THE HOMOTOPY GROUPS OF *tmf* AND OF ITS LOCALIZATIONS

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In this small survey, we present a compilation of the homotopy groups of *tmf* and of its various localizations. This work should be thought of as an exercise in "collecting some of the diffuse knowledge" from my mathematical surroundings.

1. The homotopy of *tmf*

The spectrum *tmf* is connective, which means that the ring $\pi_n(tmf)$ is zero for $n < 0$. Vaguely speaking, its homotopy ring $\pi_*(tmf)$ is an amalgam of $MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta)$, the ring of classical modular forms, and the ring $\pi_*(\mathbb{S})$ of stable homotopy groups of spheres. Namely, there are ring homomorphisms

$$\pi_*(\mathbb{S}) \longrightarrow \pi_*(tmf) \longrightarrow MF_* \tag{1}$$

that we now describe. Both maps are surprisingly close to being isomorphisms (even though $\pi_*(\mathbb{S})$ and MF_* have nothing to do with each other).

The first map (1) is the Hurewitz homomorphism: being a ring spectrum, *tmf* admits a unit map from the sphere spectrum $\mathbb{S}$. This induces a map in homotopy $\pi_*(\mathbb{S}) \to \pi_*(tmf)$, which is an isomorphism on π_0. The only torsion in $\pi_*(tmf)$ 2-torsion and 3-torsion, and so its is only at those primes that *tmf* resembles the sphere spectrum. The 3-primary part of the Hurewitz image is 72-periodic and is given by

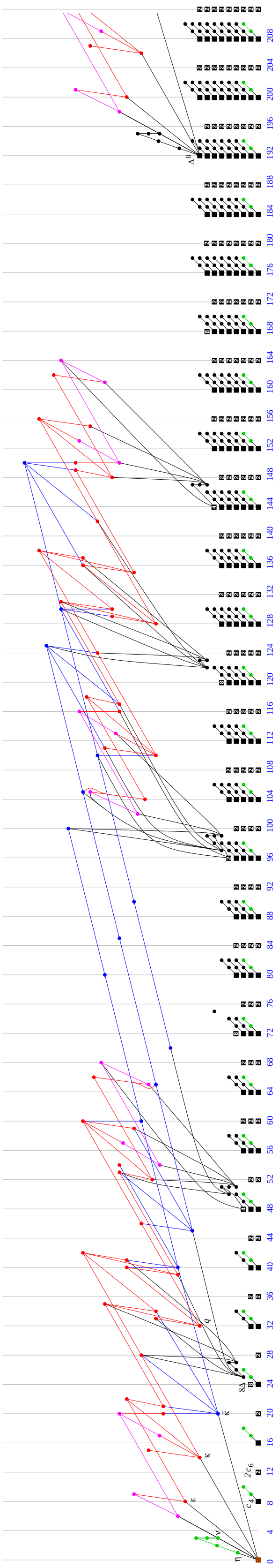
$$\operatorname{im}\Big(\pi_*(\mathbb{S}) \to \pi_*(tmf)\Big)_{(3)} = \mathbb{Z}_{(3)} \oplus \alpha\,\mathbb{Z}/3\mathbb{Z} \oplus \bigoplus_{k\geq 0} \Delta^{3k}\{\beta, \alpha\beta, \beta^2, \beta^3, \beta^4/\alpha, \beta^4\}\,\mathbb{Z}/3\mathbb{Z}, \tag{2}$$

where α has degree 3, and $\beta = \langle\alpha, \alpha, \alpha\rangle$ has degree 10. It doesn't contains all the 3-torsion of $\pi_*(tmf)$ as the classes in dimensions $27 + k\cdot 72$ and $75 + k\cdot 72$ for $k \geq 0$ are not hit by elements of $\pi_*(\mathbb{S})$. The 2-torsion of $\operatorname{im}(\pi_*(\mathbb{S}) \to \pi_*(tmf))$ is much more complicated. It exhibits very rich patterns including two distinct periodicity phenomena. The first one is a periodicity by $c_4 \in \pi_8(tmf)$ which corresponds to v_1^4, and the second one is a periodicity by $\Delta^8 \in \pi_{192}(tmf)$ which corresponds to v_2^{32}.

