TAF - 2-11-11   Beeson

Note Title   2/11/2011

Thm (Classification of Ab vties / $\bar{\mathbb{F}}_p$)

1) The natural map $\left\{\begin{matrix}\text{quasi-isogeny classes} \\ \text{of simple ab vars} / \bar{\mathbb{F}}_p\end{matrix}\right\}$

$$\downarrow$$

$\{\text{minimal } p\text{-adic types}\}$

$$A \mapsto (M, (n_x))$$

$$\begin{tikzcd}
\mathrm{End}^0(A) \arrow[r, dash] \arrow[d, dash] & E_x \arrow[d, dash, "m^2"] \\
Z(E) = M \arrow[r, dash] \arrow[d, dash, "m^+"] & M_x \arrow[d, dash, "d_x"] \\
\mathbb{Q} \arrow[r, dash] & \mathbb{Q}_p
\end{tikzcd}$$

f_x = residual gp

$\dim A(x) = n_x f_x d_x m$

defines $n_x \in \mathbb{Q}$

$$A(p) = \prod_{x \mid p} A(x)$$

A p-adic type $(M, (n_x))$ is

1) CM field M
2) (n_x) $n_x \in \mathbb{Q}$ $\forall x | p$ in M

A morphism of p-adic types

$$(M, (n_x)) \longrightarrow (M', (n'_{x'}))$$

is an embedding $M \hookrightarrow M'$

and $\forall x' | x$ a prime of M we have

$$n'_{x'} = e_{x'/x} \, n_x$$

minimal means not the target of a nontrivial morphism

B-linear ab-vars

We want $A/\bar{\mathbb{F}}_p$ s.t. $A(p)$ contains a ht n 1-dim FGL, slope $A(p) = 1/n$

$\mathrm{End}(A)_p \cong \mathrm{End}(A(p))$ = unique maximal order in a central div alg $/\mathbb{Q}_p$ of invar $1/n$.

$$\begin{array}{ll} & \mathrm{End}^0(A) = E \\ & \quad\downarrow \\ d<0 & \quad M = \mathbb{Q}(\sqrt{-d}) \qquad p = x \cdot c(x) \\ & \quad | \\ & \quad \mathbb{Q} \qquad n_x = \frac{1}{n}, \; n_{c(x)} = 1 - \frac{1}{n} \end{array}$$

In this setup $\exists !$ quasi-isog class of ab vars of dim n with CM by F ($\mathcal{O}_F \hookrightarrow \mathrm{End}^0(A)$) s.t. $A(x)$ has slope $1/n$. $(M,(n_x))/F$, $F \leq M$.

A B-linear ab var is a pair (A, i)

$$\begin{tikzcd} & \mathrm{End}^0(A) = E \arrow[d, dash] \arrow[dr, dash] & \\ \text{simple algebra } B \arrow[ur, hook, "i"] & Z(B) = F \arrow[l] \arrow[d, dash] & M \supseteq Z(E) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\left\{ \text{summands of } F \otimes_{\mathbb{Q}} M \right\} \longleftrightarrow \left\{ (A, i) \text{ s.t. } A \simeq A_0^j \right\}$$

isotypical

WHY DO WE CARE?

Tate modules

$\ell \neq \operatorname{char} k = p$

$$T_\ell(A) := \begin{cases} \varprojlim A[\ell^i] \simeq \mathbb{Z}_\ell^{2g} & g = \dim A. \\ \quad \text{if } A \text{ defined over separably closed } k = \bar{k} \\ T_\ell(A \otimes_k \bar{k}) \end{cases}$$

$$V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}$$

$$T^p(A) = \prod_{\ell \neq p} T_\ell(A)$$

Lemma $\ell \neq p = \mathrm{char}(k)$

$$\mathrm{Hom}(A,B) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}\left(T_\ell A, T_\ell B\right)$$

Conj For $\mathrm{char}(k) \neq \ell$, f.g. over prime field.

$$\mathrm{Hom}_k(A,B) \otimes \mathbb{Z}_\ell \simeq \mathrm{Hom}_{G_k}\left(T_\ell(A), T_\ell(B)\right)$$

Galois gp.

Level structures V a vector space$/\mathbb{Q}$ of dim $2g$.

$L \subseteq V$ a $\mathbb{Z}$-lattice

dim$(A) = g$

$$L^p := \prod_{\ell \neq p} L \otimes \mathbb{Z}_\ell \xrightarrow[\sim]{\eta} T^p(A)$$

choice of iso is called an integral uniformization of A

$$V^p := L^p \otimes \mathbb{Q} \xrightarrow[\simeq]{\eta} V^p(\mathbb{A})$$

rational uniformizn

$\mathrm{Aut}(V^p)$ acts on set of all rational uniformizations.

Let K^p be a subgp of $\mathrm{Aut}(V^p)$. A rational K_p-level structure $[\eta]_{K^p}$ is the K_p-orbit of a rational unifzn η