localizing subcategory, then any non-zero object $K \in \mathcal{L}$ generates $\mathcal{L}$ as $\mathrm{Loc}_{\mathcal{C}}(K) \simeq \mathcal{L}$.

These minimal localizing subcategories are tightly related to local duality, as in Theorem 2.28. In Example 2.30, we studied the local duality between p-torsion and p-complete abelian groups. By [BHV18, 2.26], we get from any local duality diagram a fracture square, which for the local duality context $(D(\mathbb{Z})_{(p)}, \mathbb{Z}_{(p)}/p)$ gives the classical arithmetic fracture square

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \mathbb{Z}_p
\end{tikzcd}$$

In particular, this decomposes the unit $\mathbb{Z}_{(p)}$ into a rational part and a p-complete part. This also extends to a general chain complex $A \in D(\mathbb{Z})_{(p)}$, where we have a homotopy pullback square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_p^\wedge \arrow[d] \\
\mathbb{Q} \otimes A \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} A_p^\wedge
\end{tikzcd}$$

where $(-)_p^\wedge$ denotes derived p-completion as in Remark 3.18. We can then wonder whether these also give our minimal localizing subcategories, which is indeed the case.

Proposition 4.4. *Let $\mathcal{L}$ be a minimal localizing subcategory of $D(\mathbb{Z})_{(p)}$. Then either $\mathcal{L} \simeq D(\mathbb{Q})$ or $\mathcal{L}$ is the category of derived p-complete objects, $\mathcal{L} \simeq D(\mathbb{Z})_p^\wedge$.*

Now, if $\mathrm{Sp}_{(p)}$ is supposed to be a homotopical enrichment, we should expect there to be an analogy of this decomposition for p-local spectra, which is indeed the case. The first to study such squares in topology was Sullivan in his 1970 MIT notes, where he constructed the analogous square for nilpotent spaces, see [Sul05, 3.20]. This was later lifted up to spectra by Bousfield in [Bou79b, 2.9], and takes the following form.

If $\mathbb{S}_{(p)}$ denotes the p-local sphere spectrum, we have a spectral artithmetic fracture square

$$\begin{tikzcd}
\mathbb{S}_{(p)} \arrow[r] \arrow[d] & \mathbb{S}_p^\wedge \arrow[d] \\
H\mathbb{Q} \arrow[r] & H\mathbb{Q} \otimes \mathbb{S}_p^\wedge
\end{tikzcd}$$

where $\mathbb{S}_p^\wedge$ denotes the p-complete sphere. This also extends to any object $X \in \mathrm{Sp}_{(p)}$, just like for $A \in D(\mathbb{Z})_{(p)}$.

We can then ask the same natural question as we did above: do these give all the minimal localizing subcategories of $\mathrm{Sp}_{(p)}$? Recall that this was indeed the case before, but now, this is no longer true. In fact, we now have an infinite sequence of minimal localizing subcategories, indexed by a natural number n, interpolating between the rational spectra $\mathrm{Sp}_{\mathbb{Q}}$ and the p-complete spectra $\mathrm{Sp}_p^\wedge$. [1]

We can identify these "intermediary" subcategories by an analysis of field objects. For $D(\mathbb{Z})_{(p)}$ there are exactly two field objects associated to $\mathbb{Z}_{(p)}$, namely $\mathbb{Q}$ and $\mathbb{F}_p$. For $Sp_{(p)}$ we have a field object for any number $n \in \mathbb{N} \cup \{\infty\}$, usually denoted $K(n)$, or $K_p(n)$ if we want to remember the prime. As we have $K(0) = H\mathbb{Q}$ and $H(\infty) = H\mathbb{F}_p$, this sequence of field objects really forms an interpolation between the two field objects coming from algebra.

Notation 4.5. The object $K_p(n)$ is called the height n Morava K-theory. Its associated minimal localizing subcategory is the Bousfield localization $\mathrm{Sp}_{K_p(n)}$.

These field objects $K_p(n)$ were constructed by Morava in the early 70's, and the categories $\mathrm{Sp}_{K_p(n)}$ have been under intense study ever since. We do not cover precise constructions here and instead refer the interested reader to [HS99].

Proposition 4.6. *Let p be a prime and n a natural number. The height n Morava K-theory spectrum $K_p(n)$ is a complex oriented $\mathbb{E}_1$-ring spectrum with coefficients*

$$K_p(n)_* := \pi_* K_p(n) \simeq \mathbb{F}_p[v_n^{\pm}],$$

with $|v_n| = 2p^n - 2$, whose associated formal group is the height n Honda formal group. Furthermore, for any two spectra $X, Y \in \mathrm{Sp}$, there is a Künneth isomorphism

$$K_p(n)_*(X \times Y) \simeq K_p(n)_* X \otimes_{K_p(n)_*} K_p(n)_* Y.$$

Remark 4.7. While the $\mathbb{E}_1$-ring structure on $K_p(n)$ can be shown to be essentially unique, it does admit uncountably many $\mathbb{E}_1$-MU-algebra structures – see [Ang11].

So, how are these new field objects related to the fracture squares above? If the $\mathrm{Sp}_{K_p(n)}$'s form minimal localizing subcategories, then we should, by the previous discussion, expect there to be an infinite sequence of pullback squares converging to $\mathbb{S}_{(p)}$. This is indeed the case.

Let $L_n := L_{K_p(0) \vee \cdots \vee K_p(n)}$. By Ravenel's smash product theorem, see [Rav92, 7.5.6], the functor $L_n \colon \mathrm{Sp}_{(p)} \longrightarrow \mathrm{Sp}_{(p)}$ is a smashing localization (Definition 2.16), hence the relevant fracture squares for the two bottom cases $n = 0$ and $n = 1$ are given by

$$\begin{tikzcd} L_1\mathbb{S} \arrow[r] \arrow[d] & L_{K(1)}\mathbb{S} \arrow[d] \\ H\mathbb{Q} \arrow[r] & H\mathbb{Q} \otimes L_{K(1)}\mathbb{S} \end{tikzcd} \qquad \begin{tikzcd} L_2\mathbb{S} \arrow[r] \arrow[d] & L_{K(2)}\mathbb{S} \arrow[d] \\ L_1\mathbb{S} \arrow[r] & L_1\mathbb{S} \otimes L_{K(2)}\mathbb{S} \end{tikzcd}$$

[1] In fact even more is true: By [Bur+23], there are at least two such infinite sequences. We can make sure that there is a single such sequence if we translate over to tensor-triangulated ideals of compact objects, but for the above exposition, we have chosen to push these details under a huge telescope-shaped rug.

making the general square have the form

$$\begin{array}{ccc} L_n\mathbb{S} & \longrightarrow & L_{K_p(n)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{n-1}\mathbb{S} & \longrightarrow & L_{n-1}\mathbb{S}\otimes L_{K_p(n)}\mathbb{S} \end{array}$$

This is called the chromatic fracture square, see for example [Hov95, 4.3]. The spectra $L_n\mathbb{S}$ assemble into a tower

$$\cdots \longrightarrow L_3\mathbb{S} \longrightarrow L_2\mathbb{S} \longrightarrow L_1\mathbb{S} \longrightarrow L_0\mathbb{S} = L_{\mathbb{Q}}\mathbb{S}$$

called the chromatic filtration, and by the chromatic convergence theorem of Hopkins-Ravenel, see [Rav92, 7.5.7], we can recover $\mathbb{S}_{(p)}$ as the limit of this diagram.

Remark 4.8. Reducing to the subcategory of $\mathrm{Sp}_{(p)}$ containing the L_n-local spectra, we should then expect there to be a local duality diagram categorifying the chromatic fracture square. This is precisely the goal of Section 4.3, but first, we need to understand this L_n-local category.

4.2 Morava E-theories

In the previous section, we obtained a localization functor L_n, which collected the information coming from height 0 up to, and including, height n. This localization is good for many purposes, but when we later want to tie the homotopy theory to algebra, we need another approach. In particular, we want a spectrum E such that localizing at E is the same as using L_n, but with some additional better properties. There are several approaches to obtaining such a spectrum E, and the goal of this short section is to give a brief overview of the ones we will need later. We will assume general knowledge about formal groups – all needed background can be found in [Rav86, Appendix 2].

Remark 4.9. Let p be a prime and k be a perfect field of characteristic p. Lubin and Tate proved in [LT66] that for any formal group law F of height n over k, there is a universal deformation $\bar{F}$ over the Lubin-Tate ring $E(k,F) = \mathbb{W}(k)[\![u_1,\dots,u_{n-1}]\!]$ of formal power series over the Witt vectors of k. Using the algebraic geometry of formal groups, Morava interpreted this universal deformation as a normal bundle over a formal neighborhood of the height n Honda formal group law, leading to a spectrum E_n^{Mor}.

Using the theory of manifolds with singularities developed by Baas-Sullivan (see [Baa73a] and [Baa73b]), Johnson and Wilson constructed in [JW75] an alternative spectrum exhibiting the same information as Morava's spectrum. Using Landweber's exact functor theorem, we can obtain a simpler description.

