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Annales scientifiques de l'É.N.S. 4e série, tome 18, nº 3 (1985), p. 437-552
<http://www.numdam.org/item?id=ASENS_1985_4_18_3_437_0>

Ann. scient. Éc. Norm. Sup.,
4ᵉ série, t. 13, 1980, p. 437 à 552.

ALGEBRAIC K-THEORY AND ETALE COHOMOLOGY

By R. W. THOMASON

Let l^ν be a prime power, and let X be a separated regular noetherian scheme in which l is invertible. If $\mathcal{O}_X$ contains a primitive l^νth root of unity, there is canonical element β in the second algebraic K-group of X with coefficients $\mathbb{Z}/l^\nu$, $K/l_2^\nu(X)$, which Bocksteins to the primitive root as an l^ν torsion class in $K_1(X)$. The graded ring $K/l_*^\nu(X)$ may be localized by inverting this β. Under a few mild extra hypotheses on X, the main Theorem 4.1 yields a strongly converging Atiyah-Hirzebruch type spectral sequence that computes $K/l_*^\nu(X)[\beta^{-1}]$ in terms of the etale cohomology of X:

$$(0.1) \qquad E_2^{p,q} = \begin{Bmatrix} H_{et}^p(X; \mathbb{Z}/l^\nu(i)), \; q=2i \\ 0, \; q \text{ odd} \end{Bmatrix} \Rightarrow K/l_{q-p}^\nu(X)[\beta^{-1}].$$

The result may be reformulated in terms of the etale or topological K-theory of Dwyer and Friedlander as giving an isomorphism between the localized algebraic and topological K-groups as in Theorem 4.11:

$$(0.2) \qquad K/l_*^\nu(X)[\beta^{-1}] \cong K/l_*^{\nu\,\mathrm{Top}}(X).$$

The result holds even without the assumption that $\mathcal{O}_X$ contains l^νth roots of unity, when β is defined as in Appendix A. There is a variant of the result for singular schemes in terms of algebraic G-theory and topological K-homology as in 2.48, 4.15-4.16.

The result is quite remarkable in that it expresses a deep and subtle link between algebraic geometry and the topology of varieties. The groups $K/l_*^\nu(X)$ are defined in terms of the category of algebraic vector bundles on X, and reflect the delicate algebraic-geometric structure of X. They carry much subtle information about intersection theory on X, and on Euler characteristics in coherent cohomology of algebraic vector bundles on X. On the other hand, $K/l_*^{\nu\,\mathrm{Top}}(X)$ is a much cruder invariant depending only on the underlying topology of X. As X runs through the moduli of K3 surfaces over the complex numbers, the rank of the image of $K/l_0^\nu(X)$ in $K/l_0^{\nu\,\mathrm{Top}}(X)$ under the forgetful map is known to take on all values from 3 to 22, despite the fact that all such surfaces are diffeomorphic ([103], IX). Nothing in the definition of $K/l_*^\nu(X)$ involves or evokes the etale topology, or leads one to expect that etale cohomology can be constructed out of $K/l_*^\nu(X)$. For varieties over the complex numbers, $K/l_*^{\nu\,\mathrm{Top}}(X)$ can be defined in a manner parallel to $K/l_*^\nu(X)$, with the category of topological vector bundles playing the role of algebraic vector bundles. However, the two categories of vector bundles are quite different: not every topological vector bundle is algebraizable, algebraic vector bundles may be isomorphic as topological bundles without being isomorphic algebraically, and not every short exact sequence of algebraic vector bundles splits algebraically, though it must split topologically. Thus $K/l_*^\nu(X)$ and $K/l_*^{\nu\,\mathrm{Top}}(X)$ look quite different. However, the theorem says that they are also quite alike, in that they differ only by β-torsion, hence only in "codimension at least one". As inverting β takes a direct limit over groups in higher degrees as in (0.3),

$$K/l_n^\nu(X)[\beta^{-1}] = \varinjlim \left(K/l_n^\nu(X) \xrightarrow{\cup\beta} K/l_{n+\deg\beta}^\nu(X) \xrightarrow{\cup\beta} \ldots\right), \tag{0.3}$$

the Theorem says that algebraic K-theory asymptotically approaches topological K-theory in high degrees. It also says that the category of algebraic vector bundles on X knows about the non-algebraizable topological vector bundles.

There are many applications of the main result. As etale cohomology is usually easy to calculate, it is usually possible to calculate the groups $K/l_*^\nu(X)[\beta^{-1}]$, which are at least close to $K/l_*^\nu(X)$ if not identical to it. Examples for curves, semi-simple algebraic groups, and smooth hypersurfaces in projective space are given in paragraph 4, along with a few arithmetic examples.

As another application, it is possible to use Theorem 4.1 to show directly that $K/l_*^\nu(X)[\beta^{-1}]$ has all the formal properties used in applications of the Dwyer-Friedlander topological K-theory. Thus it may be used to replace this construction for regular schemes, and so avoid the gruesome technicalities of etale homotopy theory. Theorem 4.11 shows that this gives exactly the Dwyer-Friedlander groups.

If $f: X \to Y$ is a proper map between regular varieties, there is an obvious commutative diagram

$$\begin{array}{ccc} K/l_*^\nu(X) & \to & K/l_*^\nu(X)[\beta^{-1}] \\ \downarrow f_* & & \downarrow f_* \\ K/l_*^\nu(Y) & \to & K/l_*^\nu(Y)[\beta^{-1}] \end{array} \tag{0.4}$$

The groups on the right of (0.4) can be computed via Theorem 4.1. This makes it possible to compute the map f_* on the right of (0.4). This f_* may be identified to the Gysin map in topological K-theory. Thus (0.4) solves the Riemann-Roch problem as a variant of Grothendieck's Riemann-Roch Theorem. For a generalization to singular varieties and a fuller discussion, *see* 4.16-4.17. This generalizes those higher Riemann-Roch theorems of Gillet and of Shekhtman that deal with the Chern character from algebraic K-theory to $\mathbb{Q}_l^\wedge$ etale cohomology.

An application related to the above is my proof of Grothendieck's absolute cohomological purity conjecture for $\mathbb{Q}_l^\wedge$ etale cohomology, as discussed in 4.18.

Known connections between zeta functions and etale cohomology may be reformulated in terms of $K/l^{\nu}_{*}(X)[\beta^{-1}]$ thanks to 4.1. This is important for calculations in arithmetic cases as in 4.7 and 4.8. It also sheds a bit of light on the results and conjectures of Beilinson, Coates, Lichtenbaum, Mazur, Soulé, and Wiles.

Most of my results concern the mod l algebraic K-groups introduced by Browder. As in A.5, there is a universal coefficient sequence

$$(0.5) \qquad 0 \to K_n(X) \otimes \mathbb{Z}/l^{\nu} \to K/l^{\nu}_n(X) \to l^{\nu}\text{-torsion in } K_{n-1}(X) \to 0.$$

See A.12 for the l-adic version. From this one sees that most of the information in $K_*(X)$ is encoded in the l-adic $K_*(X)_l^{\wedge}$ or in the system of $K/l^{\nu}_*(X)$ as ν increases: only the uniquely l-divisible subgroups of $K_*(X)$ are irretrievably lost. There is good reason to lose something, for it is well-known that the groups $K_*(X)$ violate Lefschetz's principle and that Mayer-Vietoris for closed covers and the homotopy axiom fail for singular X. Suslin has recently shown that the groups $K/l^{\nu}_*(X)$ satisfy Lefschetz's principle [117], and Weibel has proved Mayer-Vietoris and the homotopy axiom for $K/l^{\nu}_*(X)$ and singular affine X [139]. Thus these pathologies disappear mod l^{ν}. Similarly, etale cohomology exhibits pathology unless restricted to torsion coefficient sheaves. Suppose X is a projective variety over the complex numbers. Then the classical topology allows one to define an integral $K^{\mathrm{Top}}_*(X)$, and there is a Dwyer-Friedlander or forgetful map to it from $K_*(X)$. Using Hodge theory, Gillet ([44], 5.5) has shown that this map has torsion image in degrees above 0. As $K^{\mathrm{Top}}_*(X)$ is often torsion-free, this map is often zero in positive degrees. In contrast, the map from $K/l^{\nu}_*(X)$ to $K/l^{\nu\,\mathrm{Top}}_*(X)$ is a localization by 4.1, and so is highly non-trivial. In fact, a more delicate version of 4.1 shows that this map is surjective in sufficiently high degrees [129]. In integral terms, this means that the copies of $\mathbb{Z}$ in $K^{\mathrm{Top}}_n(X)$ for large n correspond not to $\mathbb{Z}$'s in $K_n(X)$, but rather *via* the universal coefficient theorem to torsion groups $\mathbb{Q}/\mathbb{Z}$ in $K_{n-1}(X)$. This is like the classical relation of $H^1(\ ;\mathbb{Z})$ of a curve to the Tate module of torsion points on the Jacobian of the curve. This relation is much easier to see working mod l^{ν}. However, a rather messy integral form of the key descent theorem is given in 2.50, and a simple rational descent Theorem is proved in 2.15-2.18.

The general outline of the proof of the main Theorem is this: First, the machinery of homological algebra must be generalized to a "homotopical algebra" that applies not only to chain complexes, but to the spaces and spectra that occur in Quillen's definition of algebraic K-theory. The foundation for this was laid by Puppe and Quillen, and a superstructure is built on it in paragraphs 5 and 1. There is a close analogy between ordinary homological algebra and this generalization. The analog of a chain complex is a spectrum in the sense of algebraic topology. The analogs of homology groups of a chain complex are homotopy groups of a spectrum. Short exact sequences of complexes correspond to fibration sequences of spectra; they yield long exact sequences of homology or homotopy groups. Quasi-isomorphism of complexes corresponds to weak homotopy equivalence of spectra, and the derived category to the stable homotopy category. The derived category treatment of homological algebra generalizes, and all fundamental results stated in this language carry over. The basic results are proved in paragraph 5, and the theory of sheaf and Čech hypercohomology of a topos with coefficients in a presheaf of spectra is developed in paragraph 1.

In paragraph 2, I prove that $K/l^\nu_*(\ )[\beta^{-1}]$ has a cohomological descent spectral sequence with respect to Čech covers for the etale topology, and also has a related descent spectral sequence for etale sheaf cohomology. The Mayer-Vietoris and localization properties of algebraic K-theory and the properties of etale hypercohomology developed in paragraph 1 allow reduction from the case of general schemes to that of local rings, and then to that of fields. The essential ingredient in handling the last case is Hilbert's Theorem 90.

In paragraph 3, I complete the proof of the main theorem by showing that the sheaf $\tilde{K}/l^\nu_*(\ )[\beta^{-1}]$ on the etale site is $\mathbb{Z}/l^\nu(i)$ or 0 depending on the degree. This amounts to showing that there is no excess in $K/l^\nu_*(R)[\beta^{-1}]$ beyond what is detected by the Dwyer-Friedlander map when R is a nice strict local hensel ring. This is done by the classical splitting principle for K_0 plus dimension-shifting techniques of Dayton and Weibel that allow one to reduce to a K_0 problem.

This is the second edition of this paper. The first was an MIT preprint of 1980. There are several significant differences between editions. The cases where $l=2$, 3 are now covered, and with fewer restrictions than announced in [126] or the *erratum* to the first edition. There was a slip in the proof of 2.13 in the first edition, which was fixed under additional hypotheses in the erratum. The proof of the main theorem in this edition avoids this difficulty entirely, but still needs the additional hypothesis that the residue fields of X not only have bounded etale cohomological dimension, but that they have a filtration like (2.112) between them and their separable closures. This is not a serious restriction as all the usual examples satisfy (2.112). In fact, there is no field known to have finite etale cohomological dimension which does not satisfy (2.112), for (2.112) is the grip by which Tsen's Theorem grasps the problem in all known proofs of finite cohomological dimensionality.

The method of proof of the descent Theorem for fields in this edition is completely new. It is shorter, more conceptual, and gives a stronger result than my older Karoubi periodicity proof. However, Karoubi periodicity applies more generally to solve other problems of a related type, and I hope to return to this subject in another paper. The reader may see a sketch of the old proof and get an idea of this general applicability from [128].

This edition also includes the results of [126], which did not appear in the first edition.

Throughout this paper, I have ignored the usual set-theoretic problems and the fact that certain "functors" are really only pseudofunctors. Rigorous correction of the first is by means of Grothendieck's method of "universes" as in [SGA 4]. The second problem is resolved by rectification of pseudofunctors into equivalent strict functors, as in [116], [122], II 4.4, or any of the other footnotes on this nuisance.

The background expected of the reader consists of comfortable familiarity with the basic ideas of homotopy theory; a fair knowledge of algebraic geometry on the level of

Hartshorne's text; thorough knowledge of Quillen's foundational paper on algebraic K-theory [97]; and a little exposure to Galois and etale cohomology on the level of [3], or the first few sections of Deligne's summary in [SGA 4 1/2].

I have received much help while doing the work reported on here. Useful suggestions, political support, and patient attention were provided by Mike Artin, Armand Borel, Bill Browder, Bill Dwyer, Eric Friedlander, Dale Husemuller, Max Karoubi, Robert Langlands, Ronnie Lee, Peter May, Vic Snaith, and Chuck Weibel. Without Dan Quillen's support this work would not have appeared. Henri Gillet deserves special thanks for his many perceptive suggestions. Christophe Soulé actually read most of the first edition with a fair amount of attention, and his remarks have been very useful. Beilinson and Shekhtman have spent their own money to keep me informed of their interesting work in K-theory; one hopes that someday they will be able to get the academic jobs their talents deserve. To all these mathematicians, I am very grateful. I also owe thanks to the institutions that have supported me during this work: M.I.T., the Institute for Advanced Study, The Johns Hopkins University, and the National Science Foundation. Anne Wolfsheimer capably converted my handwriting into typescript, so both I and the reader are much in her debt.

Suggestions for reading:

If you're an:	then read in order:
Honest man	§ 5, Appendix A, § 1, 2, 3 and 4;
Reckless cheat	§ 2, 3 and 4, bearing in mind the above analogy between homological and homotopical algebra;
Thrill seeker	§ 4.

1. Čech and sheaf hypercohomology with coefficients in a spectrum

In this section I develop the basic properties of hypercohomology of a topos with coefficients in a spectrum. I assume familiarity with the material reviewed in paragraph 5.

By a site, I mean a small category **C** which has pullbacks and is endowed with a Grothendieck topology. This consists of assigning to each object U a collection Cov(U) of families $\mathscr{V}$ of objects with maps to U. Each family $\mathscr{V}$ is thus a collection $\{V_i \to U \mid i \in I\}$. The families in Cov(U) are covers of U. The usual axioms hold: *see* [3], I 0.1 or [SGA 4], II, § 1 for a list of these. Roughly they say that a pullback of a cover is a cover, a cover of a cover is a cover, and that each U covers itself. It is often convenient to assume **C** also has a terminal object. Each site **C** has a category of presheaves $\mathbf{C}^{\wedge}$ and a category of sheaves $\mathbf{C}^{\sim}$, which is the topos associated to the site. This $\mathbf{C}^{\sim}$ is a full subcategory of $\mathbf{C}^{\wedge}$ consisting of presheaves satisfying a descent condition for covers [SGA 4], II, § 2. A topos has a sheaf cohomology and even a sort of homotopy type ([5], [36]).

Example 1.1. — Let X be a topological space and **C** the category whose objects are the open sets of X. Morphisms are inclusions. $\mathscr{V}=\{V_i \to U\}$ is a cover if $U=\bigcup V_i$. For X a scheme, this yields the Zariski site. Sheaves are sheaves in the usual sense.

Example 1.2. — Let X be a scheme, and $\mathbf{C}=\mathrm{Et}/X$ the category of schemes etale over X. An object is an etale map $u: U \to X$. The morphisms in the site are the maps $U \to U'$ that respect the maps to X. These maps $U \to U'$ are then etale. A family $\mathscr{V}=\{u_i: V_i \to U\}$ is a cover if it is faithfully flat, or equivalently if $U=\bigcup u_i(V_i)$ as sets. For details, consult [SGA 4], VII 1.2.

Example 1.3. — Let X be a separated noetherian scheme. The objects of the restricted etale site Etr/X are affine schemes U together with a map $u: U \to X$ which is separated, etale, and of finite presentation. A covering family of X is a finite collection of such $u_i: U_i \to X$, satisfying the condition that $\bigcup u_i(U_i)=X$.

Note that the U_i are also noetherian. In this site the family $\{U_i \to X\}$ may be replaced by $U=\bigsqcup U_i \to X$. Thus I may consider a covering family as a single noetherian affine scheme etale and faithfully flat over X.

Sometimes X itself is given honorary membership in this site to provide a terminal object. In any case, the site has pullbacks over X. I can also take the same site, but remove the condition that covering families be finite. Proposition 1.4 remains true.

PROPOSITION 1.4. — *For* X *a separated noetherian scheme, the sites* Et/X *and* Etr/X *have the same topos, hence the same cohomology.*

Proof. — [3], III, Thm. 1.1 or [SGA 4], VII, § 3.

I can therefore switch between these sites whenever I want.

DEFINITION 1.5. — A presheaf on a site **C** is a contravariant functor from **C** to the category of sets, or to the category of fibrant spectra as in 5.2, or to that of group objects in the category of fibrant spectra, depending on context.

Examples 1.6. — The K-theory spectra of Appendix A are examples of presheaves of fibrant spectra on Et/X. They may be replaced by weak homotopy equivalent presheaves of fibrant group spectra by 5.38-5.39.

Example 1.7. — Take any presheaf of chain complexes on a site. As in Scholium 5.32, this may be regarded as a presheaf of fibrant abelian group spectra.

CONSTRUCTION 1.8. — Let F be a presheaf on a site, and let $\mathscr{U}=\{U_i \to U \mid i \in I\}$ be a cover. The Čech complex $F^{\bullet}_{\mathscr{U}}$ is the cosimplicial set, cosimplicial fibrant spectrum, or cosimplicial group spectrum

$$F^{\bullet}_{\mathscr{U}}=\prod_{i_0 \in I} F(U_{i_0}) \rightrightarrows \prod_{(i_0,\, i_1) \in I^2} F(U_{i_0} \times_U U_{i_1}) \mathrel{\substack{\to\\ \to\\ \to}} \cdots \tag{1.1}$$

For n in Δ, $F^n_{\mathscr{U}}$, the thing in cosimplicial codimension n is the product

$$F^n_{\mathscr{U}}=\prod_{(i_0,\, i_1,\, \ldots,\, i_n) \in I^{n+1}} F(U_{i_0} \times_U U_{i_1} \times_U \ldots \times_U U_{i_n}). \tag{1.2}$$

I'll denote the factor indexed by $i_* = (i_0, i_1, \ldots, i_n)$ by $F(U(i_*))$. The coface operator $d^k: F^n_{\mathcal{U}} \to F^{n+1}_{\mathcal{U}}$ is determined as follows. Let j_* be in I^{n+2}, and let

$$d_k j_* = (j_0, j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{n+1}) \in I^{n+1}.$$

Then d^k followed by projection on the factor $F(U(j_*))$ is projection on the factor $F(U(d_k j_*))$ followed by the map

$$F(U(d_k j_*)) \to F(U(j_*)). \tag{1.3}$$

This map is obtained by applying F to the map $U(j_*) \to U(d_k j_*)$ induced by the projection $U_{j_k} \to U$.

Similarly, the codegeneracies $s^k: F^{n+1}_{\mathcal{U}} \to F^n_{\mathcal{U}}$ are such that composing s^k with projection on the factor indexed by $i_* = (i_0, \ldots, i_n) \in I^{n+1}$ yields the map which is the composite of projection on the factor indexed by $s_k i_* = (i_0, i_1, \ldots, i_k, i_k, \ldots, i_n) \in I^{n+2}$ and a map $F(U(s_k i_*)) \to F(U(i_*))$. This map is obtained by applying F to the map $U(i_*) \to U(s_k i_*)$ induced by the diagonal

$$U_{i_k} \to U_{i_k} \underset{U}{\times} U_{i_k}. \tag{1.4}$$

The cosimplicial identities (5.12) are easily but tediously checked.

For F a presheaf of abelian groups, this construction yields one of the usual forms of the Čech complex under the equivalence of cochain complexes and cosimplicial abelian groups in Scholium 5.32.

There is a canonical augmentation $F(U) \to F^{\bullet}_{\mathcal{U}}$ induced by the projections $U(i_*) \to U$. (*See* 5.26.)

DEFINITION 1.9. — For F a presheaf of fibrant spectra and $\mathcal{U}$ a cover of U, define the Čech hypercohomology spectrum of F with respect to $\mathcal{U}$ to be the homotopy limit over Δ of $F^{\bullet}_{\mathcal{U}}$:

$$\check{\mathbb{H}}^{\bullet}(\mathcal{U}; F) = \underset{\Delta}{\operatorname{holim}}\, F^{\bullet}_{\mathcal{U}} = \mathbb{H}^{\bullet}(\Delta; F^{\bullet}_{\mathcal{U}}). \tag{1.5}$$

The augmentation induces a natural map by 5.26.

$$F(U) \to \check{\mathbb{H}}^{\bullet}(\mathcal{U}; F). \tag{1.6}$$

See paragraph 5 for a discussion of homotopy limits.

For F a presheaf of abelian group spectra, or equivalently, of chain complexes, $\check{\mathbb{H}}^{\bullet}(\mathcal{U}; F)$ is equivalent to a chain complex whose homology is the usual Čech hypercohomology of F with respect to $\mathcal{U}$. In fact, it is essentially the total complex of the Čech cochain complex on the presheaf of chain complexes F. This is clear from Scholium 5.32 and Remark 1.10.

Remark 1.10. — If F' is a presheaf of fibrant group spectra, the natural map $\mathrm{Tot}(F'^{\bullet}_{\mathscr{U}}) \to \mathrm{holim}\ F'^{\bullet}_{\mathscr{U}}$ is a weak homotopy equivalence by 5.25 and 5.37. By 5.38 and 5.39 I may naturally replace any presheaf of fibrant spectra F by a weak homotopy equivalent presheaf of fibrant group spectra F'. By the above and 5.8, there is a natural chain of weak equivalences between $\check{\mathbb{H}}^{\bullet}(\mathscr{U}; F) = \mathrm{holim}\ F^{\bullet}_{\mathscr{U}}$ and $\mathrm{Tot}\ F'^{\bullet}_{\mathscr{U}}$.

DEFINITION 1.11. — Let $\mathscr{U} = \{U_i \to X \mid i \in I\}$ and $\mathscr{V} = \{V_j \to X \mid j \in J\}$ be two covers of X in a site. A map of covers $\mathscr{U} \to \mathscr{V}$ consists of a function $\varphi : J \to I$ and for each $j \in J$ a morphism $f_j : V_j \to U_{\varphi(j)}$ compatible with the projection to X. $\mathscr{V}$ is a refinement of $\mathscr{U}$ if there is a map of covers $\mathscr{U} \to \mathscr{V}$.

LEMMA 1.12. — *A map of covers $\mathscr{U} \to \mathscr{V}$ induces a cosimplicial map $F^{\bullet}_{\mathscr{U}} \to F^{\bullet}_{\mathscr{V}}$, and a map of fibrant spectra $\check{\mathbb{H}}^{\bullet}(\mathscr{U}; F) \to \check{\mathbb{H}}^{\bullet}(\mathscr{V}; F)$. Thus $\check{\mathbb{H}}^{\bullet}(\ ; F)$ is functorial in covers. (Later I'll show any two maps of covers $\mathscr{U} \rightrightarrows \mathscr{V}$ induce homotopic maps of Čech hypercohomology spectra.)*

Proof. — The cosimplicial map $F^{\bullet}_{\mathscr{U}} \to F^{\bullet}_{\mathscr{V}}$ is determined as follows. Let $j_* = (j_0, j_1, \ldots, j_n) \in J^{n+1}$. Then the projection of $F^n_{\mathscr{U}} \to F^n_{\mathscr{V}}$ on the factor indexed by j_* is defined to be the projection of $F^n_{\mathscr{U}}$ on the factor indexed by $\varphi(j_*) = (\varphi j_0, \varphi j_1, \ldots, \varphi j_n)$ followed by a map $F(U(\varphi(j_*))) \to F(V(j_*))$. This map results from applying F to the map $V(j_*) \to U(\varphi(j_*))$ induced by the $f_j : V_j \to U_{\varphi(j)}$.

To obtain the map on $\check{\mathbb{H}}^{\bullet}(\ ; F)$, apply holim to the cosimplicial map.

DEFINITION 1.13. — For **C** a site with terminal object X, the category of covers of **C** is the category of covers of the terminal object X with morphisms as in 1.11.

A category of covers may be defined without assuming **C** has a terminal object by means of the Yoneda embedding of **C** in $\mathbf{C}^{\wedge}$. (*See* [SGA 4]).

CONSTRUCTION 1.14. — Let $\mathscr{A}$ be a directed or filtering system of covers $\mathscr{U}_\alpha$, $\alpha \in \mathscr{A}$ in the site **C**. Let $F^{\bullet}_{\mathscr{A}}$ be the cosimplicial fibrant spectrum which is the colimit of the $F^{\bullet}_{\mathscr{U}_\alpha}$ along $\mathscr{A}$. Let $\check{\mathbb{H}}^{\bullet}(\mathscr{A}; F)$ be $\mathrm{holim}\ F^{\bullet}_{\mathscr{A}}$.

Choose a filtering system $\mathscr{A}$ of covers of **C** weakly cofinal in the category of covers, and define the Čech hypercohomology spectrum of **C** with coefficients in F to be

$$\check{\mathbb{H}}^{\bullet}(\mathbf{C}; F) = \check{\mathbb{H}}^{\bullet}(X; F) = \underset{\Delta}{\mathrm{holim}}\ F^{\bullet}_{\mathscr{A}}. \tag{1.7}$$

The augmentations $F(X) \to F^{\bullet}_{\mathscr{U}}$ induce natural maps

$$\begin{aligned} F(X) &\to \check{\mathbb{H}}^{\bullet}(\mathscr{A}; F), \\ F(X) &\to \check{\mathbb{H}}^{\bullet}(X; F). \end{aligned} \tag{1.8}$$

Here "weakly cofinal" means that every cover of **C** has a refinement in the filtering system. I'll show below that a different choice of weakly cofinal $\mathscr{A}$ yields a weak homotopy equivalent $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; F)$. For now define $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; F)$ using the direct system of sieves or "cribles" as in [SGA 4], II 1.1.1, II 1.3. (The site **C** may have to be enlarged to $\mathbf{C}^{\wedge}$ with a Grothendieck topology to make this work, but the bigger site has the same

Čech cohomology so this is harmless. The site Etr/X of 1.3 must be enlarged as sieves aren't finite families. The site of 1.3 with the finiteness restriction dropped and the sites of 1.1 and 1.2 have weakly cofinal systems of sieves as is.)

LEMMA 1.15. — *Let* F, G, H *be presheaves of fibrant spectra on* **C**, $\mathcal{U}$ *a cover of* **C**, *and* $\mathcal{A}$ *a filtering system of covers. Then:*

(i) $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F)$, $\check{\mathbb{H}}^{\cdot}(\mathcal{A}; F)$, $\check{\mathbb{H}}^{\cdot}(\mathbf{C}; F)$ *are fibrant spectra.*

(ii) $\check{\mathbb{H}}^{\cdot}(\mathcal{U};\ )$, $\check{\mathbb{H}}^{\cdot}(\mathcal{A};\ )$, $\check{\mathbb{H}}^{\cdot}(\mathbf{C};\ )$ *preserve finite limits and hence group objects.*

(iii) *If* $f: F \to G$ *is a weak homotopy equivalence or fibration, so are the induced maps* $\check{\mathbb{H}}^{\cdot}(\mathcal{U}; f)$, $\check{\mathbb{H}}^{\cdot}(\mathcal{A}; f)$, $\check{\mathbb{H}}^{\cdot}(\mathbf{C}; f)$.

(iv) *If* $F \to G \to H$ *is a homotopy fibre sequence, so is*

$$\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; G) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}; H),$$

and similarly for $\check{\mathbb{H}}^{\cdot}(\mathcal{A};\ )$ *and* $\check{\mathbb{H}}^{\cdot}(\mathbf{C};\ )$.

Proof. — Čech hypercohomology preserves these properties as the Čech construction 1.8, filtering colimits, and homotopy limits do. (*See* 5.5, 5.8, 5.9, 5.11, 5.12.)

PROPOSITION 1.16. — *Let* F *be a presheaf of fibrant spectra on* **C**, $\mathcal{U}$ *a cover of* **C**, *and* $\mathcal{A}$ *a filtering system of covers. There are hypercohomology spectral sequences*

$$(1.9)\qquad \left\{\begin{array}{lcl} E_2^{p,q} = \check{H}^p(\mathcal{U}; \pi_q F) & \Rightarrow & \pi_{q-p}\check{\mathbb{H}}^{\cdot}(\mathcal{U}; F), \\ & & \qquad\qquad p \geqq 0, \\ E_2^{p,q} = \check{H}^p(\mathcal{A}; \pi_q F) & \Rightarrow & \pi_{q-p}\check{\mathbb{H}}^{\cdot}(\mathcal{A}; F), \\ & & \qquad\qquad \infty > q > -\infty, \\ E_2^{p,q} = \check{H}^p(\mathbf{C}; \pi_q F) & \Rightarrow & \pi_{q-p}\check{\mathbb{H}}^{\cdot}(\mathbf{C}; F), \end{array}\right.$$

The indexing is funny so differentials run $d_r: E_r^{p,q} \to E_r^{p+r,\, q+r-1}$.

The E_2 *terms are the usual Čech cohomology groups for the presheaf of abelian groups* $\pi_q F$. *Thus* $\check{H}^p(\mathcal{A}; \pi_q F) = \varinjlim_{\alpha} \check{H}^p(\mathcal{U}_\alpha; \pi_q F)$.

The spectral sequences converge strongly if there is an N *such that* $\pi_q F = 0$ *for* $q > N$ *or if* $E_2^{p,q} = 0$ *for* $p > N$. *In general, the discussion of convergence in* 5.44-5.48 *applies to these spectral sequences.*

Proof. — The spectral sequences are special cases of the spectral sequence of 5.13 for π_* holim. This yields all but the identification of the E_2 term with Čech cohomology. This identification results from 5.31, as the complex (5.16) applied to $F^{\cdot}_{\mathcal{U}}$ or $F^{\cdot}_{\mathcal{A}}$ yields the usual Čech complex, as the functor π_q preserves products and filtering colimits.

LEMMA 1.17. — *If* F *is a presheaf of fibrant spectra such that* $\pi_q F = 0$ *for* $q > N$, *and* $\mathcal{A} = \{\mathcal{U}_\alpha\}$ *is a filtering system of covers, then the natural map*

$$(1.10)\qquad \varinjlim_{\alpha} \check{\mathbb{H}}^{\cdot}(\mathcal{U}_\alpha; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{A}; F),$$

is a weak homotopy equivalence.

Proof. — This follows by comparing the strongly converging spectral sequences of 1.16 for the two sides, as in 5.50. (The result requires some hypothesis on F or on the $\mathscr{U}_\alpha$ because of the phenomena discussed in 5.49.)

LEMMA 1.18. — *Let* $F\{n\}$ *be a tower of fibrations in the category of presheaves of fibrant spectra, with inverse limit* $F = \varprojlim_n F\{n\}$. *Then* $\check{\mathbb{H}}^{\bullet}(\mathscr{A}; F\{n\})$ *is a tower of fibrations. Suppose for all q there exists an* N_q *such that* $\pi_q F \to \pi_q F\{n\}$ *is an isomorphism for all* $n > N_q$. *Then the canonical map is a weak homotopy equivalence*

$$(1.11) \qquad \check{\mathbb{H}}^{\bullet}(\mathscr{A}; F) \xrightarrow{\sim} \varprojlim_n \check{\mathbb{H}}^{\bullet}(\mathscr{A}; F\{n\}).$$

If $\mathscr{A}$ *is a single cover* $\mathscr{U}$, (1.11) *is a weak homotopy equivalence without additional hypothesis beyond* $F = \varprojlim_n F\{n\}$.

Proof. — The first statement follows as filtering colimits over $\mathscr{A}$ and holim over Δ preserve fibrations by 5.5 and 5.9. As holim over Δ commutes with holim along a tower by 5.7, the last statement holds and the third reduces to showing (1.12) is a weak homotopy equivalence

$$(1.12) \qquad F^{\bullet}_{\mathscr{A}} = \varinjlim_\alpha \varprojlim_n F\{n\}^{\bullet}_{\mathscr{U}_\alpha} \to \varprojlim_n \varinjlim_\alpha F\{n\}^{\bullet}_{\mathscr{U}_\alpha}.$$

To check this it suffices to see this map induces an isomorphism on π_q in each cosimplicial codimension. But the hypothesis identifies π_q of both sides of (1.12) to $\pi_q F\{n\}^{\bullet}_{\mathscr{U}_\alpha}$ for any $n > N_q$.

DEFINITION 1.19. — If $\mathscr{U} = \{U_i \to U \mid i \in I\}$ and $\mathscr{V} = \{V_j \to V \mid j \in J\}$ are covers, let $\mathscr{U} \underset{X}{\times} \mathscr{V}$ be the cover $\{U_i \underset{X}{\times} V_j \to U \underset{X}{\times} V \mid (i, j) \in I \times J\}$. The axioms of Grothendieck topologies and a simple argument show this is indeed a cover of $U \underset{X}{\times} V$. If $\mathscr{V}$ is the cover $\{V = V\}$, denote $\mathscr{U} \underset{X}{\times} \mathscr{V}$ by $\mathscr{U} \underset{X}{\times} V$. If X is the terminal object, the subscript X may be dropped. Similarly, if $U = V = X$, the cover may be denoted $\mathscr{U} \times \mathscr{V}$ by abuse of notation, or by changing the site **C** to the local site **C**/U.

LEMMA 1.20. — *Let* (φ, f), (ψ, g) $\mathscr{U} \to \mathscr{V}$ *be two morphisms of covers of* X *as in* 1.11. *Then the two induced maps of* 1.12, $\check{\mathbb{H}}^{\bullet}(\mathscr{U}; F) \to \check{\mathbb{H}}^{\bullet}(\mathscr{V}; F)$ *agree in the stable homotopy category.*

Proof. — Consider the cover $\mathscr{U} \times \mathscr{U}$. There is a map of covers $\mathscr{U} \times \mathscr{U} \to \mathscr{V}$ given by $\varphi \perp \psi : J \to I \times I$ and $f_j \perp g_j : V_j \to U_{\varphi(j)} \times U_{\psi(j)}$. There are two maps $\mathscr{U} \to \mathscr{U} \times \mathscr{U}$ given by the two projections $I \times I \to I$ with $U_i \times U'_i \to U_i$ or U'_i. The two composites $\mathscr{U} \to \mathscr{U} \times \mathscr{U} \to \mathscr{V}$ are the two original maps. As $\check{\mathbb{H}}^{\bullet}(\ ; F)$ is a functor, it suffices to show the two maps $\mathscr{U} \to \mathscr{U} \times \mathscr{U}$ induce homotopic maps on $\check{\mathbb{H}}^{\bullet}(\ ; F)$.

There is a map of covers $\mathscr{U} \times \mathscr{U} \to \mathscr{U}$ given by the diagonals $I \to I \times I$ and $U_i \to U_i \times U_i$. Either of the two composites $\mathscr{U} \to \mathscr{U} \times \mathscr{U} \to \mathscr{U}$ is the identity. Thus the two maps $\check{\mathbb{H}}^{\cdot}(\mathscr{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{U} \times \mathscr{U}; F)$ will agree in the stable homotopy category if $\check{\mathbb{H}}^{\cdot}(\mathscr{U} \times \mathscr{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{U}; F)$ is a weak homotopy equivalence, for both maps will then be the unique homotopy inverse in the stable category.

To show $\check{\mathbb{H}}^{\cdot}(\mathscr{U} \times \mathscr{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{U}; F)$ is a weak homotopy equivalence, by 1.18 it suffices to do it for each Postnikov stage $F\langle n \rangle$. (*See* 5.51 for the Postnikov tower.) So replacing F by $F\langle n \rangle$, I may assume $\pi_q F = 0$ for $q > n$. Then the spectral sequences of 1.16 converge strongly. It suffices to show the map induces an isomorphism on the E_2 term of the spectral sequences. For any presheaf of abelian groups A, it is well-known that any two maps of covers induce the same map on Čech cohomology $\check{H}^*(\mathscr{U}; A) \to \check{H}^*(\mathscr{V}; A)$, e.g. [3], I 3.4 or [SGA 4], V 2.3.5. Thus $\check{H}^*(\mathscr{U} \times \mathscr{U}; A) \to \check{H}^*(\mathscr{U}; A)$ is an isomorphism with inverse either map $\check{H}^*(\mathscr{U}; A) \to \check{H}^*(\mathscr{U} \times \mathscr{U}; A)$; for both composites are the identity. Thus $\check{\mathbb{H}}^{\cdot}(\mathscr{U} \times \mathscr{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{U}; F)$ induces an isomorphism on the E_2 terms of the spectral sequences, and so is a weak homotopy equivalence as required.

It is possible, but technically gruesome to generalize the usual proof of [3], I 3.4 to produce an explicit homotopy between the two maps $\check{\mathbb{H}}^{\cdot}(\mathscr{U}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{V}; F)$.

COROLLARY 1.21. — *If $\mathscr{U}$ and $\mathscr{V}$ are covers of* X *which are each refinements of the other, then $\check{\mathbb{H}}^{\cdot}(\mathscr{U}; F)$ and $\check{\mathbb{H}}^{\cdot}(\mathscr{V}; F)$ are weak homotopy equivalent.*

Proof. — By hypothesis there are maps of covers $\mathscr{U} \to \mathscr{V}$, $\mathscr{V} \to \mathscr{U}$. By 1.20 these induce inverse weak homotopy equivalences.

LEMMA 1.22. — *Let $\mathscr{U}_\alpha$, $\alpha \in \mathscr{A}$ and $\mathscr{U}_\beta$, $\beta \in \mathscr{B}$ be two filtering systems of covers of the terminal object* X *in a site* **C**. *Suppose for each $\mathscr{U}_\alpha$ there is a $\mathscr{U}_\beta$ which refines it, and conversely that each $\mathscr{U}_\beta$ is refined by some $\mathscr{U}_\alpha$. Then for any presheaf* F *there is a natural weak homotopy equivalence between $\check{\mathbb{H}}^{\cdot}(\mathscr{A}; F)$ and $\check{\mathbb{H}}^{\cdot}(\mathscr{B}; F)$.*

Proof. — Without changing the Čech hypercohomology I may add the trivial cover of X by X as an initial object to both systems of covers $\mathscr{A}$ and $\mathscr{B}$. Let $\mathscr{A} \times \mathscr{B}$ be the filtering system of covers $\mathscr{U}_\alpha \times \mathscr{U}_\beta$ for $(\alpha, \beta) \in \mathscr{A} \times \mathscr{B}$. There are maps of filtering systems $\mathscr{A} \to \mathscr{A} \times \mathscr{B}$, $\mathscr{B} \to \mathscr{A} \times \mathscr{B}$ sending $\mathscr{U}_\alpha$ to $\mathscr{U}_\alpha \times X$, etc. It suffices to show these maps induce weak homotopy equivalences on $\check{\mathbb{H}}^{\cdot}(\ ; F)$.

Note each cover in $\mathscr{A} \times \mathscr{B}$ is refined by one in $\mathscr{A}$, and conversely. For if $\mathscr{U}_\alpha \times \mathscr{U}_\beta$ is in $\mathscr{A} \times \mathscr{B}$ and $\mathscr{U}_\beta$ is refined by $\mathscr{U}_{\alpha'}$ in $\mathscr{A}$, let $\mathscr{U}_{\alpha''}$ be a common refinement of $\mathscr{U}_\alpha$ and $\mathscr{U}_{\alpha'}$ in the filtering system $\mathscr{A}$. Then $\mathscr{U}_{\alpha''}$ refines $\mathscr{U}_{\alpha''} \times \mathscr{U}_{\alpha''}$, and so refines $\mathscr{U}_\alpha \times \mathscr{U}_{\alpha'}$ and $\mathscr{U}_\alpha \times \mathscr{U}_\beta$. Conversely, $\mathscr{U}_\alpha$ is refined by $\mathscr{U}_\alpha \times X$ in $\mathscr{A} \times \mathscr{B}$.

To show $\check{\mathbb{H}}^{\cdot}(\mathscr{A}; F) \to \check{\mathbb{H}}^{\cdot}(\mathscr{A} \times \mathscr{B}; F)$ is a weak homotopy equivalence, it suffices by 1.18 and 5.51 to do this for F replaced by each of its Postnikov stages $F\langle n \rangle$. Thus I may assume there is an n such that $\pi_q F = 0$ for $q > n$. Then by 1.17, the problem is to show (1.13) is a weak homotopy equivalence:

$$\varinjlim_{\alpha} \check{\mathbb{H}}^{\cdot}(\mathscr{U}_\alpha; F) \to \varinjlim_{(\alpha,\beta)} \check{\mathbb{H}}^{\cdot}(\mathscr{U}_\alpha \times \mathscr{U}_\beta; F). \tag{1.13}$$

As homotopy groups preserve filtering colimits, it suffices to show that (1.14) is an isomorphism.

$$(1.14)\qquad \varinjlim_{\alpha} \pi_* \check{\mathbb{H}}^{\bullet}(\mathscr{U}_\alpha; \mathrm{F}) \to \varinjlim_{(\alpha,\beta)} \pi_* \check{\mathbb{H}}^{\bullet}(\mathscr{U}_\alpha \times \mathscr{U}_\beta; \mathrm{F}).$$

The hypothesis on $\mathscr{A}$ and $\mathscr{A} \times \mathscr{B}$ do not imply that either system of covers is strictly cofinal with respect to the other in the category of all covers of $\mathbf{C}$. However, consider the poset of covers formed from the category of covers by identifying all different maps of covers $\mathscr{U} \to \mathscr{V}$ for the same $\mathscr{U}$ and $\mathscr{V}$, and identifying $\mathscr{V}$ and $\mathscr{U}$ if they refine each other. By Lemmas 1.20 and 1.21, the functor $\pi_* \check{\mathbb{H}}(\ ;\mathrm{F})$ on the category of covers factors through this poset. The cofinality of $\mathscr{A}$ and $\mathscr{A} \times \mathscr{B}$ with respect to each other in this poset (above) shows that (1.14) is an isomorphism as required.

Remark. — The Lemma 1.22 shows that I get weak homotopy equivalent $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; \mathrm{F})$ when I use any two filtering systems of covers of $\mathbf{C}$ weakly cofinal in the category of all covers to define it. Recall $\mathscr{A}$ is weakly cofinal in the category of all covers if each cover has a refinement in $\mathscr{A}$. In particular, $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; \mathrm{F})$ depends on the topos $\mathbf{C}^{\sim}$, and not on the choice of a particular site $\mathbf{C}$, as covers of X in the site are weakly cofinal in covers of X in the topos, and conversely. Thus the Čech hypercohomology of the etale site and of the restricted etale site of 1.2 and 1.3 agree.

DEFINITION 1.23. — For $\mathbf{C}$ a site, the canonical $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; \mathrm{F})$ is $\check{\mathbb{H}}^{\bullet}(\mathscr{A}; \mathrm{F})$ for the following canonical direct system of covers of $\mathbf{C}$. Let $\{\mathscr{U}_\lambda, \lambda \in \Lambda\}$ be the set of all covers of $\mathbf{C}$ in the universe. Let $\mathscr{A}$ be the directed system of finite subsets of Λ. For $\alpha = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, the λ_i distinct, define $\mathscr{U}_\alpha$, independent of the order of the λ_i up to isomorphism, by

$$(1.15)\qquad \mathscr{U}_\alpha = \mathscr{U}_{\lambda_1} \times \mathscr{U}_{\lambda_2} \times \ldots \times \mathscr{U}_{\lambda_n}.$$

If $\alpha \subseteq \alpha'$ in $\mathscr{A}$, $\mathscr{U}_\alpha \to \mathscr{U}_{\alpha'}$ is the map of covers induced by the obvious projections, or equivalently by the maps of covers $\mathrm{X} \to \mathscr{U}_\lambda$ for λ in $\alpha' - \alpha$.

Clearly $\mathscr{A}$ is weakly cofinal in the category of all covers.

This $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; \mathrm{F})$ is weak homotopy equivalent to the version using the direct system of sieves by 1.22, so 1.15 and the spectral sequence of 1.16 are valid for it.

The canonical $\check{\mathbb{H}}^{\bullet}(\mathbf{C}; \mathrm{F})$ is functorial in $\mathbf{C}$, while the sieve version is only functorial up to homotopy. I will now elaborate on this point.

1.24. Recall ([SGA 4], III 1, IV 4.9, or [3], II 4.5) that a morphism of sites $f: \mathbf{C} \to \mathbf{D}$ is a functor $f^*: \mathbf{D} \to \mathbf{C}$ (backwards!) which preserves finite limits and sends covers of any object D to covers of $f^*\mathrm{D}$. Given a presheaf F on $\mathbf{C}$, there is an induced presheaf $f_\# \mathrm{F}$ on D with $(f_\# \mathrm{F})(\mathrm{V}) = \mathrm{F}(f^* \mathrm{V})$.

CONSTRUCTION 1.25. — Let $f: \mathbf{C} \to \mathbf{D}$ be a morphism of sites, F a presheaf on $\mathbf{C}$, V an object of $\mathbf{D}$, and $\mathscr{V} = \{\mathrm{V}_i \to \mathrm{V}\}$ a cover. Let $f^*\mathscr{V} = \{f^*\mathrm{V}_i \to f^*\mathrm{V}\}$ be the induced cover of $f^*\mathrm{V}$ in $\mathbf{C}$. Then there is a natural isomorphism (1.16) induced by the isomorphism of cosimplicial fibrant spectra (1.17):

$$(1.16)\qquad \check{\mathbb{H}}^{\bullet}(\mathscr{V}; f_\# \mathrm{F}) \cong \check{\mathbb{H}}^{\bullet}(f^*\mathscr{V}; \mathrm{F}).$$

$$
(1.17)\qquad
\begin{tikzcd}
(f_* F)^{\cdot}_{\mathscr{V}} \arrow[d, phantom, "\wr\|"] & \prod\limits_{i_0} (f_* F)(V_{i_0}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, phantom, "\|"] & \prod\limits_{(i_0,\, i_1)} (f_* F)(V_{i_0} \underset{V}{\times} V_{i_1}) \cdots \arrow[d, phantom, "\|\wr"] \\
(F)^{\cdot}_{f^* \mathscr{V}} & \prod\limits_{i_0} F(f^* V_{i_0}) \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{(i_0,\, i_1)} F(f^* V_{i_0} \underset{f^* V}{\times} f^* V_{i_1}) \cdots
\end{tikzcd}
$$

If $\mathscr{A}$ is a filtering system of covers $\mathscr{V}_\alpha$ of the terminal object of **D**, let $f^* \mathscr{A}$ be the filtering system of covers $f^* \mathscr{V}_\alpha$ of the terminal object in **C**. There is then an isomorphism

$$(1.18)\qquad \check{\mathbb{H}}^{\cdot} (\mathscr{A}; f_* F) \cong \check{\mathbb{H}}^{\cdot} (f^* \mathscr{A}; F).$$

If $\mathscr{A}$ is the canonical filtering system of covers of **D** as in 1.23, then $f^* \mathscr{A}$ is a subsystem of the canonical filtering system of covers of **C**. This inclusion induces a morphism of the filtering colimit of Čech complexes 1.8. Applying holim along Δ to this morphism and composing with the isomorphism (1.18) yields a natural map (1.19)

$$(1.19)\qquad \check{\mathbb{H}}^{\cdot} (\mathbf{C}; f_* F) \to \check{\mathbb{H}}^{\cdot} (\mathbf{C}; F).$$

Thus Čech hypercohomology is functorial in the usual way with respect to sites and presheaves of fibrant spectra on them.

1.26. Given $f: \mathbf{C} \to \mathbf{D}$ a morphism of sites as above and G a presheaf of fibrant spectra on **D**, define $f^{\#} G$ by

$$(1.20)\qquad (f^{\#} G)(U) = \varinjlim_{(U/f)^{op}} G(V).$$

Here $(U/f)^{op}$ is the opposite of the category U/f whose objects are objects V of **D** together with a morphism $U \to f^* V$. The morphisms of U/f are morphisms in **D** that preserve the given maps $U \to f^*(V)$. As **D** is a site and f^* preserves fibres products, it is easy to see that $(U/f)^{op}$ is a filtering category. Thus by 5.5, $f^{\#} G$ is a fibrant spectrum, and the functor $f^{\#}$ preserves products, fibrations, weak homotopy equivalences, and homotopy fibre sequences. As π_q commutes with filtering colimits, $\pi_q f^{\#} G$ is the usual pullback of the presheaf $\pi_q G$, namely $f^{\#} \pi_q G$. The functor f_* is right adjoint to $f^{\#}$ by the usual calculation.

1.27. Before constructing sheaf hypercohomology, it is necessary to recall some facts about the points of a topos from [SGA 4], IV, § 6. The category of points of a topos $\mathbf{C}^\sim$ is the category of morphisms from the topos which is the category of sets to the topos $\mathbf{C}^\sim$. This category is equivalent to the opposite category of the category of functors $f^*: \mathbf{C}^\sim \to$ Sets such that f^* preserves colimits and finite limits. For such an f^* has a right adjoint f_*, and the pair (f^*, f_*) is a morphism of topoi.

Let f^* be a point of $\mathbf{C}^\sim$. There is a filtering diagram $\mathrm{Nbd}(f)$ in the opposite category $\mathbf{C}^{op}$ of the site. An object of $\mathrm{Nbd}(f)$ is a U in **C** with a lift of the point f^*

from $\mathbf{C}^\sim$ to $\mathbf{C}^\sim/\mathrm{U}$. A morphism in $\mathrm{Nbd}(f)^{\mathrm{op}}$ is a morphism in $\mathbf{C}$ that respects the lifted points. One has for any sheaf F in $\mathbf{C}^\sim$ the isomorphism

$$(1.21) \qquad f^*(\mathrm{F}) \cong \varinjlim_{\mathrm{U} \in \mathrm{Nbd}(f)} \mathrm{F}(\mathrm{U}).$$

If F is a presheaf, the direct limit in (1.21) is isomorphic to f^* of the sheafification $\tilde{\mathrm{F}}$ of F. Consult [SGA 4], IV 6.8 for details.

For F a set and f^* a point, the sheaf $f_*(\mathrm{F})$ in $\mathbf{C}^\sim$ is a product of copies of F indexed by the elements of $f^*(\mathrm{U})$

$$(1.22) \qquad (f_*(\mathrm{F}))(\mathrm{U}) = \prod_{f^*(\mathrm{U})} \mathrm{F}.$$

For it is easy to see this f_* is right adjoint to f^*.

