The homotopy category is a homotopy category

By

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In [4] Quillen defines the concept of a *category of models for a homotopy theory* (a *model category* for short). A model category is a category $\boldsymbol{K}$ together with three distinguished classes of morphisms in $\boldsymbol{K}$: $\boldsymbol{F}$ ("fibrations"), $\boldsymbol{C}$ ("cofibrations"), and $\boldsymbol{W}$ ("weak equivalences"). These classes are required to satisfy axioms M0—M5 of [4]. A *closed* model category is a model category satisfying the extra axiom M6 (see [4] for the statement of the axioms M0—M6).

To each model category $\boldsymbol{K}$ one can associate a category $\mathrm{Ho}\,\boldsymbol{K}$ called the *homotopy category* of $\boldsymbol{K}$. Essentially, $\mathrm{Ho}\,\boldsymbol{K}$ is obtained by turning the morphisms in $\boldsymbol{W}$ into isomorphisms.

It is shown in [4] that the category of topological spaces is a closed model category if one puts $\boldsymbol{F} = \{\text{Serre fibrations}\}$ and $\boldsymbol{W} = \{\text{weak homotopy equivalences}\}$, and takes $\boldsymbol{C}$ to be the class of all maps having a certain lifting property.

From an aesthetical point of view, however, it would be nicer to work with ordinary (Hurewicz) fibrations, cofibrations and homotopy equivalences. The corresponding homotopy category would then be the ordinary homotopy category of topological spaces, i.e. the objects would be all topological spaces and the morphisms would be all homotopy classes of continuous maps.

In the first section of this paper we prove that this is indeed feasible, and in the last section we consider the case of spaces with base points.

1. The model category structure of Top. Let Top be the category of topological spaces and continuous maps. By fibrations (cofibrations) we shall mean maps having the homotopy lifting (extension) property with respect to all spaces.

Let $\boldsymbol{F} = \{\text{fibrations}\}$, $\boldsymbol{C} = \{\text{closed cofibrations}\}$, and $\boldsymbol{W} = \{\text{homotopy equivalences}\}$.

If $i\colon A \to X$ and $p\colon E \to B$ are morphisms in Top, we shall say that i has the *left lifting property* (LLP) with respect to p, and that p has the *right lifting property* (RLP) with respect to i, if every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

in Top admits a diagonal $X \to E$.

Proposition 1. *The following relations hold between $\boldsymbol{F}$, $\boldsymbol{C}$ and $\boldsymbol{W}$.*

(a) $p \in \boldsymbol{F} \Leftrightarrow p$ *has the RLP with respect to all* $i \in \boldsymbol{C} \cap \boldsymbol{W}$.

(b) $i \in \boldsymbol{C} \Leftrightarrow i$ *has the LLP with respect to all* $p \in \boldsymbol{F} \cap \boldsymbol{W}$.

(c) $p \in \boldsymbol{F} \cap \boldsymbol{W} \Leftrightarrow p$ *has the RLP with respect to all* $i \in \boldsymbol{C}$.

(d) $i \in \boldsymbol{C} \cap \boldsymbol{W} \Leftrightarrow i$ *has the LLP with respect to all* $p \in \boldsymbol{F}$.

Proof. (a) follows from [6], Theorem 8, and the definition of fibrations.

$\Rightarrow$ in (b) follows from [6], Theorem 9. To prove $\Leftarrow$ we first note that if i has the LLP with respect to all $p \in \boldsymbol{F} \cap \boldsymbol{W}$, then i is a cofibration. It remains to show that i is closed. We may assume that i is an inclusion, $i: A \subset X$. Let

$$E = A \times I \cup X \times (0,1] \subset X \times I,$$

and define $p: E \to X$ by $p(x, t) = x$. p is a homotopy equivalence, and it is also a fibration, for given $g: Y \to E$ and $G: Y \times I \to X$ with $G_0 = pg$, we can construct a suitable lifting $\bar{G}: Y \times I \to E$ by letting

$$\bar{G}(y, t) = (G(y, t), t + (1 - t) \operatorname{pr}_I g(y)),$$

where $\operatorname{pr}_I: E \to I$ is the projection map. (This construction works for any inclusion map.)