Definition 4.10. *Let p be a prime, n a natural number and $E(n)_* := \mathbb{Z}_{(p)}[v_1,\dots,v_{n-1},v_n^{\pm}]$. The ideal $(p,v_1,\dots,v_{n-1})$ is a regular invariant ideal, meaning in particular that $E(n)_*$ is Landweber exact. In particular, there is a spectrum $E(n)$, called the height n Johnson-Wilson theory, with coefficients $E(n)_*$.*

Remark 4.11. The construction of $E(n)$ has the added benefit that quotienting by the maximal ideal $I_n = (p, v_1, \dots, v_{n-1})$ gives $E(n)_*/I_n \cong \mathbb{F}_p[v_n^{\pm}] = K_p(n)_*$. This can also be suitably interpreted as a quotient of spectra.

Definition 4.12. *An $\mathbb{E}_1$-ring spectrum R is said to be concentrated in degrees divisible by q if $\pi_k R \cong 0$ for all $k \neq 0 \bmod q$.*

Proposition 4.13. *Let p be a prime and n a natural number. Height n Johnson-Wilson theory $E(n)$ is a complex oriented, Landweber exact, $\mathbb{E}_1$-ring spectrum concentrated in degrees divisible by $2p-2$.*

Later, using a 2-periodic analogue of the universal deformation theory of Lubin and Tate, Hopkins and Miller constructed a 2-periodic $\mathbb{E}_1$-version of Morava's spectrum, which was later enhanced to an $\mathbb{E}_\infty$-ring spectrum E_n via Goerss–Hopkins theory, see [GH04] or [PV22] for a modern treatment. In essence, Hopkins–Miller constructed a functor from pairs (k, F) of a perfect field k of characteristic p, together with a choice of height n formal group law F, to even periodic ring spectra. For a specific choice of (k, F), we can summarize the properties as follows.

Proposition 4.14. *Let p be a prime, k a perfect field of characteristic p, and F a formal group law of height n over k. The spectrum $E(k, F)$ is a 2-periodic, complex oriented, Landweber exact, $\mathbb{E}_\infty$-ring spectrum, such that $\pi_0 E(k, F) = \mathbb{W}(k)[\![u_1, \dots, u_{n-1}]\!]$ and the associated formal group law is the universal deformation of F.*

Definition 4.15. *For the specific choice $(k, F) = (\mathbb{F}_{p^n}, H_n)$ we simply write $E(\mathbb{F}_{p^n}, H_n) = E_n$, and call it the height n Morava E-theory.*

Remark 4.16. One can also study maps of ring spectra $E_n \longrightarrow K_n$ such that the induced map on homotopy groups is given by taking the quotient by the maximal ideal, just as in Remark 4.11. Such spectra K_n are 2-periodic versions of Morava K-theory and have been studied, for example, in [HL17] and [BP23].

Remark 4.17. One nice benefit with E_n over $E(n)$ is that the former is $K(n)$-local, making its chromatic behavior even more interesting. In fact, the unit map $L_{K_p(n)}\mathbb{S} \longrightarrow E_n$ is a pro-Galois extension in the sense of [Rog08], where the Galois group is the extended Morava stabilizer group $\mathbb{G}_n$, see [devinatz hopkins 2004]. We can, however, fix this by instead using a completed version $\widehat{E}(n)$, often called completed Johnson-Wilson theory. It has most of the same properties as that of $E(n)$, except that it is $K_p(n)$-local and its coefficients are p-adic and I_n-complete: $\widehat{E}(n)_* \simeq \mathbb{Z}_p[v_1, \cdots, v_{n-1}, v_n^{\pm}]^{\wedge}_{I_n}$.

Remark 4.18. An $\mathbb{E}_\infty$-version of Morava's original spectrum E_n^{Mor} can be recovered from E_n by taking the homotopy fixed points with respect to the Galois action $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$. Another alternative is to use $E_n^{h\mathbb{F}_p^\times}$. This spectrum is concentrated in degrees divisible by $2p-2$, hence serves as a nice $\mathbb{E}_\infty$-version of the $\mathbb{E}_1$-ring spectrum $E(n)$. This is the model of E used, for example, in Barkan's monoidal algebraicity theory, see [Bar23].

We have now introduced several versions of E-theory, all in light of trying to understand the localization functor L_n. Hence, we round off this section by stating that the Bousfield localizations at any of the above E-theories are equivalent.

Proposition 4.19 ([Hov95, 1.12]). *Let p be a prime and n a natural number. Then there are symmetric monoidal equivalences of stable ∞-categories*

$$\mathrm{Sp}_n \simeq \mathrm{Sp}_{E(n)} \simeq \mathrm{Sp}_{E(k,F)} \simeq \mathrm{Sp}_{E_n} \simeq \mathrm{Sp}_{\widehat{E}(n)} \simeq \mathrm{Sp}_{E_n^{h\mathbb{F}_p^\times}}.$$

In fact, if E is any Landweber exact v_n-periodic spectrum, then Sp_E is equivalent to the above categories.

Notation 4.20. We will use the common notation $\mathrm{Sp}_{n,p}$ for any of the above categories.

Remark 4.21. Note that even though the different models for $\mathrm{Sp}_{n,p}$ are equivalent, some of them have non-equivalent associated module categories. For example, $\mathrm{Mod}_{E_n} \not\simeq \mathrm{Mod}_{E(n)}$, as the ring spectra E_n and $E(n)$ have different periodicity – the former is 2-periodic while the latter is $(2p^n - 2)$-periodic. Whenever such a distinction is relevant, we will make this explicit.

4.3 Monochromatic spectra and local duality

Recall from Section 4.1 that our goal is to understand the $K_p(n)$-local pieces of the category of p-local spectra, $\mathrm{Sp}_{(p)}$. By Remark 4.8, we are looking for a local duality theory that categorifies the chromatic fracture square. In this section, we construct precisely such a local duality theory, both for $\mathrm{Sp}_{n,p}$ and for modules over E for some choice of E-theory.

Definition 4.22. *A spectrum X is called n-monochromatic if it is E_n-local and E_{n-1}-acyclic. The full subcategory of n-monochromatic spectra will be denoted $\mathcal{M}_{n,p}$ and referred to as the height n monochromatic category.*

If the height is understood, we will sometimes drop the n from the notation. We have a convenient way to produce monochromatic spectra from E_n-local ones.

Definition 4.23. *Let $X \in \mathrm{Sp}_{n,p}$. The fiber of the localization $X \longrightarrow L_{n-1}X$, which we denote M_nX is called the n'th monochromatic layer of X.*

Remark 4.24. If X is a monochromatic spectrum, then it is L_{n-1}-local by definition, i.e., $L_{n-1}X \simeq 0$. Hence the fiber sequence

$$M_nX \longrightarrow X \longrightarrow L_{n-1}X$$

gives an equivalence $X \simeq M_nX$. The fully faithful inclusion $\mathcal{M}_{n,p} \longrightarrow \mathrm{Sp}_{n,p}$ has a right adjoint, given by $X \longmapsto M_nX$, which we call the monochromatization.

Proposition 4.25. *The monochromatization functor $M_n \colon \mathrm{Sp}_{n,p} \longrightarrow \mathcal{M}_{n,p}$ is a smashing colocalization.*

Proof. As far as the authors are aware, this proposition was first proved in [Bou96, Sec 6.3] in the case of finite monochromatization, i.e., the fiber functor of the finite localization L_n^f. The proof, however, uses the arguments from [Bou79a, 2.10], which also work for the non-finite case. An even simpler argument uses the before-mentioned smash product theorem, which states that the localization $L_{n-1} = L_{E_{n-1}}$ is smashing. Hence, we can compare the two fiber sequences

$$M_n\mathbb{S} \otimes X \longrightarrow X \longrightarrow L_{n-1}\mathbb{S} \otimes X \quad \text{and} \quad M_nX \longrightarrow X \longrightarrow L_{n-1}X,$$

which immediately identifies the fibers. □

We are now almost ready to construct local duality for chromatic homotopy theory. The last thing we need is the notion of a type n complex.

Definition 4.26. *A compact p-local spectrum X is said to be of type n if $K_p(n)_*X \ncong 0$ and $K_p(m)_*X \cong 0$ for all $m < n$.*

As a consequence of the thick subcategory theorem of Hopkins–Smith, [HS98, Theorem 7], such spectra exist for all primes p and natural numbers n. For example, if $n = 1$, we can choose the mod p Moore spectrum $\mathbb{S}/p$.

Construction 4.27. Let n be a non-negative integer and p a prime. For a type n spectrum $F(n)$ the L_n-localization $K = L_nF(n)$ is a compact object in $\mathrm{Sp}_{n,p}$ and hence generates a localizing tensor ideal $\mathrm{Sp}_{n,p}^{K-tors}$ in $\mathrm{Sp}_{n,p}$. By Theorem 2.28, we have a corresponding local duality diagram for the local duality context $(\mathrm{Sp}_{n,p}, K)$:

$$\begin{tikzcd}
 & \mathrm{Sp}_{n,p}^{K-loc} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Sp}_{n,p} \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathrm{Sp}_{n,p}^{K-tors} \arrow[ur, shift left] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & \mathrm{Sp}_{n,p}^{K-comp} \arrow[ul, shift left]
\end{tikzcd}$$

Even though these categories arise abstractly from the local duality process, we can luckily recognize them as familiar categories we have already encountered.

