

Tambara functors

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Abstract

We survey and extend the theory of Tambara functors. These are algebraic structures similar to Mackey functors, but with multiplicative norm maps as well as additive transfer maps, and a rule governing their interaction that is most easily formulated in an abstract categorical framework. Examples include Burnside rings, representation rings, and homotopy groups of equivariant E_∞ ring spectra in stable homotopy theory. Some other examples are related to Witt rings in the sense of Dress and Siebeneicher.

1. Introduction

In this memoir we survey the theory of Tambara functors.

We first give some motivation for these objects.

One way to think about the definition of Mackey functors (for a finite group G) is as follows. Take the Lawvere theory $\overline{\mathcal{A}}$ for (commutative) semigroups, categorify it, make it G -equivariant, decategorify, and then take the category of models for the resulting (multisorted) theory. This category is just the category of semigroup-valued Mackey functors, so it contains the more usual category of group-valued Mackey functors. All this will be explained in more detail below.

It is well-known that for any G -spectrum R (in the sense of equivariant stable homotopy theory) one has a Mackey functor $\pi_0^G R$. If R is a G -ring spectrum in the naive homotopical sense, then $\pi_0^G R$ has a certain multiplicative structure; Mackey functors with this structure are called Green rings. If R has an equivariant E_∞ structure then much more is true. In particular, there are multiplicative transfer maps, similar to the Evans norm map in ordinary group cohomology or tensor induction maps in representation theory. These interact in a complicated way with additive transfers and with restriction maps. The purpose of Tambara functors is to encapsulate these relationships.

The category of Tambara functors can be defined along the same lines as suggested above for Mackey functors. One simply starts with the Lawvere theory $\overline{\mathcal{U}}$ for semirings instead of the theory $\overline{\mathcal{A}}$ for semigroups.

Another major motivation for studying Tambara functors is the light that they shed on the theory of Witt vectors. There is a straightforward functor from C_n -Tambara functors to rings, whose left adjoint can be expressed in terms of Witt vectors of length d for all divisors d of n . For more general groups G , there is a similar relationship with the generalised Witt vector rings of Dress and Siebeneicher [8].

Tambara functors were originally introduced by Tambara, who called them TNR-functors [22]. The relationship with Witt vectors was first noticed by Brun [4], which inspired a beautiful reinterpretation of Witt theory by Elliott [9]. This in turn inspired the work leading to this memoir. During the long gestation of this work, a number of papers and preprints by Nakaoka have appeared [15–20], covering some of the same ideas; we will give detailed references in the main text.

2. The theory of semigroups

In this section we review the definition of semigroups, in a form that generalises easily to define Mackey functors and Tambara functors. For us, a *semigroup* will mean a set with a commutative and associative binary operation (denoted by $+$) and an identity element (denoted by 0). Mackey functors are usually defined in terms of abelian groups, but we will find it convenient to give a slightly more general version without additive inverses.

Received by the editor May 14, 2012.

1991 *Mathematics Subject Classification*. 19A22,20C99,20J06.

Any finite set X gives a functor $M \mapsto M^X = \text{Map}(X, M)$ from semigroups to sets. Let $\overline{\mathcal{A}}(X, Y)$ be the set of natural functions $f: M^X \rightarrow M^Y$. (We do not assume *a priori* that f is a homomorphism of semigroups, but it will turn out that this holds automatically.) This gives us a category $\overline{\mathcal{A}}$, whose objects are finite sets.

By a *span* from X to Y we mean a diagram $\omega = (X \xleftarrow{p} A \xrightarrow{q} Y)$, where A is another finite set. By an *isomorphism* from ω to another span $\omega' = (X \xleftarrow{p'} A' \xrightarrow{q'} Y)$ we mean a bijection $f: A \rightarrow A'$ with $p'f = p$ and $q'f = q$. This gives a groupoid $\mathcal{A}(X, Y)$ of spans from X to Y .

For any span ω as above, we have a map $f_\omega: M^X \rightarrow M^Y$ given by

$$f_\omega(m)(y) = \sum_{a \in q^{-1}\{y\}} m(p(a)).$$

It is easy to see that this is natural, and depends only on the isomorphism class of ω .

Alternatively, we can analyse $\overline{\mathcal{A}}(X, Y)$ using the Yoneda lemma. The semigroup \mathbb{N}^X represents the functor $M \mapsto M^X$, and it follows that

$$\overline{\mathcal{A}}(X, Y) = \text{Semigroups}(\mathbb{N}^Y, \mathbb{N}^X) = (\mathbb{N}^X)^Y = \mathbb{N}^{X \times Y}.$$

Explicitly, for any map $a: X \times Y \rightarrow \mathbb{N}$ we have a natural map $g_a: M^X \rightarrow M^Y$ given by

$$g_a(m)(y) = \sum_{x \in X} a(x, y)m(x).$$

For a span ω as above, we find that $f_\omega = g_{c(\omega)}$, where

$$c(\omega)(x, y) = |\{a \in A \mid p(a) = x \text{ and } q(a) = y\}|.$$

From this it is straightforward to check that the construction $\omega \rightarrow f_\omega$ gives a bijection from the set $\pi_0 \mathcal{A}(X, Y)$ (of isomorphism classes of spans) to $\overline{\mathcal{A}}(X, Y)$. We also see (as promised) that all natural maps $M^X \rightarrow M^Y$ are automatically homomorphisms. This means that all the morphism sets $\overline{\mathcal{A}}(X, Y)$ are semigroups in a natural way, and that composition is bilinear.

Now suppose we have two spans

$$\begin{aligned} \omega_0 &= (X_0 \xleftarrow{p_0} X_{01} \xrightarrow{q_0} X_1) \\ \omega_1 &= (X_1 \xleftarrow{p_1} X_{12} \xrightarrow{q_1} X_2). \end{aligned}$$

The above analysis implies that the composite

$$M^{X_0} \xrightarrow{f_{\omega_0}} M^{X_1} \xrightarrow{f_{\omega_1}} M^{X_2}$$

must arise from a span from X_0 to X_2 , which is unique up to isomorphism. To find the required span, we let X_{02} denote the pullback of the maps $X_{01} \xrightarrow{q_0} X_1 \xleftarrow{p_1} X_{12}$, so we have a diagram

$$\begin{array}{ccccc} & & X_{02} & & \\ & \swarrow & & \searrow & \\ & X_{01} & & X_{12} & \\ & \swarrow & & \searrow & \\ X_0 & & X_1 & & X_2 \end{array}$$

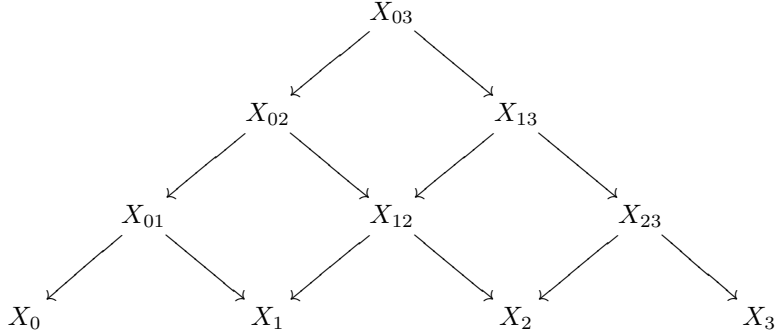
in which the middle square is cartesian. We define $\omega_1 \circ \omega_0$ to be the resulting span

$$\omega_1 \circ \omega_0 = (X_0 \leftarrow X_{02} \rightarrow X_2).$$

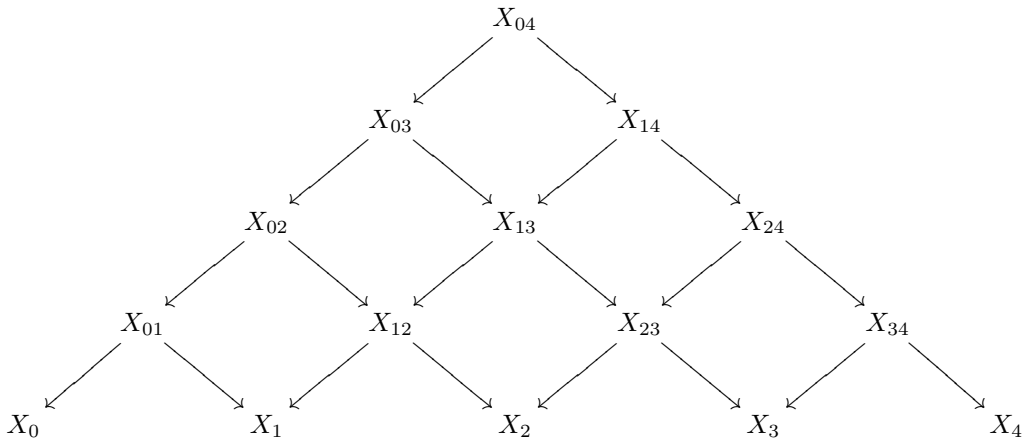
It is not hard to check that $f_{\omega_1 \circ \omega_0} = f_{\omega_{12}} f_{\omega_{01}}$ as required.

One way to encapsulate the properties of this construction is to say that we have a bicategory (as defined at [25] for example), where the 0-cells are finite sets, the 1-cells are spans, and the 2-cells are isomorphisms

of spans. To produce the required associativity isomorphisms, we need to contemplate diagrams of the form



in which all the squares are cartesian. To prove the pentagonal coherence identities for these isomorphisms, we need to consider diagrams of the form



in which all the squares are again cartesian.

Another approach is to let \mathcal{A}_k denote the set of all diagrams of this general type, involving sets X_{ij} for $0 \leq i \leq j \leq k$. To be set-theoretically respectable we should fix an infinite set U , and insist that all the sets X_{ij} should be subsets of U . One can then introduce face and degeneracy operators to make \mathcal{A}_\bullet into a simplicial set, and then check that this is an ∞ -category in the sense of Lurie [12] (or in the language of Joyal [11], a quasicategory). This idea has been developed extensively in work of Cranch [5].

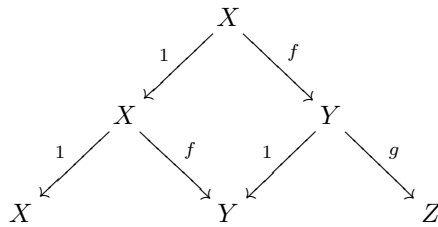
It will be convenient to introduce slightly different notation as follows. Given a map $f: X \rightarrow Y$, we put

$$T_f = (X \xleftarrow{1} X \xrightarrow{f} Y) \in \mathcal{A}(X, Y)$$

$$f^* = R_f = (Y \xleftarrow{f} X \xrightarrow{1} X) \in \mathcal{A}(Y, X).$$

We also write T_f for the corresponding isomorphism class in $\overline{\mathcal{A}}(X, Y)$, or the resulting operation $M^X \rightarrow M^Y$, and similarly for R_f .

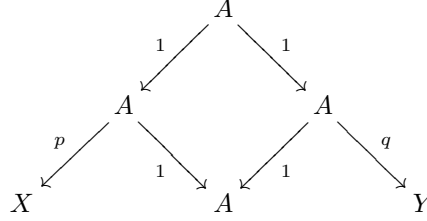
- (a) For all $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $T_{gf} = T_g T_f$. More precisely, we see by considering the diagram



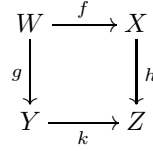
that the spans T_{gf} and T_gT_f are canonically isomorphic in $\mathcal{A}(X, Z)$, so $T_{gf} = T_gT_f$ in $\overline{\mathcal{A}}(X, Z)$ or $\text{Map}(M^X, M^Z)$.

(b) In the same sense, we have $R_{gf} = R_fR_g$, or equivalently $(gf)^* = f^*g^*$.