Since the category of fibrant spectra is closed under products and filtering colimits (*see* 5.5), the functors f_* and f^* given by (1.21) and (1.22) preserve fibrant spectra. They also commute with the homotopy group functors π_q in that $\pi_q f^* = f^* \pi_q$, etc.

Example 1.28. — Let X be a sober topological space; e.g., a Hausdorff space, or a scheme with the Zariski topology. Let $\mathbf{C}^\sim$ be its category of sheaves. Then the usual points of X are exactly the points of $\mathbf{C}^\sim$, with $x \in \mathrm{X}$ corresponding to the functor f^* that sends a sheaf F to its stalk at x, F_x. The category $\mathrm{Nbd}(f)$ is the poset of open neighborhoods of x. The functor f_* sends a set F to the corresponding skyscraper sheaf at x. For details, *see* [SGA 4], IV 7.1.

Example 1.29. — Let X be a scheme, and consider the topos of sheaves on the etale site Et/X of 1.2. Then by [SGA 4], VIII 7.9 the isomorphism classes of the points of $\mathrm{Et}/\mathrm{X}^\sim$ correspond bijectively to the points of the scheme X in the usual sense. For $x \in \mathrm{X}$, let $k(x)$ be the residue field of X at x, and let $k\overline{(x)}$ be some separably closed field containing $k(x)$. Let $f: \mathrm{Spec}(k\overline{(x)}) \to \mathrm{X}$ be the map to X factoring through $\mathrm{Spec}(k(x))$. The topos $\mathrm{Et}/k\overline{(x)}^\sim$ is the category of sets, so $f: \mathrm{Et}/k\overline{(x)}^\sim \to \mathrm{Et}/\mathrm{X}^\sim$ is a point of $\mathrm{Et}/\mathrm{X}^\sim$. The category $\mathrm{Nbd}(f)$ is the opposite of the category of schemes U etale over X with a distinguished point $\mathrm{Spec}(k\overline{(x)}) \to \mathrm{U}$. The inverse limit of this diagram of schemes is the scheme $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x})$, where $\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x}$ is the strict local henselization of X at x. Let $g: \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x}) \to \mathrm{X}$ be the canonical map. By [SGA 4], VIII 4.8 there is an isomorphism of $f^*\mathrm{F}$ with $(g^\# \mathrm{F})(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x})$ for any presheaf F on Et/X. If F is continuous in the sense of 1.42 below, $(g^\# \mathrm{F})(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x})$ is just $\mathrm{F}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},\, x})$ as explained in 1.44. *See* [SGA 4], VIII, § 3, 4 and 7, for more details about stalks in the etale topology.

1.30. A topos $\mathbf{C}^\sim$ is said to have enough points if it satisfies the condition that if a morphism γ of $\mathbf{C}^\sim$ is such that $f^*(\gamma)$ is an isomorphism for each point f, then γ is an isomorphism in $\mathbf{C}^\sim$. If $\mathbf{C}^\sim$ has enough points, then a morphism γ in $\mathbf{C}^\sim$ is a monomorphism or epimorphism respectively if and only if $f^*(\gamma)$ is such for every point f. *See* [SGA 4], IV 6.4.

The topological topoi of 1.28 and the etale topos of a scheme of 1.29 have enough points by [SGA 4], IV 7.1.6 and VIII 3.5. Most topoi of interest in topology or algebraic geometry have enough points. Some topoi arising from logic don't have enough points, and Deligne's measure-theoretic topos has no points at all ([SGA 4], IV 7.4).

CONSTRUCTION 1.31. — Let $\mathbf{C}^\sim$ be a topos with enough points. Let $\mathscr{P}$ be the set of points of $\mathbf{C}^\sim$, and let $\mathbf{P}^\sim$ be the product of the category of sets with one copy for every point in $\mathscr{P}$. Then $\mathbf{P}^\sim$ is the coproduct of the topos of sets indexed by $\mathscr{P}$. The points induce a map of topoi $p:\mathbf{P}^\sim \to \mathbf{C}^\sim$ consisting of an adjoint pair of functors p^* and p_*.

In the factor of $\mathbf{P}^\sim$ indexed by the point f, the projection of $p^*(\mathrm{F})$ is $f^*\mathrm{F}$. If the object G of $\mathbf{P}^\sim$ consists of the set G_f in the factor indexed by f, then $p_*\mathrm{G}$ is the product $\Pi f_*(\mathrm{G}_f)$. (To avoid set-theoretic problems, one must fix a site $\mathbf{C}$ for $\mathbf{C}^\sim$, and then work in a universe in which $\mathbf{C}$ is small. Let $\mathscr{P}$ be the set of points of $\mathbf{C}$, as in [SGA 4], IV 6.3. By [SGA 4], IV 6.5, $\mathscr{P}$ is a conservative family; i.e., enough points, for the topos $\mathbf{C}^\sim$ in any bigger universe. Choose a bigger universe in which $\mathscr{P}$ is small.)

Let T be the functor p_*p^* from $\mathbf{C}^\sim$ to $\mathbf{C}^\sim$. Thus TF is given by the formula (1.23) for U in $\mathbf{C}$

$$\mathrm{TF}(\mathrm{U}) = \prod_{f \in \mathscr{P}} (f_* f^* \mathrm{F})(\mathrm{U}) = \prod_{f \in \mathscr{P}} \prod_{f_*(\mathrm{U})} f^*\mathrm{F}. \tag{1.23}$$

The adjunction morphisms $\eta:\mathrm{Id}\to p_*p^*$, $\varepsilon: p^*p_* \to \mathrm{Id}$ induce natural transformations $\eta:\mathrm{Id}\to\mathrm{T}$, $\mu:\mathrm{TT}\to\mathrm{T}$ with $\mu = p_*\varepsilon p^*$. The Godement complex is the cosimplicial sheaf

$$\mathrm{T}^\bullet\mathrm{F} = \mathrm{TF} \overset{\leftarrow}{\rightrightarrows} \mathrm{TTF} \overset{\leftleftarrows}{\rightrightarrows\!\!\!\to} \mathrm{TTTF} \cdots \tag{1.24}$$

The sheaf in codimension n is $\mathrm{T}^{n+1}\mathrm{F}$. The coface map $d^i:\mathrm{T}^{n+1}\mathrm{F}\to\mathrm{T}^{n+2}\mathrm{F}$ is $\mathrm{T}^i\eta\mathrm{T}^{n+1-i}$. The codegeneracy $s^i:\mathrm{T}^{n+1}\mathrm{F}\to\mathrm{T}^n\mathrm{F}$ is $\mathrm{T}^i\mu\mathrm{T}^{n-i}$. The simplicial identities 5.12 follow from the adjunction identities and naturality of T, as in [46], Appendix. The map η induces a canonical augmentation $\eta:\mathrm{F}\to\mathrm{T}^\bullet\mathrm{F}$.

$\sqcup$Let $g:\mathbf{C}'\to\mathbf{C}$ be a morphism of sites 1.24. Let $g_*:\mathbf{C}'^\sim\to\mathbf{C}^\sim$ be the induced map of topoi. If $f_*:\mathit{Sets}\to\mathbf{C}'^\sim$ is a point of $\tilde{\mathbf{C}}'$, then g_*f_* is a point of $\mathbf{C}^\sim$. Thus g induces a function $g':\mathscr{P}'\to\mathscr{P}$ on the sets of points, and a morphism $g'_*:\mathbf{P}'^\sim\to\mathbf{P}^\sim$ of topoi compatible with g_* under p_* and p'_*. The isomorphism $g_*p'_*p'^*g^*\cong p_*g'_*g'^*p^*$ and the adjunction maps of the pairs $g^*\dashv g_*$ and $g'^*\dashv g'_*$ induce a map

$$\mathrm{T}g_* = p_*p^*g_* \to p_*g'_*g'^*p^*g_* \cong g_*p'_*p'^*g^*g_* \to g_*p'_*p'^* = g_*\mathrm{T}'$$

and so a cosimplicial map

$$\mathrm{T}^\bullet(g_*\mathrm{F}) \to g_*\mathrm{T}'^\bullet(\mathrm{F}). \tag{1.25}$$

1.32. If F is a presheaf on $\mathbf{C}$, $p^*\mathrm{F}$ still makes sense and equals $p^*\tilde{\mathrm{F}}$ of the sheafification $\tilde{\mathrm{F}}$ by 1.27. Thus (1.24) gives a cosimplicial presheaf $\mathrm{T}^\bullet\mathrm{F}$ which depends only on the sheafification $\tilde{\mathrm{F}}$ of F: $\mathrm{T}^\bullet\mathrm{F}\to\mathrm{T}^\bullet\tilde{\mathrm{F}}$ is an isomorphism.

If F is a presheaf of fibrant spectra, or of fibrant group spectra, (1.23) and the last paragraph of 1.27 show that TF is a presheaf of fibrant spectra or of fibrant group spectra respectively. The complex $T^{\cdot} F$ is then a cosimplicial presheaf of fibrant spectra. The functor $T^{\cdot}$ clearly preserves finite limits, weak homotopy equivalences, fibrations, and homotopy fibre sequences, since in each cosimplicial codimension it's a composite of functors that preserve these. Also one has the formula for homotopy groups :

$$\pi_q(T^{\cdot} F) = T^{\cdot}(\pi_q F). \tag{1.26}$$

Definition 1.33. — Let F be a presheaf of fibrant spectra on a site **C**. Suppose the topos of sheaves on **C** has enough points. Then the sheaf hypercohomology spectrum of **C** with coefficients in F is the homotopy limit along Δ of the cosimplicial fibrant spectrum $(T^{\cdot} F)(X)$. This is the Godement complex of presheaves 1.32 evaluated at the terminal object X.

$$\mathbb{H}^{\cdot}(\mathbf{C};\, F) = \underset{\Delta}{\operatorname{holim}}\, T^{\cdot} F(X) = \mathbb{H}^{\cdot}(\Delta;\, T^{\cdot} F(X)). \tag{1.27}$$

If $g : \mathbf{C} \to \mathbf{C}'$ is a morphism of sites, the induced map

$$\mathbb{H}^{\cdot}(\mathbf{C}';\, g_{*} F) \to \mathbb{H}^{\cdot}(\mathbf{C};\, F), \tag{1.28}$$

is holim along Δ of the map of Godement complexes (1.25). Thus $\mathbb{H}^{\cdot}(\ ;\ )$ is a contravariant functor on the category of sites with enough points and with presheaves of fibrant spectra.

There is a natural augmentation $\eta : F(X) \to \mathbb{H}^{\cdot}(\mathbf{C};\, F)$.

1.34. The hypothesis that **C** has enough points may be dropped by replacing the Godement resolution with the flabby resolution obtained by Barr for more general topoi. *See* [7] or [57], 8.20. I won't need this however.

Lemma 1.35. — *Let* **C** *be a site with enough points. Then* $\mathbb{H}^{\cdot}(\mathbf{C};\ )$ *on the category of presheaves of fibrant spectra preserves products, fibrations, weak homotopy equivalences, and homotopy fibre sequences.*

Proof. — Immediate from 1.32 and 5.8, 5.9, 5.11, 5.12; i.e. the usual.

Proposition 1.36. — *Let* F *be a presheaf of fibrant spectra on a site* **C**. *Suppose the topos of* **C** *has enough points* 1.30. *There is a hypercohomology spectral sequence*

$$E_2^{p,\,q} = H^p(\mathbf{C};\, \tilde{\pi}_q F) \quad \Rightarrow \quad \pi_{q-p}\mathbb{H}^{\cdot}(\mathbf{C};\, F), \qquad p \geqq 0, \quad \infty > q > -\infty. \tag{1.29}$$

The indexing is funny so differentials d_r have bidegree $(r, r-1)$.

The E_2 *term is the sheaf cohomology of the topos* $\mathbf{C}^{\sim}$ *with coefficients in the sheafification of the presheaf* $\pi_q F$.

The spectral sequence converges strongly if either there is an N *such that* $\tilde{\pi}_q F = 0$ *for* $q > N$ *or if* **C** *has bounded cohomological dimension for the sheaves* $\tilde{\pi}_* F$. *In general, the discussion of convergence in* 5.44-5.48 *applies to these spectral sequences.*

Proof. — The spectral sequence is the spectral sequence of 5.13 for π_* of holim along Δ. Everything follows from 5.13 except the identification of $E_2^{p,q}$. By 5.31, E_2^{*q} is the cohomology of the chain complex corresponding to the cosimplicial abelian group $\pi_q T^{\cdot} F(X)$. By (1.26), this is isomorphic to the cosimplicial abelian group $T^{\cdot}(\pi_q F)(X)$. By 1.32, this is isomorphic to $T^{\cdot}(\tilde{\pi}_q F)(X)$, for $\tilde{\pi}_q F$ the sheafification of $\pi_q F$. Let $p: \mathbf{P}^{\sim} \to \mathbf{C}^{\sim}$ be as in 1.31. The cosimplicial abelian group $p^* T^{\cdot}(\tilde{\pi}_q F)$ has an extra codegeneracy s^{-1}, induced by $p^* TT^n = p^* p_* p^* T^n \to p^* T^n$. By the dual of 5.21, this extra codengeneracy gives a cosimplicial homotopy contraction of $p^* T^{\cdot}(\tilde{\pi}_q F)$ to the augmentation of $p^* \tilde{\pi}_q F$. Thus the chain complex $p^* T^{\cdot} \tilde{\pi}_q F$ is an exact resolution of $p^* \tilde{\pi}_q F$. As $\mathbf{C}^{\sim}$ has enough points, p^* is conservative, so $T^{\cdot} \tilde{\pi}_q F$ is an exact resolution of $\tilde{\pi}_q F$. Any sheaf in $\mathbf{P}^{\sim}$ is flabby, as $\mathbf{P}^{\sim}$ is the coproduct of flabby topoi **Sets**, so p_* of any sheaf in $\mathbf{P}^{\sim}$ is a flabby sheaf in $\mathbf{C}^{\sim}$, by [SGA 4], V 4.9. Thus $T^{\cdot} \tilde{\pi}_q F$ is a flabby resolution of $\tilde{\pi}_q F$, in fact it's the canonical Godement resolution of [SGA 4], XVII 4.2, [46], or [57], § 8.1. As sheaf cohomology may be computed by flabby resolutions, the cohomology of the global sections of the flabby resolution, $T^{\cdot}(\tilde{\pi}_q F)(X)$ is $H^*(\mathbf{C}; \tilde{\pi}_q F)$ as required.

LEMMA 1.37. — *Let* $F\{n\}$ *be a tower of fibrations in the category of presheaves of fibrant spectra, with inverse limit* $F = \varprojlim_n F\{n\}$. *Then if* $\mathbf{C}$ *has enough points,* $\mathbb{H}^{\cdot}(\mathbf{C}; F\{n\})$ *is a tower of fibrations. Suppose that for all* q *there exists an* N *such that* $\tilde{\pi}_q F \to \tilde{\pi}_q F\{n\}$ *is an isomorphism for all* $n > N$. *Then the canonical map is a weak homotopy equivalence.*

$$(1.30)\qquad \mathbb{H}^{\cdot}(\mathbf{C}; F) \xrightarrow{\sim} \varprojlim_n \mathbb{H}^{\cdot}(\mathbf{C}; F\{n\}).$$

Proof. — The first statement follows from 1.35. As holim along Δ commutes with homotopy inverse limits, the second statement reduces to showing that $T^m F$ is weak homotopy equivalent to $\varprojlim_n T^m F\{n\}$. It suffices to show that $\pi_q T^m F = \pi_q T^m F\{n\}$ for $q > N$. But this follows from the hypothesis, (1.26), and 1.32.

Remark 1.38. — For $\mathbf{C}$ with enough points, Čech cohomology may be computed as $\check{\mathbb{H}}(\mathscr{A}; F)$ for $\mathscr{A}$ the direct system of pointed covers of $\mathbf{C}$. A pointed cover is a collection $\{U_f \mid f \in \mathscr{P}\}$ indexed by the points of $\mathbf{C}$, together with a lift of the point f to U_f; i.e. together with an element of $f^*(U_f)$. Let $T'_{\mathscr{U}}$ be $p_* p^*$ for p the map of topoi $\coprod \mathbf{C}/U_f^{\sim} \to \mathbf{C}^{\sim}$. Then the Čech complex $F^{\cdot}_{\mathscr{U}}$ of 1.8 is the complex $T'^{\cdot}_{\mathscr{U}} F$. The lift of the points provides a map to the Godement complex, $\varinjlim T'^{\cdot}_{\mathscr{U}} F \to T^{\cdot} F$. Applying holim along Δ to this provides the canonical map from Čech to sheaf hypercohomology. $\check{\mathbb{H}}^{\cdot}(\mathbf{C}; F) \to \check{\mathbb{H}}^{\cdot}(\mathbf{C}; F)$. I will not use this remark below.

PROPOSITION 1.39. — *Let* **C** *be a site consisting of coherent objects and with a terminal object, so that* $\mathbf{C}^{\sim}$ *is a coherent topos in the sense of* [SGA 4], VI 2.3, VI 1.13. *Let* F_α *be a filtering system of presheaves of fibrant spectra on* **C**, *with colimit* $\varinjlim F_\alpha = F$.

Suppose either that $\mathbf{C}^{\sim}$ *has bounded cohomological dimension for all the sheaves* $\tilde{\pi} F_{\alpha'}$, *so there is an* N *such that* $H^p(\mathbf{C}^{\sim}; \tilde{\pi}_q F_\alpha) = 0$ *for all* $p > N$ *and all* q *and* α, *or else suppose that there is an* N *such that* $\tilde{\pi}_q F_\alpha = 0$ *for all* α *and all* $q > N$. *Then the natural map* (1.31) *is a weak homotopy equivalence*

$$(1.31) \qquad \varinjlim_\alpha \mathbb{H}^{\cdot}(\mathbf{C}; F_\alpha) \xrightarrow{\sim} \mathbb{H}^{\cdot}(\mathbf{C}; \varinjlim_\alpha F_\alpha).$$

See 1.40 *for a simpler statement in the case of schemes.*

Proof. — By [SGA 4], VI 5.2, coherence yields the isomorphism

$$(1.32) \qquad \varinjlim_\alpha H^p(\mathbf{C}; \tilde{\pi}_q F_\alpha) \cong H^p(\mathbf{C}; \pi_q(\varinjlim F_\alpha)^{\sim}).$$

The left-hand side of (1.32) is the E_2 term of the direct limit spectral sequence of 1.36, which by 5.50 and the uniform bound on cohomological dimension or on q for non-zero $\tilde{\pi}_q F_\alpha$ converges strongly to $\pi_* \varinjlim \mathbb{H}(\mathbf{C}; F_\alpha)$. The right-hand side of (1.32) is the E_2 term of the strongly convergent spectral sequence of 1.36. The spectral sequence comparison theorem shows (1.31) induces an isomorphism on π_*, and so is a weak homotopy equivalence.

COROLLAIRE 1.40. — *Let* X *be a noetherian separated scheme with either the Zariski or etale topology. Suppose* X *has bounded cohomological dimension for all* $\tilde{\pi}_q F_\alpha$, *or else suppose that there exists an* N *such that for all* α *and all* $q > N$ *that* $\tilde{\pi}_q F_\alpha = 0$. *Then the natural map is a weak homotopy equivalence*

$$(1.33) \qquad \varinjlim_\alpha \mathbb{H}^{\cdot}(X; F_\alpha) \xrightarrow{\sim} \mathbb{H}^{\cdot}(X; \varinjlim F_\alpha).$$

Proof. — The etale and Zariski sites of X are coherent by [SGA 4], VI 1.6.2, VI 1.22, VI 3.10. It is easy in this case to prove (1.32) directly as direct limits are exact and commute with finite products, and the assumptions on X allow cohomology to be computed by finite Čech covers and finite hypercovers, whose complexes involve only finite products. *See* [3], II 5.4. Of course, this direct proof is really the same argument as in [SGA 4], VI. For the Zariski case, one can consult [51], III 2.9.

PROPOSITION 1.41. — *Let* $\mathbf{C}_\alpha$ *be an inverse or cofiltering system of sites. Suppose each* $\mathbf{C}_\alpha^{\sim}$ *is coherent, and the bonding maps of the system* $f_{\alpha\beta}: \mathbf{C}_\alpha^{\sim} \to \mathbf{C}_\beta^{\sim}$ *are coherent morphisms of topoi in the sense of* [SGA 4], VI 3.1. *Let* F_α *be a presheaf of fibrant spectra on* $\mathbf{C}_\alpha$, *and suppose there is a compatible family of maps* $F_\beta \to f_{\alpha\beta *} F_\alpha$ *or equivalently* $f^{\#}_{\alpha\beta} F_\alpha \to F_\beta$ *over the bonding maps* $f_{\alpha\beta}$. *Let* **C** *be the inverse limit of the sites* $\mathbf{C}_\alpha$, *and* $f_\alpha: \mathbf{C} \to \mathbf{C}_\alpha$ *the* *projections. (As a category* $\mathbf{C}$ *is the direct limit of the* $\mathbf{C}_\alpha$, *as maps of sites are backwards,* cf. 1.24.) *Let* $F = \varinjlim f^*_\alpha F_\alpha$ *be the presheaf of fibrant spectra on* $\mathbf{C}$ *induced by the* F_α *on* $\mathbf{C}_\alpha$. *Suppose finally either that there is an* N *such that for* $p > N$ *and all* q *and* α $H^p(\mathbf{C}_\alpha^\sim; \tilde{\pi}_q F_\alpha) = 0$, *or else suppose there exists an* N *such that for all* α *and all* $q > N$ *that* $\tilde{\pi}_q F_\alpha = 0$. *Then* $\tilde{\mathbf{C}}$ *or* $\tilde{\pi}_q F$ *satisfies the same condition, and the natural map* (1.34) *is a weak homotopy equivalence.*

$$\varinjlim_{\alpha} \mathbb{H}^{\cdot}(\mathbf{C}_\alpha; F_\alpha) \xrightarrow{\sim} \mathbb{H}^{\cdot}(\mathbf{C}; F). \tag{1.34}$$

See 1.45 *for a simpler statement in the case of schemes.*

Proof. – As in 1.39, the strongly converging spectral sequences 1.36, the uniform bound on p or q, and 5.50, reduce this to proving (1.35) is an isomorphism

$$\varinjlim_{\alpha} H^p(\mathbf{C}_\alpha^\sim; \tilde{\pi}_q F_\alpha) \cong H^p(\tilde{\mathbf{C}}; \tilde{\pi}_q F). \tag{1.35}$$

But this is given by [SGA 4], VI 8.7.4.

Definition 1.42. – Let F be a presheaf of fibrant spectra on the category of schemes. Let X_α be an inverse or cofiltering system of schemes with affine bonding maps $f_{\alpha\beta}: X_\alpha \to X_\beta$. Then the inverse limit scheme X exists by [EGA], IV 8.2.3. If $X_\alpha = \mathrm{Spec}(A_\alpha)$ for a direct system of rings A_α. Then $X = \mathrm{Spec}(\varinjlim A_\alpha)$.

One says F is continuous or is a finitely presented functor in the sense of Grothendieck if for any such system X_α, the natural map is a weak homotopy equivalence

$$\varinjlim_{\alpha} F(X_\alpha) \xrightarrow{\sim} F(X). \tag{1.36}$$

If one restricts F to a subcategory of the category of schemes, one has F is continuous on this subcategory if (1.36) holds for inverse systems X_α in this subcategory.

Example 1.43. – The K-theory spectra $K(\ )$, $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$, $K(\ )_K$ of Appendix A are all continuous by [97], I, § 2 and the fact smash products of spectra preserve colimits. Similary the G-theory spectra $G(\ )$, $G/l^\nu(\ )$, $G/l^\nu(\ )[\beta^{-1}]$ and $G(\ )_K$ are continuous on the category of schemes etale over a fixed X, or on any subcategory of schemes in which all morphisms are flat, so that $G(\ )$ is a presheaf. The l-adic spectra $K(\ )[\beta^{-1}]_l^\wedge$, $K(\ )_l^\wedge$, $G(\ )_l^\wedge$, $G(\ )[\beta^{-1}]_l^\wedge$ are not continuous, for the inverse limits used to define them do not commute with direct limits.

1.44. Let F be a presheaf of fibrant spectra on the category of schemes, which is continuous on the etale site Et/X. Let f be a point of the topos $\mathrm{Et}/X^\sim$ corresponding to $x \in X$. The category $\mathrm{Nbd}(f)^{\mathrm{op}}$ of 1.29 has a cofinal subcategory of affine schemes. Interpreting (1.21) in light of the continuity condition (1.36) and 1.29, one sees that the stalk of F at f is the value of F at the strict local henselization of X at x,

$$f^* F \simeq F(\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\,x})). \tag{1.37}$$

Similarly, the stalk f^*F at a point of the Zariski topos of a scheme X corresponding to $x \in X$ is the value $F(\operatorname{Spec}(\mathcal{O}_{X,x}))$ of F at the local ring of x in X, provided F is continuous on the Zariski site.

COROLLARY 1.45. — *Let* F *be a presheaf of fibrant spectra on the category of schemes which is continuous as in* 1.42. *Let* X_α *be an inverse system of schemes with affine bonding maps* $X_\alpha \to X_\beta$. *Suppose each* X_α *is noetherian and separated. Let* X *be the inverse limit scheme. Suppose either that the* X_α *have uniformly bounded Zariski* (*or etale*) *cohomological dimension for the sheaves* $\tilde{\pi}_* F$, *or else suppose there is an* N *such that for all* $q > N$, $\tilde{\pi}_q F = 0$. *Then the natural map is a weak homotopy equivalence of Zariski* (*or etale*) *sheaf hypercohomology spectra*

$$\varinjlim_{\alpha} \mathbb{H}^{\cdot}(X_\alpha; F) \xrightarrow{\sim} \mathbb{H}^{\cdot}(X; F). \tag{1.38}$$

Proof. — This is a special case of 1.41. It may be proved as 1.41 with appeal to [SGA 4], VI 8.7.4 replaced by the more pedestrian [SGA 4], VIII 5.7, or [3], III 3.9.

THEOREM 1.46. — *Let* $\mathcal{A}$ *be a filtering system of covers* $\mathcal{U}_\alpha$ *of a site* **C**. *Let* F *be a presheaf of fibrant spectra on* **C**. *Suppose that the topos* $\mathbf{C}^\sim$ *has enough points. Suppose either that there exists an* N *such that* $\tilde{\pi}_q F = 0$ *for* $q > N$ *or else that there exists an* N *such that for all* q *and for* U = X *and all* U *in covers* $\mathcal{U}_\alpha$ *in the system* $\mathcal{A}$ *that* $H^p(\mathbf{C}/U; \tilde{\pi}_q F) = 0$ *if* $p > N$.

Then the local sites **C**/U *have enough points and* $\mathbb{H}^{\cdot}(\mathbf{C}/U; F)$ *is a presheaf of fibrant spectra as* U *varies over* **C**.

The natural map (1.39) *is a weak homotopy equivalence, so* $\mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)$ *has hypercohomological descent:*

$$\mathbb{H}^{\cdot}(\mathbf{C}; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathcal{A}; \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)). \tag{1.39}$$

Proof. — The topos $\mathbf{C}^\sim/U$ has enough points as it is an exact subcategory of $\mathbf{C}^\sim$, *see* [SGA 4], IV 6.7.3 if you don't believe me.

The significant statement is (1.39). The first step in the proof is devissage to the case of F a presheaf of Eilenberg-MacLane spectra. To accomplish this, consider the Postnikov tower $F\langle n \rangle$ of 5.51. Let $\mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)\{n\}$ be the tower of fibrations $\mathbb{H}^{\cdot}(\mathbf{C}/\ ; F\langle n \rangle)$. If $\tilde{\pi}_q F = 0$ for $q > N$ the spectral sequence 1.36 shows $\mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)$ is weak homotopy equivalent to $\mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)\{n\}$ for $n \geqq N$. If N is a uniform bound on the cohomological dimension of the U, the spectral sequence 1.36 shows $\pi_q \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F \rangle n \langle) = 0$ for $n > q + N$. Here $F \rangle n \langle$ is the inverse Postnikov tower of 5.53, so $F \rangle n \langle$ is the homotopy fibre of $F \to F\langle n \rangle$. It follows that $\pi_q \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F) \to \pi_q \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)\{n\}$ is an isomorphism if $n > q + N$. The hypotheses of 1.37 and 1.18 are thus satisfied, and one gets weak homotopy equivalences

$$\mathbb{H}^{\cdot}(\mathbf{C}; F) \simeq \varprojlim_{n} \mathbb{H}^{\cdot}(\mathbf{C}; F\langle n \rangle), \tag{1.40}$$

$$\check{\mathbb{H}}^{\cdot}(\mathcal{A}; \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F)) \simeq \varprojlim_{n} \check{\mathbb{H}}^{\cdot}(\mathcal{A}; \mathbb{H}^{\cdot}(\mathbf{C}/\ ; F\langle n \rangle)). \tag{1.41}$$

The spectral sequences 1.36 and 1.16 show that the nth term in the towers of fibration in (1.40) and (1.41) has trivial homotopy groups in degrees above n. The hypercohomology functors preserve homotopy fibre sequences by 1.15 and 1.35. The tower comparison Lemma of 5.55 then shows it suffices to prove (1.39) is a weak equivalence for F such that $\pi_q F = 0$ if $q \neq n$. By 5.52, the homotopy category of such F is equivalent to the category of presheaves of abelian groups on **C**. The presheaf of abelian groups A corresponds to the canonical Eilenberg-MacLane spectrum $K(A, n)$. This accomplishes the divissage.

For F a presheaf of chain complexes, (1.39) is the classical Cartan-Leray descent spectral sequence. I will prove it from scratch however. Define functors on the category of presheaves of abelian groups A by

$$D^p(A) = \pi_{n-p} \mathbb{H}^{\cdot}(\mathbf{C}; K(A, n)), \tag{1.42}$$

$$E^p(A) = \pi_{n-p} \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathbb{H}^{\cdot}(\mathbf{C}/\ ; K(A, n))). \tag{1.43}$$

Short exact sequences of presheaves A induce homotopy fibre sequences of $K(\ , n)$'s, which are preserved by the hypercohomology constructions, and thus yield long exact sequences of homotopy groups. Thus E^* and D^* are cohomological ∂-functors on the category of presheaves.

If A is a presheaf whose sheafification $\tilde{A}$ is 0, the spectral sequence 1.36 shows that $E^*(A) = 0$ and $D^*(A) = 0$. Thus E^* and D^* are cohomological ∂-functors on the category of sheaves of abelian groups. The spectral sequences 1.36 and 1.16 also show that $E^p = D^p = 0$ for $p < 0$, and that if A is an injective sheaf, hence acyclic for Čech and sheaf cohomology, then $E^p(A) = D^p(A) = 0$ for $p > 0$ and $E^0(A) = D^0(A) = H^0(\mathbf{C}; A)$. A standard argument shows that D^* and E^* are both the right derived functor of $H^0(\mathbf{C};\ )$, and so the map $D^* \to E^*$ is an isomorphism. Thus (1.39) is a weak homotopy equivalence, as required.

COROLLARY 1.47. — *Let* X *be a noetherian scheme of finite Krull dimension. Let* F *be a presheaf of fibrant spectra on the Zariski site of* X. *Consider the presheaf that sends an open subscheme* U *to the Zariski hypercohomology spectrum* $\mathbb{H}^{\cdot}_{\mathrm{Zar}}(U; F)$. *Then for any Zariski cover* $\mathscr{U}$ *or filtering system of Zariski covers* $\mathscr{A}$ *one has a weak homotopy equivalence*

$$\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathbb{H}^{\cdot}_{\mathrm{Zar}}(\ ; F)). \tag{1.44}$$

Proof. — Each open U has Krull dimension at most that of X. By [49], III 3.6.5 or [51], III 2.7, the Zariski cohomological dimension of the U's is uniformly bounded by the Krull dimension of X. Thus the hypotheses of 1.46 are met.

COROLLARY 1.48. — *Let* X *be a separated noetherian scheme of finite Krull dimension. Let* F *be a presheaf of fibrant spectra on the etale site of* X *such that either* $\tilde{\pi}_* F$ *is a sheaf of rational vector spaces, or else is a sheaf of l-torsion abelian groups for a fixed prime l. In the latter case, suppose there is a uniform bound* M *on the etale cohomological dimension of the residue field* $k(x)$ *with respect to l-torsion sheaves for all* *points x of* X, *including the non-closed points. For example, let* X *be a scheme of finite type over an algebraically closed field, or over the ring of Gaussian integers* $\mathbb{Z}[i]$, *or over* $\mathbb{Z}$ *with* $l \neq 2$. *Then for any filtering system* $\mathscr{A}$ *of etale covers of* X, *one has a weak homotopy equivalence of etale hypercohomology spectra*

$$(1.45) \qquad \mathbb{H}^{\cdot}_{et}(X; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathbb{H}^{\cdot}_{et}(\ ; F)).$$

Similarly, if X *is a separated noetherian scheme of finite Krull dimension, and if* F *is a presheaf of fibrant spectra on the etale site of* X *such that there is an* N *such that* $\tilde{\pi}_q F = 0$ *for* $q > N$, *then* (1.45) *is a weak homotopy equivalence.*

Proof. — If U is etale over X, its Krull dimension is at most that of X. For any point $y \in U$, $k(y)$ is a finite algebraic extension of $k(x)$ for the image x of y in X, so $k(y)$ has l-torsion etale cohomological dimension at most M, by the usual Shapiro's lemma argument of [102], I, Prop. 14, or [104], III, Prop. 15. Thus U inherits the hypotheses on X. Any closed subscheme of U has residue fields which are among the residue fields of U, and any henselization of a closed subscheme has residue fields which are direct limits of finite algebraic extensions of the residue fields of U. All these fields have l-torsion cohomological dimension at most M, and $\mathbb{Q}$-module cohomological dimension 0. The induction argument of [SGA 4], X 4.1, with $\varphi(x)$ given by $M+1$ times the Krull dimension of the closure of x gives a uniform bound on the l-torsion or rational etale cohomological dimension of U etale over X. Thus the hypotheses of 1.46 are satisfied. The list of examples is justified by [SGA 4], X.

Example 1.49. — Let X be a separated noetherian scheme of finite Krull dimension. Let U and V be Zariski open subsets of X, and let $\mathscr{U} = \{U, V\}$ be the cover of $U \cup V$. Choose either the Zariski or etale site of X, and let F be a presheaf of fibrant spectra, satisfying the hypotheses of 1.48 if the etale site was chosen. Then for Zariski or etale hypercohomology respectively, there is a weak homotopy equivalence (1.46) by 1.47 or 1.48 applied to the direct system consisting of the single cover $\mathscr{U}$:

$$(1.46) \qquad \mathbb{H}^{\cdot}(U \cup V; F) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathscr{U}; \mathbb{H}^{\cdot}(\ ; F)).$$

As $U \times U = U \cap U = U$ and $V \times V = V$, the Čech complex $\pi_q \mathbb{H}^{\cdot}(\ ; F)_{\mathscr{U}}$ is highly degenerate. By [EGA], III 11.8, or [119], VIII, Application b of Lemma 1, this complex has the same cohomology as the ordered Čech complex

$$(1.47) \qquad \pi_q \mathbb{H}^{\cdot}(U; F) \oplus \pi_q \mathbb{H}^{\cdot}(V; F) \xrightarrow{d^0 - d^1} \pi_q \mathbb{H}^{\cdot}(U \cap V; F) \to 0 \to 0 \to \dots$$

In the spectral sequence 1.16 for Čech hypercohomology one has therefore $E_2^{0q} = \ker(d^0 - d^1)$, $E_2^{1q} = \operatorname{coker}(d^0 - d^1)$, and $E_2^{p,q} = 0$ if $p > 1$. This spectral sequence then collapses with $E_2 = E_\infty$. The filtration of $\pi_q \mathbb{H}(U \cup V; F)$ by $E_\infty^{0,q}$ and $E_\infty^{1,q+1}$ splices to yield the long exact Mayer-Vietoris sequence

$$(1.48) \qquad \dots \to \pi_{q+1} \mathbb{H}^{\cdot}(U \cap V; F) \xrightarrow{\partial} \pi_q \mathbb{H}^{\cdot}(U \cup V; F)$$
$$\to \pi_q \mathbb{H}^{\cdot}(U; F) \oplus \pi_q \mathbb{H}^{\cdot}(V; F) \to \pi_q \mathbb{H}^{\cdot}(U \cap V; F) \xrightarrow{\partial} \dots$$

It follows that the square (1.49) is homotopy cartesian

$$\begin{array}{ccc} \mathbb{H}^{\cdot}(U \cup V;\, F) & \to & \mathbb{H}^{\cdot}(V;\, F) \\ \downarrow & & \downarrow \\ \mathbb{H}^{\cdot}(U;\, F) & \to & \mathbb{H}^{\cdot}(U \cap V;\, F) \end{array} \tag{1.49}$$

Example 1.50. — Let L'/L be a finite Galois extension of fields with Galois group G. Consider the etale cover of $\operatorname{Spec}(L)$: $\mathscr{U}=\{\operatorname{Spec}(L') \to \operatorname{Spec}(L)\}$. The fibre product of $\operatorname{Spec}(L')$ with itself over $\operatorname{Spec}(L)$ is $\operatorname{Spec}(L' \underset{L}{\otimes} L')$. There is a canonical isomorphism

$$\kappa:\quad L' \underset{L}{\otimes} L' \xrightarrow{\cong} \prod_{G} L'. \tag{1.50}$$

The projection of κ on the factor indexed by $g \in G$ sends $l_1 \otimes l_2$ to $l_1 . g l_2$. To see κ is an isomorphism, write $L' = L[T]/f$ for an irreducible polynomial f over L. Then $L' \underset{L}{\otimes} L'$ is $L'[T]/f$. The polynomial f splits into linear factors $T - g\alpha$ over L', with α a chosen root and g ranging over G. The Chinese remainder theorem then shows (1.50) is an isomorphism. More generally, there is the isomorphism (1.51)

$$\left\{ \begin{array}{l} \kappa:\quad \overset{n+1}{\underset{L}{\otimes}} L' \xrightarrow{\sim} \prod_{G^n} L', \\ \operatorname{proj}(g_1, g_2, \ldots, g_n) \circ \kappa(l_0 \otimes l_1 \otimes \ldots \otimes l_n) = l_0 . g_1 l_1 . g_1 g_2 l_2 . \ldots . (g_1 \ldots g_n) l_n. \end{array} \right. \tag{1.51}$$

Let now F be a presheaf of fibrant spectra on the etale site of L, and suppose F takes finite coproducts of schemes to homotopy products of spectra. This is true for the K-theory spectra and for any presheaf of the form $\mathbb{H}^{\cdot}_{et}(\ ;\, F)$. Then applying F to $\operatorname{Spec}(\kappa)$ one deduces weak homotopy equivalence of the Čech complex $F^{\cdot}_{\mathscr{U}}$ of the cover $\mathscr{U}$ with the Bousfield-Kan cosimplicial complex for computing the homotopy limit of the action of G on $F(L')$ as in [16], XI, § 5

$$F^{\cdot}_{\mathscr{U}} \xrightarrow{\sim} (n \to \prod_{G^n} F(\operatorname{Spec}(L'))). \tag{1.52}$$

This induces weak homotopy equivalences.

$$\begin{array}{c} \check{\mathbb{H}}(\mathscr{U};\, F) \simeq \underset{\Delta}{\operatorname{holim}} F^{\cdot}_{\mathscr{U}} \simeq \underset{\Delta}{\operatorname{holim}} (\prod_{G} F(L')) \simeq \operatorname{Tot}(\prod_{G^{\cdot}} F(L')) \\ \wr| \\ \mathbb{H}^{\cdot}(G;\, F(L')) \end{array} \tag{1.53}$$

by 5.8 and 5.25. The cosimplicial fibrant spectrum $\prod_{G^{\cdot}} F(L')$ satisfies the hypothesis of 5.28 by [16], XI 5.3. The equivalence (1.53) identifies the spectral sequences of 1.16 and 5.13, so that one has

$$E_2^{p,\,q} = H^p(\operatorname{Gal}(L'/L);\, \pi_q F(L')) \quad \Rightarrow \quad \pi_{q-p} \check{\mathbb{H}}^{\cdot}(\mathscr{U};\, F). \tag{1.54}$$

In particular, if L has bounded etale cohomological dimension for l-torsion sheaves and $\pi_* F$ is l-torsion, 1.48 gives a weak homotopy equivalence and a spectral sequence

$$\check{\mathbb{H}}^{\cdot}(\mathrm{Gal}(L'/L);\ \mathbb{H}^{\cdot}_{et}(L';\ F)) \simeq \mathbb{H}^{\cdot}_{et}(L;\ F) \tag{1.55}$$

$$H^p(\mathrm{Gal}(L'/L);\ \pi_q \mathbb{H}^{\cdot}_{et}(L';\ F)) \ \Rightarrow\ \pi_{q-p} \mathbb{H}^{\cdot}_{et}(L;\ F). \tag{1.56}$$

Suppose now L'_α is a filtering system of finite Galois extensions of L indexed by $\mathscr{A}$. Let $L' = \varinjlim L'_\alpha$. For $\mathscr{A}$ the system of covers $\{\mathrm{Spec}(L'_\alpha) \to \mathrm{Spec}(L)\} = \mathscr{U}_\alpha$, the above argument gives a weak homotopy equivalence

$$\check{\mathbb{H}}^{\cdot}(\mathscr{A};\ F) \simeq \mathrm{Tot}(\varinjlim_{\alpha} \prod_{G^{\cdot}_\alpha} F(L_\alpha)). \tag{1.57}$$

Let $G = \varprojlim G_\alpha = \varprojlim \mathrm{Gal}(L'_\alpha/L)$ be the profinite Galois group of L′ over L. Then the spectral sequence 1.16 is identified to the profinite group cohomology spectral sequence

$$H^p(G;\ \varinjlim_{\alpha} \pi_q F(L'_\alpha)) \ \Rightarrow\ \pi_{q-p} \check{\mathbb{H}}^{\cdot}(\mathscr{A};\ F). \tag{1.58}$$

If F is a continuous presheaf 1.42, $\pi_q F(L') = \varinjlim_{\alpha} \pi_q F(L'_\alpha)$.

Under reasonable hypotheses, the above observations, 1.48, and 1.45 combine to give a weak homotopy equivalence and a spectral sequence like (1.55) and (1.56) for the profinite Galois group. This holds if F and $X = \mathrm{Spec}(L)$ satisfy 2.1.

Now let $\tilde{L}$ be the separable algebraic closure of L. Take $\mathscr{A}$ to be the direct system of all subfields L_α of $\tilde{L}$ which are finite Galois extensions of L. Then $\mathscr{A}$ is weakly cofinal 1.14 in the category of all etale covers of L. Further, $\tilde{L}$ is a strict local hensel ring as rings etale over $\tilde{L}$ are products of copies of $\tilde{L}$. Thus $\mathbb{H}^{\cdot}_{et}(\tilde{L}; F) = F(\tilde{L})$. From these remarks, 1.22, 1.48, and the profinite group analogue of (1.55), under reasonable hypotheses like 2.1, there are weak homotopy equivalences

$$\mathbb{H}^{\cdot}_{et}(L;\ F) \simeq \mathbb{H}^{\cdot}_{\mathrm{profinite}}(\mathrm{Gal}(\tilde{L}/L);\ F(\tilde{L})) \simeq \check{\mathbb{H}}^{\cdot}(\mathscr{A};\ F) \simeq \check{\mathbb{H}}^{\cdot}_{et}(L;\ F). \tag{1.59}$$

One may consult [SGA 4], VIII, § 2, for more details.

1.51. One says a presheaf of fibrant spectra F on a site **C** has cohomological descent if for all U in **C** the augmentation

$$\eta:\quad F(U) \xrightarrow{\sim} \mathbb{H}^{\cdot}(C/U^{\sim};\ F), \tag{1.60}$$

is a weak homotopy equivalence. If $\pi_q F = 0$ for $q > N$ or if everything has uniformly bounded cohomological dimension, then 1.46 shows that such an F has the property that for any cover or direct system of covers $\mathscr{A}$ of U in **C**, the augmentation is a weak homotopy equivalence

$$\eta:\quad F(U) \xrightarrow{\sim} \check{\mathbb{H}}^{\cdot}(\mathscr{A};\ F). \tag{1.61}$$

Under the usual hypotheses of 1.46, for any F, the presheaf $\mathbb{H}^{\cdot}(\mathbf{C}/\ ;\ F)$ has cohomological descent. The argument of 1.46 proves this. I like to think of presheaves with cohomological descent as "sheaves of homotopy types" and the functor sending F to $\mathbb{H}^{\cdot}(\mathbf{C}/\ ;\ F)$ as a "sheafification" functor. Lemma 1.35 then says that "sheafification" "is exact". However, one can get into trouble by carrying this too far.

It will be convenient to know that frequently etale sheaf hypercohomology can be computed by Čech covers. I begin with some preliminaries.

DEFINITION 1.52. — Let $\mathbf{C}$ be a site with finite coproducts; e. g. the etale site Et/X. Let F be a presheaf of abelian groups on $\mathbf{C}$. One says F is additive if for all coproducts U V in $\mathbf{C}$, the canonical map from $F(U \sqcup V)$ to $F(U) \times F(V)$ is an isomorphism. A presheaf of fibrant spectra is additive if for all U and V, the map (1.62) is a weak homotopy equivalence

$$F(U \sqcup V) \xrightarrow{\sim} F(U) \times F(V). \tag{1.62}$$

The K-theory spectra are additive on the category of schemes. If F is additive, so is $\pi_* F$. Any sheaf of abelian groups is additive.

PROPOSITION 1.53 (Artin, [4]). — *Let* X *be a scheme quasiprojective over some noetherian ring. Let* F *be an additive presheaf of abelian groups on the etale site of* X. *Let* $\tilde{F}$ *be the sheafification of* F. *Then the canonical map from the Čech cohomology of* F *to the sheaf cohomology of* $\tilde{F}$ *is an isomorphism*

$$\check{H}^p_{et}(X;\ F) \xrightarrow{\sim} H^p_{et}(X;\ \tilde{F}). \tag{1.63}$$

Proof. — Under the hypotheses, I may work in the restricted etale site of 1.3. For every cover in the full site is refined by one in the restricted site, so the Čech cohomology and the category of sheaves is not changed by restriction.

If F is additive, so is the presheaf of Čech cohomology groups, $\check{\mathscr{H}}^*(F)$. Suppose $\check{\mathscr{H}}^0(F)=0$. This condition is equivalent to the condition that for every U and every $x \in F(U)$ there is a finite cover of U, $\{W_i \to U\}$ such that x restricts to zero in each $F(W_i)$. If F is additive and $W = \sqcup W_i$, then x restricts to 0 in $F(W) = \prod F(W_i)$. With these observations, one sees that the argument of [4], Cor. 4.2, shows that if F is additive and $\check{\mathscr{H}}^0(F)=0$, then for all p, $\check{H}^p(X;\ F)=0$.

Now suppose F is an additive presheaf whose sheafification $\tilde{F}$ is 0. Then $\check{\mathscr{H}}^0(\check{\mathscr{H}}^0(F)) = \tilde{F}$ is 0. But $\check{\mathscr{H}}^0(F)$ is a separated presheaf, and so it injects into $\check{\mathscr{H}}^0(\check{\mathscr{H}}^0(F))=0$. Thus $\check{\mathscr{H}}^0(F)=0$, and by the above $\check{H}^*(X;\ F)=0$. The reader may consult [SGA 4], V 2.4.5, II 3.2 or [3], II, § 1 and 2 for details on $\check{\mathscr{H}}^0$ and sheafification, if required.

Now let F be any additive presheaf. Let K, I, C, be the kernal, image, and cokernal presheaves of the canonical map $F \to \tilde{F}$. There are short exact sequences of presheaves

$$\begin{gathered} 0 \to K \to F \to I \to 0, \\ 0 \to I \to \tilde{F} \to C \to 0. \end{gathered} \tag{1.64}$$

These induce long exact sequences of Čech cohomology

$$(1.65)\qquad \begin{gathered} \ldots \to \check{H}^p(X; K) \to \check{H}^p(X; F) \to \check{H}^p(X; I) \xrightarrow{\partial} \check{H}^{p+1}(X; K) \to \ldots, \\ \ldots \to \check{H}^{p-1}(X; C) \xrightarrow{\partial} \check{H}^p(X; I) \to \check{H}^p(X; \tilde{F}) \to \check{H}^p(X; C) \xrightarrow{\partial} \ldots \end{gathered}$$

Sheafification is exact, and converts $F \to \tilde{F}$ into an isomorphism. Thus $\tilde{K}=0$, $\tilde{C}=0$, and $\tilde{I}=\tilde{F}$. The presheaves K, I, C are additive as F and $\tilde{F}$ are, so $\check{H}^*(X; C)=0$, $\check{H}^*(X; K)=0$. Then the long exact sequences (1.65) yield isomorphisms.

$$(1.66)\qquad \check{H}^p(X; F) \simeq \check{H}^p(X; I) \simeq \check{H}^p(X; \tilde{F}).$$

Finally, Artin's Theorem [4], 4.2 yields an isomorphism

$$(1.67)\qquad \check{H}^p(X; \tilde{F}) \simeq H^p_{et}(X; \tilde{F}).$$

Combining (1.66) and (1.67) yields the result.

Proposition 1.54. — *Let X be a scheme quasiprojective over some noetherian ring of finite Krull dimension. Then X is noetherian, separated, and has finite Krull dimension. Suppose F is an additive presheaf of fibrant spectra as in 1.52.*

Then there is a natural chain of weak homotopy equivalences between Čech and sheaf hypercohomology

$$(1.68)\qquad \check{\mathbb{H}}^{\cdot}_{et}(X; F) \xrightarrow{\sim} \xleftarrow{\sim} \mathbb{H}^{\cdot}_{et}(X; F).$$

Proof. — Let $\mathscr{A}$ be the cofinal direct system of Čech covers as in 1.23. Let $F\langle n\rangle$ be the nth stage of the Postnikov tower, as in 5.51. Consider the diagram of augmentations

$$(1.69)\qquad \begin{array}{ccc} \check{\mathbb{H}}^{\cdot}(X; F\langle n\rangle) = \check{\mathbb{H}}^{\cdot}(\mathscr{A}; F\langle n\rangle) & \to & \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathbb{H}^{\cdot}_{et}(\ ; F\langle n\rangle)) \\ & & \uparrow \\ & & \mathbb{H}^{\cdot}_{et}(X; F\langle n\rangle) \end{array}$$

If one takes the inverse limit over n, the upper left and lower right hand corners of (1.69) are identified to the two sides of (1.68) by 1.18 and 1.37. As homotopy inverse limits or inverse limits along towers of fibrations preserve weak homotopy equivalences, it suffices to prove for each n that the maps in (1.69) are weak homotopy equivalences.

The vertical map of (1.69) is a weak equivalence by 1.46, as $\pi_q F\langle n\rangle = 0$ if $q>n$.