The second map (1) can be described as the boundary homomorphism of the elliptic spectral sequence. Under that map, a class in $\pi_n(tmf)$ maps to a modular form of weight $n/2$ (and maps to zero if n is odd). That map is an isomorphism after inverting the primes 2 and 3, which means that both its kernel and its cokernel are 2- and 3- torsion. The cokernel can be described explicitly

$$\operatorname{coker}\Big(\pi_n(tmf) \to MF_{\frac{n}{2}}\Big) = \begin{cases} \mathbb{Z}\Big/\frac{24}{\gcd(k,24)}\,\mathbb{Z}, & \text{if} \quad n = 24k \\ (\mathbb{Z}/2\mathbb{Z})^{\lceil \frac{n-8}{24} \rceil} & \text{if} \quad n \equiv 4 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Te first cyclic group is generated by Δ^k, while the second group is generated by $\Delta^a c_4^b c_6$ for integers a and b satisfying $24a+8b+12 = n$. The kernel agrees with the torsion in $\pi_*(tmf)$, and it is much more complicated since it comes from the stable homotopy groups of spheres. Its 3-primary component is at most $\mathbb{Z}/3\mathbb{Z}$ in any given degree. Its two primary component is a direct sum of $(\mathbb{Z}/2\mathbb{Z})^\ell$ for some ℓ (corresponding the v_1-periodic elements) with a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$ (corresponding the v_2-periodic elements).



The homotopy groups of *tmf* at the prime 2

The picture on the left of this page represents the homotopy ring of *tmf* at the prime 2. The vertical direction has no meaning. Bullets represent $\mathbb{Z}/2\mathbb{Z}$'s while squares represent $\mathbb{Z}_{(2)}$'s. A chain of n bullets connected by vertical lines represent a $\mathbb{Z}/2^n\mathbb{Z}$.

The bullets are named by the classes in $\pi_*(\mathbb{S})$ of which they are the image (η, ν, ε, $\bar\kappa$, κ, q are standard names), while the squares are named by their images in MF_*. The slanted lines represent multiplication by η, ν, ε, κ, and $\bar\kappa$.

The top part of the diagram is 192-periodic with polynomial generator Δ^8. We have colored the image of the Hurewitz homomorphism [conjectured by Mark Mahowald] as follows: the v_1-periodic classes are in green, and the v_2-periodic classes are in pink, red, and blue, depending on their periodicity. The green classes are v_1^4-periodic in the sphere, and, except for ν, they remain periodic in *tmf* via the identification $c_4 = v_1^4$. The v_2^8-periodic classes are pink, the v_2^{16}-periodic red, and the v_2^{32}-periodic blue. They remain periodic in *tmf* via the identification $\Delta^8 = v_2^{32}$.

The white numbers written in the squares indicate the size of $\mathrm{coker}(\pi_*(tmf) \to MF_*)_{(2)}$. The $\mathbb{Z}_{(2)}$-algebra $\pi_*(tmf)_{(2)}$ is finitely generated, with generators

degree:	1	3	8	8	12	14	20	24	25	27	32	32
name:	η	ν	c_4	ε	$\{2c_6\}$	κ	$\bar\kappa$	$\{8\Delta\}$	$\{\eta\Delta\}$	$\{2\nu\Delta\}$	q	$\{c_4\Delta\}$

36	48	51	56	60	72	80	84
$\{2c_6\Delta\}$	$\{4\Delta^2\}$	$\{\nu\Delta^2\}$	$\{c_4\Delta^2\}$	$\{2c_6\Delta^2\}$	$\{8\Delta^3\}$	$\{c_4\Delta^3\}$	$\{2c_6\Delta^3\}$
96	97	99	104	104	108	110	120
$\{2\Delta^4\}$	$\{\eta\Delta^4\}$	$\{\nu\Delta^4\}$	$\{\varepsilon\Delta^4\}$	$\{c_4\Delta^4\}$	$\{2c_6\Delta^3\}$	$\{\kappa\Delta^4\}$	$\{8\Delta^5\}$
123	128	128	132	144	147	152	156
$\{\nu\Delta^5\}$	$\{q\Delta^4\}$	$\{c_4\Delta^5\}$	$\{2c_6\Delta^5\}$	$\{4\Delta^6\}$	$\{\nu\Delta^6\}$	$\{c_4\Delta^6\}$	$\{2c_6\Delta^6\}$
168	176	180	192				
$\{8\Delta^7\}$	$\{c_4\Delta^7\}$	$\{2c_6\Delta^7\}$	$\{\Delta^8\}$				

and numerous relations. Hereafter, we list the multiplication that are not indicated our chart. On the top line are the generators, and on the bottom one are the degrees of the classes that support non-trivial multiplications by those generators:

ε	κ	$\bar\kappa$	q
1	1, 26.	1, 2, 8, 14, 15, 21, 22, 26, 98, 22, 32, 33, 34, 39, 46, 104, 110, 111, 116, 117, 118, 128, 129, 130, 135, 136, 142.	1, 3, 8, 14, 20, 21, 25, 27, 28, 34, 97, 99, 104, 110, 117, 118, 123, 124, 130.

