FUNCTORIAL PHILOSOPHY FOR FORMAL PHENOMENA
PRELIMINARY DRAFT

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Contents

1. Introduction

The purpose of this paper is to introduce the "schematic viewpoint" in algebraic topology. This seems to be the most natural framework in which to discuss the algebraic structures which arise from complex-oriented cohomology theories. Many of the parts which are original are joint work with Mike Hopkins and Matthew Ando.

We give a definition of (formal) schemes which is well adapted to the particular technicalities which arise in the study of Morava K-theory and completed E(n)-theory. We show how to interpret the generalised (co)homology of $\mathbb{C}P^\infty$, $\mathbb{Z} \times BU$, $B\Sigma_{p^m}$, projective bundles and Thom spaces of complex vector bundles, and various other spaces, using the language of formal group theory.

While we use many ideas from algebraic geometry, our examples are rather different from those usually considered by the algebraic geometer on the Clapham omnibus. It is thus difficult to extract from the literature an adequate set of foundations for our work, which cover the situations which we need to cover without prolonged discussion of phenomena which we will never encounter. This paper makes some attempt to remedy this. The reader should be warned, however, that very few of our definitions will be precisely equivalent to those used in algebraic geometry.

The sections are mostly arranged in a pedagogical order, with some more technical pieces of algebra placed at the end.

2. Notation

Given a category $\mathcal{C}$, we usually write $\mathcal{C}(X, Y)$ for the set of $\mathcal{C}$-morphisms from X to Y.

Given a spectrum E we write E^* for $E^* S^0$. If E^* has period 2 and is concentrated in even degrees we write E for E^0 and $E(X)$ for $E^0(X)$. Often, E and K will be Morava E-theory and K-theory, as explained in section 3. The prime p and the height n will be omitted from the notation. In this context, we write $E^\vee Z = \pi_0 L_K(E \wedge Z_+)$ (where Z is a space).

All vector bundles over spaces will be complex. (We will also consider vector bundles over schemes, see section 10

If R is a ring then $R\{a_i \mid i \in I\}$ means the free R-module on the indicated elements. If the a_i are already elements of an R-module M then use of this notation implicitly claims that they generate a free submodule.

3. Morava K-Theory and E-Theory

In this section, we define the cohomology theories which will provide the central examples for the rest of this paper.

Let p be a prime, and $n > 0$ an integer (called the *height*). We shall say that we are working at the chromatic prime p^n, and omit p and n from the notation almost everywhere. In particular, K and E will refer to spectra closely related to those usually called $K(n)$ and $\widehat{E(n)}$ — details are given below.

We will write $\kappa = \mathbb{F}_{p^n}$ for the finite field of order p^n. This has the form $\mathbb{F}_p[\overline{\omega}]/\overline{h}(\overline{\omega})$, for a suitable monic irreducible factor $\overline{h}$ of the cyclotomic polynomial $\Phi_{p^n-1}(x)$ in $\mathbb{F}_p[x]$. This can be lifted uniquely to give a monic irreducible factor h of Φ_{p^n-1} in $\mathbb{Z}_p[x]$, and we can define the Witt ring $W = W\mathbb{F}_{p^n}$ as $\mathbb{Z}_p[\omega]/h(\omega)$.

The ring W is a free module of rank n over $\mathbb{Z}_p$ and is a complete discrete valuation ring. Any element $x \in W$ can be written uniquely as $p^v y$ with $v \geq 0$ and $y \in W^\times$. In particular, W is local with maximal ideal (p) and residue field $W/p = \kappa$.

There is a unique map $\tau\colon \kappa \to W$ (the *Teichmüller map*) satisfying $\tau(ab) = \tau(a)\tau(b)$ and $\tau(a) = a$ (mod p). Indeed, if $\tilde{a} \in W$ is any lift of a, then the sequence $\tilde{a}^{p^{nk}}$ converges p-adically to $\tau(a)$ as $m \to \infty$. We also write $\hat{a}$ for $\tau(a)$.

Any element $a \in W$ can be written uniquely as $a = \sum_{k \geq 0} \tau(a_k)p^k$, for suitable $a_k \in \kappa$. If also $b = \sum_{k \geq 0} \tau(b_k)p^k$ and $c = a + b = \sum_{k \geq 0} \tau(c_k)p^k$ then the c_k are essentially given in terms of the a_k and b_k by the Witt addition formula. However, this fact is rarely useful in the present context.

There is a unique automorphism ϕ of W (the Frobenius automorphism) satisfying $\phi(\hat{a}) = \hat{a}^p$ for all $a \in \kappa$. This also has $\phi^n = 1$. The fixed ring of $C_n = \langle \phi \mid \phi^n = 1 \rangle$ acting on W is just $\mathbb{Z}_p$. In fact, $W \simeq \mathbb{Z}_p[C_n]$ as $\mathbb{Z}_p[C_n]$-modules (but not as rings).

We define an ungraded ring

$$E = E^0 = W[\![u_1, \ldots, u_{n-1}]\!]$$

and a graded ring

$$E^* = E[u, u^{-1}] \qquad |u| = -2$$

We also take $u_0 = p$ and $u_n = 1$ and $u_k = 0$ for $k > n$.

The coefficient ring of the Brown-Peterson spectrum is

$$BP^* = \mathbb{Z}_{(p)}[v_k \mid k > 0] \qquad |v_k| = -2(p^k - 1)$$

We define a map $BP^* \to E^*$ sending v_k to $u^{p^k - 1} u_k$. Using this, we define a functor from spectra to E_*-modules by

$$E_*(X) = E_* \otimes_{BP_*} BP_*(X)$$

The BP^*-module E^* is Landweber exact, so this functor is a homology theory, represented by a spectrum which we shall also call E. We shall refer to this as Morava E-theory. The ring E has a pleasant interpretation in terms of deformations of formal groups, which will be discussed later.

REMARK 3.1. Given two spectra E and E' and a natural isomorphism $\overline{f} \colon E_*(-) \to E'_*(-)$, there is an isomorphism $f \colon E \to E'$ of representing spectra, unique up to addition of a phantom map. In the present case, we shall see shortly that E can be written as the homotopy inverse limit of a tower of spectra with finite homotopy groups. It follows that there are no phantom maps to E, and thus that E is unique up to unique isomorphism.

This spectrum has been constructed by pure homotopy theory, so only homotopical methods (such as obstruction theory) are available to analyse it. Nonetheless, Hopkins and Miller have shown that can be made canonically into an E_∞ ring spectrum.

The ring E is a complete, regular local ring, with maximal ideal $\mathfrak{m}_E = (p = u_0, u_1, \ldots, u_{n-1})$. By iterated cofibrations, preferably carried out in the derived category of E_∞ E-modules, one can construct an algebra-spectrum K over E with $K^* = E^*/\mathfrak{m}_E = \kappa[u^{\pm 1}]$. We shall refer to this as Morava K-theory. It is a finite wedge of suspensions of the spectrum usually called $K(n)$.

More generally, given $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ we define $I_\alpha = (u_0^{\alpha_0}, \ldots u_{n-1}^{\alpha_{n-1}}) \lhd E^0$. We can construct E-algebra spectra $E_\alpha = E/I_a l$ with $E_\alpha^0 = E^0/I_\alpha$. For a cofinal family of ideals I_α there is a finite ring spectrum M_α such that $E_\alpha = E \wedge M_\alpha$ (see [3]). The original spectrum E can be recovered as $E = \operatorname*{holim}_{\leftarrow \alpha} E_\alpha$.

If $n = 1$ then $E = KU_p$, the p-adic completion of complex K-theory. Moreover, $K = KU/p$.

We make E into a topological ring by declaring $\{\mathfrak{m}^k \mid k \geq 0\}$ to be a base of neighbourhoods of zero. This actually makes E into a formal ring (see section 6). We also consider K as a formal ring with the discrete topology.

Rather than think about E and related rings directly, we shall consider the represented functors $\mathrm{spf}(E)$. This is the functor from a suitable class of topological rings to sets defined by

$$\mathrm{spf}(E)(R) = \mathrm{Hom}_{\mathrm{cts}}(E, R)$$

Details are in section 6.

We shall use the following notation:

$$\begin{aligned}
X &= \mathrm{spf}(E) \\
G &= \mathrm{spf}(E(\mathbb{C}P^\infty)) \\
X_1 &= \mathrm{spf}(W) = V(u_1, \ldots u_{n-1}) < X \\
G_1 &= G \times_X X_1 \\
X_0 &= \mathrm{spf}(\kappa) = V(\mathfrak{m}_E) < X_1 \\
G_0 &= \mathrm{spf}(K(\mathbb{C}P^\infty)) = G \times_X X_0
\end{aligned}$$

4. Schemes

Let $\mathcal{R}ings$ be the category of rings. Given a ring R we consider the functor

$$\mathrm{spec}(R) \colon \mathcal{R}ings \to \mathcal{S}ets$$

$$\mathrm{spec}(R)(S) = \mathcal{R}ings(R, S)$$

An *affine scheme* is a functor $X \colon \mathcal{R}ings \to \mathcal{S}ets$ such that $X \simeq \mathrm{spec}(R)$ for some R. We shall often say "scheme" instead of "affine scheme". Non-affine schemes do occur in topology (for example, in the theory of the period mapping or of elliptic spectra), but we will have quite enough to do without considering them here.

EXAMPLE 4.1.

$$G_m(S) = S^\times = \text{ the group of units of } S$$

$$G_m(S) \simeq \mathcal{R}ings(\mathbb{Z}[x, x^{-1}], S) \qquad \text{so} \qquad G_m \simeq \mathrm{spec}(\mathbb{Z}[x^{\pm 1}])$$

One might say that the scheme G_m is a "more natural" object than the representing ring $\mathbb{Z}[x^{\pm 1}]$. This is true to a much greater extent of many of the rings which arise in topology.

The group $G_m(S) = S^\times$ is usually called the multiplicative group of S, so we simply refer to G_m as "the multiplicative group". It arises, incidentally, in equivariant topology: $G_m = \mathrm{spec}(K_{S^1}^0)$.

EXAMPLE 4.2. A *formal group law* over a ring S is a formal power series

$$F(x, y) = \sum_{k,l \geq 0} a_{kl} x^k y^l$$

satisfying

$$\begin{aligned}
F(x, 0) &= x \\
F(x, y) &= F(y, x) \\
F(F(x, y), z) &= F(x, F(y, z))
\end{aligned}$$

We can define a scheme FGL as follows:

$$\mathrm{FGL}(S) = \{ \text{ formal group laws over } S \}$$

To see that FGL is a scheme, we consider the ring $L_0 = \mathbb{Z}[a_{kl} \mid k, l \geq 0]$ and the formal power series $F(x, y) = \sum a_{kl} x^k y^l \in L_0[\![x, y]\!]$. We then let I be the smallest ideal of L_0 such that the formal group law conditions for F are satisfied modulo I. For example, the first condition says that $a_{00} - 1 \in I$ and $a_{k0} \in I$ for $k > 0$, and the second says that $a_{kl} - a_{lk} \in I$. Finally, set $L = L_0/I$. It is easy to see that $FGL = \mathrm{spec}(L)$. The ring L is called the Lazard ring. It is usual in algebraic topology to identify L with MU_*. We shall take a slightly different point of view (explained at the end of section 14) which takes the grading into account.

We also define a *pre-scheme* to be an arbitary functor $X \colon \mathcal{R}ings \to \mathcal{S}ets$. We shall usually only use this language when we intend to prove later that X is a scheme.

Note that spec is a functor $\mathcal{R}ings^{\mathrm{op}} \to \mathcal{S}chemes$. In fact, by Yoneda's lemma, it is an equivalence of categories:
$$\mathcal{S}chemes(\mathrm{spec}(S), \mathrm{spec}(R)) \simeq \mathrm{spec}(R)(S) \simeq \mathcal{R}ings(R, S)$$

We write $\mathbb{A}^1$ for the *affine line*:
$$\mathbb{A}^1(S) = S \qquad \mathbb{A}^1 = \mathrm{spec}(\mathbb{Z}[x])$$

We write $\mathcal{O}_X = \mathcal{S}chemes(X, \mathbb{A}^1)$ so (by Yoneda again) we have $\mathcal{O}_X = R$ iff $X = \mathrm{spec}(R)$. We refer to $\mathcal{O}_X$ as the *ring of functions* on X. We shall often write $\otimes_X$ for $\otimes_{\mathcal{O}_X}$ and Hom_X for $\mathrm{Hom}_{\mathcal{O}_X}$.

EXAMPLE 4.3. There is a map $\alpha \colon G_m \times \mathrm{FGL} \to \mathrm{FGL}$ defined by
$$\alpha(u, F) = F_u \qquad\qquad F_u(x, y) = uF(x/u, y/u)$$

It is best to define this map as above, and work as far as possible with the description given, rather than trying to work out the representing map $\alpha^* \colon \mathcal{O}_{\mathrm{FGL}} \to \mathcal{O}_{G_m} \otimes \mathcal{O}_{\mathrm{FGL}}$. Sometimes one cannot avoid calculating the representing map, so we shall do this case as an example. We think of a_{ij} as a natural map $\mathrm{FGL}(R) \to R$, defined implicitly by
$$F(x, y) = \sum_{ij} a_{ij}(F) x^i y^j$$

Thus
$$F_u(x, y) = u \sum a_{ij}(F)(x/u)^i (y/u)^j = \sum u^{1-i-j} a_{ij}(F) x^i y^j$$
This shows that $a_{ij}(F_u) = u^{1-i-j} a_{ij}(F)$, in other words $\alpha^*(a_{ij}) = u^{1-i-j} a_{ij}$.

EXAMPLE 4.4. A *strictly invertible* power series over a ring S is a formal power series $f \in S[\![x]\!]$ such that $f(x) = x + O(x^2)$. This implies, of course, that f has a composition-inverse $g = f^{-1}$, so that $f(g(x)) = x = g(f(x))$. We write $\mathrm{IPS}(S)$ for the set of such f, which is easily seen to be a scheme. It is actually a group scheme, in that $\mathrm{IPS}(S)$ is a group (under composition), functorially in S.

The group IPS acts on FGL by
$$(f, F) \mapsto F_f \qquad\qquad F_f(x, y) = f(F(f^{-1}x, f^{-1}y))$$

A *strict isomorphism* between formal group laws F and G is a strictly invertible series f such that $f(F(a, b)) = G(f(a), f(b))$. Let SI be the following scheme:
$$\mathrm{SI}(S) = \{(F, f, G) \mid F, G \in \mathrm{FGL}(S) \text{ and } f \colon F \to G \text{ is a strict iso }\}$$

There is an evident composition map
$$\mathrm{SI} \times_{\mathrm{FGL}} \mathrm{SI} \to \mathrm{SI} \qquad\qquad ((F, f, G), (G, g, H)) \mapsto (F, gf, H)$$

Moreover, there is an isomorphism
$$\mathrm{IPS} \times \mathrm{FGL} \to \mathrm{SI} \qquad\qquad (F, f) \mapsto (F, f, F_f)$$

Again, one can write implicit formulae in the representing rings, but this should be avoided where possible.

The category of schemes is quite "geometric". It has an initial object $\emptyset = \mathrm{spec}(0)$ and a final object $1 = \mathrm{spec}(\mathbb{Z})$. It has coproducts and pullbacks:
$$X \sqcup Y = \mathrm{spec}\left(\mathcal{O}_X \times \mathcal{O}_Y\right)$$
$$X \times_Z Y = \mathrm{spec}\left(\mathcal{O}_X \otimes_Z \mathcal{O}_Y\right)$$

As functors, we have
$$(X \times_Z Y)(R) = X(R) \times_{Z(R)} Y(R)$$

but
$$(X \sqcup Y)(R) = \{(S, T, x, y) \mid S, T \le R \ , \ R = S \times T \ , \ x \in X(S) \ , \ y \in Y(T)\}$$

To explain this, note that an element of $(X \sqcup Y)(R)$ is (by Yoneda) a map $\mathrm{spec}(R) \to X \sqcup Y$. This will be given by a decomposition $\mathrm{spec}(R) = \mathrm{spec}(S) \sqcup \mathrm{spec}(T)$ and maps $\mathrm{spec}(S) \to X$ and $\mathrm{spec}(T) \to Y$.

More general colimits do exist in the category of schemes, but the geometric interpretation is typically bad. Part of the problem is that we consider only affine schemes. Algebraic geometers do have an extensive theory of non-affine schemes, of course, but they seem not to be very relevant in topology. Even if we allowed non-affine schemes, many problems with colimits would remain. Some of these can be resolved using the ideas of faithfully flat descent and stack theory, which we will discuss later.

An important class of delicate colimit problems which we will have to consider involves taking the quotient of a scheme X by the action of a finite group G. The functor $S \mapsto X(S)/G$ is unlikely to be a scheme. The obvious candidate for X/G is $\mathrm{spec}(\mathcal{O}_X^G)$. This gives a map $X(S)/G \to (X/G)(S)$, which is iso when S is an algebraically closed field, but not in general.

We can also do a number of things with subschemes. A *closed subscheme* of X is a scheme of the form $V(I) = \mathrm{spec}(\mathcal{O}_X/I)$ for an ideal $I \le \mathcal{O}_X$. An *open subscheme* is one of the form $D(a) = \mathrm{spec}(\mathcal{O}_X[a^{-1}])$ for some $a \in \mathcal{O}_X$, and a *locally closed subscheme* has the form $D(a) \cap V(I) = \mathrm{spec}(\mathcal{O}_X[a^{-1}]/I)$.

EXAMPLE 4.5. Suppose $X = \mathrm{spec}(k[x])$ is the affine line over a field k, and $\lambda, \mu \in k$. The closed subscheme $V(x - \lambda) = \mathrm{spec}(k[x]/(x - \lambda)) \simeq \mathrm{spec}(k)$ corresponds to the point λ of the affine line; it is natural to refer to it as $\{\lambda\}$. The closed subscheme $V((x - \lambda)(x - \mu))$ corresponds to the pair of points $\{\lambda, \mu\}$. If $\lambda = \mu$, this is to be thought of as the point λ with multiplicity two, or as an infinitesimal thickening of the point λ.

We can easily form the intersection of locally closed subschemes:
$$D(a) \cap V(I) \cap D(b) \cap V(J) = D(ab) \cap V(I + J)$$

We cannot usually form the union of open subschemes and still have an affine scheme. Again, it would be easy enough to consider non-affine schemes, but it rarely seems to be necessary. Moreover, a closed subscheme $V(a)$ determines the complementary open subscheme $D(a)$ but not conversely; $D(a) = D(a^2)$ but $V(a) \ne V(a^2)$ in general.

We say that a scheme X is *reduced* iff $\mathcal{O}_X$ has no nonzero nilpotents, and write $X_{\mathrm{red}} = \mathrm{spec}(\mathcal{O}_X/\sqrt{0})$, which is the largest reduced closed subscheme of X. Moreover, if $Y \subseteq X$ is closed then $Y_{\mathrm{red}} = X_{\mathrm{red}}$ iff $X(k) = Y(k)$ for every field k.

We define the union of closed subschemes by $V(I) \cup V(J) = V(I \cap J)$. We also define the *schematic union* by $V(I) + V(J) = V(IJ)$. This is a sort of "union with multiplicity" — in particular, $V(I) + V(I) \ne V(I)$ in general. In the previous example, we have
$$\{\lambda\} \cup \{\lambda\} = V((x - \lambda)^2)$$

which is a thickening of $\{\lambda\}$. Note that $V(IJ)_{\mathrm{red}} = V(I \cap J)_{\mathrm{red}}$.

We shall say that X is *connected* iff it cannot be split nontrivially as $Y \sqcup Z$, iff there are no idempotents in $\mathcal{O}_X$ other than 0 and 1. There is more information about this sort of question in section 32.

EXAMPLE 4.6. Let E be Morava E-theory, and G a finite group. Write $X = \mathrm{spec}(p^{-1}(E^0BG))$. **Warning:** this is not the same as $(p^{-1}E)^0BG \simeq (p^{-1}E)^0(\text{point})$. Work of Hopkins, Kuhn and Ravenel [8] implies that X has one component for each conjugacy class of Abelian p-subgroups of G. On the other hand, $\mathrm{spec}(K^0BG)$ is connected (where K is Morava K-theory).

We shall say that a scheme X is *integral* iff $\mathcal{O}_X$ is an integral domain, and that X is *irreducible* iff X_{red} is integral. We also say that X is *Noetherian* iff the ring $\mathcal{O}_X$ is Noetherian. If so, then X_{red} can be written in a unique way as a finite union $\bigcup_i Y_i$ with Y_i an integral closed subscheme. The schemes Y_i are called the *irreducible components* of X_{red}; they are precisely the schemes $V(\mathfrak{p}_i)$ for $\mathfrak{p}_i$ a minimal prime ideal of $\mathcal{O}_X$. We can also write $X = \bigcup_i X_i$ with $(X_i)_{\mathrm{red}} = Y_i$, but this decomposition is not quite unique. See [11, section 6] for this material.

Suppose that X is Noetherian and reduced, say $X = \bigcup_{i \in S} Y_i$ as above for some finite set S. Suppose $S = S' \sqcup S''$. Write $X' = \bigcup_{S'} Y_i = V(I')$, where $I' = \bigcap_{S'} \mathfrak{p}_i$, and similarly for X'' and I''. If we then write

$$\Gamma(I') = \{a \in \mathcal{O}_X \mid a(I')^N = 0 \text{ for } N \gg 0\}$$

we find that $\Gamma(I') = I''$ and thus $V(\Gamma(I')) = X''$. This construction occurs in the Greenlees-May theory of local cohomology and Tate spectra [4, 5].

EXAMPLE 4.7. Take $Z = \operatorname{spec}(k[x,y]/(xy^2))$ and set

$$X = V(y) = \operatorname{spec}(k[x])$$

$$X' = V(y^2) = \operatorname{spec}(k[x,y]/(y^2))$$

$$Y = V(x) = \operatorname{spec}(k[y])$$

Then X is the x-axis, Y is the y-axis and X' is an infinitesimal thickening of X. The schemes X and Y are integral, and X' is irreducible because $X'_{\mathrm{red}} = X$. The scheme Z is reducible, and its irreducible components are X and Y.

EXAMPLE 4.8. Let G be a finite group, and $X = \operatorname{spec}(\mathrm{H}^{2*}(BG; \overline{\mathbb{F}}_p))$. Then work of Quillen shows that X has one irreducible component for each maximal conjugacy class of elementary Abelian p-subgroups.

EXAMPLE 4.9. Let E be Morava E-theory (with height n), and suppose that A is a finite Abelian p-group. Write $A^* = \operatorname{Hom}(A, S^1)$ and $E_A = E(BA^*)$. We shall show in section 26 that $Y = \operatorname{spec}(E_A)$ has one irreducible component Y_B for each quotient A/B of rank at most n, and derive many properties of the schemes Y_B.

5. Formal Rings and Modules

Let R be a ring, and M an R-module. A *linear topology* on M is a topology such that the collection of open submodules forms a base of neighbourhoods of zero. We shall write $N \leq_O M$ to indicate that N is an open submodule. Note that if N is open and $N \leq L \leq M$ then L is a union of translates of N and thus also open. Similarly, $M \setminus N$ is open, so N is closed. Note also that the ring operations are automatically continuous for a linear topology on R. Any directed family of submodules gives rise to a linear topology in an obvious way. We say that M is *complete* with respect to a given linear topology iff

$$M = \varprojlim_{N \leq_O M} M/N$$

Suppose that M is a complete linearly-topologised A-module, and that N is a submodule. Then N inherits a linear topology in an obvious way. The closure of N is given by

$$\overline{N} = \bigcap_{L \leq_O M} N + L$$

Moreover, N is closed iff it is complete. If so, the quotient M/N also inherits a complete linear topology.

DEFINITION 5.1. A *formal ring* is a ring with a given linear topology, with respect to which it is complete.

DEFINITION 5.2. An *ideal of definition* for a formal ring R is an open ideal I such that $\{I^k \mid k \geq 0\}$ is a base of neighbourhoods of zero. Note that such a thing may or may not exist, and that in general a power of an open ideal need not be open.

EXAMPLE 5.1. Consider $K(BU) = K^0 BU = K[\![c_k \mid k > 0]\!]$, where K is Morava K-theory. We give this a topology by declaring that $\ker(K(BU) \to K(Z))$ be open for any finite complex Z and any map $Z \to BU$. In particular, the augmentation ideal J is open. Note that every open ideal contains c_k for $k \gg 0$. In particular, J^2 is not open. This shows that there is no ideal of definition.

DEFINITION 5.3. A *formal module* over a formal ring R is a complete linearly topologised R-module M such that the action map $R \times M \to M$ is continuous. (Note that this has nothing to do with the Lubin-Tate theory of formal A-modules, which are formal groups G with a map $A \to \text{End}(G)$).

If M and N are formal modules, the symbol $\text{Hom}_R(M, N)$ will always refer to *continuous* homomorphisms. Note that

$$\text{Hom}_R(M, N) = \varprojlim_\beta \varinjlim_\alpha \text{Hom}_R(M/M_\alpha, N/N_\beta)$$

where M_α and N_β run over open submodules. We give this module the topology of uniform convergence, which is defined by the family of submodules $\{\text{Hom}(M, N_\beta)\}$; this makes it into a formal module. In particular, we define the *topological dual* $M^\vee = \text{Hom}(M, R)$.

If S is a set, we write $F(S, M)$ for the formal R-module of functions $a \colon S \to R$, topologised as a product of copies of M. We say that $a \in F(S, M)$ is nullconvergent iff for all $M' \leq_O M$ we have $a_s \in M'$ for almost all $s \in S$. We write $F_0(S, M)$ for the set of nullconvergent functions. We give $F_0(S, M)$ the topology defined by the submodules $F(S, M') \cap F_0(S, M)$, where $M' \leq_O M$. One can then check that $F(S, M)^\vee$ is topologically isomorphic to $F_0(S, M^\vee)$. If M is finitely generated then $F_0(S, M)^\vee$ is algebraically isomorphic to $F(S, M^\vee)$ but has a different topology. Moreover, for all M and N there are topological isomorphisms

$$F(S, M) \widehat{\otimes}_R F(T, N) = F(S \times T, M \widehat{\otimes}_R N)$$

$$F_0(S, M) \widehat{\otimes}_R F_0(T, N) = F(S \times T, M \widehat{\otimes}_R N)$$

We say that M is *pro-free* iff it is topologically isomorphic to $F(S, R)$ for some S. If so, the functor

$$N \mapsto M \widehat{\otimes}_R N \simeq F(S, N)$$

is exact and preserves infinite products.

EXAMPLE 5.2. Suppose that Z is a spectrum and $E_0 Z$ is free over E_0 on generators indexed by a set S. Then $E(Z) \simeq F(S, E)$ and $E^\vee Z = \pi_0 L_K(E \wedge Z) \simeq F_0(S, E)$.

