TAF: Beeson on abelian varieties I

Note Title 2/4/2011

1. p-divisible gps: slopes + Newton polygons
2. Hasse invariants
3. Honda-Tate classification of ab vties in characteristic p
4. B-linear abelian gps
5. Tate modules + level structure

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1. example $[p^i] : A \to A$

$$\{\ker [p^i]\}_i =: A(p)$$

1) Def. A p-divisible gp of ht h over a scheme S is a sequence of gp schemes $/S$

$$\{1\} = G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$$

G_i is locally free of rank p^{ih} over S

$\forall i \geq 0$ $\quad G_i \longrightarrow G_{i+1} \xrightarrow{[p^i]} G_{i+1}$ is exact

2) A hom of p-div gps $f : G \longrightarrow G'$ is a compatible seq of homs $f_i : G_i \longrightarrow G_i'$

$f : G \longrightarrow G'$ is an isogeny if $\exists$ $f_i' : G_i' \longrightarrow G_i$ and $f' \circ f = [p^k]$ for some k.

3) A p-div gp is simple if it is not isogenous

to a nontrivial product. If G is simple we get a SES

$$0 \to G^{0} \longrightarrow G \longrightarrow G^{et} \longrightarrow 0$$

connected or formal part

etale or discrete

The <u>dimension</u> of a simple p-div gp G is that of G^{0}. Dimension + height are both additive under exact sequences.

4) Classification (k = field of char $p > 0$, $\bar{k} = k$)

a) G p-div gp $/k \Rightarrow G \simeq \prod_i G_i$ simple

b) simple G determined by pair (d, h)

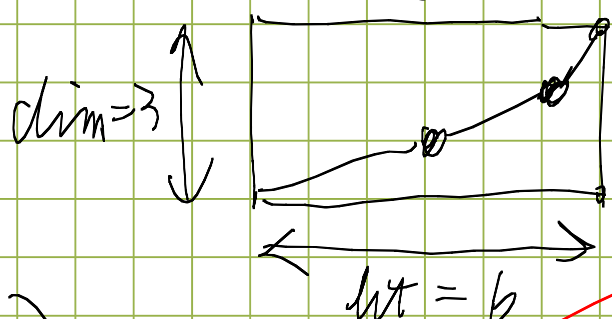
relatively prime integers $d = \dim \geq 0$

$h = \mathrm{ht} \geq 1$

with $0 \leq d/h \leq 1$. d/h is the *slope*.

This leads to a Newton polygon, e.g.

$$G = G_{1/3} \oplus G_{1/2} \oplus G$$



To be explained by Bailey

5) n.b. 1) $\lambda : A \longrightarrow A^{\vee}$ isogeny

if m is a slope, so is $1-m$.

A = abelian vty $A^{\vee}$ = dual

λ = polarization, i.e. such an isogeny

2) $A(p) = \varinjlim A[p^i]$

Let F be a number field in which $p = \prod u_i$

with $\mathcal{O}_F \hookrightarrow \mathrm{End}(A)_{(p)}$

then $A(p) = \bigoplus A(u_i)$

3) $\forall \ell$ $\mathrm{Hom}_{\mathbb{F}_q}(A, B)_\ell \simeq \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}\left(A(\ell), B(\ell)\right)$

completion at ℓ

4) For ab varieties over $\overline{\mathbb{F}}_p$

$\mathrm{Hom}(A, B)_p \xrightarrow{\simeq} \mathrm{Hom}(A(p), B(p))$

e.g. $\mathrm{End}(A)_p \simeq \mathrm{End}(A(p))$

(2) Hasse invariants

A = simple ab. vty / $\mathbb{F}_q$, $q = p^a$

$E = \mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$ fin dim div alg / $\mathbb{Q}$

E
| rank m^2

center of E $= M$ = CM field — imaginary quadratic extension

$M \supset M^+$ totally real

d | M_x

$\mathbb{Q}$ — $\mathbb{Q}_p$

$$2 \dim A \overset{?}{=} dm$$

x = prime dividing p.