Hence i has the LLP with respect to p. Now define $j: A \to E$ by $j(a) = (a, 0)$. Then $pj = 1_X i$, and consequently there is a map $f: X \to E$ extending j with $pf = 1_X$. It follows that $A = f^{-1} \operatorname{pr}_I^{-1}(0)$ is closed in X.

(c) and (d) follow from [6], Theorems 8 and 9, and (b).

Proposition 2. *Every continuous map $f: X \to Y$ can be factored as $f = pi = p'i'$, where p and p' are fibrations, i and i' are closed cofibrations, and i and p' are homotopy equivalences.*

Proof. $f = pi$: It is well known (see for instance [1], 5.27) that f can be factored $f = \pi j$, where $j: X \subset W$ imbeds X as a strong deformation retract of W, and $\pi: W \to Y$ is a fibration. As in the proof of Proposition 1 (b) let $E = X \times I \cup W \times (0,1]$ and define $i: X \to E$, $\pi': E \to W$ by $i(x) = (x, 0)$, $\pi'(w, t) = w$. Then $i(X)$ is a strong deformation retract of E and $i(X) = \operatorname{pr}_I^{-1}(0)$. It follows that i is a closed cofibration and a homotopy equivalence. π' is a fibration, and the desired factorization $f = pi$ follows, with $p = \pi\pi'$.

$f = p'i'$: To get this factorization it is sufficient to factor the fibration $p: E \to Y$ constructed above as $p = p'\bar{i}$, with $\bar{i}$ a closed cofibration and p' a fibration and a homotopy equivalence.

Let Z be the disjoint union $Y \cup E \times (0,1]$ *as a set*. Define $p': Z \to Y$, $\varphi: Z \to I$, and $\alpha: Z - Y \to E \times I$ by

$$p'(e, t) = p(e), \quad p'(y) = y,$$
$$\varphi(e, t) = t, \quad \varphi(y) = 0,$$
$$\alpha(e, t) = (e, t).$$

Then give Z the weakest topology making p', φ and α continuous. $Z - Y$ is then homeomorphic to $E \times (0, 1]$, and the only difference between Z and the mapping cylinder of p is in the topology near Y. The map $\bar{\imath}\colon E \to Z$ given by $\bar{\imath}(e) = (e, 1)$ is clearly a closed cofibration.

It only remains to show that p' is a fibration. However, one can easily verify that $p'\colon Z \to Y$ is the "generalized Whitney sum" ([3]) of the fibrations 1_Y and p, and therefore p' is a fibration.

We can now prove

Theorem 3. *The category* Top, *with the morphism classes* **F**, **C** *and* **W**, *is a closed model category.*

Proof. It is sufficient to verify M0, M2, M5 and M6. M0 and M5 are obviously true and M2 is just Proposition 2 above, while M6 follows from Proposition 1, M2 and M5.

2. Some lemmas. The following lemmas will be useful in the next section.

Lemma 4. *If $i\colon A \subset X$ is a cofibration and Y is a compact space, then the map*

$$i_{\#}\colon A^Y \subset X^Y$$

induced by i is also a cofibration (with respect to the compact-open topology).

Proof. If H, φ are as in [6], Lemma 4, then corresponding functions

$$\bar{H}\colon X^Y \times I \to X^Y \quad \text{and} \quad \bar{\varphi}\colon X^Y \to I$$

are given by

$$\bar{H}(f, t)(y) = H(f(y), t), \qquad \bar{\varphi}(f) = \sup_{y \in Y} \varphi f(y).$$

Lemma 5. *If $j\colon B \to A$ and $i\colon A \to X$ are maps such that i and ij are cofibrations, then j is also a cofibration.*

Proof. We can assume that i and j are inclusion maps. There exists a halo U around A in X together with a retraction $r\colon U \to A$. Since U is also a halo around B in X, $B \subset U$ is a cofibration ([2], Satz 2, Korollar).