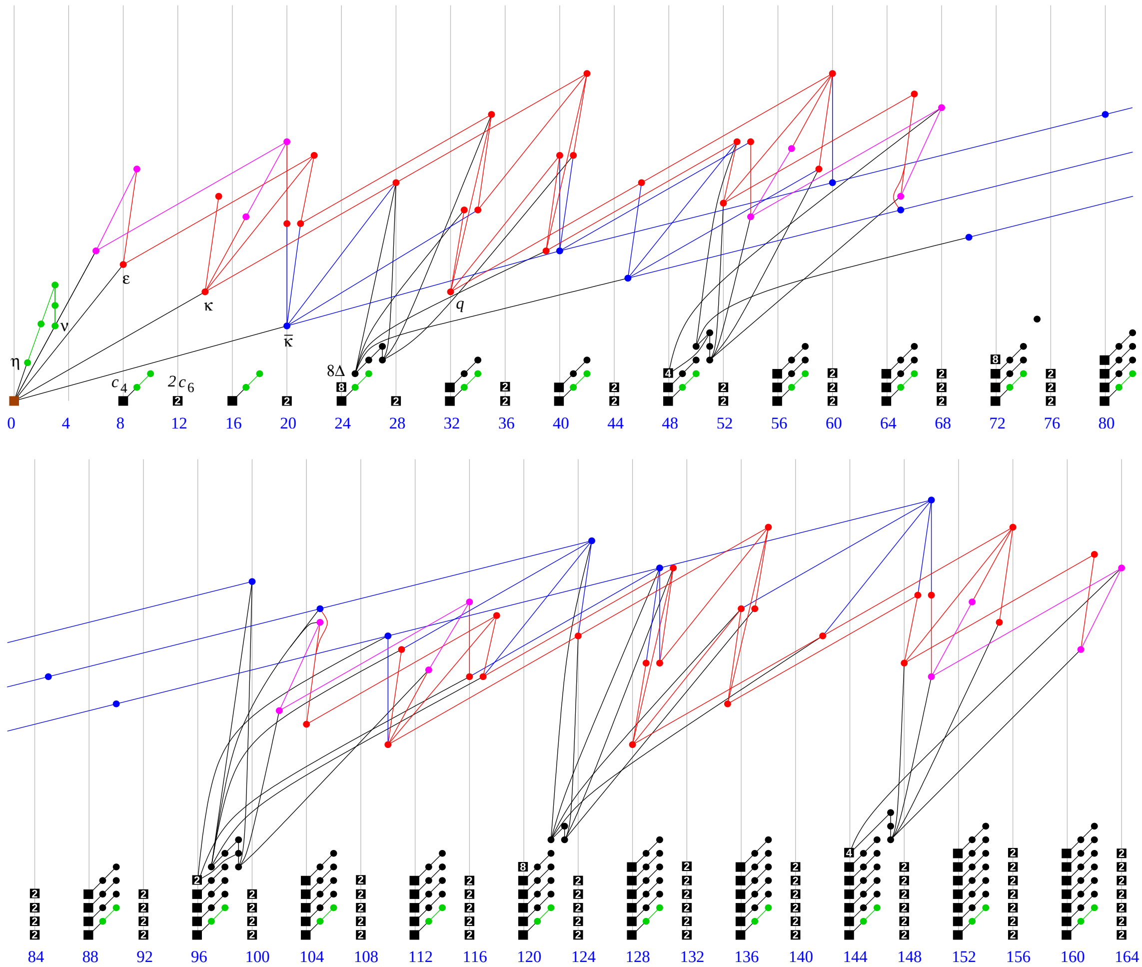
$\{\eta\Delta\}$	$\{2\nu\Delta\}$
1, 2, 3, 8, 14, 15, 20, 21, 25, 26, 27, 28, 32, 34, 35, 40, 41, 45, 50, 60, 65, 75, 80, 85, 97, 98, 99, 100, 104, 105, 110, 111, 113, 117, 122, 123, 124, 125, 128, 130, 131, 137.	1, 8, 14, 15, 25, 26, 27, 32, 33, 39, 96, 97, 98, 104, 110, 111, 122, 123, 128, 129, 135.

$\{\nu\Delta^2\}$	
1, 2, 3, 6, 8, 9, 14, 15, 17, 51, $54^\dagger$, $65^\dagger$, 96, 97, 98, 99, 102, 104, 105, 110, 111, 113, 116.	$\dagger$: $\nu\{\nu\Delta^2\} \mapsto \nu^2\{\nu\Delta^4\}$. $\kappa\{\nu\Delta^2\} \mapsto \nu\kappa\{\nu\Delta^4\}$.