Consider the horizontal map of (1.69). It induces a map of spectral sequences 1.16 for the two sides of the map. The spectral sequences converge strongly as $\pi_q F\langle n\rangle = 0$ and $\pi_q \mathbb{H}^{\cdot}_{et}(\ ; F\langle n\rangle)=0$ for $q>n$. Thus it suffices to show the map induces an isomorphism of E_2 terms of the spectral sequences. By Artin's theorem 1.53, this map is identified to

$$(1.70)\qquad H^p_{et}(X, \tilde{\pi}_q F\langle n\rangle) \to H^p_{et}(X, \tilde{\pi}_q \mathbb{H}^{\cdot}_{et}(\ ; F\langle n\rangle)).$$

This map will be an isomorphism if the map of coefficient sheaves (1.71) is an isomorphism

$$\tilde{\pi}_* \mathrm{F}\langle n\rangle \to \tilde{\pi}_* \mathbb{H}^{\cdot}_{\mathrm{et}}(\ ;\ \mathrm{F}\langle n\rangle). \tag{1.71}$$

As the etale site has enough points, by 1.29 it suffices to show (1.71) induces an isomorphism on the stalks at each point f^* of Et/X. As in 1.29, $f^*\tilde{\pi}_* \mathrm{F}\langle n\rangle$ is identified to $\pi_*(g^{\#}\mathrm{F}\langle n\rangle)(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x})$ for $g:\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x})\to \mathrm{X}$ a strict local henselization of X. This $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x})$ is the inverse limit along $\mathrm{Nbd}(f)^{\mathrm{op}}$ of schemes etale over X. The continuity of $\mathbb{H}^{\cdot}_{\mathrm{et}}(\ ;\ \mathrm{F}\langle n\rangle)$ given by 1.41 or 1.45 shows as in 1.44 that there is an isomorphism

$$(g^{\#}\pi_* \mathbb{H}^{\cdot}_{\mathrm{et}}(\ ;\ \mathrm{F}\langle n\rangle))(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x}) \simeq \pi_* \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x};\ g^{\#}\mathrm{F}\langle n\rangle). \tag{1.72}$$

By 1.29, the left side of (1.72) is f^* of the right side of (1.71). With these identifications, it suffices to prove the top horizontal arrow of (1.73) is an isomorphism.

$$\begin{array}{ccc} \pi_*(g^{\#}\mathrm{F}\langle n\rangle)(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x}) & \to & \pi_* \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x};\ g^{\#}\mathrm{F}\langle n\rangle)) \\ \downarrow & & \downarrow \\ \pi_* f^*\mathrm{F}\langle n\rangle = \pi_* f^*\mathrm{F}\langle n\rangle(k(\bar{x})) & \to & \pi_* \mathbb{H}^{\cdot}_{\mathrm{et}}(k(\bar{x});\ f^*\mathrm{F}\langle n\rangle) \end{array} \tag{1.73}$$

Here $k(\bar{x})$ is the separably closed residue field of $\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x}$, as in 1.29. We've already seen the left vertical map of (1.73) is an isomorphism. The right vertical map is an isomorphism by comparison of spectral sequences 1.36 using the acyclicity of $k(\bar{x})$ and of $\mathcal{O}^{\mathrm{sh}}_{\mathrm{X},x}$ provided by [SGA 4], VIII 4.7, 4.8. The bottom horizontal map of (1.73) is also an isomorphism by 1.36 as the etale topos of $k(\bar{x})$ is the trivial topos, the category of sets. [Thus $\mathrm{H}^p_{\mathrm{et}}(k(\bar{x});\ \tilde{\pi}_* f^*\mathrm{F}\langle n\rangle)$ vanishes for $p>0$, and takes global sections of the presheaf $\pi_* f^*\mathrm{F}\langle n\rangle$ if $p=0$.] These remarks show the top horizontal arrow of (1.73) is an isomorphism as required.

At the referee's request, I include the following:

Definition 1.55. — Let $f:\mathbf{C}\to\mathbf{D}$ be a morphism of sites with enough points. Let F be a presheaf of fibrant spectra on $\mathbf{C}$. Define $\mathbb{R}^{\cdot} f(\mathrm{F})$ to be the presheaf of fibrant spectra on $\mathbf{D}$ given by

$$\mathbb{R}^{\cdot} f(\mathrm{F})(\mathrm{U}) = \mathbb{H}^{\cdot}(\mathbf{C}/f^*\mathrm{U};\ \mathrm{F}). \tag{1.74}$$

Theorem 1.56 (Cartan-Leray). — *Let $f:\mathbf{C}\to\mathbf{D}$ be a morphism of sites and suppose the topoi $\tilde{\mathbf{C}}$, $\tilde{\mathbf{D}}$ have enough points. Let* F *be a presheaf of fibrant spectra on* $\mathbf{C}$. *Suppose either that there exists an* N *such that $\tilde{\pi}_q\mathrm{F}=0$ for $q>\mathrm{N}$, or else that there exists an* N *such that for all objects* U *in* $\mathbf{D}$, *and for all q that $\mathrm{H}^p(\tilde{\mathbf{C}}/f^*\mathrm{U};\ \tilde{\pi}_q\mathrm{F})=0$ if $p>\mathrm{N}$. Then the natural map* (1.75) *is a weak homotopy equivalence*

$$\mathbb{H}^{\cdot}(\mathbf{C};\ \mathrm{F}) \xrightarrow{\sim} \mathbb{H}^{\cdot}(\mathbf{D};\ \mathbb{R}f^{\cdot}(\mathrm{F})). \tag{1.75}$$

Proof. — Exactly as in the proof of 1.46 replacing $\check{\mathbb{H}}^{\cdot}(\mathscr{A};\ )$ with $\mathbb{H}^{\cdot}(\mathbf{D};\ )$, and $\mathbb{H}^{\cdot}(\mathbf{C}/\ ;\ \mathrm{F})$ with $\mathbb{R}f^{\cdot}(\mathrm{F})$, 1.16 with 1.36, 1.15 with 1.35, etc.

1.57. — The Theorem 1.56 applies to the map of etale sites $f: X_{et} \to Y_{et}$ for any two schemes satisfying the hypotheses of 1.48.

2. Etale cohomological descent for algebraic K-theory

In this section, I prove the key descent theorems for algebraic K-theory. The proof begins by showing that etale cohomological descent on a scheme X follows from descent on all of its local rings. This reduction results from Zariski cohomological descent for K-theory, which follows from the Mayer-Vietoris property by a theorem of Brown and Gersten. The next step shows that descent on a ring follows from descent on all its residue fields. This reduction results from induction on the Krull dimension, the localization sequence for K-theory, and the fact that etale cohomological descent implies Čech cohomological descent for all filtering systems of etale covers. One might expect to make this reduction by comparing the localization sequence with the Gysin sequence for etale cohomology. However, this Gysin sequence is not available at this stage of the argument, as one does not yet know that the etale sheaf $\tilde{K}/l^{\nu}_{*}[\beta^{-1}]$ is locally constant, nor that the required absolute cohomological purity Theorem holds in full generality. This step in the argument is the key point in proving the general cohomological purity theorem of [130].

The field case of descent is handled by an argument that interrelates cohomological descent and homological induction. First some general facts about homological induction and the hypertransfer are developed, and then the proof of descent on fields is given. The key point is the form of Hilbert's Theorem 90 which relates the units in a field L with etale cohomology by the isomorphism

$$K_1(L) \otimes \mathbb{Z}/l^{\nu} \simeq H^1_{et}(L; \mathbb{Z}/l^{\nu}(1)). \tag{2.1}$$

The machinery amplifies this connection so that its influence extends over $K/l^{\nu}(L)[\beta^{-1}]$.

This proof of descent was suggested to me by a construction of Gillet. It differs from the proof in the first edition of this paper, which depended on Karoubi periodicity, also a form of Hilbert's Theorem 90. A sketch of the Karoubi periodicity proof may be found in [128]. The new proof is more conceptual, does not rely on irksome calculation of higher order homotopy operations, and also avoids the difficulties in deducing descent for profinite Galois extensions from descent for finite Galois extensions. This last point was handled incorrectly in the first edition, and resolved by a gruesome argument in the erratum.

Usual general hypotheses 2.1. — Let X be a separated noetherian scheme of finite Krull dimension. Let F be a presheaf of fibrant spectra on the etale site of X. Suppose that F extends to a continuous presheaf of fibrant spectra on the category of schemes which are inverse limits along inverse systems with affine bonding maps of schemes finitely presented and etale over X. Here continuity is as defined in 1.42. (Techniques of [EGA], IV, §8, show that any F defined on finitely presented etale schemes over X extends to a unique continuous F as above.) Suppose that F is additive in the sense of

1.52. Fix a set of primes J such that $\pi_* F$ is a presheaf of modules over the localization $\mathbb{Z}_{(J)}$ of the ring of integers $\mathbb{Z}$ by inverting the primes not in J. Suppose there is a uniform bound on the etale cohomological dimension of all residue fields of X with respect to l-torsion sheaves for all primes l in J. Then there is a uniform bound on the etale cohomological dimension of everything in sight with respect to $\pi_* F$, as in 1.48. This long list of hypotheses is in fact satisfied in every decent situation. In particular, they hold for the K-theory spectra $K/l^\nu(\ )[\beta^{-1}]$, $G/l^\nu(\ )[\beta^{-1}]$ of Appendix A if X is an algebraic variety over an algebraically closed field of characteristic not l, or if X is separated and of finite type over $\mathbb{Z}[l^{-1}]$ and $l \neq 2$, or if X is separated and of finite type over the Gaussian integers with 2 inverted, and in many other cases too numerous to mention. In all these cases, $F(\ ) = \mathbb{H}^{\cdot}_{\mathrm{et}}(\ ;\ K/l^\nu(\ )[\beta^{-1}])$ also satisfies the hypotheses by the results of paragraph 1.

DEFINITION 2.2. — A presheaf F on the Zariski site of X has the Mayer-Vietoris property if for all Zariski open subschemes U and V of X, the square (2.2) is homotopy cartesian.

$$\begin{array}{ccc} F(U \cup V) & \rightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \rightarrow & F(U \cap V) \end{array} \tag{2.2}$$

For example, the presheaves $G(\ )$, $G/l^\nu(\ )$, $G/l^\nu(\ )[\beta^{-1}]$ of Appendix A all have the Mayer-Vietoris property. For $G(\ )$ does by [97], § 7.3.5, and the others are formed from $G(\ )$ by processes which preserve homotopy cartesian squares. For X a regular scheme, the presheaves $K(\ )$, $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$ all have the Mayer-Vietoris property, as they are weak homotopy equivalent to the corresponding G-theory presheaves by the resolution theorem, as in [97], § 7.1. For these examples I assume that X is separated and noetherian.

For X and F satisfying the usual hypotheses 2.1, $\mathbb{H}^{\cdot}_{\mathrm{et}}(\ ;\ F)$ has the Mayer-Vietoris property by 1.49.

PROPOSITION 2.3. — *Let* X *be a separated noetherian scheme of finite Krull dimension. Let* $\eta: F \rightarrow G$ *be a map of presheaves of fibrant spectra on the Zariski site of* X. *Suppose both* F *and* G *have the Mayer-Vietoris property* 2.2. *For each point* x *in* X, *let* f: $\mathrm{Spec}(k(x)) \rightarrow X$ *denote the associated inclusion. Suppose for each point* x, η *induces a weak homotopy equivalence of the stalks at* x, $\eta: f^* F \xrightarrow{\sim} f^* G$. *Then* η *is a weak homotopy equivalence of presheaves:* $\eta(U): F(U) \rightarrow G(U)$ *is a weak homotopy equivalence for all Zariski open* U *in* X.

Proof. — It suffices to show $\eta(X)$ is a weak homotopy equivalence, as the local site Zar/U inherits all the hypotheses, and so η will then be a weak equivalence on its terminal object U.

By a theorem of Brown and Gersten [20], Thms. 3 and 4, or by exercice 2.5 below, for F with the Mayer-Vietoris property there is a strongly convergent spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X;\ \tilde{\pi}_q F) \quad \Rightarrow \quad \pi_{q-p} F(X). \tag{2.3}$$

The hypothesis on the stalks of η shows that η induces an isomorphism on the Zariski sheaves associated to the presheaves of homotopy groups, $\eta : \tilde{\pi}_* F \overset{\cong}{\to} \tilde{\pi}_* G$. The spectral sequence comparison theorem then shows that $\eta : \pi_* F(X) \to \pi_* G(X)$ is an isomorphism. (The reader may review stalks in 1.28, 1.29, 1.31 if required).

COROLLARY 2.4. — *Let* X *and* F *satisfy the usual hypotheses* 2.1. *Suppose* F *has the Mayer-Vietoris property on the Zariski site of* X. *Suppose that for all local rings* $\mathcal{O}_{X, x}$ *of* X, *the augmentation is a weak homotopy equivalence*

$$(2.4) \qquad \eta : F(\mathrm{Spec}(\mathcal{O}_{X, x})) \overset{\sim}{\to} \mathbb{H}^{\bullet}_{et}(\mathrm{Spec}(\mathcal{O}_{X, x}); F).$$

Then the augmentation is a weak homotopy equivalence for X

$$(2.5) \qquad \eta(X) : F(X) \overset{\sim}{\to} \mathbb{H}^{\bullet}_{et}(X; F).$$

Proof. — This is a special case of 2.3. The continuity of F allows one to formulate the hypothesis on stalks in terms of the evaluation of η at local rings of X by 1.44. $\mathbb{H}^{\bullet}_{et}(\ ; F)$ is continuous and has the Mayer-Vietoris property by 1.45 and 1.49.

Exercise 2.5 (Optional). — Prove the theorem of Brown and Gersten. Show for a noetherian scheme X of finite Krull dimension, and for F a presheaf of fibrant spectra on the Zariski site of X such that F has the Mayer-Vietoris property, then the augmentation is a weak homotopy equivalence (2.6). Get the spectral sequence from 1.36.

$$(2.6) \qquad \eta(X) : F(X) \overset{\sim}{\to} \mathbb{H}^{\bullet}_{Zar}(X; F).$$

Hint. — The puzzling thing is how Mayer-Vietoris, a Čech cohomology condition, leads to a sheaf cohomology result. The answer is that it gives an excision condition that allows one to set up the local cohomology machine to build a Cousin resolution.

Step 1. — For Y closed in X, define $\underline{\Gamma}_Y F$ to be the homotopy fibre for each open U.

$$(2.7) \qquad \underline{\Gamma}_Y F(U) \to F(U) \to F(U-(Y \cap U)).$$

If F has the Mayer-Vietoris property, and $Y = \bar{Y} \cap V$ is locally closed, define $\underline{\Gamma}_Y F$ as the homotopy fibre

$$(2.8) \qquad \underline{\Gamma}_Y F(U) \to F(U \cap V) \to F(U \cap V - Y \cap U \cap V).$$

The Mayer-Vietoris property is as usual equivalent to an excision condition that says different choices of open V such that $Y = \bar{Y} \cap V$ yield weak homotopy equivalent $\underline{\Gamma}_Y F$. To make a canonical functorial choice, take the direct limit of the equivalent $\underline{\Gamma}_Y F$ over all such V.

Step 2. — If F has Mayer-Vietoris and Z is closed in Y, show there is a homotopy fibre sequence

$$(2.9) \qquad \underline{\Gamma}_Z F \to \underline{\Gamma}_Y F \to \underline{\Gamma}_{Y-Z} F.$$

Consider (2.10) where all rows and columns except possibly the top are homotopy fibre sequences by instances of 2.8

$$\begin{tikzcd} \underline{\Gamma}_Z F(U) \arrow[r] \arrow[d, "\wr"] & \underline{\Gamma}_Y F(U) \arrow[r] \arrow[d] & \underline{\Gamma}_{Y-Z} F(U) \arrow[d] \\ \underline{\Gamma}_Z F(U) \arrow[r] \arrow[d] & F(U \cap V) \arrow[r] \arrow[d] & F(U \cap V - Z) \arrow[d] \\ \star \arrow[r] & F(U \cap V - Y) \arrow[r, "\sim"] & F(U \cap V - Y) \end{tikzcd} \tag{2.10}$$

Then the top row of (2.10) is a homotopy fibre sequence by the usual Quetzalcoatl Lemma, e. g. [5], 1.2.

Step 3. — For p a non-negative integer, define $S^p F$ by

$$S^p F = \varinjlim_{Y} \underline{\Gamma}_Y F,$$

where the direct limit is taken over the direct system of closed subschemes Y of X of codimension at least p. Note $S^p F$ is a point if p is greater than the dimension of X. If $y \in X$ is a point of codimension p, with closure $\bar{y}$, show that (2.11) is a homotopy fibre sequence

$$S^{p+1}(\underline{\Gamma}_{\bar{y}} F) \to \underline{\Gamma}_{\bar{y}} F \to \underline{\Gamma}_y F. \tag{2.11}$$

Show that the presheaf $\pi_* \underline{\Gamma}_y F$ is a skyscraper sheaf supported at y, and so is flabby for the Zariski topology.

Conclude that for each p there is a homotopy fibre sequence

$$S^{p+1} F \to S^p F \to \bigvee_y \underline{\Gamma}_y F, \tag{2.12}$$

where the wedge of spectra is taken over the points y of codimension p. Show the presheaf (2.13) is a flabby sheaf.

$$\pi_* \bigvee_y \underline{\Gamma}_y F \simeq \bigoplus_y \pi_* \underline{\Gamma}_y F. \tag{2.13}$$

Step 4. — Show by descending induction on p that the augmentation

$$S^p F(X) \to \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X; S^p F),$$

is a weak homotopy equivalence. For p greater than the Krull dimension of X, both sides are contractible. For the flabby $\vee_y \underline{\Gamma}_y F$ in place of $S^p F$, the augmentation is a weak equivalence by collapse of the spectral sequence 1.36. The 5-lemma and the fibre sequence (2.12) then yield the induction step. For $p=0$, this yields the result claimed.

For extra credit. — Compare this with [50], IV and the construction of Quillen's spectral sequence in [97], § 7, 5.8. Note as Quillen works in an abelian category, he needs Gersten's conjecture to provide a short exact sequence of homotopy groups to play the role of the fibre sequence (2.12). Our more general context makes Gersten's conjecture irrelevant; the machine is quite happy with just (2.12) or its associated long exact sequence of homotopy groups.

DEFINITION 2.6. — Let X and F satisfy the usual hypotheses 2.1. Then F has the localization property if F extends to a contravariant functor on a subcategory of the category of schemes over X which includes all inverse limits along systems with affine bonding maps of schemes etale and finitely presented over X as in 2.1, and also includes all closed subschemes of such schemes. The subcategory must include all etale maps of schemes over X between schemes it contains, but need not include all maps of schemes over X. The main requirement is that if Z is a closed subscheme of Y with the reduced induced subscheme structure, and with open complement $Y-Z$, there is a Gysin map t and a homotopy fibre sequence

$$F(Z) \xrightarrow{t} F(Y) \to F(Y-Z). \tag{2.14}$$

The sequence (2.14) is required to be natural in that if $Y' \to Y$ is etale, and Z' is the pullback of Z to Y', then (2.15) is weak homotopy equivalent to a strictly commuting diagram of strict fibre sequences, and in particular homotopy commutes

$$\begin{array}{ccccc} F(Z) & \xrightarrow{t} & F(Y) & \to & F(Y-Z) \\ \downarrow & & \downarrow & & \downarrow \\ F(Z') & \xrightarrow{t} & F(Y') & \to & F(Y'-Z') \end{array} \tag{2.15}$$

Note Z' is etale over Z, and so is reduced. It is a closed subscheme of Y'.

F is said to have the localization property for regular schemes if the above properties hold when Y and Z are required also to be regular schemes.

If F has the localization property, (2.15) with $Y=U\cup V$, $Z=Z'=U\cup V-U$, and $Y'=V$ yields enough excision to conclude (2.2) is homotopy cartesian. Thus F has the Mayer-Vietoris property. But if F has the localization property for regular schemes, it need not have Mayer-Vietoris. Note $Z=U\cup V-V$ might not be regular.

Example 2.7. — The G-theory spectrum G( ) has the localization property by Quillen's localization theorem. The fibre sequence (2.14) is provided by [97], § 7, 3.2, and the naturality (2.15) follows from [97], § 7, 2.11 and the fact etale maps are flat and unramified. The other G-theory spectra of Appendix A, $G/l^\nu(\ )$, $G/l^\nu(\ )[\beta^{-1}]$, $G(\ )\otimes\mathbb{Q}$, $G(\ )_K$, etc., are formed from G( ) by processes which preserve homotopy fibre sequences, so they inherit the localization property.

The K-theory spectra, K( ), $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$, $K(\ )\otimes\mathbb{Q}$, $K(\ )_K$, etc., are weak homotopy equivalent to the corresponding G-theory spectra on regular schemes, by [97], § 7,1, so the K-theory spectra have the localization property for regular schemes.

PROPOSITION 2.8. — *Let* X *and* F *satisfy the usual hypotheses* 2.1. *Let* F *have the Mayer-Vietoris property* 2.2. *Suppose* F *has the localization property* 2.6, *or respectively, that* X *is regular and that* F *has the localization property for regular schemes. Suppose that for every* L *a field or Artin local ring that is etale over a residue field or Artin local* *ring which is a localization of* X (*resp., for every field* L *etale over a residue field of* X) *that* $\eta(\mathrm{L})$ *is a weak homotopy equivalence*

$$\eta(\mathrm{L}): \quad \mathrm{F}(\mathrm{L}) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(\mathrm{L}; \mathrm{F}). \tag{2.16}$$

Then F *has etale cohomological descent on* X: *for every* U *etale and finitely presented over* X

$$\eta(\mathrm{U}): \quad \mathrm{F}(\mathrm{U}) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(\mathrm{U}; \mathrm{F}), \tag{2.17}$$

is a weak homotopy equivalence.

Proof. — Note U as above satisfies 2.1 and the other hypotheses of the proposition, and its Krull dimension is at most that of X. Thus it suffices to prove $\eta(\mathrm{X})$ is a weak homotopy equivalence. I do this by induction on the Krull dimension of X.

Suppose X has Krull dimension 0. Then its local rings $\mathcal{O}_{\mathrm{X},\,x}$ are Artin local rings. If X is regular, these are residue fields. By 2.4, $\eta(\mathrm{X})$ is a weak homotopy equivalence if each $\eta(\mathcal{O}_{\mathrm{X},\,x})$ is. But this hypothesis holds by (2.16). Thus $\eta(\mathrm{X})$ is a weak homotopy equivalence.

To do the induction step, suppose X has Krull dimension N and the proposition is known for schemes of lower Krull dimension. By Corollary 2.4, $\eta(\mathrm{X})$ will be a weak equivalence if $\eta(\mathcal{O}_{\mathrm{X},\,x})$ is for all local rings $\mathcal{O}_{\mathrm{X},\,x}$ of X. Let $\mathrm{R}=\mathcal{O}_{\mathrm{X},\,x}$ be such a local ring. If R has Krull dimension less than N, $\eta(\mathrm{R})$ is a weak homotopy equivalence by induction hypothesis. It remains to consider the case where R has Krull dimension N. Let $\mathrm{Y}=\mathrm{Spec}(\mathrm{R})$, let $\mathrm{Z}=\mathrm{Spec}(k(x))$ be the closed point of Y, and let $\mathrm{U}=\mathrm{Y}-\mathrm{Z}$ be the open complement. Let $i: \mathrm{Z} \to \mathrm{Y}$, $j: \mathrm{U} \to \mathrm{Y}$ be the immersions. As Z is the unique closed point of the local ring Y, U has Krull dimension $\mathrm{N}-1$. Note that Y, U, and Z are quasiprojective over the noetherian ring R, and so $\mathbb{H}^{\cdot}_{et}(\mathrm{Y}; \mathrm{F})$ is weak equivalent to $\check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathrm{F})$ for $\mathscr{A}$ a direct system of etale covers of Y by Proposition 1.54.

The localization property (2.14) and (2.15) yields a homotopy fibre sequence of presheaves of fibrant spectra on Et/Y.

$$i_{\#}\mathrm{F} \to \mathrm{F} \to j_{\#}\mathrm{F}.$$

By 1.15 and 1.25, there is an induced diagram where all rows are homotopy fibre sequences

$$\begin{tikzcd}
\mathrm{F}(\mathrm{Z}) \arrow[r] \arrow[d, equal] & \mathrm{F}(\mathrm{Y}) \arrow[r] \arrow[d, equal] & \mathrm{F}(\mathrm{U}) \arrow[d, equal] \\
i_{\#}\mathrm{F}(\mathrm{Y}) \arrow[r] \arrow[d, "\eta(\mathrm{Z})"] & \mathrm{F}(\mathrm{Y}) \arrow[r] \arrow[d, "\eta(\mathrm{Y})"] & j_{\#}\mathrm{F}(\mathrm{Y}) \arrow[d, "\eta(\mathrm{U})"] \\
\check{\mathbb{H}}^{\cdot}(\mathscr{A}, i_{\#}\mathrm{F}) \arrow[r] \arrow[d, equal] & \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathrm{F}) \arrow[r] \arrow[d, equal] & \check{\mathbb{H}}^{\cdot}(\mathscr{A}; j_{\#}\mathrm{F}) \arrow[d, equal] \\
\check{\mathbb{H}}^{\cdot}(i^{*}\mathscr{A}; \mathrm{F}) \arrow[r] & \check{\mathbb{H}}^{\cdot}(\mathscr{A}; \mathrm{F}) \arrow[r] & \check{\mathbb{H}}^{\cdot}(j^{*}\mathscr{A}; \mathrm{F})
\end{tikzcd} \tag{2.18}$$

By the 5-Lemma, to show $\eta(\mathcal{O}_{X,\,x})=\eta(Y)$ is a weak homotopy equivalence, it suffices to show $\eta(Z)$ and $\eta(U)$ are. By the induction hypothesis, F has etale cohomological descent on the etale sites of Z and U, as these schemes have Krull dimension strictly less than N. Then by 1.51, F has descent for the direct systems of covers $i^*\mathscr{A}$, $j^*\mathscr{A}$ of Z and U. Thus $\eta(Z)$ and $\eta(U)$, and so $\eta(Y)$ are weak homotopy equivalences. This completes the induction step.

Exercise 2.9 (Optional). — Show for any closed immersion $i: Z \to Y$ and any presheaf of fibrant spectra F that there is a weak homotopy equivalence.

$$(2.19) \qquad \mathbb{H}^{\cdot}_{et}(Z; F) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(Y; i_{\#} F).$$

Hint. — Use 1.37 to reduce to case $F=F\langle n\rangle$, then appeal to the spectral sequence 1.36, and [SGA 4], VIII 5.5.

On the other hand, if i is an open immersion, the map is not in general a weak equivalence. In terms of the spectral sequence 1.36 two things go wrong. First, for $\tilde{A}$ a sheaf on the open Z, $H^*(Z; \tilde{A})$ is not isomorphic to $H^*(Y, i_* \tilde{A})$. Instead there is a Cartan-Leray spectral sequence.

$$E_2^{p,\,q}=H^p(Y; R^q i_* \tilde{A}) \quad\Rightarrow\quad H^{p+q}(Z; \tilde{A}).$$

Second, for A a presheaf, the sheafifications on Y and Z may differ; $i_* \tilde{A}$ is not $\widetilde{i_{\#}}A$. If F has descent, there is in fact a spectral sequence which is the sheafification of the presheaf of descent spectral sequences.

$$E_2^{p,\,q}=R^p i_*(\tilde{\pi}_q F) \quad\Rightarrow\quad \tilde{\pi}_{q-p} i_{\#} F.$$

Show that if F has etale cohomological descent, these two problems exactly cancel each other.

LEMMA 2.10. — *Let* L *be an Artin local ring with residue field* k. *Let* F′ *be any of the algebraic* G*-theory presheaves*, G( ), $G/l^{\nu}(\ )$, $G/l^{\nu}(\ )[\beta^{-1}]$, $G(\ )_K$, $G(\ )\otimes\mathbb{Q}$, *and let* F *be the corresponding* K*-theory presheaf*, K( ), $K/l^{\nu}(\ )$, $K/l^{\nu}(\ )[\beta^{-1}]$, $K(\ )_K$, $K(\ )\otimes\mathbb{Q}$, *respectively. Then there is a commutative diagram where the vertical maps are weak homotopy equivalences*

$$\begin{array}{ccc} F'(L) & \xrightarrow{\eta} & \mathbb{H}^{\cdot}_{et}(L; F') \\ \uparrow\wr & & \uparrow\wr \\ F(k) & \xrightarrow[\eta]{} & \mathbb{H}^{\cdot}_{et}(k; F) \end{array}$$

Thus to verify hypothesis (2.16) *of* 2.8 *for* G*-theory presheaves on Artin local rings, it suffices to verify* (2.16) *for the corresponding* K*-theory presheaves at residue fields.*

Proof. — Let R be a ring etale over L. Then R is an Artinian ring, and R modulo its nilpotent nil radical is $R\otimes_L k$. By [97], § 7, 3.1 the transfer map is a weak homotopy equivalence for $F' = G$

$$F'(R \underset{L}{\otimes} k) \xrightarrow{\sim} F'(R).$$

The other G-theory spectra inherit this property.

On the other hand [SGA 4], VIII 1.1, shows that the functor sending R to $R \underset{L}{\otimes} k$ is an equivalence of the restricted etale sites of R and k. Finally, $R \underset{L}{\otimes} k$ is etale over a field and so is regular, so the resolution theorem shows $F'(R \underset{L}{\otimes} k)$ and $F(R \underset{L}{\otimes} k)$ are weak equivalent.

Combining these facts readily yields 2.10.

2.11. Proposition 2.8 reduces the general etale cohomological descent problem to the case where $X = \mathrm{Spec}(L)$ for L a field, or at least an Artin local ring. I assume the usual hypotheses 2.1. Since L is a noetherian ring, 1.54 allows one to dispense with sheaf hypercohomology, and work with Čech hypercohomology. If L is a field, Example 1.50 shows how the etale sheaf of Čech hypercohomology may be reinterpreted as profinite group hypercohomology for the Galois group of the separable closure of L over L. If L is an Artin local ring, [SGA 4], VIII 1.1 identifies its restricted etale site to that of its residue field k, so etale hypercohomology of L is reinterpreted as Galois hypercohomology of k.

Henceforth, I will write as if L is a field to fix ideas. By 2.10, I need only consider K-theory and not G-theory.

DEFINITION 2.12. — A presheaf of fibrant spectra F on the etale site of a field (or Artin local ring) L has the weak transfer property if for every finite and etale morphism $\lambda : L_1 \to L_2$ of rings etale over L there is a morphism $\lambda_* : F(L_2) \to F(L_1)$ in addition to the usual $\lambda^* : F(L_1) \to F(L_2)$. The assignment of of λ_* to λ is to make F a covariant functor on the category of schemes etale over Spec(L) with the finite etale maps of schemes as morphisms (or equivalently, a contravariant functor on the category of rings etale over L with finite etale morphisms) with values in the stable homotopy category.

The presheaf F is required also to be continuous 1.42 and additive 1.52. If L_2 is a product $L'_2 \times L''_2$, and $\lambda : L_1 \to L_2$ decomposes as $\lambda' \times \lambda''$, then it is required that (2.20) commute in the stable homotopy category (an additive category)

$$(2.20)\qquad \begin{array}{ccc} F(L_2) & \xrightarrow{\lambda_*} & F(L_1) \\ \downarrow & \nearrow_{\lambda'_* + \lambda''_*} & \\ F(L'_2) \times F(L''_2) & & \end{array}$$

Given morphisms λ: $L_1 \to L_2$ and μ: $L_1 \to L_3$ both finite and etale, the Mackey diagram (2.21) is required to commute in the stable homotopy category

$$
\begin{tikzcd}
F(L_3 \otimes_{L_1} L_2) \arrow[r, "(1\otimes\lambda)_*"] & F(L_3) \\
F(L_2) \arrow[u, "(\mu\otimes 1)^*"] \arrow[r, "\lambda_*"'] & F(L_1) \arrow[u, "\mu^*"']
\end{tikzcd}
\tag{2.21}
$$

If $\lambda: L_1 \to L_2$ is an isomorphism, it is required that (2.22) holds in the stable homotopy category

$$\lambda_* = (\lambda^{-1})^* : F(L_2) \to F(L_1). \tag{2.22}$$

Finally, if $\lambda: L_1 \to L_2$ is etale and finite of degree n, it is required that in the stable homotopy category that $\lambda_* \lambda^* : F(L_1) \to F(L_1)$ is multiplication by n.

The reader familiar with Mackey functors (*cf.* [28], [25]) may reinterpret (2.20), (2.21), (2.22) as saying that the continuous presheaf F is a Mackey functor on the category of finite G-sets for G the profinite Galois group of L, with values in the stable homotopy category. The last requirement on $\lambda_* \lambda^*$ is special. I won't use the language of Mackey functors however.

The G-theory presheaf G( ) has the weak transfer property by [97], § 7,2. The last requirement follows from the projection formula of [97], § 7,2 and [77] 2.3, and the fact that L_2 is a free module of rank n over the local ring L_1. The presheaves $G/l^\nu(\ )$, $G/l^\nu(\ )[\beta^{-1}]$, $G(\ )_K$, $G(\ )\otimes\mathbb{Q}$ all inherit the weak transfer property. If L is a field, so all the L_1 and L_2 are products of fields and so are regular, the weakly equivalent corresponding K-theory presheaves all have the weak transfer property.

LEMMA 2.13. — *If* F *has the weak transfer property* 2.12 *and* $\lambda: L_1 \to L_2$ *is a finite etale Galois extension of rings etale over* L, *then in the stable homotopy category*

$$\lambda^* \lambda_* = \sum_G g^* : F(L_2) \to F(L_2). \tag{2.23}$$

Here the sum is over elements g of the Galois group $\mathrm{Gal}(L_2/L_1)$.

Proof. — Apply (2.21) with $L_2 = L_3$ and $\mu = \lambda$. The isomorphism (1.50) allows a reinterpretation of (2.21), as (2.24)

$$
\begin{tikzcd}
\prod_G F(L_2) \arrow[rr, bend left, "(1\otimes\lambda)_* . \kappa_* = \prod g_*"] & & \\
& F(L_2 \otimes_{L_1} L_2) \arrow[ul, "\kappa^*"', "\sim"] \arrow[r, "(1\otimes\lambda)_*"] & F(L_2) \\
& F(L_2) \arrow[uul, bend left, "\text{diagonal}"] \arrow[u, "(\lambda\otimes 1)^*"] \arrow[r, "\lambda_*"'] & F(L_1) \arrow[u, "\lambda^*"']
\end{tikzcd}
\tag{2.24}
$$

Using (2.22) $(\kappa^*)^{-1}=\kappa_*$ and (2.20), the map on the top of (2.24) is identified to the sum over g of the maps which project $\prod_{G} F(L_2)$ on the factor $F(L_2)$ indexed by g and them map it to $F(L_2)$ by g_*. Composed with the diagonal, this map is Σg_*. Appealing to (2.22) again, the map is also $\sum (g^{-1})^* = \sum g^*$.

PROPOSITION 2.14. — *Let* L *be a field, or more generally an Artin local ring, and let* F *be a presheaf of fibrant spectra on the etale site of* L *which has the weak transfer property,* 2.12. *Suppose that* F *is* $\mathbb{Q}$*-local,* i.e., *that* $\pi_* F$ *is a presheaf of vector spaces over the rationals. Then* F *has etale cohomological descent, so the augmentation is a weak homotopy equivalence*

$$(2.25)\qquad \eta:\ F(L) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(L; F).$$

Proof. — Let $\tilde{L}$ be the separable algebraic closure of L, and let L_α be a subring of $\tilde{L}$ with $\lambda: L \to L_\alpha$ a finite etale Galois extension, which has degree n. Then $\lambda_* \lambda^*: F(L) \to F(L)$ is multiplication by n by 2.12, and this is a weak homotopy equivalence as F is $\mathbb{Q}$-local. The map $\lambda^* \lambda_*: F(L_\alpha) \to F(L_\alpha)$ is the sum over the Galois group of L_α over L, $\sum g^*$, by 2.13. As F is $\mathbb{Q}$-local, one has a map $(1/n)\lambda^* \lambda_*: F(L_\alpha) \to F(L_\alpha)$. On the homotopy groups $\pi_* F(L_\alpha)$ this induces the projection on the summand fixed by the Galois group. Thus the map λ^* induces an isomorphism (2.26), with inverse induced by the restriction of $(1/n)\lambda_*$

$$(2.26)\qquad \pi_* F(L) \xrightarrow{\sim} [\pi_* F(L_\alpha)]^G = H^0(\mathrm{Gal}(L_\alpha/L);\ \pi_* F(L_\alpha)).$$

As $\mathrm{Gal}(L_\alpha/L)$ is a finite group and $\pi_* F(L_\alpha)$ is a vector space over $\mathbb{Q}$, the higher cohomology groups vanish by the classical transfer argument. Taking the limit as L_α runs over the direct system of all etale finite Galois extensions of L in $\tilde{L}$, and considering 1.50, one gets isomorphisms

$$(2.27)\qquad \begin{cases} \pi_* F(L) \cong H^0(\mathrm{Gal}(\tilde{L}/L);\ \pi_* F(\tilde{L})) \cong H^0_{et}(L;\ \pi_* F), \\ 0 = \varinjlim_{\alpha} H^p(\mathrm{Gal}(L_\alpha/L);\ \pi_* F(L_\alpha)) = H^p_{et}(L;\ \pi_* F) \qquad \text{for}\quad p>0. \end{cases}$$

These isomorphisms and the spectral sequence 1.36 or (1.56) show that (2.25) induces an isomorphism on homotopy groups, and so is a weak equivalence.

THEOREM 2.15. — *Let* X *be a separated noetherian scheme of finite Krull dimension* (*respectively, which is also regular*). *Then the presheaf of rational* G*-theory* $G(\ )\otimes\mathbb{Q}$ [*respectively,* $K(\ )\otimes\mathbb{Q}$] *has etale cohomological descent. The augmentation is a weak homotopy equivalence*

$$(2.28)\qquad \begin{cases} \eta:\ G(X)\otimes\mathbb{Q} \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(X;\ G\otimes\mathbb{Q}) \\ [\text{resp. } \eta:\ K(X)\otimes\mathbb{Q} \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(X;\ K\otimes\mathbb{Q})]. \end{cases}$$

Proof. — This follows by combining the successive reductions 2.4, 2.8, and 2.14.

COROLLARY 2.16. — *Let* X *be a separated noetherian scheme of finite Krull dimension (resp., which is regular). Let* $\mathscr{A}$ *be a direct system of etale covers* $\mathscr{U}_\alpha$*, such that for each* α *and each* $U_i \to X$ *in* $\mathscr{U}_\alpha$*,* U_i *is noetherian, or equivalently, finitely presented over* X. *Then the augmentation is a weak equivalence*

$$\begin{gathered}\eta\colon G(X)\otimes\mathbb{Q} \to \check{\mathbb{H}}^{\bullet}(\mathscr{A};\, G\otimes\mathbb{Q})\\ [\text{resp. } \eta\colon K(X)\otimes\mathbb{Q} \to \check{\mathbb{H}}^{\bullet}(\mathscr{A};\, K\otimes\mathbb{Q})].\end{gathered}\tag{2.29}$$

Proof. — This results from 2.15 and 1.46, as in 1.51.

COROLLARY 2.17. — *In the situation of* 2.15, 2.16 *there are spectral sequences*

$$\begin{gathered}E_2^{p,\,q} = H_{\text{et}}^{p}(X;\, \tilde{G}_q(\ \ )\otimes\mathbb{Q}) \;\Rightarrow\; G_{q-p}(X)\otimes\mathbb{Q}\\ [\text{resp. } E_2^{p,\,q} = H_{\text{et}}^{p}(X;\, \tilde{K}_q(\ \ )\otimes\mathbb{Q}) \Rightarrow K_{q-p}(X)\otimes\mathbb{Q}];\end{gathered}\tag{2.30}$$

$$\begin{gathered}E_2^{p,\,q} = \check{H}^{p}(\mathscr{A};\, G_q(\ \ )\otimes\mathbb{Q}) \;\Rightarrow\; G_{q-p}(X)\otimes\mathbb{Q}\\ [\text{resp. } E_2^{p,\,q} = \check{H}^{p}(\mathscr{A};\, K_q(\ \ )\otimes\mathbb{Q}) \Rightarrow K_{q-p}(X)\otimes\mathbb{Q}].\end{gathered}\tag{2.31}$$

The indexing is funny, as in 1.16, *etc. The spectral sequences* (2.30) *converge strongly, and the spectral sequences* (2.31) *do if there is an* N *such that* $E_2^{p*}=0$ *for* $p>N$.

Proof. — This follows from 2.15 and 2.16 by 1.36 and 1.16.

Exercise 2.18 (Optional). — Combine the proof of 2.14 with the etale sheafification of Quillen's construction of a Brown-Gersten type spectral sequence in [97], § 7, 5 to conclude for regular X of finite type over a field, there is an isomorphism

$$H_{\text{Zar}}^{p}(X;\, \tilde{G}_q(\ \ )\otimes\mathbb{Q}) \cong H_{\text{et}}^{p}(X;\, \tilde{G}_q(\ \ )\otimes\mathbb{Q}),\tag{2.32}$$

identifying (2.30) to the Brown-Gersten spectral sequence.

2.19. To get descent theorems for $K/l^\nu(\ \ )[\beta^{-1}]$, one needs a more sophisticated understanding of the transfer. I proceed to develop this.

LEMMA 2.20. — (Shapiro's Lemma) *Let* F *be an additive presheaf* 1.52 *of fibrant spectra on the etale site of a field* L′. *Let* L *be a field, and* $\lambda\colon L \to L'$ *a finite etale map, also denoted* $\lambda\colon \mathrm{Spec}(L') \to \mathrm{Spec}(L)$. *Let* $\lambda_\# F$ *be the induced presheaf of fibrant spectra on the etale site of* L, *as in* 1.24. *Let* L″ *be a possibly infinite etale Galois extension of* L′, *which is also Galois over* L.

Then there is a natural weak homotopy equivalence of profinite Galois hypercohomology spectra 1.50:

$$\mathbb{H}^{\bullet}(L''/L;\, \lambda_\# F) \xrightarrow{\sim} \mathbb{H}^{\bullet}(L''/L';\, F).\tag{2.33}$$

In particular, taking L″ *to be the separable algebraic alosure of* L′, *and so also of* L, *one sees that the natural map* (1.28) *is a weak homotopy equivalence*

$$\mathbb{H}_{\text{et}}^{\bullet}(L;\, \lambda_\# F) \xrightarrow{\sim} \mathbb{H}_{\text{et}}^{\bullet}(L';\, F).\tag{2.34}$$

Proof. — If R is a ring etale over L, 1.24 gives the formula

$$(\lambda_{\#} F)(R)=F(R\underset{L}{\otimes}L')=F(\operatorname{Spec}(R)\underset{\operatorname{Spec}(L)}{\times}\operatorname{Spec}(L')).$$

Let $\mathscr{A}$ be the direct system of etale covers of L by subfields L_α of L'' which are finite and etale over L. The left side of (2.33) is $\check{\mathbb{H}}^{\bullet}(\mathscr{A}; \lambda_{\#} F)$ by definition. Similarly, let $\mathscr{B}$ be the direct system of etale covers of L' by subfields L_β of L'' which are finite, etale, and Galois over L'. Then the right side of (2.33) is $\check{\mathbb{H}}^{\bullet}(\mathscr{B}; F)$. As λ is finite and etale, the systems $\lambda^* \mathscr{A}$ and $\mathscr{B}$ are weakly cofinal in each other in the sense of Lemma 1.22. For every cover of L' by L_β is refined by a cover by $L_\beta \underset{L}{\otimes} L'$; and any cover $L_\alpha \underset{L}{\otimes} L'$ of L' is refined by L_β, where L_β is the composite of the Galois conjugates of the finite extensions of L' occuring in the product of fields $L_\alpha \underset{L}{\otimes} L'$. These remarks, Lemma 1.22, and (1.18) combine to show (2.33) is a weak homotopy equivalence.

If L'' is taken to be the separable algebraic closure of L' and L, the two sides of (2.33) are identified to etale Čech hypercohomology of L and L', as the systems $\mathscr{A}$ and $\mathscr{B}$ are weakly cofinal in the category of etale covers of L and L' respectively. The weak equivalence of sheaf hypercohomology (2.34) results from (2.33) and 1.54. In this case, 1.54 may be replaced by a more elementary lemma generalizing [SGA 4], VIII 2.5 as 1.54 generalizes Artin's Theorem 1.53.

This completes the proof. Compare with [102], I 2.5 or [104], Thm. 8 of II, § 2 in light of 1.50 to see why this is the usual Shapiro's Lemma for cohomology.

2.21. The study of the transfer also requires group hyperhomology and Shapiro's Lemma for homology. Let G be a group, acting on a spectrum or prespectrum F. Consider G as a category with one object and whose morphisms are the elements of G. Then consider F as a functor from this category G into the category of prespectra. Specializing the results of 5.15-5.20 to this case, one gets a group hyperhomology prespectrum

$$\mathbb{H}_{\bullet}(G; F)=\underset{G}{\operatorname{hocolim}}\, F. \tag{2.35}$$

It preserves weak homotopy equivalences and homotopy fibre sequences in F. There is a convergent spectral sequence 5.17

$$E^2_{p,\,q}=H_p(G; \pi_q F) \quad\Rightarrow\quad \pi_{p+q}\mathbb{H}_{\bullet}(G; F). \tag{2.36}$$

The homotopy colimit $\mathbb{H}_{\bullet}(G; F)$ has a useful universal mapping property, as discussed in 5.15.

Unfortunately, $\mathbb{H}_{\bullet}(G; F)$ is in general a prespectrum and not a fibrant spectrum, even when F is a fibrant spectrum. When it is inserted as a coefficient into a hypercohomology construction that requires fibrant spectra as coefficients, $\mathbb{H}_{\bullet}(G; F)$ must be converted into a weak equivalent fibrant spectrum by the functorial process Q of 5.2 and [14]. If $\mathbb{H}_{\bullet}(G; F)$ occurs in a diagram of coefficients, one should apply Q to the entire diagram. As hypercohomology preserves weak equivalences of fibrant spectra, this process doesn't significantly change hypercohomology spectra which already had fibrant coefficients. Henceforth, I will abuse notation and treat $\mathbb{H}_{\bullet}(G; F)$ as if it were fibrant spectrum, suppressing all the Q's.

LEMMA 2.22 (Shapiro's Lemma for homology). — *Let* G *be a group with subgroup* H. *Let* F *be a prespectrum on which* H *acts, and let* $\underset{G/H}{\vee} F$ *be the induced prespectrum. The latter is a wedge of copies of* F *indexed by the cosets* G/H. *Let* $\{\sigma_i\}$ *be a set of coset representatives. Then* G *acts on* $\underset{G/H}{\vee} F$ *with* $g \in G$ *sending the factor* F *indexed by* σ_i *to that indexed by* σ_j *via* $h: F \to F$ *if* $g\sigma_i = \sigma_j h$ *with* $h \in H$.

The inclusions $H \to G$ *and* $F \to \underset{G/H}{\vee} F$ *as the factor indexed by the coset representative* 1 *induce a weak homotopy equivalence*

$$\mathbb{H}_{\bullet}(H; F) \xrightarrow{\sim} \mathbb{H}_{\bullet}(G; \underset{G/H}{\vee} F). \tag{2.37}$$

Proof. — It suffices to prove (2.37) induces an isomorphism of the E^2 terms of the converging spectral sequences 5.17:

$$H_p(H; \pi_q F) \to H_p(G; \bigoplus_{G/H} \pi_q F).$$

But this follows by the usual Shapiro's Lemma, as $\bigoplus_{G/H} \pi_q F$ is the usual induced module $\mathbb{Z}[G] \underset{\mathbb{Z}[H]}{\otimes} (\pi_q F)$. If need be, consult [19], III 6.2.

COROLLARY 2.23. — *Let* $X = \operatorname{Spec}(L)$ *be a field and let* F *be a presheaf of fibrant spectra satisfying the usual hypotheses* 2.1. *Let* L″/L *be a separable algebraic extension of fields, and let* L′/L *be a finite Galois subextension with Galois group* Gal(L′/L). *Then if* Gal(L′/L) *acts on* $F(L'' \underset{L}{\otimes} L')$ via *its action on the right factor* L′, *there is a chain of weak homotopy equivalences*

$$F(L'') \xrightarrow{\sim} \xleftarrow{\sim} \mathbb{H}_{\bullet}(\operatorname{Gal}(L'/L); F(L'' \underset{L}{\otimes} L')). \tag{2.38}$$

Proof. — By the Chinese remainder Theorem, as in 1.50, there is an isomorphism

$$\kappa : L'' \underset{L}{\otimes} L' \xrightarrow{\cong} \prod_{\operatorname{Gal}(L'/L)} L'', \qquad \kappa(l'' \otimes l') = (l'' . gl')_{g \in \operatorname{Gal}(L'/L)}. \tag{2.39}$$

As F is additive in the sense of 1.52 by 2.1, κ induces a weak equivalence of $F(L'' \underset{L}{\otimes} L')$ with the induced spectrum

$$\underset{\operatorname{Gal}(L'/L)}{\vee} F(L'') \xrightarrow{\sim} \prod_{\operatorname{Gal}(L'/L)} F(L'') \xleftarrow{\sim} F(L'' \underset{L}{\otimes} L'). \tag{2.40}$$

The corollary now results from 2.22. There is in fact a weak equivalence from the left side of (2.38) to the right induced by the insertion of L″ as a summand in (2.39).

This weak equivalence is natural with respect to Gal(L″/L′). It is natural up to homotopy for Gal(L″/L), with the appropriate action of this group on the right side of (2.38).

PROPOSITION 2.24 (Tate's appendix to [102]). — *Let* $X = \mathrm{Spec}(L)$ *be a field and* F *a presheaf of fibrant spectra satisfying the usual hypotheses* 2.1. *Let* L″/L *be an infinite Galois extension of* L *so that the profinite Galois group* Gal(L″/L) *has bounded cohomological dimension for* $\mathbb{Z}_{(J)}$*-modules, where* J *is the set of primes in* 2.1. *Let* L′/L *be a finite Galois subextension. Then there is a natural weak homotopy equivalence*

$$(2.41)\qquad \begin{array}{c} \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ \mathbb{H}^{\bullet}(L''/L;\ F(L''\underset{L}{\otimes} L'))) \\ \downarrow \wr \\ \mathbb{H}^{\bullet}(L''/L;\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ F(L''\underset{L}{\otimes} L'))) \end{array}$$

[*Here* Gal(L′/L) *acts on* $F(L''\underset{L}{\otimes} L')$ via *the right* L′, *and* Gal(L″/L) *acts* via *the left* L″ *only, leaving the right* L′ *fixed*].