REMARK 5.1. Let R be a Noetherian formal ring. Suppose that there is an ideal of definition I, and that $I \leq \mathfrak{m}$ for every maximal ideal $\mathfrak{m} \lhd R$. Then every finitely generated module is complete under the topology defined by the submodules $I^k M$. Let $f \colon M \to N$ be a map of finitely generated modules. Then f is continuous, if it is injective it is a closed embedding, and if it is surjective it is open (and thus a quotient map). The key point here is the Artin-Rees lemma, see [11, section 8].

6. Formal Schemes

The definitions in this section are not the usual ones in algebraic geometry, but they appear to be what is required for our applications.

DEFINITION 6.1. If R is a formal ring, we define a functor

$$\text{spf}(R) \colon \mathcal{R}ings \to \mathcal{S}ets \qquad \text{spf}(R)(S) = \varinjlim_{I \lhd_O R} \mathcal{R}ings(R/I, S)$$

A *formal scheme* is a functor of the form $\text{spf}(R)$ for some formal ring R. We write $\mathcal{F}ormal$ for the category of formal schemes.

Note that any ordinary scheme is a formal scheme. We shall say that a formal scheme is *discrete* iff it is actually an ordinary scheme. Any formal scheme is a directed colimit of closed inclusions of discrete schemes. Considered as a functor $\mathcal{R}ings \to \mathcal{S}ets$, it preserves finite limits.

EXAMPLE 6.1. Suppose X is a scheme and $Y = V(I)$ is a a a closed subscheme. Let $\mathcal{O}_{\widehat{X}}$ be the completion of $\mathcal{O}_X$ at I, and give it the linear topology defined by the powers of I. Then $\widehat{X} = \text{spf}\, \mathcal{O}_{\widehat{X}}$ is a formal scheme, called the *formal completion* of X along Y. It is the colimit of the schemes $V(I^k)$ in the category of formal schemes.

EXAMPLE 6.2. Consider the ring $R = \mathbb{Z}[\![x]\!]$ with the formal topology defined by powers of x. Then

$$\mathrm{spf}(R)(S) = \widehat{\mathbb{A}}^1(S) = \mathrm{Nil}(S) = \{ \text{ nilpotent elements of } S\}$$

EXAMPLE 6.3. If Z is a reasonable space, with finite subspectra Z_α, then $E(Z)$ will be the same as $\varprojlim E(Z_\alpha)$, and this will be a formal ring. More details are given in section 7.

EXAMPLE 6.4. $\mathrm{spf}\,\mathbb{Z}_p$ is a formal scheme.

EXAMPLE 6.5. Write K for p-adic complex K-theory, and consider

$$K(\mathbb{C}P^\infty) = \varprojlim_k K(\mathbb{C}P^k) = \varprojlim_l K(B\mathbb{Z}/p^l) = \mathbb{Z}_p[\![x]\!]$$

This ring has three different formal topologies, defined by ideals

$$J_{k,l} = ([p^k](x), p^l)$$

$$I_k = \ker(K(\mathbb{C}P^\infty) \to K(B\mathbb{Z}/p^k)) = ((1+x)^{p^k} - 1)$$

$$K_m = \ker(K(\mathbb{C}P^\infty) \to K(\mathbb{C}P^{m-1})) = (x^m)$$

The first of these seems most useful. It is also defined by the ideals (x^k, p^l).

Given a formal scheme X, we can form the ring

$$\mathcal{O}_X = \mathcal{F}ormal(X, \mathbb{A}^1)$$

A point $x \in X(A)$ gives a map $\hat{x}\colon \mathcal{O}_X \to A$; we write J_x for the kernel. These ideals form a directed family, and thus give rise to a linear topology on $\mathcal{O}_X$. Moreover, $\mathcal{O}_X$ is complete with this topology, so $\mathcal{O}_X$ is a formal ring. One can show that

$$\mathcal{F}ormal(X, Y) = \mathcal{F}ormal\mathcal{R}ings(\mathcal{O}_Y, \mathcal{O}_X)$$

$$\mathcal{O}_{\mathrm{spf}(R)} = R$$

so that the category of formal schemes is dual to that of formal rings.

DEFINITION 6.2. Let R be a formal ring. An element $x \in R$ is *topologically nilpotent* iff $x^n \to 0$ as $n \to \infty$, iff it is nilpotent in R/I for every $I \lhd_O R$. We write $\mathrm{Nil}(R)$ for the ideal of topologically nilpotent elements. An ideal $J \lhd R$ is *topologically nilpotent* iff $J \leq \mathrm{Nil}(R)$, and *strongly topologically nilpotent* iff for all $I \lhd_O R$ we have $J^N \leq I$ for $N \gg 0$. Note that a finitely generated ideal which is topologically nilpotent is strongly so.

Note that

$$\mathcal{F}ormal(X, \widehat{\mathbb{A}}^1) = \mathcal{F}ormal\mathcal{R}ings(\mathbb{Z}[\![x]\!], \mathcal{O}_X) = \mathrm{Nil}(\mathcal{O}_X)$$

The following is proved later as proposition 32.1.

PROPOSITION 6.1. $\mathrm{Nil}(R)$ is the intersection of all open prime ideals, and is thus closed.

We can define products and coproducts of formal schemes, with

$$\mathcal{O}_{X \sqcup Y} = \mathcal{O}_X \times \mathcal{O}_Y$$

$$\mathcal{O}_{X \times Y} = \mathcal{O}_X \widehat{\otimes} \mathcal{O}_Y = \varprojlim_{\alpha, \beta} \mathcal{O}_{X_\alpha} \otimes \mathcal{O}_{X_\beta}$$

Let X be a formal scheme. If $J \lhd \mathcal{O}_X$ is a closed ideal then $\mathcal{O}_X/J$ is a formal ring and we write $V(J) = \mathrm{spf}(\mathcal{O}_X/J)$. A formal scheme of this kind will be called a *closed formal subscheme* of X.

Most constructions with schemes can be carried over to formal schemes, by requiring rings and modules to be topologised and maps to be continuous.

We will also consider formal schemes as representable functors on the category of formally-topologised rings:

$$X(R) = \mathcal{FormalRings}(\mathcal{O}_X, R) = \mathcal{Formal}(\operatorname{spf} R, X)$$

7. Schemes in Algebraic Topology

Let E be a spectrum like Morava E-theory. The precise list of properties we require is as follows. E must be a complex-orientable two-periodic ring spectrum with $E^{\mathrm{odd}} = 0$. We must be given a directed system of ideals $I_\alpha \lhd E^0$ and algebra-spectra E_α such that $E^*_\alpha = E^*/I_\alpha$ and $E = \operatorname*{holim}_{\leftarrow \; \alpha} E_\alpha$. It follows that the I_α define a formal topology on E^0. We also require that E^0 be Noetherian, and that there be a *subsequence* of the I_α which is cofinal. This will allow us to apply the usual theory of lim and $\lim^1$ for towers. Much of the theory would go through with less restrictive assumptions, of course.

EXAMPLE 7.1. Anything obtained from Morava E-theory by killing some generators qualifies. In particular, Morava K-theory is an example, as are p-adic or mod p complex K-theory. Any two-periodic even ring spectrum E with the discrete topology also qualifies. If we take Morava E-theory at height n, invert u_m (with $m < n$) and complete at $(u_0, \ldots u_{m-1})$, we get another example.

We write $E(Z)$ for $E^0(Z)$ and E for E^0.

DEFINITION 7.1. Let Z be a CW-spectrum, and let Z_β run over the finite subspectra. Write $J_{\alpha\beta} = \ker(E(Z) \to E_\alpha(Z_\beta))$. We say that Z is *tolerable* iff $E(Z) = \lim_{\leftarrow \; \alpha,\beta} E(Z)/J_{\alpha\beta}$ and $E^1(Z) = 0$. If so, we give $E(Z)$ the formal topology defined by the submodules $J_{\alpha\beta}$. We say that Z is *decent* if $E_0 Z$ is a free E-module and $E_1 Z = 0$ — this implies that Z is tolerable.

A finite spectrum Z is tolerable iff $E^1 Z = 0$. Indeed, $[\Sigma^k Z, E/I_\alpha]$ is finite, by induction on the number of cells. Thus the tower is Mittag-Leffler and $[\Sigma^k Z, E] = \lim_{\leftarrow \; \alpha} [\Sigma^k Z, E/I_\alpha]$ by the Milnor exact sequence.

As remarked earlier, if Z is decent then $E(Z)$ is pro-free. It follows that $E(Z)\widehat{\otimes}_E E(W)$ is a cohomology theory of W, and thus that $E(Z \wedge W) = E(Z)\widehat{\otimes}_E E(W)$.

DEFINITION 7.2. Let Z be a tolerable CW complex. We then write

$$Z_E = \operatorname{spf}_E Z = \operatorname{spf}(E(Z))$$

This is a formal scheme, covariantly functorial in Z. It is also naturally polarised (see section 11).

For any finite complex and many infinite complexes, the ring $E(Z)$ will be Noetherian. The Artin-Rees lemma will then apply, and many questions about formal topologies will simplify greatly (c.f. remark 5.1).

There are various obvious possible modifications. For example, we can think about $\mathrm{H}^* Z = \mathrm{H}^*(Z; \mathbb{F}_2)$ as an ungraded commutative ring, and define Z_H to be its spectrum. We can define an action of $G_m = \operatorname{spec}(\mathbb{F}_2[u^{\pm 1}])$ on Z_H by the map

$$\alpha^* \colon \mathrm{H}^* Z \to \mathcal{O}_{G_m} \otimes \mathrm{H}^* Z$$
$$\alpha^*(\sum_k a_k) = \sum_k a_k u^k \qquad (a_k \in \mathrm{H}^k Z)$$

This action will then keep track of the grading.

In what follows, we will give (some) details of the two-periodic case, but feel free to present examples from analogous cases.

Write $X = \operatorname{point}_E = \operatorname{spf}(E^0)$, so there is a canonical map $Z_E \to X$. We now have a covariant functor from the homotopy category of (some) spaces to the (geometric) category of polarised formal schemes over X. This preserves quite a lot of structure:

$$
\begin{aligned}
(Y \sqcup Z)_E &= Y_E \sqcup Z_E \\
(Y \times Z)_E &= Y_E \times_X Z_E \qquad \text{if } Z \text{ is decent}
\end{aligned}
$$

EXAMPLE 7.2. Let K denote p-adic complex K-theory, and let A be a finite Abelian p-group, with classifying space BA. Then we have

$$(BA)_K = \text{Hom}(A^*, G_m)$$

In other words, for any formal $\mathbb{Z}_p$-algebra R we have

$$\text{Hom}(K(BA), R) = \text{Hom}(A^*, R^\times)$$

This is just a paraphrase of the well-known fact that

$$K(BA) = \mathbb{Z}_p[A^*]$$

EXAMPLE 7.3. If $E = MU[u^{\pm 1}] = \bigvee_{k \in \mathbb{Z}} \Sigma^{2k} MU$ then $MU^* \simeq E^0$ (by $a \mapsto u^{-|a|/2}a$). Thus, (by a fundamental result of Quillen) we have $X \simeq \text{spec}(MU^*) \simeq \text{FGL}$.

Many more examples will be given when we have a little more language with which to talk about them.

Note also that there are a number of rings (and therefore schemes) of topological origin which do not quite fit into the framework discussed above. For example, if Z is an H-space then we can consider $\text{spec}(E_0 Z)$. It turns out to be more useful to modify this slightly, and consider

$$E^\vee Z = \pi_0 \underset{\alpha}{\text{holim}}\, E_\alpha \wedge Z_+$$

In the case of Morava E-theory, this is the same as $\pi_0 L_K(E \wedge Z)$ (see [9]). Note that as usual we are suppressing the height n from the notation, so $L_K = L_{K(n)}$.

If Z is decent then $E^\vee Z = E_0 Z^\wedge = E(Z)^\vee$. On the other hand, if G is a finite group then the typical situation seems to be as follows. $E^0 BG$ is a free module of finite rank over E^0 and $E^{\text{odd}} BG = 0$. However, $E_0 BG$ is I-torsion and has an odd-dimensional part. On the other hand, $\pi_* L_K(E \wedge BG_+)$ is just the continuous dual of $E^* BG$. I am not sure what is the strongest theorem one can prove along these lines.

More generally, given a K-local E-algebra spectrum F we can consider $\text{spf}(F^0)$ as a scheme over $X = \text{spf}(E^0)$. By taking F to be the function spectrum $F(Z_+, E)$ or the localised smash product $L_K(E \wedge Z_+)$ we recover the previous examples.

8. Points and Sections

Let X be a scheme. An R-*valued point* of X is an element $a \in X(R)$. Such a thing corresponds naturally to a map $a' \colon Y = \text{spec}(R) \to X$. Explicitly, a' is the natural transformation

$$Y(S) = \mathcal{R}ings(R, S) \to X(S)$$

defined as follows. If $Y(S) \ni b \colon R \to S$ then $a'(b) = X(b)(a)$, the image of $a \in X(R)$ under the map $X(b) \colon X(R) \to X(S)$. From now on we shall not distinguish notationally between a and a'. We shall also refer to R-valued points as Y-based points, or points defined over Y (where $Y = \text{spec}(R)$ still), and write $a \in_R X$ or $a \in_Y X$. Such a thing should be thought of as a family of points of X indexed by Y.

We shall often talk about "a point $a \in X$" without specifying R. In that context, the word "scalar" means an element of the unspecified ring R, in other words a point of $\mathbb{A}^1$ over R. Given a point a of X and an element $f \in \mathcal{O}_X$ we write $f(a)$ for the image of a under the map $f \colon X(R) \to \mathbb{A}^1(R) = R$. If $X = \text{spec}(S)$ then we can identify $\mathcal{O}_X$ with S and so consider f as an element of S, and a as an element of $\mathcal{R}ings(S, R)$. With these identifications, we have $f(a) = a(f)$.

In these terms, we have

$$D(f)(R) = \{a \in_R X \mid f(a) \text{ is invertible }\}$$

$$V(I)(R) = \{a \in_R X \mid f(a) = 0 \text{ for all } f \in I\}$$

EXAMPLE 8.1. Let F be a point of FGL, in other words a formal group law over some ring R. We can write
$$[3](x) = F(x, F(x, x)) = 3x + u(F)x^2 + v(F)x^3 + O(x^4)$$
for certain scalars $u(F)$ and $v(F)$. This construction associates to each point $F \in \text{FGL}$ a point $v(F) \in \mathbb{A}^1$ in a natural way, thus giving an element $v \in \mathcal{O}_{FGL}$. Of course, we know that $\mathcal{O}_{FGL}$ is the Lazard ring L, which is generated by the coefficients a_{kl} of the universal formal group law
$$F_{\text{univ}}(x, y) = \sum_{k, l} a_{kl} x^k y^l$$
Using this formal group law, we find that
$$[3](x) = 3x + 3a_{11}x^2 + (a_{11}^2 + 8a_{12})x^3 + O(x^4)$$
This means that
$$v(F_{\text{univ}}) = a_{11}^2 + 8a_{12}$$
It follows for any F over any ring R that $v(F)$ is the image of $a_{11}^2 + 8a_{12}$ under the map $L \to R$ classifying F.

EXAMPLE 8.2. For any scalar a, we have a formal group law
$$H_a(x, y) = x + y + axy$$
The construction $a \mapsto H_a$ gives a natural transformation $\mathbb{A}^1(R) \to \text{FGL}(R)$, in other words a map of schemes $\mathbb{A}^1 \to \text{FGL}$. This can be thought of as a family of formal group laws, parametrised by $a \in \mathbb{A}^1$. It can also be thought of as a single formal group law over $\mathbb{Z}[a] = \mathcal{O}_{\mathbb{A}^1}$. This map $\mathbb{A}^1 \to \text{FGL}$ is actually a closed embedding, in other words an isomorphism of $\mathbb{A}^1$ with the closed subscheme $V(J)$ of FGL, where $J = (a_{kl} \mid k + l > 1)$.

EXAMPLE 8.3. The point of view described above allows for some slightly schizophrenic constructions, such as regarding the two projections $\pi_0, \pi_1 \colon X \times X \to X$ as two points of X over X^2. Indeed, this is the universal example of a scheme Y equipped with two points of X defined over Y. Similarly, we can think of the identity map $X \to X$ as the universal example of a point of X. This is analogous to thinking of the identity map of $K(\mathbb{Z}, n)$ as a cohomology class $u \in H^n K(Z, n)$; this is of course the universal example of a space with a given n-dimensional cohomology class.

Often, we will have a given base scheme X and consider various schemes $Y = \text{spec}(R)$ with given maps $Y \to X$. We refer to such a scheme Y as a scheme over X, or just an X-scheme, and write $\mathcal{S}chemes_X$ for the category of X-schemes. Given another X-scheme Z, a *section* of Z over Y is a map $a \colon Y \to Z$ such that the composite $Y \xrightarrow{a} Z \to X$ is the given map $Y \to X$. Such sections biject with $\mathcal{O}_X$-algebra maps $\mathcal{O}_Z \to \mathcal{O}_Y$. We write $a \in_{Y/X} Z$ to indicate that a is a section, and let $\Gamma(Y, Z)$ or $\Gamma(R, Z)$ denote the set of sections.

Often, we will describe a scheme Y over X by describing $\Gamma(R, Y)$ as a functor of $\mathcal{O}_X$-algebras R. More precisely, we have equivalences between the following categories:

(1) Schemes Y over X.
(2) Representable functors $Y' \colon \mathcal{O}_X\text{-Alg} \to \mathcal{S}ets$
(3) Representable functors $Y'' \colon \mathcal{S}chemes_X^{\text{op}} \to \mathcal{S}ets$
(4) $\mathcal{O}_X$-algebras R

The equivalences are given by
$$R = \mathcal{O}_Y$$
$$Y'(S) = \mathcal{O}_X\text{-Alg}(R, S) = \Gamma(S, Y)$$
$$Y''(Z) = \mathcal{S}chemes_X(Z, Y) = \Gamma(Z, Y)$$
$$Y(S) = \mathcal{R}ings(R, S) = \{(x, y) \mid x \in X(S), y \in Y'(S)\}$$
In the last equation, S is regarded as an $\mathcal{O}_X$-algebra via $x^* \colon \mathcal{O}_X \to S$.

EXAMPLE 8.4. Suppose that Y and Z are schemes over X, and that $\mathcal{O}_Y$ is a finitely generated free module over $\mathcal{O}_X$. We can define a functor $\mathrm{Map}_X(Y, Z)$ from schemes over X to sets by

$$\Gamma(W, \mathrm{Map}_X(Y, Z)) = \mathcal{S}chemes_W(W \times_X Y, W \times_X Z)$$

This is in fact representable. To see this, observe that an element of $\Gamma(W, \mathrm{Map}_X(Y, Z))$ is just an $\mathcal{O}_W$-algebra map

$$\mathcal{O}_W \otimes_X \mathcal{O}_Z \to \mathcal{O}_W \otimes_X \mathcal{O}_Y$$

or equivalently, just an $\mathcal{O}_X$-algebra map

$$\mathcal{O}_Z \to \mathcal{O}_W \otimes_X \mathcal{O}_Y$$

Write $\mathcal{O}_Y^\vee = \mathrm{Hom}_X(\mathcal{O}_Y, \mathcal{O}_X)$ and $A = \mathrm{Sym}_X[\mathcal{O}_Y^\vee \otimes_X \mathcal{O}_Z]$. Then

$$\mathcal{O}_X\text{-}\mathrm{Alg}(A, \mathcal{O}_W) = \mathcal{O}_X\text{-}\mathrm{Mod}(\mathcal{O}_Y^\vee \otimes_X \mathcal{O}_Z, \mathcal{O}_W) = \mathcal{O}_X\text{-}\mathrm{Mod}(\mathcal{O}_Z, \mathcal{O}_W \otimes_X \mathcal{O}_Y)$$

A suitable quotient B of A will pick out the algebra maps. To be more explicit, let $\{e_i\}$ be a basis for $\mathcal{O}_Y$ over $\mathcal{O}_X$, with $1 = \sum_i b_i e_i$ and $e_i e_j = \sum_k c_{ijk} e_k$. Let $\{\epsilon_i\}$ be the dual basis for $\mathcal{O}_Y^\vee$. Then B is A mod the relations

$$\epsilon_k \otimes ab = \sum_{i,j}(\epsilon_i \otimes a)(\epsilon_j \otimes b)$$

$$\epsilon_i \otimes 1 = d_i$$

More abstractly, B is the largest quotient of A such that the following diagrams commute:

We conclude that $\mathrm{Map}_X(Y, Z) = \mathrm{spec}(B)$. We also write $\underline{\mathcal{S}chemes}_X(Y, Z)$ for this scheme.

EXAMPLE 8.5. Now suppose that G and H are group schemes over X, and that $\mathcal{O}_G$ is a finitely generated free module over $\mathcal{O}_X$. We can then define a closed subscheme $\underline{\mathcal{G}roups}_X(G, H) \subseteq \underline{\mathcal{S}chemes}_X(G, H)$ such that

$$\Gamma(W, \underline{\mathcal{G}roups}_X(G, H)) = \mathcal{G}roups_W(W \times_X G, W \times_X H)$$

In particular, take $X = \mathrm{spec}(\mathbb{Z})$, and

$$G = \mu_p = \mathrm{spec}\,\mathbb{Z}[x]/(x^p - 1) \qquad H(R) = \{r \in R^\times \mid r^p = 1\}$$

$$H = \underline{\mathbb{Z}/p} = \mathrm{spec}(F(\mathbb{Z}/p, \mathbb{Z}))$$

This is the constant group scheme corresponding to $\mathbb{Z}/p$. The representing ring is the ring of functions $\mathbb{Z}/p \to \mathbb{Z}$ under pointwise multiplication, made into a Hopf algebra in the usual way. We write e_k for the function having value 1 at $k \in \mathbb{Z}/p$ and 0 elsewhere, so the e_k form a basis for $F(\mathbb{Z}/p, \mathbb{Z})$.

$$K = \underline{\mathcal{G}roups}_X(G, H) = \underline{\mathcal{G}roups}_{\mathbb{Z}}(\mu_p, \underline{\mathbb{Z}/p})$$

A point of K over a ring A is just a map of Hopf algebras

$$F(\mathbb{Z}/p, A) \to A[x]/(x^p - 1)$$

There is a trivial point of K over $\mathbb{Z}$, defined by $e_k \mapsto \delta_{k,0}$. Moreover, if ω is a primitive p'th root of unity then there is a point of K over the ring $B = \mathbb{Z}[1/p, \omega]$ defined by

$$e_k \mapsto \left(\sum_{l=0}^{p-1} \omega^{kl} x^l\right)/p$$

Together, these points give a map

$$\mathrm{spec}(\mathbb{Z} \times B) \to K$$

This can be shown to be an isomorphism.

We conclude this section with some remarks about open mappings. We have to make a slightly twisted definition, because in our affine context we do not have enough open subschemes. Suppose that $f\colon X \to Y$ is a map of spaces, and that $F \subseteq X$ is closed. We can then define $G = \{y \in Y \mid f^{-1}y \subseteq F\} = f(F^c)^c$. Clearly f is open iff (F closed implies G closed). Now suppose that $f\colon X \to Y$ is a map of schemes, and that $F \subseteq X$ is a closed subscheme. For any point $y\colon T \to Y$ of Y, so we have a closed inclusion $F_y = T \times_Y F \to X_y = T \times_Y X$. We can thus define a sub-pre-scheme G of Y by

$$G(T) = \{y \in Y(T) \mid F_y = X_y\}$$

DEFINITION 8.1. A map $f\colon X \to Y$ is *open* iff for every closed subscheme $F \subseteq X$ the prescheme G defined above is a closed subscheme of Y.

PROPOSITION 8.1. Suppose that f makes $\mathcal{O}_X$ into a free module over $\mathcal{O}_Y$. Then f is open.

PROOF. Write $A = \mathcal{O}_X$ and $B = \mathcal{O}_Y$, and choose a basis $A = B\{e_\alpha\}$. Suppose that $F = V(I)$ is a closed subscheme of X with $I = (g_\beta)$ and $g_\alpha = \sum_\beta g_{\beta\alpha} e_\alpha$. Consider a C-valued point $y^*\colon B \to C$ of Y. This will lie in $G(C)$ iff $C \otimes_B A = C \otimes_B (A/I)$, iff the image of I in $C \otimes_B A = C\{e_\alpha\}$ is zero. This image is generated by the elements $h_\beta = \sum_\alpha y^*(g_{\beta\alpha}) e_\alpha$. Thus, it vanishes iff $y^*(g_{\beta\alpha}) = 0$ for all α and β. This shows that $G = V(J)$, where $J = (g_{\beta\alpha})$. $\square$

9. Zariski Spectra and Geometric Points

If A is a ring, we define the associated *Zariski space* to be

$$\mathrm{zar}(A) = \{ \text{ prime ideals } \mathfrak{p} < A\}$$

If X is a scheme, we write $X_{\mathrm{zar}} = \mathrm{zar}(\mathcal{O}_X)$. Note that

$$V(I)_{\mathrm{zar}} = \mathrm{zar}(\mathcal{O}_X/I) = \{\mathfrak{p} \in X_{\mathrm{zar}} \mid I \leq \mathfrak{p}\}$$

$$D(a)_{\mathrm{zar}} = \mathrm{zar}(\mathcal{O}_X[a^{-1}]) = \{\mathfrak{p} \in X_{\mathrm{zar}} \mid a \notin \mathfrak{p}\}$$

$$(X \sqcup Y)_{\mathrm{zar}} = X_{\mathrm{zar}} \sqcup Y_{\mathrm{zar}}$$

There is a map

$$(X \times Y)_{\mathrm{zar}} \to X_{\mathrm{zar}} \times Y_{\mathrm{zar}}$$

but it is almost never a bijection.

Suppose that $Y, Z \leq X$ are locally closed; then

$$(Y \cap Z)_{\mathrm{zar}} = Y_{\mathrm{zar}} \cap Z_{\mathrm{zar}}$$

If Y and Z are closed then

$$(Y \cup Z)_{\mathrm{zar}} = (Y + Z)_{\mathrm{zar}} = Y_{\mathrm{zar}} \cup Z_{\mathrm{zar}}$$

We give X_{zar} the topology with closed sets $V(I)_{\mathrm{zar}}$. A map of schemes $X \to Y$ then induces a continuous map $X_{\mathrm{zar}} \to Y_{\mathrm{zar}}$.

Suppose that R is an integral domain, and that $x \in_R X$. Then x gives a map $x^* \colon \mathcal{O}_X \to R$, whose kernel $\mathfrak{p}_x$ is prime. We thus have a map $X(R) \to X_{\mathrm{zar}}$, which is natural for monomorphisms of R and arbitary morphisms of X.