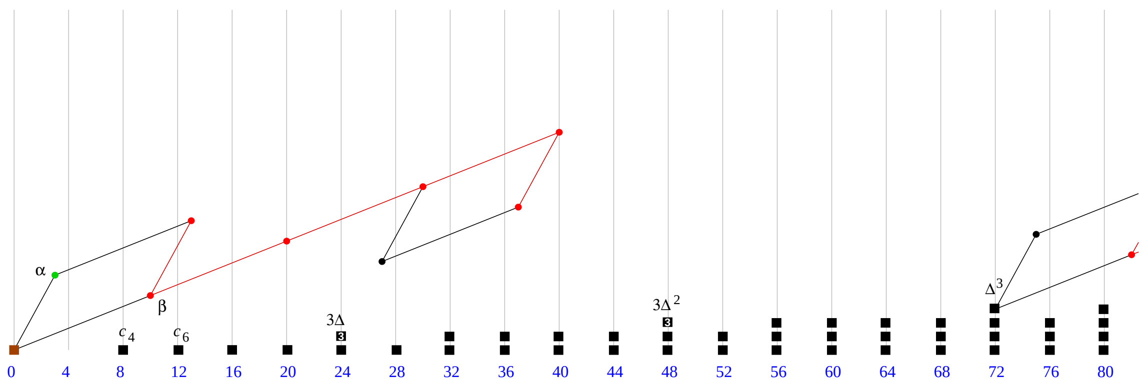
All the remaining multiplications are implied by the fact that $\pi_*(tmf) \to MF_*$ is a ring homomorphism.

To finish, we list the only two relations that are not implied by our chart: $\{\eta\Delta\}^4 = \bar\kappa^5$, $\{2\nu\Delta\}^2 = \kappa\bar\kappa^2$.

The picture of $\pi_*(tmf)_{(2)}$ from the previous page being rather small, we include here a larger version:



The homotopy groups of tmf at the prime 3 exhibit similar phenomena than at the prime 2. The picture on the bottom of this page is an illustration of $\pi_*(tmf)_{(3)}$. The bullets represent $\mathbb{Z}/3\mathbb{Z}$'s and are named after the elements of $\pi_*(\mathbb{S})$ that hit them. The squares represent $\mathbb{Z}$'s and are named after their image in MF_*. The slanted lines represent multiplication by α and β. The top part



The homotopy groups of tmf at the prime 3 of the diagram is 72-periodic, with polynomial generator Δ^3. We have drawn the image of the Hurewitz in color: in green is the unique v_1-periodic class α, and in red are the v_2-periodic classes. The latter remain periodic in *tmf* through the identification $v_2^9 = \Delta^6$ (or maybe $v_2^{9/2} = \Delta^3$?). Once again, the white numbers in the squares indicate the size of $\operatorname{coker}(\pi_*(tmf) \to MF_*)_{(3)}$. The algebra $\pi_*(tmf)_{(3)}$ is a finitely generated, with generators

degree:	3	8	10	12	24	27	32	36	48	56	60	72
name:	α	c_4	β	c_6	$\{3\Delta\}$	$\{\alpha\Delta\}$	$\{c_4\Delta\}$	$\{c_6\Delta\}$	$\{3\Delta^2\}$	$\{c_4\Delta^2\}$	$\{c_6\Delta^2\}$	$\{\Delta^3\}$

and many relations.

It is also worth wile noting that the classes in dimensions 3, 13, 20, 30 (mod 72) support non-trivial $\langle \alpha, \alpha, - \rangle$ Massey products.

When localized at a prime $p \geq 5$, the homotopy ring of *tmf* becomes isomorphic to MF_*. Since $\Delta \in MF_{12}$ is a $\mathbb{Z}_{(p)}$-linear combination of c_4^3 and c_6^2, this further simplifies to $\pi_*(tmf)_{(p)} = \mathbb{Z}_{(p)}[c_4, c_6]$.

2. Localizations of *tmf*

The periodic version of *tmf* goes by the name *TMF*. It's homotopy groups are given by

$$\pi_*(TMF) = \pi_*(tmf)\big[\{\Delta^{24}\}^{-1}\big].$$

The homotopy groups $\pi_n(TMF)$ are finitely generated, except for $n \equiv 1, 2 \pmod 8$, in which case they contain a summand isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$.