Proof. — The first step is to construct the map using the universal mapping property of homotopy colimits discussed in 5.15. Let $u : \mathbf{n} \to \mathrm{Gal}(L'/L)$ be a functor as in 5.15. Define $f(u)$

$$(2.42)\qquad f(u):\quad \mathbb{H}^{\bullet}(L''/L;\ F(L''\underset{L}{\otimes} L')) \ltimes \Delta[n] \to \mathbb{H}^{\bullet}(L''/L;\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ F(L''\underset{L}{\otimes} L'))),$$

to be the composite of the canonical map deduced from the universal mapping property of homotopy limits

$$(2.43)\qquad \mathbb{H}^{\bullet}(L''/L;\ F(L''\underset{L}{\otimes} L')) \ltimes \Delta[n] \to \mathbb{H}^{\bullet}(L''/L;\ F(L''\underset{L}{\otimes} L') \ltimes \Delta[n])$$

and the map obtained by applying $\mathbb{H}^{\bullet}(L''/L;\ \ )$ to the universal map

$$(2.44)\qquad j(u):\quad F(L''\underset{L}{\otimes} L') \ltimes \Delta[n] \to \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ F(L''\underset{L}{\otimes} L')).$$

The maps $f(u)$ satisfy the conditions of 5.15, and so induce a map (2.41).

Next, I show that (2.41) is a weak homotopy equivalence if the modules $\pi_* F(L'')$ have cohomological dimension zero for all closed subgroups of the profinite group Gal(L″/L). Then $\pi_* F(L'')$ also has cohomological dimension zero for the subgroup Gal(L″/L′). Shapiro's Lemma 2.20 and the spectral sequence 1.36 then yield isomorphisms.

$$(2.45)\qquad \pi_* \mathbb{H}^{\bullet}(L''/L,\ F(L''\underset{L}{\otimes} L')) = H^0(L''/L',\ \pi_* F(L'')).$$

By Tate's appendix [102], Lemma 1, p. I-82 or [101], IX, § 5, Thm. 8, the theory of Tate cohomology and the dimension zero hypothesis yield the results in (2.46).

$$(2.46)\qquad \begin{gathered} H_p(\mathrm{Gal}(L'/L);\ \pi_*\,\mathbb{H}^{\boldsymbol{\cdot}}(L''/L;\ F(L''\underset{L}{\otimes}L')))=0, \qquad p>0, \\ H_0(\mathrm{Gal}(L'/L);\ \pi_*\,\mathbb{H}^{\boldsymbol{\cdot}}(L''/L;\ F(L''\underset{L}{\otimes}L')))=H^0(L''/L;\ \pi_*\,F(L'')). \end{gathered}$$

Thus the spectral sequence (2.36) for the top of (2.41) collapses to

$$(2.47)\qquad \pi_*\,\mathbb{H}_{\boldsymbol{\cdot}}(\mathrm{Gal}(L'/L);\ \mathbb{H}^{\boldsymbol{\cdot}}(L''/L;\ F(L''\underset{L}{\otimes}L')))=H^0(L''/L;\ \pi_*\,F(L'')).$$

The homology Shapiro's Lemma of 2.22 and 2.23 and the spectral sequence 1.36 with the cohomological dimension zero hypothesis yield

$$(2.48)\qquad \pi_*\,\mathbb{H}^{\boldsymbol{\cdot}}(L''/L;\ \mathbb{H}_{\boldsymbol{\cdot}}(\mathrm{Gal}(L'/L);\ F(L''\underset{L}{\otimes}L'))=H^0(L''/L;\ \pi_*\,F(L'')).$$

It is left to the conscientious reader to verify that the map (2.41) indeed induces the isomorphism between (2.47) and (2.48). Since Tate's appendix works in terms of the transfer, it will be easier to verify this after reading 2.26 below. First do the case with appropriate hypertransfers, and then reformulate in terms of the equivalence of Shapiro's Lemma using the relation (2.52), and 2.28.

Finally, I prove the general case by induction on the maximum of the cohomological dimension of $\pi_*\,F(L'')$ over all closed subgroups of $\mathrm{Gal}(L''/L)$. This maximum is bounded by the cohomological dimension of $\mathrm{Gal}(L''/L)$ for $\mathbb{Z}_{(J)}$ modules by hypotheses 2.1 and the usual results on cohomological dimension of closed subgroups (e. g., [102], I 3.3 or [104], III, § 1). Let F be such that $\pi_*\,F(L'')$ has maximum cohomological dimension N, and that (2.41) is known to be a weak homotopy equivalence for maximum cohomological dimension less than N. Let $\lambda_\# F$ be the induced presheaf with $\lambda_\# F(A)=F(A\underset{L}{\otimes}L'')$. If $L''=\varinjlim L_\alpha$, L_α finite Galois over L, the additivity and continuity of F yield isomorphisms

$$(2.49)\qquad \pi_*\,F(L''\underset{L}{\otimes}L'')\cong\varinjlim_{\alpha}\pi_*\,F(L''\underset{L}{\otimes}L_\alpha)\cong\varinjlim_{\alpha}\prod_{\mathrm{Gal}(L_\alpha/L)}\pi_*\,F(L'').$$

Thus $\pi_*(\lambda_\# F)(L'')$ is the usual induced module, and has maximum cohomological dimension zero by the usual Shapiro's Lemma, [102], I, Prop. 10, [104], II, § 2, or 2.20 applied to the appropriate Eilenberg-MacLane spectrum.

The map $L\to L''$ induces a natural $F\to\lambda_\# F$. Let H be the homotopy cofibre. The long exact sequence of homotopy groups of this homotopy fibre sequence splits into short exact sequences

$$(2.50)\qquad 0\to\pi_*\,F(L'')\to\pi_*\,F(L''\underset{L}{\otimes}L'')\to\pi_*\,H(L'')\to 0.$$

This is because the map induced by multiplication $L''\otimes L''\to L''$ splits the map on the left as a map of abelian groups. It follows that $\pi_*\,H(L'')$ has maximum cohomological dimension $N-1$.

Consider the homotopy fibre sequence $F \to \lambda_{\#} F \to H$ and the map (2.41) of the induced homotopy fibre sequences of the sources and targets of (2.41). The map (2.41) is a weak homotopy equivalence for $\lambda_{\#} F$ and for H by induction hypothesis. Hence it is for F by the 5-Lemma. This completes the induction step and the proof of the Theorem.

DEFINITION 2.25. — Let F be a presheaf of fibrant spectra on the etale site of a field. F is said to have a hypertransfer if for all finite Galois extensions L'/L in the site and all algebras A over L, there is a map of prespectra

$$\text{(2.51)} \qquad T:\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ F(A \underset{L}{\otimes} L')) \to F(A).$$

These maps must satisfy several conditions. First, the map must be natural in A. Second, whenever $A = L''$ is a separable algebraic field extension of L which contains L' as a subextension, there is a homotopy commutative diagram (2.52), formed from (2.51) and the maps of Shapiro's Lemma 2.23:

$$\text{(2.52)} \qquad \begin{tikzcd}
\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ F(L'' \underset{L}{\otimes} L')) \arrow[d, "\wr"] \arrow[dr, "T"] & \\
\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ \prod\limits_{\mathrm{Gal}(L'/L)} F(L'')) & F(L'') \arrow[dl, "\sim"'] \\
\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ \bigvee\limits_{\mathrm{Gal}(L'/L)} F(L'')) \arrow[u, "\wr"] &
\end{tikzcd}$$

It follows from (2.52) and 2.23 that in this case T is a weak homotopy equivalence, the homotopy inverse to the equivalence of Shapiro's Lemma 2.23.

Finally, diagram (2.53) commutes.

$$\text{(2..53)} \qquad \begin{tikzcd}
\mathbb{H}_{\bullet}(\mathrm{Gal}(L_1/L);\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L_2/L);\ F(A \underset{L}{\otimes} L_1 \underset{L}{\otimes} L_2))) \arrow[r, "\mathbb{H}_{\bullet}(L_1/L;\ T)"] \arrow[d, "\wr"] & \mathbb{H}_{\bullet}(\mathrm{Gal}(L_1/L);\ F(A \underset{L}{\otimes} L_1)) \arrow[d, "T"] \\
 & F(A), \\
\mathbb{H}_{\bullet}(\mathrm{Gal}(L_2/L);\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L_1/L);\ F(A \underset{L}{\otimes} L_1 \underset{L}{\otimes} L_2))) \arrow[r, "\mathbb{H}_{\bullet}(L_2/L;\ T)"'] & \mathbb{H}_{\bullet}(\mathrm{Gal}(L_2/L);\ F(A \underset{L}{\otimes} L_2)) \arrow[u, "T"']
\end{tikzcd}$$

Here the isomorphism on the left of (2.53) is provided by 5.16.

Remark 2.26. — The diagram (2.52) of weak homotopy equivalences and the naturality condition of 2.25 show that the weak equivalence of Shapiro's Lemma 2.23 is in fact $\mathrm{Gal}(L''/L)$ equivariant in the homotopy category. It takes a bit of work to show this directly. Thus it is often more transparently natural to formulate Shapiro's Lemma in terms of the hypertransfer T being a weak homotopy equivalence in (2.52). Even if F does not admit a general hypertransfer, in the situation of 2.23 the folding map (2.54) which adds all summands is Gal(L′/L) invariant, and so by the universal mapping property induces a diagram of weak homotopy equivalences (2.55) just like (2.52)

$$\bigvee_{\mathrm{Gal}(L'/L)} F(L'') \to F(L''), \tag{2.54}$$

$$\begin{tikzcd} \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \bigvee_{G} F(L'')) \arrow[r, "T", "\sim"'] \arrow[d, "\wr"'] & F(L'') \\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \prod_{G} F(L'')) & \\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); F(L'' \otimes_{L} L')) \arrow[u, "\wr"] & \end{tikzcd} \tag{2.55}$$

Remark 2.27. — In (2.53) with $L_2 = L''$ containing $L_1 = L'$ as a subextension, Shapiro's Lemma shows that the top horizontal map is a weak homotopy equivalence, which by abuse of notation is the Lyndon-Hochschild-Serre equivalence

$$\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L'); F(L''))) \simeq \mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L); F(L'')). \tag{2.56}$$

Then (2.53) becomes

$$\begin{tikzcd} \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L'); F(L''))) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathbb{H}_{\cdot}(\mathrm{Gal}(L'/L);\, T)"] & \mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L); F(L'')) \arrow[d, "T"] \\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); F(L')) \arrow[r, "T"'] & F(L) \end{tikzcd} \tag{2.57}$$

The abuse of notation consists of the fact that Gal(L′/L) does not act on $\mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L'); F(L''))$, except up to coherent homotopy, but rather acts on a weak homotopy equivalent spectrum, $\mathbb{H}_{\bullet}(\mathrm{Gal}(L''/L); F(L'' \otimes_{L} L'))$.

Remark 2.28. — If F has a hypertransfer and also satisfies the conditions of Proposition 2.24, the weak homotopy equivalence (2.41) and the hypertransfer variant of the weak homotopy equivalence of Shapiro's Lemma (2.52) combine to yield the bottom horizontal map in a diagram (2.58) where the vertical maps are induced by the augmentations

$$\begin{tikzcd} \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); F(L')) \arrow[r, "T"] \arrow[d, "\mathbb{H}_{\cdot}(G,\, \eta)"'] & F(L) \arrow[d, "\eta"] \\ \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \mathbb{H}^{\bullet}(L''/L; F(L'' \otimes_{L} L'))) \arrow[r, "\sim"] & \mathbb{H}^{\bullet}(L''/L; F(L'')) \end{tikzcd} \tag{2.58}$$

By abuse and Shapiro's Lemma, the bottom map may be considered as a weak homotopy equivalence yielding the spectral sequence of Tate's appendix to [102]:

$$\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \mathbb{H}^{\bullet}(L''/L'; F(L''))) \xrightarrow{\sim} \mathbb{H}^{\bullet}(L''/L; F(L'')). \tag{2.59}$$

When F has no hypertransfer, 2.24, Shapiro's Lemma, (2.56), and abuse still provide a weak homotopy equivalence between the two sides of (2.59), which is a sort of hypertransfer up to homotopy natural in F.

Exercise 2.29 (Optional). — The stable homotopy category is an additive category, but the category of spectra is not additive before passing to the homotopy category. Show that if the category of spectra were additive, the contorted constructions of 2.24-2.28 would simplify to the usual ones.

Example 2.30. — The presheaf of K-theory prespectra K( ) admits a hypertransfer, so the discussion of 2.25-2.28 applies to it. Similarly, the presheaves $K(\ )\otimes\mathbb{Q}$, $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$, and $K(\ )_K$ all inherit a hypertransfer.

For $F=K$, the hypertransfer map (2.51) is determined by the universal mapping property of 5.15. The algebra $A\underset{L}{\otimes}L'$ is a free A-module of rank $[L':L]$, so finitely generated free or projective $A\underset{L}{\otimes}L'$-modules are finitely generated projective A-modules by neglect of structure. The forgetful functor induces the required map $f(u): K(A\underset{L}{\otimes}L')\to K(A)$ for the unique $u:\mathbf{0}\to \mathrm{Gal}(L'/L)$. A functor $u:\mathbf{1}\to\mathrm{Gal}(L'/L)$ corresponds to an element $g\in\mathrm{Gal}(L'/L)$. The action of g on $K(A\underset{L}{\otimes}L')$ is induced by the automorphism of the category of finitely generated projective modules over $A\underset{L}{\otimes}L'$ sending P to

$$g^*P=(A\underset{L}{\otimes}L'g)\underset{A\underset{L}{\otimes}L'}{\otimes}P.$$

Here $A\underset{L}{\otimes}L'g$ is $A\underset{L}{\otimes}L'$ as a right $A\underset{L}{\otimes}L'$ module *via* $A\underset{L}{\otimes}g$.

There is a natural isomorphism of A-modules $A\otimes g: P\to g^*p$. As K(A) is the spectrum associated to the symmetric monoidal category $\mathbf{K}(A)$ of finitely generated projective A-modules and isomorphisms, this natural isomorphism induces the required homotopy $f(u)$: $K(A\otimes L')\ltimes\Delta[1]\to K(A)$, e.g. by [125], 2.9, Appendix. Similarly, a functor $u:\mathbf{n}\to\mathrm{Gal}(L'/L)$ is a sequence of elements $(g_1, \ldots, g_n)$ *of* Gal(L'/L), inducing a sequence of natural isomorphisms $(A\otimes g_1, \ldots, A\otimes g_n)$, and so a homotopy $f(u)$: $K(A\otimes L')\ltimes\Delta[n]\to K(A)$ by [125], 2.9. If a sequence $v=(g'_1, \ldots, g'_k)$ is produced from a sequence $u=(g_1, \ldots, g_n)$ by inserting 1's and performing various compositions, i.e., if $v=u\varphi$ for some $\varphi:\mathbf{k}\to\mathbf{n}$, the sequence of natural isomorphisms $(A\otimes g'_1, \ldots, A\otimes g'_k)$ is produced from $(A\otimes g_1, \ldots, A\otimes g_n)$ by inserting 1's and performing compositions in the same pattern, i.e.,

$$(A\otimes g'_1, \ldots, A\otimes g'_k)=(A\otimes g_1, \ldots, A\otimes g_n).1\otimes\varphi.$$

It follows then from [125], 2.9, 3.1, 3.2 that the compatibility conditions of 5.15 hold on the induced homotopies, i.e.,

$$f(v).((g^*_{\varphi(0)-1}\cdots g^*_0)\ltimes\Delta[k])=f(u).(1\ltimes\Delta[\varphi]).$$

Thus by 5.15, this system of homotopies $f(u)$: $K(A\otimes L') \ltimes \Delta[n] \to K(A)$ for each u: $\mathbf{n} \to \mathrm{Gal}(L'/L)$ determines a map T: $\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); K(A\otimes L')) \to K(A)$ as in (2.51). Clearly T is natural in A.

To verify condition (2.53), one first notes that if one has elements $g \in \mathrm{Gal}(L_1/L)$ and $h \in \mathrm{Gal}(L_2/L)$, then the natural isomorphisms $1\otimes g\otimes 1$ and $1\otimes 1\otimes h$ commute with each other: $1\otimes g\otimes 1 . 1\otimes 1\otimes h = 1\otimes g\otimes h = 1\otimes 1\otimes h . 1\otimes g\otimes 1 : P \cong 1 \times g^* \times h^* P$. So given u: $\mathbf{n} \to \mathrm{Gal}(L_1/L)$ and v: $\mathbf{m} \to \mathrm{Gal}(L_2/L)$ one obtains a commutative $\mathbf{n}\times\mathbf{m}$ grid of symmetric monoidal natural isomorphisms of functors $\mathbf{K}(A\otimes L_1\otimes L_2) \to \mathbf{K}(A)$. By [125] 2.9, Appendix, this induces a map $f(u, v)$: $K(A\otimes L_1\otimes L_2) \ltimes \Delta[n] \ltimes \Delta[m] \to K(A)$. As each isomorphism $1\otimes 1\otimes h$ is linear over $A\otimes L_1$, the grid of natural isomorphisms factors as $\mathbf{K}(A\otimes L_1\otimes L_2) \times \mathbf{n} \times \mathbf{m} \to \mathbf{K}(A\otimes L_1) \times \mathbf{n} \to \mathbf{K}(A)$. This induces a factorization of the induced $f(u, v)$ as

$$K(A\otimes L_1\otimes L_2) \ltimes \Delta[n] \ltimes \Delta[m] \to K(A\otimes L_1) \ltimes \Delta[n] \to K(A).$$

Fixing u and n, and applying 5.15 to the system of homotopies as v and m vary, one obtains an induced factorization

$$\mathbb{H}_{\bullet}(\mathrm{Gal}(L_2/L); K(A\otimes L_1\otimes L_2)) \ltimes \Delta[n] \to K(A\otimes L_1) \ltimes \Delta[n] \to K(A).$$

The first map in this factorization is $T \ltimes \Delta[n]$ for the hypertransfer T of L_2/L, and the second map is the $f(u)$ which induces the hypertransfer for L_1/L. Hence applying 5.15 to this system of homotopies as u and n vary yields the top half of (2.53). Similarly, on reversing the order of treating u and v, one obtains the bottom half of (2.53). Thus the top and bottom arrows of (2.53) are universal with respect to the same double system of homotopies $f(u, v)$, and so they agree as claimed.

To verify (2.52) one translates the hypertransfer T *via* the homotopy equivalences to $\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L), \vee F(L''))$, and then checks it is the homotopy inverse to the Shapiro's Lemma equivalence. For the first step in translation, consider the isomorphism of (2.39) κ: $L'' \underset{L}{\otimes} L' \cong \Pi L''$, where the product is indexed by $\mathrm{Gal}(L'/L)$. The action of $\mathrm{Gal}(L'/L)$ on $L''\otimes L'$ translates *via* κ to the permutation action on $\Pi L''$. Hence the action on $\mathbf{K}(L''\otimes L')$ translates *via* the equivalence induced by κ to the permutation action on $\Pi\mathbf{K}(L'')$. The forgetful functor $\mathbf{K}(L''\otimes L') \to \mathbf{K}(L'')$ translates into the functor $\Pi\mathbf{K}(L'') \to \mathbf{K}(L'')$ sending $(P_1, \ldots, P_g)$ to $P_1 \oplus \ldots \oplus P_g$. The natural isomorphism from the forgetful functor to the forgetful functor composed with g^* translates into the canonical natural isomorphism between $P_1 \oplus \ldots \oplus P_g$ and the sum of the P's after permuting the order by action of g. By [125], the summation functor $P_1 \oplus \ldots \oplus P_g$ and permutation natural isomorphisms determine a system of homotopies $f'(u)$: $\Pi K(L'') \times \Delta[n] \to K(L'')$ that is the translation *via* the homotopy equivalence induced by κ of the system of homotopies $f(u)$. By 5.15, the $f'(u)$'s determine a

$$T' : \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \Pi K(L'')) \to K(L''),$$

that is the translation of T *via* κ. The $f'(u)$ may be restricted along the homotopy equivalence $\vee K(L'') \to \Pi K(L'')$. This yields a system of homotopies $f''(u)$: $\vee K(L'') \times \Delta[n] \to K(L'')$ and a T'': $\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \vee K(L'')) \to K(L'')$. T, T', and T'' are compatible under the vertical homotopy equivalences in (2.52). To check that (2.52) homotopy commutes, it suffices to show T'' is the homotopy inverse to the indicated Shapiro's Lemma homotopy equivalence. By 2.22, the Shapiro's Lemma equivalence $K(L'') \xrightarrow{\sim} \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \vee K(L''))$ is the composite of insertion as the summand indexed by $1 \in \mathrm{Gal}(L'/L)$, $K(L'') \to \vee K(L'')$, and the canonical map of 5.15 $j''(0)\colon \vee K(L'') \to \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); \vee K(L''))$. As T'' composed with $j''(0)$ is the summation map $f''(0)\colon \vee K(L'') \to K(L'')$, and $f''(0)$ composed with insertion is the identity map $K(L'') \to K(L'')$, the composite of T'' with the Shapiro's Lemma equivalence is the identity map. Hence T'' is the homotopy inverse, as required. This completes the verification of (2.52).

Hence the map T constructed above is a hypertransfer for $K(\ )$.

The other K-theory spectra of Appendix A, namely $K(\ )\otimes\mathbb{Q}$, $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$, $K(\ )_K$ are all formed from $K(\ )$ by smashing with a spectrum or by taking direct limits along a canonical directed system as in (A.9). As $\mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L);\ )$ commutes with both these processes by 5.20 and 5.16, these other K-theory presheaves inherit a hypertransfer.

LEMMA 2.31 (Projection formula). — *The pairing of algebraic K-theory spectra induces a commutative diagram for L'/L a finite Galois extension of fields and A and B algebras over L*

$$
(2.60)\qquad
\begin{array}{ccc}
K(B) \wedge \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); K(A \underset{L}{\otimes} L')) & \to & \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); K(B \underset{L}{\otimes} A \underset{L}{\otimes} L')) \\
\Big\downarrow{\scriptstyle 1\wedge T} & & \Big\downarrow{\scriptstyle T} \\
K(B) \wedge K(A) & \longrightarrow & K(B \underset{L}{\otimes} A)
\end{array}
$$

In particular, if $A = B$ is a commutative algebra, T is a morphism of module spectra for the ring spectrum $K(A)$. A diagram like (2.60) commutes for any induced pairing between any two of the K-theory spectra $K(\ )_K$, $K(\ )\otimes\mathbb{Q}$, $K/l^\nu(\ )$, $K/l^\nu(\ )[\beta^{-1}]$.

Proof. — The top horizontal map of (2.60) is the composite of the isomorphism (5.9) and the map of hyperhomology spectra induced by the pairing

$$K(B) \wedge K(A\otimes L') \to K(B\otimes A\otimes L').$$

The usual projection formula (A.2) and various forms of naturality give the commutativity of (2.60). If $A = B$ is commutative, multiplication is a ring map and so induces a map of spectra $K(A\otimes A) \to K(A)$. Adjoining the obvious commutative square induced by this to the right of (2.60) shows T is a module map. The last statment follows in the usual way.

PROPOSITION 2.32 (Tate). — *Let L''/L be an infinite Galois extension of fields whose profinite Galois group has cohomological dimension one for l-torsion sheaves. Let L'/L be a finite Galois subextension. There is a spectral sequence of homological type*

$$(2.61)\qquad E^2_{p,q} = H_p(\mathrm{Gal}(L'/L); H^{1-q}(L''/L'; \mathbb{Z}/l^\nu(1))) \;\Rightarrow\; H^{1-p-q}(L''/L; \mathbb{Z}/l^\nu(1)).$$

$E^2_{p,q}=0$ *if* $q \neq 0, 1$. *The spectral sequence converges strongly with* $E^3=E^\infty$. *There is a long exact sequence*

$$(2.62)\quad H_2(\mathrm{Gal}(L'/L);\ H^1(L''/L';\ \mathbb{Z}/l^\nu(1))) \xrightarrow{d_2} H_0(\mathrm{Gal}(L'/L);\ H^0(L''/L';\ \mathbb{Z}/l^\nu(1))) \to H^0(L''/L;\ \mathbb{Z}/l^\nu(1)) \to H_1(\mathrm{Gal}(L'/L);\ H^1(L''/L';\ \mathbb{Z}/l^\nu(1))) \to 0.$$

Proof. — The spectral sequence is given by Theorem 1 of Tate's appendix to [102]. Alternatively, it can be deduced from 2.24 in the form (2.59). Let μ_{l^ν} be the functor which sends a ring to the group of l^ν-th roots of unity. Let F be the presheaf of Eilenberg-MacLane spectra

$$(2.63)\qquad F(R)=K(\mu_{l^\nu}(R),\ 1).$$

Assume l is not 0 in L and that L″ contains all l^ν-th roots of unity, so that the assertions of the proposition make sense. Then $\pi_1 F(L'')=\mathbb{Z}/l^\nu(1)$. The spectral sequence of 1.36 interpreted as in 1.50 collapses as $\pi_p F=0$ for $p\neq 1$, to yield

$$(2.64)\qquad \begin{aligned} \pi_q \mathbb{H}^\cdot(L''/L';\ F(L'')) &= H^{1-q}(L''/L';\ \mathbb{Z}/l^\nu(1)),\\ \pi_q \mathbb{H}^\cdot(L''/L;\ F(L'')) &= H^{1-q}(L''/L;\ \mathbb{Z}/l^\nu(1)). \end{aligned}$$

When these values are inserted into the spectral sequence for hyperhomology 5.17 or (2.36), and the abutment interpreted by (2.59), the desired spectral sequence (2.61) results. The cohomological dimension one hypothesis is inherited by L″/L′ and yields $E^2_{p,q}=0$ for $q\neq 0$, 1 and the collapsing $E^3=E^\infty$. The long exact sequence (2.62) results from the collapsing spectral sequence *via* (2.65)

$$(2.65)\qquad \begin{array}{ccccccccc} 0 & \to & E^\infty_{0,1} & \to & H^0(L''/L;\ \mathbb{Z}/l^\nu(1)) & \to & E^\infty_{1,0} & \to & 0\\ & & \| & & & & \| & & \\ & & E^3_{0,1} & & & & E^2_{1,0} & & \\ & & \| & & & & & & \\ & & \operatorname{coker} d_2 & & & & & & \end{array}$$

One could also get the spectral sequence by considering $F=K(\mu_{l^\nu},\ 2)$, and relabeling the various homotopy groups to account for the degree shift.

COROLLARY 2.33. — *Under the hypotheses of* 2.32, *there are isomorphisms*

$$(2.66)\qquad H_0(\mathrm{Gal}(L'/L);\ H^1(L''/L';\ \mathbb{Z}/l^\nu(1)))\cong H^1(L''/L;\ \mathbb{Z}/l^\nu(1)),$$

$$(2.67)\quad d_2:\ H_{p+2}(\mathrm{Gal}(L'/L);\ H^1(L''/L';\ \mathbb{Z}/l^\nu(1))) \xrightarrow{\cong} H_p(\mathrm{Gal}(L'/L);\ H^0(L''/L';\ \mathbb{Z}/l^\nu(1))),\qquad p\geqq 1.$$

Proof. — These result from the collapse of the spectral sequence (2.61) at $E^3 = E^\infty$, and $E^\infty_{p,\,q} = 0$ for $p+q \neq 0$, 1; $q \neq 0$, 1.

DEFINITION 2.34. — Let F be a presheaf of fibrant spectra on the etale site of a field. Suppose F has a hypertransfer as in 2.25. Let L″/L be a possibly infinite Galois extension of fields in the domain of F. A family of *inductors* for $x \in \pi_n F(L)$ is a family of elements

$$\mathrm{Ind}(x,\, L_\alpha/L) \in \pi_n \mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L);\, F(L_\alpha)); \tag{2.68}$$

for every finite Galois subextension L_α/L of L″/L. If

$$T(L_\alpha/L) : \mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L); \qquad F(L_\alpha)) \to F(L)$$

is the hypertransfer, one requires that

$$T(L_\alpha/L)(\mathrm{Ind}(x,\, L_\alpha/L)) = x. \tag{2.69}$$

If L_α/L is a Galois subextension of the finite Galois subextension L_β/L of L″/L, one requires that in the instance of (2.57) that

$$\mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L);\, T(L_\beta/L_\alpha))(\mathrm{Ind}(x,\, L_\beta/L)) = \mathrm{Ind}(x,\, L_\alpha/L). \tag{2.70}$$

2.35. Note that condition (2.70) and (2.57) show that it suffices to define $\mathrm{Ind}(x,\, L_\beta/L)$ for a system of L_β cofinal in the direct system of finite Galois subextensions of L″/L. For then (2.70) uniquely determines $\mathrm{Ind}(x,\, L_\alpha)$ for all L_α. This is independent of the choice of L_β by (2.70) in the cofinal system of L_β. Condition (2.69) is met for L_α by (2.69) for L_β and (2.70) with (2.57).

LEMMA 2.36. — *Let* L *be a field of etale cohomological dimension one for l-torsion sheaves. Suppose l is not the characteristic of* L, *and that* L *contains primitive* l^νth *roots of* 1. *Let* L″ *be the separable closure of* L. *Then there is a family of inductors in the sense of* 2.34 *for the Bott element* $\beta \in K/l^\nu_2(L)$ *of Appendix* A.

Proof. — Let L_α run over the direct system of finite Galois extensions of L. Then each L_α has cohomological dimension 1 for l-torsion sheaves. As $L_\alpha \supseteq L$ contains a primitive l^ν-th root of 1, there is an isomorphism

$$H^0(L''/L_\alpha;\, \mathbb{Z}/l^\nu(1)) = H^0_{et}(L_\alpha;\, \mathbb{Z}/l^\nu(1)) = \mu_{l^\nu}(L_\alpha) \cong \mathbb{Z}/l^\nu(1). \tag{2.71}$$

Hilbert's Theorem 90 and the Kummer sequence yield Kummer's isomorphism

$$H^1_{et}(L_\alpha;\, \mathbb{Z}/l^\nu(1)) = L^*_\alpha \otimes \mathbb{Z}/l^\nu \cong K/l^\nu_1(L_\alpha). \tag{2.72}$$

Consider for now only those L_α whose degree over L is divisible by l^ν. Then the image of the restriction of the norm map

$$\mathrm{Norm}: \quad \mu_{l^\nu}(L_\alpha) \to \mu_{l^\nu}(L),$$

is 0, as the norm of any element of $\mu_{l^\nu}(L_\alpha)$ is its $[L_\alpha : L]$ power, and so 1. In Tate's spectral sequence (2.62), the image of $H_0(\mathrm{Gal}(L_\alpha/L);\, H^0(L_\alpha;\, \mathbb{Z}/l^\nu(1))$ in $H^0(L;\, \mathbb{Z}/l^\nu(1))$ is the image of the norm or transfer map from $H^0(L_\alpha;\, \mathbb{Z}/l^\nu(1))$, and so is 0. Thus for our L_α, in (2.62) d_2 is surjective, and there is an isomorphism resulting from (2.62) interpreted *via* (2.72) and (2.71)

$$(2.73)\qquad \begin{array}{c} H^0_{et}(L;\ \mathbb{Z}/l^\nu(1)) \tilde{=} H_1(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu_1(L_\alpha)) \\ \| \\ \mu_{l^\nu}(L) \end{array}$$

Let the generator β or $e^{2\pi i/l^\nu}$ of the left side of (2.73) correspond to an element $\beta(\alpha)$ in the right side. Consider now the spectral sequence (2.36) for $\pi_* \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha))$. The class $\beta(\alpha)$ in $E^2_{1,1} = H_1(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu_1(L_\alpha))$ is a permanent cycle for dimension reasons, and defines an element of $\pi_2 \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha))$ unique up to indeterminacy from the bottom filtration $E^\infty_{0,2}$.

The group $E^\infty_{0,2}$ is a quotient of $E^2_{0,2} = H_0(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu_2(L_\alpha))$. I claim this indeterminacy $E^\infty_{0,2}$ is zero. As L_α has l-torsion cohomological dimension at most 1, one has

$$(2.74)\qquad 0 = H^2_{et}(L_\alpha;\ \mathbb{Z}/l^\nu(1)) = l^\nu\text{-torsion subgroup of } \mathrm{Br}(L_\alpha).$$

By a theorem of Tate, [121], 4.4, 4.5, or by the theorem of Merkurjev and Suslin ([81], [115], Thm. 1), this implies that

$$(2.75)\qquad K_2(L_\alpha) \otimes \mathbb{Z}/l^\nu = 0.$$

Thus the universal coefficient sequence (A.6) yields

$$(2.76)\qquad K/l^\nu_2(L_\alpha) = l^\nu\text{-torsion subgroup of } K_1(L_\alpha) = \mu_{l^\nu}(L_\alpha) = \mathbb{Z}/l^\nu(1).$$

Now using (2.80) below to compare the spectral sequence (2.36) for $\pi_* \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha))$ with the Tate spectral sequence of 2.32, one gets the commutative diagram

$$(2.77)\qquad \begin{array}{ccc} H_2(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu_1(L_\alpha)) & \tilde{=} & H_2(\mathrm{Gal}(L_\alpha/L);\ H^1(L_\alpha;\ \mathbb{Z}/l^\nu(1))) \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_2} \\ H_0(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu_2(L_\alpha)) & \tilde{=} & H_0(\mathrm{Gal}(L_\alpha/L);\ H^0(L_\alpha;\ \mathbb{Z}/l^\nu(1))) \\ & & \| \\ & & H_0(\mathrm{Gal}(L_\alpha/L);\ \mu_{l^\nu}) \end{array}$$

The horizontal isomorphisms are induced by (2.72) and (2.76). The right d_2 was shown to be surjective, and (2.77) shows the left d_2 must be surjective. Thus $E^\infty_{0,2} = E^3_{0,2} = 0$, and the indeterminacy vanishes as claimed.

Thus $\beta(\alpha) \in E^\infty_{1,1}$ determines a unique element of $\pi_2 \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha))$. These elements are the members $\mathrm{Ind}(\beta,\ L_\alpha/L)$ of inductors of the Bott element. To verify this, the conditions (2.69) and (2.70) must be checked.

Under the isomorphism (2.72), the elements $\beta(\alpha)$ are a family of elements in $H_1(\mathrm{Gal}(L_\alpha/L);\ H^1(L_\alpha;\ \mathbb{Z}/l^\nu(1)))$ which modulo vanishing indeterminany $E^\infty_{0,2}=0$ determine elements of $\pi_2\,\mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ \mathbb{H}^\bullet(L_\alpha;\ K(\mathbb{Z}/l^\nu(1),\ 2)))$. Under the weak homotopy equivalence of Tate 2.24, interpreted as a hypertransfer up to homotopy as in 2.27:

$$\text{(2.78)}\qquad T\colon\ \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ \mathbb{H}^\bullet(L_\alpha;\ K(Z/l^\nu(1),\ 2)))\xrightarrow{\sim}\mathbb{H}^\bullet(L;\ K(Z/l^\nu(1),\ 2)),$$

these elements $\beta(\alpha)$ correspond to the generator β of $H^0_{\mathrm{et}}(L;\ \mathbb{Z}/l^\nu(1))=\pi_2\,\mathbb{H}^\bullet(L;\ K(\mathbb{Z}/l^\nu(1),\ 2))$. Indeed, they were defined by this condition in (2.73). Thus this second family of $\beta(\alpha)$ satisfies (2.69) to be a family of inductors of β with respect to the "hypertransfers" (2.78). As these "hypertransfers" are weak homotopy equivalences, (2.69) and compatibility (2.57) imply that the family of $\beta(\alpha)$ also satisfy (2.70).

The first family of elements $\mathrm{Ind}(\beta,\ L_\alpha/L)$ corresponds to the second family of elements $\beta(\alpha)$ under the zigzag of maps induced by applying $\mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ \ )$ to the zigzag (2.79)

$$\text{(2.79)}\qquad \begin{tikzcd} K/l^\nu(L_\alpha) \arrow[r, "\eta"] & \mathbb{H}^\bullet(L_\alpha;\ K/l^\nu(L'')) & \mathbb{H}^\bullet(L_\alpha;\ K/l^\nu(L'')\rangle 1\langle) \arrow[l] \arrow[d] \\ & & \mathbb{H}^\bullet(L_\alpha;\ (K/l^\nu(L'')\rangle 1\langle)\langle 2\rangle) \arrow[d, phantom, "\wr"] \\ & & \mathbb{H}^\bullet(L_\alpha;\ K(\mathbb{Z}/l^\nu(1),\ 2)) \end{tikzcd}$$

Here the first map η is the natural augmentation. The next two maps of (2.79) are canonical maps of the Postnikov stages as in 5.51 and 5.53. By 5.52, $(K/l^\nu(L'')\rangle 1\langle)\langle 2\rangle$ is naturally weak homotopy equivalent to an Eilenberg-MacLane spectrum $K(K/l^\nu_2(L''),\ 2)$, which is $K(\mathbb{Z}/l^\nu(1),\ 2)$ by (2.76). This yields the last equivalence of (2.79). One notes that (2.79) induces the isomorphism (2.72) on π_1, and the composite of isomorphisms (2.76) and (2.71) on π_2. It follows that the families $\mathrm{Ind}(\beta,\ L_\alpha/L)$ and $\beta(\alpha)$ correspond on the induced zigzag of group hypercohomologies. If L_β/L is a finite Galois extension with $L''\supseteq L_\beta\supseteq L_\alpha$, (2.79) induces the horizontal maps of a zigzag (2.80)

$$\begin{tikzcd} \mathbb{H}_\bullet(\mathrm{Gal}(L_\beta/L);\ K/l^\nu(L_\beta)) \arrow[r] \arrow[d, "\mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ T)"] & \circledcirc \arrow[d, "\wr"'] & \circledcirc \arrow[l] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbb{H}_\bullet(\mathrm{Gal}(L_\beta/L);\ \mathbb{H}^\bullet(L_\beta;\ K(\mathbb{Z}/l^\nu(1),\ 2))) \arrow[d, "\wr"', "\mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ T)"] \\ \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha)) \arrow[r] \arrow[d, "T"] & \circledcirc \arrow[d, "\wr"'] & \circledcirc \arrow[l] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbb{H}_\bullet(\mathrm{Gal}(L_\alpha/L);\ \mathbb{H}^\bullet(L_\alpha;\ K(\mathbb{Z}/l^\nu(1),\ 2))) \arrow[d, "\wr"', "T"] \\ K/l^\nu(L) \arrow[r, "\eta"] & \circledcirc & \circledcirc \arrow[l] \arrow[r, "\sim"] & \mathbb{H}^\bullet(L;\ K(\mathbb{Z}/l^\nu(1),\ 2)) \end{tikzcd}$$

The commutativity of the first column of (2.80) results from (2.58). The other columns commute by naturality of the homotopy transfers (2.59) with respect to maps of the coefficient systems F. As the maps in (2.79) induce isomorphisms on π_1 and π_2 by the above calculations, the horizontal maps of (2.80) induce isomorphisms on the $E^2_{2,1}$, $E^2_{1,1}$, $E^2_{0,2}$ terms of the homological induction spectral sequences (2.36). Except for the right-to-left arrow, the maps in (2.79) also induce isomorphisms on π_0. Hence the left-to-right arrows in (2.80) induce isomorphisms on $E^2_{3,0}$. The d_2 differential is the only one affecting the terms in this corner of the spectral sequence. Applying d^2 to these isomorphisms, one sees that the left-to-right arrows in (2.80) induce isomorphisms on $E^3_{1,1}=E^\infty_{1,1}$ and on $E^3_{0,2}=E^\infty_{0,2}$. Hence the left-to-right arrows induce isomorphisms on the bottom two layers of the induction spectral sequence filtration of π_2 where the family $\beta(\alpha)$ lives. Transporting the $\beta(\alpha)$ from the right to the left of (2.80) along the inverses of the left-to-right isomorphisms and along the right-to-left arrow in the middle column yields corresponding families in all columns of (2.80). The $\operatorname{Ind}(\beta, L_\alpha/L)$ form the corresponding family in the left column. After (2.78), it was proved that the $\beta(\alpha)$'s satisfied the equations (2.69) and (2.70) in the right column. Hence their homomorphic images, the $\operatorname{Ind}(\beta, L_\alpha/L)$ in the left column, also satisfy equations (2.69) and (2.70). Thus the $\operatorname{Ind}(\beta, L_\alpha/L)$ form the required family of inductors of β.

Strictly speaking, (2.80) must be replaced by a homotopy equivalent diagram that reforms the abuses of 2.27 and 2.28. This is left as an easy exercice for the conscientious reader.

The above argument proves that there is a family of inductors $\operatorname{Ind}(\beta, L_\alpha/L)$ for those L_α whose degree over L, $[L_\alpha : L]$, is divisible by l^ν. By 2.35, this family extends uniquely to a family of inductors of β for all L_α. To apply 2.35, one must verify that each finite Galois extension of L is contained in an L_α with $[L_\alpha : L]$ divisible by l^ν. It suffices to show that any finite separable field extension of L is contained in a finite separable extension L' of L with $\operatorname{Hom}(\operatorname{Gal}(L''/L'); \mathbb{Z}/l)=H^1(L'; \mathbb{Z}/l)\neq 0$. For then Galois theory yields a degree l extension of L'. Its splitting field is a finite Galois extension of L of degree divisible by l, and which contains L'. Iterating this procedure ν times yields an L_α with the required properties. Thus it suffices to show every finite separable extension L_1 of L is contained in a finite separable extension L'/L with $H^1(L'; \mathbb{Z}/l)\neq 0$. But if this failed, passage to the limit would show $H^1(L'; \mathbb{Z}/l)=0$ for any separable extension L' of L_1, in particular for the fixed field of an l-Sylow subgroup of $\operatorname{Gal}(L''/L_1)$. Then [102], I, § 4, Prop. 21 and I, § 3, Prop. 14, or [104], III, Prop. 14, 19, and Thm. 12 would show that L_1 and even L have l-torsion cohomological dimension 0, contradicting the hypothesis that L has l-cohomological dimension 1. Thus there is an L' with $H^1(L'; \mathbb{Z}/l)\neq 0$, and so the required L_α exists.

2.37. The key points in the above argument are the Kummer isomorphism (2.72) which comes from Hilbert's Theorem 90, and Tate's spectral sequence. Tate shows β is inducted by $H^1_{et}(L_\alpha; \mathbb{Z}/l^\nu(1))$, and Kummer identifies this etale cohomology group to $K/l^\nu_1(L_\alpha)$. One realizes Tate's spectral sequence as a weak equivalence of hyperhomology and hypercohomology constructions, which can be compared to a hypertransfer on all of K-theory. Using the Kummer isomorphism and the low cohomological dimension of L_α, one sees that the lower K-groups are the same as the etale cohomology groups in

Tate's spectral sequence, and the higher K-groups can't interfere to prevent the induction of β in K-theory.

To handle induction of β over L a number field is necessary to prove the descent theorem in arithmetically interesting situations. However, the higher cohomological dimension mangles the identification of lower K-groups with etale cohomology. For example, the l^ν-torsion subgroup of $\mathrm{Br}(L)$, $H^2_{et}(L; \mathbb{Z}/l^\nu(1))$, should be in $K/l^\nu_0(L)$, but instead arrives late as $H^2_{et}(L; \mathbb{Z}/l^\nu(2))$ in $K/l^\nu_2(L)$. The spectral sequences also get wilder because of the increased cohomological dimension, and the mysterious higher K-groups get new opportunites to interfere. A technical Lemma 2.38 and many auxillary spectra with various bad homotopy groups zeroed out are needed to handle this case in 2.39. A reader interested only in schemes over algebraically closed fields might prefer to skip ahead to 2.40.

LEMMA 2.38. — *Let* G *be a group acting on a fibrant spectrum* F. *Then the natural map of Postnikov stages of* 5.51 *and* 5.53 *induces a weak homotopy equivalence for any n*

$$(2.81)\qquad \mathbb{H}_\bullet(G; F\rangle n\langle)\rangle n\langle \xrightarrow{\sim} \mathbb{H}_\bullet(G; F\rangle n\langle).$$

Consequently, if F *is a presheaf and admits a hypertransfer as in* 2.25 *or hypertransfer up-to-homotopy as in* 2.28, *then* $F\rangle n\langle$ *also admits a hypertransfer up-to-homotopy as in* (2.82), *which is compatible with the original in that* (2.82) *commutes.*

$$(2.82)\qquad \begin{tikzcd}
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L); F(A \underset{L}{\otimes} L')\rangle n\langle) \arrow[ddd, bend right] \arrow[ddr, dashed] & \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L); F(A \underset{L}{\otimes} L')\rangle n\langle)\rangle n\langle \arrow[u, "\wr"'] \arrow[d] \arrow[dr] & \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L); F(A \underset{L}{\otimes} L'))\rangle n\langle \arrow[r, "T\rangle n\langle"] \arrow[d] & F(A)\rangle n\langle \arrow[d] \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L); F(A \underset{L}{\otimes} L')) \arrow[r, "T"] & F(A)
\end{tikzcd}$$

Here the vertical arrows are natural Postnikov maps and the dotted hypertransfer is only defined up to homotopy.

Similarly, for any fibrant spectrum on which G *acts, the natural map is a weak homotopy equivalence* (2.83)

$$(2.83)\qquad \mathbb{H}_\bullet(G; F)\langle n\rangle \xrightarrow{\sim} \mathbb{H}_\bullet(G; F\langle n\rangle)\langle n\rangle.$$

If F *is a presheaf with hypertransfer up-to-homotopy, there is an induced hypertransfer up-to-homotopy* (2.84)

(2.84)
$$\begin{tikzcd}
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L);\ F(A \underset{L}{\otimes} L')) \arrow[r, "T"] \arrow[d] \arrow[ddd, bend right] & F(A) \arrow[d] \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L);\ F(A \underset{L}{\otimes} L'))\langle n\rangle \arrow[r] \arrow[d, "\wr"'] & F(A)\langle n\rangle \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L);\ F(A \underset{L}{\otimes} L')\langle n\rangle)\langle n\rangle \arrow[ur, dashed] & \\
\mathbb{H}_\bullet(\mathrm{Gal}(L'/L);\ F(A \underset{L}{\otimes} L')\langle n\rangle) \arrow[u] \arrow[uur, dashed] &
\end{tikzcd}$$

Proof. — As $\pi_q F\rangle n\langle = 0$ for $q \leqq n$, the spectral sequence (2.36) reveals that $\pi_q \mathbb{H}_\bullet(G;\ F\rangle n\langle) = 0$ for $q \leqq n$. Thus $\pi_* \mathbb{H}_\bullet(G;\ F\rangle n\langle)\langle n\rangle = 0$. But by 5.53, this $\mathbb{H}_\bullet(G;\ F\rangle n\langle)\langle n\rangle$ is the stable cofibre of (2.81), so (2.81) must be a weak homotopy equivalence. Then (2.82) follows by naturality.

By 5.53 and the fact 5.19 that $\mathbb{H}_\bullet(G;\ )$ preserves homotopy fibre sequences, one sees that $\mathbb{H}_\bullet(G;\ F\rangle n\langle)$ is the homotopy fibre of (2.85)

(2.85)
$$\mathbb{H}_\bullet(G;\ F) \to \mathbb{H}_\bullet(G;\ F\langle n\rangle).$$

Thus the above vanishing result shows (2.85) induces an isomorphism on π_q for $q \leqq n$, and then that (2.83) induces an isomorphism on all π_q.

LEMMA 2.39. — *Let* L *be a field of characteristic not l, and which contains primitive l^ν-th roots of* 1. *Suppose that $\nu \geqq 2$ if $l = 2$. Let* L″ *be the cyclotomic extension of* L *obtained by adjoining all l-power roots of* 1: $L'' = L(\mu_{l^\infty})$. *Suppose that* L″ *has etale cohomological dimension at most* 1 *for l-torsion sheaves. Then the extension* L″/L *has a family of inductors of the Bott element* $\beta \in K/l^\nu_2(L)$.

Proof. — The cases not trivial or covered by 2.36 are those where L″ and L have l-torsion etale cohomological dimension exactly 1 and 2 respectively, and where $\mathrm{Gal}(L''/L) = \mathbb{Z}_l^\wedge$. Let L_n/L be the Galois subextension of degree l^n. Let L‴ be the separable closure of L.

Let $\beta: \Sigma^\infty S^2 \to K/l^\nu(L)$ be the map from the sphere spectrum shifted up two degrees that represents the Bott element. Smashing this with the mod l^ν Moore spectrum Σ^∞/l^ν, using the pairing $\Sigma^\infty/l^\nu \wedge K/l^\nu(L) \to K/l^\nu(L)$, and composing with $K/l^\nu(L) \to K/l^\nu(L')$, one produces a natural map for any L′ over L:

(2.86)
$$\beta:\quad \Sigma^\infty/l^\nu S^2 \to K/l^\nu(L) \to K/l^\nu(L').$$

One has as low degree stable homotopy groups the values

(2.87)
$$\pi_i \Sigma^\infty/l^\nu S^2 = 0, \qquad i < 2,$$
$$\pi_2 \Sigma^\infty/l^\nu S^2 = \mathbb{Z}/l^\nu.$$

The choice of a primitive l^ν-th root of 1 in defining β fixes an isomorphism $\beta\colon \mathbb{Z}/l^\nu \xrightarrow{\simeq} \mathbb{Z}/l^\nu(1)$.

Consider the diagram (2.88):

$$
\begin{tikzcd}
& & \mathbb{H}^\cdot(L''/L_n;\, \Sigma/l^\nu S^2) \\
\Sigma^\infty/l^\nu S^2 \arrow[d, "\beta"'] & \Sigma^\infty/l^\nu S^2 \rangle 0 \langle \arrow[l, "\simeq"'] \arrow[r, "\eta"] \arrow[d] & \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2) \rangle 0 \langle \arrow[u, "\wr"] \arrow[d] \\
K/l^\nu(L_n) & K/l^\nu(L_n) \rangle 0 \langle \arrow[l] \arrow[r, "\eta"] & \mathbb{H}^\cdot(L''/L_n;\, K/l^\nu(L'')) \rangle 0 \langle \arrow[d] \\
& & \mathbb{H}^\cdot(L''/L_n;\, K/l^\nu(L'') \langle 2 \rangle) \rangle 0 \langle \arrow[dr] & \\
& & & \mathbb{H}^\cdot(L'''/L_n;\, K/l^\nu(L''') \langle 2 \rangle) \rangle 0 \langle \arrow[d, phantom, "\wr\|"] \\
& & & \mathbb{H}^\cdot_{et}(L_n;\, K/l^\nu \langle 2 \rangle) \rangle 0 \langle
\end{tikzcd}
\tag{2.88}
$$

As L''/L_n has profinite Galois group of cohomological dimension 1, (2.87) and the spectral sequences of 1.50 yield

$$
\begin{aligned}
&\pi_i\, \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2) = 0, \qquad i<1,\\
&\pi_1\, \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2) = H^1(L''/L_n;\, \mathbb{Z}/l^\nu).
\end{aligned}
\tag{2.89}
$$

Thus the natural map $\mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2) \rangle 0 \langle \to \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2)$ is a weak homotopy equivalence as indicated in (2.88).