A *geometric point* of X is an element of $X(k)$, for some algebraically closed field k. Suppose that either $\mathcal{O}_X$ is a $\mathbb{Q}$-algebra, or that some prime p is nilpotent in $\mathcal{O}_X$. Let k be an algebraically closed field of the appropriate characteristic, with transcendence degree at least the cardinality of R. Then it is easy to see that $X(k) \to X_{\mathrm{zar}}$ is epi.

A useful feature of the Zariski space is that it behaves quite well under colimits. The following proposition is an example of this.

PROPOSITION 9.1. Suppose that a finite group G acts on a scheme X. Then $(X/G)_{\mathrm{zar}} = X_{\mathrm{zar}}/G$, and $(X/G)(k) = X(k)/G$ when k is an algebraically closed field.

A number of interesting things can be detected by looking at Zariski spaces. For example, X_{zar} splits as a disjoint union iff X does — see corollary 32.5.

EXAMPLE 9.1. In this example, all rings are $\mathbb{Q}$-algebras. Let $X(R)$ be the set of $n \times n$ matrices M over R with $M^2 = M$; this is a closed subscheme of $\mathbb{A}_{\mathbb{Q}}^{n^2}$. Let $X(m)$ denote the closed subscheme where $\mathrm{trace}(M) = m$. For any field $K \geq \mathbb{Q}$, elementary linear algebra gives

$$X(K) = \coprod_{m=0}^{n} X(m)(K)$$

It follows by corollary 32.5 that

$$X = \coprod_{m=0}^{n} \tilde{X}(m) \qquad \tilde{X}(m)_{\mathrm{zar}} = X(m)_{\mathrm{zar}}$$

We also use the space X_{zar} to define the *Krull dimension* of X.

DEFINITION 9.1. If there is a chain $\mathfrak{p}_0 < \ldots < \mathfrak{p}_n$ in X_{zar}, but no longer chain, then we say that $\dim(X) = n$. If there are arbitarily long chains then $\dim(X) = \infty$.

EXAMPLE 9.2. $\dim(\mathbb{Z}_p) = 1$ — the unique maximal chain is $(0) < (p)$.

EXAMPLE 9.3. If $E = W[\![u_1, \ldots, u_{n-1}]\!]$ (as in Morava E-theory) and $X = \mathrm{spf}(E)$ then $\dim(X) = n$.

EXAMPLE 9.4. $\dim(\mathrm{FGL}) = \infty$.

The appropriate generalisation to formal rings is not entirely clear. The proofs of the results in section 26 use Zariski spaces, but only for formal schemes with $\mathcal{O}_Y$ a complete Noetherian local ring. In this context all prime ideals are closed, and only the maximal ideal is open. As yet I know no applications for a more general theory.

We ought really to say something here about rigid analytic spaces (as used in [6]), but I'm not sure what.

10. Sheaves, Modules and Vector Bundles

A *sheaf* over a formal scheme X will simply mean a formal module over $\mathcal{O}_X$, in other words a complete, linearly topologised R-module such that the action $R \times M \to M$ is continuous.

A *vector bundle* or *locally free sheaf* will mean a finitely generated projective $\mathcal{O}_X$-module, with the obvious topology. If M and N are vector bundles then linear maps $M \to N$ are automatically continuous and $M \widehat{\otimes}_R N = M \otimes_R N$. The dual module $M^\vee$ is also a vector bundle and $M^{\vee\vee} = M$. If the evaluation map $M \otimes_X M^\vee \to \mathcal{O}_X$ is iso, we say that M is a *line bundle* or *invertible sheaf*. If L and M are line bundles, we often write LM for $L \otimes_X M$ and L^{-1} for $L^\vee$.

The most common situation coming from algebraic topology is that vector bundles are actually free modules and line bundles are free of rank one.

EXAMPLE 10.1. Let Z be a space, and V a vector bundle over Z with Thom space Z^V. Then $\mathbb{L}(V) = \tilde{E}(Z^V)$ is a line bundle over Z_E.

Given a vector bundle M over X, we define formal schemes $\mathbb{A}(M)$ and $\widehat{\mathbb{A}}(M)$ over X by

$$\Gamma(R, \mathbb{A}(M)) = M \widehat{\otimes}_X R$$

$$\Gamma(R, \widehat{\mathbb{A}}(M)) = M \widehat{\otimes}_X \mathrm{Nil}(R)$$

To see that these are indeed representable, write

$$\mathrm{Sym}_X[M^\vee] = \bigoplus_k \mathrm{Sym}^k_{\mathcal{O}_X}[M^\vee]$$

$$\mathrm{Sym}'_X[M^\vee] = \varprojlim_{I \lhd_O \mathcal{O}_X} \mathrm{Sym}_{\mathcal{O}_X/I}[M^\vee]/I$$

$$\mathrm{Sym}''_X[M^\vee] = \prod_k \mathrm{Sym}^k_{\mathcal{O}_X}[M^\vee]$$

Note that Sym'' is the completion of Sym or Sym' at J. Suppose that $\underline{a} \in \mathrm{Sym}''$. Then $\underline{a} \in \mathrm{Sym}$ iff $a_k = 0$ for $k \gg 0$. Moreover, $\underline{a} \in \mathrm{Sym}'$ iff $a_k \to 0$ as $k \to \infty$. More precisely, given $I \lhd_O \mathcal{O}_X$ we require that $a_l \in I\,\mathrm{Sym}^l_X[M^\vee]$ for $l \gg 0$. In particular, if $\mathcal{O}_X$ is discrete then $\mathrm{Sym}' = \mathrm{Sym}$.

It is now not hard to check that

$$\mathbb{A}(M) = \mathrm{spf}(\mathrm{Sym}'_X[M^\vee])$$

$$\widehat{\mathbb{A}}(M) = \mathrm{spf}(\mathrm{Sym}''_X[M^\vee])$$

More generally, if M is pro-free then the functor

$$\Gamma(R, \mathbb{A}(M)) = M \widehat{\otimes}_X R$$

is again a scheme, represented by $\mathrm{Sym}'_X[M^\vee]$. Here we have to build the symmetric algebra using the completed tensor product, of course.

EXAMPLE 10.2. $\mathrm{spf}(E^\vee BU) = \mathbb{A}(\tilde{E}(\mathbb{C}P^\infty))$.

If M is a sheaf over a scheme X and $x \in X(R)$ is an R-valued point of X then we write $M_x = M \otimes_{\mathcal{O}_X} R$. Here R is considered as an $\mathcal{O}_X$-module via the map $x \colon \mathcal{O}_X \to R$. Thus M_x is an R-module, which should be thought of as the fibre of M at x.

Suppose that P is a ~~vector bundle~~ finitely generated projective module over $A = \mathcal{O}_X$. We shall say that P has *constant rank* m if $P_x \simeq k^m$ for any field-valued point $x \colon A \to k$. The following proposition is partial justification for the name "vector bundle" (see also example 12.8) .

PROPOSITION 10.1. We can canonically write $X = \coprod_{m=0}^n X_m$ such that P has constant rank m on X_m, and X_m is an open and closed subscheme.

PROOF. We have $P = E.A^n$ for some matrix $E \in M_n(A)$ with $E^2 = E$. For sets $S, T \subseteq \{1, \ldots n\}$ with $|S| = |T| = m$ we write E_{ST} for the $m \times m$ minor of E indexed by $S \times T$ and E'_{ST} for the complementary $(n-m) \times (n-m)$ minor of $1 - E$. We also write $a_{ST} = \det(E_{ST})\det(E'_{ST})$. We let I_m denote the ideal generated by the elements a_{ST} for which $|S| = |T| \neq m$, and write $X_m = V(I_m)$. Suppose that $f \colon A \to k$ where k is a field. Elementary linear algebra applied to the matrix $f(E)$ assures us that $f(a_{ST}) \neq 0$ for some S, T with $|S| = |T| = \mathrm{rank}(f(E))$. It follows that $X(k) = \coprod_m X_m(k)$. We deduce using corollary 32.5 that there are ideals I'_m with the same radical as I_m, such that the corresponding schemes X'_m partition X. It is easy to see that P has constant rank m on X'_m. (see also example 9.1. $\square$

11. Polarised Schemes

DEFINITION 11.1. A *polarised scheme* is a scheme X equipped with a free line bundle L, in other words a free module L of rank one over $\mathcal{O}_X$. A morphism $(X, L) \to (Y, M)$ is a map $f \colon X \to Y$ together with an isomorphism $L \simeq f^*M$. Equivalently, a morphism is a pullback square

$$
\begin{array}{ccc}
L & \xrightarrow{\;\bar{f}\;} & M \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & Y
\end{array}
$$

If X is a polarised scheme, then there is a noncanonical isomorphism $u \colon L \to \mathcal{O}_X$ (so $u \in L^{-1}$).

The category of polarised schemes is equivalent to that of graded rings R^* such that $R^{\mathrm{odd}} = 0$ and there exists a unit $u \in R^{-2}$. The equivalence is just $R^{2k} = L^{\otimes k}$, where the tensor product is taken over $R^0 = \mathcal{O}_X$ and $L^{\otimes(-k)} = \mathrm{Hom}_{R^0}(L^{\otimes k}, R_0)$.

In particular, if E is a two-periodic ring spectrum then $X = \mathrm{spf}(\pi_0 E)$ has a natural polarisation. If E is p-adic K-theory then $L^{-1} = \mathbb{Z}_p u$, and the action of the Adams operations is $\psi^k u = k^{-1} u$ (for $k \in \mathbb{Z}_p^\times$). This shows that although L is a trivial line bundle, it is not equivariantly trivial for an important group of automorphisms.

12. Faithful Flatness and Descent

Let $f \colon A \to B$ be a map of rings. We say as usual that f is *flat* iff B is flat as an A-module via f. We say that f (or B) is *faithfully flat* iff it is flat and also satisfies the following equivalent conditions:

  (1) $B \otimes_A M = 0$ implies $M = 0$.
  (2) If C_* is a complex of A-modules and $B \otimes_A C_*$ is acyclic then C_* is acyclic.
  (3) The map of Zariski spaces $\mathrm{zar}(B) \to \mathrm{zar}(A)$ is surjective.

Let $f \colon X \to Y$ be a map of affine schemes. We say that f is (faithfully) flat iff $f^* \colon \mathcal{O}_Y \to \mathcal{O}_X$ is.

EXAMPLE 12.1. An open inclusion $D(a) \to X$ (where $a \in \mathcal{O}_X$) is always flat. If $a_1, \ldots, a_m \in \mathcal{O}_X$ generate the unit ideal then $\coprod_k D(a_k) \to X$ is faithfully flat.

EXAMPLE 12.2. If D is a divisor on G over Y (see section 17) then $D \to Y$ is faithfully flat.

EXAMPLE 12.3. Looking forward to section 14, suppose that G and H are formal groups over X. Suppose that $\mathcal{O}_X$ is a complete local ring, and write $X_0 = \mathrm{spf}(\mathcal{O}_X/\mathfrak{m})$. Suppose that $q \colon G \to H$ is not zero on X_0. We shall see later that q is faithfully flat.

EXAMPLE 12.4. Let G be a finite group, X a G-space and V a complex vector bundle over X. Suppose that $\mathrm{Flag}(V)^G \to X^G$ is surjective. Then $(\mathrm{Flag}(V)^G)_E \to (X^G)_E$ is faithfully flat [8].

PROPOSITION 12.1. The composite of two faithfully flat maps is faithfully flat. If $X \to Y$ is faithfully flat and $Z \to Y$ is arbitary then $X \times_Y Z \to Z$ is faithfully flat. ☐

PROPOSITION 12.2. If $X \to Y$ is faithfully flat, then the diagram

$$ X \times_Y X \rightrightarrows X \to Y $$

is a coequaliser in the category of affine schemes, and this remains true after pulling back along an arbitrary map $Z \to Y$.

PROOF. Write $A = \mathcal{O}_Y$ and $B = \mathcal{O}_X$, so we need to show that the following diagram is an equaliser:

$$ A \to B \rightrightarrows B \otimes_A B $$

$$ a \to f(a) \qquad\qquad b \to \begin{cases} b \otimes 1 \\ 1 \otimes b \end{cases} $$

As B is faithfully flat over A, it is enough to check that the diagram is an equaliser after tensoring by B over A. Indeed, this makes it a split equaliser:

$$B = B \otimes_A A \leftarrow B \otimes_A B \leftarrow B \otimes_A B \otimes_A B$$

$$bb' \leftarrow b \otimes b' \qquad\qquad bb' \otimes b'' \leftarrow b \otimes b' \otimes b''$$

For the last part, we need only recall that $Z \times_Y X \to Z$ is again faithfully flat. $\square$

Now suppose that $f\colon X \to Y$ is faithfully flat, and that M is a sheaf over X. We will need to know when M descends to a sheaf over Y, in other words when there is a sheaf N on Y such that $M \simeq f^*N$. There is an entirely parallel theory for schemes U over X; one asks whether they have the form $V \times_Y X$ for some scheme V over Y.

DEFINITION 12.1. *Descent data* for a sheaf M over X consists of a collection of maps $\theta_{a,b}\colon M_a \to M_b$ for any pair of points a, b of X with $f(a) = f(b)$. These maps are required to be natural in (a, b), and to satisfy the cocycle conditions $\theta_{a,a} = 1$ and $\theta_{a,c} = \theta_{b,c} \circ \theta_{a,b}$.

REMARK 12.1. The universal example of such a pair a, b consists of the two projections $\pi_0, \pi_1\colon X \times_Y X \to X$. It is thus enough to specify a map $\theta\colon \pi_0^* M \to \pi_1^* M$. Similarly, the cocycle conditions can be checked over X and $X \times_Y X \times_Y X$ respectively.

REMARK 12.2. Note also that the cocycle conditions imply $\theta_{a,b} \circ \theta_{b,a} = 1$, so $\theta_{a,b}$ is iso.

See section 27 for another interpretation of this.

DEFINITION 12.2. Descent data as above are *effective* iff there is a sheaf N over Y and an isomorphism $\phi\colon M \simeq f^*N$ such that

$$\theta_{a,b} = (M_a \xrightarrow{\phi} N_{f(a)} = N_{f(b)} \xrightarrow{\phi^{-1}} M_b)$$

PROPOSITION 12.3 (FAITHFULLY FLAT DESCENT). If f is faithfully flat, then descent data are always effective. Moreover, the functor f^* gives an equivalence between sheaves over Y and sheaves over X with given descent data. The inverse sends M to the equaliser N of the following diagram:

$$N \to M \rightrightarrows \mathcal{O}_Y \otimes_X M$$

$$m \to \begin{cases} \pi_1^* m \\ \theta \pi_0^* m \end{cases}$$

In other words, the sections of N biject with sections s of M such that $s(b) = \theta_{a,b} s(a)$ whenever $f(a) = f(b)$.

We shall say that a statement holds *locally in the flat topology* or *fpqc locally* if it is true after pulling back along a faithfully flat map. (fpqc stands for fidèlement plat et quasi-compact; the compactness condition is automatic for affine schemes). Suppose that a certain statement S is true whenever it holds fpqc-locally. We then say that S is an fpqc-local statement.

REMARK 12.3. Let X be a topological space. We say that a statement S holds locally on X iff there is an open covering $X = \bigcup_i U_i$ such that S holds on each U_i. Write $Y = \coprod_i U_i$, so $Y \to X$ is a coproduct of open inclusions and is surjective. We could call such a map an "disjoint covering map". We would then say that S holds locally iff it holds after pulling back along a disjoint covering map. One can get many analogous concepts varying the class of maps in question. For example, we could use covering maps in the ordinary sense. In the category of compact smooth manifolds, we could use submersions. This is the conceptual framework in which the above definition is supposed to fit.

EXAMPLE 12.5. Suppose that N is a sheaf on Y which vanishes fpqc-locally. This means that there is a faithfully flat map $f\colon X \to Y$ such that $f^*N = \mathcal{O}_X \otimes_Y N = 0$. By the very definition of faithful flatness, this implies that $N = 0$. In other words, the vanishing of N is an fpqc-local condition. Similarly, suppose that $n \in N$ vanishes fpqc-locally, so that $f^*n = 0 \in M = f^*N$. By proposition 12.3, we know that there is an equaliser diagram

$$N \to M \rightrightarrows \pi_1^* M$$

In particular, $f^*\colon N \to M$ is mono and thus $n = 0$. Thus, the vanishing of n is also an fpqc-local condition.

EXAMPLE 12.6. Flatness is itself an fpqc-local property. Indeed, suppose that $X \to Y$ is such that $X \times_Y Z \to Z$ is flat for some faithfully flat map $Z \to Y$. One can then show easily from the definitions that $X \to Y$ is flat.

EXAMPLE 12.7. Any open inclusion $D(a) \to X$ is clearly flat, as is any finite coproduct $\coprod_{i \in I} D(a_i) \to X$ of such. Using the third criterion for faithful flatness, we see that this map is faithfully flat iff no maximal ideal contains all the a_i, iff $X(k) = \bigcup_i D(a_i)(k)$ whenever k is a field, iff $(a_i \mid i \in I) = A$.

EXAMPLE 12.8. Suppose that P is a vector bundle over $X = \mathrm{spec}(A)$. We keep the notation of proposition 10.1 (and its proof). Suppose that P has constant rank m. The claim is that P is fpqc-locally free of rank m. To see this, write $Y = \coprod_{|S|=|T|=m} D(a_{ST})$. This is clearly flat over X, and P is free over $D(a_{ST})$ (because $A^m \xrightarrow{j_S} A^n \xrightarrow{E} P$ is iso) and hence over Y. Moreover, the map $Y \to X$ is faithfully flat because $X(k) = \bigcup_{S,T} D(a_{ST})(k)$ when k is a field.

EXAMPLE 12.9. Any monic polynomial $f \in A[x]$ can be factored as a product of linear terms, locally in the flat topology. Indeed, suppose $f = \sum_0^m (-1)^{m-k} a_{m-k} x^k$ with $a_0 = 1$. It is well known that $B = \mathbb{Z}[x_1, \ldots x_m]$ is free of rank $m!$ over $C = B^{\Sigma_m} = \mathbb{Z}[\sigma_1, \ldots \sigma_m]$. A basis is given by the monomials $x^\alpha = \prod x_k^{\alpha_k}$ for which $\alpha_k < k$. We can map C to A by sending σ_k to a_k, and then observe that $D = B \otimes_C A$ is free and thus faithfully flat over A. Clearly $f(x) = \prod_k (x - x_k)$ in $D[x]$, as required.

EXAMPLE 12.10. Let $f\colon G \to H$ be a homomorphism of group schemes over X. Suppose that the kernel K is faithfully flat over X, and that f is fpqc-locally surjective. This means that the identity point $1 \in_X X$ is in the image of f after pulling back along a faithfully flat map $Y \to X$. In other words, the homomorphism $G \times_X Y \to H \times_X Y$ admits a non-additive section, so that $G \times_X Y \simeq H \times_X K \times_X Y$ as schemes over Y. It follows that $G \times_X Y \to H \times_X Y$ is faithfully flat, and finally that f itself is faithfully flat. This has applications to various maps of the connective covers $BU\langle m \rangle$.

13. Constant and Étale Schemes

Let X be a formal scheme and S a set. We define $F(S, \mathcal{O}_X)$ to be the ring of functions $S \to \mathcal{O}_X$, with the product topology. We write

$$\underline{S}_X = \mathrm{spf}(F(S, \mathcal{O}_X))$$

Such a scheme is called a *constant scheme* over X. It is the S-fold coproduct of copies of X, in the sense that

$$\mathcal{S}chemes(\underline{S}_X, Y) = \mathrm{Map}(S, \mathcal{S}chemes(X, Y))$$

$$\mathcal{S}chemes_X(\underline{S}_X, Y) = \mathrm{Map}(S, \Gamma(X, Y))$$

Moreover, we have $Y \times_X \underline{S}_X = \underline{S}_Y$. In particular, $\underline{S}_X = \underline{S} \times X$, where $\underline{S} = \mathrm{spf}(F(S, \mathbb{Z}))$.

We shall say that a map $f\colon X \to Y$ is an *étale covering* iff it becomes constant after pulling back along a faithfully flat map $Z \to Y$.

EXAMPLE 13.1. Let $K \to L$ be a finite separable field extension; then $\mathrm{spec}(L) \to \mathrm{spec}(K)$ is an étale covering. Indeed, by the theorem of the primitive element, we can write $L = K[x]/f(x)$ for some irreducible, separable polynomial $f(x)$. Let K' be the splitting field of f over K, so that K' is clearly faithfully flat over K. Moreover, $f(x) = \prod_{i \in I}(x - a_i) \in K'[x]$ with $a_i \neq a_j$ for $i \neq j$. It follows by the Chinese remainder theorem that

$$K' \otimes_K L = K'[x]/f(x) \simeq F(I, K') \qquad g(x) \mapsto (i \mapsto g(a_i))$$

We can also interpret I as $K\text{-Alg}(L, K')$; in this picture, the map

$$K' \otimes_K L \to F(I, K')$$

sends $a \otimes b$ to $(\sigma \mapsto a\sigma(b))$.

EXAMPLE 13.2. More generally, let A be an arbitrary ring and $f(x)$ a monic polynomial of degree m over A. We would like to know when $A[x]/f(x)$ is étale over A. Let B be a faithfully flat extension of A in which f factors as $\prod_{k=0}^{m-1}(x - b_k)$. We define the discriminant of f as $\Delta = \prod_{k \neq l}(b_k - b_l)$. This is a symmetric polynomial in the roots b_k, and thus lies in A and is independent of the choice of B and the factorisation. Suppose that Δ is invertible. As in the previous example, we have a map $B[x]/f(x) \to B^m$. If we use the basis $\{x^k \mid 0 \leq k < m\}$ for $B[x]/f(x)$ and the obvious basis for B^m then the matrix of this map is the Vandermonde matrix $M = (a_k^l)_{0 \leq k, l < m}$. It is well known that $\det(M)^2 = \Delta$, so our map is an isomorphism and $A[x]/f(x)$ is étale. It is also not hard to establish the converse, so that $A[x]/f(x)$ is étale iff Δ is a unit.

PROPOSITION 13.1. The composite of two étale coverings is an étale covering. If $X \to Y$ is an étale covering and $Z \to Y$ is arbitary then $X \times_Y Z \to Z$ is an étale covering. If $X \to Y$ is an étale covering then $X = \emptyset$ or $X \to Y$ is faithfully flat.

More generally, we would like to define $\underline{S}$ where S is a topological space. Let us suppose that S is Hausdorff, and that the compact open subsets form a basis (e.g. S could be discrete or profinite). This seems to cover the cases we need. We then let $C(S, \mathcal{O}_X)$ be the ring of continuous functions $S \to \mathcal{O}_X$. Let T be an compact open subspace of S, and $I \lhd \mathcal{O}_X$ an open ideal. Note that $\mathcal{O}_X/I$ is discrete. It follows that the map $C(S, \mathcal{O}_X) \to C(T, \mathcal{O}_X/I)$ is epi; we write $W(T, I)$ for the kernel. We give $C(S, \mathcal{O}_X)$ the compact-open topology, which is just the same as that defined by the ideals $W(T, I)$. Finally, we define $\underline{S} = \mathrm{spf}(C(S, \mathbb{Z}))$ and $\underline{S}_X = \mathrm{spf}(C(S, \mathcal{O}_X))$.

LEMMA 13.2. If $A \to B$ is a continuous map of formal rings then $C(S, A) \widehat{\otimes}_A B = C(S, B)$

PROOF. Let I run over open ideals of A, J over open ideals of B, and T over compact open subspaces of S. Write J^* for the preimage of J in A, which is an open ideal. Note that B/J is discrete, so that a continuous map $T \to B/J$ is locally constant, and can be written more or less uniquely as a B/J-linear combination of characteristic functions of open-and-closed subsets of T. It follows that when $I \leq J^*$ we have

$$C(T, A/I) \otimes_A B/J \simeq C(T, B/J)$$

The definition of the completed tensor product is

$$C(S, A) \widehat{\otimes}_A B = \varprojlim_{T, I, J} C(S, A)/W(T, I) \otimes_A B/J = \varprojlim_{T, I, J} C(T, A/I) \otimes_A B/J$$

The family of triples (T, I, J) with $I \leq J_*$ is cofinal, so

$$C(S, A) \widehat{\otimes}_A B = \varprojlim_{T, J} C(T, B/J)$$

We have $S = \varinjlim T$ and $B = \varprojlim B/J$ as spaces, so $\varprojlim_{T, J} C(T, B/J) = C(S, B)$. This gives a bijection $C(S, A) \widehat{\otimes}_A B \simeq C(S, B)$, natural in all variables. Using naturality, one can see that it is a homeomorphism. □

COROLLARY 13.3. $\underline{S}_X = \underline{S} \times X$.

Now let X and Y be formal schemes. The discrete closed subschemes $X' \subseteq X$ biject with open ideals $I \lhd_O \mathcal{O}_X$ via $X' = V(I)$.

Write $\tilde{S}$ for the constant functor $\mathcal{R}ings \to \mathcal{S}ets$, sending everything to S (this is not a scheme unless $S = \emptyset$). Thus, natural transformations $\alpha \colon \tilde{S} \times X \to Y$ biject with functions $S \to \mathcal{F}ormal(X, Y)$. We say that such a map is a *proper action* iff for all compact open $T \subseteq S$ and all discrete closed $X' \le X$ the map $T \times X' \to Y$ factors through a discrete closed subscheme $Y' \le Y$.

PROPOSITION 13.4. *There are natural bijections between*

(1) *continuous maps $S \times \mathcal{O}_Y \to \mathcal{O}_X$ which are ring homomorphisms in the second variable.*
(2) *proper actions $\tilde{S} \times X \to Y$*
(3) *maps of formal schemes $\underline{S} \times X \to Y$*

PROOF. Let us abbreviate $A = \mathcal{O}_X$, $B = \mathcal{O}_Y$. Let I, J and T run over open ideals in A or B, or compact open subspaces of S respectively. Let $\alpha \colon S \times B \to A$ be a ring map in the second variable. I claim that α is continuous iff for all T and I there exists J such that $\alpha(T \times J) \subseteq I$. Indeed, suppose α is continuous. Then $\alpha^{-1}(I)$ is open, and contains $T \times \{0\}$. As T is compact, the "tube lemma" says that $T \times J \subseteq \alpha^{-1}(I)$ for some open neighbourhood J of 0 in B, wlog an ideal. The converse is easy. It is immediate from the definitions that it is also equivalent for α to give a proper action $\tilde{S} \times X \to Y$. Thus, (1) and (2) are equivalent. Next, recall that $C(S, -)$ is right adjoint to $S \times (-)$ (because S is locally compact), and that $\mathcal{O}_{\underline{S} \times X} = C(S, \mathcal{O}_X)$. It follows easily that (1) is equivalent to (3). $\square$

EXAMPLE 13.3. If Σ is the Morava stabiliser group, E is Morava E-theory and Z is a tolerable space then

$$\mathrm{spf}(\pi_0 L_K(E^{\wedge (n+1)} \wedge Z)) = \underline{\Sigma}^n \times \mathrm{spf}(E^\vee Z)$$

14. Formal Groups

A *group scheme* over a scheme X is an X-scheme $G \to X$ which is a commutative group in the category of X-schemes. In other words, it must be provided with a zero-section $X \xrightarrow{0} G$, an addition $G \times_X G \xrightarrow{\mu} G$ and an inversion $G \xrightarrow{\chi} G$, satisfying the obvious identities. It is equivalent to require that $\mathcal{O}_G$ be a bicommutative Hopf algebra over $\mathcal{O}_X$.