(c) Any span $(X \xleftarrow{p} A \xrightarrow{q} Y)$ can be expressed in $\overline{\mathcal{A}}(X, Y)$ as T_qR_p , as we see by considering the diagram

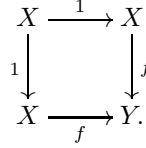


(d) For any cartesian square



we have $T_gR_f = R_kT_h \in \overline{\mathcal{A}}(X, Y)$.

(e) For any bijection $f: X \rightarrow Y$ we have $T_f = R_f^{-1}$. This can be seen as a special case of (d), for the diagram



Next, consider the canonical inclusions

$$X \xrightarrow{i} X \amalg Y \xleftarrow{j} Y.$$

These give a diagram as follows in $\overline{\mathcal{A}}$:

$$X \xleftarrow{R_i} X \amalg Y \xrightarrow{R_j} Y.$$

We claim that this is a product diagram, or in other words that for every diagram

$$X \xleftarrow{\alpha} U \xrightarrow{\beta} Y$$

there is a unique morphism $\phi \in \overline{\mathcal{A}}(U, X \amalg Y)$ with $R_i\phi = \alpha$ and $R_j\phi = \beta$. Indeed, we can represent α and β by spans as follows:

$$\alpha = [U \xleftarrow{p} A \xrightarrow{q} X]$$

$$\beta = [U \xleftarrow{r} B \xrightarrow{s} Y].$$

Let $(p, r): A \amalg B \rightarrow U$ be the map given by p on A and by r on B . We then find that the morphism

$$\phi = [U \xleftarrow{(p,r)} A \amalg B \xrightarrow{q \amalg s} X \amalg Y]$$

has the required property. We also note that the maps

$$M^X \xleftarrow{R_i} M^{X \amalg Y} \xrightarrow{R_j} M^Y$$

are just the obvious projections, so they again give a product diagram (in either the category of sets, or the category of semigroups).

A similar argument shows that the maps $X \xrightarrow{T_i} X \amalg Y \xleftarrow{T_j} Y$ give a coproduct diagram. Similarly, the empty set is both initial and terminal in $\overline{\mathcal{A}}$. Indeed, if we let $z: \emptyset \rightarrow X$ be the inclusion, then $\overline{\mathcal{A}}(\emptyset, X) = \{T_z\}$ and $\overline{\mathcal{A}}(X, \emptyset) = \{R_z\}$. Using this, we see that $\overline{\mathcal{A}}$ is a semiadditive category, as discussed in Appendix A. (The

arguments are essentially the same as in the more familiar additive case, discussed in [13, Chapter VIII] for example). In terms of spans, the addition is

$$[X \xleftarrow{p} A \xrightarrow{q} Y] + [X \xleftarrow{p'} A' \xrightarrow{q'} Y] = [X \xleftarrow{(p,p')} A \amalg A' \xrightarrow{(q,q')} X'].$$

Alternatively, we can regard each object $X \in \overline{\mathcal{A}}$ as a semigroup object: the unit is $T_2: \emptyset \rightarrow X$, and the addition is $T_s: X \amalg X \rightarrow X$, where s is the function $X \amalg X \rightarrow X$ given by the identity on both copies of X .

We can recover the category of semigroups from $\overline{\mathcal{A}}$, as follows:

PROPOSITION 2.1. *The category of semigroups is equivalent to the category of product-preserving functors from $\overline{\mathcal{A}}$ to the category of sets.*

This should really be seen as a basic part of the setup of algebraic theories in the sense of Lawvere [1, Chapter 3; 24]. In Lawvere's terminology, the category $\overline{\mathcal{A}}$ is the theory of semigroups.

PROOF. If A is a semigroup then for each finite set X we have a set A^X , and for each element $f \in \overline{\mathcal{A}}(X, Y)$ we have (tautologically) a map $f_A: A^X \rightarrow A^Y$. Moreover, these compose in the obvious way, so we have a functor $E_A: \overline{\mathcal{A}} \rightarrow \text{Sets}$ sending X to A^X . As $X \amalg Y$ is the product of X and Y in $\overline{\mathcal{A}}$ and $A^{X \amalg Y} = A^X \times A^Y$ we see that E_A preserves products.

For the opposite construction, it will be convenient to identify numbers with sets by the usual rule $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$ and so on. For any semigroup U we have natural maps $U^0 \xrightarrow{\eta_U} U^1 \xleftarrow{\sigma_U} U^2$ given by $\eta(0) = 0$ and $\sigma(u_0, u_1) = u_0 + u_1$. These can be regarded as morphisms $0 \xrightarrow{\eta} 1 \xleftarrow{\sigma} 2$ in $\overline{\mathcal{A}}$.

Now suppose we start with a product-preserving functor $F: \overline{\mathcal{A}} \rightarrow \text{Sets}$. Put $A = F(1)$. Any finite set X can be thought of as $\coprod_{x \in X} 1$, and disjoint unions are products in $\overline{\mathcal{A}}$, so the natural map $F(X) \rightarrow \prod_{x \in X} F(1) = A^X$ is bijective. Thus, the maps $F(0) \xrightarrow{F(\eta)} F(1) \xleftarrow{F(\sigma)} F(2)$ give maps $1 \xrightarrow{0} A \xleftarrow{+} A^2$. We claim that these make A into a semigroup. We will prove the associativity axiom as an example. All semigroups are associative, so the two natural operations $(u, v, w) \mapsto \sigma(\sigma(u, v), w)$ and $(u, v, w) \mapsto \sigma(u, \sigma(v, w))$ are the same, so the following diagram commutes in $\overline{\mathcal{A}}$:

$$\begin{array}{ccc} 3 & \xrightarrow{\sigma \times 1} & 2 \\ 1 \times \sigma \downarrow & & \downarrow \sigma \\ 2 & \xrightarrow{\sigma} & 1. \end{array}$$

We can apply F to this and use the natural isomorphisms $F(k) \rightarrow F(1)^k = A^k$ to obtain a commutative square

$$\begin{array}{ccc} A^3 & \xrightarrow{(+)\times 1} & A^2 \\ 1 \times (+) \downarrow & & \downarrow + \\ A^2 & \xrightarrow{+} & A. \end{array}$$

This proves that addition is associative in A , and the other semigroup axioms can be handled in the same way.

We leave it to the reader to check that these two constructions give an inverse pair of functors between the relevant categories. \square

We next discuss quotients of semigroups. This is a little more subtle than the corresponding construction with groups, but the relevant concepts are a standard part of universal algebra.

DEFINITION 2.2. Let M be a semigroup. A *congruence* on M is a subset $E \subseteq M \times M$ that is both a subsemigroup of $M \times M$ and an equivalence relation on M .

PROPOSITION 2.3.

(a) For any morphism $\phi: M \rightarrow M'$ of semigroups there is a congruence $\text{eqker}(\phi)$ on M given by

$$\text{eqker}(\phi) = \{(a, b) \in M \times M \mid \phi(a) = \phi(b) \in M'\}.$$

- (b) For any congruence E on M there is a unique way to make the quotient set M/E into a semigroup such that the projection $\pi: M \rightarrow M/E$ is a semigroup morphism.
- (c) If $E \leq \text{eqker}(\phi)$ then there is a unique semigroup morphism $\bar{\phi}: M/E \rightarrow M'$ with $\phi = \bar{\phi}\pi$, but if $E \not\leq \text{eqker}(\phi)$ then there is no such morphism.
- (d) The diagonal $\Delta = \{(m, m) \mid m \in M\}$ is a congruence with $M/\Delta = M$. The whole set $M \times M$ is a congruence with $M/(M \times M) = 0$.
- (e) The intersection of any family of congruences is a congruence. (In particular, the intersection of the empty family is $M \times M$.)
- (f) For any subset $F \subseteq M \times M$, there is a smallest congruence containing F , namely the intersection of the family of all congruences that contain F .

PROOF. Left to the reader. □

REMARK 2.4. If M has additive inverses and so is actually a group, then it is easy to check that every congruence has the form

$$E_N = \{(a, b) \in M \times M \mid a - b \in N\}$$

for some subgroup $N \leq M$, and thus that congruences biject with subgroups.

REMARK 2.5. Let F be a subset of $M \times M$. Put

$$\begin{aligned} E(0) &= \{(a, b) \in M \times M \mid a = b \text{ or } (a, b) \in F \text{ or } (b, a) \in F\} \\ E(2k+1) &= \{(a+a', b+b') \mid (a, b) \in E(2k) \text{ and } (a', b') \in E(2k)\} \\ E(2k+2) &= \{(a, b) \mid \text{there exists } u \in M \text{ with } (a, u) \in E(2k+1) \text{ and } (u, b) \in E(2k+1)\} \\ E &= \bigcup_n E(n). \end{aligned}$$

Then one can check that E is the smallest congruence containing F , as in part (f) of the above proposition. We suspect that there is no substantially simpler construction. In particular, we suspect that it is necessary to alternate infinitely many times between operations designed to make E a subsemigroup and operations designed to make E transitive. However, we have not constructed explicit examples to support this.

DEFINITION 2.6. For any subsemigroup $N \leq M$ we let E_N denote the smallest congruence containing $N \times 0$, and we write M/N for M/E_N .

DEFINITION 2.7. Let M be any semigroup with an action of a finite group G . We put $F = \{(m, gm) \mid m \in M \text{ and } g \in G\}$, and let E be the smallest congruence containing F . We then put $M_G = M/E$, and call this the *coinvariant semigroup* for the action.

REMARK 2.8. It is clear by construction that a homomorphism $\phi: M \rightarrow M'$ factors through M/N iff $\text{eqker}(\phi) \supseteq N \times 0$ iff $\phi(N) = 0$ (and the factorisation is unique if it exists). Similarly, ϕ factors through M_G iff $\phi(gm) = \phi(m)$ for all $g \in G$ and $m \in M$.

REMARK 2.9. It is possible to have $M/N = 0$ even when $N \neq M$. For example, for any $a \in \mathbb{N}$ the set

$$U_a = \{n \in \mathbb{N} \mid n = 0 \text{ or } n \geq a\}$$

is a subsemigroup of \mathbb{N} . In \mathbb{N}/U_a we have $0 \sim a$ so $k \sim k+a$ for all $k \in \mathbb{N}$, but also $k+a \in U_a$ so $k \sim 0$. This shows that $\mathbb{N}/U_a = 0$. However, the inclusion $U_a \rightarrow \mathbb{N}$ is not an epimorphism in the category of semigroups. To see this, let E_0 denote the set of pairs $((i, j), (k, l)) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $\max(i, j) \geq a$ and $\max(k, l) \geq a$ and $i+j = k+l$. One can check that $E = \Delta \cup E_0$ is a congruence on \mathbb{N}^2 ; let $\pi: \mathbb{N}^2 \rightarrow Q = \mathbb{N}^2/E$ be the resulting quotient morphism. We have morphisms $\alpha, \beta: \mathbb{N} \rightarrow Q$ given by $\alpha(n) = \pi(n, 0)$ and $\beta(n) = \pi(0, n)$. These agree on U_a , but not on all of \mathbb{N} , as required.

3. Mackey functors

Let G be a finite group. Everything in the previous section can be done G -equivariantly. Explicitly, we introduce a bicategory \mathcal{A}_G , where the 0-cells are finite G -sets, the 1-cells are diagrams $(X \xleftarrow{p} A \xrightarrow{q} Y)$ of finite G -sets, and the 2-cells are the evident equivariant isomorphisms. We can then decategorify this: we form a category $\overline{\mathcal{A}}_G$ whose objects are finite G -sets, with morphisms $\overline{\mathcal{A}}_G(X, Y) = \pi_0 \mathcal{A}_G(X, Y)$. For us, a *Mackey functor* will mean a product-preserving functor from $\overline{\mathcal{A}}_G$ to Sets (these are semi-Mackey functors in the terminology of Nakaoka). We write Mackey_G for the category of Mackey functors.