Fix a prime p, and let $K(n)$ denote the nth Morava K-theory at that prime (p is omitted from the notation). We can then consider the $K(n)$-localization $L_{K(n)}tmf$ of the spectrum *tmf*. The spectrum $L_{K(0)}tmf$ is simply the rationalisation of *tmf* (and doesn't depend on p). Its homotopy ring is therefore given by

$$\pi_*(L_{K(0)}tmf) = \pi_*(tmf) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The homotopy groups of $L_{K(1)}tmf$ are easiest to describe at the primes 2 and 3. In those cases, they are given by

$$\begin{aligned} \pi_*(L_{K(1)}tmf) &= \Big(\pi_*(KO)[j^{-1}]\Big)^{\hat{}}_p \\ &= \pi_*(KO)^{\hat{}}_p\, \langle j^{-1} \rangle \end{aligned} \qquad p = 2 \text{ or } 3. \tag{3}$$

Here, the notation $R\langle x \rangle$ refers to powers series $\sum_{k=0}^\infty a_k x^k$ whose coefficients $a_k \in R$ tend to zero p-adically as $k \to \infty$. The variable is called j^{-1} because its inverse corresponds to the j-invariant of elliptic curves. The reason why (3) is simpler at $p = 2$ and $p = 3$ is that at those primes there is a unique supersingular elliptic curve, and that its j-invariant is equal to zero. For general prime $p \geq 3$, let $\alpha_1, \ldots, \alpha_n$ denote the supersingular j-values. Each element α_i is a priori only an element of $\mathbb{F}_{p^n}$ (actually in $\mathbb{F}_{p^2}$), however, their union $S := \{\alpha_1, \ldots, \alpha_n\}$ is always a scheme defined over $\mathbb{F}_p$. Let $\tilde{S}$ denote any sheme over $\mathbb{Z}$ whose reduction mod p is S. The homotopy groups of $L_{K(1)}tmf$ are then given by

$$\pi_*(L_{K(1)}tmf) = \Big(\text{functions on } \mathbb{P}^1 \setminus \tilde{S}\Big)^{\hat{}}_p\big[b^{\pm 1}\big], \qquad p \geq 3,$$

where b is in degree 4.

The homotopy ring of $L_{K(2)}tmf$ is the completion of $\pi_* TMF$ at the ideal generated by p and by the Hasse invariant E_{p-1}:

$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p, E_{p-1})}, \qquad p \text{ arbitrary}.$$

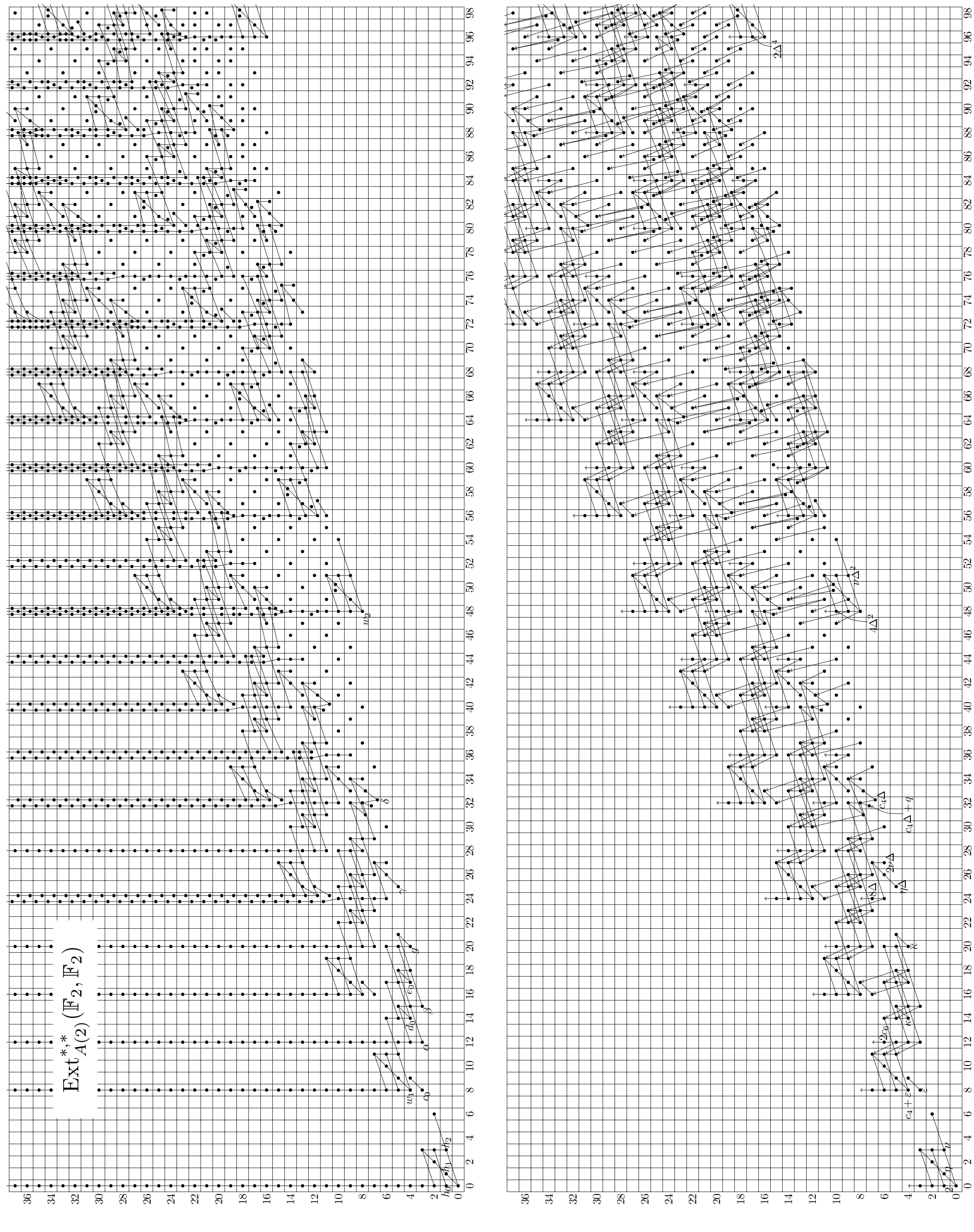
The latter is a polynomial in c_4 and c_6 whose zeroes correspond to the supersingular elliptic curves. Once again, given the fact that there is a unique supersingular elliptic curve at $p = 2$ and 3, the above formula simplifies to