Such a group scheme G should be thought of as a "bundle of groups" over X. Note that this structure makes $\Gamma(Y, G)$ into an honest group for any X-scheme Y.

Similarly, we can define formal group schemes G over a formal scheme X (in shorthand, formal groups G/X), or over an ordinary scheme X. We are mostly interested in smooth, commutative one-dimensional formal group schemes, so we shall just refer to these as "formal groups":

DEFINITION 14.1. Let X be a formal scheme. Define $\widehat{\mathbb{A}}^1_X = \mathrm{spf}\, \mathcal{O}_X[\![x]\!]$, where the topology is generated by ideals $I[\![x]\!] + (x^k)$ with I open in $\mathcal{O}_X$. A *formal group* over X is a formal group scheme G over X which is isomorphic to $\widehat{\mathbb{A}}^1_X$ as a pointed formal scheme over X. In other words, there is required to be an element $x \in \mathcal{O}_G$ such that

(1) $x(0) = 0$
(2) $\mathcal{O}_G = \mathcal{O}_X[\![x]\!]$
(3) The topology is generated by ideals $I[\![x]\!] + (x^k)$.

Note that $x(0) \in \mathcal{O}_X$ is the value of the function $x \in \mathcal{O}_G$ at the point 0 of G over X, or equivalently the image of x under the map $\mathcal{O}_G \to \mathcal{O}_X$ induced by the zero-section map $X \to G$. In other words, the condition $x(0) = 0$ just means that x lies in the augmentation ideal. An element x as above is called a *coordinate* on G.

Suppose that H/Y is a formal group and $f \colon X \to Y$. The pullback $f^*H = X \times_Y H$ is then a formal group over X in an obvious way. We will say that f^*H is obtained from H by a *base change*. We will also sometimes think of f as a point of Y defined over X. In that case we would probably call it y instead of f, and write H_y for y^*H.

We shall need to consider various kinds of morphisms of formal groups.

DEFINITION 14.2. Let G/X and H/Y be formal groups, and suppose $f\colon X \to Y$. A morphism of formal groups over f is a map of formal schemes $\tilde{f}\colon G \to H$ such that the following diagrams commute:

$$
\begin{array}{ccc}
G & \xrightarrow{\tilde{f}} & H \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
G \times_X G & \xrightarrow{\tilde{f} \times_x \tilde{f}} & H \times_Y H \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\
G & \xrightarrow{\tilde{f}} & H
\end{array}
$$

If $X = Y$ then a *morphism from G to H* will mean a morphism over 1_X unless otherwise stated. In general, a morphism over f from G to H is equivalent to a morphism over 1_X from G to f^*H. We shall say that a morphism $\tilde{f}$ over f is a *fibrewise isomorphism* iff the left hand square above is a pullback, iff the resulting map $G \to f^*H$ is iso.

DEFINITION 14.3. A map $q\colon G \to H$ of formal groups over X is an *isogeny* iff it makes $\mathcal{O}_G$ into a finitely generated projective module over $\mathcal{O}_H$.

PROPOSITION 14.1. Suppose that $\mathcal{O}_Y$ is a complete local ring, and that $f\colon G \to H$ is a nonzero map of formal groups over Y. Then Y is an isogeny. $\square$

We also define a *curve* on G to be a map $\gamma\colon \widehat{\mathbb{A}}^1_X \to G$ of pointed X-schemes (so $\gamma(0) = 0$). We say that a curve γ is *basic* if it is an isomorphism, so that the inverse map $x = \gamma^{-1}$ is a coordinate.

Suppose that x a coordinate on G. As

$$\mathcal{O}_{G \times_X G} = \mathcal{O}_G \widehat{\otimes}_X \mathcal{O}_G = \mathcal{O}_X[\![x, y]\!]$$

we have

$$\mu^*(x) = F(x, y) = \sum_{k,l} a_{kl} x^k y^l$$

for uniquely defined elements $a_{kl} \in \mathcal{O}_X$. It is easy to see that F is actually a formal group law over X.

Suppose that Y is an X-scheme, and that $u, v \in \Gamma(Y, G)$, so we can form the sum $u + v \in \Gamma(Y, G)$ using the group structure on G. We can also evaluate the function x at the points $u, v, u + v$ of G over Y to get functions $x(u)$ etc. on Y. One can see by unraveling the definitions that these satisfy

$$x(u + v) = \sum a_{kl} x(u)^k x(v)^l$$

This is the most natural way to think about the formal group law in our schematic picture.

Similarly, for each integer n there is a power series $[n](x)$ such that

$$x(nu) = [n](x(u))$$

REMARK 14.1. To construct definite examples of formal groups, we often start with a formal group law F and then define $G(R)$ to be $\widehat{\mathbb{A}}^1(R) = \operatorname{Nil}(R)$ with the group structure given by F. We shall write $(\widehat{\mathbb{A}}^1, F)$ for this formal group. It is convenient to forget that G has the same underlying scheme as $\widehat{\mathbb{A}}^1$, and to write γ and x for the (identity) maps $\widehat{\mathbb{A}}^1 \xrightarrow{\gamma} G \xrightarrow{x} \widehat{\mathbb{A}}^1$. Thus, γ is a curve on G and x is a coordinate. If we write $a, b \in G$ and then refer to $a + b$, we will always mean $F(a, b)$. If we want to refer to the ordinary sum, we shall write $x(a) + x(b)$.

Write $J = \{f \in \mathcal{O}_G \mid f(0) = 0\}$ for the augmentation ideal. Then $\omega_G = J/J^2$ is a free module of rank one over $\mathcal{O}_X$, in other words a line bundle over X. It should be thought of as the fibrewise cotangent space of G at the zero section. There is more about this in section 16. We also define the Lie algebra of G as $LG = \omega_G^\vee$.

Let X be a polarised formal scheme, and G a formal group over X. As $\widehat{\mathbb{A}}(L) \simeq \widehat{\mathbb{A}}^1$, there certainly exist pointed isomorphisms $y\colon G \to \widehat{\mathbb{A}}(L)$. Any such map gives an obvious identification $Ly\colon LG \simeq L$ (or equivalently, $L^\vee \simeq \omega_G$).

DEFINITION 14.4. Let X be a polarised formal scheme. A *polarised formal group* over X is a formal group together with an isomorphism $\lambda\colon LG \simeq L$. An *orientation* of a polarised formal group is an isomorphism $y\colon G \to \widehat{\mathbb{A}}(L)$ of pointed formal schemes over X such that $Ly = \lambda\colon LG \simeq L$.

DEFINITION 14.5. A *strict isomorphism* of polarised formal groups is a map $f\colon G \to H$ such that $\lambda_H \circ Lf = \lambda_G\colon LG \to L$. (Note that such a map gives an isomorphism $LG \to LH$ and thus is iso, as the name suggests).

Now let E be a two-periodic ring spectrum, giving rise to a polarised scheme (X, L). Note that there is a canonical identification $L^{-1} = E^{-2} = \tilde{E}^0 \mathbb{C}P^1$. We shall say that E is *complex-orientable* iff $\tilde{E}^0 \mathbb{C}P^\infty \to \tilde{E}^0 \mathbb{C}P^1$ is surjective. Suppose so. It is then well known that $E(\mathbb{C}P^\infty)$ is a Hopf algebra over E of the right kind for $G = \mathrm{spf}\, E(\mathbb{C}P^\infty)$ to be a formal group over X. Moreover, there is a canonical isomorphism $\omega_G \simeq \tilde{E}^0 \mathbb{C}P^1 \simeq L^{-1}$, so G is canonically polarised.

An orientation of G corresponds to a choice of generator $y \in \tilde{E}^2 \mathbb{C}P^\infty$ which restricts to the canonical class in $\tilde{E}^2 \mathbb{C}P^1 = L^{-1}$. In other words, y is a complex orientation of the spectrum E in the usual (strict) sense, corresponding to a map of ring spectra $MU \to E$.

This brings us to the following question: what functor does the ring MU^* represent ? In the present picture, the answer is as follows. Given a polarised scheme (X, L), write

$$\mathrm{OFG}(X, L) = \{\text{ iso classes of oriented formal groups over } X\}$$

The isomorphisms here are supposed to respect the orientation; in particular, there are no automorphisms. Suppose (G, y) is such an oriented formal group. We can then write

$$y(a + b) = \sum_{k,l} \alpha_{kl} y(a)^k y(b)^l$$

for uniquely determined elements $\alpha_{kl} \in L^{1-k-l}$. These elements depend only on the isomorphism class of (G, y). They satisfy the Lazard identities (c.f. example 4.2), and thus give a map of graded rings $MU^* \to \mathcal{O}_X^* = \bigoplus_{k \in \mathbb{Z}} L^k$. We can interpret Quillen's theorem as saying that

$$\mathrm{OFG}(X, L) \simeq \mathcal{G}raded\mathcal{R}ings(MU^*, \mathcal{O}_X^*)$$

This has a number of advantages over the usual description. Firstly, it exhibits the rôle of the grading; this becomes important when one considers unstable operations. Secondly, it is more compatible with the available descriptions of the homotopy rings of other spectra, such as elliptic homology or Morava E-theory. In each case, the homotopy ring classifies isomorphism classes of polarised formal groups or elliptic curves with some extra structure.

Write $\Gamma(Y, \mathrm{Orient}(G))$ for the set of orientations on $G \times_X Y$. It is not hard to see that this is a scheme over X; in fact

$$\mathrm{Orient}(G) = \mathrm{spf}(E^\vee MU)$$

One might ask for a spectrum whose homotopy classifies isomorphism classes of formal groups with no extra structure. It is not hard to see, however, that this functor (call it FG, say) is not representable. Indeed, the map $\mathrm{FG}(\mathbb{Z}) \to \mathrm{FG}(\mathbb{Q}) = \{G_a\}$ is not even injective.

More generally, one only expects the functor

$$R \mapsto \{\text{ iso classes of widgets over } R\}$$

to be representable in a reasonable way if widgets have no automorphisms. This leads to discussion of stacks, moduli problems and Adams-type spectral sequences, which may or may not end up in this document.

15. More about Morava E-Theory

Let us resume the notation of section 3. We thus have a formal group $G = (\mathbb{C}P^\infty)_E$ over a base $X = \text{point}_E$. The group G thus comes equipped with a p-typical basic curve $\gamma \colon \widehat{\mathbb{A}}^1 \to G$, and thus a basis $e = L\gamma$ of LG. There is a unique curve η with $\eta = pe$ and $\eta(s+t) = \eta(s) + \eta(t)$. In language which may be more familiar, this is just the statement that $\exp_F(px)$ is integral (and even divisible by p). The generators u_k are characterised by

$$\gamma(pt) = \eta(t) + \sum_{k=1}^n \gamma(u_k t^{p^k})$$

As well as the coordinate $x = \gamma^{-1}$, there is another coordinate x_{Ando} constructed by Matthew Ando, which is more convenient for many purposes — see section 29.

If we restrict everything to X_0 then we have $\gamma(pt) = \gamma(t^{p^n})$ so $[p](x_0) = x_0^{p^n}$. On the other hand, if we work over $D(p) = \text{spec}(p^{-1}E) \subset X$ then η becomes a basic curve and there is a coordinate l (the *logarithm*) with $l(\eta(t)) = pt$ and thus $l(a + b) = l(a) + l(b)$. An explicit formula can be given for l in terms of x. Over X_1 it becomes quite simple:

$$l_1(a) = \sum_k x_1(a)^{p^{nk}}/p^k$$

More conceptually, we have

$$l(a) = \lim_{k \to \infty} p^{-k} x(p^k a)$$

Note that we have done something a bit funny here. If we rationalise the pro-system $\{E/\mathfrak{m}^k\}$ then we get 0, because $p \in \mathfrak{m}$. Thus, in order to get a nontrivial answer we have to replace $\text{spf}(E)$ by $\text{spec}(E)$ before rationalising. However, the whole point about the ring E is that it represents a nice functor in the category of formal schemes (see section 23), so we certainly want to work with $\text{spf}(E)$ most of the time. See also section 31 in this regard.

Next, we consider the endomorphisms of G. As G is an Abelian group object, its endomorphisms form a ring. By proposition 16.1, we have a monomorphism

$$\text{End}(G) \to \text{End}(LG) = \mathcal{O}_X$$

It is not hard to show that the map $\mathbb{Z} \to \text{End}(G)$ extends uniquely to give an isomorphism $\mathbb{Z}_p \to \text{End}(G)$. We write n_G for the image of $n \in \mathbb{Z}_p$ in $\text{End}(G)$.

Let $\alpha \in W$ be such that $\alpha^{p^n} = \alpha$. Using the formula for l_1, we can see that $l_1(\alpha a) = \alpha l_1(a)$ and thus $F(\alpha x, \alpha y) = \alpha F(x, y)$ over X_1. Using this, we obtain a map $\alpha_G \colon G_1 \to G_1$. Using the fact that W is generated by $\mathbb{Z}_p$ together with such elements α, we obtain a map $W \to \text{End}(G_1)$, which is an isomorphism.

Over G_0, proposition 16.1 does not apply, so the map $\text{End}(G_0) \to \text{End}(LG_0)$ need not be (and is not) injective. Indeed, there is an endomorphism F defined by $F\gamma_0(t) = \gamma_0(t^p)$. The full endomorphism ring is isomorphic to the noncommutative ring

$$D = W\langle S\rangle/(S^n - p, Sa - a^\phi S)$$

(with S acting as F). Any element a of D_n can be written uniquely as

$$a = \sum_{k=0}^\infty a_k S^k \qquad a_k \in W \qquad a_k^{p^n} = a_k$$

or

$$a = \sum_{l=0}^{n-1} b_l S^l \qquad\qquad b_k \in W$$

This is invertible iff $a_0 \neq 0$ iff $b_0 \in W^\times$. The corresponding automorphism a_G of G_0 is given by

$$a_G \gamma_0(t) = \sum_k \gamma_0(a_k t^{p^k}) = \sum_l b_l \gamma_0(t^{p^l})$$

The group $\Sigma = D^\times \simeq \mathrm{Aut}(G_0)$ is the (non-strict) Morava stabiliser group.

16. Differential Geometry of Formal Groups

A formal group should be thought of as the analogue of a Lie group in the category of schemes, so we need to understand some differential geometry. The *cotangent bundle* is just the module $\Omega_{G/X}$ of Kähler differentials. To define this, we let I be the kernel of the multiplication map

$$\mathcal{O}_G \otimes_X \mathcal{O}_G \to \mathcal{O}_G$$

We regard this as an $\mathcal{O}_G$-module using the left factor. We set

$$\Omega_{G/X} = I/I^2$$

If $f \in \mathcal{O}_G$ we define

$$df = f \otimes 1 - 1 \otimes f + I^2 \in \Omega_{G/X}$$

and check easily that

$$d(fg) = d(f)g + f d(g)$$

It follows that if x is a coordinate then $\Omega_{G/X}$ is a free module over $\mathcal{O}_G$ on one generator.

This construction can be interpreted as follows. The closed subscheme $V(I) \subset G \times_X G$ is just the diagonal $\Delta_{G/X}$, in other words the set of pairs of points (a,b) of G in the same fibre such that $a = b$. The slightly thicker subscheme $V(I^2)$ is the set of pairs (a,b) where a and b are "infinitesimally close to first order". A form $\omega \in \Omega_{G/X}$ is just a function on $V(I^2)$ which vanishes on Δ. The form df corresponds to the function $(a,b) \mapsto f(a) - f(b)$.

We write TG for the dual bundle $\Omega_{G/X}^\vee$. The Lie algebra of G is the pullback of TG along the zero-section: $LG = 0^* TG$. One checks that $LG^\vee$ is just J/J^2, where $J = \{f \in \mathcal{O}_G \mid f(0) = 0\}$ is the augmentation ideal.

We next need to understand how to define invariant differentials on G, and show that the space ω_G of such is isomorphic to $LG^\vee$ (as in the case of Lie groups).

If we are prepared to work "synthetically" with infinitesimal neighbourhoods, this is easy. A form ω on G is a function on $V(I^2)$ such that $\omega(a,a) = 0$. We say that such a form is invariant iff $\omega(a+c, b+c) = \omega(a,b)$ for any a,b,c where a and b are close to first order. It is clear that such an ω is freely and uniquely determined by the function ω_0 on $V(J^2)$ sending c to $\omega(0,c)$. This is just the image of ω in $J/J^2 = LG^\vee$, or equivalently the value of ω at the zero section.

To make this more concrete, choose a coordinate x on G and write F for the resulting formal group law. We need to find the invariant form ω on G with $\omega_0 = d_0 x \in LG^\vee$. As $\omega_G = \mathcal{O}_G\{dx\}$, we must have $\omega = g(x)dx$ for some $g \in \mathcal{O}_X[\![x]\!]$ with $g(0) = 1$. As a function on $V(I^2)$ we have

$$\omega(a,b) = g(x(a))(x(a) - x(b))$$

Write $D_2 F$ for the partial derivative of F with respect to the second variable. If $c \in V(J^2)$ we have $x(c)^2 = 0$ so

$$\omega(a, a+c) = g(x(a)) \, [x(a) - F(x(a), x(c))] = -g(x(a)) \, D_2 F(x(a), 0) \, x(c)$$

On the other hand, we are supposed to have

$$\omega(a, a+c) = \omega(0,c) = -x(c)$$

It follows that

$$g(x) = D_2 F(x,0)^{-1} \qquad\qquad \omega = dx/D_2 F(x,0)$$

PROPOSITION 16.1. Suppose that $\mathcal{O}_X$ is torsion-free, and that G and H are formal groups over X. Then $L\colon \mathrm{Hom}(H,G) \to \mathrm{Hom}_X(LH, LG)$ is injective.

PROOF. Suppose that $f: G \to H$ has $Lf = 0$. Let x be a coordinate on G, and $\omega = g(x)dx$ the corresponding invariant differential. Let y be a coordinate on H, so $f^*x = u(y)$ for some $u \in \mathcal{O}_X[\![y]\!]$ with $u(0) = 0$. Then $f^*\omega$ is an invariant differential on H which vanishes (because $Lf = 0$) at zero, so it vanishes everywhere. Thus

$$0 = f^*(g(x)dx) = g(u(y))u'(y)dy$$

Because $g(x)$ is invertible, this means that $u' = 0$. Because $\mathcal{O}_X$ is torsion-free, this means that $u = 0$ and thus $f = 0$ $\square$

17. Divisors and Bundles

Let G/X be a formal group. An *effective divisor* of degree m on G over an X-scheme Y is a closed formal subscheme $D < G \times_X Y$ such that $\mathcal{O}_D$ is a free module of rank m over $\mathcal{O}_Y$.

Let x be a coordinate on G, and let D be such a divisor. Note that $\mathcal{O}_D$ is a quotient of $\mathcal{O}_G \otimes_X \mathcal{O}_Y = \mathcal{O}_Y[\![x]\!]$. Write

$$f(t) = f_D(t) = \sum_{k=0}^{m} c_{m-k}t^k \qquad c_0 = 1$$

for the (monic) characteristic polynomial of the $\mathcal{O}_Y$-linear endomorphism of $\mathcal{O}_D$ given by multiplication by x. By Cayley-Hamilton, we see that $f(x) = 0$ in $\mathcal{O}_D$. It follows that

$$\mathcal{O}_D = \mathcal{O}_Y[\![x]\!]/f(x) = \mathcal{O}_Y[\![x]\!]/\sum_{k=0}^{m} c_{m-k}x^k$$

for uniquely determined elements $c_k \in \mathcal{O}_Y$. We refer to $f(x)$ as the *equation* of D.

LEMMA 17.1. c_k is topologically nilpotent for $k > 0$.

PROOF. Let $\mathfrak{p}$ be an open prime ideal in $\mathcal{O}_X$. By lemma 32.1, we need only check that $c_k \in \mathfrak{p}$, or equivalently that $f(t)$ becomes t^m over $\mathcal{O}_X/\mathfrak{p}$. As $x \in \mathrm{Nil}(\mathcal{O}_G)$ and $\mathcal{O}_G \to \mathcal{O}_D$ is continuous, we know that multiplication by x is a nilpotent endomorphism of $\mathcal{O}_X/\mathfrak{p} \otimes_X \mathcal{O}_D \simeq (\mathcal{O}_X/\mathfrak{p})^m$. Some standard linear algebra over the field of fractions of $\mathcal{O}_X/\mathfrak{p}$ assures us that the characteristic polynomial can only be t^m, as required. $\square$

EXAMPLE 17.1. If E is complex oriented then $\mathbb{C}P_E^m = \mathrm{spf}(\mathcal{O}_G/x^{m+1})$ is a divisor on G over X.

EXAMPLE 17.2. Suppose that E is Morava K-theory or E-theory of height n at a prime p. We have a long fibration sequence

$$\mathbb{Z}/p^m \to S^1 \xrightarrow{f} S^1 \to B\mathbb{Z}/p^m \to \mathbb{C}P^\infty \xrightarrow{Bf} \mathbb{C}P^\infty$$

where $f(z) = z^{p^m}$. We pick out from this the circle bundle

$$S^1 \to B\mathbb{Z}/p^m \to \mathbb{C}P^\infty$$

The Euler class is just $[p^m](x) = (p_G^m)^*x$. This can be written (by the Weierstrass preparation theorem) as $g(x)u(x)$ where $u(x)$ is an invertible power series and $g(x)$ is a monic polynomial of degree p^{nm}, congruent to $x^{p^{nm}}$ modulo the maximal ideal of E. In particular, $[p^m](x)$ is not a zero-divisor, so the Gysin sequence for our circle bundle is just a short exact sequence

$$E(B\mathbb{Z}/p^m) \leftarrow E(\mathbb{C}P^\infty) \xleftarrow{\times [p^m](x)} E(\mathbb{C}P^\infty)$$

It follows that

$$E^*B\mathbb{Z}/p^m = E^*[\![x]\!]/[p^m](x) = E^*[\![x]\!]/g(x)$$

This is the cokernel in the category of Hopf algebras of the map $(p_G^m)^*: E(\mathbb{C}P^\infty) \to E(\mathbb{C}P^\infty)$. We conclude that $G(m) = (B\mathbb{Z}/p^m)_E$ is a divisor of degree p^{nm} on G over X, and is also the kernel of $p_G^m: G \to G$. This map is in fact an isogeny. This means that it is about as surjective as a map of schemes can be without being

split, so G should be thought of as a divisible sort of group. In fact, $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ is a good model to have in mind.

EXAMPLE 17.3. Let Z be a space, and V a complex vector bundle of dimension m over Z. We can define the associated projective bundle

$$P(V) = \{(z, L) \mid z \in Z ,\ L \leq V_z \text{ a line }\}$$

(where "line" means "one-dimensional subspace"). There is a tautological line bundle $L(V)$ over $P(V)$, defined by

$$L(V)_{(z,L)} = L$$

This is classified by a map $P(V) \to \mathbb{C}P^\infty$. We thus obtain a map $P(V) \to \mathbb{C}P^\infty \times Z$ and hence $P(V)_E \to G \times_X Z_E$. It is a standard fact that

$$E^* P(V) = E^* Z[\![x]\!]/\sum c_{m-k} x^k$$

where the coefficients c_{m-k} are the Chern classes of V (and x is identified with the Euler class of $L(V)$). This shows that $D(V) = P(V)_E$ is a divisor of degree m on G over Z_E.

EXAMPLE 17.4. Consider the diagonal subscheme $\Delta < G \times_X G$. This can be considered as a divisor on G defined over G, or as the family of one-point divisors $[a]$ parametrised by $a \in G$. If we write $\mathcal{O}_{G \times_X G} = \mathcal{O}_X[\![x, y]\!]$ in the obvious way, then $J_\Delta = (x - y)$. Let $L \to \mathbb{C}P^\infty$ be the universal bundle. Then $D(L) = \Delta$.

EXAMPLE 17.5. Let $a_1, \dots, a_m$ be sections of G over an X-scheme Y, so that $x(a_k) \in \mathcal{O}_Y$. We can define ideals

$$J_k < \mathcal{O}_{G \times_X Y} \qquad\qquad J_k = \{f \mid f(a_k) = 0\} = (x - x(a_k))$$

and thus a divisor

$$D = \mathcal{O}_{G \times_X Y}/\prod_k J_k = \mathcal{O}_{G \times_X Y}/\prod_k (x - x(a_k))$$

Note that this is actually independent of the choice of coordinate x. We write $[a_1, \dots a_m]$ or $\sum_k [a_k]$ for this divisor. Note that $\{a_k\}$ is not really a single point, but should be thought of as

$$\{a_k\} = \{(a_k(y), y) \mid y \in Y\} \subset G \times_X Y$$

Given a divisor D, a list of sections giving rise to it as above is called a *full set of points* for D. If ζ is a vector bundle of dimension m over Z then a splitting of ζ as a sum of line bundles gives rise to a full set of sections of the associated divisor. See section 20 for some related but more general definitions.

We would like to be able to define virtual divisors, to be compared with virtual bundles. We can do this as follows. Let Y be a formal scheme, and H a formal group over Y. If x and y are two coordinates on H then $x = uy$ for a unit $u \in \mathcal{O}_H^\times$. Thus, the ring

$$\mathcal{M}_H = \varprojlim_{I \lhd_O \mathcal{O}_Y} x^{-1}(\mathcal{O}_Y/I \widehat{\otimes}_Y \mathcal{O}_H)$$

is invariantly defined. Any element $f \in \mathcal{M}_H$ has the form $\sum_{k \in \mathbb{Z}} a_k x^k$ with $a_k \in \mathcal{O}_Y$ and $a_k \to 0$ as $k \to -\infty$.

DEFINITION 17.1. A *Cartier divisor* on H over Y is an element of $\mathcal{M}_H^\times/\mathcal{O}_H^\times$.

If we need to distinguish between Cartier divisors and divisors as defined previously, we shall refer to the latter as effective Weil divisors. The effective Weil divisors form an Abelian semigroup with cancellation. This can be embedded in a group in the usual way. We refer to elements of this group as Weil divisors.

If D is an effective Weil divisor of degree m, then $J_D = \ker(\mathcal{O}_H \to \mathcal{O}_D)$ is a Cartier divisor. This construction gives a homomorphism from the group of all Weil divisors to that of Cartier divisors. This is iso if Y is a connected informal scheme, but not in general.