As before, the inclusions $X \xrightarrow{i} X \amalg Y \xleftarrow{j} Y$ give a product diagram $X \xleftarrow{R_i} X \amalg Y \xrightarrow{R_j} Y$ in $\overline{\mathcal{A}}_G$, and also a coproduct diagram $X \xrightarrow{T_i} X \amalg Y \xleftarrow{T_j} Y$, and the empty set is both initial and terminal. This makes $\overline{\mathcal{A}}_G$ a semiadditive category. If M is a Mackey functor we thus have $M(X \amalg Y) \simeq M(X) \times M(Y)$, and we can apply M to the semigroup structure maps $\emptyset \xrightarrow{T_z} X \xleftarrow{T_s} X \amalg X$ to make $M(X)$ into a semigroup in a natural way.

EXAMPLE 3.1. Let A be a semigroup with an action of G . We would like to define a Mackey functor cA by $cA(X) = \text{Map}_G(X, A)$ (so $cA(G/H) = A^H$). Given any equivariant span $\omega = (X \xleftarrow{p} A \xrightarrow{q} Y)$, we define $f_\omega: \text{Map}(X, A) \rightarrow \text{Map}(Y, A)$ by the usual rule

$$f_\omega(u)(y) = \sum_{q(a)=y} u(p(a)).$$

This commutes with the G -action and so restricts to give a map $f_\omega: cA(X) \rightarrow cA(Y)$. This depends only on the equivariant isomorphism class of ω and is compatible with composition of spans, so the construction $[\omega] \mapsto f_\omega$ makes cA into a Mackey functor as desired.

If G acts trivially on A we just have $cA(X) = \text{Map}(X/G, A)$. In particular, we can identify $cA(G/H)$ with A , and for any $f: G/K \rightarrow G/H$ the resulting map $R_f: cA(G/H) \rightarrow cA(G/K)$ is just the identity $A \rightarrow A$. For this reason, Mackey functors of this type are called *constant* Mackey functors. However, this name is somewhat misleading because the map $T_f: cA(G/K) \rightarrow cA(G/H)$ is $|H|/|K|$ times the identity on A , rather than the identity itself. (Note here that the existence of f means that K is conjugate to a subgroup of H , so $|H|/|K|$ is an integer.)

EXAMPLE 3.2. Now let B be a semigroup without G -action, and define $N(X) = \text{Map}(X^G, B)$ (where X^G is the subset of G -fixed points in X). The construction $X \mapsto X^G$ preserves the pullbacks used to define composition of spans, and it also preserves disjoint unions, so it gives a product-preserving functor $\overline{\mathcal{A}}_G \rightarrow \overline{\mathcal{A}}$. Given this, there is an evident way to regard N as a Mackey functor. Explicitly, for any equivariant span $\omega = (X \xleftarrow{p} A \xrightarrow{q} Y)$, we define $f_\omega: N(X) \rightarrow N(Y)$ by

$$f_\omega(u)(y) = \sum_{a \in A^G, q(a)=y} u(p(a)).$$

EXAMPLE 3.3. For any finite G -set U we have a representable Mackey functor $H_U: \overline{\mathcal{A}}_G \rightarrow \text{Sets}$ given by $H_U(X) = \overline{\mathcal{A}}_G(U, X)$. We also write $A(X)$ for $H_1(X)$, which is the set of isomorphism classes of spans $(1 \xleftarrow{p} T \xrightarrow{q} X)$. It is clear here that p gives no information, so $A(X)$ is the set of isomorphism classes of finite G -sets equipped with a map to X , which is known as the *Burnside semigroup* of X .

EXAMPLE 3.4. Let E be a G -equivariant spectrum in the sense of stable homotopy theory. For any finite G -set X we have another G -spectrum $\Sigma_G^\infty(X_+)$, and we write $\pi_0^G(E)(X)$ for the set of homotopy classes of maps $\Sigma_G^\infty(X_+) \rightarrow E$. It is well known that $\pi_0^G(E)$ is a Mackey functor in a natural way. We will give a proof in Section 8, in a form that is convenient for generalisation to the Tambara framework.

PROPOSITION 3.5. *Let $\overline{\mathcal{AO}}_G$ denote the full subcategory of $\overline{\mathcal{A}}_G$ whose objects are the orbits G/H for all subgroups $H \leq G$. Then Mackey_G is equivalent to the category of preadditive functors from $\overline{\mathcal{AO}}_G$ to the category of semigroups.*

PROOF. As every finite G -set is a disjoint union of orbits, we see that every object in $\overline{\mathcal{A}}_G$ can be expressed as a product (or equivalently, a coproduct) of objects in $\overline{\mathcal{AO}}_G$. The claim therefore follows from Proposition A.13 (in Appendix A). \square

REMARK 3.6. Using this idea, we can also regard Mackey_G as the category of algebras for a coloured or multisorted Lawvere theory, with one colour for each subgroup of G . (Such theories are discussed in [2, Chapter II], for example).

From either of the above points of view, it is important to understand $\overline{\mathcal{A}}_G(G/H, G/K)$. There are three basic kinds of elements:

- (a) Suppose that $K \leq H \leq G$, so there is an evident projection $p: G/K \rightarrow G/H$. We put

$$\begin{aligned} T_K^H &= T_p \in \overline{\mathcal{A}}_G(G/K, G/H) \\ R_K^H &= R_p \in \overline{\mathcal{A}}_G(G/H, G/K). \end{aligned}$$

- (b) For any $g \in G$ and $H \leq G$ we define $c_g^H: G/H \rightarrow G/gHg^{-1}$ by

$$c_g^H(xH) = xHg^{-1} = xg^{-1}(gHg^{-1}).$$

We also define $C_g^H = T_{c_g^H} = R_{c_g^H}^{-1} \in \overline{\mathcal{A}}_G(G/H, G/gHg^{-1})$.

Any element of $\overline{\mathcal{A}}_G(G/H, G/K)$ can be represented by a span $\omega = (G/H \xleftarrow{p} A \xrightarrow{q} G/K)$. We can decompose A as a disjoint union of orbits, each orbit gives a span from G/H to G/K , and ω is the sum of these terms. We can therefore focus on the case where A itself is an orbit. As G acts transitively on G/H , the map p is necessarily surjective, so we can choose $a \in A$ with $p(a) = H$. This identifies A with G/L for some subgroup $L \leq H$, and p with the canonical projection $G/L \rightarrow G/H$. As q is also surjective we will have $q(xL) = K$ for some $x \in G$. As the coset $xL \in G/L$ is fixed by the subgroup xLx^{-1} and q is equivariant, we must have $xLx^{-1} \leq K$. We therefore have maps

$$G/H \xrightarrow{R_L^H} G/L \xrightarrow{C_x^L} G/xLx^{-1} \xrightarrow{T_{xLx^{-1}}^K} G/K$$

in $\overline{\mathcal{A}}_G$, and it is straightforward to check that the composite is ω . This allows us to express an arbitrary element of $\overline{\mathcal{A}}_G(G/H, G/K)$ as a sum of composites of our basic operators.

These satisfy relations as follows:

- (a) For $L \leq K \leq H \leq G$ it is clear that $T_K^H T_L^K = T_L^H: G/L \rightarrow G/H$ and $R_L^K R_K^H = R_L^H: G/H \rightarrow G/L$.
(b) It is also clear that $C_f^{gHg^{-1}} C_g^H = C_{fg}^H: G/H \rightarrow G/fgH(fg)^{-1}$.
(c) By regarding C_g^H as $T_{c_g^H}$ we see that the left hand square below commutes. By regarding it as $R_{c_g^H}^{-1}$, we see that the right hand square commutes.

$$\begin{array}{ccc} G/K & \xrightarrow{C_g^K} & G/gKg^{-1} \\ T_K^H \downarrow & & \downarrow T_{gKg^{-1}}^{gHg^{-1}} \\ G/H & \xrightarrow{C_g^H} & G/gHg^{-1} \end{array} \qquad \begin{array}{ccc} G/K & \xrightarrow{C_g^K} & G/gKg^{-1} \\ R_K^H \uparrow & & \uparrow R_{gKg^{-1}}^{gHg^{-1}} \\ G/H & \xrightarrow{C_g^H} & G/gHg^{-1}. \end{array}$$

- (d) Now consider a composite $G/H \xrightarrow{T_L^H} G/L \xrightarrow{R_L^K} G/K$ (where $H, K \leq L$). Let A denote the pullback $(G/H) \times_{G/L} (G/K)$, so $R_L^K T_L^H$ is represented by an evident span $(G/H \xleftarrow{p} A \xrightarrow{q} G/K)$, so it can be written as a sum of terms indexed by the G -orbits in A . We can let $K \times H$ act on L by $(k, h).l = klh^{-1}$, and thus decompose L as $\coprod_{t \in T} KtH$ for some subset $T \subseteq L$. For each $t \in T$ we have a point $\phi(t) = (H, t^{-1}K) \in A$, whose isotropy group is $M_t = H \cap tKt^{-1}$. We find that each orbit in A contains precisely one of the points $\phi(t)$, and that $p(\phi(t)) = H$ and $q(\phi(t)) = K$. Note that the conjugate $M'_t = tM_t t^{-1}$ is $tHt^{-1} \cap K$, and in particular is contained in K . From this we deduce the double coset formula: $R_L^K T_L^H$ is the sum over $t \in T$ of the composites

$$G/H \xrightarrow{R_{M_t}^H} G/M_t \xrightarrow{C_t^{M_t}} G/M'_t \xrightarrow{T_{M'_t}^K} G/K.$$

CONSTRUCTION 3.7. Given spans

$$\begin{aligned}\alpha &= (X \xleftarrow{p} A \xrightarrow{q} Y) \in \mathcal{A}_G(X, Y) \\ \alpha' &= (X' \xleftarrow{p'} A' \xrightarrow{q'} Y') \in \mathcal{A}_G(X', Y')\end{aligned}$$

we write $\alpha \times \alpha'$ for the product span

$$\alpha \times \alpha' = (X \times X' \xleftarrow{p \times p'} A \times A' \xrightarrow{q \times q'} Y \times Y') \in \mathcal{A}_G(X \times X', Y \times Y').$$

As composition of spans is defined using pullbacks, and pullbacks commute with products, we see that there are natural isomorphisms

$$(\beta \times \beta') \circ (\alpha \times \alpha') \simeq (\beta \circ \alpha) \times (\beta' \circ \alpha').$$

We can thus define a functor $\times: \overline{\mathcal{A}}_G \times \overline{\mathcal{A}}_G \rightarrow \overline{\mathcal{A}}_G$ by $(X, X') \mapsto X \times X'$ on objects, and $([\alpha], [\alpha']) \mapsto [\alpha \times \alpha']$ on morphisms. This gives a symmetric monoidal structure on the category $\overline{\mathcal{A}}_G$.

REMARK 3.8. It is clear that for any maps $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ of finite G -sets, we have $R_f \times R_{f'} = R_{f \times f'}$ and $T_f \times T_{f'} = T_{f \times f'}$. Using the natural isomorphisms

$$(A \amalg B) \times (C \amalg D) \simeq (A \times C) \amalg (A \times D) \amalg (B \times C) \amalg (B \times D)$$

we also see that for $u, v \in \overline{\mathcal{A}}_G(X, Y)$ and $u', v' \in \overline{\mathcal{A}}_G(X', Y')$ we have

$$(u + v) \times (u' + v') = (u \times u') + (u \times v') + (v \times u') + (v \times v'),$$

so the product functor is bilinear.