Now consider a commutative diagram

$$(1) \qquad \begin{array}{ccc} B & \xrightarrow{F} & Y^I \\ {\scriptstyle j}\cap & & \downarrow{\scriptstyle \pi_0} \\ A & \xrightarrow{f} & Y \end{array}$$

where $\pi_0(\omega) = \omega(0)$. The diagram

$$(2) \qquad \begin{array}{ccc} B & \xrightarrow{F} & Y^I \\ \cap & & \downarrow{\scriptstyle \pi_0} \\ U & \xrightarrow{fr} & Y \end{array}$$

is also commutative, and since $B \subset U$ is a cofibration, (2) admits a diagonal $G\colon U \to Y^I$. $G|A$ is then a diagonal in (1). It follows that j is a cofibration.

Recall that a well-pointed space is a space X together with a base point $* \in X$ such that the inclusion map $\{*\} \subset X$ is a closed cofibration.

Lemma 6. *Consider a pullback diagram*

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

where p is a fibration. Suppose that E, B, B' are well-pointed and that f and p respect base points. Then E' (with the obvious base point) is also well-pointed. In particular, the fiber of p is well-pointed.

Proof. Consider the sequence

$$\{*\} \overset{j}{\subset} F \overset{i}{\subset} E' \overset{f'}{\longrightarrow} E$$

where $*$ is the base point of E' and F is the fiber over the base point of B'. By [6], Theorem 12, i and $f'i$ are cofibrations. Since E is well-pointed, $(f'i)j$ is a cofibration; hence, by Lemma 5 above, j is a cofibration. It follows that ij is a cofibration.

3. The pointed case. Let Top* be the category of pointed spaces and continuous base point preserving maps. All base points will be denoted by $*$. Fibrations and cofibrations in Top* are defined exactly as in Top, except that all maps and homotopies are required to respect the base points. From now on homotopies, fibrations, etc. in Top will be referred to as *free* homotopies, fibrations, etc.

It is clear that if a map $i\colon A \to X$ in Top* is a free cofibration (that is, when considered as a map in Top), then it is a cofibration in Top*. On the other hand, a fibration in Top* is also a free fibration.

Just as in the free case one can prove that all cofibrations in Top* are imbeddings. Also, if the base point of X is closed, an inclusion $A \subset X$ in Top* is a cofibration if and only if $(X \times 0 \cup A \times I)/* \times I$ is a retract (and hence a strong deformation retract) of $X \times I/* \times I$ (A need not be closed). The arguments are similar to those in [5] and [6].

We shall need the following result, analogous to [6], Lemma 4.

Proposition 7. *Let $i\colon A \subset X$ be an inclusion map in* Top*, *and suppose that there exists a continuous function $\psi\colon X \to I$ such that $\psi^{-1}(0) = \{*\}$. Then i is a cofibration if and only if there exist a continuous function $\varphi\colon X \to I$ with $A \subset \varphi^{-1}(0)$, and a homotopy $H\colon X \times I \to X$* rel A *such that $H_0 = 1_X$ and $H(x, t) \in A$ whenever*

$$\operatorname{Min}(t, \psi(x)) > \varphi(x).$$

If such φ and H exist they can be chosen in such a way that $\varphi(x) \leqq \psi(x)$ for all $x \in X$.

Proof. Suppose first that i is a cofibration. Let

$$K = \{(x, t) \in X \times 0 \cup A \times I \mid t \leqq \psi(x)\}$$

and define $\varrho: X \times 0 \cup A \times I \to K$ by

$$\varrho(x, t) = (x, \mathrm{Min}(t, \psi(x))).$$

Since i is a cofibration, ϱ extends to $r: X \times I \to K$. φ and H can then be defined by

$$\varphi(x) = \sup_{t \in I} (\mathrm{Min}(t, \psi(x)) - \mathrm{pr}_I\, r(x, t)), \qquad H(x, t) = \mathrm{pr}_X\, r(x, t).$$

It is clear that φ and H have the desired properties.