Suppose that Y is a connected formal scheme, so that $\mathcal{O}_Y$ has no nontrivial idempotents. We say that $g(x) \in \mathcal{M}_H$ is *holomorphic at infinity* if it can be written as $g(x) = \sum_{k \le 0} a_k x^k$, and then we write $g(\infty)$ for a_0.

Using corollary 33.3, we see that any Cartier divisor has a unique representative of the form $x^n g(x)$, where $n \in \mathbb{Z}$, $g(x)$ is holomorphic at infinity and $g(\infty) = 1$. We refer to n as the *degree* of the divisor.

18. Classification of Divisors

Let G be a formal group over X. Consider the m'th tensor power $R = \widehat{\bigotimes}_{\mathcal{O}_X}^m \mathcal{O}_G$ and the symmetric subring $S = R^{\Sigma_m}$. If x is a coordinate on G then

$$R = \mathcal{O}_X[\![x_1, \dots, x_m]\!]$$

$$S = \mathcal{O}_X[\![c_1, \dots, c_m]\!]$$

where the c_k are (up to sign) the elementary symmetric functions of the x_k (and we take $c_0 = 1$).

It is clear that $\mathrm{spf}(R)$ is the m-fold product $G_X^m = G \times_X \dots G$. The k'th projection $a_k \colon G_X^m \to G$ is a section of G over G_X^m. As in example 17.5, these sections give a divisor D_0 on G over G_X^m. In fact, $D_0 = \mathrm{spf}\, R[\![x]\!]/\sum c_{m-k} x^k$, so D_0 is obtained by pulling back a divisor $D = \mathrm{spf}\, S[\![x]\!]/\sum c_{m-k} x^k$ over $Y = \mathrm{spf}(S)$.

Now let D' be a divisor on G over an arbitary formal X-scheme Y'. Then

$$\mathcal{O}_{D'} = \mathcal{O}_{Y'}[\![x]\!]/\sum c'_{m-k} x^k$$

for uniquely determined topologically nilpotent coefficients $c'_l \in \mathrm{Nil}(\mathcal{O}_{Y'})$. There is a map $\mathcal{O}_Y \to \mathcal{O}_{Y'}$ sending c_l to c'_l, and thus a corresponding map $Y' \to Y$. This is clearly the unique map $Y' \to Y$ for which the pullback of D is D'. This construction gives a natural bijection

$$\{\text{divisors of degree } m \text{ on } G \text{ over } Y'\} = \mathcal{F}ormal_X(Y, Y')$$

It follows that the functor

$$\mathrm{Div}_m^+(R) = \{(f, D) \mid f \colon \mathrm{spf}(R) \to X \,,\, D \text{ a divisors of degree } m \text{ on } G \text{ over } R\}$$

is actually a formal scheme; in fact $\mathrm{Div}_m^+ = Y$.

Topologically, of course, we have

$$\mathrm{Div}_m^+ = BU(m)_E$$

Moreover, $BU(m)$ classifies bundles, Div_m^+ classifies divisors, we know how to construct divisors from bundles, and everything is compatible in the evident sense.

It is tempting to interpret the above construction as saying that $\mathrm{Div}_m^+ = G_X^m/\Sigma_m$. This is true in the sense that Div_m^+ is the categorical quotient of G_X^m by the action of Σ_m in the category of formal schemes, and this is useful for constructing maps out of Div_m^+ (e.g. proposition 18.1, or the definition of convolution below). However, the functor Div_m^+ is rather poorly related to the functor $R \mapsto G(R)_{X(R)}^m/\Sigma_m$.

We can make $\{\mathrm{Div}_m^+\}_{m \ge 0}$ into a graded semiring in the category of schemes, as follows. Given a divisor D we write J_D for the ideal such that $D = \mathrm{spf}(\mathcal{O}_{G \times_X Y}/J_D)$. Given a coordinate x, we know that $J_D = (f)$ for some monic polynomial f, so J_D is actually a free module of rank one over $\mathcal{O}_{G \times_X Y}$. We define addition of divisors by $J_{D+D'} = J_D J_{D'}$. This defines a map

$$\mathrm{Div}_m^+ \times_X \mathrm{Div}_n^+ \to \mathrm{Div}_{m+n}^+$$

Suppose now that we have full sets of sections $\{a_1, \dots a_m\}$ and $\{b_1, \dots b_n\}$ for D and D'. It is easy to see that $D + D' = \{a_1, \dots a_m, b_1, \dots b_n\}$. In this context, we can also define the *convolution*

$$D * D' = \{a_k + b_l \mid 1 \le k \le m, 1 \le l \le n\}$$

Here $a_k + b_l$ refers to the addition in the group G, of course. Using the description $\mathrm{Div}_m^+ = G_X^m / \Sigma_m$ and working with the universal case, one can construct a unique map $\mathrm{Div}_m^+ \times_X \mathrm{Div}_n^+ \to \mathrm{Div}_{mn}^+$ giving rise to this convolution.

This semiring structure also arises from the maps $BU(m) \times BU(n) \to BU(m+n)$ and $BU(m) \times BU(n) \to BU(mn)$ classifying direct sum and tensor product of vector bundles.

Arguments similar to the above show that

$$BU_E = \mathrm{Div}_0 = \{ \text{ Cartier divisors of degree 0 } \}$$

$$(\mathbb{Z} \times BU)_E = \mathrm{Div} = \{ \text{ Cartier divisors } \}$$

Note that the natural inclusion is

$$BU(n) \to \{n\} \times BU \to \mathbb{Z} \times BU$$

Another interpretation of BU_E is the following. Further explanation of the statement is given in the proof.

PROPOSITION 18.1. $\mathrm{Div}_0 = BU_E$ the free group scheme over X on the underlying pointed X-scheme of $G = \mathbb{C}P_E^\infty$. Similarly, Div is the free ring scheme over X generated by the group scheme G. Moreover, the above remains true after pullback along an arbitrary map $Y \to X$.

PROOF. First, let us make the claim more concrete. There is a map $j \colon G \to \mathrm{Div}_0$ sending a point a of G to the divisor $[a] - [0]$. This corresponds to the usual map $\mathbb{C}P^\infty \to \{0\} \times BU$, classifying the reduced canonical bundle $L - 1$. Let H be a commutative group in the category of formal schemes over X (not necessarily of dimension one), and let $f \colon G \to H$ be a map of formal schemes with $f(0) = 0$. The claim is that there is a unique factorisation $f = g \circ j$ with $g \colon \mathrm{Div}_0 \to H$ a group map. To prove this, observe that the map

$$G^m \xrightarrow{f^m} H^m \xrightarrow{\text{sum}} H$$

factors through a map

$$g_m \colon G^m / \Sigma_m = \mathrm{Div}_m^+ \to H \ .$$

There are maps

$$j_m \colon \mathrm{Div}_m^+ \to \mathrm{Div}_0 \qquad\qquad D \mapsto D - m[0]$$

corresponding to the usual maps $BU(m) \to BU$. It is easy to see that $E(BU) = \varprojlim E(BU(m))$ as formal rings, so $\mathrm{Div}_0 = \varinjlim \mathrm{Div}_m^+$ as formal schemes. The maps g_m thus fit together to give a map $g \colon \mathrm{Div}_0 \to H$ as required. By merely changing the notation, we see that the same holds after base change to Y. For the second claim, we note that Div is a ring scheme over X under addition and convolution of divisors, corresponding to direct sum and tensor product of bundles. The unit element is the one-point divisor $[0]$. The map $j' \colon G \to \mathrm{Div}$ sending a to $[a]$ is a homomorphism from G to Div (considered as a semigroup under convolution). Suppose that R is another ring scheme over X, and that $f' \colon G \to R$ is a homomorphism in the same sense. The claim is that $f' = g' \circ j'$ for a unique map $g' \colon \mathrm{Div} \to R$ of ring schemes. To see this, define $f \colon G \to R$ by $f(a) = f'(a) - 1$ and use the first part to get a map $g \colon \mathrm{Div}_0 \to R$ of (additive) group schemes. The map $\mathrm{Div}_0 \to \mathrm{Div}_m$ sending D to $D + m[0]$ is iso, so we can define $g' \colon \mathrm{Div} \to R$ by $g'(D + m[0]) = g(D) + m$. One can check that this works. $\square$

REMARK 18.1. There is a certain confusion among the people about the above result. It is widely held that "$E^* BU$ is the Witt Hopf algebra" and that "the Witt Hopf algebra classifies curves". Both of these things are true up to unnatural isomorphism. If we choose a basic curve $\gamma \colon \widehat{\mathbb{A}}^1 \simeq G$, then the above gives an isomorphism

$$\mathcal{G}roups_X(BU_E, H) = \mathrm{Map}_X^0(G, H) \simeq \mathrm{Map}_X^0(\widehat{\mathbb{A}}^1, X) = \mathrm{Curves}(H)$$

However, because of the arbitrary choice of γ we cannot expect this identification to commute with most interesting operations. Moreover, it is telling us about curves on any group H, and not especially about curves on G.

EXAMPLE 18.1. We could, nonetheless, take $H = G$, and ask for a map of groups $\mathrm{Div}_0 \to G$ extending the inclusion $j_0 : a \mapsto [a] - [0]$. The resulting map is just $(B \det)_E : BU_E \to BS_E^1$. It sends a degree zero divisor $\sum_i n_i[a_i]$ (with $\sum_i n_i = 0$) to $\sum_i n_i a_i$.

There is an evident partial order on divisors, defined by $D' \leq D$ iff $D = D' + D''$ for some (necessarily unique) D'', iff $f_{D'}$ divides f_D. This order is particularly useful in the light of the following proposition.

PROPOSITION 18.2. Suppose that D and D' are divisors on G defined over an X-scheme Y. There is then a closed subscheme $Z \leq Y$ which is universal for the condition $D' \leq D$. In other words, a map $w : W \to X$ has $w^*D' \leq w^*D$ iff w factors through Z.

PROOF. Suppose D has degree m. We can then write

$$f_{D'}(x) = \sum_{k=0}^{m-1} a_k x^k \pmod{f_D}$$

for uniquely determined coefficients a_k. It follows that

$$f_{w^*D'}(x) = \sum_{k=0}^{m-1} w^* a_k x^k \pmod{f_{w^*D}}$$

Thus, we can take $I = (a_k \mid 0 \leq k < m)$ and $Z = V(I) = \mathrm{spf}(\mathcal{O}_Y/I)$. $\square$

See section 26 for an interesting application.

The scheme of divisors is functorial in two different ways. First, suppose that $q : G \to H$ is an map of formal groups over X. Using the description $\mathrm{Div}_m^+(G) = G_X^m/\Sigma_m$, we see immediately that q induces a map $q_* : \mathrm{Div}_m^+(G) \to \mathrm{Div}_m^+(H)$. Moreover, this gives a map of ring schemes.

Now suppose that q is an isogeny of degree d, and that D is a divisor on H of degree m. We can then form the pullback

$$
\begin{array}{ccc}
q^*D & \longrightarrow & D \\
\downarrow & & \downarrow \\
G & \xrightarrow{q} & H
\end{array}
$$

We see that $q^*D \to D$ is free of degree d, and thus that q^*D is a divisor of degree md on H. This gives a map $q^* : \mathrm{Div}(H) \to \mathrm{Div}(G)$, which is q_*-linear in the sense that

$$q^*(D * q_* D') = (q^*D) * D'$$

Given a divisor D over Y, the ideal J_D is free of rank one over $\mathcal{O}_{G \times_X Y}$. It can thus be thought of as a line bundle over $G \times_X Y$; in this guise we shall refer to it as $\mathcal{O}(D)$.

REMARK 18.2. There are apparently two possible meanings for $q^*\mathcal{O}(D)$. On the one hand, $\mathcal{O}(D)$ is a line bundle over H, so we can form the pullback $L = \mathcal{O}_G \otimes_H \mathcal{O}(D)$. On the other hand, q^* is a map $\mathcal{O}_H \to \mathcal{O}_G$ and $\mathcal{O}(D)$ is an ideal in $\mathcal{O}_H$ so we can form the image $M = q^*(\mathcal{O}(D))$, which is an $\mathcal{O}_H$-submodule of $\mathcal{O}_G$. Applying flatness of q to the inclusion $\mathcal{O}(D) \to \mathcal{O}_H$, one can see that $L \simeq \mathcal{O}_G M$. Now let f be a generator of $\mathcal{O}(D)$. Thinking of f as a map $H \to \widehat{\mathbb{A}}^1$, we see that $D = f^*[0]$. It follows that $q^*D = (f \circ q) * [0]$ and thus that $f \circ q = q^*f \in M$ generates $\mathcal{O}(q^*D)$. In conclusion,

$$q^*\mathcal{O}(D) = \mathcal{O}_G . \mathrm{image}(\mathcal{O}(D) \xrightarrow{q^*} \mathcal{O}(G)) = \mathcal{O}(q^*D)$$

19. Cohomology of Thom Spectra

We also have an algebraic analogue of the Thom space of a vector bundle. Let D be a divisor on G over Y. Pulling back the line bundle $\mathcal{O}(D)$ on $G \times_X Y$ by the zero section $0 \colon Y \to G \times_X Y$, we obtain a line bundle over Y:
$$\mathbb{L}(D) = 0^* \mathcal{O}(D) = J_D / J_{\{0\}+D}$$
It is not hard to show that $\mathbb{L}(D + D') = \mathbb{L}(D) \otimes \mathbb{L}(D')$. If $Y = Z_E$ and $D = D(V) = P(V)_E$ then $\mathbb{L}(D) = \tilde{E}^* Z^V$, the reduced cohomology of the Thom space of V. This follows from the cofibration
$$P(V) \to P(V \oplus \mathbb{C}) \to Z^V$$
We also write $\mathbb{L}(V)$ for $\mathbb{L}(D(V))$.

We write Δ for the diagonal divisor on $G \times_X G$ (see example 17.4). In terms of topology, we have $\Delta = D(L)$, where L is the universal line bundle over $\mathbb{C}P^\infty$.

We also write $*$ for convolution of divisors (so that $D(V \otimes W) = D(V) * D(W)$) and $\overline{D}$ for the image of D under the negation map $(-1)_G \colon G \to G$.

PROPOSITION 19.1. $\mathbb{L}(D * \Delta) = \mathcal{O}(\overline{D})$

PROOF. First note that if $f(x) \in \mathcal{O}_{G \times_X Y}$ generates J_D then $f(x -_F y) \in \mathcal{O}_{G \times_X Y \times_X G}$ generates $J_{D*\Delta}$. This implies that
$$\mathbb{L}(D * \Delta) = \frac{\mathcal{O}_Y[\![x, y]\!] f(x -_F y)}{\mathcal{O}_Y[\![x, y]\!] x f(x -_F y)}$$
Next, consider the embedding $Y \times_X G \to G \times_X Y \times_X G$ by $(y, b) \mapsto (0, y, b)$. This induces a map $\mathcal{O}_{G \times_X Y \times_X G} \to \mathcal{O}_{Y \times_X G}$, sending x to 0. It is easy to see that this induces $\mathbb{L}(D * \Delta) \simeq (f([-1]_F(y))) = J_{\overline{D}}$ as required. $\square$

There is a small point of compatibility to be dealt with here. Pulling back the above statement along the zero map $Y \to Y \times_X G$, we obtain an isomorphism $\mathbb{L}(D) \simeq 0^* \mathcal{O}(\overline{D})$. On the other hand, we have $\mathbb{L}(D) = 0^* \mathcal{O}(D)$ by definition. One can check that the resulting isomorphism $0^* \mathcal{O}(D) \simeq 0^* \mathcal{O}(\overline{D})$ is just induced by the negation map $(-1)_G$.

The Thom module for the trivial line bundle over a point is just $J_0 / J_0^2 = \omega_G$. This is of course compatible the isomorphism $\tilde{E}(1^{\mathbb{C}}) = \tilde{E}(\mathbb{C}P^1) = E^{-2} = L^{-1} = \omega_G$.

20. Norms and Full Sets of Points

Suppose that $f \colon X \to Y$ is a finite free map, in other words it makes $\mathcal{O}_X$ into a finitely generated free module (of rank m, say) over $\mathcal{O}_Y$. We can then define a (nonadditive) *norm map* $N_f = N_{X/Y} \colon \mathcal{O}_X \to \mathcal{O}_Y$, by letting $N_f(u)$ be the determinant of multiplication by u, considered as an $\mathcal{O}_Y$-linear endomorphism of $\mathcal{O}_X$.

PROPOSITION 20.1.
  (1) $N_{X/Y}(uv) = N_{X/Y}(u) N_{X/Y}(v)$
  (2) $N_{Y/Z} \circ N_{X/Y} = N_{X/Z}$
  (3) $N_{X/Y} f^*(w) = w^n$
Moreover suppose we have a pullback diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ a\ } & X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
W & \xrightarrow{\ b\ } & Y
\end{array}
$$

If f is a finite free map then so is g, and $N_g \circ a^* = b^* \circ N_f$. $\square$

EXAMPLE 20.1. Suppose $C = B[t]/g(t)$, for some monic polynomial g of degree m. By using the basis $\{1, t, \ldots t^{m-1}\}$ for C over B, we find that $N_{C/B}(t) = (-1)^m g(0)$.

More generally, given $Z \to Y$ we define

$$N^Z_{X/Y} = N_{X \times_Y Z/Z} \colon \mathcal{O}_X \otimes_Y \mathcal{O}_Z \to \mathcal{O}_Z$$

In particular, we can consider $Z = \mathbb{A}^1_Y = \operatorname{spec} \mathcal{O}_Y[t]$, take $u \in \mathcal{O}_Y$ and consider the characteristic polynomial

$$P_{X/Y}(u, t) = N^Z_{X/Y}(t - u) \in \mathcal{O}_X[t]$$

Similarly, we define $P^Z_{X/Y}(u, t)$ for arbitary $Z \to Y$ and $u \in \mathcal{O}_X \otimes_Y \mathcal{O}_Z$.

EXAMPLE 20.2. Let B and C be as in example 20.1. I claim that $N^{B[s]}_{C/B}(t - s) = g(s)$. To see this, take $u = t - s$, so that

$$B[s] \otimes_B C = C[s] = B[s, t]/g(t) = B[s][u]/g(s + u)$$

Note that $h(u) = g(s + u)$ is a monic polynomial of degree m in u over $B[s]$, so example 20.1 gives

$$N^{B[s]}_{C/B}(t - s) = N_{C[s]/B[s]}(u) = h(0) = g(s)$$

DEFINITION 20.1. Suppose Z is a scheme over Y. We say that a list $(a_1, \dots a_n) \in \Gamma(Z, X)^n$ is a *full set of points* of X (over Z) iff for all $g \colon W \to Z$ and $u \in \mathcal{O}_{W \times_Y X}$ we have

$$N^W_{X/Y}(u) = \prod_i u(a_i)$$

It is equivalent to require that for all W and u we have

$$P^W_{X/Y}(u, t) = \prod_i (t - u(a_i))$$

In the last two equations, a_i really means $g^* a_i \in \Gamma(W, X)$.

LEMMA 20.2. Suppose $C = B[t]/g(t)$, for some monic polynomial g of degree m, and write $X = \operatorname{spec}(C)$ and $Y = \operatorname{spec}(B)$. Then $\{b_1, \dots b_m\}$ is a full set of sections iff $g(s) = \prod_i (s - b_i^* t)$.

PROOF. Necessity follows immediately from example 20.2. For sufficiency, write b_i for $b_i^* t$ and suppose that $g(s) = \prod_i (t - b_i)$. Consider an element $f(t) = \sum_{i=0}^{m-1} a_i t^i \in A \otimes_B C$, for some B-algebra A. We are required to show that $N^A_{C/B}(f) = \prod_i f(b_i)$. By an evident naturality argument, we may assume that

$$A = B = \mathbb{Z}[a_0, \dots, a_{m-1}, b_1, \dots, b_m]$$

We only need to verify that certain polynomial expressions in the a's and b's are equal, so we are free replace B by a larger ring. In particular, we can invert the discriminant of g (c.f. example 13.2). In this context, the map

$$B[t]/g(t) \to B^m \qquad t \mapsto (b_1, \dots b_m)$$

is iso, and the claim is trivial. $\square$

EXAMPLE 20.3. Suppose that D is an effective divisor of degree m on a formal group G/X, and that $a_1, \dots, a_m$ are sections of G. By choosing a coordinate on G and appealing to lemma 20.2, we see that the a_i form a full set of sections for D iff $D = \sum_i [a_i]$. This shows that definition 20.1 generalises that in example 17.5.

Next, consider the X-prescheme defined by

$$\Gamma(T, \operatorname{FSP}(D/X)) = \{ \text{ full sets of points for } D \text{ over } T \}$$

This is actually a scheme, in fact a closed subscheme of the m-fold fibre product D^m_X. Put $x_k = x(a_k)$ and let

$$d_k \in \mathcal{O}_{D^m_X} = \mathcal{O}_X[x_1, \dots, x_m]/(f(x_1), \dots f(x_m))$$

be the coefficient of t^{m-k} in $\prod_i (t - x(a_i))$. If we write $J = (d_1 - c_1, \dots d_m - c_m)$, then $\operatorname{FSP}(D/X) = V(J)$.

EXAMPLE 20.4. Let Z be a space and V a vector bundle over Z. Then

$$\mathrm{FSP}(D(V)/Z_E) = \mathrm{Flag}(V)_E$$

This follows easily from the universal case, in which $Z = BU(m)$ and V is the canonical bundle. We take $EU(m)$ to be the space of m-frames in $\mathbb{C}^\infty$, so that $BU(m) = EU(m)/U(m)$ is the Grassmannian of m-dimensional subspaces. Let $T(m) = (S^1)^m \leq U(m)$ be the maximal torus. With this model it is clear that $\mathrm{Flag}(V) = EU(m)/T(m) = BT(m) \simeq (\mathbb{C}P^\infty)^m$. Thus $Z_E = \mathrm{Div}_m^+ = G_X^m/\Sigma_m$, $D(V)$ is the universal divisor over Div_m^+, and $\mathrm{Flag}(V)_E = G_X^m = \mathrm{FSP}(D(V))$.

EXAMPLE 20.5. Let $q\colon G \to H$ be an isogeny of formal groups over X, with kernel K. Suppose that $\{a_1,\dots,a_n\}$ is a full set of sections for K (they need not form a group). Write τ_i for the translation map $b \mapsto b + a_i$ of G. Then $q^* N_q f = \prod_i \tau_i^* f$.

PROOF. Let $\sigma\colon K \times_X G \to G$ be the addition map, so we have a pullback square

$$
\begin{array}{ccc}
K \times_X G & \xrightarrow{\ \sigma\ } & G \\
\ \downarrow{\scriptstyle \pi} & & \ \downarrow{\scriptstyle q} \\
G & \xrightarrow{\ q\ } & H
\end{array}
$$

Write $\tilde{a}_i = a_i \times_X 1_G \colon G \to K \times_X G$. These maps form a full set of sections for $\pi\colon K \times_X G \to G$, and $\sigma \circ \tilde{a}_i = \tau_i$. Thus

$$q^* N_q f = N_\pi \sigma^* f = \prod_i \tilde{a}_i^* \sigma^* f = \prod_i \tau_i^* f$$

□

EXAMPLE 20.6. This example uses concepts defined in section 26 below. Consider the finite free map $\mathrm{Level}(A,G) \to X$, where $\mathrm{rank}(A) \leq n$. This has no sections. Suppose that $p^m A = 0$, so there exist monomorphisms $\alpha\colon A \to \Lambda_m$. Consider the pulled-back map

$$\mathrm{Level}(A,G) \times_X \mathrm{Level}(\Lambda_m, G) \to \mathrm{Level}(\Lambda_m, G)$$

For each monomorphism α as above, we get a section

$$(\phi \circ \alpha, \phi) \leftarrow \phi$$

These sections form a full set, although the corresponding map

$$\coprod_\alpha \mathrm{Level}(\Lambda_m, G) \to \mathrm{Level}(A,G) \times_X \mathrm{Level}(\Lambda_m, G)$$

is only an isomorphism rationally.

EXAMPLE 20.7. We can generalise the part of example 20.3 as follows. Let $Y \to X$ be an arbitrary finite free map, and define the X-prescheme $\mathrm{FSP}(Y/X)$ by

$$\Gamma(T, \mathrm{FSP}(Y/X)) = \{\text{ full sets of points for } Y \text{ over } T \}$$

This is again a closed subscheme of Y_X^m.

PROOF. The condition for an m-tuple $\underline{a}$ of sections to be a full set involves an element f of $\mathcal{O}_{Y' \times_X Y}$. The universal example of a scheme Y' over X with m sections and a function f is $\tilde{Y} = Y_X^m \times_X \mathbb{A}(\mathcal{O}_Y)$. In other words, for any point $x \in X$

$$\tilde{Y}_x = \{(\underline{a}, f) \mid \underline{a} \in Y_x^m \ , \ f \colon Y_x \to \mathbb{A}^1\}$$

Note that $\mathcal{O}_{\tilde{Y}}$ is free over $\mathcal{O}_{Y_X^m}$, so that the projection $\tilde{Y} \to Y_X^m$ is an open map by proposition 8.1. We have a function $g = N_{Y/X}^{\tilde{Y}}(f) - \prod f(a_i) \in \mathcal{O}_{\tilde{Y}}$. The fibre of $V(g)$ over a point $\underline{a} \in Y_x^m$ is

$$V(g)_{\underline{a}} = \{f \colon Y_x \to \mathbb{A}^1 \mid N(f) = \prod_i f(a_i)\}$$

Thus

$$\Gamma(T, \mathrm{FSP}(Y/X)) = \{\underline{a} \in \Gamma(T, Y_x^m) \mid V(g)_{\underline{a}} = \tilde{Y}_{\underline{a}}\}$$

It follows (from the definition 8.1 of an open map) that this is a closed subscheme as claimed. To make this more explicit, write

$$A = \mathcal{O}_X \qquad\qquad B = \mathcal{O}_Y$$
$$C = B^{\otimes_A m} = \mathcal{O}_{Y_X^m}$$
$$D = C \otimes_A \mathrm{Sym}_A[B^\vee] = \mathcal{O}_{\tilde{Y}}$$

The inclusion $B^\vee \to \mathrm{Sym}_A[B^\vee]$ gives by adjunction an element of $B \otimes \mathrm{Sym}_A[B^\vee]$ and thus (under the obvious inclusion) an element $f \in B \otimes_A D$. Taking the determinant of this as D-linear endomorphism of $B \otimes_A D$ gives an element $Nf \in D$. On the other hand, we have m inclusions $a_k \colon B \to C$, each giving rise to a map

$$B \otimes_A D = B \otimes_A C \otimes_A \mathrm{Sym}_A[B^\vee] \to C \otimes_A \mathrm{Sym}_A[B^\vee]$$

The image of f is naturally denoted $f(a_k)$. We can now form $g = Nf - \prod f(a_k) \in D$. Expanding this in terms of a basis for D over C, we get various coefficients $g_\alpha \in C$. Finally, $\mathrm{FSP}(Y/X) = \mathrm{spec}(C/(g_\alpha))$. □

Our next task is to extend the norm construction to line bundles and their sections. First, we describe the motivating example. Let X be a scheme, S a finite set, and $Y = S \times X$. A line bundle L over Y just consists of a family $\{L_s\}_{s \in S}$ of line bundles over X. We can define $N_{Y/X}L = \bigotimes_s L_s$, considered as a bundle over X. Given a section $a = (a_s)_{s \in S}$ of L, we define a section $N_{Y/X}a$ of $N_{Y/X}L$ in the evident way.