DEFINITION 3.9. Given Mackey functors M and N , we write $M \boxtimes N$ for the Day tensor product of M and N . This was originally defined in [6], and the construction is reviewed in Appendix A. It is characterised by the fact that morphisms $M \boxtimes N \rightarrow P$ biject with natural maps $M(X) \times N(Y) \rightarrow P(X \times Y)$ for $(X, Y) \in \overline{\mathcal{A}}_G^2$. This gives a biadditive symmetric monoidal structure on Mackey $_G$.

There is a tautological map $M(X) \times N(Y) \rightarrow (M \boxtimes N)(X \times Y)$. We write $m \boxtimes n \in (M \boxtimes N)(X \times Y)$ for the image of a pair $(m, n) \in M(X) \times N(Y)$ under this map.

REMARK 3.10. The functor $M \boxtimes N: \overline{\mathcal{A}}_G \rightarrow \text{Sets}$ is defined as a left Kan extension, as explained in Appendix A. There is a subtlety that becomes important in the Tambara context. The box product would usually be defined as a Kan extension of a certain functor taking values in semigroups, but it is shown in the appendix that we can form the Kan extension in the category of sets instead and it automatically has the required semigroup structure (which is unusual for colimit constructions).

By unwrapping the usual construction of the Kan extension as a colimit over a comma category, we obtain the following description. Given a span $\alpha \in \mathcal{A}(X, Y)$, we will write f_α for the resulting map $M(X) \rightarrow M(Y)$.

- (a) Every element of $(M \boxtimes N)(X)$ has the form $f_\alpha(m \boxtimes n)$ for some span $\alpha \in \mathcal{A}_G(U \times V, X)$ and some elements $m \in M(U)$ and $n \in N(V)$.
- (b) Suppose we have spans $\lambda \in \mathcal{A}_G(U', U)$ and $\mu \in \mathcal{A}_G(V', V)$. Then for any $m' \in M(U')$ and $n' \in N(V')$ we have

$$f_{\alpha \circ (\lambda \times \mu)}(m' \boxtimes n') = f_\alpha(f_\lambda(m') \boxtimes f_\mu(n')).$$

- (c) All identities between elements of type (a) can be deduced by chaining together identities of type (b).

The above description is somewhat cumbersome; we now outline a slightly different description that is easier to use.

DEFINITION 3.11. Given $m \in M(X)$ and $n \in N(X)$ we put $m \otimes n = R_\delta(m \boxtimes n)$, where $\delta: X \rightarrow X \times X$ is the diagonal map.

The operation \boxtimes on element can be expressed in terms of the operation \otimes as follows:

LEMMA 3.12. *Let $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ be the projections. Then for $m \in M(X)$ and $n \in N(Y)$ we have $m \boxtimes n = (R_p(m)) \otimes (R_q(n))$.*

PROOF. By definition we have

$$(R_p(m)) \otimes (R_q(n)) = R_\delta(R_p(m) \boxtimes R_q(n)).$$

By the naturality properties of the \boxtimes pairing we also have

$$R_p(m) \boxtimes R_q(n) = R_{p \times q}(m \boxtimes n).$$

Note also that $(p \times q) \circ \delta: X \times Y \rightarrow X \times Y$ is just the identity, so $R_\delta R_{p \times q}$ is the identity on $(M \boxtimes N)(X \times Y)$. The claim follows by combining these facts. \square

We also have a Frobenius reciprocity formula as follows:

LEMMA 3.13. *Suppose we have a map $f: W \rightarrow X$ of finite G -sets.*

- (a) *For all $m' \in M(W)$ and $n \in N(X)$ we have $T_f(m' \otimes R_f(n)) = T_f(m') \otimes n \in (M \boxtimes N)(X)$.*
- (b) *For all $m \in M(X)$ and $n' \in N(W)$ we have $T_f(R_f(m) \otimes n') = m \otimes T_f(n') \in (M \boxtimes N)(X)$.*

PROOF. For part (a), it is straightforward to check that the square

$$\begin{array}{ccccc} W & \xrightarrow{\delta} & W \times W & \xrightarrow{1 \times f} & W \times X \\ f \downarrow & & & & \downarrow f \times 1 \\ X & \xrightarrow{\delta} & X \times X & & X \times X \end{array}$$

is cartesian, so we have

$$T_f R_\delta R_{1 \times f} = R_\delta T_{f \times 1}: (M \boxtimes N)(W \times X) \rightarrow (M \boxtimes N)(X).$$

We now apply this to the element $m' \boxtimes n \in (M \boxtimes N)(W \times X)$. By the naturality properties of the \boxtimes pairing we have $R_{1 \times f}(m' \boxtimes n) = m' \boxtimes R_f(n)$ and $T_{f \times 1}(m' \boxtimes n) = T_f(m') \boxtimes n$. We therefore get $T_f R_\delta(m' \boxtimes R_f(n)) = R_\delta(T_f(m') \boxtimes n)$, or in other words $T_f(m' \otimes R_f(n)) = T_f(m') \otimes n$. The proof for (b) is similar. \square

This can be sharpened as follows.

PROPOSITION 3.14. *Fix Mackey functors M and N and a finite G -set X . Let \mathcal{E} be the set of quadruples (U, p, m, n) where $p: U \rightarrow X$ is an equivariant map, and $(m, n) \in M(U) \times N(U)$. Let E be the smallest equivalence relation on \mathcal{E} such that*

- (a) *For all $U' \xrightarrow{r} U \xrightarrow{q} X$ and $(m', n) \in M(U') \times N(U)$ we have*

$$(U', qr, m', R_r(n))E(V, q, T_r(m'), n)$$
- (b) *For all $U' \xrightarrow{r} U \xrightarrow{q} X$ and $(m, n') \in M(U) \times N(U')$ we have*

$$(U, qr, R_r(m), n')E(V, q, m, T_r(n')).$$

Define $\epsilon: \mathcal{E} \rightarrow (M \boxtimes N)(X)$ by $\epsilon(U, p, m, n) = T_p(m \otimes n)$. Then ϵ induces a bijection $\mathcal{E}/E \rightarrow (M \boxtimes N)(X)$.

PROOF. First, it is clear from Lemma 3.13 that ϵ respects the equivalence relation E and so induces a map $\mathcal{E}/E \rightarrow (M \boxtimes N)(X)$. Next, let \mathcal{F} be the set of systems

$$y = (U, V, W, i, j, p, m, n)$$

where the first six entries give an equivariant span diagram $\alpha = (V \times W \xleftarrow{(i,j)} U \xrightarrow{p} X)$ and $(m, n) \in M(V) \times N(W)$. Let F be the equivalence relation on \mathcal{F} that is implicit in Remark 3.10. In more detail, suppose we have spans $\lambda = (V' \xleftarrow{q} A \xrightarrow{r} V)$ and $\mu = (W' \xleftarrow{s} B \xrightarrow{t} W)$ and that

$$\alpha \circ (\lambda \times \mu) = (V' \times W' \xleftarrow{(i',j')} U' \xrightarrow{p'} X).$$

Suppose we also have an element $(m', n') \in M(V') \times N(W')$, and thus elements

$$\begin{aligned} z &= (U, V, W, i, j, p, T_r R_q(m'), T_t R_s(n')) \\ z' &= (U', V', W', i', j', p', m', n') \end{aligned}$$

in \mathcal{F} ; then zFz' , and F is the smallest equivalence relation with this property.

We have a map $\xi: \mathcal{F} \rightarrow (M \boxtimes N)(X)$ given by

$$\xi(U, V, W, i, j, p, m, n) = T_p R_{(i,j)}(m \boxtimes n),$$

and Remark 3.10 tells us that this induces a bijection $\bar{\xi}: \mathcal{F}/F \rightarrow (M \boxtimes N)(X)$.

Now define $\phi: \mathcal{F} \rightarrow \mathcal{E}$ by

$$\phi(U, V, W, i, j, p, m, n) = (U, p, R_i(m), R_j(n)).$$

We can write (i, j) as $(i \times j) \circ \delta$ so

$$R_{(i,j)}(m \boxtimes n) = R_\delta R_{i \times j}(m \boxtimes n) = R_\delta(R_i(m) \boxtimes R_j(n)) = R_i(m) \otimes R_j(n).$$

Using this we see that $\epsilon\phi = \xi$ (and it follows that ϵ is surjective).

We next claim that ϕ induces a map $\mathcal{F}/F \rightarrow \mathcal{E}/E$. It will be enough to show that in the notation used to introduce the relation F , we have $\phi(z)E\phi(z')$. Here the definition of z' involves the composite $\alpha \circ (\lambda \times \mu)$. To analyse this, we first construct pullback squares as follows:

$$\begin{array}{ccccc} U' & \xrightarrow{t'} & \tilde{A} & \xrightarrow{i^*} & A \\ r' \downarrow & & \downarrow r^* & & \downarrow r \\ \tilde{B} & \xrightarrow{t^*} & U & \xrightarrow{i} & V \\ j^* \downarrow & & \downarrow j & & \\ B & \xrightarrow{t} & W & & \end{array}$$

We then put

$$\begin{aligned} k &= r^*t' = t^*r': U' \rightarrow U \\ p' &= pk = pr^*t' = pt^*r': U' \rightarrow X \\ i' &= qi^*t': U' \rightarrow V' \\ j' &= sj^*r': U' \rightarrow W'. \end{aligned}$$

This gives a diagram

$$\begin{array}{ccccc} & & & & U' \\ & & & & \swarrow \scriptstyle p' \\ & & & & A \times B \\ & & \swarrow \scriptstyle (i', j') & \swarrow \scriptstyle (i^*t', j^*r') & \downarrow \scriptstyle k \\ & & V' \times W' & & U \\ & \swarrow \scriptstyle q \times s & \swarrow \scriptstyle r \times t & \swarrow \scriptstyle (i, j) & \downarrow \scriptstyle p \\ & & V \times W & & X \end{array}$$

One can check that the middle square is a pullback, so the diagram exhibits the span

$$V' \times W' \xleftarrow{(i', j')} U' \xrightarrow{p'} X$$

as the composite $\alpha \circ (\lambda \times \mu)$, so our notation is consistent with that used previously. We now have

$$\phi(z') = (U', p', R_{i'}(m'), R_{j'}(n')) = (U', pr^*t', R_{qi^*t'}(m'), R_{sj^*r'}(n')) = (U', pr^*t', R_{t'}R_{qi^*}(m'), R_{sj^*r'}(n')).$$

Now put $y_1 = (\tilde{A}, pr^*, R_{qi^*}(m'), T_{t'}R_{sj^*r'}(n'))$. By clause (b) in the definition of the relation E , we have $\phi(z')Ey_1$. On the other hand, because the square defining U' is a pullback, we have $T_{t'}R_{r'} = R_{r^*}T_{t^*}$, so $y_1 = (\tilde{A}, pr^*, R_{qi^*}(m'), R_{r^*}T_{t^*}R_{sj^*}(n'))$. Using this description together with clause (a), we get y_1Ey_2 , where $y_2 = (U, p, T_{r^*}R_{qi^*}(m'), T_{t^*}R_{sj^*}(n'))$. Next, the square defining \tilde{A} is also a pullback, so $T_{r^*}R_{i^*} = R_iT_r$. Using this and the corresponding fact for \tilde{B} we obtain $y_2 = (U, p, R_iT_rR_q(m'), R_jT_tR_s(n'))$. Now inspection of the definitions shows that $y_2 = \phi(z)$, so $\phi(z) = \phi(z')$ as required. There is thus an induced operation $\bar{\phi}: \mathcal{F}/F \rightarrow \mathcal{E}/E$ with $\bar{\epsilon}\bar{\phi} = \bar{\xi}$ as claimed. As $\bar{\xi}$ is bijective we see that $\bar{\phi}$ is injective and $\bar{\epsilon}$ is surjective.