Proposition 4.28. *There are symmetric monoidal equivalences of stable ∞-categories*

$$(1)\ \mathrm{Sp}_{n,p}^{K-tors} \simeq \mathcal{M}_{n,p}, \quad (2)\ \mathrm{Sp}_{n,p}^{K-loc} \simeq Sp_{n-1,p} \quad \text{and} \quad (3)\ \mathrm{Sp}_{n,p}^{K-comp} \simeq Sp_{K_p(n)}.$$

These equivalences are classical, but we recall their arguments for the reader's convenience and for building intuition.

Proof. By definition $\mathcal{M}_{n,p}$ is the full subcategory of L_{n-1}-acyclics in $\mathrm{Sp}_{n,p}$ and M_n coincides with the L_{n-1}-acyclification. By [HS99, 6.10] L_{n-1}-localization is the finite localization away from $K = L_nF(n)$, which proves equivalence (2). This also means that the L_{n-1}-acyclics are precicely the objects in $\mathrm{Loc}^{\otimes}_{\mathrm{Sp}_{n,p}}(K)$, which by definition is $\mathrm{Sp}_{n,p}^{K-tors}$. This gives the equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{K-tors}$ and $\Gamma \simeq M_n$, which proves (1). One can also see this by the fact that M_n preserves compact objects, as it is smashing by Proposition 4.25, which also implies that $\mathcal{M}_{n,p}$ is closed under colimits. The compact objects $L_nX \in \mathrm{Sp}_{n,p}$ for X any finite spectrum of type $\geqslant n$ are also monochromatic, as $E_{n-1*}L_nX \cong E_{n-1*}X \cong 0$, and do in fact generate $\mathcal{M}_{n,p}$ under colimits.

The equivalence in (3) follows from [BHV18, 2.34], which shows that Λ can be identified with the Bousfield localization L_K whenever the set of compact objects in a local duality context $(\mathcal{C}, \mathcal{K})$ consists of a single element $\mathcal{K} = \{K\}$. Note that this localization L_K is not the same as the functor L, which we earlier denoted by $L_{\mathcal{K}}$. Since Bousfield localizations are symmetric monoidal, this proves (3). ☐

Remark 4.29. The equivalence $\mathrm{Sp}_{n,p}^{K-tors} \xrightarrow{\simeq} \mathrm{Sp}_{n,p}^{K-comp}$ is then given by the adjoint pair $(L_{K_p(n)} \dashv M_n)$, which recovers the symmetric monoidal equivalence $\mathcal{M}_{n,p} \simeq Sp_{K_p(n)}$ of [HS99, 6.19].

Remark 4.30. The local duality diagram from Construction 4.27 gives via [BHV18, 2.26] precisely the chromatic fracture square, as wanted in Remark 4.8.

Remark 4.31. By Remark 2.25, all the categories in the above diagram are compactly generated. But, the unit $L_{K_p(n)}\mathbb{S}$ in $\mathrm{Sp}_{K_p(n)}$ is not compact, so by Remark 2.7 the compact objects and the dualizables might differ. The same is then necessarily true for $\mathcal{M}_{n,p}$.

We have a similar construction for the case of modules over E_n, which we will need in Section 6.

Construction 4.32. Let n be a non-negative integer, p a prime, and $E = E_n$ the height n Morava E-theory at the prime p. The object E/I_n is compact in Mod_E and generates a localizing tensor ideal $\mathrm{Mod}_E^{I_n-tors}$. By Theorem 2.28, we have a corresponding local duality diagram for the local duality context $(\mathrm{Mod}_E, E/I_n)$:

$$\begin{tikzcd}
 & \mathrm{Mod}_E^{I_n-loc} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Mod}_E \arrow[u, shift left, "L"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "\Lambda"] & \\
\mathrm{Mod}_E^{I_n-tors} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{Mod}_E^{I_n-comp} \arrow[ul, shift left]
\end{tikzcd}$$

Just as in Construction 4.27 there are equivalences

$$\mathrm{Mod}_E^{K-tors} \simeq \mathcal{M}_n\mathrm{Mod}_E \tag{2}$$

$$\mathrm{Mod}_E^{K-loc} \simeq L_{n-1}\mathrm{Mod}_E \tag{3}$$

$$\mathrm{Mod}_E^{K-comp} \simeq L_{K_p(n)}\mathrm{Mod}_E \tag{4}$$

where (2) is the full subcategory of monochromatic E-modules, (3) is the full subcategory of E_{n-1}-local E-modules and (4) is the full subcategory of $K_p(n)$-local E-modules.

4.4 Hopf algebroids revisited

In Section 3, we surveyed some results on Hopf algebroids and their associated derived categories and local duality theories, but we did not give any examples. In this short subsection, we relay some constructions and facts about Hopf algebroids coming from homotopy theory, which will serve as the connection between the world of homotopy theory and the world of algebra.

Construction 4.33. Let R be a ring spectrum. Associated to R we have an R-homology functor defined by $R_*(-) := \pi_*(R \otimes -)$. We denote $R_* := R_*(\mathbb{S})$ and $R_*R := R_*(R)$, which we for now assume are both commutative (graded) rings. From the unit map $\mathbb{S} \longrightarrow R$, the multiplication map $\mu\colon R \otimes R \longrightarrow R$ and the twist map $\tau\colon R \otimes R \longrightarrow R \otimes R$ we get maps on R_*-homology

1. $\eta_L\colon R_* \longrightarrow R_*R$, from the identification $R \otimes \mathbb{S} \simeq R$
2. $\eta_R\colon R_* \longrightarrow R_*R$, from the identification $\mathbb{S} \otimes R \simeq R$
3. $\varepsilon\colon R_*R \longrightarrow R$, from μ
4. $c\colon R_*R \longrightarrow R_*R$, from τ
5. $R_*(R \otimes R) \longrightarrow R_*R$, from μ

We have a comparison map $R_*R \otimes_{R_*} R_*R \longrightarrow R_*(R \otimes R)$, which is an isomorphism in nice cases – for example, if R_*R is a flat module over R_*. If this is the case we can extend the map $R_*R \longrightarrow R_*(R \otimes R)$ through the above isomorphism to get a coassociative comultiplication $\Delta\colon R_*R \longrightarrow R_*R \otimes_{R_*} R_*R$ as well as a multiplication map $\nabla\colon R_*R \otimes_{R_*} R_*R \longrightarrow R_*R$ from the fifth map in the above list. The relations on ring spectra also induce relations on the pair (R_*, R_*R), like coassociativity, counitality, and the antipode relation.

Remark 4.34. If R_* is a field object, for example, $K_p(n)$ or $H\mathbb{F}_p$, then the operations described above, together with the associated relations, make (R_*, R_*R) into a Hopf algebra. In particular, the left and right unit maps are equal: $\eta_L = \eta_R$.

Definition 4.35. *A ring spectrum R is called flat if R_*R is a flat module over R_*. We say R is of Adams type if it can be written as a filtered colimit $R \simeq \operatorname{colim}_\alpha R_\alpha$, where each R_α is a finite spectrum such that R_*R_α is a finitely generated projective R_*-module and the natural map $R^*R_\alpha \longrightarrow \operatorname{Hom}_{R_*}(R_*R_\alpha, R_*)$ is an isomorphism.*

In particular, all Adams type ring spectra are flat, as the filtered colimit $R \simeq \operatorname{colim}_\alpha R_\alpha$ gives a filtered colimit $R_*R \cong \operatorname{colim}_\alpha R_*R_\alpha$ of projective objects.

Most of the following examples were given by Adams in [Ada95, III.13.4], except for $K_p(n)$, which was not discovered yet.

Example 4.36 ([Hov04, 1.4.7, 1.4.9]). The ring spectra MU, MSp, KO, $H\mathbb{F}_p$, $K_p(n)$, $E(n)$, E_n are all of Adams type. We also have the following class of examples: if R is Adams type, then any Landweber exact R-algebra is also Adams type.

Recall the definition of an Adams Hopf algebroid in Definition 3.7. The following proposition is standard – see, for example, [Hov04, 1.4.6].

Proposition 4.37. *Let R be a flat ring spectrum such that R_*R is a commutative ring. Then, the pair (R_*, R_*R) is a Hopf algebroid. If R is Adams, then (R_*, R_*R) is an Adams Hopf algebroid.*

The following proposition is what allows us to translate the homotopy theoretical information from Sp into an algebraic setting.

