Topological Automorphic Forms Seminar

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First Lecture 1/28/11

Goal: Compute $\pi_k^{st} = \pi_{n+k} S^n$ the stable homotopy groups of the n-sphere. (π_k^{st} is independent of n for n large enough.) $n > k+1$

These groups will be abelian.

Recall $k=0$, $\pi_n S^n = \mathbb{Z}$.

2 approaches:

1) Pontryagin: (1930s) Consider $f: S^{n+k} \to S^n$, deform so f is smooth.
if x is regular,
$f^{-1}(B_\varepsilon(x))$ determines f up to homotopy.
"framed cobordism"

2) Serre (1950s):
Use Eilenberg-MacLane spaces $K(A,n)$

$$\pi_* K(A,n) = \begin{cases} A & * = n \\ 0 & \text{---} \end{cases}$$

The circle S^1 is a $K(\mathbb{Z},1)$.

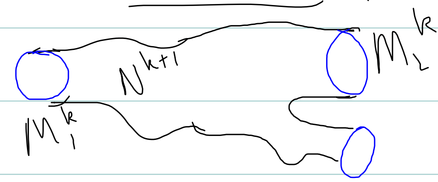
I. Examples of stable stems.

k:	0	1	2	3	4	11	12
π_k^{st}:	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}/504$	$\mathbb{Z}/2 \oplus \mathbb{Z}/480$

Pontryagin: $\underset{\text{group}}{\Omega^{fr}_{k,n}} \to \pi_{n+k} S^n$ homomorphism. $\Omega^{fr}_{k,n}$ is manifolds up to cobordism.

Thom: This map is an isomorphism.

M_1^k and M_2^k mflds are cobordant if $\exists N^{k+1}$ s.t. $M_1^k \amalg M_2^k = \partial N^{k+1}$



Cobordism invariants

Defn: A cobordism invariant with values in an abelian gp A is a group homomorphism

$$\pi_{n+k} S^n \longrightarrow A.$$

Question: Do there exist geometric interpretations of these invariants?

Ex: $k=0$, $f: S^n \to S^n$.
invariant = # of pts in $f^{-1}(x)$ for some regular $x \in S^n$
framed 0-mfld

This invariant is the degree of f. For $k=1$, yes, for $k=2$, Kervaire invariant.

Groups A that support invariants.
Adams (60s): "d" and "e" invariants.
Eg. $\pi^{st}_{8n} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ misses one summand.
Stong interpreted this using spin cobordism.

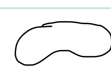
k	8	9	10	13	14	15
missing summand in geometric invariants	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$SU(3)$	$U(3)$	$Sp(2)$	$Sp(1) \times Sp(2)$	G_2	$U(1) \times G_2$

In each case there is a missing Lie group associated

II. Tools

Thm: $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$ for $\underbrace{k < n-1}_{\text{Stable range of dimensions}}$

Given M^n, the suspension $(\Sigma M)^{n+1}$ is the double cone:



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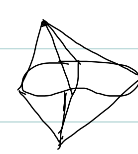

ΣM

then $\Sigma S^n = S^{n+1}$

Defn: A spectrum E is a sequence of spaces $\{E_n\}_{n=0}^{\infty}$ and structure maps $s_n^E : \Sigma E_n \to E_{n+1}$ whose adjoints $t_n^E : E_n \to \Omega E_{n+1}$ are homeomorphisms.

$\uparrow$ Loop space

Example: $E_n = S^n$. $\quad s_n : \Sigma S^n \xrightarrow{=} S^{n+1}$ $\quad$ Spectrification $\downarrow$

except $t_n : S^n \to \Omega S^{n+1}$ is not a homeomorphism.

Define a spectrum $\tilde{E}$ by $\tilde{E}_n = \varinjlim \Omega^k S^{n+k}$

Then $\tilde{E} = S^0$ is the "sphere spectrum"

Given E, ① $\pi_k E = \pi_{n+k} E_n$ (independent of n)

and for X a space, $E^k(X) = [X, E_k]$

$$E_k(X) = \varinjlim_n \pi_{n+k}(E_n \wedge X)$$

$\uparrow$ smash product

Thm (Brown representability)

Given a cohomology theory E (functor to graded rings)

$\exists$ a representing spectrum.