By Tate's Theorem interpreted as in 2.28, there is a weak homotopy equivalence

$$
T\colon \mathbb{H}_\cdot(\mathrm{Gal}(L_n/L);\, \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2)) \xrightarrow{\sim} \mathbb{H}^\cdot(L''/L;\, \Sigma^\infty/l^\nu S^2).
\tag{2.90}
$$

The canonical element β in $\pi_2\, \mathbb{H}^\cdot(L''/L;\, \Sigma^\infty/l^\nu S^2)$ thus has a family of inductors in the extension L''/L. This family yields a family of inductors of β in all the functors on the right hand side of (2.88) by naturality of the homotopy hypertransfer of 2.28 and Lemma 2.38 which allows chopping down to $\rangle 0 \langle$. Considering the hyperhomology spectral sequence (2.36) in light of (2.89), one sees that the inductors of β must lie in the bottom two layers $E^\infty_{0,2}$ and $E^\infty_{1,1}$ of filtration of $\pi_2\, \mathbb{H}_\cdot(\mathrm{Gal}(L_n/L);\, \mathbb{H}^\cdot(L''/L_n;\, \Sigma^\infty/l^\nu S^2))$. By naturality of the spectral sequence (2.36), the same is true for every functor on the right side of (2.88).

As L''' is separably closed, $L'''^{*}\otimes\mathbb{Z}/l^{\nu}=0$, so $K/l_1^{\nu}(L''')=0$. As in (2.76), $K/l_2^{\nu}(L''')=\mathbb{Z}/l^{\nu}(1)$. These values and the spectral sequences of 1.50 yield

$$(2.91)\quad \begin{cases} \pi_i\,\mathbb{H}^{\cdot}(L'''/L_n;\ K/l^{\nu}(L''')\langle 2\rangle)\rangle 0\langle =0; & i\neq 1,\ 2,\\ \pi_1\,\mathbb{H}^{\cdot}(L'''/L_n;\ K/l^{\nu}(L''')\langle 2\rangle)\rangle 0\langle \cong H^1_{et}(L_n;\ \mathbb{Z}/l^{\nu}(1)),\\ \pi_2\,\mathbb{H}^{\cdot}(L'''/L_n;\ K/l^{\nu}(L''')\langle 2\rangle)\rangle 0\langle \cong H^0_{et}(L_n;\ \mathbb{Z}/l^{\nu}(1))\cong\mathbb{Z}/l^{\nu}(1). \end{cases}$$

The map (2.92) induced by (2.88) and (2.91)

$$(2.92)\quad \begin{array}{c} K/l_1^{\nu}(L_n)\cong\pi_1\,K/l^{\nu}(L_n)\rangle 0\langle \to\pi_1\,\mathbb{H}^{\cdot}(L'''/L_n;\ K/l^{\nu}(L''')\langle 2\rangle)\rangle 0\langle \\ \hfill \wr\Vert \qquad\qquad \\ \hfill H^1_{et}(L_n;\ \mathbb{Z}/l^{\nu}(1)) \quad \end{array}$$

is the isomorphism of Hilbert's Theorem 90. Thus comparing the spectral sequences (2.36) for

$$\mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ K/l^{\nu}(L_n)\rangle 0\langle) \quad\text{and}\quad \mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ \mathbb{H}^{\cdot}_{et}(L_n;\ K/l^{\nu}\langle 2\rangle)\rangle 0\langle),$$

one sees that the $E^2_{1,1}$ terms are the same. As for $s\geqq 0$, $E^2_{3+s,\,-s}=0$ in both spectral sequences because $\rangle 0\langle$ cuts off π_n for $n\leqq 0$, no non-zero differentials can enter or leave $E^2_{1,1}$.

Thus the $E^2_{1,1}=E^{\infty}_{1,1}$ terms are isomorphic for the two spectral sequences. For $n\geqq\nu$ an argument similar to that establishing (2.73) shows $E^{\infty}_{0,2}=0$ in the spectral sequence for $\mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ \mathbb{H}^{\cdot}_{et}(L_n;\ K/l^{\nu}\langle 2\rangle)\rangle 0\langle)$. Thus the map from

$$\pi_2\,\mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ K/l^{\nu}(L_n)\rangle 0\langle) \text{ to } \pi_2\,\mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ \mathbb{H}^{\cdot}_{et}(L_n;\ K/l^{\nu}\langle 2\rangle)\rangle 0\langle)$$

is surjective on the bottom two layers of the filtration where the family of inductors of β lives. Thus these elements may be lifted to $\pi_2\,\mathbb{H}_{\cdot}(\mathrm{Gal}(L_n/L);\ K/l^{\nu}(L_n)\rangle 0\langle)$ with the lifting uniquely defined module indeterminacy $E^{\infty}_{0,2}$. For the cofinal system of L_n with $n\geqq\nu$, the indeterminacy is in fact all of $E^{\infty}_{0,2}$. By 2.35, one need only consider these cofinal values of n.

It remains to be shown that one can choose the liftings so that (2.69) and (2.70) hold, so that the lifted elements form a family of inductors of β. As they are lifted from a family of inductors of β, the elements satisfy (2.70) modulo indeterminacy $E^{\infty}_{0,2}$ and satisfy (2.69) modulo the kernal of

$$(2.93)\quad \pi_2\,K/l^{\nu}(L)\rangle 0\langle \to\pi_2\,\mathbb{H}^{\cdot}(L'''/L;\ K/l^{\nu}(L''')\langle 2\rangle)\rangle 0\langle.$$

The calculation (2.91) identifies (2.93) to the right map of the short exact universal coefficient sequence (2.94) for $L=L_0=L_n$

$$(2.94)\quad 0\to K_2(L_n)\otimes\mathbb{Z}/l^{\nu}\to K/l_2^{\nu}(L_n)\to\mathbb{Z}/l^{\nu}(1)\to 0.$$

Thus (2.69) holds modulo $K_2(L)\otimes\mathbb{Z}/l^{\nu}=K_2(L_0)\otimes\mathbb{Z}/l^{\nu}$.

For $n\geqq\nu$, an argument similar to that establishing (2.73) and (2.77) shows that the $H_0(\mathrm{Gal}(L_n/L);\ \mathbb{Z}/l^{\nu}(1))$ quotient of the $E^2_{0,2}$ term $H_0(\mathrm{Gal}(L_n/L);\ K/l_2^{\nu}(L_n))$ is hit by the d_2 differential modulo the image of $H_0(\mathrm{Gal}(L_n/L);\ K_2(L_n)\otimes\mathbb{Z}/l^{\nu})$. Thus the indeterminacy $E^{\infty}_{0,2}$ is a quotient of $K_2(L_n)\otimes\mathbb{Z}/l^{\nu}$ *via* the surjection

$$K_2(L_n)\otimes\mathbb{Z}/l^{\nu}\to H_0(\mathrm{Gal}(L_n/L);\ K_2(L_n)\otimes\mathbb{Z}/l^{\nu}).$$

Thus it will follow that the liftings can be chosen so that (2.69) and (2.70) hold provided the hypertransfers induce surjections on indeterminacy, which in turn follows if for all $m \geqq n \geqq 0$, the transfer map (2.95) is a surjection

$$(2.95) \qquad T: \ K_2(L_m) \otimes \mathbb{Z}/l^\nu \twoheadrightarrow K_2(L_n) \otimes \mathbb{Z}/l^\nu.$$

The Theorem of Merkurjev and Suslin ([81], [115]) identifies (2.95) to the transfer on etale cohomology (2.96)

$$(2.96) \qquad T: \ H^2_{et}(L_m; \mathbb{Z}/l^\nu(2)) \to H^2_{et}(L_n; \mathbb{Z}/l^\nu(2)),$$

(For the most important cases where L is finite over $\mathbb{Q}$ or $\mathbb{Q}_l^\wedge$, one may also consult the paper of Tate [121].) As L_n and L_m have l-torsion etale cohomological dimension 2, this map is the transfer in the top dimension, and so is surjective by [102], I, § 3.3, Lemma 4. Thus the hypertransfer induces a surjection on indeterminacies, and by induction on n one may choose liftings of the family of inductors to $\pi_2 \mathbb{H}_\bullet(\mathrm{Gal}(L_n/L); K/l^\nu(L_n) \rangle 0 \langle)$ so that (2.69) and (2.70) are satisfied. The liftings form a family of inductors of β. The image of this family in $\pi_2 \mathbb{H}_\bullet(\mathrm{Gal}(L_n/L); K/l^\nu(L_n))$ is the family of inductors of β claimed by the statement of this Lemma. This completes the proof.

PROPOSITION 2.40. — *Let* L *be a field of characteristic not* l. *Let* L''/L *be an infinite Galois extension. Suppose* L *contains a subfield* M *which contains primitive* l^ν*-th roots of* 1. *Suppose* $L'' = LM''$ *for a Galois extension* M''/M, *and that* $M'' \cap L = M$, *so that* $\mathrm{Gal}(L''/L)$ *is isomorphic to* $\mathrm{Gal}(M''/M)$ *via restriction. Suppose that either* M *has* l*-torsion etale cohomological dimension* 1 *and* M'' *is the separable closure of* M, *or else that* $M'' = M(\mu_{l^\infty})$ *is the cyclotomic extension of* M *and that* M'' *has* l*-torsion etale cohomological dimension at most one. If* $l = 2$, *suppose* $\nu \geqq 2$.

Let A *be an algebra over* L, *and* $x \in K/l^\nu_n(A)$. *Then there is a family of inductors for* $\beta \cup x \in K/l^\nu_{n+2}(A)$ *in the extension* L''/L.

Proof. — By 2.37 or 2.39, $\beta \in K/l^\nu_2(M)$ has a family of inductors in the extension M''/M. As $\mathrm{Gal}(M''/M) = \mathrm{Gal}(L''/L)$, Galois theory shows that fields L_α between L'' and L are induced by fields M_α between M'' and M. Then by naturality of the hypertransfer, the inductors $\mathrm{Ind}(\beta, M_\alpha/M)$ map to a family of inductors $\mathrm{Ind}(\beta, L_\alpha/L)$ for β in the extension L''/L.

The projection formula 2.31 shows that $x \cup \mathrm{Ind}(\beta, L_\alpha/L)$ is a family of inductors for x in L''/L.

2.41. We shall see later that for the most interesting fields L, the extension of L to its separable closure may be filtered by subextensions with each step of the type 2.40.

Thus 2.40 yields surjectivity of the hypertransfer after inverting β for enough finite field extensions to allow one to deduce surjectivity of the hypertransfer after inverting β for a fairly general finite Galois extension. One should be able to dualize in some sense to get injectivity after inverting β for the augmentation from $K/l^\nu(L)$ into hypercohomology $\mathbb{H}^\bullet(\mathrm{Gal}(L'/L); K/l^\nu(L'))$. In fact, this works much better than one first suspects.

Proposition 2.42. — *Let* L *be a field of characteristic not* l, *and which contains primitive* l^ν*-th roots of* 1. *Let* $\nu \geqq 2$ *if* $l=2$. *Let* L''/L *be an infinite Galois extension induced by* M''/M *as in* 2.40. *Then for any* L*-algebra* A *there is a homotopy commutative diagram of fibrant spectra*

$$(2.97)\qquad \begin{tikzcd} \Omega^2 K/l^\nu(A) \arrow[r, "\eta"] & \Omega^2 \mathbb{H}^{\bullet}(L''/L; K/l^\nu(A \underset{L}{\otimes} L'')) \\ K/l^\nu(A) \arrow[u, "\cup\beta"] \arrow[r, "\eta"'] & \mathbb{H}^{\bullet}(L''/L; K/l^\nu(A \underset{L}{\otimes} L'')) \arrow[u, "\cup\beta"'] \arrow[ul, "\varphi(L''/L;\, A)"'] \end{tikzcd}$$

If $l=2$ *or* 3, β *may be replaced by any of the* x*'s of* A.11 *provided* Ω^2 *is replaced by an appropriate* Ω^n, $x \in K/l^\nu_n(L)$.

The diagram (2.97) *may even be considered as a homotopy commutative diagram in the homotopy category of strict functors from the category of* L*-algebras to that of fibrant spectra. That is, the maps in the diagram* (2.97) *and the homotopies that make it homotopy commute may be chosen to be strictly natural in the algebra* A.

Proof. — First one does the case where K/l^ν has a natural associative ring structure, so assume that either $l>3$, or that l^ν is divisible by 9, or that l^ν is divisible by 16. (Consult Appendix A.)

Let L_α run over the directed system of finite Galois extensions of L contained in L''. Then by 1.17 and 1.18 with $F\{n\}=F\langle n\rangle$, interpreted as in 1.50, there is a weak equivalence (2.98)

$$(2.98)\qquad \mathbb{H}^{\bullet}(L''/L; K/l^\nu(A \underset{L}{\otimes} L'')) \simeq \underset{\underset{n}{\longleftarrow}}{\operatorname{holim}}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathbb{H}^{\bullet}(L_\alpha/L; K/l^\nu(A \underset{L}{\otimes} L'')\langle n\rangle).$$

There is a dualizing map (2.99) into the mapping spectrum of 5.34

$$(2.99)\qquad K/l^\nu(A \underset{L}{\otimes} L_\alpha) \to \operatorname{Map}_*(K/l^\nu(L_\alpha), K/l^\nu(A)) \qquad \text{(Gillet duality).}$$

Under the adjunction (5.25), the map (2.99) corresponds to the map (2.100).

$$(2.100)\qquad K/l^\nu(A \underset{L}{\otimes} L_\alpha) \wedge K/l^\nu(L_\alpha) \overset{\cup}{\to} K/l^\nu(A \underset{L}{\otimes} L_\alpha) \overset{\lambda_*}{\to} K/l^\nu(A).$$

Here λ_* is the transfer map. If A is a commutative ring, $K/l^\nu(A)$ is a homotopy associative ring spectrum, and (2.99) and (2.100) are $K/l^\nu(A)$ module maps by associativity and the projection formula of A.3 for λ_*. For general A, (2.100) and (2.99) respect the pairing with $K/l^\nu(B)$ into $K/l^\nu(A \otimes B)$ in the obvious sense. The maps (2.100) and (2.99) are strictly natural in A. They fail to be $\operatorname{Gal}(L_\alpha/L)$-equivariant only because the transfer map λ_* is not strictly equivariant. As the construction of the hypertransfer T in 2.30 shows that the transfer $\lambda_* = f(\mathbf{0})$ is equivariant up to a coherent system of higher homotopies, the maps (2.99) and (2.100) are $\operatorname{Gal}(L_\alpha/L)$-equivariant up to a coherent system of higher homotopies. The reader who wishes an explanation of this concept may consult [132]. The usual rectification techniques allow one to replace (2.100) naturally by a weak equivalent diagram of strictly equivariant maps.

As $K/l^\nu(L_\alpha)$ is a connective spectrum, (2.100) and (2.99) induce maps of Postnikov stages

$$K/l^\nu(A \underset{L}{\otimes} L_\alpha)\langle n\rangle \wedge K/l^\nu(L_\alpha) \to K/l^\nu(A)\langle n\rangle, \tag{2.101}$$

$$K/l^\nu(A \underset{L}{\otimes} L_\alpha)\langle n\rangle \to \mathrm{Map}_*(K/l^\nu(L_\alpha),\ K/l^\nu(A)\langle n\rangle). \tag{2.102}$$

Justification is provided by an argument like 2.38, with the Atiyah-Hirzebruch spectral sequence for spectra of [1], III, § 7, replacing the spectral sequence (2.36).

The map (2.102) is Gal (L_α/L)-equivariant up to coherent homotopy, and so induces a map on homotopy limits

$$\underset{\underset{n}{\leftarrow}}{\lim}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathbb{H}^\cdot(L_\alpha/L;\ K/l^\nu(A \underset{L}{\otimes} L_\alpha)\langle n\rangle) \tag{2.103}$$
$$\to \underset{\underset{n}{\leftarrow}}{\lim}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathbb{H}^\cdot(L_\alpha/L;\ \mathrm{Map}(K/l^\nu(L_\alpha),\ K/l^\nu(A)\langle n\rangle)).$$

There is a "universal coefficient" weak homotopy equivalence given by 5.35

$$\underset{\underset{n}{\leftarrow}}{\lim}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathbb{H}^\cdot(L_\alpha/L;\ \mathrm{Map}_*(K/l^\nu(L_\alpha),\ K/l^\nu(A)\langle n\rangle) \tag{2.104}$$
$$\wr|$$
$$\underset{\underset{n}{\leftarrow}}{\lim}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathrm{Map}_*(\mathbb{H}_\cdot(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha)),\ K/l^\nu(A)\langle n\rangle).$$

The inductors of β give a homotopy compatible family of maps

$$\mathrm{Ind}(\beta,\ L_\alpha/L): \Sigma^\infty S^2 \to \mathbb{H}_\cdot(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha)). \tag{2.105}$$

These maps are strictly natural in A, since they have nothing to do with A.

Evaluation on $\mathrm{Ind}(\beta,\ L_\alpha/L)$ induces a natural map

$$\underset{\underset{n}{\leftarrow}}{\lim}\ \underset{\underset{\alpha}{\longrightarrow}}{\lim}\ \mathrm{Map}_*(\mathbb{H}_\cdot(\mathrm{Gal}(L_\alpha/L);\ K/l^\nu(L_\alpha)),\ K/l^\nu(A)\langle n\rangle)$$
$$\downarrow \tag{2.106}$$
$$\underset{\underset{n}{\leftarrow}}{\lim}\ \mathrm{Map}_*(\Sigma^\infty S^2;\ K/l^\nu(A)\langle n\rangle) \simeq \underset{\underset{n}{\leftarrow}}{\lim}\ \Omega^2 K/l^\nu(A)\langle n\rangle \simeq \Omega^2 K/l^\nu(A)$$

The composition of (2.98), (2.103), (2.104), and (2.106) is the required map $\varphi(L''/L;\ A)$ of (2.97). The map $\varphi(L''/L;\ A)$ is strictly natural in A. If A is commutative, it is a map of $K/l^\nu(A)$ modules up to homotopy. If A is not commutative, the diagram (2.107) homotopy commutes.

$$(2.107)\qquad \begin{array}{ccc} \mathbb{H}^{\bullet}(L''/L;\, K/l^\nu(A \underset{L}{\otimes} L'')) \wedge K/l^\nu(B) & \longrightarrow & \mathbb{H}^{\bullet}(L''/L;\, K/l^\nu(B \underset{L}{\otimes} A \underset{L}{\otimes} L'')) \\ \downarrow{\scriptstyle \phi(A)\wedge 1} & & \downarrow{\scriptstyle \phi(B\bullet A)} \\ \Omega^2 K/l^\nu(A) \wedge K/l^\nu(B) & \longrightarrow & \Omega^2 K/l^\nu(B \underset{L}{\otimes} A) \end{array}$$

The next step is to check that the left triangle of (2.97) commutes. As $\varphi.\eta$ is natural in A, and is a map of $K/l^\nu(A)$-modules or at least respects the pairing as in (2.107), the map $\varphi.\eta$ sends $a=a\cup 1$ to $\varphi.\eta(a)=a\cup\varphi.\eta(1)$. Thus $\varphi.\eta$ is cup product with $\varphi.\eta(1)$. I claim that $\varphi.\eta(1)$ is the Bott element β. By naturality, it suffices to do this for $A=L$. By (2.100) the element 1 in $K/l_0^\nu(L)$ goes to the transfer map λ_* in $\pi_0\,\mathrm{Map}_*(K/l^\nu(L_\alpha),\, K/l^\nu(L))$ under the composition of the map $\lambda^*: K/l^\nu(L)\to K/l^\nu(L_\alpha)$ followed by the duality map $K/l^\nu(L_\alpha)\to \mathrm{Map}_*(K/l^\nu(L_\alpha),\, K/l^\nu(L))$ of (2.99). I generalize this, taking Galois equivariance up to coherent homotopy into account. The map λ^* and its homotopies induce the augmentation $\eta: K/l^\nu(L)\to \mathbb{H}^{\bullet}(L_\alpha/L;\, K/l^\nu(L_\alpha))$. Composing this η with $\mathbb{H}^{\bullet}(L_\alpha/L;\ )$ of the duality map (2.99) and following this with the universal coefficient equivalence (2.104) yields a map

$$K/l^\nu(L)\to \mathrm{Map}_*(\mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L);\, K/l^\nu(L_\alpha));\, K/l^\nu(L)).$$

Just as 1 goes to the transfer λ_* above, here 1 goes to the map $\mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L);K/l^\nu(L_\alpha))\to K/l^\nu(L)$ induced by λ_* and its coherent system of equivariance homotopies, that is, to the hypertransfer T. (This may be verified in detail by chasing the universal mapping properties of homotopy limits and colimits like $\mathbb{H}^{\bullet}(\mathrm{Gal}(L_\alpha/L);\ )$ and $\mathbb{H}_{\bullet}(\mathrm{Gal}(L_\alpha/L);\ )$ as given in [16], XI, 3.4 and 5.15, or by using the techniques of [125] and [128] to build categorical models and analogs for the above constructions and applying the calculus of 2-categories to produce the system of homotopies.) As $\eta(1)$ goes to the hypertransfer T, (2.98)-(2.106) shows that $\varphi\eta(1)$ is the result of evaluating the hypertransfer T on the inductors of β (2.105). By definition, this result is $\beta=\varphi\eta(1)$, as claimed. Thus $\varphi\eta$ is cup product with β as required.

To check that the right triangle of (2.97) commutes, let $A_\alpha^{\bullet}$ be the cosimplicial algebra

$$(2.108)\qquad A_\alpha^{\bullet}=A\underset{L}{\otimes}L_\alpha \rightrightarrows A\underset{L}{\otimes}L_\alpha\underset{L}{\otimes}L_\alpha \overset{\rightarrow}{\rightrightarrows} A\underset{L}{\otimes}L_\alpha\underset{L}{\otimes}L_\alpha\underset{L}{\otimes}L_\alpha \overset{\rightrightarrows}{\rightrightarrows}\cdots$$

For $A=L$, Spec of (2.108) is the simplicial Čech complex of the etale cover $\mathscr{U}_\alpha=\{\mathrm{Spec}(L_\alpha)\to\mathrm{Spec}(L)\}$. Thus by 1.50, there is a weak homotopy equivalence

$$(2.109)\qquad \underset{\Delta}{\mathrm{holim}}\ \underset{\alpha}{\varinjlim}\ F(A_\alpha^{\bullet})\simeq \underset{\Delta}{\mathrm{holim}}\ \underset{\alpha}{\varinjlim}\ F_{\mathscr{U}_\alpha}\simeq \mathbb{H}^{\bullet}(L''/L;\, F(A\underset{L}{\otimes}\ )).$$

By strict naturality of φ with respect to A, there is a cosimplicial $\varphi(A^{\bullet}_{\alpha})$. Naturality and the homotopy commutativity of the left triangle of (2.97) yield a homotopy commutative diagram (2.110) induced by φ and augmentations

(2.110)
$$\begin{tikzcd}
\Omega^2 K/l^{\nu}(A) \arrow[rr] & & \operatorname*{holim}_{\Delta} \varinjlim_{\alpha} \Omega^2 K/l^{\nu}(A^{\bullet}_{\alpha}) & \\
& \mathbb{H}^{\bullet}(L''/L;\, K/l^{\nu}(A \otimes_L L'')) \arrow[ul, "\varphi"] \arrow[rr, "\sim"] & & \operatorname*{holim}_{\Delta} \varinjlim_{\alpha} \mathbb{H}^{\bullet}(L''/L;\, K/l^{\nu}(A^{\bullet}_{\alpha})) \arrow[ul, "\operatorname*{holim}_{\Delta}\varphi"'] \\
K/l^{\nu}(A) \arrow[uu, "\cup\beta"] \arrow[ur] \arrow[rr] & & \operatorname*{holim}_{\Delta} \varinjlim_{\alpha} K/l^{\nu}(A^{\bullet}_{\alpha}) \arrow[uu, "\cup\beta"'] \arrow[ur, "\sim"] & \\
& & \mathbb{H}^{\bullet}(L''/L;\, K/l^{\nu}(A \otimes_L L'') \arrow[u, phantom, "\wr"] &
\end{tikzcd}$$

The two indicated maps are weak homotopy equivalences as hypercohomology has cohomological descent, i. e., by 1.46 with **C** the classifying topos of the profinite Galois group of L''/L and $\mathscr{A}$ the cofinal system of covers of L by L_{α}. The details are similar to those of the proof of 1.54. Note $\mathrm{Gal}(L''/L)$ has l-torsion cohomological dimension 1 by hypothesis.

When one identifies the weak equivalent spectra in (2.110), it collapses to the required diagram (2.97).

It is clear that β may be replaced in the above argument by any element x in $K/l^{\nu}_{*}(L)$ which has a family of inductors in L''/L. By 2.40, this is true for any x which is divisible by β, and in particular for those x of A.8, which are powers of β.

Finally, if $l=2$ or 3, and l^{ν} is only divisible by 4 or 3 respectively, the non-associativity of the ring spectrum $K/l^{\nu}(L)$ may prevent $\varphi\eta$ from being a map of $K/l^{\nu}(L)$ modules. This causes a problem in proving that $\varphi\eta$ is cup product with β. However, the maps composed to get φ are maps of $K(L)$ modules and respect the pairing with $K(A)$. This is enough to construct a homotopy commutative diagram (2.111)

(2.111)
$$\begin{tikzcd}
\Omega^2 K/l^{\nu}(A) & \\
K(A) \arrow[u, "\cup\beta"] \arrow[r, "\eta"] & \mathbb{H}^{\bullet}(L''/L;\, K/(A \otimes_L L'') \arrow[ul, "\varphi(L''/L;\, A)"']
\end{tikzcd}$$

It is easy to verify that $\varphi\eta$ is cup product with $\varphi\eta(1)=\beta$. Then (2.111) may be smashed with the mod l^{ν}-Moore spectrum Σ^{∞}/l^{ν}, and composed with the pairing $\Sigma^{\infty}/l^{\nu} \wedge \Omega^2 K/l^{\nu}(A) \to \Omega^2 K/l^{\nu}(A)$. This yields the left triangle of (2.97). The proof is then completed as above.

THEOREM 2.43. — *Let* L *be a field of characteristic not* l. *Let* L'' *be the separable closure of* L. *Suppose there is a sequence of subfields* (*a Tate-Tsen filtration*)

$$L''=L_n \supseteqq L_{n-1} \supseteqq \dots \supseteqq L_1 \supseteqq L_0 = L, \tag{2.112}$$

such that L_1/L_0 *is the cyclotomic extension of* L *with* $L_1 = L(\mu_{l^\infty})$ *and for* $i \geqq 1$, L_{i+1}/L_i *is a Galois extension induced as in* 2.40 *by* M''/M' *with* M'' *the separable closure of the field* M' *of l-torsion etale cohomological dimension at most* 1.

Let A *be a algebra over* L. *If* $l=2$, *let* l^ν *be divisible by* 4 *and assume* L *contains a primitive square root of* -1.

Then the augmentation is a weak homotopy equivalence

$$\begin{array}{c} \eta : K/l^\nu(A)[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(L;\, K/l^\nu(A \otimes_L \ )[\beta^{-1}]) \\ \| \\ \mathbb{H}^{\cdot}_{et}(L''/L;\, K/l^\nu(A \otimes_L L'')[\beta^{-1}]) \end{array} \tag{2.113}$$

Proof. — By the 5-Lemma and the Bockstein fibration sequence (A.10) it suffices to do the cases $\nu=1$ for l odd and $\nu=2$ for $l=2$.

Suppose l is odd and L does not contain a primitive l-th root of 1. Let $L'=L(\mu_l)$ so $[L' : L]=l-1$. Then the transfer gives a commutative diagram

$$\begin{tikzcd} K/l^\nu(A)[\beta^{-1}] \arrow[r, "\eta(A)"] \arrow[d] \arrow[dd, bend right=60, "l-1"'] & \mathbb{H}^{\cdot}_{et}(L;\, K/l^\nu(A \otimes_L \ )[\beta^{-1}]) \arrow[d] \arrow[dd, bend left=60, "l-1"] \\ K/l^\nu(A \otimes_L L')[\beta^{-1}] \arrow[r, "\eta(A\otimes L')"] \arrow[d, "\lambda_*"] & \mathbb{H}^{\cdot}_{et}(L;\, K/l^\nu(A \otimes_L L' \otimes_L \ )[\beta^{-1}]) \arrow[d] \\ K/l^\nu(A)[\beta^{-1}] \arrow[r, "\eta(A)"'] & \mathbb{H}^{\cdot}_{et}(L;\, K/l^\nu(A \otimes_L \ )[\beta^{-1}]) \end{tikzcd} \tag{2.114}$$

Shapiro's Lemma 2.20 gives a weak equivalence

$$\mathbb{H}^{\cdot}_{et}(L;\, K/l^\nu(A \otimes_L L' \otimes_L \ )[\beta^{-1}]) \simeq \mathbb{H}^{\cdot}_{et}(L';\, K/l^\nu(A \otimes_L L' \otimes_{L'} \ )[\beta^{-1}]). \tag{2.115}$$

This equivalence identifies the middle map of (2.114) to $\eta(A \otimes_L L')$. As multiplication by the mod l unit $l-1=[L' : L]$ is an isomorphism on $K/l^\nu(A)[\beta^{-1}]$, $\eta(A)$ is a retract of $\eta(A \otimes_L L')$. Hence it suffices to show that $\eta(A \otimes_L L')$ is a weak homotopy equivalence. Thus one may replace L by L', and assume that L contains a primitive l-th root of 1 if l is odd. For $l=2$, L contains a primitive 4-th root of one by hypothesis.

Now for each $i=0, 1, \dots, n-1$, I claim that the map (2.116) is a weak homotopy equivalence.

$$\eta(L_{i+1}/L_i \otimes A) : K/l(A \otimes_L L_i)[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(L_{i+1}/L_i;\, K/l(A \otimes_L L_{i+1})[\beta^{-1}]). \tag{2.116}$$

In the case where $\mathrm{Gal}(L_{i+1}/L_i)$ has l-torsion etale cohomological dimension 0, $\mathrm{Gal}(L_{i+1}/L_i)$ is pro l-torsion free by [102], I, 3.3, Cor. 2 to Lemma 4 or [104], III, § 1, Prop. 16. Thus L_{i+1}/L_i is a direct limit of finite subextensions L'/L_i of degree $[L' : L_i]$ prime to l. Then by a transfer argument as above the natural map of $\eta(L_{i+1}/L_i \otimes A)$ to $\eta(L_{i+1}/L' \otimes A)$ is split by $[L' : L_i]^{-1}$ times the transfer from L' to L_i. Thus $\eta(L_{i+1}/L_i \otimes A)$ is a retract of $\eta(L_{i+1}/L' \otimes A)$, and even of the limit of the $\eta(L_{i+1}/L' \otimes A)$ as L' goes to L_{i+1}. By 1.41 this limit is $\eta(L_{i+1}/L_{i+1} \otimes A)$, which is the obvious weak equivalence :

$$(2.117)\quad K/l(A \underset{L}{\otimes} L_{i+1})[\beta^{-1}] = \mathbb{H}^{\cdot}(\mathrm{Gal}(L_{i+1}/L_{i+1}) = \{1\};\, K/l(A \underset{L}{\otimes} L_{i+1})[\beta^{-1}]).$$

Thus $\eta(L_{i+1}/L_i \otimes A)$ is a retract of a weak homotopy equivalence, and so is a weak equivalence.

In the remaining case where $\mathrm{Gal}(L_{i+1}/L_i)$ has l-torsion etale cohomological dimension 1, 2.42 says that (2.116) is a weak homotopy equivalence with homotopy inverse $\beta^{-1}.\varphi(L_{i+1}/L_i;\, A \otimes L_i)$. Note that $\mathbb{H}^{\cdot}_{et}(L_{i+1}/L_i;\ )$ commutes with direct limits, e. g., by 1.39. Thus the direct limit (A.9) that inverts β may be taken inside or outside $\mathbb{H}^{\cdot}_{et}(L_{i+1}/L_i;\ )$.

In both cases of (2.116), K/l should be replaced by $K/4$ if $l=2$, and β should be replaced by that power x of β that lifts to the associative and commutative rings $K/16$ and $K/9$ if $l=2$ or 3 respectively. This is as in Appendix A.

Using (2.116), one proves by descending induction on i, starting from $i=n-1$, that (2.118) is a weak homotopy equivalence

$$(2.118)\quad \begin{array}{ccc} \eta : K/l(A \underset{L}{\otimes} L_i)[\beta^{-1}] & \xrightarrow{\sim} & \mathbb{H}^{\cdot}_{et}(L_i;\, K/l(A \otimes L_i \underset{L_i}{\otimes}\ )[\beta^{-1}]) \\ & & \wr\| \\ & & \mathbb{H}^{\cdot}_{et}(L_n/L_i;\, K/l(A \underset{L}{\otimes} L_n)[\beta^{-1}]) \end{array}$$

For (2.118) for i and $i+1$ fit into a commutative diagram with the weak equivalences (2.116) and (1.39) or (1.55), where the bottom map is an equivalence by induction hypothesis

$$\begin{array}{ccc} K/l(A \underset{L}{\otimes} L_i)[\beta^{-1}] & \longrightarrow & \mathbb{H}^{\cdot}_{et}(L_i;\, K/l(A \underset{L}{\otimes} L_n)[\beta^{-1}]) \\ \wr\downarrow & & \downarrow\wr \\ \mathbb{H}^{\cdot}_{et}(L_{i+1}/L_i;\, K/l(A \underset{L}{\otimes} L_{i+1})[\beta^{-1}]) & \xrightarrow{\sim} & \mathbb{H}^{\cdot}_{et}(L_{i+1}/L_i;\, \mathbb{H}^{\cdot}_{et}(L_{i+1};\, K/l(A \underset{L}{\otimes} L_n)[\beta^{-1}])) \end{array} \qquad (2.119)$$

For $i=0$, (2.118) is (2.113) for $\nu=1$ (or $\nu=2$ if $l=2$). This is the weak equivalence required above. This completes the proof.

COROLLARY 2.44. – *Let* L *be a field of characteristic not l. If $l=2$, suppose that* L *contains a square root of* -1. *Suppose* L *has finite transcendence degree over a field k, where either k is separably closed, or has l-torsion etale cohomological dimension at most* *one, or* $k=\mathbb{Q}$, *or* $k=\mathbb{Q}_p^\wedge$ *for some prime* p *possibly equal to* l, *or* $k=\mathbb{F}_p$, $\mathbb{F}_p(t)$, *or* $\mathbb{F}_p((t))$. *Then for any algebra* A *over* L, *the augmentation is a weak homotopy equivalence* (2.113).

Proof. — In all these cases, one may construct a sequence of subfields like (2.112) in the separable closure L″ of L. Let $\bar{L}'$ denote the algebraic closure of a field L′. If L is algebraic over $k(t_1, \ldots, t_n)$ with k separably closed, let L_i be the subfield of L″ generated by $\overline{k(t_1, \ldots, t_i)}(t_{i+1}, \ldots, t_n) \cap L''$ and L_{i-1} in L″, for $i=1, \ldots, n$. Then $L_0=L$, $L_n=L''$, and L_{i+1}/L_i is generated by a separable subextension of $\overline{k(t_1, \ldots, t_i)(t_{i+1})}/\overline{k(t_1, \ldots, t_i)}(t_{i+1})$, which has l-torsion cohomological dimension 1 by Tsen's Theorem. *See* [102], II, 4.2 or [104], Thms. 24 and 28 of IV for more details.

If k has etale cohomological dimension 1 for l-torsion sheaves, let L_1/L_0 be induced by $k(\mu_{l^\infty})/k$, and L_2/L_1 be induced by $k^s/k(\mu_{l^\infty})$ for k^s the separable closure. Then find a sequence between L″ and L_2 as in the above paragraph. This case includes that of $k=\mathbb{F}_p$. The case $k=\mathbb{F}_p$ includes the case $k=\mathbb{F}_p(t)$ as L of finite transcendence degree over $\mathbb{F}_p(t)$ also has finite transcendence degree over $\mathbb{F}_p$: this case is idiotic.

The other global field case $k=\mathbb{Q}$ is handled by noting that $\mathbb{Q}(\mu_{l^\infty})$ has l-torsion etale cohomological dimension one by [102] proof of Lemma 1, II, 4.4. Similarly $\mathbb{Q}_l^\wedge(\mu_{l^\infty})$ has l-torsion etale cohomological dimension 1 by [102], II, § 5, Lemma 3. For $p\neq l$, the l-torsion etale cohomological dimension of $\mathbb{Q}_p^\wedge(\mu_{l^\infty})$ and of $\mathbb{F}_p((t))(\mu_{l^\infty})$ is at most one greater than that of $\mathbb{F}_p(\mu_{l^\infty})$ by [102], II, 4.3, Prop. 12. But an easy elementary argument shows that $\mathbb{F}_p(\mu_{l^\infty})$ has no algebraic extensions of degree l, and so has l-torsion etale cohomological dimension 0. Thus for $k=\mathbb{Q}$, $\mathbb{Q}_l^\wedge$, $\mathbb{Q}_p^\wedge$, $\mathbb{F}_p((t))$, $k(\mu_{l^\infty})$ has dimension 1. Let L_1/L_0 be induced by $k(\mu_{l^\infty})/k$, and find a sequence between L″ and L_1 by the preceding paragraph.

THEOREM 2.45. — *Let* X *be a separated noetherian scheme of finite Krull dimension. Let* l^ν *be a prime power, and suppose* l *is invertible in* X. *Suppose there is a uniform bound on the* l*-torsion etale cohomological dimension of all residue fields of* X, *even at the non-closed points. Suppose that all residue fields of* X *admit a sequence like* (2.112), *e.g., that they are of finite transcendence degree over* $\mathbb{Q}$, $\mathbb{Q}_l^\wedge$, $\mathbb{Q}_p^\wedge$, $\mathbb{F}_p$, $\mathbb{F}_p((t))$, *or over a field* k *which is separably closed, or at least has* l*-torsion cohomological dimension which is at most one. If* $l=2$, *assume that* X *is over* $\mathrm{Spec}(\mathbb{Z}[i])$, *for* i *the square root of* -1.

Finally suppose that X *is regular.*

Then the natural augmentation is a weak homotopy equivalence

$$K/l^\nu(X)[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(X; K/l^\nu(\ )[\beta^{-1}]). \tag{2.120}$$

There is a strongly convergent spectral sequence

$$E_2^{p,q}=H^p_{et}(X; \tilde{K}/l^\nu_q[\beta^{-1}]) \quad\Rightarrow\quad K/l^\nu_{q-p}(X)[\beta^{-1}]. \tag{2.121}$$

Proof. — The Theorem results from 2.2, 2.4, 2.7, 2.8, 2.43, and 2.44. Thus one reduces to the case where X is a local ring by the Brown-Gersten spectral sequence, then to the case where X is a field by the localisation Theorem. The hyped-up Hilbert's Theorem 90 of 2.43 proves (2.120) in this case. The trick of A.13 for $l=2$ removes the hypothesis that $\nu\geqq 2$ in 2.43.

The spectral sequence (2.121) is deduced from (2.120) *via* 1.36.

COROLLARY 2.46. — *Let l be a prime, and let X be a separated regular scheme of finite type over either $\mathbb{Z}[l^{-1}]$, $\mathbb{Q}$, $\mathbb{Q}_p^{\wedge}$, $\mathbb{Q}_l^{\wedge}$, $\mathbb{F}_p((t))$, or $\bar{k}$, for $\bar{k}$ a separably closed field of characteristic not l. If $l=2$, assume that X is a scheme over $\mathbb{Z}[l^{-1}, i]$ also. Then* (2.120) *is a weak homotopy equivalence and the spectral sequence* (2.121) *converges. Similarly if X is the inverse limit of a system of schemes etale over a fixed regular scheme satisfying the above conditions.*

Proof. — Obvious special case of 2.45.

THEOREM 2.47. — *Let X satisfy all the hypotheses of* 2.45, *except that X may not be regular. Then the augmentation which is a natural transformation of functors contravariant for flat morphisms of schemes, is a weak homotopy equivalence*

$$G/l^\nu(X)[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\bullet}_{et}(X; G/l^\nu(\ )[\beta^{-1}]). \tag{2.122}$$

There is a strongly convergence spectral sequence, natural with respect to flat maps.

$$E_2^{p,q} = H^p_{et}(X; \tilde{G}/l^\nu_q[\beta^{-1}]) \Rightarrow G/l^\nu_{q-p}(X)[\beta^{-1}]. \tag{2.123}$$

In particular, any X as in 2.46, *except possibly not regular, has a weak equivalence* (2.122) *and a spectral sequence* (1.123).

Proof. — The proof is as in 2.45, using also 2.10 and 2.11 to pass to the case X is a field from the case where X is an Artin local ring.

THEOREM 2.48. — *Let X be as in* 2.47. *Let $\mathscr{A}$ be a filtering system of etale covers $\mathscr{U}_\alpha$ of* X. *Then the natural augmentation is a weak equivalence*

$$G/l^\nu(X)[\beta^{-1}] \to \check{\mathbb{H}}^{\bullet}(\mathscr{A}; G/l^\nu(\ )[\beta^{-1}]), \tag{2.124}$$

and there is spectral sequence, which converges strongly if $\mathscr{A}$ has finite Čech cohomological dimension

$$E_2^{p,q} = \check{H}^p(\mathscr{A}; G/l^\nu_q[\beta^{-1}]) \Rightarrow G/l^\nu_{q-p}(X)[\beta^{-1}]. \tag{2.125}$$

If in addition, X *is regular*, G/l^ν *may be replaced by* K/l^ν.

Proof. — This results from 2.45, 2.47, and 1.46 or 1.48.

2.49. The rational Theorem 2.15 and the l-adic Theorems 2.45 and 2.47 yield an integral Theorem for the $K(\ )_K$ of A.14.

THEOREM 2.50. — *Let X be a noetherian separated scheme of finite Krull dimension. Let J be a set of primes. Suppose there is a uniform bound on the l-torsion etale cohomological dimension for all residue fields of X and all primes l in J. Suppose the other conditions of* 2.45 *are met for all l in* J.

Then for X regular, the augmentation is a weak homotopy equivalence

$$K(X)_K \otimes \mathbb{Z}_{(J)} \xrightarrow{\sim} \mathbb{H}^{\bullet}_{et}(X; K(\ )_K \otimes \mathbb{Z}_{(J)}). \tag{2.126}$$

For X *possibly singular, a similar statement is true for* G*-theory.*

Proof. — By the arithmetic homotopy fibre square for spectra of [12], 2.9 and the 5-Lemma, it suffices to prove that (2.126) induces equivalences on the $\mathbb{Q}$-localization and the p-adic completion for all primes p.

The hypotheses on cohomological dimension and the arguments of [SGA 4], X show that X has finite cohomological dimension for $\mathbb{Z}_{(J)}$ modules. Then by 1.39 or 1.40, the direct limit process that inverts all integers commutes with $\mathbb{H}^{\cdot}_{et}(X;\ )$. On the other hand, $K(\ )_K \otimes \mathbb{Q} = K(\ ) \otimes \mathbb{Q}$ by A.14. Thus the $\mathbb{Q}$-localization of (2.126) is identified to the weak equivalence (2.28).

The p-adic completion of (2.126) is the homotopy inverse limit over ν of (2.126) smashed with a $\bmod p^\nu$ Moore spectrum. For p not in J, both sides of (2.126) are uniquely p-divisible, and so smashing $\bmod p^\nu$ produces 0. For $l=p$ in J, the $\mathbb{Z}_{(J)}$ localization is irrelevant $\bmod l^\nu$, and the $\bmod l^\nu$ smash of (2.126) is identified with the weak equivalence (2.120), using A.14.

Thus (2.126) is a weak homotopy equivalence as required.

3. The etale sheaf of coefficients $\tilde{K}/l^\nu_*[\beta^{-1}]$

Theorem 3.1. — *Let* X *be a regular noetherian scheme in which l is invertible, and which satisfies the other hypotheses of* 2.45. *Then the sheaf of localized algebraic* K*-groups in the etale topos of* X *is given by* (3.1) *and* (3.2)

$$\tilde{K}/l^\nu_q[\beta^{-1}] = \begin{cases} \mathbb{Z}/l^\nu(i), & q=2i, \\ 0, & q \text{ odd}, \end{cases} \tag{3.1}$$

$$\tilde{K}/l^\nu_*[\beta^{-1}] = \mathbb{Z}/l^\nu[\beta, \beta^{-1}], \qquad \deg \beta = 2. \tag{3.2}$$

See remark 3.34.

The proof of 3.1 occupies the entire section. The basic idea is to reglobalize it to an assertion about K_0 of algebraic simplicial complexes over a strict local hensel ring, and then to prove this assertion by the splitting principle for algebraic vector bundles.

3.2. The element β^i defines a map $\mathbb{Z}/l^\nu(i) \to \tilde{K}/l^\nu_{2i}[\beta^{-1}]$. To prove 3.1 it suffices to show this map is an isomorphism and that the sheaf $\tilde{K}/l^\nu_{2i+1}[\beta^{-1}]$ is 0. It suffices to check this at stalks by 1.30. By 1.29, it suffices to prove 3.3.

Theorem 3.3. — *Let* R *be a regular noetherian strict local hensel ring in which l is invertible, and which satisfies the other hypotheses of* 2.45. *Then*

$$K/l^\nu_*(R)[\beta^{-1}] = \mathbb{Z}/l^\nu[\beta, \beta^{-1}]. \tag{3.3}$$

Lemma 3.4 (Snaith). — *Let* R *be a strict local hensel ring which is noetherian and in which l is invertible. Then* R *contains all l-power roots of* 1. *Consider the construction of* A.4 *and* A.7. *Then*

$$\pi_* \Sigma^\infty/l^\nu(BGL_1(R) \sqcup *)[b^{-1}] \simeq \mathbb{Z}/l^\nu[b, b^{-1}]. \tag{3.4}$$

Note R *is not assumed to be regular.*

Proof. — As R is strict local hensel and l is invertible, $GL_1(R)$ is the sum of the l-torsion group of l-power roots of unity $\mu_{l^\infty} \simeq \mathbb{Q}/\mathbb{Z}_{(l)}$, and a uniquely l-divisible group of units. Hence by [16], VI, 5.1 one has weak homotopy equivalences of l-adic completions of spaces

$$(3.5)\qquad BGL_1(R)_l^\wedge \simeq (B\mu_{l^\infty})_l^\wedge \simeq (B\,\mathbb{Q}/\mathbb{Z}_{(l)})_l^\wedge \simeq K(\mathbb{Z}, 2)_l^\wedge \simeq \mathbb{C}P_l^{\infty\,\wedge}.$$

Under the equivalences (3.5), the element b of A.7 corresponds to the generator of $\pi_2\,\mathbb{C}P^\infty$, as one sees by applying $H_2(\ ;\mathbb{Z}/l^\nu)$ to the maps (3.5) and using the mod l Hurewicz Theorem.

For any nilpotent space Z, the map $Z \to Z_l^\wedge$ induces an isomorphism on mod l homology. Hence $\Sigma^\infty(Z \sqcup *) \to \Sigma^\infty(Z_l^\wedge \sqcup *)$ induces an isomorphism on mod l homology, and $\Sigma^\infty/l^\nu(Z \sqcup *) \to \Sigma^\infty/l^\nu(Z_l^\wedge \sqcup *)$ induces an isomorphism on integral homology and so is a weak homotopy equivalence. Combining this and (3.5), one gets weak homotopy equivalences

$$\begin{aligned}(3.6)\quad & \Sigma^\infty/l^\nu(BGL_1(R) \sqcup *)[b^{-1}] \\ & \qquad \simeq \Sigma^\infty/l^\nu(BGL_1(R)_l^\wedge \sqcup *)[b^{-1}] \simeq \Sigma^\infty/l^\nu(\mathbb{C}P^\infty \sqcup *)[b^{-1}].\end{aligned}$$

Snaith has proved that the inclusion $\mathbb{C}P^\infty \to BU(1) \to BU$ of spaces induces an equivalence of spectra ([108], or [107])

$$(3.7)\qquad \Sigma^\infty(\mathbb{C}P^\infty \sqcup *)[b^{-1}] \simeq BU.$$

The homotopy groups of BU are known by Bott periodicity, one has $\pi_* BU/l^\nu = \mathbb{Z}/l^\nu[\beta, \beta^{-1}]$. Combining this with (3.6) and the smash of (3.7) with a mod l^ν Moore spectrum yields (3.4).

LEMMA 3.5 (Dwyer, Friedlander, Snaith, Thomason). — *For* R *as in* 3.3, *the map* γ *of* A.4 *and the Dwyer-Friedlander map* ρ *of* [31], [35], § 2, [30] *have a weak homotopy equivalence as a composite* $\rho\gamma$. *Thus* $\mathbb{Z}/l_*^\nu[\beta, \beta^{-1}]$ *splits off* $K/l_*^\nu(R)[\beta^{-1}]$. *To prove* 3.3, *it suffices to show that* $\pi_*\gamma$ *is surjective.*

$$(3.8)\qquad \begin{tikzcd}
 & K/l^\nu(R)[\beta^{-1}] \arrow[dr, "\rho"] & \\
\Sigma^\infty/l^\nu(BGL_1(R) \sqcup *)[b^{-1}] \arrow[ur, "\gamma"] \arrow[ddrr, "\sim"'] & & K/l^{\nu\,\mathrm{Top}}(R) \arrow[d, "\wr"] \\
 & & (BGL_{R\,\mathrm{et}})/l^\nu \arrow[d, "\wr"] \\
 & & BU/l^\nu
\end{tikzcd}$$

Proof. — As the strict local hensel ring R has the etale homotopy type of a point, its Dwyer-Friedlander topological K-theory spectrum is that of a point. Thus

$K/l^{\nu\mathrm{Top}}(R) \simeq BU/l^\nu$. The composite of this equivalence with $\rho\gamma$ is the composite of equivalences (3.6) and (3.7). Hence $\rho\gamma$ is a weak equivalence. For more details, one may consult [31].

3.6. To show that $\pi_* \gamma$ is surjective one reduces to the splitting principle for K_0. The reduction requires development of excision properties and Mayer-Vietoris spectral sequences for closed covers. This is applied to certain algebraic simplicial complexes. I pause to develop these subjects. In the discussion 3.7-3.29 R will be a regular noetherian ring in which l is invertible, and which satisfies the other hypotheses of 2.45. For now, R need not be henselian.

3.7. Let M be a finite simplicial complex with vertices $x_0, \ldots, x_m$. Then M has a poset of faces **M**, whose objects σ are the non-degenerate simplices of M, and which has a map $\sigma \to \sigma'$ iff σ' is a face of σ. There is an algebra $\mathscr{M}$ of R-valued polynomials in the barycentric co-ordinates of M. This is given as

$$(3.9) \qquad \mathscr{M} = R[x_0, x_1, \ldots, x_m]/(x_0 + \ldots + x_m - 1, \text{ certain monomials}),$$

where a monomial $x_{i_1} \ldots x_{i_k}$ is in the ideal if and only if the vertices $x_{i_1}, \ldots, x_{i_k}$ do not span a simplex of M. Let M′ be the affine scheme Spec$(\mathscr{M})$. Then M′ is a union of linear subspaces in $\mathbb{A}_R^{m+1}$ that meet in the cofiguration M.