In the opposite direction, we define $\psi: \mathcal{E} \rightarrow \mathcal{F}$ by

$$\psi(U, p, m, n) = (U, U, U, 1, 1, p, m, n).$$

As

$$\epsilon(U, p, m, n) = T_p R_\delta(m \boxtimes n) = T_p R_{(1,1)}(m \boxtimes n)$$

we see that $\xi\psi = \epsilon$. It is also clear that $\phi\psi = 1: \mathcal{E} \rightarrow \mathcal{E}$. Now put $\bar{\psi} = \bar{\xi}^{-1}\bar{\epsilon}: \mathcal{E}/E \rightarrow \mathcal{F}/F$. Using the bijectivity of $\bar{\xi}$, we find that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{E} & \twoheadrightarrow & \mathcal{E}/E & \xrightarrow{\bar{\epsilon}} & (M \boxtimes N)(X) \\ \psi \downarrow & & \bar{\psi} \downarrow & & \parallel \\ \mathcal{F} & \twoheadrightarrow & \mathcal{F}/F & \xrightarrow[\simeq]{\bar{\xi}} & (M \boxtimes N)(X) \\ \phi \downarrow & & \bar{\phi} \downarrow & & \parallel \\ \mathcal{E} & \twoheadrightarrow & \mathcal{E}/E & \xrightarrow{\bar{\epsilon}} & (M \boxtimes N)(X) \end{array}$$

We now claim that for any element $y = (U, V, W, i, j, p, m, n)$ we have $yF\psi(\phi(y))$. Indeed, $\psi(\phi(y))$ is the system $(U, U, U, 1, 1, p, R_i(m), R_j(n))$. To show that this is F -related to y it will suffice to exhibit spans $\lambda \in \mathcal{A}_G(V, U)$ and $\mu \in \mathcal{A}_G(W, U)$ with

$$\begin{aligned} T_p \circ R_{(1,1)} \circ (\lambda \times \mu) &= T_p R_{(i,j)} \\ f_\lambda(m) &= R_i(m) \\ f_\mu(n) &= R_j(n). \end{aligned}$$

Clearly we can just take $\lambda = R_i$ and $\mu = R_j$. It follows that $\bar{\phi}$ and $\bar{\psi}$ are inverse to each other, as required. \square

We next investigate the relationship between Mackey_G and the more obvious category of semigroups with a G -action.

DEFINITION 3.15. We write Semigroups_G for the category of semigroups with an action of G . We note that for each $g \in G$ we have a G -set automorphism $\rho(g): G \rightarrow G$ given by $\rho(g)(x) = xg^{-1}$. Thus, for any Mackey functor M , we have semigroup maps $T_{\rho(g)} = R_{\rho(g)}^{-1}: M(G) \rightarrow M(G)$, which we can use to give a G -action on $M(G)$. We write ω for the resulting functor $\text{Mackey}_G \rightarrow \text{Semigroups}_G$.

For later use, it will be helpful to understand the left and right adjoints of ω . One possible approach is as follows: we let $\overline{\mathcal{FA}}_G$ denote the full subcategory of $\overline{\mathcal{A}}_G$ whose objects are the free finite G -sets. Note that any G -set that admits an equivariant map to a free G -set is itself automatically free. Thus, if X and Y are free and $(X \leftarrow A \rightarrow Y)$ is a span diagram then A is free as well. Moreover, we have $\text{Map}_G(G, G) = \text{Aut}_G(G, G) \simeq G$. Given these facts, one can check that Semigroups_G is equivalent to the category of product-preserving functors from $\overline{\mathcal{FA}}_G$ to sets. From this point of view (which is used in [4]), the functor ω becomes the restriction functor associated to the inclusion $\overline{\mathcal{FA}}_G \rightarrow \overline{\mathcal{A}}_G$ of coloured theories, and there is a general theory giving left adjoints for such functors. However, we prefer to give more direct and explicit constructions.

PROPOSITION 3.16. *If we define $cA(X) = \text{Map}_G(X, A)$ as in Example 3.1, we get a functor $c: \text{Semigroups}_G \rightarrow \text{Mackey}_G$ that is right adjoint to ω . Moreover, the counit $\epsilon: \omega cA \rightarrow A$ is an isomorphism, so c is a full and faithful embedding.*

PROOF. We define a semigroup isomorphism

$$\epsilon: \omega cA = cN(G) = \text{Map}_G(G, A) \rightarrow A$$

by $\epsilon(u) = u(1)$. In the opposite direction, we define $\eta: M \rightarrow c\omega(M)$ as follows. Consider a finite G -set X and a point $x \in X$. This gives an equivariant map $\hat{x}: G \rightarrow X$ by $\hat{x}(g) = gx$, and this in turn gives a map $R_{\hat{x}}: M(X) \rightarrow M(G) = \omega M$. We can thus define

$$\eta: M(X) \rightarrow \text{Map}(X, M(G))$$

by $\eta(m)(x) = R_{\hat{x}}(m)$. We have $\widehat{g\hat{x}} = \hat{x} \circ \rho(g)^{-1}$, and using this we see that η lands in $\text{Map}_G(X, M(G)) = c\omega(M)(X)$. Now consider a map $f: X \rightarrow Y$ of finite G -sets. We claim that the following diagram commutes:

$$\begin{array}{ccccc} M(X) & \xrightarrow{T_f} & M(Y) & \xrightarrow{R_f} & M(X) \\ \eta \downarrow & & \eta \downarrow & & \eta \downarrow \\ c\omega M(X) & \xrightarrow{T_f} & c\omega M(Y) & \xrightarrow{R_f} & c\omega M(X). \end{array}$$

For the right-hand square, we have

$$(\eta R_f(n))(x) = R_{\hat{x}} R_f(n) = R_{f\hat{x}}(n) = R_{\widehat{f(x)}}(n) = \eta(n)(f(x)) = (R_f \eta(n))(x).$$

Now consider the left hand square. For any point $y \in Y$, we have a pullback square

$$\begin{array}{ccc} G \times f^{-1}\{y\} & \xrightarrow{\text{proj}} & G \\ p \downarrow & & \downarrow \hat{y} \\ X & \xrightarrow{f} & Y, \end{array}$$

where $p(g, x) = gx = \hat{x}(g)$. It follows that for $m \in M(X)$ we have

$$R_{\hat{y}} T_f(m) = T_{\text{proj}} R_p(m) = \sum_{x \in f^{-1}\{y\}} R_{\hat{x}}(m),$$

and it follows that the left square commutes. This means that η is a morphism of Mackey functors. Next, we claim that the standard triangular diagrams

$$\begin{array}{ccc} \omega M & \xrightarrow{\omega \eta_M} & \omega c\omega M \\ & \searrow & \downarrow \epsilon_{\omega M} \\ & & \omega M \end{array} \quad \begin{array}{ccc} cN & \xrightarrow{\eta_{cN}} & c\omega cN \\ & \searrow & \downarrow c\epsilon_N \\ & & cN \end{array}$$

commute. This can be proved by unwinding the definitions, and is left to the reader. Thus, η and ϵ are the unit and counit of an adjunction, as claimed.

We have seen already that $\epsilon: \omega cA \rightarrow A$ is an isomorphism. Together with the adjunction this gives

$$\text{Mackey}_G(cA', cA) \simeq \text{Semigroups}_G(A', \omega cA) \simeq \text{Semigroups}_G(A', A).$$

It is standard and straightforward that the isomorphism arising here is inverse to the map induced by c , and we conclude that c is full and faithful. \square

DEFINITION 3.17. Let A be a semigroup with an action of G . For any finite G -set X we let G act on $\text{Map}(X, A)$ by $(gu)(x) = g.u(g^{-1}x)$ as usual, and using this we can construct a coinvariant quotient $\text{Map}(X, A)_G$ as in Definition 2.7. We define $dA(X) = \text{Map}(X, A)_G$. Given any equivariant span $\omega = (X \xleftarrow{p} A \xrightarrow{q} Y)$, we define $f_\omega: \text{Map}(X, A) \rightarrow \text{Map}(Y, A)$ by the usual rule

$$f_\omega(u)(y) = \sum_{q(a)=y} u(p(a)).$$

This commutes with the G -action and so induces a map $f_\omega: dA(X) \rightarrow dA(Y)$ of coinvariant quotients. It is clear that this makes dA into a Mackey functor.

REMARK 3.18. Let $\pi: X \rightarrow X/G$ denote the obvious quotient map, which gives a map $T_\pi: \text{Map}(X, A) \rightarrow \text{Map}(X/G, A)$. If G acts trivially on A then it is not hard to check that T_π induces an isomorphism $dA(X) = \text{Map}(X, A)_G \rightarrow \text{Map}(X/G, A)$. However, if G acts nontrivially on A then T_π does not interact well with the actions.

REMARK 3.19. One can also check that $dA(G/H) \simeq A_H$. More generally, any finite G -set X can be written in the form $X \simeq \coprod_{i=1}^r G/H_i$, and then we have $dA(X) \simeq \bigoplus_i A_{H_i}$.

PROPOSITION 3.20. *The functor $d: \text{Semigroups}_G \rightarrow \text{Mackey}_G$ is left adjoint to ω . Moreover, the unit map $\eta: A \rightarrow \omega dA$ is an isomorphism, so d is a full and faithful embedding.*

PROOF. First, we define maps

$$A \xrightarrow{\zeta} \text{Map}(G, A) \xrightarrow{\sigma} A$$

by $\sigma(u) = \sum_{x \in G} x.u(x^{-1})$ and

$$\zeta(a)(x) = \begin{cases} a & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\sigma\zeta = 1$. Next, recall that the standard action of G on $\text{Map}(X, A)$ is $(gu)(x) = g.u(g^{-1}x)$. Applying this in the case $X = G$ we get

$$\sigma(gu) = \sum_{x \in G} xg.u(g^{-1}x^{-1}) = \sum_{y \in G} y.u(y^{-1}) = \sigma(u).$$

It follows that there is a unique map

$$\bar{\sigma}: \text{Map}(G, A)_G = dA(G) = \omega d(A) \rightarrow A$$

satisfying $\bar{\sigma}\pi = \sigma$ (where $\pi: \text{Map}(G, A) \rightarrow \text{Map}(G, A)_G$ is the usual quotient map). We write

$$\eta = \pi\zeta: A \rightarrow \text{Map}(G, A)_G = dA(G) = \omega dA,$$

so $\bar{\sigma}\eta = 1$. Next, note that $g.\zeta(a)$ is the map $G \rightarrow A$ sending g to ga and all other points to zero. It follows that an arbitrary element $u \in \text{Map}(G, A)$ can be written as $u = \sum_g g.\zeta(g^{-1}u(g))$. As $\pi(g.v) = \pi(v)$ it follows that

$$\pi(u) = \sum_g \pi(\zeta(g^{-1}u(g))) = \eta \left(\sum_g g^{-1}u(g) \right) = \eta\sigma(u) = \eta\bar{\sigma}\pi(u).$$

This implies that $\eta\bar{\sigma} = 1$, so $\bar{\sigma}$ is an inverse for η and η is an isomorphism.

Next, for any $x \in X$ we have a map $\hat{x}: G \rightarrow X$ sending g to gx , so we can define

$$\epsilon_0: \text{Map}(X, M(G)) \rightarrow M(X)$$

by $\epsilon_0(u) = \sum_{x \in X} T_{\hat{x}}(u(x))$. After recalling that G acts on $M(G)$ via the maps $T_{\rho(g)}$ and that $\hat{x} \circ \rho(g) = \widehat{g^{-1}x}$ we find that $\epsilon_0(g.u) = \epsilon_0(u)$, so there is a unique homomorphism $\epsilon: d\omega M(X) = \text{Map}(X, M(G))_G \rightarrow M(X)$ with $\epsilon\pi = \epsilon_0$.