Now let $f \colon Y \to X$ be an arbitary finite free map, of degree m say. Given a line bundle L over Y, we define

$$N_f L = N_{Y/X}L = \mathrm{Hom}_X (\textstyle\bigwedge_X^m \mathcal{O}_Y, \bigwedge_X^m L)$$

Let l be a section of L. By regarding it as an $\mathcal{O}_X$-linear map $l \colon \mathcal{O}_Y \to L$, we get an section $N_f(l)$ of $N_f L$. Here are some properties of the functor on line bundles; analogous things are true for the map on sections.

(1) $N_f L$ is a line bundle over X.
(2) $N_f \mathcal{O}_Y = \mathcal{O}_X$ and $N_f(L \otimes_Y M) = N_f(L) \otimes_X N_f(M)$.
(3) $N_f(f^* N) = N^{\otimes m}$.
(4) Suppose that $\{a_1, \dots, a_m\}$ is a full set of sections. Then $N_f L \simeq \bigotimes_k a_k^* L$.

PROPOSITION 20.3. Let $q \colon G \to H$ be an isogeny of formal groups over X, with kernel $K \in \mathrm{Div}(G)$. Suppose $D \in \mathrm{Div}(G)$. Then $N_q \mathcal{O}(D) = \mathcal{O}(q_* D)$, and $q^* N_q \mathcal{O}(D) = q^* \mathcal{O}(q_* D) = K * D$.

PROOF. This is an equality between two ideals in $\mathcal{O}_H$. As q is faithfully flat, it suffices to show that $q^* N_q \mathcal{O}_D = q^* \mathcal{O}(q_* D)$. Write K for the kernel of q. By making a faithfully flat base change to give K a full set of sections, and using a pullback diagram as in example 20.5, we see that

$$q^* N_q \mathcal{O}(D) = \mathcal{O}(K * D)$$

Now suppose $D = [a]$. Note that $K = q^*[0]$, so

$$q^*[q(a)] = q^* \tau_{-q(a)}^*[0] = \tau_{-a}^* q^*[0] = [a] * K$$

After giving D a full set of points and extending this in the evident way, we conclude that $q^* \mathcal{O}(q_* D) = \mathcal{O}(K * D)$ also. $\square$

21. Subgroup Divisors

A *subgroup divisor* on G over Y is a divisor H on G over Y which is a subgroup-scheme of $G \times_X Y$ over Y. It is equivalent to require that $\mathcal{O}_H$ be a quotient Hopf algebra of $\mathcal{O}_G \otimes_X \mathcal{O}_Y$ over $\mathcal{O}_Y$, which is free of finite rank over $\mathcal{O}_Y$.

EXAMPLE 21.1. Let G_0 be the formal group arising from Morava K-theory, so $\mathcal{O}_{G_0} = \kappa[\![x]\!]$ and $\mathcal{O}_{G_0^2} = \kappa[\![x,y]\!]$. This is a Hopf algebra with $\psi(x) = F(x,y) = x + y \pmod{xy}$. Note every element of $\mathcal{O}_{G_0}$ has the form $x^m u$ for a unit $u \in \mathcal{O}_{G_0}^\times$, so that every divisor defined over κ has the form $\mathrm{spf}(\kappa[\![x]\!]/x^m)$ for some m. For this to be a subgroup divisor, we must have

$$\psi(x^m) = F(x,y)^m = 0 \pmod{x^m, y^m}$$

If we work mod $(x,y)^{m+1}$ then $F(x,y)^m = (x + y)^m$ and this will only vanish mod (x^m, y^m) if m is a power of p. On the other hand, as the coefficients of F lie in $\mathbb{F}_p \leq \kappa$, we have

$$F(x,y)^{p^k} = F(x^{p^k}, y^{p^k}) = 0 \pmod{x^{p^k}, y^{p^k}}$$

It follows that the subgroup divisors defined over κ are precisely the divisors $\mathrm{spf}(\kappa[\![x]\!]/x^{p^k})$.

EXAMPLE 21.2. Let G_a be the (informal) additive group over $\mathbb{F}_p$, so that

$$G_a = \mathrm{spec}(\mathbb{F}_p[x]) \qquad G_a^2 = \mathrm{spec}(\mathbb{F}_p[x,y]) \qquad \psi(x) = x + y$$

A degree m divisor defined over an $\mathbb{F}_p$-algebra R has the form $\mathrm{spec}(R[x]/f(x))$ where $f(x)$ is monic of degree m and $f(x + y) = 0 \pmod{f(x), f(y)}$, say $f(x + y) = g(x,y)f(x) + h(x,y)f(y)$. We can make this representation unique by requiring that g have degree less than m in y. Considering the coefficient of $x^k y^l$ with $l \geq m$ we conclude that h is constant. A similar argument shows that g is constant. By working mod x or y we find that $g = h = 1$, and thus $f(x + y) = f(x) + f(y)$. From this we find that it is neccessary and sufficient for f to have the form $\sum_{k=0}^{r} a_k x^{p^k}$ with $m = p^r$ and $a_r = 1$.

EXAMPLE 21.3. Let G_m be the (informal) multiplicative group, so that

$$G_m = \mathrm{spec}(\mathbb{Z}[u^{\pm 1}]) \qquad G_m^2 = \mathrm{spec}(\mathbb{Z}[u^{\pm 1}, v^{\pm 1}]) \qquad \psi(u) = uv$$

Let H be a closed subgroup of G_m over R such that $\mathcal{O}_H$ is free of rank r over R. Let f be the characteristic polynomial of u on $\mathcal{O}_H$, so f is a monic polynomial of degree r and $f(u) = 0$ in $\mathcal{O}_H$. Moreover, $f(0)$ is the determinant of u on $\mathcal{O}_H$ and thus a unit. It follows that

$$\mathcal{O}_H = R[u^{\pm 1}]/f(u) = R\{1, u, \ldots u^{r-1}\}$$

The condition for H to be a subgroup scheme is that $f(uv) = 0 \pmod{f(u), f(v)}$. Using the obvious bases, it is not hard to conclude that $f(u) = u^r - 1$. In particular, H is already defined over $\mathbb{Z}$ rather than R.

EXAMPLE 21.4. If E is Morava E-theory then the divisor $G(r) = (B\mathbb{Z}/p^r)_E = \mathrm{spf}\, \mathcal{O}_G/[p^r](x) = \ker(p_G^r)$ is actually a subgroup divisor — see example 17.2.

EXAMPLE 21.5. Consider the Morava K theory formal group G_0/X_0. It is a finite (but arduous) computation to classify subgroup divisors on G_0 over X_0-schemes. For example, if $n = p = 2$, then a subgroup divisor of degree 4 over a κ-algebra R has equation

$$x^4 + b^5 x^3 + bx^2 + (b^3 + b^6)x$$

with $b^7 = 0$. I have Mathematica programs to assist with such calculations, which I would be happy to distribute or discuss.

Suppose that H is a subgroup divisor on G of degree r; let us say for simplicity that it is defined over X. We can define a quotient group G/H as follows. There are two maps

$$\alpha, \beta \colon \mathcal{O}_G \to \mathcal{O}_{G \times_X H}$$

$$(\alpha f)(g, h) = f(g + h) \qquad\qquad (\beta f)(g, h) = f(g)$$

We define $\mathcal{O}_{G/H}$ to be the equaliser, i.e. $\mathcal{O}_{G/H} = \{f \in \mathcal{O}_G \mid \alpha f = \beta f\}$. Finally, we set $G/H = \mathrm{spf}(\mathcal{O}_{G/H})$, and write $q \colon G \to G/H$ for the projection.

Suppose that x is a coordinate on G. We can define a function y on G/H as follows. Observe that β makes $\mathcal{O}_{G \times_X H}$ into a free module of rank r over $\mathcal{O}_G$. Multiplication by $\alpha(x)$ is an $\mathcal{O}_G$-endomorphism of $\mathcal{O}_{G \times_X H}$ with this module structure. Let $\tilde{y} \in \mathcal{O}_G$ be the determinant of this endomorphism. It can be shown that $\alpha \tilde{y} = \beta \tilde{y}$, so $\tilde{y}$ comes from an element $y \in \mathcal{O}_{G/H}$. Suitably interpreted (see section 20), one can say that

$$y(qa) = \prod_{b \in H} x(a + b)$$

$$y(c) = \prod_{qa = c} x(c)$$

It can also be shown that $\mathcal{O}_{G/H} = \mathcal{O}_X[\![y]\!]$, and that this is a sub-Hopf algebra of $\mathcal{O}_G$. In other words, G/H is a quotient formal group of G. Moreover, $\mathcal{O}_G$ is free of rank r over $\mathcal{O}_{G/H}$, so that the map $G \to G/H$ is faithfully flat.

Another useful fact is that the "multiplication by r" map $H \xrightarrow{r} H$ is always just the zero map. This is reasonable, because r should be thought of as the order of the finite group H. Moreover, if the base ring $\mathcal{O}_X$ is p-local then r is a power of p.

Recall (see section 18) that the divisors of degree m themselves form a scheme $\mathrm{Div}_m^+ = \mathrm{spf}\,\mathcal{O}_X[\![c_1, \dots, c_m]\!]$. It is not hard to see that there are certain power series $f_i(\underline{c})$ in the variables c_k such that a divisor with parameters c_k is a subgroup divisor iff $f_i(\underline{c}) = 0$. It follows the the functor on $\mathcal{O}_X$-algebras

$$\Gamma(\mathrm{Sub}_m, R) = \{ \text{ subgroup schemes of degree } m \text{ on } G \text{ over } R \}$$

is a scheme, represented by the $\mathcal{O}_X$-algebra $\mathcal{O}_X[\![c_1, \dots, c_m]\!]/(f_i)$. Of course, it is the empty scheme unless $m = p^k$ for some k, by one of the remarks above.

We have a good understanding of this scheme in the case of Morava E-theory. To explain this, consider the ring $R_k = E(B\Sigma_{p^k})$. A *partition subgroup* means a subgroup of the form $\Sigma_i \times \Sigma_j$ with $i + j = p^m$ and $i, j > 0$. We write I for the ideal in R_k generated by transfers from partition subgroups. We also let V denote the standard permutation representation of Σ_{p^k}; note that this has a trivial summand after restriction to a partition subgroup. The Euler class $e(V - 1)$ thus restricts to zero on any such subgroup, implying that $I \leq J = \mathrm{ann}(e(V - 1))$. A construction involving the E_∞ structure on E gives a map $Y = \mathrm{spf}(EB\Sigma_{p^m}/I) \to \mathrm{Sub}_{p^k}$. We shall publish elsewhere a proof of the following result (the algebraic input is provided by [12]).

THEOREM 21.1. With notation as above, we have $I = J$ and $Y = \mathrm{Sub}_{p^k}$. Moreover, $\mathcal{O}_Y = R/I$ is a finitely generated free module over $\mathcal{O}_X$.

22. Cohomology of Abelian Groups

In this section, we suppose we keep the notations of section 3.

Let A be a finite Abelian p-group. We write $I = \mathbb{Q}_p/\mathbb{Z}_p < S^1$ and

$$A^* = \mathrm{Hom}(A, I) = \mathrm{Hom}(A, S^1)$$

Note that $A = A^{**}$. We define a formal pre-scheme $\mathrm{Hom}(A, G)$ over X by

$$\Gamma(Y, \mathrm{Hom}(A, G)) = \mathrm{Hom}(A, \Gamma(Y, G))$$

PROPOSITION 22.1. *The pre-scheme* $\mathrm{Hom}(A, G)$ *is a scheme. It is represented by the ring* $E(BA^*)$, *which is a free module over* E *of rank* $|A|^n$.

PROOF. Suppose $a \in A$, and let $f \colon E(BA^*) \to R$ be a map of E-algebras. The element a gives a map $A^* \to S^1$ and thus a map $Ba \colon BA^* \to \mathbb{C}P^\infty$. Let $\phi(a) \in \Gamma(R, G) = E\text{-Alg}(E(\mathbb{C}P^\infty), R)$ be the composite $E(\mathbb{C}P^\infty) \xrightarrow{Ba^*} E(BA^*) \xrightarrow{f} R$. This construction gives a map

$$\phi \colon A \to \Gamma(R, G)$$

or in other words a section of $\mathrm{Hom}(A, G)$ over R. This gives a map of pre-schemes

$$\alpha_A \colon \mathrm{spf}(E(BA^*)) \to \mathrm{Hom}(A, G)$$

It is easy to see directly that

$$\mathrm{Hom}(A \oplus A', G) = \mathrm{Hom}(A, G) \times_X \mathrm{Hom}(A', G)$$

Recall from example 17.2 that $E(B\mathbb{Z}/p^m) = E[\![x]\!]/[p^m](x)$ is free of finite rank over E. This implies (by a Künneth argument) that

$$\mathrm{spf}\, E(B(A \oplus A')^*) = \mathrm{spf}\, E(BA^*) \times_X \mathrm{spf}\, E(BA'^*)$$

It also implies that

$$\Gamma(Y, (\mathrm{spf}\, EB\mathbb{Z}/p^m)) = \{a \in \Gamma(Y, G) \mid pa = 0\} = \Gamma(Y, \mathrm{Hom}(\mathbb{Z}/p^m, G))$$

It follows that α_A is iso for all A.   □

23. Deformations of Formal Groups

Consider the Morava formal group G_0/X_0. The scheme X_0 is about as small as a scheme can get, so we can think about the problem of extending G_0 over a larger base Y. More generally, we can pull back G_0 along a map $Y_0 \to X_0$ and then try to extend it to a "thicker" scheme $Y \supseteq Y_0$. A *deformation* of G_0/X_0 is such an extension; we shall be more precise in a moment.

We shall initially define deformations only over a restricted class of base schemes Y. We shall then show that there is a universal example in this restricted setting. We can if we wish extend the definition by requiring that this example remain universal (as one does in extending K-theory to infinte complexes).

We shall say that a formal scheme Y is *local* iff $\mathcal{O}_Y$ is a complete local ring, topologised by powers of the maximal ideal. If so, we write $Y_0 = \mathrm{spec}(\mathcal{O}_Y/\mathfrak{m})$, and call this the *special fibre* of Y. We shall say that Y is *semilocal* iff it is a finite disjoint union of local formal schemes. If so, we define Y_0 in the obvious way. If H is a formal group over a semilocal formal scheme, we define $H_0 = H \times_Y Y_0$.

DEFINITION 23.1. A *deformation* of G_0/X_0 consists of a semilocal formal scheme Y together with a formal group H/Y and a fibrewise isomorphism as follows.

$$\begin{array}{ccc} H_0 & \xrightarrow{\tilde{f}_0} & G_0 \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow[f_0]{} & X_0 \end{array}$$

A *morphism of deformations* is a fibrewise isomorphism of formal groups making the obvious diagram commute.

Note that the Morava E-theory formal group G/X is by construction a deformation of G_0/X_0.

PROPOSITION 23.1. *An endomorphism of a deformation* H/Y *which is the identity on* Y *is the identity.*

PROOF. We may assume that Y is local (rather than just semilocal). The claim is true (by the definition of a map of deformations) over $Y_0 = \mathrm{spf}(\mathcal{O}_Y/\mathfrak{m})$. Choose a coordinate y on H compatible in the evident sense with the usual coordinate x_0 on G_0. Thus $y(pa) = y(a)^q + v(y(a))$ where $q = p^n$ and $v \in \mathfrak{m}[\![y]\!]$. Let f be the endomorphism, so $y(f(a)) = y(a) + u(y(a))$ for some $u \in \mathfrak{m}[\![y]\!]$. Suppose $u \in \mathfrak{m}^k[\![y]\!]$. We find that

$$
\begin{aligned}
y(pf(a)) &= (y(a) + u(y(a))^q + h(y(a) + u(y(a)))\\
&= y(a) + h(y(a))\\
&= y(pa) \pmod{\mathfrak{m}^{k+1}}
\end{aligned}
$$

On the other hand, $y(pf(a)) = y(f(pa))$ because f is a homomorphism. Thus $y(f(pa)) = y(pa)$ over $V(\mathfrak{m}^{k+1})$, but $a \mapsto pa$ is faithfully flat (see example 12.3) so $y(f(b)) = y(b)$ over $V(\mathfrak{m}^{k+1})$. $\square$

The above is a prerequisite for a strong classification theory for deformations, and it at least makes plausible the following theorem.

THEOREM 23.2. G/X is terminal in the category of deformations of G_0/X_0. In other words, given a deformation H/Y, there is a map $g\colon Y \to X$ and an isomorphism $H \simeq g^*G$ compatible with the given isomorphism $(H_0, Y_0) \simeq (G_0, X_0)$.

Because of this, we refer to G as the universal deformation of G_0, and to X as the (Lubin-Tate) universal deformation space.

We can reinterpret the above slightly as follows. Let Y be a semilocal formal scheme, and consider the category of deformations of G_0/X_0 with base Y. We can write a typical object as $(H, f_0, \tilde{f}_0)$. The morphisms are required to be the identity on Y. Write $\mathrm{Def}(Y)$ for the set of isomorphism classes. There is then a natural isomorphism

$$
\mathcal{F}ormal(Y, X) \simeq \mathrm{Def}(Y)
$$

24. The Action of the Morava Stabiliser

In this section (as in the last) we use the notation of the "Morava situation". We shall show that there is a unique action of the Morava stabiliser group $\Sigma \simeq \mathrm{Aut}(G_0)$ on the universal deformation G/X extending the action on G_0/X_0. The action is (highly) nontrivial on X, so this does not contradict our earlier claim that $\mathrm{End}(G) = \mathbb{Z}_p$.

The topological significance is as follows. As mentioned in section 3, Hopkins and Miller have shown that E can be made into an E_∞ ring spectrum in such a way that Σ acts by E_∞ ring maps on the nose. This gives an action of Σ on G/X, extending the action on G_0/X_0. The algebraic argument given below shows that this characterises it uniquely, so that calculation of the action becomes a problem in pure algebra.

Suppose $a \in \Sigma$, and write a_{G_0} for the corresponding automorphism of G_0. Then a acts on the functor $Y \mapsto \mathrm{Def}(Y)$ by

$$
(H, f_0, \tilde{f}_0) \mapsto (H, f_0, \tilde{f}_0 \circ a_{G_0}^{-1})
$$

It therefore acts also on the representing object X.

We can be slightly more explicit and also more precise. We can present G/X as a deformation of G_0/X_0 in a twisted way, using the embedding

$$
\begin{array}{ccccc}
G_0 & \xrightarrow{a_{G_0}^{-1}} & G_0 & \longrightarrow & G\\
\downarrow & & \downarrow & & \downarrow\\
X_0 & \xrightarrow{\;1\;} & X_0 & \longrightarrow & X
\end{array}
$$

Because G/X (with the usual deformation structure) is terminal, the above fits into an expanded diagram as follows.

$$
\begin{array}{c}
\text{(cube diagram: } G \xleftarrow{\ a_G\ } G,\quad G_0 \xrightarrow{\ a_{G_0}^{-1}\ } G_0,\quad X_0 \xrightarrow{\ 1\ } X_0,\quad X \xleftarrow{\ a_X\ } X)
\end{array}
\tag{1}
$$

or equivalently

$$
\begin{array}{c}
\text{(cube diagram: } G \xrightarrow{\ a_G\ } G,\quad G_0 \xrightarrow{\ a_{G_0}\ } G_0,\quad X_0 \xrightarrow{\ 1\ } X_0,\quad X \xrightarrow{\ a_X\ } X)
\end{array}
\tag{2}
$$

It follows that Σ acts on G/X as claimed.

We can give more explicit descriptions of this action in a variety of cases.

(1) The subgroup $\mathbb{Z}_p{}^\times \leq \Sigma$ (which is actually the centre) acts trivially on X. The action of $\mathbb{Z}_p{}^\times$ on G is just the obvious one discussed in section 15.

(2) Using the universal property of G/X, one can show that the action of $W^\times \leq \Sigma$ extends the action on $G_1 \to X_1$ discussed in section 15. In fact, $W^\times$ is the largest subgroup of Σ which preserves X_1, and it acts as the identity on X_1.

(3) For general $a = \sum_k a_k S^k$ we have

$$
u_k = a_0^{-1} \sum_{k=l+m} a_l^{p^m} a_X^*(u_m) \pmod{\mathfrak{m}^2}
$$

so

$$a_X^* \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = a_0 \begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0^p & 0 & 0 \\ a_2 & a_1^p & a_0^{p^2} & 0 \\ a_3 & a_2^p & a_1^{p^2} & a_0^{p^3} \end{pmatrix}^{-1} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (\mathrm{mod}\ \mathfrak{m}^2)$$

(4) There are uniquely determined functions $t_k \colon \Sigma \to \mathcal{O}_X$ such that

$$a_G^*(x) = \sum_k^{a_X^* F} t_k(a) x^{p^k}$$

These functions satisfy

$$\sum_k^{a_G^* F} t_k(a)([p]_F(x))^{p^k} = [p]_{a_X^* F} \left(\sum_l^{a_X^* F} t_l(a) x^{p^l} \right)$$

In principle one can determine $a_X^*(u_k)$ and $t_k(a)$ by expanding this out and comparing coefficients of x.

Further information can be obtained from the Dieudonné module of G (see [6]).

Because Σ acts on G/X, it also acts on the line bundle ω_G of invariant differentials. On the other hand, because Σ acts on the spectrum E, it acts on $E^{-2} = \tilde{E}^0 \mathbb{C}P^1 = \omega_G$. It can be shown that these two actions are the same.

25. Quotient Groups as Deformations

Let Y be a semilocal formal scheme. Suppose we have a point $a \in \mathrm{Def}(Y)$. This means that we have an isomorphism class of deformations H/Y of G_0/X_0. By the classification theory, this corresponds to a map $a \colon Y \to X$; each representative H is uniquely isomorphic to $G_a = a^* G$. We shall mostly use the notation G_a. The picture is that G is a bundle of groups over the scheme $X = \mathrm{Def}$, that a is a point of X, and that G_a is the fibre of G over the point a.

By assumption, Y is a semilocal formal scheme, but we shall argue as though it were local as this will involve no loss of generality. This means that $\mathcal{O}_{Y_0}$ is field; we are given a map $Y_0 \to X_0$, so it must have characteristic p. There is thus a *Frobenius map* $F_{Y_0} \colon Y_0 \to Y_0$, given by $F_{Y_0}^* u = u^p$ for $u \in \mathcal{O}_{Y_0}$. Similarly, there is a map F_{H_0} of formal groups covering F_{Y_0}.

Now suppose we also have a subgroup divisor $K < H$, of degree p^m say, so we can form the quotient group H/K. The claim is that this group can also be considered as a deformation of G_0. Indeed, example 21.1 tells us that $K_0 = Y_0 \times K$ is just $\mathrm{spf}\,\mathcal{O}_{H_0}/(y^{p^m})$. This implies that $\mathcal{O}_{H/K_0} = \mathcal{O}_{Y_0}[\![z]\!]$, where z can be identified with y^{p^m} under the embedding $\mathcal{O}_{H/K_0} \to \mathcal{O}_{H_0}$. This in turn means that $F_{H_0}^m$ induces a map $(H/K)_0 = H_0/K_0 \to H_0$ covering $F_{Y_0}^m$. This gives the required fibrewise isomorphism:

$$\begin{array}{ccccc} (H/K)_0 = H_0/K_0 & \xrightarrow{F^m} & H_0 & \xrightarrow{\tilde{f}_0} & G_0 \\ \downarrow & & \downarrow & & \downarrow \\ Y_0 & \xrightarrow{F^m} & Y_0 & \xrightarrow{f_0} & X_0 \end{array}$$

We need to check that the left hand square is indeed a pullback, which is essentially clear once the notation is straight. Write $L = \mathcal{O}_{Y_0}$, and write $\psi \colon L \to L$ for the map $a \mapsto a^{p^m}$ induced by $F_{Y_0}^m$. The claim is now that the map

$$L \otimes_{L,\psi} L[\![y]\!] \to L[\![y^{p^m}]\!] \qquad a \otimes u \mapsto au^{p^m}$$

is iso. This is a special case of the fact that an embedding $L \to M$ together with the map $y \to y^{p^m}$ induces an isomorphism

$$M \otimes_L L[\![y]\!] \simeq M[\![y^{p^m}]\!]$$

The fact that $L = L$ just confuses the issue.

Anyway, the above diagram exhibits H/K as a deformation of G_0. We therefore get a new map $b\colon Y \to X$, and a uniquely determined isomorphism $H/K = G_a/K \to G_b$. In other words, we get an exact sequence of group schemes over Y:

$$K \to a^*G \xrightarrow{q} b^*G$$

On the special fibre, we have $b_0 = F^m \circ a_0$ and $q_0 = a_0^*(F^m)$.

Conversely, suppose we have a semilocal scheme Y and two points a, b of X defined over Y. We say that a map $q\colon a^*G \to b^*G$ is an F^m-*isogeny* if it behaves as above on the special fibre. Using Weierstrass preparation, we see that q is indeed an isogeny. Moreover, there is a subgroup divisor $K < a^*G$ such that q factors as $a^*G \to a^*G/K \xrightarrow{\sim} b^*G$. Using the classification theorem for deformations, we see that (a, K) determines (a, q, b) and *vice-versa*. We conclude that the following functor is isomorphic to Sub_{p^m}, and hence is a formal scheme:

$$\mathrm{Isog}_m^F(Y) = \{(a, q, b) \mid a, b \in \mathrm{Def}(Y), q\colon G_a \to G_b \text{ an } F^m\text{-isogeny }\}$$

REMARK 25.1. Note that $F_{X_0}^n = 1$ and $p_{G_0} = F^n\colon G_0 \to G_0$ (because the formal group law for K has $[p](x) = x^{p^n}$). This implies that $p_G\colon G \to G$ is an F^n-isogeny. This fact is somewhat accidental, however. All formal groups of height n over an algebraically closed field, but the choice of a representative formal group over $\mathbb{F}_p$ or $\mathbb{F}_{p^n}$ is arbitary. Some choices satisfy $p_{G_0} = F^n$, but most do not. In particular, if $n = 2$ and G_0 is the formal group associated to a supersingular elliptic curve over $\mathbb{F}_{p^2}$ then this need not hold. It does hold if $n = 1$ and G_0 is the formal multiplicative group, however.