$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p, c_4)} \qquad p = 2, 3.$$

For $n > 2$, the localization $L_{K(n)}tmf$ is trivial, and thus satisfies $\pi_*(L_{K(n)}tmf) = 0$.

3. The Adams spectral sequence

The Adams spectral sequence for tmf is a spectral sequence that converges converges to $\pi_*(tmf)^\wedge_p$ for any given prime p. Its E_2 page is given by $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$, where A_p^{tmf} is a finite dimensional



$\mathbb{F}_p$-algebra which is a *tmf*-analog of the Steenrod algebra.

$$A_p^{tmf} := \hom_{tmf-\text{modules}}(H\mathbb{F}_p, H\mathbb{F}_p).$$

At the prime 2, the natural map $A_2^{tmf} \to A \equiv A_2$ to the Steenrod algebra is injective. Its image is the subalgebra $A(2) \subset A$ generated by Sq^1, Sq^2 and Sq^4. That algebra is of dimention 64 over $\mathbb{F}_2$, and defined by the relations

$$Sq^1Sq^1 = 0, \quad Sq^2Sq^2 = Sq^1Sq^2Sq^1,$$
$$Sq^1Sq^4 + Sq^4Sq^1 + Sq^2Sq^1Sq^2 = 0, \text{ and}$$
$$Sq^4Sq^4 + Sq^2Sq^2Sq^4 + Sq^2Sq^4Sq^2 = 0.$$

By the change of rings theorem, the Adams spectral sequence for *tmf* can then be identified with the classical Adams spectral sequence

$$\begin{aligned} E_2 &= \mathrm{Ext}_A\big(H^*(tmf), \mathbb{F}_2\big) \\ &= \mathrm{Ext}_A\big(A/\!\!/A(2), \mathbb{F}_2\big) \Rightarrow \pi_*(tmf)\hat{\,}_2. \end{aligned}$$

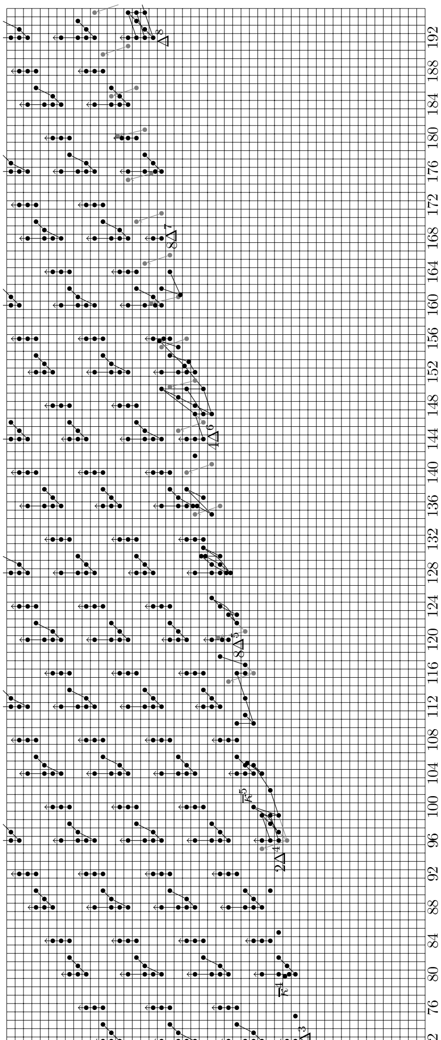
The bigraded ring $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the classes:

(4)

bidegree:	(0,1)	(1,1)	(3,1)	(8,4)	(8,3)	(12,3)	(14,4)
name:	h_0	h_1	h_2	w_1	c_0	α	d_0
	(15,3)	(17,4)	(20,4)	(25,5)	(32,7)	(48,8)	
	β	e_0	g	γ	δ	w_2	

subject to the following complete set of relations:

$h_0h_1 = 0$, $h_1h_2 = 0$, $h_0^2h_2 = h_1^3$, $h_0h_2^2 = 0$, $h_2^3 = 0$, $h_0c_0 = 0$, $h_1^2c_0 = 0$, $h_2c_0 = 0$, $c_0^2 = 0$, $c_0d_0 = 0$, $c_0e_0 = 0$, $c_0g = 0$, $c_0\alpha = h_0^2g$, $c_0\beta = 0$, $h_0^2d_0 = h_2^2w_1$, $h_1d_0 = h_0^2\beta$, $h_2d_0 = h_0e_0$, $d_0^2 = w_1g$, $d_0g = e_0^2$, $h_0\alpha d_0 = h_2\beta w_1$, $\alpha^2d_0 = \beta^2w_1$, $\beta d_0 = \alpha e_0$, $h_1e_0 = h_2^2\alpha$, $h_2e_0 = h_0g$, $\beta e_0 = \alpha g$, $h_1g = h_2^2\beta$, $h_2g = 0$, $e_0g = \alpha\gamma$, $g^2 = \beta\gamma$, $h_1\alpha = 0$, $h_1\beta = 0$, $h_2\alpha = h_0\beta$, $h_0\beta^2 = 0$, $h_2\beta^2 = 0$, $\alpha^4 = h_0^4w_2 + g^2w_1$, $h_0\gamma = 0$, $h_1^2\gamma = h_2\alpha^2$, $h_2\gamma = 0$, $c_0\gamma = h_1\delta$, $d_0\gamma = \alpha^2\beta$, $e_0\gamma = \alpha\beta^2$, $g\gamma = \beta^3$, $\gamma^2 = h_1^2w_2 + \beta^2g$, $h_0\delta = h_0\alpha g$, $h_1^2\delta = h_0d_0g$, $h_2\delta = 0$, $c_0\delta = 0$, $d_0\delta = 0$, $e_0\delta = 0$, $g\delta = 0$, $\alpha\delta = 0$, $\beta\delta = 0$ $\gamma\delta = h_1c_0w_2$, $\delta^2 = 0$.



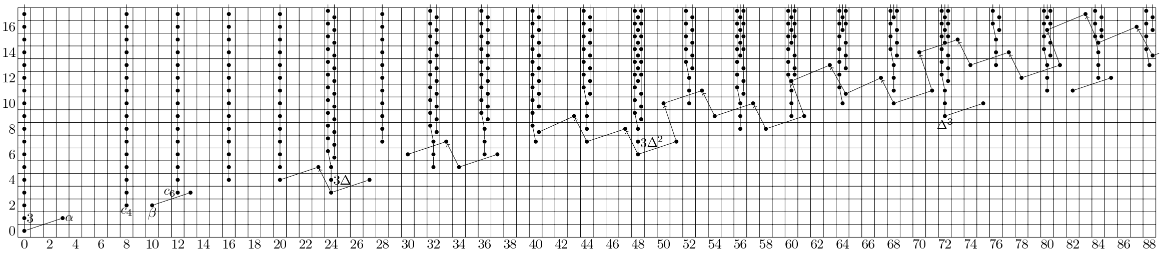
Above: The E_∞ page of the Adams spectral sequence for *tmf* at the prime 2.
Previous Page: The E_2 page with classes named as in (4) ; The differentials.

The charts on the previous page only go until half the periodicity. Therefore, we have also included in the picture of the E_∞ page the most important differential in dimensions ≥ 96.

At the prime 3, the map from A_3^{tmf} to the Steenrod algebra is no longer injective. Indeed, the algebra A_3^{tmf} is 24 dimensional, while its image in the Steenrod algebra is the 12 dmensional subalgerba generated by β and $\mathcal{P}^1$. Naming its generators by their image in A_3, the following relations define A_3^{tmf}:

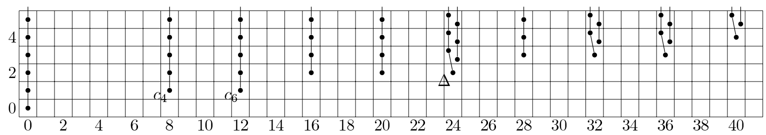
$$\beta^2 = 0, \quad (\mathcal{P}^1)^3 = 0,$$
$$\beta\,\mathcal{P}^1\beta\,\mathcal{P}^1 + \mathcal{P}^1\beta\,\mathcal{P}^1\beta = \beta\,(\mathcal{P}^1)^2\beta.$$

Note that the relation $\beta(\mathcal{P}^1)^2 + \mathcal{P}^1\beta\mathcal{P}^1 + (\mathcal{P}^1)^2\beta = 0$ holds in A_3, but not in A_3^{tmf}.