Now consider a map $f: X \rightarrow Y$ of finite G -sets. We claim that the following diagram commutes:

$$\begin{array}{ccccc} d\omega M(X) & \xrightarrow{T_f} & d\omega M(Y) & \xrightarrow{R_f} & d\omega M(X) \\ \epsilon \downarrow & & \epsilon \downarrow & & \downarrow \epsilon \\ M(X) & \xrightarrow{T_f} & M(Y) & \xrightarrow{R_f} & M(X). \end{array}$$

It will clearly suffice to show that the related diagram

$$\begin{array}{ccccc} \text{Map}(X, M(G)) & \xrightarrow{T_f} & \text{Map}(Y, M(G)) & \xrightarrow{R_f} & \text{Map}(X, M(G)) \\ \epsilon_0 \downarrow & & \epsilon_0 \downarrow & & \downarrow \epsilon_0 \\ M(X) & \xrightarrow{T_f} & M(Y) & \xrightarrow{R_f} & M(X) \end{array}$$

commutes. For the left square we note that $(T_f u)(y) = \sum_{f(x)=y} u(x)$ and $\widehat{f(x)} = f \circ \hat{x}$ so

$$\begin{aligned} \epsilon_0 T_f(u) &= \sum_{y \in Y} T_{\hat{y}}((T_f u)(y)) = \sum_{y \in Y} T_{\hat{y}} \left(\sum_{f(x)=y} u(x) \right) \\ &= \sum_{x \in X} T_{\widehat{f(x)}}(u(x)) = T_f \left(\sum_{x \in X} T_{\hat{x}}(u(x)) \right) = T_f \epsilon_0(u). \end{aligned}$$

For the right square, we recall that there is a cartesian square

$$\begin{array}{ccc} G \times f^{-1}\{y\} & \xrightarrow{\text{proj}} & G \\ p \downarrow & & \downarrow \hat{y} \\ X & \xrightarrow{f} & Y, \end{array}$$

giving $R_f T_{\hat{y}} = \sum_{f(x)=y} T_{\hat{x}}: M(G) \rightarrow M(X)$. Using this we get

$$R_f \epsilon_0(u) = \sum_{y \in Y} R_f T_{\hat{y}} u(y) = \sum_{y \in Y} \sum_{x \in f^{-1}\{y\}} T_{\hat{x}} u(y) = \sum_{x \in X} T_{\hat{x}} u(f(x)) = \epsilon_0 R_f(u).$$

This proves that ϵ is a morphism of Mackey functors. We again leave the reader to check the triangular diagrams

$$\begin{array}{ccc} dA & \xrightarrow{d\eta_A} & d\omega dA \\ & \searrow & \downarrow \epsilon_{dA} \\ & & dA \end{array} \quad \begin{array}{ccc} \omega M & \xrightarrow{\eta_{\omega M}} & \omega d\omega M \\ & \searrow & \downarrow \omega \epsilon_M \\ & & \omega M \end{array}$$

showing that we have an adjunction. We saw earlier that $\eta: A \rightarrow \omega dA$ is an isomorphism, so we have

$$\text{Mackey}_G(dA', dA) \simeq \text{Semigroups}_A(A', \omega dA) \simeq \text{Semigroups}_G(A', A),$$

showing that d is full and faithful. \square

4. Mackey functors for the group of order two

By way of example, we will study the case where $G = \{1, \chi\}$ with $\chi^2 = 1$. The simplest (and most standard) approach would be to use Proposition 3.5. Although we will mention this in passing, we will mostly focus on a different construction which generalises more easily to the nonadditive context of Tambara functors.

When G acts on a set X , we will usually write \bar{x} for $\chi.x$.

DEFINITION 4.1. A *Mackey pair* consists of semigroups A and B , together with an action of G on A by semigroup maps, and semigroup maps $A \xrightarrow{\text{trace}} B \xrightarrow{\text{res}} A$ satisfying

$$\begin{aligned} \text{trace}(0) &= 0 \\ \text{trace}(a_0 + a_1) &= \text{trace}(a_0) + \text{trace}(a_1) \\ \text{trace}(\bar{a}) &= \text{trace}(a) \\ \overline{\text{res}(b)} &= \text{res}(b) \\ \text{res}(\text{trace}(a)) &= a + \bar{a}. \end{aligned}$$

We write MP for the category of Mackey pairs.

CONSTRUCTION 4.2. Given a G -Mackey functor M , put $A = M(G/1) = M(G)$ and $B = M(G/G) = M(1)$ (so both of these are semigroups). Next, as G is commutative we see that $\chi: G \rightarrow G$ is a G -map, with $\chi = \chi^{-1}$, so $T_\chi = R_\chi: A \rightarrow A$. We use this map to define an action of G on A by semigroup maps.

The projection $\epsilon: G \rightarrow 1$ gives maps

$$\begin{aligned} \text{res} &= \epsilon^*: B \rightarrow A \\ \text{trace} &= T_\epsilon: A \rightarrow B. \end{aligned}$$

We will prove that this gives an equivalence between Mackey functors and Mackey pairs. The first thing to check is that we at least have a functor.

PROPOSITION 4.3. *The above construction gives a faithful functor $F: \text{Mackey}_G \rightarrow \text{MP}$.*

PROOF. We first need to check that the construction gives a Mackey pair. We have already seen that A and B have natural semigroup structures such that res and trace are semigroup homomorphisms. By applying M to the identity $\epsilon\chi = \epsilon$ we see that $\overline{\text{res}(b)} = \text{res}(b)$, so the image of res lies in the subsemigroup $A^G = \{a \in A \mid \bar{a} = a\}$. Similarly, we can apply T to the identity $\epsilon = \epsilon\chi$ to see that $\text{trace}(a) = \text{trace}(\bar{a})$. Next, it is straightforward to check that the diagram

$$\begin{array}{ccc} G \amalg G & \xrightarrow{1 \amalg \chi} & G \amalg G & \xrightarrow{s} & G \\ \downarrow s & & & & \downarrow \epsilon \\ G & \xrightarrow{\epsilon} & & & 1 \end{array}$$

is a pullback. By the Mackey property, it follows that

$$R_\epsilon T_\epsilon = T_s T_{1 \amalg \chi} R_s: M(G) \rightarrow M(G),$$

or in other words that $\text{res}(\text{trace}(a)) = a + \bar{a}$ for all $a \in A$. There is thus an evident way to make F into a functor $\text{Mackey}_G \rightarrow \text{MP}$.

To check that F is faithful, suppose we have morphisms $\phi, \psi: M \rightarrow M'$ with $M(\phi) = M(\psi)$, or equivalently $\phi_1 = \psi_1$ and $\phi_G = \psi_G$. Let X be an arbitrary finite G -set. Let n be the number of free orbits, and let m be the number of fixed points, so X is isomorphic to the disjoint union of n copies of G and m copies of 1. This gives commutative diagrams

$$\begin{array}{ccc} M(X) & \xrightarrow{\cong} & A^n \times B^m \\ \phi_X \downarrow & & \downarrow \phi_G^n \times \phi_1^m \\ M'(X) & \xrightarrow{\cong} & (A')^n \times (B')^m \end{array} \quad \begin{array}{ccc} M(X) & \xrightarrow{\cong} & A^n \times B^m \\ \psi_X \downarrow & & \downarrow \psi_G^n \times \psi_1^m \\ M'(X) & \xrightarrow{\cong} & (A')^n \times (B')^m. \end{array}$$

As $\phi_G = \psi_G$ and $\phi_1 = \psi_1$ we deduce that $\phi_X = \psi_X$, as required. \square

It is now not hard to describe the structure of the semigroups $\overline{\mathcal{A}}_G(G/H, G/K)$ for $H, K \in \{1, G\}$ and then use Proposition A.13 to see that F is an equivalence. However, we will construct the inverse functor in a more explicit way.

CONSTRUCTION 4.4. Let $P = (A, B)$ be a Mackey pair. For any finite G -set X , we put

$$EP(X) = \{(u, v) \in \text{Map}_G(X, A) \times \text{Map}(X^G, B) \mid u(x) = \text{res}(v(x)) \text{ for all } x \in X^G\},$$

so we have a cartesian square

$$\begin{array}{ccc} EP(X) & \longrightarrow & \text{Map}(X^G, B) \\ \downarrow & & \downarrow \text{res}_* \\ \text{Map}_G(X, A) & \xrightarrow{\rho} & \text{Map}(X^G, A^G). \end{array}$$

Now suppose we have a G -equivariant map $f: X \rightarrow Y$. We define $R_f = f^*: EP(Y) \rightarrow EP(X)$ by $R_f(m, n) = (m \circ f, n \circ f^G)$ (where $f^G: X^G \rightarrow Y^G$ is just the restriction of f).

After some further discussion and examples we will define maps $T_f: EP(X) \rightarrow EP(Y)$ which will make EP into a Mackey functor. First, however, we mention an approach that does *not* work. The constructions $X \mapsto \text{Map}_G(X, A)$ and $X \mapsto \text{Map}(X^G, A^G)$ and $X \mapsto \text{Map}(X^G, B)$ define Mackey functors as

in Examples 3.1 and 3.2. However, the restriction map $\rho: \text{Map}_G(X, A) \rightarrow \text{Map}(X^G, A^G)$ is not a morphism of Mackey functors, so we do not have a pullback square in Mackey_G as one might naively expect.

EXAMPLE 4.5. Suppose that G acts freely on X , so X is a disjoint union of n copies of G say. Then $X^G = \emptyset$, so $\text{Map}(X^G, A^G)$ and $\text{Map}(X^G, B)$ are singletons, so $EP(X) = \text{Map}_G(X, A) \simeq A^n$. More precisely, the defining pullback square for $EP(X)$ has the form

$$\begin{array}{ccc} EP(X) & \longrightarrow & 1 \\ \simeq \downarrow & & \downarrow \text{res}_* \\ \text{Map}_G(X, A) & \xrightarrow{\rho} & 1. \end{array}$$

In particular, for the case $X = G$ we have $EP(G) = A$.

EXAMPLE 4.6. Now suppose instead that G acts trivially on X and $|X| = k$. Then $X = X^G$ and $\text{Map}_G(X, A) = \text{Map}(X, A^G) = \text{Map}(X^G, A^G)$. It follows that $EP(X) = \text{Map}(X, B) = B^k$. More precisely, the defining pullback square for $EP(X)$ has the form

$$\begin{array}{ccc} EP(X) & \xrightarrow{\simeq} & \text{Map}(X, B) \\ \downarrow & & \downarrow \text{res}_* \\ \text{Map}(X, A^G) & \xrightarrow{=} & \text{Map}(X, A^G). \end{array}$$

In particular, for the case $X = 1$ we have $EP(1) = B$.

CONSTRUCTION 4.7. Suppose we have a G -equivariant map $f: X \rightarrow Y$. Note that $f^{-1}(Y^G)$ will contain X^G , and possibly some free orbits as well. If we choose a point in each such free orbit, we get a decomposition $f^{-1}(Y^G) = X^G \amalg X_1 \amalg \overline{X_1}$ say.

We define $T_f: EP(X) \rightarrow EP(Y)$ by $T_f(u, v) = (m, n)$, where

$$\begin{aligned} m(y) &= \sum_{x \in f^{-1}\{y\}} u(x) \\ n(y) &= \sum_{x_0 \in X^G \cap f^{-1}\{y\}} v(x_0) + \sum_{x_1 \in X_1 \cap f^{-1}\{y\}} \text{trace}(u(x_1)). \end{aligned}$$

To see that this does indeed define an element of $EP(Y)$, observe that when $x_1 \in X_1$ we have $\overline{u(x_1)} = u(\overline{x_1})$, so when $y \in Y^G$ we have

$$\begin{aligned} \text{res}(n(y)) &= \sum_{x_0 \in X^G \cap f^{-1}\{y\}} \text{res}(v(x_0)) + \sum_{x_1 \in X_1 \cap f^{-1}\{y\}} \text{res}(\text{trace}(u(x_1))) \\ &= \sum_{x_0 \in X^G \cap f^{-1}\{y\}} u(x_0) + \sum_{x_1 \in X_1 \cap f^{-1}\{y\}} (u(x_1) + \overline{u(x_1)}) \\ &= \sum_{x_0 \in X^G \cap f^{-1}\{y\}} u(x_0) + \sum_{x_1 \in X_1 \cap f^{-1}\{y\}} u(x_1) + \sum_{\overline{x_1} \in \overline{X_1} \cap f^{-1}\{y\}} u(\overline{x_1}) \\ &= m(y). \end{aligned}$$

Using the identity $\text{trace}(u(\overline{x_1})) = \text{trace}(\overline{u(x_1)}) = \text{trace}(u(x_1))$ we also see that the construction is independent of the choice of X_1 .