We also define

$$\mathrm{Isog}_*^F = \coprod_{m \geq 0} \mathrm{Isog}_m^F$$

This gives a category scheme, i.e. a functor from schemes to categories. The object-scheme is $X = \mathrm{Def}$ and the morphism-scheme is Isog_*^F. We will develop a little general theory of such categories in section 27, and apply these ideas to cohomology operations in section 28.

26. Level Structures

Let A be a finite Abelian p-group, and ϕ a point of $\mathrm{Hom}(A, G)$ defined over an X-scheme Y, so $\phi\colon A \to \Gamma(Y, G)$. We need a good notion of what it means for ϕ to be injective. The naive definition has bad technical properties; in particular, it does not give a subscheme of $\mathrm{Hom}(A, G)$.

To formulate a better notion, we need some definitions. First, we set $A_1 = \{a \in A \mid pa = 0\}$. For each $a \in A$ we have a section $\phi(a)$ of G and hence a divisor $[\phi(a)]$. We write $[\phi] = \sum_{a \in A}[\phi(a)]$, which is a divisor of degree $|A|$ on G over Y. Similarly, we set $[\phi]_1 = \sum_{a \in A_1}[\phi(a)]$. It turns out that the following conditions are equivalent:

(1) $[\phi]$ is a subgroup divisor.
(2) $[\phi]_1 \leq G(1) = \ker(G \xrightarrow{p} G)$.
(3) $[p](x)$ is divisible by $\prod_{a \in A_1}(x - x(\phi(a)))$.

We say that ϕ is a *level structure* or *level-A structure* iff these conditions are satisfied. More generally, we say that ϕ is a p^k-*fold level structure* iff the following equivalent conditions hold:

(1) $p^k[\phi]_1 \leq G(1)$
(2) $[p](x)$ is divisible by $\prod_{a \in A_1}(x - x(\phi(a)))^{p^k}$.

We write
$$\text{Level}_k(A, G) \subseteq \text{Level}_0(A, G) = \text{Level}(A, G) \subseteq \text{Hom}(A, G)$$

for the corresponding subfunctors. Using the second criterion for a level structure and proposition 18.2, we see that they are closed subschemes: there are quotient rings $D_A D_A^k$ of $E(BA^*) = \mathcal{O}_{\text{Hom}(A,G)}$ such that $\text{Level}_k(A, G) = \text{spf}(D_A^k)$. These rings will be described in more detail later.

First, however, we make some remarks which will relate to power operation in Morava E-theory. Write a for the map $\text{Level}(A, G) \to X$. There is thus a level structure ϕ (the universal example of such a thing) on a^*G. This gives rise to a subgroup divisor $K = [\phi]$. As in section 25, we have a map $b\colon \text{Level}(A, G) \to X$ classifying G/K, and thus an isogeny $q\colon a^*G \to b^*G$ with kernel K.

To return to the construction of D_A, recall that $[p](x)$ can be factored as $g(x)u(x)$, where $g(x)$ is a monic polynomial of degree p^n and $u(x)$ is invertible. Using this, we see that $\text{Level}_k(A, G) = \emptyset$ if $\text{rank}(A) > n - k$ (where $\text{rank}(A)$ is the number of cyclic factors).

EXAMPLE 26.1. Suppose $Y = \text{spf}(R)$, where R is an integral domain in which $p \neq 0$. It turns out that a map $\phi\colon A \to \Gamma(Y, G)$ is a level structure iff it is injective.

EXAMPLE 26.2. Suppose $Y = X_0 = \text{spf}(\kappa)$. Then $\Gamma(Y, G) = 0$, so the only possible ϕ is the zero map. Moreover, we have $[p](x) = x^{p^n}$ in $\kappa[\![x]\!]$. This implies that ϕ is a p^k-fold level structure iff $\text{rank}(A) \leq n - k$.

EXAMPLE 26.3. Suppose $n = 1$, so that $G = \hat{G}_m$ is the formal multiplicative group, and $[p^r](x) = (1 + x)^{p^r} - 1$. We know that $\text{Level}(A, G) = \emptyset$ if $\text{rank}(A) > 1$, which leaves only the case $A = \mathbb{Z}/p^r$. In this case, a map $\phi\colon A \to \Gamma(Y, G)$ sends the generator to a root α of $[p^r](x)$. One can check that this is a level structure iff α is a root of $[p^r](x)/[p^{r-1}](x) = f(x)$. Thus $\text{Level}(A, G) = \text{spf}(\mathbb{Z}_p[x]/f(x))$.

EXAMPLE 26.4. Suppose $n = p = 2$ and $A = \{0, a, b, c = a + b\} \simeq (\mathbb{Z}/2)^2$. Suppose that $p = u_1 = 0$ in the E-algebra $\mathcal{O}_Y$ (so that Y lies entirely over the special fibre $X_0 \subset X$). Suppose $\phi\colon A \to \Gamma(Y, G)$. Write $\alpha = x(\phi(a))$ and so on. One can show that $[2](x) = x^4$ and $x +_F y = x + y + x^2 y^2 \pmod{x^4, y^4}$ over $\mathcal{O}_Y$. It follows that $\alpha^4 = [2](\alpha) = x(\phi(2a)) = 0$ and similarly $\beta^4 = \gamma^4 = 0$. Moreover, $c = a + b$ implies that $\gamma = \alpha +_F \beta = \alpha + \beta + \alpha^2 \beta^2$. For ϕ to be a level structure, we require that x^4 be divisible by
$$x(x - \alpha)(x - \beta)(x - \gamma)$$

It follows that this product must actually be equal to x^4, and hence that $(x - \alpha)$ divides x^3, and hence that $\alpha^3 = 0$. Similarly, $\beta^3 = 0$ and so the product can be expanded as
$$\alpha\beta(\alpha + \beta)x + (\alpha^2 + \alpha\beta + \beta^2)x^2 + \alpha^2\beta^2 x^3 + x^4 = x^4$$

This says that $\alpha\beta(\alpha + \beta) = (\alpha^2 + \alpha\beta + \beta^2) = \alpha^2\beta^2 = 0$. In fact, it is equivalent to require only that $\alpha^2 + \alpha\beta + \beta^2 = 0$, as the other relations follow easily from this. We conclude that ϕ is a level structure iff
$$\alpha^3 = \beta^3 = \alpha^2 + \alpha\beta + \beta^2 = 0$$

This means that
$$\text{Level}(A, G) \times_X X_0 = \text{spf}(\kappa[\alpha, \beta]/(\alpha^3, \beta^3, \alpha^2 + \alpha\beta + \beta^2))$$

We next turn to the definition and properties of the rings D_A and D_A^k. We first give a simple definition and then a more complicated one. Unfortunately it seems necessary to use the more complicated version to establish the properties of D_A and hence justify the simple version.

Because $\text{spf}(E_A) = \text{Hom}(A, G)$ we have a universal map $\phi\colon A \to \Gamma(E_A, G)$. We can thus think of A as a subset of E_A via $a \mapsto x(\phi(a))$. Write $J = \sum_{a \neq 0} \text{ann}(a) \lhd E_A$; then $D_A = E_A/J$. Unfortunately, this definition is hard to work with, because annihilators are not stable under change of base; if $a \in R$ and $f\colon R \to S$ then $\text{ann}(f(a))$ may be much larger than $f(\text{ann}(a))$.

For the more complicated version, we choose generators $\{e_0, \ldots e_{r-1}\}$ for A, so that $A = \langle a_k \mid p^{m_k} a_k = 0\rangle$ with $m_0 \geq m_1 \ldots \geq m_{r-1}$. We may assume wlog that $r \leq n$, as otherwise $D_A = 0$. We write $A(k) = \langle a_0, \ldots a_{k-1} \leq A\rangle$ and define $D_{A(k)}$ recursively. Set $A(k)' = \{a \in A(k) \mid p^{m_k}a = 0\}$. Over $D_{A(k)}$ it works out that $[p^{m_k}](x)$ is divisible by $\prod_{a \in A(k)}(x - x(\phi(a)))$. Moreover, the quotient has a Weierstrass factorisation

as $g(x)u(x)$, where $g(x)$ is a monic polynomial of degree $p^{nm_k} - |A(k)'|$. We take $D_{A(k+1)} = D_{A(k)}[\alpha]/g(\alpha)$, where $\alpha = x(\phi(e_k))$.

The ring D_A has many good properties. It is a complete regular local ring of Krull dimension n, and thus a unique factorisation domain, and thus is integrally closed in its field of fractions. It is a quotient of

$$W[\![e_0, \ldots e_{r-1}, u_r, \ldots u_{n-1}]\!]$$

by a principal prime ideal. The prime p lies in the square of the maximal ideal of D_A (provided $A \neq 0$), and the role of the single relation is to write p in terms of the generators e_i and u_j. Moreover, D_A is a finitely generated free module over E.

As an important special case, take $\Lambda_m = (\mathbb{Z}/p^m)^n$ and $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n = \bigcup_m \Lambda_m$. We write $D_m = D_{\Lambda_m}$ and $D_\infty = \bigcup_m D_m$. Note that D_m is obtained from E by adjoining a full set Λ_m of roots for $[p^m](x)$ (or equivalently, for the associated Weierstrass polynomial).

We can do a certain amount of Galois theory with these rings. Write K and K_A for the fields of fractions of E and D_A. Write also $\text{Mon}(A, B)$ for the set of monomorphisms from A to B.

(1) $\text{Hom}_E(D_A, D_B) = \text{Hom}_K(K_A, K_B) = \text{Mon}(A, B)$.
(2) K_m is Galois over K with group $\text{Aut}(\Lambda_m)$.
(3) The rank of D_A over E is $|\text{Mon}(A, \Lambda)|$.

Similar constructions produce the rings D_A^k. Suppose that $\phi \colon A \to \Gamma(Y, G)$ is a p^k-fold level structure. It is immediate from the definition that x^{p^k} divides $[p](x) = \exp_F(px) +_F u_1 x^p +_F \ldots u_{n-1} x^{p^{n-1}} +_F x^{p^n}$ in $\mathcal{O}_Y[\![x]\!]$, and thus that $p = u_0 = u_1 \ldots = u_{k-1} = 0$ in $\mathcal{O}_Y$. Thus, to construct D_A^k, we start with $E' = E/I_k = \kappa[\![u_k, \ldots u_{n-1}]\!]$, where $I_k = (u_0, \ldots u_{k-1})$. We then adjoin p^k'th roots of the remaining generators to get a new ring E'' (thus making a totally inseparable extension). Over E'' we have $[p^m](x) = f(x)^{p^k}$ for some power series f, and we can adjoin roots of f to get D_A^k just as we adjoined roots of $[p](x)$ to get D_A. The resulting ring D_A^k is again a complete regular local ring.

For each subgroup $B \leq A$ we have closed subschemes

$$\text{Level}(A/B, G) \leq \text{Hom}(A/B, G) \leq \text{Hom}(A, G)$$

These are the irreducible components of $\text{Hom}(A, G)$, and

$$\text{Hom}(A, G) = \text{Hom}(A, G)_{\text{red}} = \bigcup_B \text{Level}(A/B, G)$$

If we look only at what happens above the open subscheme $D(p) = \text{spec}(p^{-1}E) < X$, then this union is disjoint:

$$\text{Hom}(A, G) \times_X D(p) = \coprod_B \text{Level}(A/B, G) \times_X D(p)$$

In terms of rings, we have a monomorphism

$$E_A \to \prod_B D_{A/B}$$

which is an isomorphism after inverting p.

More generally, consider the subschemes $X(k) = \text{spf}(E/I_k) \leq X$ and $X(k)' = X(k) \cap D(u_k)$. We find that

$$(\text{Hom}(A, G) \times_X X(k))_{\text{red}} = \bigcup_B \text{Level}_k(A/B, G)$$

Moreover, the union is disjoint over $X(k)'$. The schemes $\text{Level}_k(A/B, G)$ for varying k and B are precisely the irreducible closed subschemes of $\text{Hom}(A, G)$ invariant under the action of the Morava stabiliser group $\Sigma = \text{Aut}(G_0)$.

27. Category Schemes

A *category scheme* is a category object in the category of schemes, or equivalently, a functor C from rings to categories, such that $\mathrm{obj}(C)$ and $\mathrm{mor}(C)$ are schemes. We shall also refer to such a thing as an *internal category* in the category of schemes.

EXAMPLE 27.1. The category of formal group laws and strict isomorphisms (more precisely, the functor $R \mapsto (\mathrm{FGL}(R), \mathrm{SI}(R))$ is an internal category.

EXAMPLE 27.2. Let $q \colon X \to Y$ be a map of schemes. We can define an internal category $C = C(q)$ by $\mathrm{obj}(C) = X$ and $\mathrm{mor}(C) = X \times_Y X$. Thus, given points $a, b \in X$ there is a unique map $a \to b$ if $qa = qb$, and no maps otherwise.

EXAMPLE 27.3. We can define four different category schemes with object scheme the Lubin-Tate deformation space $X = \mathrm{Def}$:

(1) $\mathrm{mor}(\mathrm{DefIso})(a, b) = \{$ isomorphisms $G_a \to G_b\}$
(2) $\mathrm{mor}(\mathrm{DefHom})(a, b) = \{$ homomorphisms $G_a \to G_b\}$
(3) $\mathrm{mor}(\mathrm{Isog}^F_*)(a, b) = \{F^*\text{-isogenies } q \colon G_a \to G_b\}$
(4) $\mathrm{mor}(\mathrm{Isog})(a, b) = \{$ all isogenies $G_a \to G_b\}$

We need to see that these really are representable. Firstly, let Σ denote the Morava stabiliser group. It can be shown that

$$\mathrm{DefIso} = \underline{\Sigma} \times \mathrm{Def} = \mathrm{spf}(C(\Sigma, E)) = \mathrm{spf}(E^\vee E)$$

Next, DefHom can be constructed as a closed subscheme of $\mathrm{Map}_X(G, G)$ by techniques which should by now be familiar. We conjecture that it arises in topology as follows. Let $\underline{E}_0$ be the 0'th space in the Ω-spectrum for E. Then $A = \pi_0 L_K(E \wedge \underline{E}_0)$ is supposed to be a Hopf ring, so that the $*$-indecomposables $\mathrm{Ind}(A)$ form a ring under the circle product. We are supposed to get

$$\mathrm{DefHom} = \mathrm{spf}(\mathrm{Ind}(A))$$

This can be proved modulo technicalities about formal topologies. Next, we know from section 25 that $\mathrm{Isog}^F_* \simeq \coprod_k \mathrm{Sub}_{p^k}$ is a scheme, and it was explained in theorem 21.1 how it arises in topology. We can rephrase this a little to make it look more like the case of DefHom. Consider $E(DS^0)$, where

$$DS^0 = \bigvee_k D_k S^0 = \coprod_k B\Sigma_k$$

This can also be made into a Hopf ring, as explained in section 28 below. We have

$$\mathrm{Isog}^F_* = \mathrm{spf}(\mathrm{Ind}\, E(DS^0))$$

Finally, let $f \colon G_a \to G_b$ be an isogeny of deformations, with kernel K say. Arguing much as in section 25, we get a unique F^m-isogeny $G_a \to G_c$ with kernel K and an isomorphism $G_c \to G_b$, whose composite is f. This shows that $\mathrm{Isog} = \mathrm{Isog}^F_* \times_X \mathrm{DefIso}$, which is a scheme as required.

DEFINITION 27.1. An *internal functor* on an internal category C is a scheme F over $\mathrm{obj}(C)$ with the following extra structure. For any points $a, b \in_Y \mathrm{obj}(C)$ and any C-morphism $u \colon a \to b$ we are given a map $F_u \colon F_a \to F_b$. These maps satisfy $F_1 = 1$ and $F_{uv} = F_u F_v$. Moreover, if we pull a, b and u back along a map $f \colon Y' \to Y$ (so that $F_{f^*a} = u^* F_a$ automatically) then $F_{f^*u} = f^* F_u$. By analysing the universal example, it is equivalent to require a map $F \times_{\mathrm{obj}(C)} \mathrm{mor}(C) \to F$ satisfying some not-quite-so-obvious conditions.

EXAMPLE 27.4. In a suitable technical setting, $\mathrm{spec}(MU^*Z)$ becomes an internal functor on FGL.

EXAMPLE 27.5. An internal functor on $C(q)$ is just a map $Z \to X$ with descent data. Thus, if q is faithfully flat, then internal functors on $C(q)$ are equivalent to schemes over Y.

28. Cohomology Operations

Let E and K be Morava E-theory and K-theory of height n, say. Let Z be a tolerable space, so that Z_E is a formal scheme over $X = \mathrm{Def}$. We would like to understand the extra structure which this has because of the action of various kinds of operations on $E(Z)$. More generally let F be a K-local E-algebra spectrum. For example, we could have $F = F(Y_+, E)$ where Y is a tolerable space, or $F = L_K(E \wedge MU)$. Write $Z = \mathrm{spf}(F^0)$, which is a formal scheme over X.

CONJECTURE 28.1. Assuming various things about flatness, projectivity, and completeness, we have actions as follows.

(1) Z is an internal functor for DefIso.
(2) If $F = F(Y_+, E)$ then Z is an internal functor for DefHom.
(3) If F is an E_∞ E-algebra then Z is an internal functor for Isog.
(4) If $F = L_K(E \wedge Y_+)$ and Y is a decent infinite loop space then Z has a covariant action of Isog and a contravariant action of DefHom.

All of these things are almost certainly true, with a suitable choice of technical details about formal topologies. However, I have not yet pinned these details down. The last statement follows from the others. There should be some sort of compatibility statement (which would be the analogue of the Nishida relations) but I'm not sure what it should say. Note that none of the above accounts for nonadditive unstable operations, although we shall say something about nonadditive operations when we outline the construction the actions mentioned above.

EXAMPLE 28.1. The scheme G itself is an internal functor for DefHom, as are things constructed from it like $\mathrm{Div}_m^+ = BU(m)_E$ and $\mathrm{Hom}(A, G) = (BA^*)_E$. The scheme $\mathrm{Orient}(G) = \mathrm{spf}(E^\vee MU)$ has (as predicted) no contravariant action of DefHom, because of the requirement that an orientation be an *isomorphism* $G \to \widehat{\mathbb{A}}(L)$. It does have a covariant action of Isog, however, given by the norm construction discussed in section 29 below. Divisors can be pushed forwards by an arbitrary map, or pulled back by an isogeny; this gives two actions on $\mathrm{Div} = (\mathbb{Z} \times \ BU)_E$, as predicted.

We next indicate the construction of the action of Isog. Because DefIso already acts, it is enough to construct an action of Isog_*^F. We shall ignore some technicalities about topologies.

It is traditional to construct power operations one symmetric group at a time, giving a (nonadditive) map $P_k \colon F^0 \to F(B\Sigma_k)$. We will instead outline an approach which uses all symmetric groups simultaneously, and which compares very nicely with the Boardman-Johnson-Wilson [1] approach to unstable operations. Unfortunately, this approach does introduce extra technicalities about formal topologies, which we have not yet got around to resolving.

There is a total extended power functor $DX = \bigvee_{k \geq 0} D_k X$ from spectra to spectra, given on spaces by $D_k X = E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k}$. In particular, $D_k S^0 = (B\Sigma_k)_+$. One also knows that $D(X \vee Y) \simeq DX \wedge DY$. The obvious maps $S^0 \to S^0 \vee S^0 \to S^0$ thus give maps

$$DS^0 \xrightarrow{\phi} DS^0 \wedge DS^0 \xrightarrow{\mu} DS^0$$

The components can be described as follows.

$$\phi_{k,l} \colon B\Sigma_k \times B\Sigma_l \to B\Sigma_{k+l}$$

is the usual map, and

$$\nu_{k,l} \colon B\Sigma_{k+l} \to B\Sigma_k \times B\Sigma_l$$

is the transfer (this is not obvious, but it is stated in [2] and proved in [10]). We also consider the diagonal map

$$\psi_k \colon B\Sigma_k \to B\Sigma_k \times B\Sigma_k$$

It turns out that $E(DS^0)$ becomes a Hopf ring with star and circle products given by ϕ and ψ, and coproduct induced by μ. Moreover, the E_∞ structure of E gives a map $\theta \colon DE \to E$ with various good properties. To

be more precise, D is a monad on spectra and θ makes E into a D-algebra. Given $u \in E$ (i.e. $u\colon S^0 \to E$) we can define $[u] = \theta \circ Du \in E(DS^0)$. It turns out that $[u] * [v] = [u + v]$ and $[u] \circ [v] = [uv]$, so that $E(DS^0)$ is a Hopf ring over the ring-ring $E[E]$.

We now dualise. It can be shown that $E(DS^0)$ is pro-free and $E^\vee DS^0 = E(DS^0)^\vee$. We thus discover that $T = \mathrm{spf}(E^\vee DS^0)$ is an E-algebra object in the category of schemes over $X = \mathrm{spf}(E)$, in other words a functor from formal E-algebras to E-algebras. It should be possible to introduce a natural topology on $T(R)$, in a way which will make T into a comonad.

Now let F be a K-local, E_∞ E-algebra spectrum. Using the fact that F is K-local, one can show that $F^0(DS^0) = E(DS^0)\widehat{\otimes}_E F^0$. We propose to define a coaction of our comonad on F^0, in other words a map of E^0-algebras

$$F^0 \to E\text{-Alg}(E^\vee DS^0, F^0)$$

The right hand side is contained in

$$E\text{-Mod}(E^\vee DS^0, F^0) = E(DS^0)\widehat{\otimes}_E F^0 \simeq F(DS^0)$$

In this guise, the map $F^0 = F(S^0) \to F(DS^0)$ just sends $u\colon S^0 \to F$ to $\theta \circ D(u)$.

We can now "linearise" this construction. For any ungraded Hopf ring A over E, we define

$$\mathrm{GL}(A) = \mathrm{GL}_E(A) = \{\text{group-like elements}\} = \{a \in A \mid \epsilon(a) = 1 \text{ and } \psi(a) = a \otimes a\}$$

This is a ring, with addition given by the star product and multiplication by the circle product. Moreover, $\mathrm{spf}(A^\vee)$ is a ring scheme, so $E\text{-Alg}(A^\vee, R)$ is a ring; this is just the same as $\mathrm{GL}_R(R\widehat{\otimes}_E A)$. Also, the star-indecomposables in A form a ring under ordinary addition and circle product, and $\mathrm{Ind}(R\widehat{\otimes}_E A) = R\widehat{\otimes}_E \mathrm{Ind}(A)$. There is a ring homomorphism

$$\mathrm{GL}(A) \to \mathrm{Ind}(A) \qquad a \mapsto a - [0] = a - 1$$

Combining this with our power operation, we get a map

$$F^0 \to T(F^0) = \mathrm{GL}(F^0\widehat{\otimes}_E E(DS^0)) \to F^0\widehat{\otimes}_E \mathrm{Ind}(E(DS^0))$$

This turns out to be a ring map. If we recall that the star product in $E(DS^0)$ is given by transfers, and use theorem 21.1, we conclude that $\mathrm{spf}\,\mathrm{Ind}(E(DS^0)) = \mathrm{Isog}_*^F$. We have thus constructed a map

$$\mathrm{spf}(F^0) \times_X \mathrm{Isog}_*^F \to \mathrm{spf}(F^0)$$

After chasing many more diagrams, we conclude that this makes $\mathrm{spf}(F^0)$ into an internal functor as claimed.

The Boardman-Johnson-Wilson theory, suitably adapted, should make the functor

$$T'\colon R \mapsto E\text{-Alg}(E(\underline{E}_0), R)$$

into a comonad in the category of formal E-algebras. Recall that G is a functor from formal E-algebras to groups. The results of [7], suitably adapted, should show that T' is the initial example of a representable functor from formal E-algebras to formal E-algebras equipped with a map $G \to G \circ T'$. One can show purely algebraically that such an example exists, and is a comonad. If Z is a decent space then $E(Z)$ is T'-coalgebra. Linearising this coaction makes Z_E into an internal functor for DefHom. This part of the theory is joint work with Paul Turner.

We now present a different version of power operations which is in some ways less satisfying, but which does have the advantage that more details are in place.

Let A be a finite Abelian p-group, with dual A^*. We write $D_{A^*}Z$ for the A^*-extended power of a spectrum Z. We also write $D_A = \mathcal{O}_{\mathrm{Level}(A,G)}$ as in section 26, and apologise profusely for any confusion caused by this. If Z is a space then

$$D_{A^*}Z = EA^* \times_{A^*} Z^{\wedge(A^*)}$$

The usual construction gives a map

$$F^0 = [S^0, F] \to [D_{A^*}S^0, DF] \to [D_{A^*}S^0, F] = F(BA^*)$$

This is additive mod transfers from proper subgroups of A^*. Using the fact that F is K-local, one can show that the right hand side is just $E(BA^*) \otimes_E F^0$. Combining this with the map $E(BA^*) \to D_A$ (which kills the transfers) we get a ring map

$$F^0 \to D_A \otimes_E F^0$$

or equivalently

$$\mathrm{spf}(F^0) \leftarrow \mathrm{Level}(A, G) \times_X \mathrm{spf}(F^0)$$

Here, the fibre product is formed using the obvious map $\mathrm{Level}(A, G) \to X$, which we shall call a. Performing the above construction with $F = E$ and with $F = F(\mathbb{C}P_+^\infty, E)$, we get a new map $b \colon \mathrm{Level}(A, G) \to X$ and a map of schemes $q \colon a^*G \to b^*G$. One can check that this is an isogeny of formal groups, that the kernel is the subgroup divisor K defined by the level structure, and that the map is a power of Frobenius mod the maximal ideal of D_A. This is enough to prove that q and b are the same as the maps constructed algebraically in section 26.

More generally, suppose we have a short exact sequence $A \to B \to C$. This gives a dual sequence $C^* \to B^* \to A^*$ and hence a map $B^* \to A^* \wr C^*$. Now, given an element $u \in F(BC^*) = [D_{C^*}S^0, F]$ we can form the composite

$$B(B^*)_+ \to B(A^* \wr C^*)_+ = D_{A^*}D_{C^*}S^0 \xrightarrow{D_{A^*}u} D_{A^*}F \xrightarrow{\theta} F$$

This gives rise to a ring map $D_C \otimes_E F \to D_B \otimes_E F$.

To interpret this, define a new category $\mathrm{Level\,Def}(Y)$ as follows. The objects are triples (a, A, ϕ), where $a \in \mathrm{Def}(Y)$ and $\phi \colon A \to G_a$ is a level structure. The morphisms are diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\phi \downarrow & & \downarrow \psi \\
G_a & \xrightarrow[q]{} & G_b
\end{array}
$$

in which q is an F^*-isogeny, and $\ker(q) = [\phi(\ker(f))]$ as divisors. This defines a category scheme, provided that we restrict to a small skeleton of the category of finite Abelian p-groups (or avoid the company of obsessive set-theorists).