The constructions are all functorial in the simplicial complex M.

The proof of [23], 4.8 shows that $\mathscr{M}$ is the limit along **M** of the functor from **M** to the category of R-algebras that sends a simplex σ to the ring of barycentric co-ordinates on σ. It also results from the proof of [23], 4.8 in the work of Dayton and Weibel that if $N \subsetneq M$ is any subcomplex, then $\mathscr{M} \to \mathscr{N}$ is surjective and a GL fibration. If M is formed from N by attaching a simplex, this follows immediately from the last paragraph of the proof of [23], 4.8. The obvious induction then proves it for general subcomplexes. This is the key step in setting up the Mayer-Vietoris spectral sequence of [23], 4.8 for the KV-theory of M′. The observations of this paragraph will allow one to recover their theorem from 3.17 below.

Definition 3.8. — Consider a class of squares of schemes and immersions (3.10)

$$(3.10) \qquad \begin{array}{ccc} X & \xrightarrow{i'} & Z \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow[i]{} & W \end{array}$$

A contravariant functor F into the category of fibrant spectra has the Mayer-Vietoris property for the class if each square is in the domain of F and if the induced square of fibrant spectra (3.11) is homotopy cartesian

$$(3.11) \qquad \begin{array}{ccc} F(W) & \longrightarrow & F(Z) \\ \downarrow & \square & \downarrow \\ F(Y) & \longrightarrow & F(X) \end{array}$$

F has the connective Mayer-Vietoris property for the class if the map from F(W) into the homotopy pullback of F(Z) and F(Y) over F(X) induces an isomorphism on π_k at least for $k \geqq 0$.

Example 3.9. — For the class of squares (3.10) with Y and Z Zariski open subschemes of $W = Y \cup Z$, and with $X = Y \cap Z$, this is the Mayer-Vietoris property of 2.2. For this section, the interest is in squares (3.10) where the immersions are closed, not open.

Example 3.10. — Let R be a regular noetherian scheme. Consider the class of squares (3.10) corresponding to a square of simplicial complexes as in 3.7. Suppose that Y and Z are subcomplexes of the finite complex W. Suppose $Y \cup Z = W$ and $Y \cap Z = X$. Then by 3.7 the corresponding maps of rings are surjections and GL fibrations $\mathscr{Y} \to \mathscr{X}$, $\mathscr{Z} \to \mathscr{X}$. Further, $\mathscr{W}$ is the pullback of $\mathscr{Y}$ and $\mathscr{Z}$ over $\mathscr{X}$. Thus by [59], [60], Appendice 7, or [39], 2.10, Karoubi-Villamayor K-theory has the Mayer-Vietoris property.

Example 3.11. — Consider the class of squares (3.10) which consist of closed immersions of affine schemes corresponding to a pullback square of rings. Suppose l is invertible in these rings. As the immersions are closed, the corresponding maps of rings are surjective. A Theorem of Weibel, [139], 1.3 says that K/l^ν has the connective Mayer-Vietoris property for this class of squares. It follows that $K/l^\nu[\beta^{-1}]$ has the full Mayer-Vietoris property for this class.

Note that as the maps of rings are surjective, the assertion that the ring $\mathcal{O}_W$ is the pull-back of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ over $\mathcal{O}_X$ is just the assertion that the sequence (3.12) of $\mathcal{O}_W$-modules is exact

$$0 \to \mathcal{O}_W \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_X \to 0. \tag{3.12}$$

This condition is preserved by tensoring over $\mathcal{O}_W$ with any flat $\mathcal{O}_W$-algebra. Thus if $W' \to W$ is affine and flat, e.g., affine and etale, the pullback to W′ from W of any square (3.10) in this class is also in this class.

Example 3.12. — Let (3.10) be a square for which $K/l^\nu[\beta^{-1}]$ has the Mayer-Vietoris property. Let ε be a vector bundle on W and consider the induced square of projective space bundles $\mathbb{P}(\varepsilon)$. Suppose W is quasicompact. Then $K/l^\nu[\beta^{-1}]$ has the Mayer-Vietoris property for the square of projective space bundles. For by [97], § 8, 2.1, $K/l^\nu(\mathbb{P}(\varepsilon))[\beta^{-1}]$ is the product of rank ε copies of $K/l^\nu[\beta^{-1}]$ of the base.

3.13. As complete flag bundles can be built by successive construction of projective space bundles as in [SGA 6], VI, 3.1, a result like 3.12 holds for them.

LEMMA 3.14. — *Let (3.10) be a diagram of closed immersions. Let W be a separated noetherian scheme. Let F be such that for every affine W′ etale over W, F has the Mayer-Vietoris property for the pull-back of (3.10) to W′. Then $\mathbb{H}^{\bullet}_{et}(\ ;F)$ has the Mayer-Vietoris property for the original square. That is, (3.13) is homotopy cartesian:*

$$\begin{array}{ccc} \mathbb{H}^{\bullet}_{et}(W;F) & \to & \mathbb{H}^{\bullet}_{et}(Z;F) \\ \downarrow & & \downarrow \\ \mathbb{H}^{\bullet}_{et}(Y;F) & \to & \mathbb{H}^{\bullet}_{et}(X;F) \end{array} \tag{3.13}$$

Proof. — By hypothesis, there is a homotopy cartesian square of presheaves of fibrant spectra on the restricted etale sites

$$(3.14)\qquad \begin{array}{ccc} F & \to & j_* F \\ \downarrow & & \downarrow \\ i_* F & \to & j i'_* F \end{array}$$

Applying $\mathbb{H}^{\cdot}_{et}(W;\ )$ to (3.14) yields a homotopy cartesian square. By (2.19), the resulting diagram is weak equivalent to (3.13).

PROPOSITION 3.15. — *Let* M′ *be a scheme associated to a finite simplicial complex* M *as in* 3.7. *Let* F *have the Mayer-Vietoris property* (*resp., the connective Mayer-Vietoris property*) *for squares* (3.10) *of simplicial subcomplexes of* M *with* $W=Y\cup Z$ *and* $X=Y\cap Z$. F *restricts to a functor from the category of simplicies* **M** *to that of fibrant spectra. Then the natural augmentation is a weak homotopy equivalence* (*resp., induces an isomorphism on* π_k *for* $k\geqq 0$)

$$(3.15)\qquad F(M')\xrightarrow{\sim}\mathbb{H}^{\cdot}(\mathbf{M};\ F)=\operatorname*{holim}_{\mathbf{M}} F.$$

There is a spectral sequence which converges strongly (*resp., to those* π_k *for which* $k\geqq 0$)

$$(3.16)\qquad E_2^{p,q}=H^p(\mathbf{M};\ \pi_q F)\ \Rightarrow\ \pi_{q-p}F(M').$$

If $\pi_q F$ *is a constant functor, the* E_2 *term is isomorphic to cohomology of the simplicial complex* M *with coefficients in* $\pi_q F$

$$(3.17)\qquad H^p(\mathbf{M};\ \pi_q F)\simeq H^p(M;\ \pi_q F).$$

Proof. — First suppose that M is a simplex $\Delta[n]$. Then the top cell is an initial object of the category **M**. The inverse limit along **M** is given by evaluation at this initial object. Hence $\varprojlim$ is exact, and its right derived functors $H^p(\mathbf{M};\ )$ vanish for $p>0$. The spectral sequence of 5.13 collapses for **M**, yielding (3.15).

To prove (3.15) for more general M, one inducts on the number of simplices in M. To do the induction step, let M be formed from a subcomplex N by attaching a simplex $\Delta[n]$ along $\partial\Delta[n]=\Delta[n]\cap N$. One then appeals to the 5-Lemma and the diagram (3.18), in which the left column is a homotopy fibre sequence by the Mayer-Vietoris hypothesis and the right column is by Lemma 3.16 below.

$$(3.18)\qquad \begin{array}{ccc} F(M') & \longrightarrow & \mathbb{H}^{\cdot}(\mathbf{M};\ F) \\ \downarrow & & \downarrow \\ F(\Delta[n])\times F(N') & \xrightarrow{\sim} & \mathbb{H}^{\cdot}(\mathbf{\Delta[n]};\ F)\times\mathbb{H}^{\cdot}(\mathbf{N};\ F) \\ \downarrow & & \downarrow \\ F(\partial\Delta[n]) & \xrightarrow{\sim} & \mathbb{H}^{\cdot}(\partial\mathbf{\Delta[n]};\ F) \end{array}$$

The spectral sequence (3.16) is just the spectral sequence of 5.13 interpreted in light of (3.15). Note that as **M** is a finite poset the canonical complex of [16], XI, 6.2 for computing $H^*(\mathbf{M};\ )$ has a finite complex as its noncodegenerate subcomplex. Thus **M** has finite cohomological dimension.

The statement (3.17) results from the fact that the canonical complex of [16], XI, 6.2 for computing $H^*(\mathbf{M}; \pi_q F)$ is isomorphic to the canonical complex for computing cohomology with coefficients in $\pi_q F$ of the simplicial nerve of $\mathbf{M}$, $N\mathbf{M}$. But this simplicial set is just the barycentric subdivision of the simplicial complex M. The reader may also consult [38], App. II, 3.3, or [97], § 1.

LEMMA 3.16. — *Let* (3.19) *be a cartesian square of categories. Suppose that i and j are fully faithful embeddings of subcategories which are cosieves: that is, suppose any morphism* $Y \to W$ *in* $\mathbf{W}$ *with source* Y *in* $\mathbf{Y}$ *is in fact a morphism in the subcategory* $\mathbf{Y}$, *and similarly for* $\mathbf{Z} \subseteq \mathbf{W}$. *Suppose that any object of* $\mathbf{W}$ *is either in* $\mathbf{Y}$ *or in* $\mathbf{Z}$.

$$\begin{tikzcd} \mathbf{X} \arrow[r, hook, "i'"] \arrow[d, hook, "j'"'] & \mathbf{Z} \arrow[d, hook, "j"] \\ \mathbf{Y} \arrow[r, hook, "i"] & \mathbf{W} \end{tikzcd} \tag{3.19}$$

For example, let (3.19) *be the square of posets of simplices of a square of simplicial complexes satisfying the hypotheses of* 3.15.

Then the square (3.20) *is homotopy cartesian for any functor* F *from* $\mathbf{W}$ *into the category of fibrant spectra*

$$\begin{tikzcd} \mathbb{H}^{\cdot}(\mathbf{W}; F) \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}(\mathbf{Z}; F) \arrow[d] \\ \mathbb{H}^{\cdot}(\mathbf{Y}; F) \arrow[r] & \mathbb{H}^{\cdot}(\mathbf{X}; F) \end{tikzcd} \tag{3.20}$$

Proof. — Consider the canonical cosimplicial fibrant spectrum whose Tot is $\mathbb{H}^{\cdot}(\mathbf{W}; F)$, as in [16], XI, § 5. This cosimplicial spectrum is

$$\prod_{\mathbf{W}}{}^* F, \qquad (\prod_{\mathbf{W}}{}^* F)^n = \prod_{W_0 \leftarrow \dots \leftarrow W_n} F(W_0). \tag{3.21}$$

The spectrum in codimension n is a product indexed by the set of n-simplices of the nerve of $\mathbf{W}$, i.e., by the set of sequences of morphisms $W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \dots \leftarrow W_n$ in $\mathbf{W}$. Under the hypotheses, such a sequence lies entirely in $\mathbf{Y}$ or in $\mathbf{Z}$, depending on whether W_n lies in $\mathbf{Y}$ or $\mathbf{Z}$. As $\mathbf{X} = \mathbf{Y} \cap \mathbf{Z}$ is also a cosieve in $\mathbf{W}$, the sequence lies in $\mathbf{X}$ if and only if W_n does. Thus the indexing set of n-simplices of the nerve of $\mathbf{W}$ is the union of the sets of n-simplices of $\mathbf{Y}$ and $\mathbf{Z}$, with intersection the set of n-simplices of $\mathbf{X}$. It follows that the square (3.22) is a pullback square of cosimplicial fibrant spectra

$$\begin{tikzcd} \Pi^*_{\mathbf{W}} F \arrow[r] \arrow[d] & \Pi^*_{\mathbf{Z}} F \arrow[d] \\ \Pi^*_{\mathbf{Y}} F \arrow[r] & \Pi^*_{\mathbf{X}} F \end{tikzcd} \tag{3.22}$$

Further, the maps in (3.22) may easily be checked to be fibrations of cosimplicial fibrant objects in the sense of [16], X, § 4.6. Applying Tot to (3.22) yields (3.20), which is a homotopy cartesian square by [16], X.

An alternate proof is to use the devissage technique of 5.52. As $\mathbb{H}^{\cdot}(\mathbf{K}; \ )$ commutes with holims along Postnikov towers by 5.7, one reduces first to the case $F = F\langle n \rangle$, and then to the case where F is a presheaf of Eilenberg-MacLane spectra $K(\tilde{\pi}, n)$ as in 5.52. Then the spectral sequence 5.13 shows that (3.20) is homotopy cartesian if and only if the usual Mayer-Vietoris theorem holds for the cohomology of the square (3.19) with coefficients in the presheaf of abelian groups $\tilde{\pi}$. This may be proved by considering canonical resolutions as above, or by citing [64], § 1.

Example 3.17 (Dayton, Weibel [23]). — Let R be a regular noetherian ring, and let M′ be the scheme over Spec(R) associated to a finite simplicial complex as in 3.7. Then Karoubi-Villamayor K-theory, KV, satisfies the hypotheses of 3.15 by 3.10.

Recall that for any ring A, $KV_0(A)$ is the usual $K_0(A)$. As R is regular, $KV_q(R[t_1, \ldots, t_n]) = KV_q(R)$ is Quillen's $K_q(R)$ for all q. Thus the spectral sequence (3.16) may be interpreted *via* (3.15) as a spectral sequence

$$H^p(M; K_q(R)) \Rightarrow KV_{q-p}(M'). \tag{3.23}$$

Let M be a n-sphere S^n. As a point is a retract of S^n, the spectral sequence (3.23) for M a point splits off the spectral sequence for $M = S^n$. Thus the latter spectral sequence collapses to yield

$$K_0(S^n) = KV_0(S^n) = K_0(R) \oplus K_n(R). \tag{3.24}$$

This allows one to shift problems about higher K-theory to problems about the classical K_0.

Example 3.18. — With R a regular noetherian ring containing $1/l$, combining 3.15 with 3.11 as above yields isomorphisms:

$$K/l^\nu_p(S^n) = K/l^\nu_p(R) \oplus K/l^\nu_{p+n}(R), \qquad p \geq 0, \tag{3.25}$$

$$K/l^\nu_p(S^n)[\beta^{-1}] = K/l^\nu_p(R)[\beta^{-1}] \oplus K/l^\nu_{p+n}(R)[\beta^{-1}] \quad \text{all } p. \tag{3.26}$$

Example 3.19. — Let M be a mod l^ν Moore complex; i.e., the cofibre of a map of degree l^ν from S^n to S^n. Then for any spectrum Z one has weak equivalent homotopy fibre sequences

$$\begin{tikzcd} \mathrm{Map}_*(M, Z) \arrow[r] \arrow[d, "\wr"] & \mathrm{Map}_*(S^n, Z) \arrow[r] \arrow[d, "\wr"] & \mathrm{Map}_*(S^n, Z) \arrow[d, "\wr"] \\ \Omega^{n+1}(Z/l^\nu) \arrow[r] & \Omega^n Z \arrow[r, "l^\nu"] & \Omega^n Z \end{tikzcd} \tag{3.27}$$

Thus $\pi_0 \mathrm{Map}_*(M, Z)$ is $\pi_{n+1} Z/l^\nu$.

On the other hand, if Z is regarded as a constant functor from **M** to the category of fibrant spectra, the formulae of [16], XI simplify to show that the homotopy limit $\mathbb{H}^{\cdot}(\mathbf{M}; Z)$ is the spectrum of unbased maps from the nerve of **M** to Z. As the nerve of **M** is the barycentric subdivision of M, one has a weak homotopy equivalence

$$\mathbb{H}^{\cdot}(\mathbf{M}; Z) \simeq \mathrm{Map}_*(M \sqcup *, Z) \simeq Z \times \mathrm{Map}_*(M, Z). \tag{3.28}$$

There is an isomorphism

$$\pi_p \mathbb{H}^{\cdot}(\mathbf{M}; Z) = \pi_p Z \oplus \pi_{p+n+1} Z/l^\nu. \tag{3.29}$$

In fact the spectral sequence (3.16) yields the universal coefficient sequence

$$\begin{array}{lccccc} (3.30) & 0 \to H^{n+1}(M;\, \pi_{p+n+1} Z) & \to \pi_{p+n+1} Z/l^\nu \to & H^n(M;\, \pi_{p+n} Z) \to 0 \\ & \| & & \| \\ & (\pi_{p+n+1} Z) \otimes \mathbb{Z}/l^\nu & & \mathrm{Tor}^1_{\mathbb{Z}}(\pi_{p+n} Z,\, \mathbb{Z}/l^\nu) \\ & & & \| \\ & & & (\pi_{p+n} Z) * \mathbb{Z}/l^\nu \end{array}$$

The identifications in (3.30) result from the long exact cohomology sequence of the cofibre sequence $S^n \to S^n \to M$.

Combining these remarks with 3.17 and 3.18, one obtains isomorphisms

$$(3.31) \qquad K_0(M') = K_0(R) \oplus K/l^\nu_{n+1}(R),$$

$$\begin{aligned} (3.32) \quad K/l^\nu_p(M') &= K/l^\nu_p(R) \oplus \pi_{p+n+1}(\Sigma^\infty/l^\nu \wedge K/l^\nu(R)) \\ &= K/l^\nu_p(R) \oplus (K/l^\nu_{p+n+1}(R) \otimes \mathbb{Z}/l^\nu) \oplus (K/l^\nu_{p+n}(R) * \mathbb{Z}/l^\nu) \\ &= K/l^\nu_p(R) \oplus K/l^\nu_{p+n+1}(R) \oplus K/l^\nu_{p+n}(R) \quad \text{for } p \geqq 0 \text{ and } \nu \geqq 2 \text{ if } l=2. \end{aligned}$$

$$(3.33) \quad K/l^\nu_p(M')[\beta^{-1}] = K/l^\nu_p(R)[\beta^{-1}] \oplus K/l^\nu_{p+n+1}(R)[\beta^{-1}] \oplus K/l^\nu_{p+n}(R)[\beta^{-1}]$$

for all p, and with $\nu \geqq 2$ if $l=2$.

PROPOSITION 3.20. — *Let* R *be a regular noetherian ring in which l is invertible, and which satisfies the other hypotheses of* 2.45. *Let* M *be a finite simplicial complex, and* M′ *the associated scheme over* R *as in* 3.7. *Let $p : P \to M'$ be the projective space bundle $\mathbb{P}(\varepsilon)$ or complete flag bundle Flag(ε) of a vector bundle ε on* M′. *Even though* P *is not usually regular, the natural augmentation will be a weak equivalence*

$$(3.34) \qquad K/l^\nu(P)[\beta^{-1}] \xrightarrow{\sim} \mathbb{H}^{\cdot}_{et}(P;\, K/l^\nu(\ )[\beta^{-1}]).$$

The map (3.34) *is similarly a weak equivalence for any* P *built up from $\mathbb{P}(\varepsilon)$ by successively taking projective space bundles.*

Proof. — Let F be the contravariant functor from the category of subcomplexes of M to that of fibrant spectra given by $F(N) = K/l^\nu(p^{-1}(N))[\beta^{-1}]$. Let

$$F'(N) = \mathbb{H}^{\cdot}_{et}(p^{-1}(N);\, K/l^\nu(\ )[\beta^{-1}]).$$

Then both F and F′ have the Mayer-Vietoris property for certain squares of subcomplexes by 3.12, 3.13, 3.11, and 3.14. Then by 3.15, the vertical arrows in (3.35) are weak equivalences.

$$(3.35) \qquad \begin{array}{ccc} K/l^\nu(P)[\beta^{-1}] & \longrightarrow & \mathbb{H}^{\cdot}_{et}(P;\, K/l^\nu(\ )[\beta^{-1}]) \\ \| & & \| \\ F(M) & \longrightarrow & F'(M) \\ \wr\downarrow & & \wr\downarrow \\ \mathbb{H}^{\cdot}(M;\, F) & \longrightarrow & \mathbb{H}^{\cdot}(M;\, F') \end{array}$$

If σ is a simplex of M, corresponding to a hyperplane $\mathbb{A}^n_{\mathbf{R}}$ in M′, then $p^{-1}(\sigma)=p^{-1}(\mathbb{A}^n_{\mathbf{R}})$ is regular and satisfies all the other conditions of 2.45. Thus by Theorem 2.45, $\mathrm{F}(p^{-1}(\sigma)) \to \mathrm{F}'(p^{-1}(\sigma))$ is a weak homotopy equivalence. By 5.8, the bottom horizontal map of (3.35) is then a weak equivalence. Thus the top horizontal map of (3.35) is also a weak equivalence, as claimed.

I will need some constructions to control the line bundles into which I shall split the problem.

CONSTRUCTION 3.21. — For X a scheme, let BGL_1 be the presheaf of fibrant spaces on the restricted etale site of X, 1.3, that sends an affine $\mathrm{U}=\mathrm{Spec}(\mathrm{A})$ to the classifying space of the group of units of A. There is a natural map from BGL_1 to the zeroth space of $\Sigma^\infty(\mathrm{BGL}_1 \sqcup *)$.

One may form $\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)$, a fibrant space by the methods of paragraph 1. If one considers BGL_1 as the zeroth space of an Eilenberg-MacLane spectrum, $\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)$ as a space is the zeroth space of the etale hypercohomology spectrum. From 1.36, one calculates homotopy groups

$$(3.36)\qquad \left\{\begin{array}{l} \pi_0\, \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)=\mathrm{H}^1_{\mathrm{et}}(\mathrm{X};\ \mathrm{GL}_1)=\mathrm{Pic}(\mathrm{X}),\\ \pi_1\, \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)=\mathrm{H}^0_{\mathrm{et}}(\mathrm{X};\ \mathrm{GL}_1),\\ \pi_q\, \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)=0 \qquad \text{for} \quad q>1. \end{array}\right.$$

The natural map from BGL_1 to the zeroth space of $\Sigma^\infty(\mathrm{BGL}_1\ \ *)$ induces natural maps of spectra

$$(3.37)\qquad \left\{\begin{array}{l} \Sigma^\infty(\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1)\sqcup *) \to \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \Sigma^\infty(\mathrm{BGL}_1 \sqcup *)),\\ \Sigma^\infty/l^\nu(\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{H};\ \mathrm{BGL}_1)\sqcup *) \to \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \Sigma^\infty/l^\nu(\mathrm{BGL}_1 \sqcup *)). \end{array}\right.$$

CONSTRUCTION 3.22. — For X a scheme, let $\mathrm{Div}(\mathrm{X})$ be the classifying space of the category of line bundles on X and their isomorphisms. Then $\mathrm{Div}(\ \ )$ is a presheaf of fibrant spaces on any scheme. It is the zeroth space of a generalized Eilenberg-MacLane spectrum, $\mathrm{K}(\mathrm{Pic}(\ \ ), 0) \times \mathrm{K}(\mathrm{GL}_1(\ \ ), 1)$.

On the category of affine schemes there is a natural map $\mathrm{BGL}_1(\ \ ) \to \mathrm{Div}(\ \ )$, which identifies BGL_1 to the classifying space of the subcategory of trivial line bundles and their isomorphisms. If every line bundle on X is trivial; e. g., if X is a local ring, then this map is a weak equivalence. As both functors $\mathrm{BGL}_1(\ \ )$ and $\mathrm{Div}(\ \ )$ are continuous, 1.29, 1.30, 1.45, 1.44, and 1.36 show that the induced map of hypercohomology spaces is a weak equivalence

$$(3.38)\qquad \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{BGL}_1) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{Div}).$$

Combining (3.38), the calculation of homotopy groups (3.36), and a similar calculation for $\mathrm{Div}(\ \ )$, one sees that the augmentation is a weak homotopy equivalence of spaces

$$(3.39)\qquad \mathrm{Div}(\mathrm{X}) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{Div}).$$

There is a natural map from $\mathrm{Div}(X)$ to the zeroth space of the K-theory spectrum $K(X)$

$$\mathrm{Div}(X) \to K(X). \tag{3.40}$$

For X affine, the zeroth space of $K(X)$ is the group completion of the E_∞ space which is the disjoint union over all isomorphism classes of finitely generated projective modules P of $B\,\mathrm{Aut}(P)$. $\mathrm{Div}(X)$ is the disjoint union over those P of rank 1, and (3.40) is the inclusion followed by the canonical group completion map.

For general schemes X, the zeroth space of $K(X)$ is the loop space of the classifying space of Quillen's Q category of the exact category of vector bundles on X. By adjointness of loops and suspension, to give (3.40) is equivalent to giving a map from $\Sigma\,\mathrm{Div}(X)$ to the classifying space of the Q-category. But $\Sigma\,\mathrm{Div}(X)$ is the classifying space of the suspension of the category of line bundles and isomorphisms. This suspended category has as objects 0 and the line bundles L. The morphisms are the isomorphisms of line bundles and two special morphisms for each L, $0 \twoheadleftarrow L$ in the northern hemisphere and $0 \rightarrowtail L$ in the south. These morphisms go from 0 to L. This category includes into the Q-category in the obvious way, inducing the desired map.

The map (3.40) induces a function $\mathrm{Pic}(X) \to K_0(X)$ on taking π_0. This function sends a isomorphism class of line bundles L to the class [L] of the rank 1 vector bundle L in $K_0(X)$.

LEMMA 3.23. — *The constructions and maps of* A.4, A.7, 3.21, *and* 3.22 *combine to yield a commutative diagram of spaces* (3.41), *natural in* X *for any separated noetherian scheme* X. *The indicated maps are weak homotopy equivalence:*

$$\begin{tikzcd}
\mathrm{Div}(X) \arrow[r] \arrow[d, "\wr"] & K(X) \arrow[r] \arrow[d] & K/l^\nu(X)[\beta^{-1}] \arrow[d] \\
\mathbb{H}^{\cdot}_{et}(X;\,\mathrm{Div}) \arrow[r] & \mathbb{H}^{\cdot}_{et}(X;\,K) \arrow[r] & \mathbb{H}^{\cdot}_{et}(X;\,K/l^\nu(\ )[\beta^{-1}]) \\
\mathbb{H}^{\cdot}_{et}(X;\,BGL_1) \arrow[r] \arrow[u, "\wr"] & \mathbb{H}^{\cdot}_{et}(X;\,\Sigma^\infty(BGL_1 \sqcup *)) \arrow[r] \arrow[u, "\gamma"] & \mathbb{H}^{\cdot}_{et}(X;\,\Sigma^\infty/l^\nu(BGL_1 \sqcup *)[b^{-1}]) \arrow[u, "\gamma"]
\end{tikzcd} \tag{3.41}$$

Proof. — Combine 3.21, 3.22 and obvious naturality statements.

3.24. The last preliminaries are to construct topological K-theory and to prove the splitting principle.

For X a separated noetherian scheme in which l is invertible, define $K/l^{\nu\,\mathrm{Top}}(X)$ by (3.42)

$$K/l^{\nu\,\mathrm{Top}}(X) = \mathbb{H}^{\cdot}_{et}(X;\,\Sigma^\infty/l^\nu(BGL_1 \sqcup *)[b^{-1}]). \tag{3.42}$$

By 3.4, 1.36, and 1.44, there is a spectral sequence (3.43) which converges strongly if X has finite etale cohomological dimension

$$E_2^{p,q} = \left\{ \begin{matrix} H^p_{et}(X;\,\mathbb{Z}/l^\nu(i)),\ q=2i \\ 0,\ q \text{ odd} \end{matrix} \right\} \Rightarrow \pi_{q-p} K/l^{\nu\,\mathrm{Top}}(X). \tag{3.43}$$

The map γ of A.4 and 3.5 induces a map

$$
(3.44)\qquad
\begin{array}{c}
\gamma: \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \Sigma^{\infty}/l^{\nu}(\mathrm{BGL}_1 \sqcup *)[b^{-1}]) \to \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ \mathrm{K}/l^{\nu}[\beta^{-1}]) \\
\| \\
\mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{X})
\end{array}
$$

The map $\rho\gamma$ is a weak equivalence of $\mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{X})$ with the Dwyer-Friedlander topological K-theory spectrum of X by [31].

LEMMA 3.25. — *Let X be a separated noetherian scheme in which l is invertible. Suppose there is a uniform bound on the l-torsion etale cohomological dimension of all schemes etale over X; e.g., that X satisfies the hypotheses of 2.1 or 2.47. Let ε be a vector bundle of rank n on X, and $\mathbb{P}(\varepsilon)$ the associated projective space bundle. Let $[\mathcal{O}(-1)]$ be the image in $\pi_0 \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathbb{P}(\varepsilon))$ of the class of $\mathcal{O}(-1)$ in $\mathrm{Pic}(\mathbb{P}(\varepsilon))$ under the map of (3.41). Assume that if $l=2$, then either X is over $\mathbb{Z}[1/2, \sqrt{-1}]$ or else $\nu \geqq 2$. Then cup product with $([\mathcal{O}]-[\mathcal{O}(-1)])^i$ for $i=0, 1, \ldots, n-1$ defines a weak homotopy equivalence*

$$
(3.45)\qquad \bigvee_1^n \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{X}) \xrightarrow{\sim} \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathbb{P}(\varepsilon)).
$$

Proof. — Consider the map of presheaves of fibrant spectra on the etale site of X, which for U etale over X maps $\bigvee_1^n \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{U})$ to $\mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathbb{P}(\varepsilon|\mathrm{U}))$ as in (3.45). Applying $\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ )$ to this map of presheaves gives (3.45) up to weak equivalence by (3.42) and the descent theorem 1.48, 1.56, 1.57.

By 1.36, it suffices to show that this map of presheaves induces an isomorphism on sheaves of homotopy groups to deduce that $\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{X};\ )$ of it, and so (3.45), is a weak equivalence. It suffices to check this on stalks by 1.30. By 1.41 and 1.44, this amounts to showing that (3.45) is a weak equivalence if X is a noetherian strict local hensel ring.

If $\mathrm{X}=\mathrm{Spec}(\mathrm{R})$, R a noetherian strict local hensel ring with l^{-1}, then ε is a trivial vector bundle, and $\mathbb{P}(\varepsilon)$ is $\mathbb{P}^{n-1}_{\mathrm{R}}$. The standard calculation, [SGA 5], VII, 2.2.2 shows that

$$
(3.46)\qquad \mathrm{H}^*_{\mathrm{et}}(\mathbb{P}^{n-1}_{\mathrm{R}},\ \mathbb{Z}/l^{\nu}) = \mathbb{Z}/l^{\nu}[\mathrm{T}]/\mathrm{T}^n,
$$

where T has degree 2, and is the Chern class of $[\mathcal{O}]-[\mathcal{O}(-1)]$. The spectral sequence (3.43) for $\mathbb{P}^{n-1}_{\mathrm{R}}$ thus collapses as everything is in even bidegree. The E^{∞} term, and thus $\mathrm{K}/l^{\nu\,\mathrm{Top}}_*(\mathbb{P}^{n-1}_{\mathrm{R}})$ is a free module over $\mathbb{Z}/l^{\nu}[\beta, \beta^{-1}] = \mathrm{K}/l^{\nu\,\mathrm{Top}}_*(\mathrm{R})$ with basis $\mathrm{T}^i = ([\mathcal{O}]-[\mathcal{O}(-1)])^i$ for $i=0, 1, \ldots, n-1$. Thus (3.45) is a weak equivalence in this case. This completes the proof of the Theorem.

LEMMA 3.26 (Quillen). — *Let X be a separated noetherian scheme. Let ε be a vector bundle of rank n on X. If $l=2$, assume that either $\nu \geqq 2$ or else that X is over $\mathbb{Z}[\sqrt{-1}]$.*

Then cup product with the classes $([\mathcal{O}]-[\mathcal{O}(-1)])^i$ *for* $i=0, 1, \ldots, n-1$ *defines weak homotopy equivalences*

$$(3.47)\qquad \begin{cases} \bigvee_1^n \mathrm{K}(\mathrm{X}) \xrightarrow{\sim} \mathrm{K}(\mathbb{P}(\varepsilon)), \\ \bigvee_1^n \mathrm{K}/l^\nu(\mathrm{X}) \xrightarrow{\sim} \mathrm{K}/l^\nu(\mathbb{P}(\varepsilon)), \\ \bigvee_1^n \mathrm{K}/l^\nu(\mathrm{X})[\beta^{-1}] \xrightarrow{\sim} \mathrm{K}/l^\nu(\mathbb{P}(\varepsilon))[\beta^{-1}]. \end{cases}$$

Proof. — The first equivalence results from an easy change of basis and [97], § 8, 2.1. The others follow immediately.

LEMMA 3.27 (Splitting principle). — *Let* R *be a regular noetherian ring of finite Krull dimension in which* l *is invertible, and which satisfies the other hypotheses of* 2.45. *Let* M *be a finite simplicial complex, and* M′ *the associated scheme over* R *as in* 3.7. *Let* $\mathrm{P} \to \mathrm{M}'$ *be a scheme quasiprojective over* M′ *which is built by iteratively taking the associated projective space bundles of vector bundles. Let* ε *be a vector bundle on* P, *and let* $\mathrm{P}' = \mathbb{P}(\varepsilon) \to \mathrm{P}$ *be the associated projective space bundle.*

In diagram (3.48), *the indicated maps are weak homotopy equivalences. The vertical arrows are monomorphisms, which split naturally with respect to the horizontal arrows. In particular coker* $\pi_* \gamma(\mathrm{P})$ *maps to coker* $\pi_* \gamma(\mathrm{P}')$ *by a split monomorphism.*

$$(3.48)\qquad \begin{array}{ccccc} \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{P}') & \xrightarrow{\gamma(\mathrm{P}')} & \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{P}'; \mathrm{K}/l^\nu(\ )[\beta^{-1}]) & \xleftarrow{\sim} & \mathrm{K}/l^\nu(\mathrm{P}')[\beta^{-1}] \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{K}/l^{\nu\,\mathrm{Top}}(\mathrm{P}) & \xrightarrow{\gamma(\mathrm{P})} & \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{P}; \mathrm{K}/l^\nu(\ )[\beta^{-1}]) & \xleftarrow{\sim} & \mathrm{K}/l^\nu(\mathrm{P})[\beta^{-1}] \end{array}$$

Proof. — The left horizontal maps of (3.48) are weak equivalences by 3.20. The left and right vertical arrows are inclusions of the summand indexed by $[\mathcal{O}]=1=([\mathcal{O}]-[\mathcal{O}(-1)])^0$ by 3.25 and 3.26, and so are split by the projections off the summands indexed by $([\mathcal{O}]-[\mathcal{O}(-1)])^i$ for $i=1, 2, \ldots, n-1$. The map γ is natural with respect to this splitting as γ preserves the classes of the line bundles $[\mathcal{O}]$ and $[\mathcal{O}(-1)]$ by 3.23.

COROLLARY 3.28. — *Let* R *be a regular noetherian ring satisfying the other hypotheses of* 2.45. *Let* M *be a finite simplicial complex, and* M′ *the associated scheme over* R *as in* 3.7. *Let* ε *be a vector bundle on* M′, *and let* $p : \mathrm{Flag}(\varepsilon) \to \mathrm{M}'$ *be the complete flag bundle. Then* $p^* \varepsilon$ *has a canonical filtration by subbundles, with canonical line bundles as filtration quotients. Thus* $p^*[\varepsilon]$ *in* $\mathrm{K}_0(\mathrm{Flag}(\varepsilon))$ *is a sum of line bundle classes.*

Further, the left and right vertical maps in (3.49) *are monomorphisms which split naturally with respect to* γ. *Thus* p^* *induces a monomorphism of coker* $\pi_* \gamma(\mathrm{M}')$ *into coker* $\pi_* \gamma(\mathrm{Flag}(\varepsilon))$.

$$(3.49)\qquad \begin{array}{ccccc} K/l^{\nu\,\mathrm{Top}}(\mathrm{Flag}(\varepsilon)) & \xrightarrow{\gamma} & \mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{Flag}(\varepsilon);\, K/l^{\nu}(\ \ )[\beta^{-1}]) & \xleftarrow{\sim} & K/l^{\nu}(\mathrm{Flag}(\varepsilon))[\beta^{-1}] \\ \uparrow & & \uparrow & & \uparrow \\ K/l^{\nu\,\mathrm{Top}}(M') & \xrightarrow{\gamma} & \mathbb{H}^{\cdot}_{\mathrm{et}}(M';\, K/l^{\nu}(\ \ )[\beta^{-1}]) & \xleftarrow{\sim} & K/l^{\nu}(M')[\beta^{-1}] \end{array}$$

Proof. — The fact that $p^*[\varepsilon]$ splits is standard, for example *see* [SGA 6], VI, 4.7. It is also standard that $\mathrm{Flag}(\varepsilon)$ is built up from M′ by iteratively taking projective space bundles of vector bundles, *see* [SGA 6], VI, 4.2. Thus the assertions about (3.49) follow from an easy induction using 3.27.

LEMMA 3.29. — *Let* R *be a regular noetherian ring in which* l *is invertible, and which satisfies the other hypotheses of* 2.45. *Then the map* γ *of* 3.24 *is a surjection*

$$(3.50)\qquad \begin{array}{ccc} K/l^{\nu\,\mathrm{Top}}_{*}(R) & \xrightarrow{\qquad\gamma\qquad} & K/l^{\nu}_{*}(R)[\beta^{-1}] \\ \wr\| & & \wr\| \\ \pi_*\,\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{Spec}(R);\, \Sigma^{\infty}/l^{\nu}(BGL_1 \sqcup *)[b^{-1}]) & \to & \pi_*\,\mathbb{H}^{\cdot}_{\mathrm{et}}(\mathrm{Spec}(R);\, K/l^{\nu}(\ \ )[\beta^{-1}]) \end{array}$$

Proof. — Let x be an element of $\pi_* K/l^{\nu}(R)[\beta^{-1}]$. I will show that it is in the image of γ. As $b=\beta$ is a unit on both sides of (3.50), I may clear denominators from x and assume x is in the image of $\pi_n K/l^{\nu}(R)$. Multiplying by β again, I may assume that $n \geqq 2$. Let M be the mapping cone of a degree l^{ν} map from S^{n-1} to S^{n-1}. Then M is a mod l^{ν} Moore space. Let M′ be the corresponding scheme over R, as in 3.7. Under the isomorphism (3.31), x in $K/l^{\nu}_n(R)$ corresponds to an element x in $K_0(M')$. Under the isomorphism (3.33) and the canonical map $K_0(M') \to \pi_0 K/l^{\nu}(M')[\beta^{-1}]$, x corresponds to x in the summand $K/l^{\nu}_n(R)[\beta^{-1}]$ of $\pi_0 K/l^{\nu}(M')[\beta^{-1}]$. $\pi_0 K/l^{\nu\,\mathrm{Top}}(M')$ decomposes like $\pi_0 K/l^{\nu}(M')[\beta^{-1}]$ in (3.33). This follows as in 3.17 from 3.15 with $F = K/l^{\nu\,\mathrm{Top}}$. $K/l^{\nu\,\mathrm{Top}}$ has the required Mayer-Vietoris property by 3.14. The map γ respects the decompositions (3.33). Thus to show x is in the image of γ in (3.50), it suffices to show x in $\pi_0 K/l^{\nu}(M')[\beta^{-1}]$ is in the image of $\pi_0 K/l^{\nu\,\mathrm{Top}}(M')$ under γ.

Let the element in $K_0(M')$ corresponding to x be $[\varepsilon]-[\mathcal{O}^n]$, ε a vector bundle. Let $p : \mathrm{Flag}(\varepsilon) \to M'$ be the complete flag bundle corresponding to ε. Then $p^*(x)$ in $K_0(\mathrm{Flag}(\varepsilon))$ is a sum of line bundles. Thus $p^*(x)$ in $\pi_0 K/l^{\nu}(\mathrm{Flag}(\varepsilon))[\beta^{-1}]$ is a sum of elements in the image of $\mathrm{Pic}(\mathrm{Flag}(\varepsilon))$. By the commutative diagram of 3.23, $p^*(x)$ is thus a sum of elements in the image of γ, and so $p^*(x)$ is in the image of $\pi_0 K/l^{\nu\,\mathrm{Top}}(\mathrm{Flag}(\varepsilon))$ under γ. By the splitting principle 3.28, it follows that x in $\pi_0 K/l^{\nu}(M')[\beta^{-1}]$ is in the image of γ. This completes the proof.

3.30. The proof of 3.1 is now easily completed. By 3.3 and 3.5, it suffices to show $\pi_* \gamma$ is surjective if R is a strict local hensel ring satisfying the hypotheses of 2.45. This is done by 3.29.

3.31. The rough idea of the splitting principle has more applications than the particular incarnation above. The rough idea is this. Suppose one has a natural transformation

$\gamma_* : F_* \to K'_*$ which one wishes to prove surjective. Here K'_* is some kind of K-theory or something like it. By Mayer-Vietoris for F_* and K'_*, one reduces surjectivity of $\gamma : F_*(R) \to K'_*(R)$ to proving that γ hits a given element x in $K'_0(M')$, for some construction M'. One reduces to the case where x is the class of a vector bundle ε under a canonical map from the classical $K_0(M')$ to the given $K'_0(M')$. One proves a splitting principle to reduce the problem to $K'_0(\mathrm{Flag}(\varepsilon))$, and x a sum of line bundles. Then one is done if the canonical function $\mathrm{Pic} \to K_0 \to K'_0$ factors through γ.

Exercise 3.32. — Let R be a regular noetherian ring in which l is invertible, and which satisfies the other hypotheses of 2.45. Represent the mod l^ν Adams map A of A.14 by a simplicial map of mod l^ν Moore spaces

$$P_0 \xleftarrow{A} \Sigma^{l^{\nu-1}(l-1)} P_1 \xleftarrow{\Sigma^{l^{\nu-1}(l-1)}A} \Sigma^{2l^{\nu-1}(l-1)} P_2 \leftarrow \dots \tag{3.51}$$

Form the corresponding inverse system of schemes over R as in 3.7. For l odd, use A.14 to show that

$$K/l^\nu_*(R)[\beta^{-1}] = \varinjlim_i K_0(\Sigma^{il^{\nu-1}(l-1)} P_i)/K_0(R), \tag{3.52}$$

where the maps in the direct system are induced by A. This requires that P_0 be chosen to have dimension congruent to $* \bmod l^{\nu-1}(l-1)$.

Use (3.52) to reinterpret Theorem 4.11 in terms of an equivalence (3.53)

$$\varinjlim_i K_0(\Sigma^{il^{\nu-1}(l-1)} P_i)/K_0(R) \simeq K/l^{\nu\,\mathrm{Top}}_*(\mathrm{Spec}(R)). \tag{3.53}$$

Note that P_i over $\mathrm{Spec}(R)$ is induced by base-change from a universal one over $\mathrm{Spec}(\mathbb{Z})$. Use this to interpret P_i as a mod l^ν suspension of $\mathrm{Spec}(R)$. Interpret (3.53) as saying that topological K-theory is to algebraic K-theory as stable homotopy theory is to homotopy theory. Argue for the slogan: stable algebraic geometry is determined by topology.

Exercise 3.33 (Optional). — Let F be a functor from the category of noetherian schemes to that of spectra. Suppose that F satisfies the Mayer-Vietoris property for regular schemes, and satisfies the projective space bundle property 3.26. Suppose also that F/l^ν satisfies Mayer-Vietoris for the closed covers of 3.10, 3.11. Suppose finally that there is a natural transformation $F(\ ) \to K(\ )$ such that for R a regular local ring $\pi_1 F(R) \to \pi_1 K(R) = R^*$ and $\pi_0 F(R) \to \pi_0 K(R) = \mathbb{Z}$ are isomorphisms.

(*a*) Note by 2.5 that F has Zariski cohomological descent.

(*b*) Show the map (3.40) factors $\mathrm{Div}(\ ) \to F(\ ) \to K(\ )$.

Note that the analogs of (3.38) and (3.39) for $\mathbb{H}^\cdot_{\mathrm{Zar}}(\ ,\ )$ are true.

(*c*) Replacing $\mathbb{H}^\cdot_{\mathrm{et}}$ by $\mathbb{H}^\cdot_{\mathrm{Zar}}$ in the argument of paragraph 3, show that $F/l^\nu_*(R) \to K/l^\nu_*(R)$ is surjective for R regular.

(*d*) Use the fact that the image of Milnor K-theory in algebraic K-theory is a proper sub-group for a general field to refute the common expectation that there is a global theory F which extends Milnor K-theory of local rings and which has the good properties of algebraic K-theory.

Remark 3.34. — While I was finishing this paper, a stronger version of Theorem 3.1 was proved by combining results of Suslin, Gabber, Gillet, and myself. The end results are that by Suslin ([117], [118]), the groups $K/l^\nu_n(\bar{k})$ of a separably closed field of characteristic not l are $Z/l^\nu(i)$ if $n=2i\geqq 0$, and 0 for other n. By unpublished work of Gabber, or by Gillet-Thomason [45] in the geometric case, the groups $K/l^\nu_n(R)$ of a strict local Hensel ring R in which l is a unit are equal to the $K/l^\nu_n(k)$ of its separably closed residue field. Thus Theorem 3.1 is true in non-negative degrees without inverting β.

It follows that inverting β has no effect on $H^p_{et}(X; \tilde{K}/l^\nu_q)$ for $q\geqq 0$. Hence (2.120) and (2.126) may be rephrased as the existence of spectral sequences

$$(3.54)\qquad \begin{cases} E_2^{p,q}=H^p_{et}(X;\tilde{K}_q) \Rightarrow K_{q-p}(X)_K, & q-p\geqq 0,\\ E_2^{p,q}=H^p_{et}(X;\tilde{K}/l^\nu_q) \Rightarrow K/l^\nu_{q-p}(X)[\beta^{-1}], & q-p\geqq 0.\end{cases}$$

Thus inverting β is the cheapest possible price to pay to get etale cohomological descent.

Unfortunately, to get this in the non-geometric cases requires the general rigidity theorem of Gabber for Hensel pairs in mixed characteristic, and it seems unlikely that Gabber will ever write this up. (In proof: see Gabber's letter to Karoubi.)

4. Main results and examples of calculations; applications to zeta functions, Riemann-Roch, and cohomological purity

As a result of the labor in the other sections, I obtain:

THEOREM 4.1. — *Let* X *be a separated noetherian regular scheme of finite Krull dimension. Let* l^ν *be a prime power, and suppose* l *is invertible in* X. *Suppose there is a uniform bound on the l-torsion etale cohomological dimension of all residue fields of* X, *even at the non-closed points. Suppose that all residue fields of* X *admit a sequence of subfields like* (2.112), *e.g., that they are of finite transcendence degree over a local, global, or separably closed field. If* $l=2$, *assume that* X *contains a square root of* -1.

Then there is a strongly convergent spectral sequence (4.1) *with differentials* d_r *of bidegree* $(r, r-1)$

$$(4.1)\qquad E_2^{p,q}=\begin{Bmatrix} H^p_{et}(X;\mathbb{Z}/l^\nu(i)), & q=2i\\ 0, & q \text{ odd}\end{Bmatrix} \Rightarrow K/l^\nu_{q-p}(X)[\beta^{-1}].$$

Proof. — Combine 2.45 and 3.1.

Remark 4.2. — The hypotheses of 4.1 are met if X is any separated regular scheme of finite type over $\mathbb{Z}[1/l]$, or $\mathbb{Q}$, or $\mathbb{F}_p$ with $p\neq l$, or $\mathbb{F}_p[[t]]$ with $p\neq l$, or $\mathbb{F}_p((t))$ with $p\neq l$, or $\mathbb{Z}_p^\wedge$ with $p \neq l$, or $\mathbb{Q}_p^\wedge$, or over $\bar{k}$ a separably closed field of characteristic not l. If $l=2$, one must add the requirement that -1 has a square root in $\mathcal{O}_X$.

Remark 4.3. — In the spectral sequence (4.1), $E_r^{p,q} \neq 0$ implies that q is even. Hence only for odd r can be differential d_r be non-zero. Thus the spectral sequence (4.1) collapses if X has cohomological dimension at most 2, or if X has cohomology only in even dimensions.

Considering the action of the Adams operations on algebraic K-theory in a manner parallel to Soulé's paper [114], 3.2-3.3, one finds an integer $M(d)$ depending only on the etale cohomological dimension of X such that $M(d)\,d_r = 0$ for all r. Thus the spectral sequence (4.1) degenerates modulo torsion of bounded order independent of l and ν. Thus in situations where the l-adic cohomology in the sense of [SGA 5] is defined and is torsion free, the l-adic variant of (4.1) collapses, and hence so does (4.1).

Example 4.4. — Let $\bar{k}$ be a separably closed field of characteristic not l. Let C be a connected, proper, and smooth curve over $\bar{k}$ of genus g. Then

$$K/l^\nu_n(C)\,[\beta^{-1}] = \begin{cases} \mathbb{Z}/l^\nu \oplus \mathbb{Z}/l^\nu, & n \text{ even}, \\ \bigoplus^{2g} \mathbb{Z}/l^\nu, & n \text{ odd}. \end{cases} \tag{4.2}$$

If C′ is such a curve minus $k+1$ closed points, then

$$K/l^\nu_n(C')\,[\beta^{-1}] = \begin{cases} \mathbb{Z}/l^\nu, & n \text{ even}, \\ \bigoplus^{2g+k} \mathbb{Z}/l^\nu, & \text{n odd}, \end{cases} \tag{4.3}$$

Proof. — This follows from the collapsing spectral sequence (4.1) and standard calculations of the etale cohomology of a curve, e. g., [SGA 4], IX, 4.7.

Example 4.5. — Let R be the ring of integers in a number field. Assume that R contains primitive l^ν-th roots of 1, and a square root of -1 if $l=2$. Let k be the number of distinct primes of R lying over l. Then

(4.4)

$$K/l^\nu_n(R\,[l^{-1}])\,[\beta^{-1}] = \begin{cases} (\mathrm{Pic}(R\,[l^{-1}]) \otimes \mathbb{Z}/l^\nu) \oplus \bigoplus^{k} \mathbb{Z}/l^\nu, & n \text{ even}, \\ (GL_1(R\,[l^{-1}]) \otimes \mathbb{Z}/l^\nu) \oplus l^\nu\text{-torsion in } \mathrm{Pic}(R\,[l^{-1}]), & n \text{ odd}, \end{cases}$$

Proof. — This follows from the collapse of the spectral sequence (4.1) and the calculations of Artin and Verdier in [6]. Here GL_1 is the group of units and Pic is the ideal class group. The l^ν-torsion in the Brauer group of $R\,[l^{-1}]$ accounts for $k-1$ of the $\mathbb{Z}/l^\nu$ factors for n even.