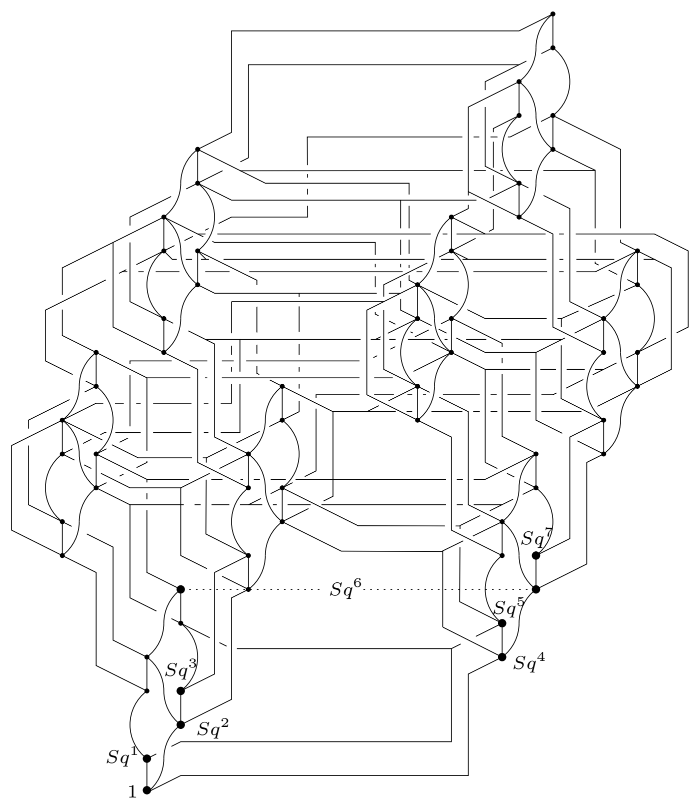
The Adams spectral sequence $\mathrm{Ext}_{A_3^{tmf}}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \pi_*(tmf)^\wedge_3$.

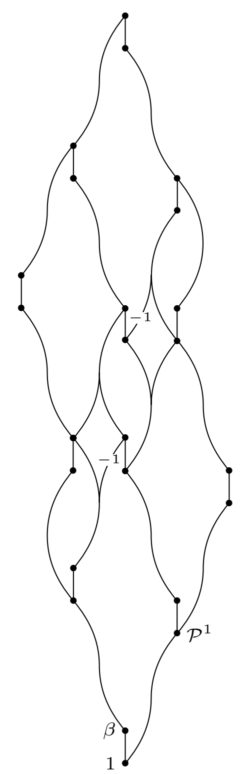
At primes $p > 3$, the algebra A_p^{tmf} is an exterior algebra on generators in degrees 1, 9, and 13. The ring $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$ a polynomial algebra on classes in bidegrees $(0,1)$, $(8,1)$, and $(12,1)$ and the Adams spectal sequence for *tmf* collapses.



The Adams spectral sequence for *tmf* at a prime $p > 3$.

It is interesting to note that regardless of the prime, the algebra A_p^{tmf} has its top dimensional class in degree 23. Below, we picture the algebras A_p^{tmf} for the primes 2 and 3:





4. Acknowledgements

I am seriously indebted to many people for the completion of this survey on the homotopy groups of *tmf*. First of all, I would like to thank Tilman Bauer for spending an enormous amount of time with me sorting out the multiplicative structure of $\pi_*(tmf)$. Based on a computer program of Christian Nassau, Tilman is also the one who created the first draft of the $\mathrm{Ext}_{A(2)}$ charts. I should thank Mark Behrens for sending me his calculations of the Adams spectral sequence, and for answering many questions about the multiplicative structure of $\pi_*(tmf)$. The defining relations of $\mathrm{Ext}_{A(2)}$ were taken from a calculation of Peter May, who was made available to me by Mark Behrens. Next, I would like to thank Mark Mahowald for some very informative email exchanges. I am especially grateful to him for explaining me how to detect v_2^{32}-periodic families of elements in the image of Hurewitz homomorphism, and for agreeing to make the following conjecture:

Conjecture (Mark Mahowald)**.** *The image of the Hurewitz homomorphism* $\pi_*(\mathbb{S}) \to \pi_*(tmf)$ *is given by the classes drawn in color in the pictures on page 3.*

I thank Niko Nauman for useful conversations. I am indebted to Mike Hopkins for putting me in contact with Mark Mahowald. Finally, I should also thank Mike Hill, who was the first to figure out the structure of A_3^{tmf}, for the numerous conversations that we had while we were graduate students at MIT.