Conversely, if φ and H are given, we can define a retraction

$$r: X \times I/* \times I \to (X \times 0 \cup A \times I)/* \times I$$

by

$$r(x, t) = \begin{cases} (H(x, t), 0), & t\psi(x) \leqq \varphi(x), \\ (H(x, t), t - \varphi(x)/\psi(x)), & t\psi(x) > \varphi(x). \end{cases}$$

Our main interest will be in the full subcategory Top^w of well-pointed spaces, rather than the whole category Top^*. It is an easy consequence of the product theorem for cofibrations ([6], Theorem 6) that the mapping cylinder in Top^* of a map between well-pointed spaces is well-pointed. Consequently, a map in Top^w is a cofibration in Top^w if and only if it is a cofibration in Top^*.

Dually, it follows from Lemmas 4 and 6 above that the mapping track of a map in Top^w is well-pointed, and so a map in Top^w is a fibration in Top^w if and only if it is a fibration in Top^*.

A necessary and sufficient condition for a pointed space X to be well-pointed is that there exist a $\psi: X \to I$ with $\psi^{-1}(0) = \{*\}$ and a homotopy $P: X \times I \to X$ with $P_0 = 1_X$ and $P(x, t) = *$ when $t > \psi(x)$ ([6], Lemma 4). Let us call (P, ψ) a *well-pointing couple* for X.

Lemma 8. *If $A \subset X$ is a cofibration in* Top^w *there exists a well-pointing couple (P, ψ) for X such that $P(A \times I) \subset A$.*

Proof. Let (P_X, ψ_X) and (P_A, ψ_A) be well-pointing couples for X and A, respectively. Choose $\varphi: X \to I$ and $H: X \times I \to X$ satisfying the conditions of Proposition 7 with respect to ψ_X, and define $\alpha: X - \{*\} \to I$ by $\alpha(x) = 1 - \varphi(x)/\psi_X(x)$. Since $A \subset X$ is a cofibration, P_A extends to a homotopy $\bar{P}: X \times I \to X$ with $\bar{P}_0 = 1_X$. Also, ψ_A can be extended to a continuous $\bar{\psi}: X \to I$ by putting

$$\bar{\psi}(x) = \begin{cases} \alpha(x)\, \psi_A H(x, 1) + \varphi(x), & \varphi(x) < \psi_X(x), \\ \psi_X(x), & \varphi(x) = \psi_X(x). \end{cases}$$

(Recall that $H(x, 1) \in A$ when $\varphi(x) < \psi_X(x)$.)

We then have $\bar{\psi}^{-1}(0) = \{*\}$. For, if $x \neq *$ and $\bar{\psi}(x) = 0$, we should have

$$\psi_A H(x, 1) = \varphi(x) = 0.$$

But this would imply $H(x, 1) = *$ and $x \in \bar{A}$. However, it is clear that

$$\psi_X(x') = \psi_X H(x', 1) \quad \text{for all} \quad x' \in \bar{A},$$

and therefore $\psi_X(x) = 0$, contradicting the assumption that $x \neq *$.

The required couple (P, ψ) is now given by

$$P(x, t) = \begin{cases} \bar{P}(x, t/\overline{\psi}(x)), & t < \overline{\psi}(x), \\ P_X(\bar{P}(x, 1), t - \overline{\psi}(x)), & t \geqq \overline{\psi}(x), \end{cases}$$

$$\psi(x) = \operatorname{Min}(1, \overline{\psi}(x) + \psi_X \bar{P}(x, 1)).$$

We shall use this lemma to prove

Proposition 9. *A map $i: A \to X$ in* Top^w *is a cofibration if and only if it is a free cofibration.*

Proof. Only "only if" needs proof. Suppose, then, that $i: A \subset X$ is a cofibration in Top^w. (For simplicity we assume that i is an inclusion.) Let (P, ψ) be a well-pointing couple as described in Lemma 8, and then let H and φ be as in Proposition 7.

