TAF-3-25-11 Larson on TAF Ch. 5

Note Title 3/25/2011

Recall

(A, i) := B-linear ab vty

where B is a simple $\mathbb{Q}$-alg

$i: B \longrightarrow \mathrm{End}^0(A)$ = quasi endomorphisms of A

$= \mathbb{Q} \otimes \mathrm{End}(A)$

$*$: involution on B with certain properties.

$\lambda_L : A \longrightarrow A^\vee$ compatible polarizn

$A^\vee$ = dual of A

where

- λ_L is a symm isogeny determined by and ample line bundle L. "Symm" means

$\lambda_L^\vee = \lambda_L$, i.e. λ_L is self-dual.

- The λ-Rosati invol on $\mathrm{End}^0(A)$ restricts to $*$ on B

$$\left.\begin{array}{l} \dagger : \mathrm{End}^0(A) \longrightarrow \mathrm{End}^0(A) \\ \quad f^\dagger = \lambda^{-1} \circ f^\vee \circ \lambda \end{array}\right\} \lambda\text{-Rosati invol.}$$

Def ① Two polarizations λ, λ' are equiv if

$\exists\, m, n \geq 0$ s.t. $\lambda_{L^{\otimes n}} = \lambda'_{L'^{\otimes m}}$

A polarizn is determined by its line bundle L.

(2) A *weak polzn* is an equiv class of polzns.

(3) For weak polzns λ and $\bar{\lambda}$, λ is *similar* to $\bar{\lambda}$ if $\alpha^*\lambda$ is equiv to $\bar{\lambda}$ for an isogeny $\alpha : A \to A$.

Motivation

(1) $\left\{\begin{array}{l}\text{compatible weak} \\ \text{polzns on } A \\ \text{up to similarity}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\ker \text{ of} \\ H^1(\mathbb{Q}, H_{(A,-i,\lambda)}) \to H^1(\mathbb{R}, H_{(A,-i,\lambda)})\end{array}\right\}$

$\updownarrow$

$$\left\{\begin{array}{c}\text{alternating non-degen} \\ \text{*-hermitian forms} \\ \lambda\langle -,-\rangle \text{ on } V_\ell(A)\end{array}\right\}$$

② Elements in $H^1(\mathbb{Q}_\ell, GU_{V_\ell(A)}) \leftrightarrow \left\{\begin{array}{c}\text{similitude classes of forms} \\ \lambda\langle -,-\rangle\end{array}\right\}$

<u>Notation</u>

$F = \mathbb{Q}(\delta) \quad \delta^2 = d \in \mathbb{Z}^-$

$c =$ conjugation on F

$B =$ CSA$/F$ of dim n^2

$*$ = positive invol on B of the second kind
$(*|_F = c)$

$V = B$ as a left B-module

$C = \mathrm{End}_B(V) = B^{op}$

Forms (2 types)

① $\langle -,- \rangle : V \otimes_{\mathbb{Q}} V \longrightarrow \mathbb{Q}$

nondeg, alternating, $*$ hermitian

i.e. $\langle bv, w \rangle = \langle v, b^* w \rangle$

Def Two such forms $\langle -,- \rangle$ and $\langle -,- \rangle'$ are similar if $\exists\, \alpha \in C$ and $\nu(\alpha) \in \mathbb{Q}^\times$ s.t.

$$\langle \alpha v, \alpha w \rangle' = \nu(\alpha) \langle v, w \rangle \quad \forall\, v, w \in V$$

If $\nu(\alpha) = 1$ then the forms are isometric.

Exercise $\langle -,- \rangle$ induces an involution ι on C defined by $\langle c v, w \rangle = \langle v, c^\iota w \rangle$ and similar forms induce the same ι, the Rosati involn.