Remark 4.6. — To compute $K/l^\nu_*(\ )\,[\beta^{-1}]$ for a curve C over $\mathbb{F}_q$ or for a general ring of integers in a number field, one adjoins all roots of unity, applies 4.4 or 4.5 to this cyclotomic extension, and then cuts down to the original case by the cohomological descent spectral sequence 2.48 with $\mathcal{A}$ the direct system of cyclotomic subextensions. The E_2 term of this spectral sequence is given by Galois cohomology of the cyclotomic Galois group acting on the groups $K/l^{\nu}_{*}(\ )[\beta^{-1}]$ of the extension. Information about this action is often contained in an L-function. In particular, one has:

Example 4.7. — Let X be a geometrically connected smooth curve over $\mathbb{F}_q$, with l prime to q. If $l=2$, assume 4 divides $q-1$. Let $\bar{X}=X\underset{\mathbb{F}_q}{\otimes}\bar{\mathbb{F}}_q$. The cohomology of the group $\mathbb{Z}^{\wedge}=\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the finite modules $K/l^{\nu}_{*}(\bar{X})[\beta^{-1}]$ is the same as the cohomology of $\mathbb{Z}$ by [102], I-15. Fitting the canonical two step resolution for computing the cohomology of $\mathbb{Z}$ ([21], X, 5) into the cohomological descent spectral sequence 2.48 yields a long exact sequence (4.5):

$$(4.5)\qquad \ldots \to K/l^{\nu}_{n}(\bar{X})[\beta^{-1}] \xrightarrow{1-\varphi} K/l^{\nu}_{n}(\bar{X})[\beta^{-1}] \to K/l^{\nu}_{n-1}(X)[\beta^{-1}] \to \ldots$$

Here φ is the arithmetic Frobenius, the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

By 4.4, the groups $K/l^{\nu}_{n}(\bar{X})[\beta^{-1}]$ are finite. Hence so are the groups $K/l^{\nu}_{n}(X)[\beta^{-1}]$. Thus the Mittag-Leffler condition is satisfied, and one may obtain an exact sequence of l-adic groups by taking inverse limits in (4.5) and appealing to (A.12).

Suppose now that X is a ring of integers in a function field, i.e., an affine curve satisfying the above conditions.

Then by the Riemann Hypothesis as proved by Weil for curves, $1-\varphi$ is injective for $n \geqq 2$ in the l-adic version of (4.5), and so it breaks up into short exact sequences

$$(4.6)\qquad \left\{ \begin{array}{l} 0 \to K_{n+1}(\bar{X})[\beta^{-1}]_l^{\wedge} \xrightarrow{1-\varphi} K_{n+1}(\bar{X})[\beta^{-1}]_l^{\wedge} \to K_n(X)[\beta^{-1}]_l^{\wedge} \to 0, \\ \qquad\qquad n \geqq 2. \end{array} \right.$$

Thus for $n \geqq 2$, the order of $K_n(X)[\beta^{-1}]_l^{\wedge}$ is the determinant of $1-\varphi$ as a number in $\mathbb{Z}_l^{\wedge}$. On the other hand, Grothendieck's formula for the L-function of X gives it as a ratio of determinants of $1-tF$, where F is the geometric frobenius acting on etale cohomology with compact supports. Comparing these by Poincaré duality, Theorem 4.1 linking $K/l^{\nu}_{*}(\ )[\beta^{-1}]$ with etale cohomology, and the relations between arithmetic and geometric frobenii yields the equality of l-adic valuations of the ratio of orders of K-groups and a value of the zeta function of X

$$(4.7)\qquad \left| \frac{\# K_{2i-2}(X)[\beta^{-1}]_l^{\wedge}}{\# K_{2i-1}(X)[\beta^{-1}]_l^{\wedge}} \right|_l = |\zeta(X, 1-i)|_l, \qquad i \geqq 2.$$

Example 4.8. — Let l be an odd prime. As $K/l^{\nu}_{*}(\mathbb{F}_l)=0$ for $*>0$ by [96], $K/l^{\nu}_{*}(\mathbb{F}_l)[\beta^{-1}]=0$. Then the localization sequence shows that $K/l^{\nu}_{*}(\mathbb{Z})[\beta^{-1}]$ and $K/l^{\nu}_{*}(\mathbb{Z}[l^{-1}])[\beta^{-1}]$ are isomorphic. These groups are given in terms of etale cohomology by the collapsing spectral sequence (4.1). The orders of certain of these cohomology groups are given in terms of the zeta function by the Main Conjecture proved by Mazur and Wiles. For partial results and more details one can consult the published works of Soulé, Coates, and Lichtenbaum, e.g. ([112], [113], [114], [65], [66]). The end result of this analysis is that for i even and greater than 1, the groups $K_{2i-2}(\mathbb{Z})[\beta^{-1}]_l^\wedge$ and $K_{2i-1}(\mathbb{Z})[\beta^{-1}]_l^\wedge$ are finite. The ratio of their orders is given by the zeta function and the i-th Bernoulli number B_i

$$\frac{\# K_{2i-2}(\mathbb{Z})[\beta^{-1}]_l^\wedge}{\# K_{2i-1}(\mathbb{Z})[\beta^{-1}]_l^\wedge} = |\zeta(\mathbb{Z};\, 1-i)|_l = \left|\frac{B_i}{i}\right|_l, \tag{4.8}$$

for $i \geqq 2$ even, l odd.

Since the groups $K_*(\mathbb{Z})$ are finitely generated, the surjectivity result of [30] and the equivalences (A.16) imply that there is an isomorphism for l odd and $n \geqq 0$:

$$K_n(\mathbb{Z})[\beta^{-1}]_l^\wedge \simeq K_n(\mathbb{Z})_K \otimes \mathbb{Z}_l^\wedge. \tag{4.9}$$

Here $K(\ )_K$ is the Bousfield localization of A.14. Thus modulo powers of 2, one has

$$\frac{\# K_{2i-2}(\mathbb{Z})_K}{\# K_{2i-1}(\mathbb{Z})_K} = \zeta(\mathbb{Z},\, 1-i), \qquad i \geqq 2, \text{ even.} \tag{4.10}$$

As $K(\ )_K$ is rationally the same as $K(\ )$ by (A.16), the groups $K_*(\ )_K$ give the same higher regulators ([8], [9]) as the K groups. The torsion in $K_*(\ )_K$ is related to etale cohomology and values of the zeta function in a reasonable way. And $K_*(\ )_K$ inherits most of the formal properties of $K_*(\ )$. It appears that $K(\ )_K$ is a more amenable replacement for $K(\ )$ in its conjectural role in number theory.

Example 4.9. — Let $\bar{k}$ be a separably closed field of characteristic not l. Let $X \subseteqq \mathbb{P}_{\bar{k}}^{m+1}$ be a smooth hypersurface of degree d and dimension m. Then if m is even

$$K_n(X)[\beta^{-1}]_l^\wedge = \begin{cases} \bigoplus^{m+1} \mathbb{Z}_l^\wedge \oplus \bigoplus^{(d-1)[(d-1)^{m+1}+1]/d} \mathbb{Z}_l^\wedge, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases} \tag{4.11}$$

If m is odd

$$K_n(X)[\beta^{-1}]_l^\wedge = \begin{cases} \bigoplus^{m+1} \mathbb{Z}_l^\wedge, & n \text{ even,} \\ \bigoplus^{(d-1)[(d-1)^{m+1}-1]/d} \mathbb{Z}_l^\wedge, & n \text{ odd.} \end{cases} \tag{4.12}$$

Proof. — By [SGA 7], XI, 1.6, the l-adic etale cohomology of a complete intersection such as X is torsion-free. Thus as in 4.3, the spectral sequence (4.1) collapses. Thus for X a complete intersection, $K_*(X)[\beta^{-1}]_l^\wedge$ is a direct sum of as many $\mathbb{Z}_l^\wedge$ as in the sum of the l-adic cohomology groups of X of the same parity as $*$. This rank is invariant under smooth deformation of X by smooth and proper base change. One deforms X until it lifts to characteristic 0, and by another appeal to smooth proper base change one reduces the computation of the rank to the case where X is over the complex numbers. The ranks are then given in terms of Hodge numbers of X. Appealing to the formulas of Hirzebruch for these ([54], 22.1.1, [SGA 7], XI, 2.3) and specializing them to our case yields the result.

Remark 4.10. — The method of 4.9 yields a calculation of $K_*(X)[\beta^{-1}]_l^\wedge$ for any smooth complete intersection over $\bar{k}$.

THEOREM 4.11. — *Let* X *be as in* 4.1. *Let* $K/l^{\nu\,\mathrm{Top}}(X)$ *denote the topological or "etale"* K*-theory spectrum of Dwyer and Friedlander* ([30], [31], [34], [35]). *Then the Dwyer-Friedlander map is a weak homotopy equivalence*

$$\rho:\ K/l^{\nu}(X)[\beta^{-1}] \xrightarrow{\sim} K/l^{\nu\,\mathrm{Top}}(X). \tag{4.13}$$

Proof. — For X a strict local hensel ring as in 4.1, the map ρ is a weak equivalence by 3.1 and 3.5. Thus on the etale site of a general X satisfying 4.1, ρ induces an isomorphism of sheaves of homotopy groups (4.14) by 1.44 and 1.30

$$\rho:\ K/l^{\nu}_{*}(\ \ )[\beta^{-1}] \xrightarrow{\cong} K/l^{\nu\,\mathrm{Top}}_{*}(\ \ ). \tag{4.14}$$

This deduction requires that the functor $K/l^{\nu\,\mathrm{Top}}_{*}(\ \ )$ is continuous on the etale site of X in the sense of 1.42. But the hypotheses of 4.1 and [SGA 4], X give a uniform bound on the l-torsion etale cohomological dimension of schemes etale over X. The continuity of $K/l^{\nu\,\mathrm{Top}}_{*}(\ \ )$ is then proved using the uniformly converging Atiyah-Hirzebruch spectral sequence of [30] as in the proof of 1.41 above.

Now consider the diagram (4.15)

$$\begin{tikzcd} K/l^{\nu}(X)[\beta^{-1}] \arrow[r, "\rho"] \arrow[d, "\wr"'] & K/l^{\nu\,\mathrm{Top}}(X) \arrow[d, "\wr"'] \\ \mathbb{H}^{\cdot}_{et}(X;\ K/l^{\nu}(\ \ )[\beta^{-1}]) \arrow[r, "\mathbb{H}_{et}(X;\ \rho)"'] & \mathbb{H}^{\cdot}_{et}(X;\ K/l^{\nu\,\mathrm{Top}}(\ \ )) \end{tikzcd} \tag{4.15}$$

The bottom map of (4.15) is a weak equivalence by (4.14) and the strongly converging spectral sequence of 1.36. The left vertical map is a weak equivalence by 4.1. The right vertical map should be a weak equivalence just because cohomology theories have cohomological descent. Technicalities force a slightly more complex proof of this, given in [31], Thm. 9.

From these facts, it follows that the top map of (4.15) is a weak equivalence as required.

Remark 4.12. — The Theorem concerns the non-connective version of topological K-theory, and not the truncated version sometimes considered.

In early versions of [30], $K/l^{\nu\,\mathrm{Top}}(X)$ and the map ρ were defined only for quasiprojective schemes over some base, or for affine schemes. The diagram (4.15) can be used to define these quantities in general given them on the affine schemes of the restricted etale site of X as in 1.3.

Remark 4.13. — For X an algebraic variety over the complex numbers, $K/l^{\nu\,\mathrm{Top}}(X)$ is equivalent to the classical mod l^{ν} topological K-theory spectrum of X as a space with the analytic topology, as implicit in [34], [35], or explicit in [36], 13.10.

Example 4.14. — Let $\bar{k}$ be a separably closed field of characteristic not l. Let G be a semisimple and simply connected algebraic group over $\bar{k}$. Let G have rank r.

Let $\Lambda^*[b_1, \ldots, b_r]$ be the $\mathbb{Z}/l^\nu$ exterior algebra on r generators of degree 1 corresponding to basic representations as in [55]. Consider this as a $\mathbb{Z}/2$-graded algebra with parts of even and odd degrees. Then

$$K/l^\nu_n(G)[\beta^{-1}] = \begin{cases} \Lambda^*[b_1, \ldots, b_r]_{\text{even}}, & n \text{ even}, \\ \Lambda^*[b_1, \ldots, b_r]_{\text{odd}}, & n \text{ odd}, \end{cases} \tag{4.16}$$

$$K/l^\nu_n(G)[\beta^{-1}] \simeq \bigoplus^{2^{r-1}} \mathbb{Z}/l^\nu \quad \text{all } n. \tag{4.17}$$

Proof. — By 4.1, [SGA 4], XVI, 1.6 = Lefschetz's principle for etale cohomology, and the not-quite-proper base change Theorem of [37], Thm. 1 or [36], 8.8, one reduces to the case where $\bar{k}$ is the complex numbers. By 4.13 and 4.11, it suffices to show that (4.16) gives the values of the topological K-groups of G. As the other factors in the Iwasawa decomposition of G are contractible, G is homotopy equivalent to its maximal compact subgroup. The result follows from Hodgkin's calculation of the K-theory of such a compact Lie group *in* [55]. Note as Hodgkin points out, the Atiyah-Hirzebruch spectral sequence does not always collapse in this case.

Remark 4.15. — Suppose X is not necessarily regular, but otherwise satisfies the hypotheses of 4.1. The functor $G/l^\nu(\ )[\beta^{-1}]$ on such X has etale cohomological descent by 2.47. By [97], § 7, # 2 augmented by [43], 4.1, G/l^ν and so $G/l^\nu(\ )[\beta^{-1}]$ is a covariant functor up to homotopy with respect to proper maps. One might expect then that $G/l^\nu(\ )[\beta^{-1}]$ is a sort of generalized Borel-Moore homology theory. However, there is no known definition of homology for such a broad class of schemes. Homology is defined in terms of cohomology and either Alexander duality or trivial duality with respect to cohomology with compact supports. These definitions require fixing a regular base scheme S and considering only schemes separated and of finite type over S. Under these circumstances, one has the expected comparison theorem. For example, *see* [127] and 4.16.

Remark 4.16. — Let X be a not necessarily regular scheme that satisfies the other hypotheses of 4.1. Let $X \subsetneq Y$ be a closed immersion, where Y is regular and satisfies the hypotheses of 4.1. Then one has a homotopy commutative diagram (4.18). The indicated maps are homotopy equivalences by 4.11. The first two rows are homotopy fibre sequences by Quillen's localization theorem. The last row is a fibre sequence by definition of $K/l^\nu_X{}^{\text{Top}}(Y)$

$$\begin{tikzcd}
G/l^\nu(X) \arrow[r] \arrow[dd] & G/l^\nu(Y) \arrow[r] \arrow[d, "\wr"'] & G/l^\nu(Y-X) \arrow[d, "\wr"'] \\
 & K/l^\nu(Y) \arrow[r] \arrow[d] & K/l^\nu(Y-X) \arrow[d] \\
G/l^\nu(X)[\beta^{-1}] \arrow[r] \arrow[d] & K/l^\nu(Y)[\beta^{-1}] \arrow[r] \arrow[d, "\wr"] & K/l^\nu(Y-X)[\beta^{-1}] \arrow[d, "\wr"] \\
K/l^\nu_X{}^{\text{Top}}(Y) \arrow[r] & K/l^{\nu\,\text{Top}}(Y) \arrow[r] & K/l^{\nu\,\text{Top}}(Y-X).
\end{tikzcd} \tag{4.18}$$

By the 5-Lemma, the lower left hand map is a weak homotopy equivalence

$$G/l^{\nu}(X)[\beta^{-1}] \xrightarrow{\sim} K/{}^{\nu}_{X}{}^{\mathrm{Top}}(Y). \tag{4.19}$$

This proves the claim of [127], 1.6. In [127], it was shown that $K/{}^{\nu}_{X}{}^{\mathrm{Top}}(Y)$ was independent of the choice of Y smooth over an affine base S satisfying 4.1, provided that all schemes considered are quasi-projective over S. On the category of such schemes, [127] shows that $K/{}^{\nu}_{X}{}^{\mathrm{Top}}(Y)=G/l^{\nu\,\mathrm{Top}}(X)$ is a reasonable definition of topological K-homology theory by Alexander duality. If S is the complex numbers, this is equivalent to the classical theory with "locally compact supports".

Remark 4.17. — For $f: X \to Y$ any proper map of schemes that are not necessarily regular, but otherwise satisfy the hypotheses of 4.1, one has the trivially homotopy commutative diagram (4.20)

$$\begin{array}{ccccc} G/l^{\nu}(X) & \longrightarrow & G/l^{\nu}(X)[\beta^{-1}] & = & G/l^{\nu\,\mathrm{Top}}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*[\beta^{-1}]} & & \downarrow{\scriptstyle f_*} \\ G/l^{\nu}(Y) & \longrightarrow & G/l^{\nu}(Y)[\beta^{-1}] & = & G/l^{\nu\,\mathrm{Top}}(Y) \end{array} \tag{4.20}$$

For general X, we take $G/l^{\nu}(X)[\beta^{-1}]$ as the definition of $G/l^{\nu\,\mathrm{Top}}(X)$. If X is regular, this is the usual $K/l^{\nu}(X)[\beta^{-1}]\simeq K/l^{\nu\,\mathrm{Top}}(X)$. For many singular X the definition is justified by [127] and 4.16, and in general has the expected formal properties.

This trivially commutative diagram is the Riemann-Roch theorem. The Riemann-Roch theorems of Gillet and of Shekhtman ([42], [106]) for the map from G/l^{ν}_{*} to etale cohomology may be easily deduced by purely topological calculations of the behavior of Chern classes under topological Gysin maps as in [32], I.D.2. In reasonable situations where one has an alternative definition of $G/l^{\nu\,\mathrm{Top}}(\ \ )$ and of the Gysin map, [127] shows that they agree with the current one.

The central application of the Riemann-Roch Theorem is to calculate Euler characteristics of coherent modules on X. This proceeds by taking Y a point and f the projection and noting that f_* sends the G_0 class of a module to its Euler characteristic. Thus one wants to calculate $f_*: G_0(X) \to G_0(Y)=\mathbb{Z}$. It suffices to do this mod l^{ν} for all ν. As $\mathbb{Z}/l^{\nu}=G/l^{\nu}_{0}(Y)$ injects into $G/l^{\nu\,\mathrm{Top}}_{0}(Y)$ for Y any field, it suffices to calculate $f_*: G/l^{\nu\,\mathrm{Top}}_{0}(X) \to G/l^{\nu\,\mathrm{Top}}_{0}(Y)$. But this can be done, essentially because the groups can be calculated in terms of topological information. This kind of application does not require comparison of f_* with any classical topological Gysin map *a priori*. Thus 4.1 without [127] gives a solution to the Riemann-Roch problem of calculating Euler characteristics of coherent modules in topological terms. In fact, the link between algebraic geometry and algebraic topology provided by 4.1 is both stronger and deeper than classical Theorems of Riemann-Roch type.

Remark 4.18. — If X and Y are regular schemes as in 4.1, and $X \to Y$ is a closed immersion, then 4.11 or 4.1 and Quillen's localization Theorem for algebraic K-theory yields a homotopy fibre sequence

$$K/l^{\nu\,\mathrm{Top}}(X) \to K/l^{\nu\,\mathrm{Top}}(Y) \to K/l^{\nu\,\mathrm{Top}}(X-Y). \tag{4.21}$$

This yields the long exact Gysin sequence for topological K-theory. This does not require the usual hypothesis that X and Y are a smooth pair over some base scheme S. Thus it is an absolute cohomological purity result for topological K-theory. By $\mathbb{Q}_l^\wedge$ degeneration of the Atiyah-Hirzebruch spectral sequence as in 4.3, this yields a proof of Grothendieck's absolute cohomological purity conjecture for $\mathbb{Q}_l^\wedge$-etale cohomology of schemes satisfying the hypotheses of 4.1. (*See* [130] for more details.)

5. Homotopy limits and homotopy colimits for spectra

In this section, I recount some basic results about spectra in the simplicial setting, and extend the results of Bousfield and Kan [16] to homotopy limits and colimits of diagrams of spectra. The results presented here are easy extensions of results in the references. I will assume the reader has some familiarity with simplicial homotopy theory and homotopy limits and colimits of simplicial sets as developed in Part II of [16].

I begin by recalling from Bousfield and Friedlander [14] some notions concerning spectra. Let S^n be the standard simplicial n-sphere, defined as an n-simplex with its boundary collapsed to a point for $n \geqq 1$, and as two points for $n=0$. Let $\mathrm{Map}_*(X, Y)$ be the function complex of based maps $X \to Y$ for pointed simplicial sets X and Y. $\mathrm{Map}(X, Y)$ will be the function complex of all maps.

For X a pointed simplicial set, let ΣX be $S^1 \wedge X$, and ΩX be $\mathrm{Map}_*(S^1, X)$. Σ is left adjoint to Ω. If X is fibrant so is ΩX, and then it is weak equivalent to the loop space on X.

DEFINITION 5.1. — A prespectrum X is a sequence of pointed simplicial sets X_n for non-negative integers n, together with structure maps $\Sigma X_n \to X_{n+1}$. A map of prespectra is a sequence of maps $f_n : X_n \to Y_n$ such that the obvious diagram involving Σf_n, f_{n+1}, and the two structure maps commutes on the nose.

The structure maps $\Sigma X_n \to X_{n+1}$ may be equally well described by their adjoints $X_n \to \Omega X_{n+1}$.

The category of prespectra is just what Bousfield and Friedlander call the category of spectra. They show it is a closed model category. Using their "stable" closed model category structure, one shows that all the usual results about topological prespectra can be translated into this simplicial setting. In particular, the homotopy category is the usual stable category. There are infinite loop space machines which take symmetric monoidal categories and manufacture spectra, just as in the topological case.

There is a particularly nice class of prespectra.

DEFINITION 5.2. — A fibrant spectrum is a prespectrum such that each X_n is a fibrant simplicial set, and the structure maps $X_n \to \Omega X_{n+1}$ are weak equivalences.

These objects are the fibrant Ω-spectra of [14]. There is a functor that associates to every prespectrum X a fibrant spectrum QX and a natural weak equivalence $X \to QX$. One version of Q replaces X_n by the direct limit of Kan complexes $\varinjlim_k \Omega^k \mathrm{Ex}^\infty X_{n+k}$. Thus prespectra may be replaced by equivalent fibrant spectra.

5.3. The homotopy groups of a prespectrum are given for all integers k as a direct limit of homotopy groups of the component simplicial sets

$$\pi_k X = \varinjlim_n \pi_{k+n} X_n. \tag{5.1}$$

The direct limit is over the system of bounding maps (5.2)

$$\pi_{k+n} X_n \to \pi_{k+n} \Omega X_{n+1} \xrightarrow{\cong} \pi_{k+n+1} X_{n+1}, \qquad n+k \geqq 0. \tag{5.2}$$

If X is a fibrant spectrum, $\pi_k X$ is isomorphic to the homotopy group of the zeroth space $\pi_k X_0$, provided k is non-negative. For k negative, $\pi_k X$ is $\pi_{k+n} X_n$ for any n so that $k+n$ is non-negative: still assuming that X is a fibrant spectrum.

5.4. A map $f: X \to Y$ of prespectra is a weak (homotopy) equivalence if it induces an isomorphism on homotopy groups. A map of fibrant spectra is a weak equivalence if each f_n is a weak equivalence of simplicial sets. For a map between connective fibrant spectra (those with 0 homotopy in negative degrees), this is the same as requiring that f_0 be a weak equivalance. A map f of fibrant spectra is a fibration if each f_n is a fibration of simplicial sets. A prespectrum is cofibrant if each structure map $\Sigma X_n \to X_{n+1}$ is a monomorphism of simplicial sets.

LEMMA 5.5. — *The category of fibrant spectra is closed under filtering colimits; e.g., direct limits. A filtering colimit of fibrations is a fibration; of weak equivalences is a weak equivalence.*

Proof. — $\Omega = \mathrm{Map}_*(S^1, \ )$ commutes with filtering colimits as S^1 is finite. The filtering colimit of fibrations or of weak equivalences of simplicial sets is a fibration or weak equivalence respectively. The homotopy groups of a filtering colimit of simplicial sets are the filtering colimits of the homotopy groups of the simplicial sets. With these observations, the proof is easy.

5.6. Let F be a functor from a small category **K** to the category of fibrant spectra. For each non-negative integer n, one has a diagram of n-th spaces of the spectra, F_n. For each object K of **K**, $F_n(K)$ is a fibrant simplicial set. Following Bousfield and Kan ([16], XI, 3.2) one forms the homotopy limit of this diagram, holim F_n. By [16], XI, 5, holim F_n is a fibrant simplicial set. Using the definition of holim as an end, i.e., as a function space of a diagram ([16], XI, 3.1, 3.3), and the definition of Ω as $\mathrm{Map}_*(S^1, \ )$, one easily shows that there is a natural isomorphism

$$\operatorname*{holim}_{\mathbf{K}} \Omega F_n \cong \Omega \operatorname*{holim}_{\mathbf{K}} F_n. \tag{5.3}$$

Using the fact that Ω preserves fibrant simplicial sets and [16], XI, 5.6, one sees that the weak equivalences $F_n \xrightarrow{\sim} \Omega F_{n+1}$ induce weak equivalences

$$\underset{\mathbf{K}}{\operatorname{holim}} F_n \xrightarrow{\sim} \underset{\mathbf{K}}{\operatorname{holim}} \Omega F_{n+1} \cong \Omega \underset{\mathbf{K}}{\operatorname{holim}} F_{n+1}. \tag{5.4}$$

Thus the holim F_n together with the structure maps (5.4) form a fibrant spectrum, holim F. This spectrum is the homotopy limit of the diagram of spectra F. I also denote it by $\mathbb{H}^{\cdot}(\mathbf{K}; F)$, and think of it as a fibrant spectrum valued hypercohomology of $\mathbf{K}$ with coefficients in the functor F. It is a covariant functor of F, and is contravariant in $\mathbf{K}$. It inherits all the basic properties of holim for simplicial sets, including the universal mapping property [16], XI, 3.4. (The reader may consult Gray, [48] for a discussion of this mapping property.) These basic properties may be checked on each component simplicial set holim F_n of the spectrum, using [16]. In particular, one has the following.

LEMMA 5.7. — *Let* F *be a functor from* $\mathbf{I} \times \mathbf{J}$ *into the category of fibrant spectra. Then there are natural isomorphisms:*

$$\underset{\mathbf{J}}{\operatorname{holim}} (J \to \underset{\mathbf{I}}{\operatorname{holim}} (I \to F(I, J))) \cong \underset{\mathbf{I}\times\mathbf{J}}{\operatorname{holim}} F \cong \underset{\mathbf{I}}{\operatorname{holim}} (I \to \underset{\mathbf{J}}{\operatorname{holim}} (J \to F(I, J))).$$

Proof. — [16], XI, 4.3, or the Fubini theorem for ends [48], [68], IX, 8.

LEMMA 5.8. — *Let* $F \to G$ *be a natural transformation of functors from* $\mathbf{K}$ *into the category of fibrant spectra. Suppose that for each* K *in* $\mathbf{K}$, *the map* $F(K) \to G(K)$ *is a weak equivalence. Then the induced map* holim $F \to$ holim G *is a weak equivalence.*

Proof. — [16], XI, 5.6.

LEMMA 5.9. — *Let* $F \to G$ *be a natural transformation of functors from* $\mathbf{K}$ *into the category of fibrant spectra. Suppose each* $F(K) \to G(K)$ *is a fibration. Then* holim $F \to$ holim G *is a fibration.*

Proof. — [16], XI, 5.5.

5.10. For any diagram F of prespectra, one may form a prespectrum holim F with structure maps (5.4). If F is not a diagram of fibrant spectra, Lemma 5.8 may fail. The important Proposition 5.13 may also fail. However, Lemma 5.7 remains true for such F. One also has:

LEMMA 5.11. — *For any fixed* $\mathbf{K}$, holim *considered as a functor from the category of diagrams of prespectra to the category of prespectra preserves limits. In particular, it preserves products, pullbacks, and kernals. Further* holim *preserves cotensors: if* X *is any based simplicial set and* $\mathrm{Map}_*(X, ?)$ *is the prespectrum of maps from* X *to a prespectrum* ? ([95], II, 1.3, Def. 3; [29], I, 2.1), *then there is a natural isomorphism*

$$\mathrm{Map}_*(X, \underset{\mathbf{K}}{\operatorname{holim}} F) \cong \underset{\mathbf{K}}{\operatorname{holim}} \mathrm{Map}_*(X, F). \tag{5.5}$$

Proof. — This reduces immediately to proving the analogue for diagrams of simplicial sets. Then it is an easy calculation from the definition of holim as an end; the essential point being that the function space, limit, and end constructions in the definition of holim all preserve limits and cotensors. Alternatively, one can use the universal mapping property of holims, [16], XI, 3.4, or [48]. The cotensor statement is the first remark of [16], XI, 7.6, this much requires no fibrancy.

LEMMA 5.12. — *Let* $\mathrm{G} \to \mathrm{H}$ *be a natural transformation of functors from* $\mathbf{K}$ *into the category of fibrant spectra. Suppose that for each* K *in* $\mathbf{K}$, $\mathrm{G}(\mathrm{K}) \to \mathrm{H}(\mathrm{K})$ *is a fibration with fibre* $\mathrm{F}(\mathrm{K})$. *Then* $\operatorname{holim} \mathrm{F}$ *is the fibre of the fibration* $\operatorname{holim} \mathrm{G} \to \operatorname{holim} \mathrm{H}$. *Thus,* holim *preserves fibre sequences, and even homotopy fibre sequences of fibrant spectra.*

Proof. — $\mathrm{F}(\mathrm{K})$ is a fibrant spectrum as $\mathrm{G}(\mathrm{K}) \to \mathrm{F}(\mathrm{K})$ is a fibration of fibrant spectra. The last statement follows from the rest by 5.8. The rest follows by 5.9 and 5.11.

The next proposition justifies thinking of $\operatorname{holim} \mathrm{F} = \mathbb{H}^{\cdot}(\mathbf{K}; \mathrm{F})$ as hypercohomology.

PROPOSITION 5.13. — *Let* F *be a functor from* $\mathbf{K}$ *to the category of fibrant spectra. Then there is a natural spectral sequence abutting to the homotopy groups of* $\operatorname{holim} \mathrm{F} = \mathbb{H}^{\cdot}(\mathbf{K}; \mathrm{F})$

$$\mathrm{E}_2^{p,q} \Rightarrow \pi_{q-p} \mathbb{H}^{\cdot}(\mathbf{K}; \mathrm{F}).$$

Here $\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathbf{K}; \pi_q \mathrm{F})$, $p \geqq 0$, $-\infty < q < \infty$ *is the cohomology of the category* $\mathbf{K}$ *with coefficients in the functor* $\pi_q \mathrm{F}$. *The groups* $\mathrm{E}_2^{p,q}$ *with* $p<0$ *are zero. The r-th differential is*

$$d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q+r-1}.$$

(*Note that this indexing is abnormal; the differentials don't go to the usual place!*) *The convergence of this spectral sequence is discussed below in* 5.44-5.48. *It converges strongly if* $\mathbf{K}$ *has finite cohomological dimension or if there is an* N *such that* $\pi_q \mathrm{F} = 0$ *for* $q > \mathrm{N}$.

Proof. — This spectral sequence is the direct limit of the spectral sequences for $\operatorname{holim} \mathrm{F}_n$ produced by Bousfield and Kan in [16], XI, § 7. The direct limit is over the system (5.4) with induced system of homotopy groups (5.2). There is a fringe effect and trouble with non-abelian low degree homotopy groups in the spectral sequence of [16], XI, § 7; in the direct system these troubles move progressively lower in degree and disappear in our limit. (*See* [20], § 3, Thm. 5, Remark 1.) Alternatively, one may derive the spectral sequence from a tower of fibrations of fibrant spectra as below, mimicking the construction in [16]. There is no fringe effect as the long exact sequence of homotopy groups associated to a fibre sequence of spectra can be continued into negative dimensional homotopy groups. These two approaches clearly yield the same spectral sequence. Proof of the convergence statments is deferred till 5.48.

5.14. The homology and cohomology of a category with coefficients in a functor F are the derived functors of inductive and projective limits evaluated at F. This generalizes the notion of the homology and cohomology of a group with coefficients in a module. The reader may consult [16], XI, 6, [97], § 1, p. 91, [SGA 4], [53], XI, 6, [63], [99], [136], if she is curious.

5.15. Let F be a functor from a small category **K** to the category of prespectra. Following Bousfield and Kan, [16], XII, § 2, one may form the homotopy colimit of the diagram of n-th spaces, hocolim F_n. One shows that suspension commutes with formation of homotopy colimits, so that there is an isomorphism

$$\Sigma \operatorname*{hocolim}_{\mathbf{K}} F_n \cong \operatorname*{hocolim}_{\mathbf{K}} \Sigma F_n. \tag{5.6}$$

The hocolim F_n and the structure maps (5.7) form a prespectrum, hocolim F:

$$\Sigma \operatorname*{hocolim}_{\mathbf{K}} F_n \cong \operatorname*{hocolim}_{\mathbf{K}} \Sigma F_n \to \operatorname*{hocolim}_{\mathbf{K}} F_{n+1}. \tag{5.7}$$

I also denote the prespectrum hocolim F by $\mathbb{H}_{.}(K; F)$ and think of it as the prespectrum valued hyperhomology of **K** with coefficients in F. This construction is covariant as a functor in **K** and in F. It has a universal mapping property. Let **n** be the category with objects 0, 1, 2, ..., n, and with a unique morphism $i \to j$ if $i \leqq j$. A functor $u : \mathbf{n} \to \mathbf{K}$ is equivalent to a string of n composible morphisms in **K**, $u(0) \to u(1) \to \ldots \to u(n)$. A compatible family of morphisms is a system of maps of prespectra

$$f(u): \quad F(u(0)) \ltimes \Delta[n] \to Z,$$

indexed by all non-negative integers n and all functors $u : \mathbf{n} \to \mathbf{K}$. The maps of prespectra must satisfy the condition that for all functors $\varphi : \mathbf{k} \to \mathbf{n}$,

$$f(u) . F(u(0)) \ltimes \Delta[\varphi] = f(u\varphi) . F(u(0 \to \varphi(0))) \ltimes \Delta[k].$$

There is a universal compatible family $j(u)$ into $\operatorname*{hocolim}_{\mathbf{K}} F$, and maps of prespectra $f: \operatorname*{hocolim}_{\mathbf{K}} F \to Z$ corresponding bijectively to compatible families $f(u)$ in such a way that $f(u) = f . j(u)$.

The reader may consult [125] for details. This reference contains many basic facts about homotopy colimits and shows how to build categorical models for them.

LEMMA 5.16. — *Let* F *be a functor from* $\mathbf{I} \times \mathbf{J}$ *into the category of prespectra. Then there are natural isomorphisms:*

$$\operatorname*{hocolim}_{\mathbf{J}} (\mathbf{J} \to \operatorname*{hocolim}_{\mathbf{I}} F(\mathbf{I}, \mathbf{J})) \cong \operatorname*{hocolim}_{\mathbf{I} \times \mathbf{J}} F \cong \operatorname*{hocolim}_{\mathbf{I}} (\mathbf{I} \to \operatorname*{hocolim}_{\mathbf{J}} F(\mathbf{I}, \mathbf{J})).$$

Proof. — [16], XII, 3.3, or [125], 3.5.

PROPOSITION 5.17. — *Let* F *be a functor from* **K** *to the category of prespectra. Let* E_* *be a homology theory. Then there is natural spectral sequence*

$$E^2_{p,q} = H_p(\mathbf{K}; E_q F) \quad \Rightarrow \quad E_{p+q} \mathbb{H}_{.}(\mathbf{K}; F), \qquad p \geqq 0, \quad -\infty < q < \infty.$$

The r-th differential is

$$d_r: \quad E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

The spectral sequence always converges completely, in that the $\tilde{E}^{p,q}_\infty$ *are filtration quotients of an exhaustive complete Hausdorff filtration of* $E_{p+q}\mathbb{H}_\bullet(\mathbf{K}; F)$. *To get strong convergence with filtrations of finite length, it suffices that* $\mathbf{K}$ *have finite homological dimension or that there is an* N *such that* $E_q F = 0$ *for* $q < N$.

Proof. — The spectral sequence is the direct limit over the system (5.7) of the spectral sequences of [16], XII, 5.7. Note as $H_*(\mathbf{K};\ )$ is the left derived functor of colimit along $\mathbf{K}$ and direct limit is an exact functor on the category of abelian groups; $H_*(\mathbf{K};\ )$ commutes with direct limits. Thus one has an isomorphism

$$\varinjlim_n H_*(\mathbf{K}; \tilde{E}_{q+n} F_n) \cong H_*(\mathbf{K}; \varinjlim_n \tilde{E}_{q+n} F_n) \cong H_*(\mathbf{K}; E_q F). \tag{5.8}$$

This justifies the description of the E^2 term. The convergence statements result from those of [16] and [10].

LEMMA 5.18. — *Let* $F \to G$ *be a natural transform of functors from* $\mathbf{K}$ *into the category of prespectra. Suppose that for each* K *in* $\mathbf{K}$, *that* $F(K) \to G(K)$ *is a weak equivalence of prespectra, i.e., it induces an isomorphism on homotopy groups. Then the map* hocolim $F \to$ hocolim G *is a weak equivalence.*

Proof. — First note that the maps $F_n(K) \to G_n(K)$ needn't be homotopy equivalences until one passes to the limit in n, so this Lemma can't be deduced from the analogous Lemma for simplicial sets.

However if there is an N such that for all K, $\pi_q F(K) = 0$ and $\pi_q G(K) = 0$ for all $q < N$, the claimed result follows from comparison of the strongly convergent spectral sequences for stable homotopy given by Proposition 5.17 with $E_* = \pi_*$.

The connective Postnikov tower of 5.53 below, shows that $F \to G$ is the direct limit as N goes to $-\infty$ of weak equivalences of prespectra satisfying the above condition, $F\rangle N \langle \to G \rangle N \langle$. The general result follows from the special case above by passage to the limit, using 5.16 and the fact that direct limits are homotopy colimits (e.g., [16], XII, 3.5, or [125], 3.8) to show hocolim commutes with direct limits.

LEMMA 5.19. — *Let* $F \to G \to H$ *be a sequence of natural transformations of functors from* $\mathbf{K}$ *into the category of prespectra. Suppose for each* K, $F(K) \to G(K) \to H(K)$ *is a homotopy fibre or cofibre sequence naturally in* K, *i.e., there is a natural transformation from* F *into the homotopy fibre of* $G \to H$ *or from* H *into the homotopy cofibre of* $F \to G$ *which is a weak equivalence for each* K. *Then* hocolim F $\to$ hocolim G $\to$ hocolim H *is a homotopy fibre and cofibre sequence.*

Proof. — By Lemma 5.16, hocolim commutes with formation of the canonical homotopy cofibre, as this is a homotopy colimit (e.g., [125], 3.7). Then by 5.17, hocolim preserves homotopy cofibre sequences. It also preserves homotopy fibre sequences, as these are the same as homotopy cofibre sequences for prespectra (e.g. [1], Part III, 3.10).

LEMMA 5.20. — *Let* F *be a functor from* **K** *into the category of prespectra. Let* Z *be a prespectrum. Then there is a natural isomorphism* (5.9)

$$\mathrm{Z} \wedge \operatorname*{hocolim}_{\mathbf{K}} \mathrm{F} \cong \operatorname*{hocolim}_{\mathbf{K}} \mathrm{Z} \wedge \mathrm{F}. \tag{5.9}$$

Proof. — Use the fact that the functor $\mathrm{Z} \wedge ?$ is left adjoint to $\mathrm{Map}_*(\mathrm{Z}, ?)$ and the universal mapping property of hocolim, 5.15 or [125] 3.13, to show that the two sides of (5.9) have the same universal mapping property, and so are isomorphic.

5.21. Let Δ^{op} be the category whose diagrams are simplicial objects; i.e., the opposite category of the skeletal category of finite ordered sets $\mathbf{n} = \{0, 1, 2, \ldots, n\}$. Let F be a simplicial prespectrum; i.e., a functor from Δ^{op} into the category of prespectra. Then each F_n is a simplicial set, and has a diagonalization $|\mathrm{F}_n|$. These fit together to form a prespectrum as n varies, $|\mathrm{F}|$. By [16], XII, 3.4, there is a natural weak equivalence of prespectra

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathrm{F} \xrightarrow{\sim} |\mathrm{F}|. \tag{5.10}$$

Let $\Delta^{+\mathrm{op}}$ be the opposite category of the skeletal category of based finite total orders $\mathbf{n} = \{-1, 0, 1, \ldots, n\}$ for $n = -1, 0, 1, \ldots$ The set $\mathbf{n}$ has basepoint -1. There is a canonical inclusion $\Delta^{\mathrm{op}} \to \Delta^{+\mathrm{op}}$ sending $\mathbf{n}$ to $\mathbf{n}$. A functor F from $\Delta^{+\mathrm{op}}$ to the category of prespectra consists of a simplicial prespectrum $\{\mathrm{F}(n), n \geqq 0\}$ augmented to a prespectrum $\mathrm{F}(-1)$, and with extra degeneracies s_{-1} that exhibit $\mathrm{F}(-1)$ as a simplicial deformation retract of the simplicial prespectrum. In this case there is an augmentation which is a weak equivalence of prespectra

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathrm{F} \xrightarrow{\sim} |\mathrm{F}| \xrightarrow{\sim} \mathrm{F}(-1). \tag{5.11}$$

This follows for instance from the spectral sequence 5.17. The homology $\mathrm{H}_*(\Delta^{\mathrm{op}}; \pi_q \mathrm{F})$ is the homology of a chain complex corresponding to the simplicial abelian group $\pi_q \mathrm{F}$ by [16], XII, 5.6, and [71], § 22. The extra degeneracies s_{-1} combine with the usual ones to give a simplicial homotopy that contracts this chain complex to $\pi_q \mathrm{F}(-1)$. Then in the spectral sequence of 5.17, $\mathrm{E}^2_{p,q} = 0$ if $p \neq 0$, while $\mathrm{E}^2_{0,q} = \pi_q \mathrm{F}(-1)$. The spectral collapses, and so converges strongly by 5.17.

Alternatively, one may show that the simplicial deformation retraction of F down to $\mathrm{F}(-1)$ induces a deformation retraction of $|\mathrm{F}|$ down to $\mathrm{F}(-1)$, as in [73], 11.10.

For an explicit description of $\Delta^{+\mathrm{op}}$ *see* [125], pp. 1597-1598. The formula for the deformation retraction in the notation of [125] and [73], 9.1 is $h_i = (s_{-1})^{i+1} (d_0)^i$.

5.22. The other construction I want to steal from [16] is that of the total spectrum of a cosimplicial fibrant spectrum. Let Δ be the skeletal category of finite ordered sets and monotone maps. A cosimplicial object in a category is a functor from Δ to that category. Equivalently, a cosimplicial object is a sequence of objects X^n for each $n=0, 1, 2, \ldots$ in Δ, with coface and codegeneracy maps

$$d^i : \ X^{n-1} \to X^n, \qquad s^i : \ X^{n+1} \to X^n, \qquad 0 \leqq i \leqq n.$$

These must satisfy the cosimplicial identities:

$$\begin{gathered} d^j d^i = d^i d^{j-1}, \qquad i<j, \\ s^j d^i = \begin{cases} d^i s^{j-1}, & i<j, \\ 1, & i=j, j+1, \\ d^{i-1} s^j, & i>j+1, \end{cases} \\ s^j s^i = s^{i-1} s^j, \qquad i>j. \end{gathered} \tag{5.12}$$

Definition 5.23. — A cosimplicial prespectrum is a cosimplicial fibrant spectrum if each X^l is a fibrant spectrum. It is a fibrant cosimplicial fibrant spectrum if in addition each cosimplicial n-th space X_n is a fibrant cosimplicial simplicial set in the sense of [16], X, 4.6.

Definition 5.24. — For X a cosimplicial prespectrum, the total prespectrum Tot X is the prespectrum whose n-th space is the Bousfield-Kan total complex of the cosimplicial space X_n, (*see* [16], X, 3). The structure maps are the composites of the maps induced by $X_n \to \Omega X_{n+1}$ and the canonical isomorphism

$$\operatorname{Tot}(\Omega X_{n+1}) = \operatorname{Tot}(\operatorname{Map}_*(S^1, X_{n+1})) \cong \operatorname{Map}_*(S^1, \operatorname{Tot} X_{n+1}) = \Omega \operatorname{Tot} X_{n+1}. \tag{5.13}$$

A map $X \to Y$ of cosimplicial fibrant spectra is a fibration if for each n, $X_n \to Y_n$ is a fibration of cosimplicial spaces in the sense of [16], X, 4.6. A map of cosimplicial fibrant spectra is a weak equivalence if each $X_n \to Y_n$ is a weak equivalence in the sense of [16], that is, if each $X_n^l \to Y_n^l$ is a weak equivalence of simplicial sets. This condition is equivalent to each $X^l \to Y^l$ being a weak equivalence of fibrant spectra.

Lemma 5.25. — *For* X *a cosimplicial prespectrum, there is a natural map of prespectra*

$$\operatorname{Tot} X \to \operatorname*{holim}_{\Delta} X = \mathbb{H}^{\cdot}(\Delta; X). \tag{5.14}$$

If X *is a fibrant cosimplicial fibrant spectrum, this map is a weak equivalence.*

Proof. — [16], XI, 4.4.

5.26. An augmentation of a cosimplicial prespectrum X is a prespectrum Z and a map $\varepsilon : Z \to X^0$ such that $d^0 \varepsilon = d^1 \varepsilon$. Equivalently, one may regard Z as a cosimplicial prespectrum where all s^i and d^i are identity maps, then an augmentation is a cosimplicial map $\varepsilon : Z \to X$. Then $Z = \operatorname{Tot} Z$ and the map (5.14) with $\operatorname{Tot} \varepsilon$ induces an augmentation

$$Z = \operatorname{Tot} Z \to \operatorname{Tot} X \to \operatorname*{holim}_{\Delta} X. \tag{5.15}$$

Lemma 5.27. — *The functor* Tot *preserves limits. Further* Tot *preserves the cotensor* $\operatorname{Map}_*(X, \ )$ *for any space* X.

Proof. — This immediately reduces to the analogous properties of Tot on the category of cosimplicial simplicial sets. These follow from the definition in [16] of Tot as an end or function space, as in Lemma 5.11.

LEMMA 5.28. — *Let* X^{**} *be a bicosimplicial prespectrum. Then there is a natural isomorphism*

$$\mathrm{Tot}(p \to \mathrm{Tot}(q \to X^{pq})) \cong \mathrm{Tot}(q \to \mathrm{Tot}(p \to X^{p,q})).$$

Proof. — Ends commute ([68], IX, 8) and Tot preserves cotensors. The result follows from these observations and the construction of Tot in [16].

PROPOSITION 5.29. — *Let* X *be a fibrant cosimplicial fibrant spectrum. Then there is a natural half-plane spectral sequence abutting to* $\pi_* \mathrm{Tot}\, X$

$$E_1^{p,q} \quad \Rightarrow \quad \pi_{q-p} \mathrm{Tot}\, X.$$

The groups $\pi_q X^*$ *form a cosimplicial abelian group, and the* E_1 *term is given by*

$$E_1^{p,q} = \begin{cases} \pi_q X^p \cap \ker s^0 \cap \ker s^1 \cap \ldots \cap \ker s^{p-1} & \textit{for} \quad p \geqq 0, \\ 0, \quad p<0. \end{cases}$$

The indexing is peculiar, so differentials run

$$d_r: \quad E_r^{p,q} \to E_r^{p+r,\, q+r-1}.$$

Convergence is discussed below in 5.44-5.48. *If there exists an* N *such that* $\pi_q X^p = 0$ *for* $q > N$, *then the spectral sequence converges strongly.*

Proof. — As does 5.13, this follows from the analogous Theorems in [16], especially [16], X, § 6.

5.30. The spectral sequence of 5.29 specializes to that of 5.13 under "cosimplicial replacement of diagrams", as in [16], XI 7, XI 5. On the other hand, it may be deduced from the spectral sequence of a tower of fibrations 5.43, as in [16], X, 6.1.

LEMMA 5.31. — *In the spectral sequence of* 5.29, $E_2^{p,q}$ *is the p-th cohomology group of the cochain complex*

$$\pi_q X^0 \xrightarrow{\partial} \pi_q X^1 \xrightarrow{\partial} \pi_q X^2 \xrightarrow{\partial} \ldots, \tag{5.16}$$

where

$$\partial: \quad \pi_q X^n \to \pi_q X^{n+1} \qquad \textit{is} \quad \sum_{i=0}^{n} (-1)^i \pi_q (d^i).$$

This is also true for the E_2 *term of the spectral sequence of* 5.13 *for* holim *along* Δ.

Proof. — This in none other than [16], X, 7.2. The equivalence of the cohomology of the normalized and unnormalized chain complexes of [16], X, 7.1 is obtained from [71], 22.3 applied to simplicial objects in the opposite category of abelian groups, then dualized to a statement about cosimplicial abelian groups. Recall that dualizing an abelian category converts sums to intersections and images to kernals.

The last statement follows by compatibility of the two spectral sequences, [16], XI, 7.5. (Note it is not necessary to assume that X is a *fibrant* cosimplicial fibrant spectrum to apply the last statement of Lemma 5.31 to 5.13.)

Scholium of Great Enlightenment 5.32. — The category of simplicial abelian groups and simplicial homomorphisms is equivalent to the category of non-negatively graded chain complexes as was proved by Kan and Dold ([26], [61], [22], 5.3, [71], 22.4). (Here chain complex means topologist's and not algebraic geometer's chain complex, so ∂ *lowers* degrees by 1.) Under this equivalence a simplicial abelian group A_* corresponds to the chain complex (5.17)

$$(5.17) \qquad \dots \xrightarrow{d_0-d_1+d_2-d_3} A_2/\mathrm{im}\, s_0+\mathrm{im}\, s_1 \xrightarrow{d_0-d_1+d_2} A_1/\mathrm{im}\, s_0 \xrightarrow{d_0-d_1} A_0.$$

The homotopy groups of the simplicial abelian group correspond to the homology groups of the chain complex ([71], 22.3). A simplicial abelian group may be delooped by applying the bar construction, yielding another simplicial abelian group. Thus a simplicial abelian group gives rise to a connective spectrum of simplicial abelian groups. Note that simplicial groups are always fibrant, and Ω preserves products, and so preserves simplicial abelian groups. The category of spectra of simplicial abelian groups and homomorphisms is equivalent to the category of chain complexes which are not necessarily 0 in negative degrees. Up to homotopy, looping and delooping an abelian group spectrum corresponds to shifting the gradings of the chain complex up or down. The homotopy groups of the abelian group spectrum are isomorphic to the homology groups of the corresponding chain complex. The homotopy category formed by inverting the weak equivalences of abelian group spectra is the category formed by inverting homology isomorphisms of chain complexes. This is the derived category of abelian groups, the natural home of homological algebra as formulated by Verdier. This category is treated in [93], [50], [SGA 4 1/2] C.D., and is used extensively by Grothendieck and his school. Quillen was led to the idea of a closed model category [95] in developing an analogous formalism for unstable homotopy theory, with the intent to develop a homotopy theory for commutative algebras for use in deformation theory [56].