Write $Z = \mathrm{spf}(F^0)$ as before, and $\tilde{Z} = \mathrm{Level\,Def} \times_{\mathrm{Def}} Z$. The extended power construction defined above makes $\tilde{Z}$ into an internal functor on $\mathrm{Level\,Def}$. In other words, given a morphism (f, q) in $\mathrm{Level\,Def}$ as above, we get a map

$$Z_{f,q} \colon Z_a \to Z_b$$

satisfying the obvious conditions.

Let Z be an internal functor for Isog_*^F and D a divisor of degree m on G over Z. Thus $D < Z \times_X G$ is a subscheme of an internal functor, so it makes sense to require it to be a subfunctor. We shall say that D is an *equivariant divisor* if this is so. Let us make this condition more explicit. Write π for the map $Z \to X$ and G_a for $G_{\pi a}$. Suppose that $a, b \in Z$, that $q \colon G_a \to G_b$ is an F^*-isogeny, and that $Z_q(a) = b$. In future, we shall simply write $q \colon a \mapsto b$ for this. We then have two degree-m divisors $D_a < G_a$ and $D_b < G_b$, so it makes sense to ask that $q_*D_a = D_b$. This holds for all $q \colon a \mapsto b$ iff D is an equivariant divisor. If so, the map $q^* \colon \mathcal{O}_{G_b} \to \mathcal{O}_{G_a}$ restricts to give a map (c.f. proposition 20.3)

$$\mathcal{O}(D_b) \to q^*\mathcal{O}(D_b) = q^*\mathcal{O}(q_*D_a) = \mathcal{O}(K * D_a)$$

This kind of structure arises in topology as follows. Let V be the standard permutation representation of Σ_m, which we can regard as a vector bundle over $B\Sigma_m$. Let W be another vector bundle over a tolerable space Z. A small generalisation of our previous construction gives a (nonadditive) map

$$\tilde{E}(Z^W) \to \tilde{E}((B\Sigma_m \times Z)^{V \otimes W})$$

After passing to the quotient by proper transfers, we hope to find that $D(V) = K$ is the universal subgroup defined over $\mathrm{Sub}_m(G)$. Assuming this, we find that $D(V \otimes W) = K * D(W)$ and that $D(W)$ is an equivariant divisor on Z_E. The power operation on

$$\tilde{E}((Z \times \mathbb{C}P^\infty)^{W \otimes L}) = \mathbb{L}(D(W) * \Delta) = \mathcal{O}(\overline{D(W)})$$

can be identified with the map $q^* \colon \mathcal{O}(\overline{D}_b) \to \mathcal{O}(K * \overline{D}_a)$. The operation on $\tilde{E}(Z^W)$ itself can be obtained by pulling back along the zero section.

29. The Ando Orientation

Now suppose that y is an orientation of the universal deformation G/X. Then y corresponds to a ring map $MU \to E$. One would like to know when this is a map of H_∞ ring spectra.

Let us first reinterpret the notion of an orientation. Let $G \xrightarrow{\pi} X$ be a polarised formal group.

DEFINITION 29.1. A k-*rigid line bundle* on G is a line bundle M together with an isomorphism $\lambda \colon L^k \simeq 0^*M$. A *rigid section* of M is an isomorphism $\mu \colon \pi^*L^k \simeq M$ extending λ. We write $\Gamma_{\mathrm{rig}}(M)$ for the set of rigid sections. Let N be a line bundle over X and $m \colon \pi^*N \to M$ an isomorphism. We then write $\Delta(m) = \lambda m(0)^{-1}m \colon \pi^*L^k \to M$. This is clearly a rigid section.

Now let $J = \mathcal{O}([0])$ be the augmentation ideal in $\mathcal{O}_G$. The polarisation makes this into a 1-rigid line bundle, and $\Gamma_{\mathrm{rig}}(J) = \mathrm{Orient}(G)$.

Consider again an orientation y of the universal deformation. Let $q \colon G_a \to G_b$ be an F^m-isogeny. We thus have orientations y_a and y_b. We can think of $N_q y_a$ as a map from $\pi^*L^{p^m}$ to

$$N_q J_a = N_q \mathcal{O}_{G_a}[0] = \mathcal{O}_{G_b}(q_*[0]) = J_b$$

It thus makes sense to consider $\Delta(N_q y_a) \in \mathrm{Orient}(G_b)$.

DEFINITION 29.2. y is an *Ando orientation* iff $\Delta(N_q y_a) = y_b$ for all a, b and q as above.

THEOREM 29.1. y is an Ando orientation iff the map $y \colon MU \to E$ is H_∞.

There is a simpler and closely related notion for coordinates.

DEFINITION 29.3. A coordinate x on the universal deformation is an *Ando coordinate* iff $N_q x_a = x_b$ for all a, b and q as above.

THEOREM 29.2. It is equivalent to require that $N_{p_G} x = x$. Any coordinate x_0 on G_0 extends uniquely to an Ando coordinate x on G. In fact, if x is any coordinate on G then $N_{p_G}^k x$ converges to the Ando coordinate y which agrees with x on G_0.

REMARK 29.1. We have used here the fact that $p_{G_0} = F^n$ and thus p_G is an F^n-isogeny (c.f. remark 25.1). The theorem becomes a little more complex without this.

If x is a coordinate on G then $\Delta(x)$ is an orientation. If x is an Ando coordinate then $\Delta(x)$ is an Ando orientation. Conversely, any Ando orientation is $\Delta(x)$ for some Ando coordinate x.

EXAMPLE 29.1. Let G be the formal multiplicative group (defined over $\tilde{\mathbb{Z}}_p$), and x the usual coordinate $x(u) = u - 1$. Let ζ be a primitive p'th root of 1, and write $A = \mathbb{Z}_p[\zeta]$, which is faithfully flat over $\mathbb{Z}_p$. After pulling back to A, we have a full set of points $\{1, \zeta, \dots \zeta^{p-1}\}$ for $G(1) = \ker(p_G)$. This gives

$$(N_{p_G})f(u^p) = \prod_{k=0}^{p-1} f(u\zeta^k)$$

In particular (if $p > 2$),

$$(N_{p_G})x(u^p) = \prod (u\zeta^k - 1) = u^p - 1 = x(u^p)$$

Thus $N_{p_G} x = x$.

EXAMPLE 29.2. Let G_0 be the Morava K-theory formal group over $X_0 = \mathrm{spf}(\kappa)$, where $\kappa = \mathbb{F}_{p^n}$. Let $x = x_0$ be the usual p-typical coordinate, so that $x(pa) = x(a)^{p^n}$ by definition. It is then easy to see that $f(pa) = f(a)^{p^n}$ for any function $f = \sum_k \alpha_k x^k \in \mathcal{O}_G$. Moreover, $G_0(1) = \mathrm{spf}(\kappa[x]/x^{p^n})$ has $\{0, \ldots 0\}$ (with p^n entries) as a full set of points. It follows that $(N_{p_G} f)(pa) = f(a)^{p^n} = f(pa)$ for any $f \in \mathcal{O}_G$.

EXAMPLE 29.3. More generally, suppose that G is a formal group over a complete local ring A, of residue characteristic $p > 2$. Suppose that a coordinate x has the property that $x(pa) = f(x(a))$ for some (monic) Weierstrass polynomial f, of degree p^n say. Then $N_{p_G}(x) = x$. To see this, write $y = f(x)$ and $B = A[\![y]\!]$ and $g(t) = f(t) - y \in B[t]$. Then $C = A[\![x]\!] = B[x]/g(x)$, so $N_{C/B}(x) = (-1)^{p^n} g(0)$. Because $f(0) = 0$ and p is odd, we get $N_{C/B}(x) = y$. To compute $N_{p_G}(x)$, we have to identify C with B via $x \mapsto x \circ p_G = y$, so $N_{p_G}(x) = x$ as claimed.

30. Cartier Duality

Let H be a commutative formal group scheme over a formal scheme X. We do not assume that H is a one-dimensional formal group. Suppose that $\mathcal{O}_H$ is a pro-free $\mathcal{O}_X$-module. The duality then works well enough to make $\mathcal{O}_H^\vee$ into a topological Hopf algebra (in the obvious sense involving the completed tensor product). Thus $DH = \mathrm{spf}(\mathcal{O}_H^\vee)$ is again a formal group scheme over X.

Let $\widehat{G}_m$ be the formal multiplicative group:

$$\widehat{G}_m(R) = 1 + \mathrm{Nil}(R) \leq R^\times$$

$$\widehat{G}_m = \mathrm{spf}\left(\mathbb{Z}[u^{\pm 1}]^\wedge_{(u-1)}\right) = \mathrm{spf}(\mathbb{Z}[\![x]\!]) \qquad x = u - 1$$

PROPOSITION 30.1. $DH = \underline{\mathcal{G}roups}(H, \widehat{G}_m)$. In other words, for any formal X-scheme Y we have

$$\Gamma(Y, DH) = \mathcal{G}roups_Y(H \times_X Y, \widehat{G}_m \times Y)$$

PROOF. A map of schemes over Y $f\colon H \times_X Y \to \widehat{G}_m \times Y$ is just the same as a map of schemes $H \times_X Y \to \widehat{G}_m \subset \mathbb{A}^1$, which is just the same as an element $f' \in \mathcal{O}_H \widehat{\otimes}_X \mathcal{O}_Y$ such that $f' - 1$ is topologically nilpotent. Note that $\mathcal{O}_H \widehat{\otimes}_X \mathcal{O}_Y$ is a Hopf algebra over $\mathcal{O}_Y$. It is easy to see that f is a map of group schemes iff $\epsilon(f') = 1$ and $\psi(f') = f' \widehat{\otimes} f'$. Because $\mathcal{O}_H$ is pro-free, we have

$$\mathcal{O}_H \widehat{\otimes}_X \mathcal{O}_Y \simeq \mathcal{O}_X\text{-Mod}(\mathcal{O}_H^\vee, \mathcal{O}_Y) = \mathcal{O}_X\text{-Mod}(\mathcal{O}_{DH}, \mathcal{O}_Y)$$

Write f'' for the map $\mathcal{O}_{DH} \to \mathcal{O}_Y$ corresponding to f'. Again, one sees that this is a ring map iff $\epsilon(f') = 1$ and $\psi(f') = f' \widehat{\otimes} f'$. This gives the required bijection. $\square$

We shall often just write $DH = \mathrm{Hom}(H, \widehat{G}_m)$.

EXAMPLE 30.1. Take $X = \mathrm{spf}(\mathbb{Z}_p)$ and $H = \widehat{G}_m \times X$, so that $DH = \mathrm{End}(H)$. There is a continuous function $\chi\colon \mathbb{Z}_p \to \mathcal{O}_H = \mathbb{Z}_p[\![x]\!]$ defined by $\chi(n) = (1+x)^n$. Using $C(\mathbb{Z}_p, \mathcal{O}_X) = C(\mathbb{Z}_p, \mathbb{Z}_p) \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_X$, we obtain an adjoint map $\xi\colon \mathcal{O}_H^\vee = \mathcal{O}_{DH} \to C(\mathbb{Z}_p, \mathbb{Z}_p)$. This can be shown to be a topological isomorphism.

EXAMPLE 30.2. Let H be the constant group $\underline{A} = \mathrm{spf}(F(A, \mathbb{Z}))$, where A is a finite Abelian group. Then $DH = \mathrm{spf}(\mathbb{Z}[A])$.

EXAMPLE 30.3. Let X be the Lubin-Tate deformation space, and $H = \mathrm{Div}_0 = BU_E$. Then

$$D(\mathrm{Div}_0) = \mathrm{spf}(E^\vee BU) = \mathrm{Hom}(\mathrm{Div}_0, \widehat{G}_m) = \mathrm{Map}_0(G, \widehat{G}_m)$$

Here $\mathrm{Map}_0(G, \widehat{G}_m)$ means the scheme of maps preserving the zero section. The last equality comes from proposition 18.1.

31. Barsotti-Tate Groups

Let G be a formal group over a formal scheme X. Suppose that $p_G \colon G \to G$ is an isogeny of degree $p^n > 1$. It follows that p is topologically nilpotent in $\mathcal{O}_X$. We write $G(m) = \ker(p_G^m)$; this is a finite free group scheme over X of degree p^{nm}. These groups fit together into an inductive system of closed inclusions

$$G(*) = (G(1) \to G(2) \to G(3) \to \dots)$$

This is called the Barsotti-Tate group (or BT-group) associated to G.

More generally, one can define a Barsotti-Tate group over X to be a system $H(*) = (H(1) \to H(2) \to \dots)$ of closed inclusions of finite group schemes, such that there are short exact sequences

$$H(k) \to H(k+l) \xrightarrow{p^k} H(l)$$

The point is that $G(*)$ behaves differently from G under change of base. We started with a formal scheme X, with ring of functions $\mathcal{O}_X$. We can forget the topology on $\mathcal{O}_X$ and consider the informal scheme $\tilde{X} = \mathrm{spec}(p^{-1}\mathcal{O}_X)$. This is crude, of course. There is no sensible linear topology on $p^{-1}\mathcal{O}_X$, but there is a sensible topology of a more general kind (consider the case $\mathcal{O}_X = \mathbb{Z}_p$). However, we have not as yet developed the relevant theory. We can pull back $G(*)$ to get a BT-group $\tilde{G}(*)$ over $\tilde{X}$ (provided we allow ourselves a little latitude with the algebraic geometers' definition of a BT-group). It turns out that $\tilde{G}(m)$ is an étale torsion group. If G/X is the Lubin-Tate universal deformation, then $\tilde{G}(m)$ becomes isomorphic to the constant group $\Lambda_m = (\mathbb{Z}/p^m)^n$ after a faithfully flat base change.

Suppose, on the other hand, that we want to pull back G to get a formal group over $\tilde{X}$. We cannot just take $p^{-1}\mathcal{O}_G$, because this is not a formal power series ring over $p^{-1}\mathcal{O}_X = \mathcal{O}_{\tilde{X}}$. Instead, we have to complete again at (x) after rationalising. This gives a formal group $\tilde{G}$. This is now a formal group over a rational ring, so it has a logarithm and is isomorphic to the additive group. It therefore has no torsion subgroups, and the associated BT-group is zero.

Consider again the universal deformation G/X, of height n. Write $I_m = (u_0, \dots u_{m-1}) \lhd \mathcal{O}_X$ and $R = (u_m^{-1}R)_{I_m}^\wedge$. We make this a formal ring, with ideal of definition I_m, and write $Y = \mathrm{spf}(R)$. We can pull back $G(*)$ to get a BT-group $H(*)$ over Y. It turns out that this fits into an extension

$$H(*)_{\inf} \to H(*) \to H(*)_{\text{ét}}$$

Here $H(l)_{\inf}$ is local (or "infinitesimal") and $H(l)_{\text{ét}}$ is étale. (The extension is analogous to the sequence $G_1 \to G \to \pi_0 G$ where G is a Lie group.) The infinitesimal part has degree p^{lm}. The étale part has degree $p^{l(n-m)}$, and becomes isomorphic to $(\mathbb{Z}/p^{n-m})^l$ after faithfully flat extension.

It seems that the BT-group is the natural home of "chromatic fringe phenomena" such as the Greenlees-May generalised Tate cohomology and the root invariant.

32. Nilpotents, Idempotents and Connectivity

Recall that for a formal ring R, $\mathrm{Nil}(R)$ is the set of topologically nilpotent elements.

PROPOSITION 32.1. $\mathrm{Nil}(R)$ is the intersection of the open prime ideals of R, and thus is closed.

PROOF. It is easy to reduce the first statement to the discrete case, which is theorem 1.2 of [11]. The second statement follows as open ideals are closed. $\square$

PROPOSITION 32.2 (IDEMPOTENT LIFTING). Suppose that $e \in R/\mathrm{Nil}(R)$ is idempotent. Then there is a unique idempotent $\tilde{e} \in R$ lifting e.

PROOF. It is enough to prove this mod each open ideal I, so we may assume that R is discrete. Choose a (not necessarily idempotent) lift of e to R, call it e, and write $f = 1 - e$. We know that ef is nilpotent, say $e^n f^n = 0$. Define

$$c = e^n + f^n - 1 = e^n + f^n - (e+f)^n$$

This is visibly divisible by ef, hence nilpotent; thus $e^n + f^n = 1 + c$ is invertible. Define

$$\tilde{e} = e^n/(1+c) \qquad\qquad \tilde{f} = f^n/(1+c) = 1 - \tilde{e}$$

Then $\tilde{e}$ is an idempotent lifting e. If $\tilde{e}_1$ is another such then $\tilde{e}_1\tilde{f}$ is idempotent. It lifts $ef = 0$, so it is also nilpotent. It follows that $\tilde{e}_1\tilde{f} = 0$ and $\tilde{e}_1 = \tilde{e}\tilde{e}_1$. Similarly, $\tilde{e} = \tilde{e}\tilde{e}_1$. $\square$

PROPOSITION 32.3. Suppose that $e \in R$ is idempotent, and $f = 1 - e$. Then

$$eR = R/f = R[e^{-1}] = \{a \in R \mid fa = 0\}$$

This ring has the same topology as a subspace or as a quotient of R and R is homeomorphic to $eR \times fR$.

PROOF. This is all fairly trivial. $\square$

THEOREM 32.4 (CHINESE REMAINDER THEOREM). Suppose that $\{I_\alpha\}$ is a finite family of ideals in R, which are pairwise coprime (i.e. $I_\alpha + I_\beta = R$ when $\alpha \neq \beta$). Then

$$R/\bigcap_\alpha I_\alpha = \prod_\alpha R/I_\alpha$$

PROOF. [11, Theorems 1.3,1.4] $\square$

COROLLARY 32.5. Suppose that $\mathrm{zar}(R) = \coprod_\alpha \mathrm{zar}(R/I_\alpha)$ (a finite coproduct). Then there are unique ideals $J_\alpha \leq I_\alpha \leq \sqrt{J_\alpha}$ such that $R \simeq \prod_\alpha R/J_\alpha$.

PROOF. Proposition 32.1 implies that $\bigcap_\alpha I_\alpha$ is nilpotent. If $\alpha \neq \beta$ then no prime ideal contains $I_\alpha + I_\beta$, so $I_\alpha + I_\beta = R$. Now use the chinese remainder theorem, followed by idempotent lifting. $\square$

33. The Weierstrass Preparation Theorem

In this section, we assemble some results about the structure of rings of formal power series.

THEOREM 33.1 (WEIERSTRASS PREPARATION). Let R be a formal ring, and $f(x) = \sum_k c_k x^k \in R[\![x]\!]$ a formal power series. Suppose that c_k is topologically nilpotent for $k < n$, and that c_n is a unit. Regard $R[\![x]\!]$ as an algebra over $R[\![y]\!]$ via $y \mapsto f(x)$. Then the map

$$\alpha\colon R[\![y]\!]^n \to R[\![x]\!] \qquad\qquad \underline{a} \mapsto \sum_{k=0}^{n-1} a_k x^k$$

is a homeomorphism.

PROOF. Write $A = R[\![y]\!]$ and $B = R[\![x]\!]$. Set $I = (a_k \mid k < n)$. This is a topologically nilpotent ideal, so that for any $J \lhd_O R$ we have $I^N \leq J$ for $N \gg 0$. It is also finitely generated, which implies that $J^m A = J^m[\![x]\!]$ (and similarly for B). Without loss of generality, we may assume that $c_n = 1$. For $0 \leq k < n$ and $l \geq 0$ write $z_{kl} = x^k f(x)^l$, so that $z_{kl} = x^{k+nl} \pmod{J, x^{k+nl+1}}$. Given an R-module M consider the map

$$\prod_{k,l} M \to M[\![x]\!] \qquad\qquad \underline{a} \mapsto \sum_{k,l} a_{kl} z_{kl}$$

It is clear that this is iso if $JM = 0$, given the form of z_{kl} mod J. It then follows inductively for $M = R/J^k$. As J is topologically nilpotent and R is complete, we have $R = \varprojlim_k R/J^k$, so our map is iso for $M = R$. This implies that our original map α is also iso. We still need to prove that it is a homeomorphism. Suppose $I \lhd_O R$ and $m \geq 0$, so that $U = (x^m, I)B$ is a basic open neighbourhood of zero in B. If $J^l \leq I$ then $\alpha((y^{lm}, I)A^n) \subseteq U$, so α is continuous. Next, observe that x^n is a unit multiple of $f(x)$ in B/JB. It follows that x is nilpotent in $B/(I, y^m)$ for any $I \lhd_O R$ and $m \geq 0$; say $x^l \in (I, y^m)$. Moreover, taking $M = I$ above, we have $\alpha(IA^n) = IB$. It follows that $\alpha((I, y^m)A^n) \supseteq (I, x^l)B$, and thus α is an open map. $\square$

COROLLARY 33.2. If f is as above then there is a unique way to write $f(x) = g(x)u(x)$ where $g(x)$ is a monic polynomial of degree n and $u(x)$ is invertible. Moreover, $g(x) = x^n \pmod{J}$ and neither f nor g is a zero-divisor in $R[\![x]\!]$.

PROOF. It is immediate from the theorem that $R[\![x]\!]/f(x) = R\{x^k \mid 0 \leq k < n\}$. Thus, there is a unique way to write $x^n \bmod f(x)$ as an R-linear combination of x^k for $k < n$, in other words there is a unique monic polynomial $g(x)$ of degree n dividing $f(x)$. Next, observe that x^n divides $f(x) \bmod J$; by an obvious uniqueness argument, $g(x) = x^n \pmod{J}$. Thus, $g(x)$ is invertible in $x^{-1}R/J[\![x]\!]$, and thus also in $x^{-1}R/J^m[\![x]\!]$. It follows easily that $g(x)$ is not a zero-divisor in $R[\![x]\!]$, so there is a unique $u(x)$ such that $f(x) = g(x)u(x)$. Moreover, $u(x)$ is visibly invertible mod the topologically nilpotent ideal J, so it is invertible. $\square$

Next, we consider the ring

$$C = (x^{-1}R[\![x]\!])^{\wedge} = \varprojlim_{I} x^{-1}R/I[\![x]\!]$$

This is the ring of series $f(x) = \sum_{k \in \mathbb{Z}} a_k x^k$ such that $a_k \to 0$ as $k \to \infty$. We shall say that $f \in C$ is *holomorphic at zero (resp. infinity)* if $a_k = 0$ for $k < 0$ (resp. $k > 0$).

COROLLARY 33.3. Suppose that $f(x) = \sum a_k x^k \in C$ has $a_n \in R^{\times}$, and $a_k \in \mathrm{Nil}(R)$ for $k < n$. Then $f \in C^{\times}$. Moreover, $f(x)$ can be written uniquely as $x^n g(x)u(x)$ with

(1) $g(x)$ holomorphic at infinity
(2) $g(\infty) = 1$
(3) $u(x)$ holomorphic at zero
(4) $u(0) \in R^{\times}$.

PROOF. Without loss of generality, $n = 0$. Suppose $I \lhd R$ is open, and write $f_I(x)$ for the image of f in $x^{-1}R/I[\![x]\!]$. Then theorem 33.1 applies to $x^m f_I(x)$ for some $m \gg 0$. This gives a unique factorisation $f_I = g_I u_I$ as described above (with g_I equal to x^{-m} times the polynomial provided by the theorem). These pass to the limit as required. $\square$

PROPOSITION 33.4. Suppose that R is connected, in other words it has no idempotents other than 0 and 1. Suppose that $f \in C^{\times}$. Then f is as in corollary 33.3

PROOF. Suppose $f = \sum_k a_k x^k$ and $f^{-1} = g = \sum_l b_l x^l$. Write $J_n = (a_k \mid k < n) + (b_l \mid l < -n)$. Suppose that $I \lhd_O R$. There exists $K > 0$ such that $a_k, b_k \in I$ for $k < -K$. Suppose that $\mathfrak{p} \lhd R$ is prime, and that $\mathfrak{p} \geq I$. For some n we have $a_k \in \mathfrak{p}$ for $k < n$ but $a_n \notin \mathfrak{p}$. Similarly $b_l \in \mathfrak{p}$ for $l < m$ but $b_m \notin \mathfrak{p}$. This means that mod $\mathfrak{p}$ we have $fg = a_n b_m x^{n+m}$ plus higher terms, with $a_n b_m \neq 0$. This means that $m = -n$ and $|n| \leq K$, and that $J_n \leq \mathfrak{p}$; but $J_l \not\leq \mathfrak{p}$ for $l \neq n$. It follows that the ideals J_n are pairwise coprime mod I. Their intersection is contained in all prime ideals $\mathfrak{p} \geq I$, hence is nilpotent mod I. The Chinese remainder theorem together with idempotent lifting gives a canonical splitting

$$R/I = \prod_n R(I, n)$$

such that $a_n b_{-n}$ is invertible in $R(I, n)$ and nilpotent in all the other factors. This splitting is actually finite, because $R(I, n) = 0$ when $|n| > K$. In the limit, we get

$$R = \prod_n \varprojlim_{I} R(I, n)$$

As R is connected, we have $R = \varprojlim_{I} R(I, n)$ for some n. It follows that $a_n b_{-n}$ is invertible in R, hence in R/I, so that $R/I = R(I, n)$ for all I. This implies that a_k is topologically nilpotent for $k < n$. $\square$

Of course, if R is not connected, we may get different n's on different components.

34. Dictionary

$\mathrm{spf}(E)$	The universal deformation space $\mathrm{Def} = X$
$\mathrm{spf}(K)$	The special fibre $X_0 < X$
$\mathbb{C}P_E^\infty$	The universal deformation G over X
$(B\mathbb{Z}/p^m)_E$	$G(m) = \ker(p_G^m \colon G \to G)$
$K(\mathbb{Z}/p^m, l)_E$	$\bigwedge^l G(m)$
$(BA^*)_E$	$\mathrm{Hom}(A, G)$
$\mathrm{spf}(E(BA^*)/\sum_{a \neq 0} \mathrm{ann}(x(a)))$	$\mathrm{Level}(A, G)$
$BU(m)_E$	$\mathrm{Div}_m^+(G)$
BU_E	$\mathrm{Div}_0(G)$
$(\mathbb{Z} \times BU)_E$	$\mathrm{Div}(G)$
$\mathrm{spf}(E^\vee \mathbb{C}P^\infty)$	$\mathrm{Hom}(G, \widehat{G}_m)$
$\mathrm{spf}(E^\vee(\mathbb{Z} \times BU))$	$\mathrm{Map}(G, \widehat{G}_m)$
$\mathrm{spf}(E^\vee BU)$	$\mathrm{Map}_0(G, \widehat{G}_m)$
$\mathrm{spf}(E^\vee MU)$	$\mathrm{Orient}(G)$
$E^\vee E$	The pro-constant stabiliser group $\underline{\Sigma} = \mathrm{DefIso}$

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