Define $H': X \times I \to X$ and $\varphi': X \to I$ by

$$H'(x, t) = \begin{cases} P(H(x, t), \operatorname{Min}[t, \varphi(x)/\psi(x)]), & x \neq *, \\ *, & x = *, \end{cases}$$

$$\varphi'(x) = \varphi(x) - \psi(x) + \sup_{t \in I} \psi H(x, t).$$

H' and φ' then satisfy the conditions of [6], Lemma 4, and it follows that i is a free cofibration.

The dual statement is also true:

Proposition 10. *A map $p: E \to B$ in* Top^w *is a fibration if and only if it is a free fibration.*

Proof. It follows from [5], Theorem 4 that if p is free fibration, then it has the pointed homotopy lifting property with respect to all well-pointed spaces.

It is also true that a map in Top^w is a homotopy equivalence if and only if it is a free homotopy equivalence ([1], 2.18).

Theorem 11. *The category* Top^w, *with the classes of (pointed) fibrations, closed cofibrations, and homotopy equivalences, satisfies the axioms* M1—M6.

Proof. This follows from Theorem 3 and Propositions 9 and 10. It is not hard to show that the constructions in the proof of Proposition 2, when performed on well-pointed spaces, yield well-pointed spaces.

One could hardly expect M0 to hold in Top^w, but Top^w does have sums, finite products, pullbacks of fibrations, pushouts of cofibrations, smash products, suspensions, loop spaces, etc., and this goes a long way.

A simple consequence of the product theorem for cofibrations and Proposition 9 is the following "smash product theorem".

Proposition 12. *If $A \subset X$ and $B \subset Y$ are cofibrations in* Top^w *and at least one of them is closed, then*

$$X \wedge B \cup A \wedge Y \subset X \wedge Y$$

is also a cofibration in Top^w. ($X \wedge B$ and $A \wedge Y$ should here be given the subspace topology induced by $X \wedge Y$.)

Remark. Results analogous to 10, 11 and 12 above hold in the larger category Top^0 consisting of all pointed spaces X for which there exist functions $\psi: X \to I$ with $\psi^{-1}(0) = \{*\}$, but the proofs become a bit more complicated.

One could also try to generalize the results of this section in a different direction. Instead of well-pointed spaces consider the category Cof^K of closed cofibrations under a fixed space K, that is, the objects of Cof^K are closed free cofibrations $K \to X$ and the morphisms are commutative triangles (thus, $\mathrm{Top}^w = \mathrm{Cof}^*$). The analogues of Lemmas 6 and 8 and Propositions 9 and 10 are easily proved, but when we try to prove the corresponding version of Theorem 11, we encounter a little difficulty in showing that the constructions in the proof of Proposition 2 do not take us outside Cof^K. The problem is that it is not clear that Cof^K has path spaces, which are necessary for the construction of the mapping track W. The natural candidate for the path space of $i: K \to X$ is $i_{\#} s: K \to K^I \to X^I$, where $s: K \to K^I$ sends each point of K to the constant path at that point. $K \to X^I$ is then an object of Cof^K if and only if s is a closed free cofibration. This is equivalent to the existence of a continuous $\varphi: K^I \to I$ with $\varphi^{-1}(0) = s(K)$. (See [7] for an example of a path-connected compact Hausdorff space which admits no such φ.) If such a function exists, then Theorem 11 holds for the category Cof^K.

A sufficient condition for the existence of such a $\varphi: K^I \to I$ is that there exist a continuous $\delta: K \times K \to I$ with $\delta^{-1}(0) = \{(k, k) \mid k \in K\}$. This condition is satisfied, for instance, for all metric spaces and all CW-complexes.

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