By Lemmas 5.11 and 5.27 holim and Tot preserve products and zero, and so preserve abelian group spectra. A simplicial abelian group is always fibrant; similarly a cosimplicial abelian group spectrum is always a fibrant cosimplicial fibrant spectrum by [16], X, 4.9. It is natural to ask what the constructions holim and Tot become when restricted to the derived category.

For $\mathbf{K}$ a small category, a functor F from $\mathbf{K}$ into the category of abelian group spectra is a functor from $\mathbf{K}$ into the category of chain complexes. Then holim $F = \mathbb{H}^{\bullet}(\mathbf{K}; F)$ is the chain complex which is the evaluation at F of the total right derived functor of the limit along $\mathbf{K}$. The k-th homotopy group of $\mathbb{H}^{\bullet}(\mathbf{K}; F)$ is the $-k$ hypercohomology group of $\mathbf{K}$ with coefficients in the system F. The spectral sequence of 5.13 is a reindexing of the hypercohomology spectral sequence. This is usually written as

$$E_2^{p,q} = H^p(\mathbf{K}; H^q(F)) \Rightarrow H^{p+q}(\mathbf{K}; F), \tag{5.18}$$

with $H^*(F)$ the cohomology of F considered as a cochain complex by reindexing. In terms of the homology of the chain complex this is

$$H^{-q}(F) = H_q(F) = \pi_q(F). \tag{5.19}$$

Under this identification, the differentials and grading in the spectral sequence 5.13 translate into the usual form for the hypercohomology spectral sequence.

In particular, for $\mathbf{K} = \mathbf{G}$ a group, F is a chain complex on which G acts, $\pi_* \mathbb{H}^{\cdot}(G; F)$ is the hypercohomology of G with coefficients in F, and 5.13 is the usual group hypercohomology spectral sequence.

The situation with respect to Tot is similar. Dually to the theorem that the category of simplicial objects in an abelian category is equivalent to the category of non-negative chain complexes there, one has that the category of cosimplicial objects is equivalent to that of non-negative cochain complexes. The category of cosimplicial abelian group spectra is equivalent to a category of bicomplexes. The functor Tot corresponds to a functor sending a bicomplex to its total complex. The spectral sequence 5.29 becomes the usual spectral sequence for the homology of a bicomplex.

The homotopy colimit along $\mathbf{K}$ is up to homotopy taking hyperhomology of $\mathbf{K}$. The details of the correspondence here are a bit awkward.

Thus holim, Tot, and hocolim in the category of prespectra are generalizations of the fundamental constructions of homological algebra in the category of abelian spectra. The properties I have noted above are generalizations of the fundamental results of homological algebra. This generalization of homology theory for chain complexes to homotopy theory for spectra allows one to use homological ideas in algebraic K-theory. The possibility of using homotopy theory as a substitute for homological techniques in K-theory was the reason algebraic topology was introduced into algebraic K-theory by Swan and Quillen ([120], [97]) when chain complex techniques failed to yield reasonable higher K-groups. This point of view was anticipated by Dold and Puppe [27].

The homotopy category of spectra is a triangulated category ([92], [93], [52]), in fact it was the original example of a triangulated category given by Puppe. Because much of the Verdier treatment of homological algebra works in any triangulated category, it applies directly to the stable category. Using [14] and [16], one may show holims and hocolims induce the appropriate total right and left derived functors on the stable category. This shows such derived functors exist, which is the only thing in [SGA 4 1/2] C.D. that may fail for a general triangulated category. This captures the substance of homological algebra. The long exact homotopy sequence of a homotopy fibre sequence and the spectral sequences 5.13, 5.17, and 5.29 allow one to display this substance in its usual clothing of abelian categories and exact sequences.

LEMMA 5.33 (Diagonalization Lemma). — *Let* $F : \Delta \times \Delta \to \text{Spectra}$ *be a bicosimplicial fibrant spectrum. Let* $d : \Delta \to \Delta \times \Delta$ *be the diagonal, and let* $\operatorname{diag} F$ *be the cosimplicial fibrant spectrum* $F d$. *Then* d *induces a natural weak equivalence*

$$\underset{\Delta \times \Delta}{\operatorname{holim}} F \xrightarrow{\sim} \underset{\Delta}{\operatorname{holim}} (\operatorname{diag} F). \tag{5.20}$$

Combined with 5.7, *this yields a natural weak equivalence*

$$\underset{\Delta}{\operatorname{holim}} (p \to \underset{\Delta}{\operatorname{holim}} (q \to F(p, q))) \xrightarrow{\sim} \underset{\Delta}{\operatorname{holim}} (p \to F(p, p)). \tag{5.21}$$

Proof. — As usual, the result will follow once the analogue for cosimplicial fibrant simplicial sets is proved. By appeal to [16], XI, 9.2, it suffices to show that $d : \Delta \to \Delta \times \Delta$ is left cofinal in the sense of [16]. This is, I must show that for each object (p, q) of $\Delta \times \Delta$ that the comma category $d/(p, q)$ of d objects over (p, q) is contractible. An object of this category is an n in Δ and a pair of morphisms $n \to p$, $n \to q$ in Δ. A morphism in the comma category is a morphism $m \to n$ is Δ compatible with the maps to p and q. The objects may be identified to maps $\Delta[n] \to \Delta[p] \times \Delta[q]$; i.e., to the simplices of $\Delta[p] \times \Delta[q]$. A morphism is a simplicial operator carrying one simplex to another. Thus $d/(p, q)$ is the category of simplices of $\Delta[p] \times \Delta[q]$. By a Theorem of Quillen, [56], V, 3.3, the nerve of this category is weak equivalent to $\Delta[p] \times \Delta[q]$, and so is contractible. Alternatively, in the terminology of [123], $d/(p, q)$ is the Grothendieck construction $\Delta^{op} \int G$, where $G : \Delta^{op} \to \underline{\text{Cat}}$ sends n to the category with only identity maps and whose objects are the n-simplices of $\Delta[p] \times \Delta[q]$. Then the nerve $NG(n)$ is the constant simplicial set with one vertex for each n-simplex of $\Delta[p] \times \Delta[q]$. By [123], 1.2 and [16], XII, 4.3 there are weak equivalences

$$N(d/(p, q)) \cong N\left(\Delta^{op} \int G\right) \simeq \underset{\Delta^{op}}{\operatorname{hocolim}} NG \simeq \operatorname{diag} NG \cong \Delta[p] \times \Delta[q] \simeq *. \tag{5.22}$$

5.34. I need certain facts about mapping spectra. For X a prespectrum and N an integer, then $X[N]$ is the shifted prespectrum whose n-th space is

$$X[N]_n = X_{N+n}, \tag{5.23}$$

together with the obvious structure maps. The prespectrum $X[N]$ is weak equivalent to $\Omega^{-N} X$ and $\Sigma^N X$ for any integer N. For X, Y prespectra let $\operatorname{Map}_*(X, Y)$ be the prespectrum whose n-th space is the Bousfield-Friedlander simplicial set of maps of prespectra ([14], 2.1), $\operatorname{Map}_*(X, Y[n])$. A p-simplex of this simplicial set is a map of prespectra $X \wedge (\Delta[p] \sqcup *) \to Y[n]$. The structure maps

$$\operatorname{Map}_*(X, Y[n]) \to \Omega \operatorname{Map}_*(X, Y[n+1]), \tag{5.24}$$

are induced by the obvious weak equivalences of prespectra $Y[n] \to \Omega Y[n+1]$. The mapping prespectrum $\operatorname{Map}_*(X, Y)$ is a covariant functor of Y and a contravariant functor of X. If Z is a based simplicial set, there is a natural enriched adjunction isomorphism

$$(5.25) \qquad \mathrm{Map}_*(X \wedge Z, Y) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Z, Y)).$$

If X is a cofibrant prespectrum and Y is a fibrant spectrum, then $\mathrm{Map}_*(X, Y)$ is a fibrant spectrum whose π_0 is the set of maps from X to Y in the stable category. All these assertions are easy consequences of the corresponding results for simplicial mapping spaces of prespectra in [14].

PROPOSITION 5.35 (Universal coefficient Theorem). — *Let* F *be a functor from the small category* **K** *to the category of cofibrant prespectra. Let* Z *be a prespectrum. There is a natural isomorphism of prespectra, which are fibrant spectra if* Z *is:*

$$(5.26) \qquad \mathrm{Map}_*(\mathbb{H}_\bullet(\mathbf{K}; F), Z) \cong \mathbb{H}^\bullet(\mathbf{K}^{op}, \mathrm{Map}_*(F; Z)).$$

Proof. — This follows by comparing the universal mapping property of a homotopy limit to the dual universal mapping property of a homotopy colimit in light of the adjointness of $\wedge F$ and $\mathrm{Map}_*(F, \ )$. Alternatively, it follows by adapting the proof of the analogous theorem for simplicial sets, [16], XII, 4.1.

5.36. In the Lemmas on Tot, the hypothesis that X is a fibrant cosimplicial fibrant spectrum occurs repeatedly. Tot does not have good homotopy properties for X not satisfying this condition. I'll now show how to get X that do satisfy it.

LEMMA 5.37. — *If* X *is a group object in the category of cosimplicial fibrant spectra, it is a fibrant cosimplicial fibrant spectrum. If* $f: X \to Y$ *is a homomorphism of group objects, and for each* k *and* n *the map of simplicial groups* $X_n^k \to Y_n^k$ *is a surjection, then* f *is a fibration of cosimplicial fibrant spectra.*

Proof. — This follows from [16], X, 4.9.

LEMMA 5.38. — *There is a functor which associates to each fibrant spectrum* X *a group object in the category of spectra* X′. *There is a sequence of functors and natural weak equivalences of fibrant spectra connecting* X *to* X′.

Proof. — The corresponding result in Kan's category of simplicial spectra was proved by Kan as Corollary 5.6 in [62]. The Lemma follows using the chain of equivalences between the Bousfield-Friedlander category of prespectra and Kan's category, [14], 2.5. As right adjoint functors preserve products and the terminal object, they preserve group objects. Geometric realization also preserves finite products and the terminal object, so it preserves group objects. Thus the functors of [14] take a group object in Kan's category to a group object in the Bousfield-Friedlander category of prespectra. As all the adjunction maps in [14], 2.5 are weak equivalences in the stable sense of 5.4, if one takes a prespectrum X, pushes it into Kan's category, converts it into a group spectrum, and then brings it back, the end result is connected to the original X by a chain of prespectra and natural weak equivalences. If all the original n-th spaces X_n are fibrant, so are all the n-th spaces in the chain. Some of the prespectra Y in the chain might not be fibrant spectra because $Y_n \to \Omega Y_{n+1}$ might fail to be a weak equivalence. To avoid this, replace all Y in the chain with Y′,

$$(5.27) \qquad Y'_n = \varinjlim_k \Omega^k Y_{k+n},$$

with the maps in the system given by the structure maps. This is the simplicial analogue of [72], Thm. 6. As the homotopy groups of a direct limit of simplicial sets are the direct limits of the homotopy groups, the resulting Y′ is a spectrum which is a weak equivalent to the original prespectrum Y. If all Y_n are fibrant, Y′ is a fibrant spectrum, and it is a group object if Y is. This replacement yields the chain of weak equivalences of fibrant spectra that is sought.

5.39. The naturality in 5.38 allows one to replace diagrams of fibrant spectra by weak equivalent diagrams of fibrant group spectra. In particular, a cosimplicial fibrant spectrum may be replaced by a weak equivalent cosimplicial group spectrum, which by 5.37 is a fibrant cosimplicial fibrant spectrum.

Note also by 5.11 and 5.27 that holim and Tot preserve group spectra, and so preserve this guarantee of fibrancy.

5.40. The last results collected in this section concern towers of fibrations, Postnikov towers, the spectral sequence of a tower, and the convergence problem for such spectral sequences.

A tower of fibrations is a sequence of fibrations of fibrant spectra $X(n)$,

$$(5.28) \qquad \ldots \to X(n) \to X(n-1) \to \ldots \to X(1) \to X(0) \to X(-1) = *.$$

An extended tower continues the sequence with $X(n)$ for negative n, requiring $\varinjlim X(n)$ to be contractible in place of $X(-1)$.

The sequence of k-th spaces $X_k(\ )$ of a tower is a tower of fibrations in the sense of [16], IX. Let X be the fibrant spectrum $\varprojlim X(n)$; it is weak equivalent to the holim of the inverse system.

LEMMA 5.41. — *For* $X(\ )$ *a tower of fibrations and all* k, *there is a natural short exact Milnor sequence*

$$0 \to \varprojlim_n{}^1 \pi_{k+1} X(n) \to \pi_k (\varprojlim_n X(n)) \to \varprojlim_n \pi_k X(n) \to 0.$$

Proof. — This follows from [16], IX, 3.1, or from [10], 2.2.

Recall that if $\pi_{k+1} X(n)$ satisfies the Mittag-Leffler condition as n varies, then $\varprojlim^1$ vanishes.

5.42. Let $F(n)$ be the fibre of $X(n) \to X(n-1)$. There is a long exact sequence of homotopy groups

$$(5.29) \qquad \ldots \to \pi_k F(n) \xrightarrow{\gamma} \pi_k X(n) \xrightarrow{\alpha} \pi_k X(n-1) \xrightarrow{\beta} \pi_{k-1} F(n) \xrightarrow{\gamma} \ldots$$

Arguing as in [16], IX, § 4, one may produce an exact couple from these sequences for various n, and then a spectral sequence by the usual techniques. Because the exact sequence (5.29) is not amputated at π_0 but extends to π_k for $k<0$, and because all the π_k are abelian groups, this spectral sequence does not have the fringe troubles afflicting the analogous spectral sequence for spaces of Bousfield and Kan ([16], IX, 4). This is why it is simpler to do K-theory with spectra rather than with spaces.

These observations prove

LEMMA 5.43. — *For a tower of fibrations* X( ) *with fibres* F( ) *as in* 5.42, *there is a spectral sequence with*

$$E_1^{p,q}=\begin{cases}\pi_{q-p}F(p), & p\geqq 0,\\ 0, & p<0.\end{cases}$$

The differentials run

$$d_r:\quad E_r^{p,q}\to E_r^{p+r,q+r-1}.$$

The differential d_r is induced by $\beta.(a^{r-1})^{-1}.\gamma$ in terms of the maps in (5.29). *In particular, $d_1=\beta\gamma$.*

The spectral sequence abuts to $\pi_*\varprojlim X(n)$.

$$E_1^{p,q}\quad\Rightarrow\quad \pi_{q-p}\varprojlim_n X(n).$$

Convergence is discussed below 5.44-5.48. *There is a similar spectral sequence for extended towers of fibrations in which p is allowed to be negative.*

5.44. The spectral sequence for Tot of 5.29 is the spectral sequence of the tower (5.30) as in [16], X, 6.1

$$\ldots\to\mathrm{Tot}_{n+1}(X)\to\mathrm{Tot}_n(X)\to\ldots\to\mathrm{Tot}_0(X)\to *. \tag{5.30}$$

The spectral sequence of 5.13 for holim is obtained from that of 5.29 by cosimplicial replacement of diagrams as in [16], XI, 7.1. Thus the spectral sequence of 5.43 encompasses those of 5.13 and 5.29, and it suffices to discuss convergence for it.

The theory of convergence of such spectral sequences is due to Boardman, and his treatment has finally appeared in type [10]. Another briefer treatment is [34], Appendix. I follow the treatment of Bousfield and Kan in [16], IX, 5.

5.45. Let $X(n)$ be a tower of fibrations, with $X=\varprojlim X(n)$. Define

$$\begin{cases}\pi_k X(n)^{(r)}=\text{image}\,[\pi_k X(n+r)\to\pi_k X(n)],\\ Q_n\pi_k X=\text{image}\,[\pi_k X\to\pi_k X(n)].\end{cases}\tag{5.31}$$

There are natural maps $Q_n\pi_k X\to Q_{n-1}\pi_k X$, $Q_n\pi_k X\to\pi_k X(n)^{(r)}$. Define

$$e^{p,p+k}=\ker\,[Q_p\pi_k X\to Q_{p-1}\pi_k X].\tag{5.32}$$

In the spectral sequence of 5.43, note $E_{r+1}^{p,q}=\ker d_r$ is a subobject of $E_r^{p,q}$ if $p<r$, as $E_r^{p-r,q+r+1}=0$ for $p-r<0$. Let $E_\infty^{p,q}$ be the inverse limit of the system (5.33)

$$\ldots \subseteq E_r^{p,q} \subseteq \ldots \subseteq E_{p+2}^{p,q} \subseteq E_{p+1}^{p,q}. \tag{5.33}$$

Then using 5.41 and [16], IX, 2.2; IX, 3.4, one shows that the natural maps induce isomorphisms for all r

$$\varprojlim_n Q_n \pi_k X \cong \varprojlim_n \pi_k X(n)^{(r)}. \tag{5.34}$$

The natural maps also induce an inclusion (5.35) as follows from the description of $E_r^{p,q}$ in terms of the $\pi_{q-p} X(p)^{(r)}$ in [16], IX

$$e^{p,p+k} \subseteq E_\infty^{p,p+k}. \tag{5.35}$$

It is in this sense that the spectral sequence abuts to $\pi_* X$: the $Q_n \pi_* X$ are the cokernals of a filtration of $\pi_* X$, and the filtration quotients e^{**} are subobjects of E_∞^{**}.

Definition 5.46. — For any k, the spectral sequence converges completely to $\pi_k X$ if:

(i) $\varprojlim_n{}^1 \pi_{k+1} X(n)=0$.

(ii) $e^{p,p+k}=E_\infty^{p,p+k}$ for all p.

The spectral sequence converges completely if these conditions hold for all k.

The spectral sequence converges strongly if in addition:

(iii) For each (p, q) there exists an r with $E_r^{p,q}=E_\infty^{p,q}$.

(iv) For each k, $E_\infty^{p,p+k}=0$ except for finitely many p.

Note condition (i) implies by 5.41 and (5.34) with $r=0$ that

$$\pi_k X = \varprojlim_n Q_n \pi_k X, \tag{5.36}$$

so that the filtration of $\pi_k X$ is complete and Hausdorff.

Proposition 5.47 (Boardman). — *In the spectral sequence of a tower of fibrations, for each k the condition*

$$\varprojlim_r{}^1 E_r^{p,p+k}=0 \quad \textit{for all} \quad p, \tag{5.37}$$

is equivalent to the combination of the two conditions

$$\begin{gathered} \varprojlim_n{}^1 \pi_k X(n)=0, \\ e^{p,p+k}=E_\infty^{p,p+k} \quad \textit{for all} \quad p. \end{gathered} \tag{5.38}$$

In particular, if

(5.39) $$\varprojlim_r{}^1 E_r^{p,\,p+k}=0=\varprojlim_r{}^1 E_r^{p,\,p+k+1} \quad \textit{for all } p,$$

then the spectral sequence converges completely to $\pi_k X$.

Proof. — *See* [16], IX, 5.4, or for a more detailed proof, either [34], Appendix, or [10], 9.3.

LEMMA 5.48. — *If* $d_r=0$ *for all* $r \geqq N$ *in the spectral sequence, then* $E_N^{**}=E_\infty^{**}$ *and the spectral sequence converges completely. The spectral sequence converges strongly if one has either*

(i) $\exists\, a$, $E_2^{p,q}=0$ *for all* q *unless* $0 \leqq p \leqq a$;

(ii) $\exists\, b$, $E_2^{p,q}=0$ *for all* p *unless* $q \leqq b$ (*and* $p \geqq 0$).

The filtration of $\pi_k X$ *has length at most* a *in case* (i) *and at most* $b-k$ *in case* (ii).

Proof. — The first statement follows immediately from 5.47. If (i) or (ii) holds, the usual calculation reveals that the d_r differentials into or out of $E_r^{p,q}$ are zero if $r>p$ and either $r>b-q+1$ or $r>a-p$. Thus for r large enough for p and q, $E_r^{p,q}=E_\infty^{p,q}$ and so $\varprojlim_r{}^1 E_r^{p,q}=0$. Now 5.47 yields complete convergence. Also $E_\infty^{p,\,p+k}$ contributes a non-zero filtration quotient to $\pi_k X$ only if $0 \leqq p \leqq a$ or $0 \leqq p \leqq b-k$. Thus in case (i) or (ii) hold, convergence is strong and the filtration has bounded length.

5.49. Let $X_\alpha(n)$ be a direct system of towers of fibrations indexed by α running over a directed category. Then $X(n)=\varinjlim_\alpha X_\alpha(n)$ is also a tower of fibrations. As direct and inverse limits do not in general commute, it will not generally be true that the canonical map

(5.40) $$\varinjlim_\alpha \varprojlim_n X_\alpha(n) \to \varprojlim_n \varinjlim_\alpha X_\alpha(n)$$

is a weak equivalence. The spectral sequence of the tower $X(n)$ is the direct limit of the spectral sequences of the towers $X_\alpha(n)$. However, the former spectral sequence may not converge completely even if all the sepctral sequences for the X_α do, as convergence is expressed by conditions on inverse limits which are not preserved under direct limits. Even if the spectral sequence does converge completely, it converges to $\pi_* X$ which is not necessarily $\varinjlim_\alpha \pi_* X_\alpha$. A uniform convergence condition does allow passage to the direct limit however. In particular.

LEMMA 5.50. — *Let* $X_\alpha(n)$ *be a direct system of towers of fibrations with direct limit tower* $X(n)=\varinjlim_\alpha X_\alpha(n)$. *Suppose there exists an* a *or* b *independent of* α *such that the* *spectral sequence of each tower* $X_\alpha(n)$ *satisfies condition* (i) *or* (ii) *of Lemma* 5.48. *Then the spectral sequence of the tower* $X(n)$ *satisfies the same condition. All the spectral sequences converge strongly. The map* (5.40) *is a weak equivalence.*

Proof. — All but the last statement follows immediately from 5.48 and 5.49. As the spectral sequence for $X(n)$ is the direct limit of the spectral sequences for $X_\alpha(n)$, an easy induction using (5.32) and 5.46 (ii) shows that for all k and n

$$\varinjlim_{\alpha} Q_n \pi_k X_\alpha \cong Q_n \pi_k X. \tag{5.41}$$

But for $n \geqq a+1$ in case (i) and for $n \geqq b-k+1$ in case (ii), (5.36), 5.48, 5.46 (ii), and (5.32) identify the isomorphism (5.41) to an isomorphism

$$\varinjlim \pi_k X_\alpha \cong \pi_k X. \tag{5.42}$$

This isomorphism shows that (5.40) is a weak equivalence.

LEMMA 5.51. — *There is a functor associating to each fibrant spectrum* X *an extended tower of fibrations*

$$\ldots \to X\langle 2\rangle \to X\langle 1\rangle \to X\langle 0\rangle \to X\langle -1\rangle \to \ldots \tag{5.43}$$

such that $X = \varprojlim X\langle n\rangle$, $\pi_q X \to \pi_q X\langle n\rangle$ *is an isomorphism if* $q \leqq n$, *and* $\pi_q X\langle n\rangle = 0$ *if* $q > n$. *The functor* $X \mapsto X\langle n\rangle$ *preserves products, filtering colimits, and weak equivalences.*

Proof. — Check that Moore's functorial construction of the Postnikov tower for fibrant simplicial sets preserves products and filtering colimits. Details of this construction may be found in [87], or [71], § 8. For X a fibrant spectrum, let $X\langle n\rangle$ be the prespectrum whose k-th space is $X_k\langle n+k\rangle$, the $n+k$-th stage in the functorial Postnikov tower for X_k. The induced map $X_k\langle n+k\rangle \to \Omega X_{k+1}\langle n+k+1\rangle$ is a weak equivalence, so $X\langle n\rangle$ is a fibrant spectrum. The other assertions follow easily from the properties of simplicial Postnikov towers.

5.52. For X an abelian group spectrum, the $X\langle n\rangle$ are abelian group spectra. Under the correspondence of Scholium 5.32, the Postnikov tower is the canonical good filtration of a chain complex, the σ not τ of [50], p. 69 or the τ not σ of [SGA 4], XVII, 1.1.13.

For a general fibrant spectrum X, the fibre of $X\langle n\rangle \to X\langle n-1\rangle$ is an Eilenberg-MacLane spectrum, and hence an abelian group spectrum. The Postnikov tower may be used in devissage arguments to show that the general behavior of Tot and $\mathbb{H}^{\cdot}(\mathbf{K};\ )$ on spectra is determined by the classical behavior on abelian groups.

To apply this devissage method, it must be shown for F a functor from **K** into the category of spectra such that each F(K) has only one-zero homotopy group, $\pi_n F(K)$, then there is a natural weak equivalence from F to a functor F′ from **K** into the category of abelian group fibrant spectra. For **K** a point, this is standard, but the general case cannot be deduced from this. Instead, one proceeds as follows.

For X_n a simplicial set, let $\mathbb{Z}\otimes X_n$ be the free simplicial abelian group on X_n, and $\mathbb{Z}X_n$ the kernal of the augmentation $\mathbb{Z}\otimes X_n \to \mathbb{Z}\otimes pt$. The homotopy groups of $\mathbb{Z}X_n$ are the reduced homology groups of the simplicial set X_n. For X_n based, there is a natural map $X_n \to \mathbb{Z}X_n$ sending x to $1\otimes x - 1\otimes *$. This map has an obvious universal mapping property for maps of X into simplicial abelian groups. (*See* [16], I, § 2 for a similar construction.)

For X a spectrum, let ZX be the prespectrum whose n-th space is $\mathbb{Z}X_n$, and whose structure map

$$\mathbb{Z}X_n \to \mathbb{Z}\Omega X_{n+1} \to \Omega\mathbb{Z}X_{n+1}, \tag{5.44}$$

is the simplicial abelian group homomorphism induced by applying $\mathbb{Z}$ to the structure map of X and the map given by the universal mapping property applied to Ω on the universal map $X_{n+1} \to \mathbb{Z}X_{n+1}$. Taking the direct limit over the structure maps as in (5.27) one obtains a fibrant abelian group spectrum $Z'X$ and a map of spectra $X \to Z'X$. The homotopy groups of $Z'X$ are the homology groups of the spectrum X, and the map $\pi_* X \to \pi_* Z'X = H_*(X)$ is the Hurewicz map.

The construction is strictly functorial. So if F is a functor from **K** to the category of fibrant spectra, $Z'F$ is a functor from **K** to the category of fibrant abelian group spectra, and $F \to Z'F$ is a natural transformation. If F has only one non-vanishing homotopy group $\pi_n F(K)$, the Hurewicz Theorem applied pointwise shows $\pi_n F \to \pi_n Z'F$ is an isomorphism. Further, $F \to (Z'F)\langle n \rangle$ is a weak equivalence, and $(Z'F)\langle n \rangle$ is a diagram of fibrant abelian group spectra as sought.

Under the equivalence of the category of fibrant abelian group spectra and the category of chain complexes of 5.32, the fibrant abalian group spectra with non-zero homotopy groups only in degree n correspond to chain complexes with homology only in degree n. It is well-known that the corresponding subcategory of the derived category of an abelian category $\mathscr{A}$ is equivalent to the category $\mathscr{A}$ *via* the homology functor H_n. *See* [50], I, 7.2 for some details.

Thus the functor π_n induces an equivalence from the homotopy category of functors from **K** into the category of fibrant spectra with π_n as the only non-zero homotopy group to the abelian category of functors from **K** into the category of abelian groups. The inverse equivalence sends an abelian-group-valued functor A to the canonical Eilenberg-MasLane spectrum $K(A, n)$. Note this equivalence sends short exact sequences of abelian-group-valued functors to homotopy fibre sequences of spectra.

5.53. The dual of the Postnikov tower is sometimes useful. For X a fibrant spectrum, let $X\rangle n\langle$ be the fibre of the map $X \to X\langle n \rangle$ of 5.51. Then $\pi_q X\rangle n\langle \to \pi_q X$ is an isomorphism if $q>n$, and $\pi_q X\rangle n\langle = 0$ if $q \leqq n$. X is the direct limit of the $X\rangle n\langle$ as n goes to $-\infty$. The functor $X \mapsto X\rangle n\langle$ preserves products, filtering colimits, and weak equivalences. All this is immediate from 5.51.

5.54. Consider the spectral sequence of a tower of fibrations in 5.43,

$$\begin{aligned} &E_1^{p,q} \Rightarrow \pi_{q-p} \varprojlim X(n), \\ &d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}. \end{aligned} \tag{5.45}$$

Pick any $s \geqq 1$ and set

$$(5.46) \qquad \begin{cases} \varepsilon_{s-1+r}^{p,q} = E_r^{p+(s-1)(q-p),\,q+(s-1)(q-p)}, \\ \varepsilon_{s-1+r}^{p+(s-1)(p-q),\,q+(s-1)(p-q)} = E_r^{p,q}. \end{cases}$$

Then the differential d_r of (5.45) is identified to a differential $d_{r+s-1} : \varepsilon^{p,q} \to \varepsilon^{p+r+s-1,\,q+r+s-2}$. In fact the spectral sequence (5.45) is reindexed to a spectral sequence

$$(5.47) \qquad \varepsilon_s^{p,q} \quad \Rightarrow \quad \pi_{p-q} \varprojlim X(n).$$

The spectral sequence, the filtration on $\pi_* \varprojlim X(n)$, and the underlying exact couple are all unchanged by this reindexing, only the labels and not the real objects are changed.

It will be useful to reindex 5.43 when its E_1 term has the form expected of an E_2 term. As an example apply $\mathbb{H}^\bullet(\mathbf{K};\ )$ to a Postnikov tower of functors $X\langle n \rangle$. $\mathbb{H}^\bullet(\mathbf{K}; X\langle n \rangle)$ is a tower of fibrations with inverse limit $\mathbb{H}^\bullet(\mathbf{K}; X)$ by 5.9 and 5.7. The $E_1^{p,q}$ term of the spectral sequence 5.43 for this tower is by 5.13

$$E_1^{p,q} = H^{2p-q}(\mathbf{K}; \pi_p X).$$

Reindexing this spectral sequence to begin with an E_2 term yields the usual spectral sequence 5.13.

The discussion of convergence in 5.44-5.50 remains valid for reindexed spectral sequences, as nothing essential is changed by the reindexing. In particular, since 5.48 is valid with any E_r term replacing E_2, it is valid for reindexed spectral sequences. Similarly 5.50 still holds.

The question of convergence of the general spectral sequence of an extended tower of fibrations is the delicate problem of convergence of whole-plane spectral sequences as discussed by Boardman in [10], § 10. However the vanishing of homotopy groups in a range for the extended Postnikov tower and for many towers derived from it constrain its spectral sequence to some half-plane. Reindexing then results in a usual half-plane spectral sequence for which the convergence results of 5.44-5.50 are valid.

LEMMA 5.55. — *Let $X(n) \to X'(n)$ be a map between towers of fibrations. Suppose it induces a weak equivalence of fibres* 5.42 *for all $n : F(n) \xrightarrow{\sim} F'(n)$. If the towers are extended towers, assume that for each k there is an N such that for all $n < N$, $\pi_k X(n) = \pi_k X'(n) = 0$. Then the induced map $\varprojlim X(n) \to \varprojlim X'(n)$ is a weak equivalence.*

Proof. — By induction on n, starting with $n = -1$ or $n = N - 1$ respectively, the long exact sequences (5.29) and the hypothesis on $F(n)$ imply that $\pi_k X(n) \to \pi_k X'(n)$ is an isomorphism. The result follows by the Milnor sequence 5.41.

5.56. Lemma 5.55 is a form of the spectral sequence comparison theorem for the spectral sequence 5.43. Amazingly, it doesn't require that the spectral sequence completely converges. *See* Boardman's paper [10], 8.2 for a similar phenomenon. The hypothesis of 5.55 for extended towers is usually met by towers derived from a Postnikov tower. This lemma is very useful in devissage arguments.

APPENDIX A

K-THEORY SPECTRA AND BOTT ELEMENTS

A.1. For X a scheme, consider the Quillen Q-categories constructed from the exact categories of vector bundles and of coherent modules on X. The direct sum operation on the exact categories induces a symmetric monoidal structure on the Q-categories. Feeding this input into an infinite loop space machine (e. g., [74], [100], or [125], Appendix) one obtains a spectrum. The loops on this spectrum is denoted $K(X)$ for the vector bundle case, and $G(X)$ for the coherent module case. The homotopy groups of this spectrum are Quillen's higher K-groups and G or K′-groups of X respectively.

If $X = \operatorname{Spec}(R)$ is the spectrum of a ring, the spectrum $K(X)$ may be produced directly by applying an infinite loop space machine to the symmetric monoidal category of projective R-modules and isomorphisms.

Up to homotopy, the spectra obtained are independent of the choice of machine by May's uniqueness Theorem ([76], [77], 4.3). The two ways of constructing $K(X)$ for X affine agree by the infinite loop version of Quillen's $\Omega BQ = +$ theorem, [125], § 5. Choose a machine so $K(X)$ and $G(X)$ are both fibrant spectra and cofibrant prespectra.

A.2. The tensor product of vector bundles and the tensor product of a vector bundle with a coherent module are exact functors in both variables, and so induce pairings of spectra

$$\begin{cases} K(X) \wedge K(X) \to K(X), \\ K(X) \wedge G(X) \to G(X). \end{cases} \tag{A.1}$$

If X is affine, $K(X)$ is even an E_∞-ring spectrum. In any case $K(X)$ is a homotopy commutative and associative ring spectrum. *See* [137], [75] corrected by [78], [77], [140], and [134] for details.

There is a map $K(X) \to G(X)$ induced by the forgetful functor. If X is separated and regular noetherian, this map is a weak homotopy equivalence by [57], § 7.1.

A.3. $K(X)$ is a contravariant functor of X, so a morphism of schemes $f: Y \to X$ induces a map of spectra $K(X) \to K(Y)$ which is strictly compatible with the pairing (A.1), and so also compatible with the pairing up to homotopy. $G(X)$ is a covariant functor of X with respect to finite maps of schemes, and is a functor up to homotopy with respect to proper maps ([97], § 7, # 2, augmented by [43], Thm. 4.1). For $f: Y \to X$ finite, the diagram (A.2) strictly commutes.

$$\begin{tikzcd}
 & K(Y) \wedge G(Y) \arrow[r] & G(Y) \arrow[dd, "f_*"] \\
K(X) \wedge G(Y) \arrow[ur, "f^* \wedge 1"] \arrow[dr, "1 \wedge f_*"'] & & \\
 & K(X) \wedge G(X) \arrow[r] & G(X)
\end{tikzcd} \tag{A.2}$$

This is the projection formula of [77], 2.3.

A.4. If R is a ring, $K(R)=K(\mathrm{Spec}(R))$ results from applying a machine to the bisymmetric monoidal category of finitely generated projective R-modules. This category has as a subcategory the free symmetric monoidal category on $GL_1(R)$. This inclusion is even a map of bisymmetric monoidal categories

$$\gamma: \coprod_n \Sigma_n \times GL_1(R)^n \to \coprod_n GL_n(R) \to \coprod_P \dot{A}ut(P). \tag{A.3}$$

It induces a map of spectra, in fact a E_∞-ring map as γ preserves tensor products as well as sums. Using the generalized Barratt-Priddy-Quillen-Segal Theorem (e. g. [125], Lemma 2.5), one sees that this map is a map of ring spectra from the suspension spectrum of $BGL_1(R)$.

$$\gamma: \Sigma^\infty(BGL_1(R) \sqcup *) \to K(R). \tag{A.4}$$

A.5. The mod n Moore spectrum Σ^∞/n is the cofibre in the stable homotopy category of multiplication by n on the sphere spectrum Σ^∞. For any spectrum Z, one has a cofibre or fibre sequence

$$\begin{array}{ccccc} \Sigma^\infty \wedge Z & \xrightarrow{n} & \Sigma^\infty \wedge Z & \xrightarrow{r} & \Sigma^\infty/n \wedge Z \\ \wr\| & & \wr\| & & \\ Z & & Z & & \end{array} \tag{A.5}$$

The long exact homotopy sequence of (A.5) yields the universal coefficient exact sequence (A.6)

$$0 \to (\pi_p Z)\otimes\mathbb{Z}/n \to \pi_p(\Sigma^\infty/n \wedge Z) \to \mathrm{Tor}^1_{\mathbb{Z}}(\pi_{p-1} Z, \mathbb{Z}/n) \to 0. \tag{A.6}$$

This sequence splits if n is odd or if 4 divides n by [2], 2.7.

The spectrum Σ^∞/n is a wedge of Σ^∞/l^ν for the primary factors of n, thus we may as well restrict n to be a prime power, l^ν.

Let $K/l^\nu(X)$, $G/l^\nu(X)$ be $K(X)\wedge\Sigma^\infty/l^\nu$, $G(X)\wedge\Sigma^\infty/l^\nu$ respectively. The homotopy groups of these spectra are the K and G groups of X with coefficients $\mathbb{Z}/l^\nu$; $K/l^\nu_*(X)$, $G/l^\nu_*(X)$. By an easy S-duality argument, these are canonically equivalent to the groups of homotopy classes of maps from $\Sigma^\infty/l^\nu S^*$ into $K(X)$ or $G(X)$. This alternate definition often appears in the literature, e. g. [89].

A.6. If $l>3$; Σ^∞/l^ν, and so $K/l^\nu(X)$, is a homotopy associative and commutative ring spectrum by [2] (or *see* [89], 8.5, 8.6 and take S-duals).

If $l=3$, and 9 divides l^ν; Σ^∞/l^ν, and so $K/l^\nu(X)$, are homotopy associative and commutative by [131], Thm. 6.

If $l=2$, and 16 divides l^ν; Σ^∞/l^ν and $K/l^\nu(X)$ are homotopy associative and commutative by [90].

The spectra $\Sigma^\infty/3$ and $K/3(X)$ have a unital multiplication but associativity may fail. $\Sigma^\infty/4$ and $K/4(X)$ have a unital multiplication but associativity and commutativity may fail. The mod 2 spectrum $K/2(X)$ may not even have a multiplication! These results are found in [2]. This mess at the primes 2 and 3 provides a counterexample to the teleological argument.

A.7. Consider $R=\mathbb{Z}[e^{2\pi i/l^\nu}]$ where $\nu=1$ if $l>3$, $\nu=2$ if $l=3$, and $\nu=4$ if $l=2$. There is a map of ring spectra

$$(A.7)\qquad \Sigma^\infty/l^\nu(B\,\mu_{l^\nu}\sqcup *)\to\Sigma^\infty/l^\nu(BGL_1(R)\sqcup *)\xrightarrow{\gamma}K/l^\nu(R).$$

The class $e^{2\pi i/l^\nu}$ is an l^ν torsion class in $\pi_1(B\,\mu_{l^\nu})$, and is the Bockstein of a unique class b in mod $l^\nu\,\pi_2$ of $B\,\mu_{l^\nu}$. This class stabilizes to an element in $\pi_2\Sigma^\infty/l^\nu(B\,\mu_{l^\nu}\sqcup *)$. Under the ring map (A.7), it goes to a β in $K/l^\nu_2(R)$. This is the Bott element.

The spectrum $K/l^{\nu k}(R)$ has a Bockstein filtration with filtration quotients $K/l^\nu(R)$. This gives a Bockstein spectral sequence for $\pi_* K/l^{\nu k}(R)$. If $l>3$ so $\nu=1$, this is the usual Bockstein spectral sequence as in [89], § 12, [17], § 5. For $l=2$, 3 it has been modified to have coarser filtration quotients and good ring structure. The differentials are derivations, and it follows that the $l^{\nu(k-1)}$*st* power of β in $K/l^\nu_*(R)$ is the reduction of a class x in $K/l^{\nu k}_*(R)$. The class x is not well-defined, but any other such x' differs by a class which is divisible by l^ν. Thus multiplication by x or by x' induce the same map on the $\mathbb{Z}/l^\nu$ filtration quotients of $K/l^{\nu k}_*(R)$, and so x' becomes a unit in $K/l^{\nu k}_*(R)[x^{-1}]$, and conversely. For any scheme X over $\mathbb{Z}[e^{2\pi i/l^\nu}]=R$, the element x has an image x in $K/l^{\nu k}_*(X)$. The localization $K/l^{\nu k}_*(X)[x^{-1}]$ does not depend on the ambiguity between x and x'.

Represent x by a map $\Sigma^N\to K/l^{\nu k}(R)$. This induces a map (A.8)

$$(A.8)\quad \Sigma^N\wedge K/l^{\nu k}(X)\xrightarrow{x\wedge 1}K/l^{\nu k}(R)\wedge K/l^{\nu k}(X)\xrightarrow{f^*\wedge 1}K/l^{\nu k}(X)\wedge K/l^{\nu k}(X)\xrightarrow{\text{mult.}}K/l^{\nu k}(X).$$

Let $K/l^{\nu k}(X)[\beta^{-1}]$ be the direct limit of the system whose bonding map is the adjoint of (A.8):

$$(A.9)\qquad K/l^{\nu k}(X)\xrightarrow{\cup x}\Omega^N K/l^{\nu k}(X)\xrightarrow{\Omega^N(\cup x)}\Omega^{2N}K/l^{\nu k}(X)\to\dots$$

The result doesn't depend up to homotopy on the choice of x over x'. One gets a canonical choice of $K/l^{\nu k}(X)[\beta^{-1}]$ by inverting all possible choices of x in a multidimensional version of (A.9). The homotopy groups of $K/l^{\nu k}(X)[\beta^{-1}]$ are the ring of mod $l^{\nu k}$ K-groups of X localized by inverting x.

The choices of x are compatible under reduction, so that there is a Bockstein fibration sequence

$$(A.10)\qquad K/l^{\nu(k-1)}(X)[\beta^{-1}]\to K/l^{\nu k}(X)[\beta^{-1}]\to K/l^\nu(X)[\beta^{-1}].$$

A.8. If $l\geqq 3$, $R=\mathbb{Z}[e^{2\pi i/l^\nu}]$ with $\nu=1$ is a finite extension of $\mathbb{Z}$ of degree $l-1$ and group of automorphisms over $\mathbb{Z}$ given by $\mathbb{Z}/l-1$. These automorphisms permute the roots of unity, and so move b and β in $K/l_2(R)$. An easy calculation shows that β^{l-1} is invariant, for $\mathbb{Z}/l-1$ acts on β by multiplication by a unit in $\mathbb{Z}/l$, and these are all $l-1$st roots of 1. One may choose an invariant $\hat{x}$ in $K/l^\nu_*(R)$ that reduces to β^{l-1} mod l by averaging over $\mathbb{Z}/l-1$. Then an easy transfer argument based on the fact that the degree $l-1$ is a unit in $\mathbb{Z}/l^\nu$ shows that $\hat{x}$ is the image of an element $\hat{x}$ in $K/l^\nu_*(\mathbb{Z})$. *See* [17], § 2, § 3 for details.

Now for any scheme X and $l>3$, define $K/l^\nu(X)[\beta^{-1}]$ by inverting $\hat{x}$ as in A.7. This agrees with the $K/l^\nu(X)[\beta^{-1}]$ already constructed when A.7 applies as inverting β and inverting the power β^{l-1} give the same result. Again, the homotopy groups of $K/l^\nu(X)[\beta^{-1}]$ are the localization of the ring $K/l^\nu_*(X)$ by inverting $\hat{x}$. The Bockstein fibration sequence (A.10) is valid here.

A.9. If X is a scheme over $\mathbb{Z}[e^{2\pi i/l^\nu}]$, the l^ν-torsion class $e^{2\pi i/l^\nu}$ induces a class β in $K/l^\nu_2(X)$ as in A.7. For $l>3$, or for $l=3$ and ν divisible by 2, or for $l=2$ and ν divisible by 4, this class reduces to the β of A.7 in $K/l_2(X)$, or $K/9_2(X)$, or $K/16_2(X)$ respectively. This follows from the naturality of the universal coefficient sequence (A.5), (A.6) with respect to changes in l^ν. Thus multiplication by β on $K/l^\nu(X)[\beta^{-1}]$ is a homotopy equivalence.

A.10. Under the various conditions of A.7, A.8, A.9, one may also localize the module spectrum $G/l^\nu(X)$ to get $G/l^\nu(X)[\beta^{-1}]$.

A.11. For $l=2$ or 3, given an element x in $K/l^\nu_*(X)$ one can still form $K/l^\nu(X)[x^{-1}]$ by formula (A.9). Right multiplication by x on $K/l^\nu(X)$ induces a map on $K/l^\nu(X)[x^{-1}]$ which is an automorphism. This much does not need associativity or commutativity of the multiplication, and suffices to allow the proofs of 2.42 and 2.43 to work. I need only find an appropriate x to invert which has a family of inductors mod l in the situation of 2.42. Thus I want an x in $K/3^\nu_*(\mathbb{Z})$ or in $K/4^\nu_*(\mathbb{Z}[i])$ which has reduction in $K/3_*(\mathbb{Z}[e^{2\pi i/3}])$ or in $K/4_*(\mathbb{Z}[i])$ which is divisible by the Bott element β of A.7.

These elements are provided by Bockstein spectral sequence and transfer arguments as in A.7 and A.8. Roughly x is a lift of some multiple product of β with itself. The problem is that one must worry whether there is enough associativity and commutativity left so that l^ν-th powers are cycles for the Bockstein derivations. It turns out that this is so, (*cf.* [17], 5.4). In applying Browder's theorem for $l=2$ note that the Hopf map η goes to the class of -1 in $K_1(\mathbb{Z}[i])$, and that this class is zero mod 2 as $-1=i^2$.

A.12. The l-adic K theory spectrum is defined as the homotopy inverse limit of the tower of K/l^ν spectra under the reduction maps

$$\text{(A.11)}\qquad \begin{aligned} K(X)_l^\wedge &= \underset{\substack{\leftarrow\\ \nu}}{\operatorname{holim}}\, K/l^\nu(X),\\ G(X)_l^\wedge &= \underset{\substack{\leftarrow\\ \nu}}{\operatorname{holim}}\, G/l^\nu(X). \end{aligned}$$

By the Milnor sequence 5.41 and [16], VI, 5.1 there are short exact sequences

$$\text{(A.12)}\qquad 0\to \underset{\substack{\leftarrow\\ \nu}}{\lim}{}^1\, K/l^\nu_{n+1}(X)\to \pi_n(K(X)_l^\wedge)\to \underset{\substack{\leftarrow\\ \nu}}{\lim}\, K/l^\nu_n(X)\to 0,$$

$$\text{(A.13)}\qquad 0\to \operatorname{Ext}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}_{(l)}, K_n(X))\to \pi_n(K(X)_l^\wedge)\to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}_{(l)}, K_{n-1}(X))\to 0$$

and similarly for $G(X)_l^\wedge$.

To define $K(X)[\beta^{-1}]_l^\wedge$ one takes a similar inverse limit of $K/l^\nu(X)[\beta^{-1}]$, after fussing about the compatibility of the β at different levels l^ν. In general $K/l^\nu(X)[\beta^{-1}]$ is periodic of period $2(l-1)l^{\nu-1}$. As ν goes to infinity in the tower, $K(X)[\beta^{-1}]_l^\wedge$ need not be periodic at all. If X is over a ring containing all l-power roots of 1 however, $K(X)[\beta^{-1}]_l^\wedge$ is periodic of period 2.

One sometimes wishes to consider the pro-system $\{K/l^\nu(X)[\beta^{-1}]\}$ instead of taking the inverse limit. In the usual geometric situations the pro-system of homotopy groups is Artin-Rees-Mittag-Leffler, and so this version of l-adic theory sheafifies well, and taking stalks and other geometrically interesting direct limits works reasonably well. The l-adic theory of the preceeding paragraph runs into trouble here because direct and inverse limits do not commute. *See* [SGA 5], V and VI.

A.13. The l-adic spectra $K(X)_l^\wedge$ and $K(X)[\beta^{-1}]_l^\wedge$ may be formed by taking homotopy inverse limits over the mod l^ν spectra where ν is required to be divisible by any fixed integer. In particular for $l=2$, we may take the inverse limit over spectra mod powers of 4. For any ν we still have

$$\text{(A.14)}\qquad \begin{cases} K(X)_l^\wedge \wedge \Sigma^\infty/l^\nu \simeq K/l^\nu(X), \\ (K(X)[\beta^{-1}]_l^\wedge) \wedge \Sigma^\infty/l^\nu \simeq K/l^\nu(X)[\beta^{-1}], \\ (K(X)_l^{\mathrm{Top}\wedge}) \wedge \Sigma^\infty/l^\nu \simeq K/l^{\nu\mathrm{Top}}(X). \end{cases}$$

Thus if the Dwyer-Friedlander map

$$\text{(A.15)}\qquad \rho:\quad K/l^\nu(X)[\beta^{-1}] \to K/l^{\nu\mathrm{Top}}(X)$$

is a weak equivalence for a cofinal system of l^ν, say those divisible by 4 if $l=2$, then the l-adic version of (A.15) is a weak equivalence. It then follows (A.15) is a weak equivalence for all ν. This is useful for $l=2$, 3 in light of the mess above.

A.14. There is a more sophisticated way to "invert β" in $K(X)$ simultaneously for all primes l by a Bousfield localization of the spectrum. Let $K(X)_K$, $G(X)_K$ denote the localizations of the spectra $K(X)$, $G(X)$ with respect to topological K-homology as in [12], [13]. The localizations may be obtained by smashing the original spectrum with the K-localization of the sphere spectrum. One has

$$\text{(A.16)}\qquad \begin{cases} K(X)_K \otimes \mathbb{Q} \simeq K(X) \otimes \mathbb{Q}, \\ K(X)_K \wedge \Sigma^\infty/l^\nu \simeq (K/l^\nu(X))_K \simeq K(X) \wedge (\Sigma^\infty/l^\nu)_K \simeq K(X) \wedge (\Sigma^\infty/l^\nu)[A^{-1}]. \end{cases}$$

Here A is the Adams map $\Sigma^\infty/l^\nu(S^n) \to \Sigma^\infty/l^\nu$ which is a K-equivalence. For l odd, A has degree $n=2(l-1)l^{\nu-1}$. For l odd and X a scheme over $\mathbb{Z}[1/l]$, Dwyer and Snaith have shown that A goes to a power of the Bott element in $K/l^\nu(X)$. Thus $K/l^\nu(X)_K \simeq K/l^\nu(X)[\beta^{-1}]$, and $G/l^\nu(X)_K \simeq G/l^\nu(X)[\beta^{-1}]$. Consult [12] and [110], § 3 for details. This construction is also treated by Waldhausen in [135]. I would conjecture that the same holds for $l=2$, but I have not gone through the calculations to check this.

For $l=2$ and $\sqrt{-1}$ in $\mathcal{O}_X$, the result that A is a power of the Bott element has been proven by Zaldiver, a student of Snaith [142], verifying my conjecture.

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(Manuscrit reçu le 2 mai 1984,
révisé le 19 février 1985.)

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