PROPOSITION 4.8. *The above definitions make EP into a Mackey functor.*

PROOF. Suppose we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. It is clear that $R_{gf} = R_f R_g$. We must check that we also have $T_{gf} = T_g T_f$. Consider a pair $(u, v) \in EP(X)$, so $T_f(u, v) = (m, n)$ and $T_g(m, n) = (p, q)$ say. For

the first component, we have

$$\begin{aligned}
m(y) &= \sum_{x \in f^{-1}\{y\}} u(x) \\
p(z) &= \sum_{y \in g^{-1}\{z\}} m(y) = \sum_{y \in g^{-1}\{z\}} \sum_{x \in f^{-1}\{y\}} u(x) \\
&= \sum_{x \in (gf)^{-1}\{z\}} u(x),
\end{aligned}$$

which is the same as the first component in $T_{gf}(u, v)$. The second component requires more work. Fix a point $z \in Z^G$. Put $Y_0 = Y^G \cap g^{-1}\{z\}$ and choose $Y_2 \subseteq Y$ such that $g^{-1}\{z\} = Y_0 \amalg Y_2 \amalg \overline{Y_2}$. Then put $Z_0 = X^G \cap f^{-1}(Y_0)$, and choose $X_1 \subseteq X$ such that $f^{-1}(Y_0) = X_0 \amalg X_1 \amalg \overline{X_1}$. Put $X_2 = f^{-1}(Y_2)$ and observe that $\overline{X_2} = f^{-1}(\overline{Y_2})$ and so

$$(gf)^{-1}\{z\} = X_0 \amalg X_1 \amalg X_2 \amalg \overline{X_1} \amalg \overline{X_2}.$$

$$\begin{array}{ccccc}
X_0 & & & & \\
& \searrow f & & & \\
X_1 \amalg \overline{X_1} & \xrightarrow{f} & Y_0 & & \\
& & \searrow g & & \\
X_2 \amalg \overline{X_2} & \xrightarrow{f} & Y_2 \amalg \overline{Y_2} & \xrightarrow{g} & \{z\}
\end{array}$$

Now, for $y_0 \in Y_0$ we have

$$n(y_0) = \sum_{x_0 \in X_0 \cap f^{-1}\{y_0\}} v(x_0) + \sum_{x_1 \in X_1 \cap f^{-1}\{y_0\}} \text{trace}(u(x_1)).$$

It follows that

$$\begin{aligned}
q(z) &= \sum_{y_0 \in Y_0} n(y_0) + \sum_{y_2 \in Y_2} \text{trace}(m(y_2)) \\
&= \sum_{x_0 \in X_0} v(x_0) + \sum_{x_1 \in X_1} \text{trace}(u(x_1)) + \sum_{x_2 \in X_2} \text{trace}(u(x_2)),
\end{aligned}$$

which is the same as the second component in $T_{gf}(u, v)(z)$, as required.

Now suppose we have a cartesian square

$$\begin{array}{ccc}
W & \xrightarrow{f} & X \\
g \downarrow & & \downarrow h \\
Y & \xrightarrow{k} & Z
\end{array}$$

We claim that $T_g R_f = R_k T_h: EP(X) \rightarrow EP(Y)$. To see this, consider a pair $(u, v) \in EP(X)$, so $T_h(u, v) = (p, q)$ say, and $R_k T_h(u, v) = (p \circ k, q \circ k^G)$. The cartesian property means that f induces a bijection $g^{-1}\{y\} \rightarrow h^{-1}\{k(y)\}$ for all $y \in Y$. This means that

$$p(k(y)) = \sum_{x \in h^{-1}\{k(y)\}} u(x) = \sum_{w \in g^{-1}\{y\}} u(f(w)),$$

so $p \circ k$ is also the first component in $T_g R_f(u, v)$. Next, consider a point $y \in Y^G$. Put $W_0 = W^G \cap g^{-1}\{y\}$ and choose $W_1 \subseteq W$ such that $g^{-1}\{y\} = W_0 \amalg W_1 \amalg \overline{W_1}$. Put $X_i = f(W_i)$; the cartesian property means

that f induces bijections $W_i \rightarrow X_i$, and that $h^{-1}\{k(y)\} = X_0 \amalg X_1 \amalg \overline{X_1}$. We thus have

$$\begin{aligned} q(k(y)) &= \sum_{x_0 \in X_0} v(x_0) + \sum_{x_1 \in X_1} \text{trace}(u(x_1)) \\ &= \sum_{w_0 \in W_0} v(f(w_0)) + \sum_{w_1 \in W_1} \text{trace}(u(f(w_1))), \end{aligned}$$

so $q \circ k$ is also the second component in $T_g R_f(u, v)$. This proves that $T_g R_f = R_k T_h$, so we have a Mackey functor as claimed. \square

THEOREM 4.9. *The functor $F: \text{Mackey}_G \rightarrow \text{MP}$ is an equivalence, with inverse given by E .*

PROOF. There is an evident way to define E on morphisms, giving a functor $E: \text{MP} \rightarrow \text{Mackey}_G$. It is clear from Examples 4.5 and 4.6 that $FEP = (EP(G), EP(1)) = (A, B) = P$, so $FE = 1$.

In the opposite direction, suppose we start with a Mackey functor M and put $A = M(G)$ and $B = M(1)$, so $FM = (A, B)$. Consider an arbitrary finite G -set X . For $x \in X$ we have a map $\hat{x}: G \rightarrow X$ given by $\hat{x}(1) = x$ and $\hat{x}(\chi) = \bar{x}$. This gives a map $R_{\hat{x}}: M(X) \rightarrow M(G) = A$, and by combining these we get a map $\alpha: M(X) \rightarrow \text{Map}(X, A)$. It is straightforward to check that this actually lands in $\text{Map}_G(X, A)$. On the other hand, for $x \in X^G$ we have a map $\check{x}: 1 \rightarrow X$ sending 0 to x , and thus $R_{\check{x}}: M(X) \rightarrow M(1) = B$. By combining these we get a map $\beta: M(X) \rightarrow \text{Map}(X^G, B)$. Using $\hat{x} = \check{x}\epsilon$ we see that the square

$$\begin{array}{ccc} M(X) & \xrightarrow{\beta} & \text{Map}(X^G, B) \\ \alpha \downarrow & & \downarrow \\ \text{Map}_G(X, A) & \longrightarrow & \text{Map}(X^G, A^G) \end{array}$$

commutes, so we have a natural map from $M(X)$ to the pullback $EP(X) = EFM(X)$. This is a morphism of Mackey functors, which is an isomorphism for $X = G$ or $X = 1$. As Mackey functors convert disjoint unions to products, we see that $M \simeq EFM$ as required. \square

5. The theory of semirings

We now repeat much of section 2, for semirings rather than semigroups. For us, a *semiring* will mean a semigroup with a second commutative, associative and unital binary operation (written as multiplication) that distributes over addition. In particular, we assume that $0a = 0$ for all a (which is the distributivity rule for the sum of no terms).

- EXAMPLE 5.1.** (a) Any commutative ring is of course also a semiring.
(b) \mathbb{N} is a semiring under the usual operations. We can also define polynomial semirings $\mathbb{N}[t]$ or $\mathbb{N}[t_1, \dots, t_r]$ in an evident way.
(c) For any semigroup A we have a semigroup semiring $\mathbb{N}[A]$. This is freely generated as a semigroup by elements $[a]$ for all $a \in A$, with the multiplication rule $[a][b] = [a + b]$. The polynomial semiring $\mathbb{N}[t_1, \dots, t_r]$ can be identified with $\mathbb{N}[\mathbb{N}^r]$.
(d) Let G be a finite group, and let $A(G)$ denote the set of isomorphism classes of finite G -sets. This is a semiring under the operations $[X] + [Y] = [X \amalg Y]$ and $[X][Y] = [X \times Y]$. We call this the *Burnside semiring* of G .
(e) Let G be a finite group, and let $R(G)$ denote the set of isomorphism classes of complex representations of G . This is a semiring under the operations $[V] + [W] = [V \oplus W]$ and $[V][W] = [V \otimes W]$. We call this the *representation semiring* of G .
(f) Consider a set E consisting of elements ϵ^n for all $n \in \mathbb{Z}$, together with two more elements 0 and α . We write 1 for ϵ^0 . We can make this a semiring by the following rules

+	α	ϵ^n	0
α	α	α	α
ϵ^m	α	$\epsilon^{\min(n,m)}$	ϵ^m
0	α	ϵ^n	0

\cdot	α	ϵ^n	0
α	α	α	0
ϵ^m	α	ϵ^{n+m}	0
0	0	0	0

Alternatively, we can regard α as $\epsilon^{-\infty}$ and 0 as ϵ^∞ and then we have $\epsilon^n + \epsilon^m = \epsilon^{\min(n,m)}$ for all n and m , and also $\epsilon^n \epsilon^m = \epsilon^{n+m}$ provided that we interpret $\infty + (-\infty)$ as ∞ . The real point about this example is as follows. Let C_* be a graded vector space over \mathbb{Q} , and put $B(C_*) = \{n \mid C_n \neq 0\}$ and $\beta(C_*) = \epsilon^{\inf(B(C_*))} \in E$ (with the convention $\inf(\emptyset) = \infty$). We then have $\beta(C_* \oplus D_*) = \beta(C_*) + \beta(D_*)$ and $\beta(C_* \otimes D_*) = \beta(C_*)\beta(D_*)$. For more general graded abelian groups we still have $\beta(C_* \oplus D_*) = \beta(C_*) + \beta(D_*)$ and $\beta(C_* \otimes D_*) \leq \beta(C_*)\beta(D_*)$, but the inequality can be strict if C_* and D_* have torsion at different primes.

Any finite set X gives a functor $R \mapsto R^X = \text{Map}(X, R)$ from semirings to sets. Let $\overline{\mathcal{U}}(X, Y)$ be the set of natural maps $R^X \rightarrow R^Y$.

REMARK 5.2. In contrast with the case of semigroups, there are many natural maps $R^X \rightarrow R^Y$ that are not semiring homomorphisms, for example the map $f: R^2 \rightarrow R^2$ given by $f(x, y) = (y^2 + 1, x^2 + 1)$.

DEFINITION 5.3. For any function $f: X \rightarrow Y$ we define three different operations as follows:

- (a) We have a map $T_f: R^X \rightarrow R^Y$ given by $T_f(r)(y) = \sum_{f(x)=y} r(x)$.
- (b) We have another map $N_f: R^X \rightarrow R^Y$ given by $N_f(r)(y) = \prod_{f(x)=y} r(x)$.
- (c) We have a map $R_f = f^*: R^Y \rightarrow R^X$ given by $R_f(r)(x) = r(f(x))$, or equivalently $f^*(r) = r \circ f$.