Proposition 4.38. *Let R be an Adams type ring spectrum. Then the functor $R_*(-)$ takes values in the Grothendieck abelian category $Comod_{R_*R}$. In particular, given any spectrum X, then R_*X has a coassociative and counital coaction $R_*R \longrightarrow R_*X \otimes_{R_*} R_*R$.*

Remark 4.39. We don't need the Adams type condition in order for R_*X to be a comodule, but in this case, $Comod_{R_*R}$ is not Grothendieck.

In Section 4.2 we developed serveral versions of E-theory, and by Proposition 4.19 all the corresponding E-local categories are equivalent. The same occurs for the categories of comodules associated to the Adams Hopf algebroid (E_*, E_*E).

Proposition 4.40 ([HS05a, 4.2])**.** *Let p be a prime and n a positive natural number. Then the categories of comodules over the Hopf algebroids associated to E_n, $E(n)$ and $A = E_n^{h\mathbb{F}_p^\times}$ are equivalent:*

$$\mathrm{Comod}_{E_{n*}E_n} \simeq \mathrm{Comod}_{E(n)_*E(n)} \simeq \mathrm{Comod}_{A_*A}.$$

Notation 4.41. We will use the common notation Comod_{E_*E} for any of the above categories.

5 Exotic algebraic models

We now have two sets of local duality diagrams, one coming from chromatic homotopy theory and one from the homological algebra of Adams Hopf algebroids. We also have a way to pass between them by using Proposition 4.38. In particular, if we let $E = E_n$ be height n Morava E-theory at a prime p, then we have the E-homology functor $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ converting between homotopy theory and algebra. We can, in some sense, say that E_* approximates homotopical information by algebraic information.

The goal of this section is to set up an abstract framework for studying how good such approximations are. The version we recall below was developed in [PP21], taking inspiration from [Fra96] and [Pst23].

5.1 Adapted homology theories

Recall from Definition 3.37 that a locally graded category is a category $\mathcal{D}$ together with a choice of autoequivalence $[1]\colon \mathcal{D} \to \mathcal{D}$. All stable ∞-categories are locally graded by the suspension, and the categories of graded modules (comodules) over a graded ring (Hopf algebroid) are locally graded by the shift functor $(M[1])_k = M_{k-1}$.

Definition 5.1. *Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category and $\mathcal{A}$ an abelian category with a local grading* [1]. *A functor $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is called a homology theory if:*

1. *H is additive*
2. *for a cofiber sequence $X \to Y \to Z$ in $\mathcal{C}$, then $HX \to HY \to HZ$ is exact in $\mathcal{A}$*
3. *there is a natural isomorphism $H(\Sigma X) \equiv (HX)[1]$ for any $X \in \mathcal{C}$.*

Remark 5.2. The first two axioms make H a homological functor, while the last makes H into a locally graded functor, i.e., a functor that preserves the local grading.

Example 5.3. Let R be a ring spectrum. Then the functor $\pi_*\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R_*}$ defined as $\pi_* M = [\mathbb{S}, M]_*$ is a homology theory.

Example 5.4. Let R be a ring spectrum. The functor $R_*(-)\colon \mathrm{Sp} \longrightarrow \mathrm{Mod}_R$, defined as the composition

$$\mathrm{Sp} \xrightarrow{R\otimes(-)} \mathrm{Mod}_R \xrightarrow{\pi_*} \mathrm{Mod}_{R_*},$$

is a homology theory. If R is of Adams type, then $R_*(-)$ naturally lands in the subcategory Comod_{R_*R} by Proposition 4.38.

Definition 5.5. *A homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is conservative if it reflects isomorphisms, meaning that if a map $C \xrightarrow{f} D$ gives an isomorphism $HC \xrightarrow{Hf} HD$ if and only if f is an equivalence.*

Example 5.6. The functor $\pi_*\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_{R_*}$ is always a conservative homology theory. The functor R_* is not conservative in general. It can, however, be made conservative by restricting to the R-local objects Sp_R instead of the whole category Sp.

Remark 5.7. Reflecting isomorphisms is a very important property to have, as it allows us to check equivalences using algebraic tools, which are usually simpler than homotopical ones. For this reason, we will only be working with the above "local" version of R_*-homology, i.e. the restricted functor $R_*\colon \mathrm{Sp}_R \longrightarrow \mathrm{Mod}R_*$.

Definition 5.8. *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a homology theory and J an injective object in $\mathcal{A}$. An object $\bar{J} \in \mathcal{C}$ is said to be an injective lift of J if it represents the functor*

$$\mathrm{Hom}_{\mathcal{A}}(H(-), J)\colon \mathcal{C}^{op} \longrightarrow \mathcal{A}b$$

in the homotopy category $h\mathcal{C}$, i.e. $\mathrm{Hom}_{\mathcal{A}}(H(-), J) \cong [-, \bar{J}]$. We call $\bar{J}$ a faithful lift if the map $H(\bar{J}) \longrightarrow J$ coming from the identity on $\bar{J}$ is an equivalence.

Definition 5.9. *A homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is said to be adapted if $\mathcal{A}$ has enough injectives, and for any injective $J \in \mathcal{A}$ there is a faithful lift $\bar{J} \in \mathcal{C}$.*

Example 5.10. We again return to our two guiding examples π_* and R_*. The former is an adapted homology theory, with faithful lifts provided by Brown representability. The latter also has injective lifts by Brown representability, but they are not faithful in general, as can be seen by the following argument.

Let $R = H\mathbb{F}_p$, which gives the standard mod p homology $H\mathbb{F}_{p*} = H(-;\mathbb{F}_p)\colon \mathrm{Sp} \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}$. Standard mod p singular cohomology is represented by $H\mathbb{F}_p$, which, together with the universal coefficient theorem, gives equivalences

$$[X, H\mathbb{F}_p] \simeq H^*(X;\mathbb{F}_p) \simeq \mathrm{Hom}_{\mathbb{F}_p}(H_*(X;\mathbb{F}_p), \mathbb{F}_p).$$

Now, the one-dimensional vector space $\mathbb{F}_p$ is an injective object, and the above equivalences show that $H\mathbb{F}_p$ is an injective lift of $\mathbb{F}_p$. But, there is not an equivalence between $H\mathbb{F}_{p*}H\mathbb{F}_p$ and $\mathbb{F}_p$, as the former is equivalent to the mod p dual Steenrod algebra $\mathcal{A}_p$.

But, as noted in Proposition 4.38, we should think of R_*-homology as landing in Comod_{R_*R}. By passing to this more restricted subcategory with more structure, we get by Proposition 3.11 that $\mathbb{F}_p$ is no longer injective, and reducing the image of $H_*(-;\mathbb{F}_p)$ to $\mathrm{Comod}_{\mathcal{A}_p}$ ensures that any injective lift is faithful. In fact, by [PP21, 3.27, 3.28] something more general is true.

Proposition 5.11. *Let R be an Adams-type ring spectrum and $R_*\colon \mathrm{Sp}_R \longrightarrow \mathrm{Mod}_{R_*}$ the associated homology theory. Then, there is an essentially unique factorization*

$$\mathrm{Sp}_R \xrightarrow{\bar{R}_*} \mathrm{Comod}_{R_*R} \xrightarrow{U} \mathrm{Mod}_{R_*}$$

such that $\bar{R}_*$ is adapted (in particular, having faithful injective lifts), and U is an exact functor.

Notation 5.12. We will denote $R_*\colon \mathrm{Sp}_R \longrightarrow \mathrm{Comod}_{R_*R}$ for the conservative adapted homology theory associated with an Adams-type ring spectrum R.

Remark 5.13. The definition of an adapted homology theory H states that for any injective $J \in \mathcal{A}$, there is some object $\bar{J} \in \mathcal{C}$ together with an equivalence $[X, \bar{J}] \simeq \mathrm{Hom}_{\mathcal{A}}(HX, J)$. Because $\mathcal{A}$ has enough injective objects, we can use these equivalences to approximate homotopy classes of maps by repeatedly mapping into injective envelopes. This gives precisely an associated Adams spectral sequence for the homology theory H. In fact, Patchkoria and Pstrągowski proved that there is a bijection between adapted homology theories and Adams spectral sequences, see [PP21, 3.24, 3.25]. The constructioin of the Adams spectral sequence associated to an adapted homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is given in [PP21, 2.24], or alternatively as a totalization spectral sequence in [PP21, 2.27].

In certain situations, which in particular apply to us, this H-Adams spectral sequence converges and has a description in simple terms. This holds, for example, in the cases where the abelian category $\mathcal{A}$ has a finite cohomological dimension.

Definition 5.14. *Let $\mathcal{A}$ be a locally graded abelian category with enough injectives. Then the cohomological dimension of $\mathcal{A}$ is the smallest integer d such that $\mathrm{Ext}^{s,t}_{\mathcal{A}}(-,-) = 0$ for all $s > d$.*

Lemma 5.15 ([PP21, 2.24, 2.25])**.** *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be an adapted homology theory such that $\mathcal{A}$ has a finite cohomological dimension. The associated H-Adams spectral sequence converges and has the signature*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(HX, HY) \implies [X, Y]_*.$$

Example 5.16. Let n be an integer, p a prime such that $p > n+1$ and $E_{n,p}$ the associated Morava E-theory. Then by [Pst21, 2.5] the category Comod_{E_*E} has cohomological dimension n^2+n.