M^+ = subfield of M fixed by cx conjugation $\langle c \rangle$.

E is determined up to M-algebra iso by (local) Hasse invariant in $\mathbb{Q}/\mathbb{Z}$ which determines $E \otimes_M M_x$ for each prime x

1) x complex : $\mathrm{inv}_x E = 0$ quaternions

2) x real $\mathrm{inv}_x E = \begin{cases} 1/2 \Longrightarrow E \otimes_M M_x \simeq \mathbb{H} \\ 0 \Longrightarrow \quad M_2(\mathbb{R}) \end{cases}$

3) x finite $\mathrm{inv}_x = s/m = \dfrac{v_x(\text{alg int gen for } M/\mathbb{Q})}{v_x(q)}$

$E \otimes_M M_x \simeq$ cyclic alg $(W/M_x, \phi, \pi^s)$

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③ Honda-Tate classification of simple ab var/$\mathbb{F}_q$.

a) $\left\{ \begin{array}{l} \text{quasi-isogeny} \\ \text{classes of simple} \\ \text{ab vties}/\mathbb{F}_q \end{array} \right\} \longleftrightarrow \{\text{Weil } q\text{-integers}^{+}\}/\sim$

$Z(\mathrm{End}^0 A)$

where $\pi \sim i(\pi')$ $\forall i : M \longrightarrow M'$

A Weil q-integer is $\pi \in \mathcal{O}_M$ (ring of integers in M)
s.t. $\forall\, M \hookrightarrow \mathbb{C}$
$$\pi \longmapsto \alpha \quad \text{with} \quad |\alpha| = q^{1/2}$$
for ex. $\pi = \mathrm{Frob} \in \mathrm{End}(A) \cap M$

We have no examples, alas.

b) Under
$$\mathbb{F}_q \longrightarrow \mathbb{F}_{q^s}$$
$$[\pi] \longmapsto [\pi^s]$$

$$\left\{ \begin{array}{l} \text{quasi-isog.} \\ \text{classes of simple} \\ \text{vars } / \bar{\mathbb{F}}_p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{minimal} \\ p\text{-adic} \\ \text{type} \end{array} \right\}$$

$$\begin{tikzcd}
 & \mathrm{End}^{\circ}(A) = E \arrow[r, dash] \arrow[d, dash] & E_x \arrow[d, dash, "m^2"] \\
 & Z(E) = M \arrow[r, dash] \arrow[dl, dash] \arrow[dd, dash] & M_x \arrow[dd, dash, "d_x = e_x f_x"] \\
M^+ \arrow[dr, dash] & & \\
 & \mathbb{Q} \arrow[r, dash] & \mathbb{Q}_x
\end{tikzcd}$$

choose $x \mid p$

$$f_x = [\mathcal{O}_{M,x}/x : \mathbb{Z}_p/p]$$

residue field

1) $M_p = \prod_{x \mid p} M_x$

2) $A(p) = \bigoplus_{x \mid p} A(x)$

3) $E_p = \prod_{x \mid p} E_x$

$$\underbrace{\mathrm{End}^0(A)_\lambda}_{E_\lambda \,=\, \text{simple local algebra}} \cong \mathrm{End}^0(A(x))$$

$$\Downarrow$$

$$\text{slope of } A(x) = s_x$$

If $$p = \underbrace{x_1^{e_1} \cdots x_k^{e_k}}_{\text{distinct imaginary factors}} \; \underbrace{c(x_1)^{e_1} \cdots c(x_k)^{e_k}}_{\text{their conjugates}} \; \overbrace{x_1'^{e_1'} \cdots x_l'^{e_l'}}^{\text{distinct real factors}}$$

We will assume WLOG that $s_{x_i} \le s_{c(x_i)} = 1 - s_{x_i}$ and $s_{x_1'} = 1/2$.

Writing

$\dim \hat{A}(\chi) = n_\chi \, [\mathbb{Q}_p(\chi) : \mathbb{Q}_p] \, m$ for $n_\chi \in \mathbb{Q}$

A p-adic type is a CM field

$(M, (n_\chi))$