② $(-,-) : V \otimes_{\mathbb{Q}} V \to F$ is non-degen, $*$-herm and $*$-symmetric, i.e. F-linear in first var and $(v, w) = \overline{(w, v)}$

Lemma 7.1.12 $\exists$ 1-1 corresp between

$$\left\{\begin{array}{l}\text{n.d. } * \text{-herm} \\ \text{alt forms} \\ \langle -, - \rangle \text{ on } V\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{nondeg } * \text{-herm} \\ * \text{-symm forms} \\ (-, -) \text{ on } V\end{array}\right\}$$

Proof Given $\langle -, - \rangle$, form

$$(v, w) = \langle \delta v, w \rangle + \delta \langle v, w \rangle$$

Given $(-, -)$ form

$$\langle v, w \rangle = \frac{1}{2d} \mathrm{Tr}_{F/\mathbb{Q}} \, \delta (v, w)$$

Lemma 5.13: Let $\langle -,- \rangle$ be as above

(a) $\exists!$ $\beta \in B = V$ satisfying $\beta^* = -\beta$ and encodes $\langle x,y \rangle = \mathrm{Tr}_{F/\mathbb{Q}} \mathrm{Tr}_{B/F}(x \beta y^*)$

(b) let γ be β in $C = B^{op}$. The involn ι on C induced by $\langle -,- \rangle$ is $c^\iota = \gamma^{-1} c^* \gamma$.

<u>Cor 5.1.14</u> $\exists$ $\xi \in B$ s.t. $\xi^* = \xi$ that encodes $(-,-)$ assoc. to $\langle -,- \rangle$:

$$(x,y) = \mathrm{Tr}_{B/F}(x \xi y^*)$$

Pf $\xi = 2\delta\beta$.

Def Involns ι and ι' on G are equiv if $\exists\, c \in C^\times$ so that $c\, x^{\iota}\, c^{-1} = (c x c^{-1})^{\iota'}$.

Prop 5.1.7 The assoc $\langle -, - \rangle \rightsquigarrow \iota$ induces a 1-1 correspt

$$\left\{ \begin{array}{l} \text{similitude classes} \\ \text{of n.d. } *\text{-herm} \\ \text{alt forms on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{equiv classes of} \\ \text{involns of } 2^{\text{nd}} \\ \text{kind on } G \cong B^{op} \end{array} \right\}.$$

<u>Lemma 5.1.6</u> Let ι be an involn of 2^{nd} kind on C.

Then $\exists \gamma \in C^{\times}$ s.t. $\gamma^{*} = -\gamma$ and that gives ι

via

$$x^{\iota} = \gamma^{-1} x^{*} \gamma$$

and γ is unique up to $\mathbb{Q}^{\times}$-multiples.

<u>Pf of Lemma</u>: Skolem-Noether thm

$\Rightarrow$ all auts of C are inner auts

$\Rightarrow \exists \alpha \in C^{\times}$ (unique up to $F^{\times}$-multiple)

s.t. $(x^{*})^{\iota} = \alpha^{-1} x \alpha$ for $x \in C$.

Thus $(x)^{\iota} = \alpha^{-1} x^{*} \alpha$

Since ι is an involn

$$x = (x^{\iota})^{\iota} = (\alpha^{-1}\alpha^{*})\, x\, (\alpha^{-1}\alpha^{*})^{-1}$$

$$\Longrightarrow \alpha^{-1}\alpha^{*} \in F^{\times}$$

Hilbert's Thm 90 $\Longrightarrow \exists\, a \in F^{\times}$ s.t.

$\gamma = a\alpha$ satisfies $\gamma^{*} = -\gamma$ where a is unique up to $\mathbb{Q}^{\times}$-mult.

Pf of 5.1.5: $\left(\langle -, - \rangle \rightsquigarrow \iota \right)^{-1}$
antisymmetric equiv

Given $\langle -,- \rangle$, $\exists\, \beta$ s.t.

$$\langle x, y \rangle = \langle x, y \rangle_\beta = \mathrm{Tr}_{F/\mathbb{Q}}\, \mathrm{Tr}_{B/F}(x \beta y^*)$$

Let γ be β in C. Then the associated ι

is $\iota_\gamma = \gamma^{-1} x^* \gamma$.

Conversely, $\Rightarrow \exists\, \gamma \in C^\times$ unique up to $\mathbb{Q}^\times$-mult

s.t. $x^\iota = x^{\iota_\gamma} = \gamma^{-1} x^* \gamma$ and $\gamma^* = -\gamma$

Let β be γ in B

$\rightsquigarrow$ alternating form $\langle -,- \rangle_\beta$

__Claim:__ $\langle -,- \rangle_\beta$ similar $\langle -,- \rangle_{\beta'}$ iff the assoc. involns τ_γ and $\tau_{\gamma'}$ are equiv.

QED