Remark 5.17. The H-Adams spectral sequence exists and could also converge in examples where $\mathcal{A}$ is not of finite cohomological dimension. But, we will only be interested in cases where it does, hence a more restricted Lemma 5.15.

Construction 5.18. Let R be an Adams-type ring spectrum such that π_*R is concentrated in degrees divisible by some positive number $q+1$, i.e., $\pi_m R = 0$ for all $m \neq 0 \bmod q+1$. Any comodule M in the category Comod_{R_*R} splits uniquely into a direct sum of subcomodules $\bigoplus_{\phi \in \mathbb{Z}/q+1} M_\phi$ such that M_ϕ is concentrated in degrees divisible by ϕ. Such a splitting induces a decomposition of the full subcategory of injective objects

$$\mathrm{Comod}^{inj}_{R_*R} \simeq \mathrm{Comod}^{inj}_{R_*R,0} \times \mathrm{Comod}^{inj}_{R_*R,1} \times \cdots \times \mathrm{Comod}^{inj}_{R_*R,q}$$

where the category $\mathrm{Comod}^{inj}_{R_*R,\phi}$ denotes the full subcategory spanned by injective comodules concentrated in degrees divisible by ϕ.

Let $h_k\mathcal{C}$ denote the homotopy k-category of $\mathcal{C}$, obtained by $k+1$-truncating all the mapping spaces in $\mathcal{C}$. The lift associated with each injective via the Adapted homology theory R_* allows us to construct a partial inverse to R_*, called the Bousfield functor β^{inj} in [PP21]. It is a functor $\beta^{inj}\colon \mathrm{Comod}^{inj}_{R_*R} \longrightarrow h_{q+1}\mathrm{Sp}^{inj}_R$, where the latter category is the homotopy $(q+1)$-category of the full subcategory of Sp_R containing all spectra X such that R_*X is injective and $[X,Y] \to \mathrm{Hom}_{R_*R}(R_*X, R_*Y)$ is a bijection for all $Y \in \mathrm{Sp}_R$.

In order to mimic this behavior for a general adapted homology theory, Franke introduced the notion of a splitting of an abelian category.

Definition 5.19 ([Fra96])**.** *Let $\mathcal{A}$ be an abelian category with a local grading* $[1]$. *A splitting of $\mathcal{A}$ of order $q+1$ is a collection of Serre subcategories $\mathcal{A}_\phi \subseteq \mathcal{A}$ indexed by $\phi \in \mathbb{Z}/(q+1)$ satisfying*

1. $[k]\mathcal{A}_n \subseteq \mathcal{A}_{n+k \bmod (q+1)}$ *for any $k \in \mathbb{Z}$, and*
2. *the functor $\prod_\phi \mathcal{A}_\phi \longrightarrow \mathcal{A}$, defined by $(a_\phi) \mapsto \oplus_\phi a_\phi$, is an equivalence of categories.*

Example 5.20. As we saw above in Construction 5.18, the category of comodules over an Adams Hopf algebroid (R_*, R_*R), where R_* is concentrated in degrees divisible by $q+1$, has a splitting of order $q+1$. This, then, also holds for the discrete Hopf algebroid (R_*, R_*), giving the module category Mod_{R_*} a splitting of order $q+1$ as well.

Example 5.21. In the case $R = E(1)$ this has been written out in detail in [BR11, Section 4]. The Serre subcategories are all copies of the category of p-local abelian groups together with Adams operations ψ^k for $k \neq 0$ in $\mathbb{Z}_{(p)}$. The shift leaves the underlying module unchanged, but changes the Adams operation.

Notation 5.22. We will say that objects $A \in \mathcal{A}_\phi$ are of pure weight ϕ.

Remark 5.23. Just as for Comod_{R_*R}, a splitting of order $q+1$ of a locally graded abelian category $\mathcal{A}$ is enough to define, for any adapted homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$, a partial inverse Bousfield functor β^{inj}.

5.2 Exotic homology theories

In order to make some statements about exotic equivalences a bit simpler to write down and read, we introduce the concept of exotic adapted homology theories. Note that this is not the way similar results are phrased in [PP21], but the notation serves as a shorthand for the criteria that they use.

Definition 5.24. *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be an adapted homology theory. We say H is k-exotic if H is conservative, $\mathcal{A}$ has finite cohomological dimension d and a splitting of order $q+1$ such that $k = d+1-q > 0$.*

Remark 5.25. By Lemma 5.15, the H-Adams spectral sequence associated with any k-exotic homology theory is automatically convergent.

The remarkable thing about a k-exotic homology theory $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ is that it forces the stable ∞-category $\mathcal{C}$ to be approximately algebraic. Intuitively: As the order of the splitting is greater than the cohomological dimension, the H-Adams spectral sequence is very sparse and well-behaved. There is a partial inverse of H via the Bousfield functor $\beta\colon \mathcal{A}^{inj} \to h_k\mathcal{C}^{inj}$, which forces a certain subcategory of a categorified deformation of H to be equivalent to both $h_k\mathcal{C}$ and $h_k D^{per}(\mathcal{A})$.

Theorem 5.26 ([PP21, 7.56])**.** *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a k-exotic homology theory. Then there is an equivalence of homotopy k-categories $h_k\mathcal{C} \simeq h_k D^{per}(\mathcal{A})$.*

There are several interesting examples of homology theories satisfying Theorem 5.26, see Section 8 in [PP21]. We highlight again our two guiding examples but focus specifically on certain Morava E-theories, see Section 4.2.

Example 5.27 ([PP21, 8.7]). Let p be a prime, n be a non-negative integer, and E a height n Morava E-theory concentrated in degrees divisible by $2p-2$. If $k = 2p-2-n > 0$, then the functor $\pi_*\colon \mathrm{Mod}_E \longrightarrow \mathrm{Mod}_{E_*}$ is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Mod}_E \simeq h_k D^{per}\mathrm{Mod}_{E_*}.$$

Notation 5.28. For the following example and the rest of the paper, we follow the notation of [BSS20], [BSS21] and [Bar23] and denote the category $D^{per}(\mathrm{Comod}_{E_*E})$ by $\mathrm{Fr}_{n,p}$.

Example 5.29 ([PP21, 8.13]). Let p be a prime, n be a non-negative integer, and E any height n Morava E-theory. If $k = 2p-2-n^2-n > 0$, then the functor $E_n\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Sp}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}.$$

Remark 5.30. As noted in [BSS20, 5.29], this equivalence is strictly exotic for all $n \geqslant 1$ and primes p. In other words, it can never be made into an equivalence of stable ∞-categories. In particular, the mapping spectra in $\mathrm{Fr}_{n,p}$ are $H\mathbb{Z}$-linear, while the mapping spectra in $\mathrm{Sp}_{n,p}$ are only $H\mathbb{Z}$-linear for $n = 0$.

Definition 5.31. *Let $H\colon \mathcal{C} \longrightarrow \mathcal{A}$ be a k-exotic homology theory. The category $D^{per}(\mathcal{A})$ is called an exotic algebraic model of $\mathcal{C}$ if the equivalence $h_k\mathcal{C} \simeq h_k D^{per}(\mathcal{A})$ can not be enhanced to an equivalence of ∞-categories $\mathcal{C} \simeq D^{per}(\mathcal{A})$.*

Remark 5.32. The existence of an exotic algebraic model for a stable ∞-category $\mathcal{C}$ implies that the category is not rigid. See [BRW23] for a great exposé on different levels of algebraicity for stable ∞-categories. This means, in particular, that there cannot exist a k-exotic homology theory with source Sp or $\mathrm{Sp}_{(p)}$ as these are all rigid for all primes, see [Sch07], [SS02] and [Sch01]. The same holds for $\mathrm{Sp}_{1,2}$, as this is rigid by [Roi07], and similarily for $\mathrm{Sp}_{K_2(1)}$ by [Ish19]. This shows that being k-exotic is quite a strong requirement.

6 Algebraicity for monochromatic modules

We now turn to proving our two main results, namely that monochromatic homotopy theory is algebraic at large primes. In this section, we prove this for modules, and in Section 7, we will prove it for all monochromatic spectra. In essence, by Example 5.27 we know that the homotopy groups functor $\pi_*\colon \mathrm{Mod}_E \longrightarrow \mathrm{Mod}_{E_*}$ is a good approximation when the prime p is large compared to the height n, and the goal of this section is to prove that it remains a good approximation when we restrict to monochromatic E-modules as in Construction 4.32.

We prove this in three steps, which are essentially just checking that the functor π_* restricted to monochromatic modules is k-exotic (Definition 5.24) for a given value of k, i.e., that it is a conservative adapted homology theory, that we have finite cohomological dimension and a splitting.

For the rest of this section, we assume that E is a version of height n Morava E-theory at the prime p that is concentrated in degrees divisible by $2p-2$, for example, $E(n)$ or $E_n^{h\mathbb{F}_p^\times}$.