Ex: The spectrum MU is the "complex cobordism spectrum"

- It is "complex oriented"
- closely related to formal group laws.

Sketch of tmf

Given $E/\mathbb{F}_4$ supersingular

F a formal gp law $/\mathbb{F}_4$.

Γ be a formal gp law $/\mathbb{F}_4$ the "Honda FGL"

$\mathrm{Aut}(\Gamma) = \mathbb{S}_2$ (profinite)

Theory of Lubin-Tate, the group law of Γ lifts to FGL/W called $\tilde{\Gamma}$
where W is a graded ring (Witt ring of $\mathbb{F}_4$), and $W = \pi_* E_2$.
for some spectrum E
It turns out this E_2 is complex oriented.

$\mathbb{S}_2 \circlearrowright \pi_* E_2$

$\cup$

finite $G = \mathrm{Aut}(F)$ where F was FGL assoc to E. ← the ell curve

Goerss, Hopkins, Miller $\Rightarrow$ $G \circlearrowright E$ in some meaningful way.
↑ the spectrum

homotopy fixed points of G

$TMF = E_2^{hG}$

tmf = (−1)-connected cover of TMF. This is a spectrum with all of its homotopy groups in positive dimension.

$\Delta^8 \in \pi_* tmf \xrightarrow[\text{homomorphism}]{\text{edge}}$ certain ring of modular forms

$\downarrow$ invert

$\pi_* TMF$

Hopkins/Mahowald: The edge homomorphism is not surjective, eg,
$\Delta \notin \mathrm{Im}(\text{edge})$. 8Δ, $4\Delta^2$, $2\Delta^4$, Δ^8, etc are.

There is a map $S^0 \to tmf$
(like: $\mathbb{Z} \to$ more complicated ring)

$\pi_* S^0 \to \pi_* tmf$
This is an isomorphism for $* \leq 6$
nonzero on 6 Liegroup classes from above (see blue ink)
$\pi_* S^0$ for $* < 60$ are accounted for.

IV. Chromatic approach & TAF: study π_k^{st} one prime at a time.

X = finite spectrum (e.g., $X = S^0$ – the suspension spectrum of a finite CW-complex)

fix prime p. Chromatic tower:

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0$$

Thm (Hopkins & Ravenel): $\varprojlim X_n \simeq X_{(p)}$ (Localizations at p)

$\pi_* X_{(p)} \cong \pi_* X \otimes \mathbb{Z}_{(p)}$

Cor: $\pi_* X_{(p)} = \varprojlim_n \pi_* X_n$.

$$\text{Fiber } \underset{\substack{\uparrow \\ n^{\text{th}} \text{ chromatic layer}}}{M_n X} \to X_n \to X_{n-1}$$

$$M_n X \simeq M_n(\underbrace{X_{K(n)}}_{\substack{K(n)\text{-local sphere} \\ \text{when } X = S^0.}})$$

The first chromatic layer is well-understood:

$$\underset{\substack{\uparrow \\ \text{Its homotopy is} \\ \text{unknown}}}{S_{K(1)}} \to KO_{(p)} \xrightarrow{\psi^\ell - 1} KO_{(p)}$$

For $n=2$,

$$\underset{\substack{\uparrow \\ \text{Not } S_{K(2)}}}{Q(\ell)} \simeq \mathrm{Tot}\left(TMF_p \Rightarrow \begin{matrix} TMF_0(\ell)_p \\ \times \\ TMF \end{matrix} \Rightarrow TMF_0(\ell)_p \right)$$

$D_{K(2)} Q(\ell) \to S_{K(2)} \to Q(\ell)_{K(2)}$

	$n=1$	$n=2$	$n \geq 2$
E	KO	TMF	?
Q	$S_{K(1)}$	$Q(\ell)$	?

TAF