Note Title 4/2/2010

Winfree: Sheaves on functors

Report on Strickland's paper
"Formal schemes + formal groups"

Def. $\mathfrak{X}$ = category of covariant functors
$$\text{Ring} \longrightarrow \text{Set}$$
comm with unit

=: Schemes

$\mathfrak{X}$ = cat of Representable functors =: affine schemes

Example: $\mathbb{A}^1 : \text{Ring} \to \text{Set}$ forgetful functor

$\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$

$\operatorname{Spec} \mathbb{Z}[t](R) = \operatorname{Ring}(\mathbb{Z}[t], R) = R$

Def For $X \in \mathfrak{X}$ we have $\mathcal{O}_X = \mathcal{F}(X, \mathbb{A}^1)$

Prop There is an equivalence

$$\mathfrak{X} \longleftrightarrow \operatorname{Ring}^{op}$$

$$X \longmapsto \mathcal{O}_X$$

$$\operatorname{Spec} A \longleftarrow A$$

Ex $\mathbb{G}_m \in \mathfrak{X}$ with $\mathbb{G}_m(R) = R^\times$

$$\mathcal{O}_{\mathbb{G}_m} = \mathbb{Z}[x^{\pm 1}]$$

m for multiplicative

Ex $FGL(R)$ = formal gp laws $/R$

$\mathcal{O}_{FGL} = L =$ Lazard ring

Quillen showed $\pi_* MU = L$

Using $\underline{\mathfrak{X}} \longleftrightarrow \mathrm{Ring}^{op}$ we produce limits and colimits

(1) limits in $\underline{\mathfrak{X}}$ (corr. to colimits in Ring)

* $X, Y \in \underline{\mathfrak{X}}$ then

$$(X \times Y)\, R = X(R) \times Y(R)$$

and $\mathcal{O}_{X \times Y} = \mathcal{O}_X \otimes \mathcal{O}_Y$

(Note: terminal object in $\underline{\mathfrak{X}}$ is $\mathrm{Spec}\,\mathbb{Z} = 1$)

* given $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$

$X, Y, Z \in \underline{\mathfrak{X}}$

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

PULLBACK

$$(X \times_Z Y)\, R = X(R) \otimes_{Z(R)} Y(R)$$

$$\mathcal{O}_{X \times_Z Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Y$$

Example Let $\mathrm{IPS}(R)$ = invertible (under comp) power series $/R$

$$f(x) = ux + \cdots \qquad u = \text{unit}.$$

$\mathrm{SIPS}(R)$ = the above with $u = 1$

$$\mathcal{O}_{\mathrm{IPS}} = \mathbb{Z}[b_0^{\pm 1}, b_1, b_2, \ldots]$$

$$\mathcal{O}_{\mathrm{SIPS}} = \mathbb{Z}[b_1, b_2, \ldots]$$

$$FI(R) = \{ (F, f, G) \mid F(x,y) = f^{-1}(G(f(x), f(y))) \}$$

f is iso between FGLs F and G

IPS and FIPS acts on FGL,

$$IPS \times FGL \longrightarrow FGL$$

$$(f, F) \longmapsto F_f \quad \text{where} \quad F_f(x,y) = f(F(f^{-1}(x), f^{-1}(y)))$$

Note that $FI = FGL \times IPS$

$$(F, f, F_f) \longleftarrow\!\!| \ (F, f)$$

so $SFI = FGL \times SIPS$

$$\mathcal{O}_{SFI} = MU_*(MU)$$

Colimits in $\overline{X}$ (corr to limits in Ring)

* Given $X, Y \in \overline{X}$ we have $X \amalg Y$ with

$$\mathcal{O}_{X \amalg Y} = \mathcal{O}_X \times \mathcal{O}_Y$$

$$X \amalg Y(R) = \left\{ (S, T, x, y) \;\middle|\; \begin{array}{l} S, T \leq R,\ R = S \times T \text{ and} \\ x \in X(S),\ y \in Y(T) \end{array} \right\}$$

$$= \overline{X}(\operatorname{Spec} R, X \amalg Y)$$

$$= \operatorname{Ring}(\mathcal{O}_X \times \mathcal{O}_Y, R)$$

<u>Example</u>: $\operatorname{Mult}(n)(R) = \left\{ \begin{array}{l} \phi(u) \in R[u] \text{ of deg} \leq n \\ \phi(1) = 1 \text{ and } \phi(uv) = \phi(u)\phi(v) \end{array} \right\}$

so $\phi(u) = \sum_{i=0}^{n} e_i u^i$ with

$\phi(1) = 1$ and $\phi(uv) = \phi(u)\phi(v)$

gives $e_i e_j = \delta_{ij} e_j$ so $\mathrm{Mult}(n) = \coprod_{i=0}^{n} 1$

Ex $\mathrm{Mult}(n)(X) = \left\{ \text{splittings of } X = \coprod_{0 \le i \le n} X_i \right\}$

<u>subschemes</u>

① Given an ideal $I \leq \mathcal{O}_X$ we define

$$V(I)(R) = \left\{ x \in X(R) \mid f(x) = 0 \quad \forall f \in I \right\}$$

Note that $f \in \mathbb{X}(X, \mathbb{A}^1)$ so f gives a map

$$X(R) \longrightarrow \mathbb{A}^1(R) = R.$$

These are the closed subschemes with

$$V(I) = \operatorname{spec} \mathcal{O}_X / I$$

(2) for $f \in \mathcal{O}_X$ we define

$$D(f)(R) = \{ x \in X(R) \mid f(x) \in R^\times \}$$

$$D(f) = \operatorname{spec} \mathcal{O}_X[1/f]$$

Points and Sheaves:

Def ① A *point* of X is an elt $x \in X(R)$ for some R.