The following lemma is the I_n-torsion version [BF15, 3.14], and the proof is similar.

Lemma 6.1. *Let p be a prime and n a natural number. Then an E-module M is monochromatic if and only if π_*M is I_n-torsion.*

Proof. Let $X \in \mathrm{Mod}_E^{I_n-tors}$. By [BHV18, 3.19] there is a strongly convergent spectral sequence of $E(n)_*$-modules with signature

$$E_2^{s,t} = (H_{I_n}^{-s}\pi_*X)_t \implies \pi_{s+t}M_nX,$$

where $H_{I_n}^{-s}$ denotes local cohomology. By [BS12, 2.1.3(ii)] the E_2-page consist of only I_n-power torsion modules. As $\mathrm{Mod}_{E_*}^{I_n-tors}$ is abelian, it is closed under quotients and subobjects, as as the higher pages are created from the E_2-page using quotients and subobjects, they must also consist of only I_n-power torsion modules. In particular, the E_∞-page is all I_n-power torsion. By Grothendieck's vanishing theorem, see for example [BS12, 6.1.2], $H^s_{I_n}(-) = 0$ for $s > n$, hence the abutment of the spectral sequence $\pi_* M_n X$ is a finite fintration of I_n-power torsion E_*-modules, and is therefore itself an I_n-power torsion module. Since X was assumed to be monochromatic, i.e. $X \in \mathrm{Mod}_E^{I_n-tors}$, we have $\pi_* M_n X \cong \pi_* X$, and thus $\pi_* X \in \mathrm{Mod}_{E_*}^{I_n-tors}$.

Assume now $X \in \mathrm{Mod}_E$ such that its homotopy groups are I_n-power torsion. Monochromatization gives a map $\phi\colon M_n X \longrightarrow X$, and as $\pi_* M_n X$ is I_n-power torsion this map factors on homotopy groups as

$$\pi_* M_n X \longrightarrow H^0_{I_n} \pi_* X \longrightarrow \pi_* X,$$

where the first map is the edge morphism in the above-mentioned spectral sequence. As $\pi_* X$ was assumed to be I_n-power torsion we have $\pi_* X \cong H^0_{I_n} \pi_* X$, and $H^s_{I_n} \pi_* X \cong 0$ for $s > 0$. Hence the spectral sequence collapses to give the isomorphism $\pi_* M_n X \cong H^0_{I_n} \pi_* X$, which shows that $\pi_* \phi$ is an isomorphism. As π_* is conservative ϕ was already an isomorphism, hence $X \in \mathrm{Mod}_E^{I_n-tors}$. ∎

Lemma 6.2. *Let p be a prime and n a natural number. Then the functor*

$$\pi_*\colon \mathrm{Mod}_E^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$$

is a conservative adapted homology theory.

Proof. We first note that the functor $\pi_*\colon \mathrm{Mod}_E \longrightarrow \mathrm{Mod}_{E_*}$ is a conservative adapted homology theory. By Lemma 6.1 its restriction to $\mathrm{Mod}_E^{I_n-tors}$ lands in $\mathrm{Mod}_{E_*}^{I_n-tors}$, hence autmoatically $\pi_*\colon \mathrm{Mod}_E^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$ is a conservative homology theory.

Let J be an injective I_n-power torsion E-module. By Lemma 3.26 and Remark 3.27 we can assume $J = T^{E_*}_{I_n} Q$ for an injective E-module Q. Since π_* is adapted on Mod_E we can chose a faithful injective lift $\bar{J}$ of J to Mod_E, and since $\bar{J}$ was assumed to have I_n-torsion homotopy groups we know by Lemma 6.1 that $\bar{J} \in \mathrm{Mod}_E^{I_n-tors}$. In particular, we have faithful lifts for any injective in $\mathrm{Mod}_{E_*}^{I_n-tors}$, which means that $\pi_*\colon \mathrm{Mod}_E^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$ is adapted. ∎

Lemma 6.3. *Let p be a prime and n a natural number. Then the category $\mathrm{Mod}_{E_*}^{I_n-tors}$ has cohomological dimension n.*

Proof. Note first that the category Mod_{E_*} has cohomological dimension n. By Lemma 3.26 and Remark 3.27, this implies that the cohomological dimension of $\mathrm{Mod}_{E_*}^{I_n-tors}$ can't be greater than n, so it remains to prove that it is exactly n. We prove this by computing an $\mathrm{Ext}^n_{E_*}$ group that is non-zero.

By [HS99, A.2(d)] we have $L_0 M \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), M)$ for any E_*module M. In other words, this states that the derived completion of an E_*-module is the n'th derived functor of maps from the I_n-local cohomology of E_* into M. Choosing $M = E_*/I_n$ we get

$$L_0(E_*/I_n) \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), E_*/I_n).$$

As any bounded I_n-torsion E_*-module is I_n-adically complete we have, as remarked in [BH16, 1.4], that $L_0(E_*/I_n) \cong E_*/I_n$. The local cohomology of E_* is also I_n-torsion, in particular $H^n_{I_n} E_* = E_*/I_n^\infty$. Hence we have

$$\mathrm{Ext}^n_{E_*}(E_*/I_n^\infty, E_*/I_n) \cong E_*/I_n \ncong 0,$$

showing that there are two I_n-power torsion E_*-modules with non-trivial n'th Ext, which concludes the proof. ☐

Lemma 6.4. *Let p be a prime and n a natural number. Then, the category $\mathrm{Mod}_{E_*}^{I_n-tors}$ has a splitting of order $2p-2$.*

Proof. By [PP21, 8.1] the category Mod_{E_*} has a splitting of order $2p-2$. We will use this to induce a splitting on $\mathrm{Mod}_{E_*}^{I_n-tors}$. In particular, we define the pure weight ϕ component of $\mathrm{Mod}_{E_*}^{I_n-tors}$, denoted $\mathrm{Mod}_{E_*,\phi}^{I_n-tors}$, to be the essential image of $T_{I_n}^{E_*} \colon \mathrm{Mod}_{E_*} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$ restricted to the pure weight ϕ component $\mathrm{Mod}_{E_*,\phi}$. We claim that this defines a splitting of order $2p-2$ on $\mathrm{Mod}_{E_*}^{I_n-tors}$. For the claim to be true, we need to check the axioms in Definition 5.19: (1) that the pure weight components are Serre subcategories, (2) that they are shift-invariant and (3) that they form a decomposition of $\mathrm{Mod}_{E_*}^{I_n-tors}$.

The first point, (1), follows from the fact that $\mathrm{Mod}_{E_*,\phi}$ is a Serre subcategory, and being I_n-power torsion is a property closed under subobjects, quotients, and extensions. Hence also $\mathrm{Mod}_{E_*,\phi}^{I_n-tors}$ is a Serre subcategory.

For (2), we note that we have a diagram of adjoint functors

$$\begin{tikzcd}
\mathrm{Mod}_{E_*} \arrow[r, shift left, "{[1]}"] \arrow[d, shift left, "T_{I_n}^{E_*}"] & \mathrm{Mod}_{E_*} \arrow[l, shift left] \arrow[d, shift left, "T_{I_n}^{E_*}"] \\
\mathrm{Mod}_{E_*}^{I_n-tors} \arrow[r, shift left, "{[1]}"] \arrow[u, shift left, "i"] & \mathrm{Mod}_{E_*}^{I_n-tors} \arrow[l, shift left] \arrow[u, shift left, "i"]
\end{tikzcd}$$

which is commutative from bottom left to top right. Here $[1]$ denotes the local grading on $\mathrm{Mod}_{E_*}^{I_n-tors}$. We want the diagram to commute from top left to bottom right, which can be obtained by the dual Beck-Chevalley condition. This reduces to checking $[-1] \circ i \simeq i \circ [-1]$, which is true due to the commutativity and the fact that $[1]$ and $[-1]$ are autoequivalences. Hence we have $[1] \circ T_{I_n}^{E_*} \simeq T_{I-n}^{E_*} \circ [1]$. In fact, the diagram is commutative in all possible directions. This means that for any I_n-power torsion E_*-module M of pure weight ϕ, we have

$$[k]M \cong [k]T_{I_n}^{E_*} M \cong T_{I_n}^{E_*}[k]N \in \mathrm{Mod}_{E_*,\phi+k \bmod 2p-2}^{I_n-tors}$$

as $[k]M \in \mathrm{Mod}_{E_*,\phi+k \bmod 2p-2}$.

For the final point (3), note that any subcategory of a product category is a product of subcategories. Hence, the $\mathrm{Mod}_{E_*}^{I_n-tors}$ splits as a product of the pure weight components. In particular, the functor

$$\prod_{\phi \in \mathbb{Z}/(2p-2)} \mathrm{Mod}_{E_*,\phi}^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$$

defined by $(M_\phi) \longmapsto \bigoplus_\phi M_\phi$ is an equivalence of categories. ☐

Remark 6.5. This is the part where it was important we chose a version of Morava E-theory that is concentrated in degrees divisible by $2p-2$. If we instead chose a 2-periodic E-theory, for example E_n, then neither Mod_{E_*} nor $\mathrm{Mod}_{E_*}^{I_n-tors}$ would have a splitting of the above degree.