② Define the category $\mathrm{Points}(X)$ as

objects $= \{ (R, x) : x \in X(R) \}$

morphisms are as follows

$(R, x) \xrightarrow{f} (S, y)$ is ring map $R \xrightarrow{f} S$

with $X(f)(x) = y$

Def A sheaf M over $X \in \mathcal{J}$ is given by the following data:

1) $\forall$ (R, x) in $\mathrm{Points}(X)$ we have an R-module M_x

2) For all morphisms f in $\mathrm{Points}(X)$
$$(R, x) \xrightarrow{f} (S, y)$$
there is an isomorphism of S-modules
$$\theta(f, x) : S \otimes_R M_x \longrightarrow M_y$$

satisfying

(1) $\theta(id, x) = id$

(2) for maps $(R,x) \xrightarrow{f} (S,y) \xrightarrow{g} (T,z)$

$$\begin{tikzcd} \theta(gf,x) = T\otimes_R M_x = T\otimes_S S\otimes_R M_x \arrow[d, "1\otimes\theta(f,x)"] \\ T\otimes_S M_y \arrow[d, "\theta(g,y)"] \\ M_z \end{tikzcd}$$

$\mathrm{Sheaves}_X$ = category of sheaves over X

Ex Let $\mathcal{O}$ be the sheaf over X given by

$$\mathcal{O}_x = R \quad \text{for } x \in X(R)$$

Ex $M, N \in \mathrm{Sheaves}_X$

$$(M \oplus N)_x = M_x \oplus N_x \quad \text{and} \quad (M \otimes N)_x = M_x \otimes_R N_x$$

Ex Define a sheaf N over $\mathbb{A}^1$ by $N_x = R/(x)$

We produce the equivalence

$$\mathrm{Sheaves}_X \longleftrightarrow \mathrm{Mod}_{\mathcal{O}_X} \quad \text{for } X \in \overline{\underline{X}}$$

The maps:

Def (1) $N \in \mathrm{Mod}_{\mathcal{O}_X}$ then define a sheaf $\tilde{N}$ over X

by $\tilde{N}_x = R \otimes_{\mathcal{O}_X} N$ for $x \in X(R)$

② Define for $M \in \text{Sheaves}_X$, $\mathbb{A}(M) \in \mathcal{F}_X$

$\mathcal{F}_X = \left\{ \begin{array}{l} \text{category of scheme } Y \text{ with map } Y \to X \\ \text{and morphisms with resp to those maps} \end{array} \right\}$

$\mathbb{A}(M)(R) = \coprod_{x \in X(R)} M_x$ and for $R \xrightarrow{f} S$

we get a map

$$\mathbb{A}(M)(R) \longrightarrow \mathbb{A}(M)(S)$$

$$M_x \longmapsto M_{f(x)}$$

$$m \longmapsto \theta(f,x)(1 \otimes m)$$

$$\mathbb{A}(M)(R) = \coprod_{x \in X(R)} M_x \longrightarrow X(R)$$

③ Define $\Gamma(X, M)$ an $\mathcal{O}_X$-module as

$$\Gamma(X, M) = \mathcal{F}_X(X, \mathbb{A}(M))$$

This gives a functor

$$\mathrm{Sheaves}_X \longrightarrow \mathrm{Mod}_{\mathcal{O}_X}$$

$$M \longmapsto \Gamma(X, M)$$

<u>Notes:</u>

① $\Gamma(X, M)$ is an $\mathcal{O}_X$-module. Note that $\mathbb{A}(\mathcal{O}) = \mathbb{A} \times X$

$$\mathbb{A}(\mathcal{O}) = \coprod_{x \in X(R)} R \quad \text{and} \quad (\mathbb{A} \times X)(R) = R \times X(R)$$

so $\Gamma(X, \mathcal{O}) = \mathfrak{F}(X, \mathbb{A} \times X)$

$$= \begin{tikzcd} X \arrow[rr, "1"] \arrow[dr, "id"'] & & \mathbb{A} \times X \arrow[dl, "proj"] \\ & X & \end{tikzcd}$$

$$= X \xrightarrow{f} \mathbb{A} = \mathcal{O}_X$$

We have

$\mathcal{O} \to \mathcal{O} \otimes M$ for $M \in$ sheaves$_X$

which gives $\Gamma(X, \mathcal{O}) \longrightarrow \Gamma(X, M)$

$$\Gamma(X,\mathcal{O}) = \mathcal{O}_X$$

② $\Gamma(X, M)$ can be viewed as the "sheaf" of sections of M.

$$
\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}"'] & \mathbb{A}(M) \arrow[d, "\mathrm{proj}"] \\
& X
\end{tikzcd}
$$