We can now summarize the above discussion with the first of our main results.

Theorem 6.6 (Theorem C)**.** *Let p be a prime, n a natural number, and E a version of height n Morava E-theory concentrated in degrees divisible by $2p-2$. If $k = 2p-2-n > 0$, then the functor*

$$\pi_* \colon \mathrm{Mod}_E^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$$

is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Mod}_E^{I_n-tors} \simeq h_k D^{per}(\mathrm{Mod}_{E_*}^{I_n-tors}).$$

In particular, monochromatic E-modules are exotically algebraic at large primes.

Proof. By Lemma 6.3 the cohomological dimension of $\mathrm{Mod}_{E_*}^{I_n-tors}$ is n, and by Lemma 6.4 we have a splitting on $\mathrm{Mod}_{E_*}^{I_n-tors}$ of order $2p-2$. Hence, by Lemma 6.2 the functor

$$\pi_* \colon \mathrm{Mod}_E^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}$$

is a k-exotic homology theory for $k = 2p-2-n > 0$, which gives an equivalence

$$h_k\mathrm{Mod}_E^{I_n-tors} \simeq h_k D^{per}(\mathrm{Mod}_{E_*}^{I_n-tors})$$

by Theorem 5.26. ☐

We can also phrase this dually in terms of $K_p(n)$-local E-modules.

Corollary 6.7. *Let p be a prime, n a positive integer. Let further $K_p(n)$ be the height n Morava K-theory at the prime p and E be a height n Morava E-theory at p concentrated in degrees divisible by $2p-2$. If $k = 2p-2-n > 0$, then we have a k-exotic algebraic equivalence*

$$h_k L_{K_p(n)}\mathrm{Mod}_E \simeq h_k D^{per}(\mathrm{Mod}_{E_*})^{I_n-comp}.$$

In particular, $K_p(n)$-local E-modules are exotically algebraic at large primes.

Proof. The equivalence is constructed from the equivalences obtained from Construction 4.32, Theorem 6.6, Theorem 3.49 and Construction 3.47. In particular, we have

$$\begin{aligned} h_k\mathrm{Mod}_E^{I_n-comp} &\overset{4.32}{\simeq} h_k\mathrm{Mod}_E^{I_n-tors} \\ &\overset{6.6}{\simeq} h_k D^{per}(\mathrm{Mod}_{E_*}^{I_n-tors}) \\ &\overset{3.49}{\simeq} h_k D^{per}(\mathrm{Mod}_{E_*})^{I_n-tors} \\ &\overset{3.47}{\simeq} h_k D^{per}(\mathrm{Mod}_{E_*})^{I_n-comp}, \end{aligned}$$

where we have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. $\square$

Now, let HE_* be the Eilenberg-MacLane spectrum of E_*. By [Lur17, 7.1.1.16] there is a symmetric monoidal equivalence $D(E_*) \simeq \mathrm{Mod}_{HE_*}$ and we can form a local duality diagram for Mod_{HE_*} corresponding to Construction 3.32 for the discrete Hopf algebroid (E_*, E_*). By arguments similar to Lemma 6.1 and Lemma 6.2 one can show that the homotopy groups functor $\pi_*\colon \mathrm{Mod}_{HE_*} \longrightarrow \mathrm{Mod}_E$ restricts to a conservative adapted homology theory

$$\pi_* \mathrm{Mod}_{HE_*}^{I_n-tors} \longrightarrow \mathrm{Mod}_{E_*}^{I_n-tors}.$$

In the same range as Theorem 6.6 this is then automatically also k-exotic. We can then combine the algebraicity for $\mathrm{Mod}_E^{I_n-tors}$ and Mod_{HE_*} to get the following statement.

Corollary 6.8. *Let $k = 2p-2-n > 0$. Then, there is an exotic equivalence*

$$h_k \mathrm{Mod}_E^{I_n-tors} \simeq h_k \mathrm{Mod}_{HE_*}^{I_n-tors}.$$

7 Algebraicity for monochromatic spectra

Having proven that monochromatic E-modules are algebraic at large primes, we now turn to the larger category of all monochromatic spectra $\mathcal{M}_{n,p}$ with the same goal. The strategy is exactly the same as in Section 6: we first prove that the conservative adapted homology theory $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ of Example 5.29, restricts to a conservative adapted homology theory on $\mathcal{M}_{n,p}$, before proving that $\mathrm{Mod}_{E_*}^{I_n-tors}$ has a splitting and finite cohomological dimension, making sure that we have a k-exotic homology theory.

The interesting thing about this bigger case is that E_* is k-exotic in a better range for $\mathcal{M}_{n,p}$ compared to $\mathrm{Sp}_{n,p}$. For Theorem 6.6, the range stayed the same, which shows that E-modules form a much simpler theory than all of $\mathrm{Sp}_{n,p}$ – which is, of course, to be expected.

Lemma 7.1. *Let $X \in \mathrm{Sp}$. Then $X \in \mathcal{M}_{n,p}$ if and only if $E_*X \in \mathrm{Comod}_{E_*E}^{I_n-tors}$.*

Proof. Assume first that $X \in \mathcal{M}_{n,p}$. We have $E \otimes X \in \mathrm{Mod}_E^{I_n-tors}$ as

$$E \otimes X \simeq E \otimes M_n X \simeq M_n E \otimes X,$$

where the last equivalence follows from M_n being smashing. In particular, the restricted functor $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ factors through $\mathrm{Mod}_E^{I_n-tors}$. By Lemma 6.1 and Lemma 3.25 this means that E_*X is an I_n-power torsion E_*E-comodule.

For the converse, assume that we have $X \in \mathrm{Sp}_{n,p}$ such that $E_*X \in \mathrm{Comod}_{E_*E}^{I_n-tors}$. Using the monochroimatization functor we obtain a comparison map $M_n X \longrightarrow X$, which induces a map on E-modules $E \otimes M_n X \longrightarrow E \otimes X$. This map is an isomorphism on homotopy groups, as E_*X was assumed to be I_n-power torsion. As E_* is conservative on $\mathrm{Sp}_{n,p}$, the original comparison map $M_n X \longrightarrow X$ was an isomorphism, meaning that $X \in \mathcal{M}_{n,p}$. $\square$

Lemma 7.2. *Let p be a prime and n a natural number. Then, the functor*

$$E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-tors}$$

is a conservative adapted homology theory.

Proof. First note that the image of the functor $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ restricted to $\mathcal{M}_{n,p}$ is contained in $\mathrm{Comod}_{E_*E}^{I_n-tors}$ by Lemma 7.1. The functor $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-tors}$ is then automatically a conservative homology theory. The category $\mathrm{Comod}_{E_*E}^{I_n-tors}$ has enough injectives by Lemma 3.26(2). Hence, it only remains to prove that we have faithful lifts for all injective objects.

Let J be an injective in $\mathrm{Comod}_{E_*E}^{I_n-tors}$. Following [BHV20, 3.16] we can by Lemma 3.26(3) and Remark 3.27 assume that $J = T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q)$ for some injective E_*-module Q. Since being torsion is a property of the underlying module, and the forgetful functor ε_* is conservative, we have an isomorphism $T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q) \cong E_*E \otimes_{E_*} T_{I_n}^{E_*} Q$. By Lemma 3.26(1) the functor $T_{I_n}^{E_*}$ preserves injectives, hence J is also injective in Comod_{E_*E}.

Now, E_* has faithful injective lifts from Comod_{E_*E} to $\mathrm{Sp}_{n,p}$ by Proposition 5.11, hence there is a lift $\bar{J}$ such that $[X, \bar{J}] \simeq \mathrm{Hom}_{E_*E}(E_*X, J)$ and $E_*\bar{J} \simeq J$. By Lemma 7.1 $\bar{J} \in \mathcal{M}_{n,p}$ as J was assumed to be I_n-power torsion, hence we have found our faithful injective lift. $\square$

Lemma 7.3. *Let p be a prime and n a natural number such that $p > n + 1$. Then the category $\mathrm{Comod}_{E_*E}^{I_n-tors}$ has cohomological dimension n^2.*

Proof. The proof follows [Pst21, 2.5] closely, which is itself a modern reformulation of [Fra96, 3.4.3.9]. We start by defining good targets to be I_n-power torsion comodules N such that $\mathrm{Ext}_{E_*E}^{s,t}(E_*/I_n, N) = 0$ and good sources to be I_n-power torsion comodules M such that $\mathrm{Ext}_{E_*E}^{s,t}(M, N)$ for all I_n-torsion comodules N.

By the Landweber filtration theorem, see for example [HS99, 5.7], we know that any finitely presented comodule M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{s-1} \subseteq M_s = M,$$

where $M_r/M_{r-1} \cong E_*/I_{j_r}[t_r]$ and $j_r \leqslant n$. When M is I_n-power torsion we get $j_r = n$ for all r, as noted in [HS99, 4.3]. By Morava's vanishing theorem, see for example [Rav86, 6.2.10], we have $\mathrm{Ext}_{E_*E}^{s,t}(E_*, E_*/I_n) = 0$ for all $s > n^2$, which by the long exact sequence in Ext-groups also give $\mathrm{Ext}_{E_*E}^{s,t}(E_*/I_n, E_*/I_n) = 0$ for $s > n^2$. By using the Landweber filtration, this implies that any finitely presented I_n-power torsion comodule is a good target. By Lemma 3.29 any I_n-power torsion comodule is a filtered colimit of finitely presented ones, and as $\mathrm{Ext}_{E_*E}^{s,t}(E_*/I_n, -)$ commutes with colimits this implies that any I_n-power torsion comodule is a good target.

Note that the above argument also proves that E_*/I_n is a good source, which by the Landweber filtration argument implies that any finitely presented I_n-torsion comodule is a good source.

By Lemma 3.29, the category $\mathrm{Comod}_{E_*E}^{I_n-tors}$ is generated under filtered colimits by finitely presented ones. Hence, we can apply [Pst21, 2.4] to any injective resolution

$$0 \longrightarrow M \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots$$

to get that the map $J_{n^2} \longrightarrow \mathrm{Im}(J_{n^2} \to J_{n^2+1})$ is a split surjection, and that the object $\mathrm{Im}(J_{n^2} \to J_{n^2+1})$ is injective. Hence, any injective resolution can be modified to have length n^2, which concludes the proof. ☐

Remark 7.4. This is where we obtain the better range compared to $\mathrm{Sp}_{n,p}$, as the non-torsion category Comod_{E_*E} has cohomological dimension n^2+n, as seen in Example 5.16.

Remark 7.5. When $p < n+1$, the category Comod_{E_*E} is not of finite cohomological dimension. By Lemma 3.26(1) the functor $T_{I_n}^{E_*E}\colon \mathrm{Comod}_{E_*E} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-tors}$ preserves injectives, which means that also $\mathrm{Comod}_{E_*E}^{I_n-tors}$ does not have finite cohomological dimension.

Lemma 7.6. *Let p be a prime, n a natural number, and E any height n Morava E-theory. Then, the category $\mathrm{Comod}_{E_*E}^{I_n-tors}$ has a splitting of order $2p-2$.*

Proof. All of the height n Morava E-theories give equivalent categories of comodules, see Proposition 4.40. Hence we can chose a version concentrated in degrees divisible by $2p-2$. The category Comod_{E_*E} has a splitting of order $2p-2$ by [PP21, 8.13]. The proof of the induced splitting on the I_n-torsion category is then identical to Lemma 6.4. ☐

We can now summarize the above results with our second main result, which is the monochromatic analogue of Example 5.29.

Theorem 7.7 (Theorem B)**.** *Let p be a prime, n a natural number, and E any height n Morava E-theory. If $k = 2p-2-n^2 > 0$, then the restricted functor $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-tors}$ is k-exotic. In particular, there is an equivalence*

$$h_k\mathcal{M}_{n,p} \simeq h_k D^{per}(E_*E^{I_n-tors}),$$

meaning that monochromatic homotopy theory is exotically algebraic at large primes.

Proof. By Lemma 7.3, the cohomological dimension of $\mathrm{Comod}_{E_*E}^{I_n-tors}$ is n^2 and by Lemma 7.6 we have a splitting of order $2p-2$. The restricted functor E_* is then by Lemma 7.2 k-exotic whenever $k = 2p-2-n^2 > 0$, which by Theorem 5.26 finishes the proof. ☐

Remark 7.8. Note that the unrestricted functor $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ is not k-exotic unless $k > n$, hence Theorem 7.7 is stronger than just restricting Example 5.29 to the relevant subcategories.

Remark 7.9. By Theorem 3.49 there is an equivalence $D^{per}(E_*E^{I_n-tors}) \simeq \mathrm{Fr}_{n,p}^{I_n-tors}$ and by Proposition 4.28 there is an equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n-tors}$, so the equivalence in Theorem 7.7 can alternatively be written as

$$h_k\mathrm{Sp}_{n,p}^{I_n-tors} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-tors}$$

for $k = 2p-2-n^2 > 0$. This is more in line with thinking about Theorem 7.7 as "coming from" the chromatic algebraicity of Example 5.29 on localizing ideals, except with a better bound on k. This formulation is perhaps also easier to connect to the limiting case $p \to \infty$ as described using ultraproducts in [BSS21], which can be stated informally as

$$\lim_{p\to\infty} \mathrm{Sp}_{n,p}^{I_n-tors} \simeq \lim_{p\to\infty} \mathrm{Fr}_{n,p}^{I_n-tors}.$$

Via local duality (Theorem 2.28), we obtain the associated exotic algebraicity statement for the category of $K_p(n)$-local spectra.

Theorem 7.10 (Theorem A)**.** *Let p be a prime and n a natural number. Let further $K_p(n)$ be height n Morava K-theory at the prime p and E be any height n Morava E-theory at p. If $k = 2p-2-n^2 > 0$, then we have a k-exotic algebraic equivalence*

$$h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-comp}.$$

Proof. As we did in Corollary 6.7, we construct the equivalence from a sequence of equivalences coming from Theorem 2.28 and Theorem 7.7. More precisely we use equivalences coming from Construction 4.27, Theorem 7.7, Theorem 3.49 and Construction 3.47, which give

$$\begin{aligned} h_k\mathrm{Sp}_{K_p(n)} &\overset{4.27}{\simeq} h_k\mathcal{M}_{n,p} \\ &\overset{7.7}{\simeq} h_kD^{per}(\mathrm{Comod}_{E_*E}^{I_n-tors}) \\ &\overset{3.49}{\simeq} h_k\mathrm{Fr}_{n,p}^{I_n-tors} \\ &\overset{3.47}{\simeq} h_k\mathrm{Fr}_{n,p}^{I_n-comp}, \end{aligned}$$

where we again have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. □

Remark 7.11. By Proposition 4.28 we can also phrase this as $h_k\mathrm{Sp}_{n,p}^{I_n-comp} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-comp}$.

7.1 Some remarks on future work

The reason why Theorem 5.26 works so well, is that there is a deformation of stable ∞-categories lurking behind the scenes. One does not need this in order to apply the theorem, but it is there regardless. In the case of a Morava E-theory $E = E_n$, the deformation associated with the adapted homology theory $E_*\colon \mathrm{Sp}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}$ is equivalent to the category of hypercomplete E-based synthetic spectra, $\widehat{\mathrm{Syn}}_E$, introduced in [Pst23]. Our restricted homology theory $E_*\colon \mathcal{M}_{n,p} \longrightarrow \mathrm{Comod}_{E_*E}^{I_n-tors}$ should then be associated to a deformation $\widehat{\mathrm{Syn}}_E^{I_n-tors}$ coming from a local duality theory for $\widehat{\mathrm{Syn}}_E$, in the sense that there is a diagram of stable ∞-categories

$$\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n-tors} \xleftarrow{\tau^{-1}} \widehat{\mathrm{Syn}}_E^{I_n-tors} \xrightarrow{\tau\sim 0} \mathrm{Fr}_{n,p}^{I_n-tors}.$$

Since E_* is adapted on $\mathcal{M}_{n,p}$, we abstractly know that there is a deformation $D^\omega(\mathcal{M}_{n,p})$ arising out of the work of Patchkoria-Pstrągowski in [PP21], called the perfect derived category. This should give an equivalent "internal" approach to I_n-torsion synthetic spectra, much akin to how we have equivalences $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n-tors}$ and $D(E_*E)^{I_n-tors} \simeq D(E_*E^{I_n-tors})$.

In [Bar23], Barkan provides a monoidal version of Theorem 5.26 by using filtered spectra. His deformation $\mathscr{E}_{n,p}$ is equivalent to $\widehat{\mathrm{Syn}}_E$, which hints that there should be a monoidal version of Theorem 7.7 as well. We originally intended to incorporate such a result into this paper but decided against it in order to keep it deformation-theory-free. We do, however, state the conjectured monoidal result, which we will pursue in future work.

Conjecture 7.12. Let p be a prime and n a natural number. If k is a positive natural number such that $2p-2 > n^2+(k+3)n+k-2$, then we have an equivalence $h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-tors}$ of symmetric monoidal stable ∞-categorires.

By the monoidality of local duality, see Remark 2.29, this would give a similar statement for the $K_p(n)$-local category, i.e. a symmetric monoidal equivalence $h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\mathrm{Fr}_{n,p}^{I_n-comp}$.

Since E-based synthetic spectra are categorifications of the E-Adams spectral sequence, one should expect the above-mentioned local duality for $\widehat{\mathrm{Syn}}_E$ to give a category $\widehat{\mathrm{Syn}}_E^{I_n-comp}$, which categorifies the $K_p(n)$-local E-Adams spectral sequence. We plan to study such categorifications of the $K_p(n)$-local E-Adams spectral sequence in future work joint with Marius Nielsen.

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