Note that R_f and T_f only use the additive semigroup structure, so their properties and interactions are the same as in Section 2. Similarly, R_f and N_f only use the multiplicative semigroup structure, so their properties and interactions are the same as in Section 2 up to a slight change of notation. In more detail:

- (a) For all $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $T_{gf} = T_g T_f$ and $N_{gf} = N_g N_f$ and $R_{gf} = R_f R_g$.
- (b) For any cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

we have $T_f R_g = R_k T_h$ and $N_f R_g = R_k N_h$.

- (c) As a special case of (b), for any bijection $f: X \rightarrow Y$ we have $N_f = T_f = R_f^{-1}$.

This just leaves the problem of understanding how maps of the form N_f interact with maps of the form T_g , which comes down to exploring the combinatorics of expanding products of sums as sums of products.

First, however, we will study the set $\overline{\mathcal{U}}(X, Y)$ of all natural operations. By the Yoneda Lemma, we have

$$\overline{\mathcal{U}}(X, Y) = \text{Map}(Y, \mathbb{N}[t_x \mid x \in X]) = \text{Semirings}(\mathbb{N}[t_y \mid y \in Y], \mathbb{N}[t_x \mid x \in X]).$$

Now suppose we have a decomposition $Y = \coprod_{a \in A} Y_a$. The inclusions $i_a: Y_a \rightarrow Y$ give morphisms $R_{i_a}: Y \rightarrow Y_a$ in $\overline{\mathcal{U}}$, which induce maps $(R_{i_a})_*: \overline{\mathcal{U}}(X, Y) \rightarrow \overline{\mathcal{U}}(X, Y_a)$, which we can combine to give a single natural map

$$\overline{\mathcal{U}}(X, Y) \rightarrow \prod_{a \in A} \overline{\mathcal{U}}(X, Y_a).$$

Using the description $\overline{\mathcal{U}}(X, Y) = \text{Map}(Y, \mathbb{N}[t_x \mid x \in X])$ we see that this map is a natural isomorphism. This means that Y is the categorical product in $\overline{\mathcal{U}}$ of the objects Y_a .

PROPOSITION 5.4. *The category of semirings is equivalent to the category of product-preserving functors from $\overline{\mathcal{U}}$ to the category of sets.*

PROOF. Essentially the same as for Proposition 2.1. □

The description $\overline{\mathcal{U}}(X, Y) = \text{Map}(Y, \mathbb{N}[t_x \mid x \in X])$ gives a canonical semiring structure on the set $\overline{\mathcal{U}}(X, Y)$. It is generated by elements t_x and e_y (corresponding to the natural maps $t_x(f)(z) = f(x)$ and $e_y(f)(z) = \delta_{yz}$) subject only to the relations $e_y e_z = \delta_{yz} e_z$ and $\sum_y e_y = 1$. This semiring structure behaves well with respect to functions $X \rightarrow X'$, but not with respect to arbitrary morphisms $X \rightarrow X'$ in $\overline{\mathcal{U}}$. Thus, we cannot say that Y is a semiring object in $\overline{\mathcal{U}}$.

The idea of Tambara theory is to introduce a bicategory \mathcal{U} whose 0-cells are finite sets, such that $\mathcal{U}(X, Y)$ is a groupoid with $\pi_0\mathcal{U}(X, Y) = \overline{\mathcal{U}}(X, Y)$.

Specifically, we take $\mathcal{U}(X, Y)$ to be the category of diagrams of the form

$$X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y,$$

where A and B are finite sets. We call these diagrams *bispans*. The morphisms are commutative diagrams

$$\begin{array}{ccccccc} X & \longleftarrow & A & \longrightarrow & B & \longrightarrow & Y \\ \parallel & & \downarrow \alpha & & \downarrow \beta & & \parallel \\ X & \longleftarrow & A' & \longrightarrow & B' & \longrightarrow & Y \end{array}$$

in which α and β are bijections. Given a bispan

$$\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y) \in \mathcal{U}(X, Y)$$

as above, we define $f_\omega = T_r N_q R_p: R^X \rightarrow R^Y$, or more explicitly

$$f_\omega(s)(y) = \sum_{b \in r^{-1}\{y\}} \prod_{a \in q^{-1}\{b\}} s(p(a)).$$

Alternatively, in terms of our generators t_x and e_y for the semiring $\overline{\mathcal{U}}(X, Y)$, we have

$$f_\omega = \sum_y e_y \sum_{b \in r^{-1}\{y\}} \prod_{a \in q^{-1}\{b\}} t_{p(a)} = \sum_{b \in B} e_{r(b)} \prod_{a \in q^{-1}\{b\}} t_{p(a)}.$$

One can check that this construction gives a bijection $\pi_0\mathcal{U}(X, Y) \rightarrow \overline{\mathcal{U}}(X, Y)$. Indeed, a typical monomial in $\overline{\mathcal{U}}(X, Y)$ has the form $e_y \prod_x t_x^{n(x)}$ for some $y \in Y$ and $n \in \mathbb{N}^X$. Thus a typical element of $\overline{\mathcal{U}}(X, Y)$ has the form

$$g = \sum_{(y,n)} m(y,n) e_y \prod_x t_x^{n(x)}$$

for some map $m: Y \times \mathbb{N}^X \rightarrow \mathbb{N}$ of finite support. Now put

$$A = \{(y, n, i, x, j) \in Y \times \mathbb{N}^X \times \mathbb{N} \times X \times \mathbb{N} \mid i < m(y, n), j < n(x)\}$$

$$B = \{(y, n, i) \in Y \times \mathbb{N}^X \times \mathbb{N} \mid i < m(y, n)\}$$

$$p(y, n, i, x, j) = x$$

$$q(y, n, i, x, j) = (y, n, i)$$

$$r(y, n, i) = y.$$

This gives a bispan $\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$ with $f_\omega = g$, and it is not hard to check that any other bispan with this property is isomorphic to ω .

In particular, if we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, the operation $N_g T_f: R^X \rightarrow R^Z$ must come from a bispan. We can construct one as follows.

DEFINITION 5.5. For any maps $X \xrightarrow{g} Y \xrightarrow{h} Z$ we define a bispan

$$\Delta(g, h) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$$

by

$$B = \{(z, s) \mid z \in Z, s: h^{-1}\{z\} \rightarrow X, gs = 1_{h^{-1}\{z\}}\}$$

$$A = Y \times_Z B = \{(y, s) \mid y \in Y, s: h^{-1}\{h(y)\} \rightarrow X, gs = 1_{h^{-1}\{h(y)\}}\}$$

$$p(y, s) = s(y)$$

$$q(y, s) = (h(y), s)$$

$$r(z, s) = z.$$

We call this the *distributor* for (g, h) .

PROPOSITION 5.6. *In the above context, we have $N_h T_g = f_{\Delta(g,h)} = T_r N_q R_p$.*

PROOF. First note that all the sets X, Y, A and B have compatible maps to Z , and everything happens independently over the different points of Z . Because of this, we can easily reduce to the special case where Z is a single point. In that context we just have $B = \{s: Y \rightarrow X \mid gs = 1_Y\}$ and $A = Y \times B$. Now put $X_y = g^{-1}\{y\} \subseteq X$, so the requirement $gs = 1_Y$ just means that $s(y) \in X_y$ for all y . We thus have $B = \prod_y X_y$ and $A = Y \times \prod_y X_y$. Now consider an element $u \in R^X$. The image $N_h T_g(u) \in R$ is given by $\prod_y \sum_{x \in X_y} u(x)$. If we expand this out in the obvious way we get $\sum_{s \in B} \prod_{y \in Y} u(s(y))$, and by unwinding the definitions we see that this is $T_r N_q R_p(u)$, as required. \square

For another perspective on the above construction, define $m: A \rightarrow Y$ by $m(y, s) = y$. We then have a diagram D as follows, in which the right hand square is cartesian.

$$\begin{array}{ccc} & A & \xrightarrow{q} B \\ & \downarrow m & \downarrow r \\ X & \xrightarrow{g} Y & \xrightarrow{h} Z. \end{array}$$

Let $\mathcal{D}(g, h)$ denote the category of all diagrams D' like

$$\begin{array}{ccc} & A' & \xrightarrow{q'} B' \\ & \downarrow m' & \downarrow r' \\ X & \xrightarrow{g} Y & \xrightarrow{h} Z, \end{array}$$

where the bottom row is the same as before, and the right hand square is cartesian. A morphism from D' to D'' consists of maps $A' \rightarrow A''$ and $B' \rightarrow B''$ with the evident commutativity properties. The following result gives a more abstract characterisation of D .

PROPOSITION 5.7. *The object D is terminal in $\mathcal{D}(g, h)$.*

PROOF. Consider another object D' as above. Let $b' \in B'$. Put $z = r'(b') \in Z$ and $Y_1 = h^{-1}\{z\} \subseteq Y$ and $A_1 = (q')^{-1}\{b'\} \subseteq A'$. As the square in D' is assumed to be cartesian, the map $m': A' \rightarrow Y$ must restrict to give a bijection $m_1: A_1 \rightarrow Y_1$. We thus have a map $s = p'm_1^{-1}: Y_1 \rightarrow X$. As D' commutes we have $gp' = m'$ and so $gs = 1_{Y_1}$, which means that $(z, s) \in B$. We can thus define $\beta: B' \rightarrow B$ by $\beta(b') = (z, s)$. Next, given $a' \in A'$ we can put $b' = q'(a') \in B'$ and then define z, Y_1, A_1, m_1 and s as before. We then have $a' \in A_1$ so $m_1(a')$ is defined and is an element of Y_1 . We thus have a point $\alpha(a') = (m_1(a'), z, s) \in A$. This construction gives a map $\alpha: A' \rightarrow A$, and it is straightforward to check the equations

$$m\alpha = m' \quad p\alpha = p' \quad q\alpha = \beta q' \quad r\beta = r'.$$

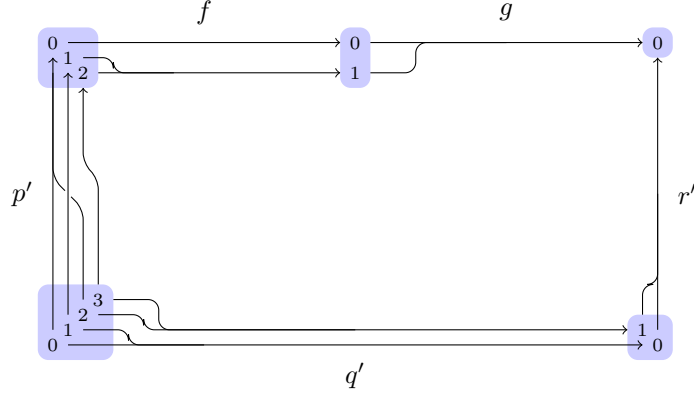
This means that the pair (α, β) gives a morphism $D' \rightarrow D$. Suppose we have another morphism, say (α^*, β^*) . Consider a point $b' \in B'$, and put $(z, s) = \beta^*(b') \in B$. As $r\beta^* = r'$ we see that $z = r'(b')$. Now consider a point $y \in h^{-1}\{z\}$, so $s(y) \in X$. By the pullback property for D' , there is a unique element $a' \in (q')^{-1}\{b'\}$ with $m'(a') = y$. The point $\alpha^*(a') \in A$ has $q(\alpha^*(a')) = \beta^*(q'(a')) = \beta^*(b') = (z, s)$ and $m(\alpha^*(a')) = m'(a') = y$ so we must have $\alpha^*(a') = (y, z, s)$. This means that $s(y) = p(y, z, s) = p(\alpha^*(a')) = p'(a') = p'm_1^{-1}(y)$. We now see that $s = p'm_1^{-1}$ as before, so $\beta^* = \beta$. Given this, it is also clear from the above equations that $\alpha^* = \alpha$. \square

It is instructive to recover the distributivity axiom $a_0(a_1 + a_2) = a_0a_1 + a_0a_2$ directly from the above construction. Let $S: \bar{U} \rightarrow \text{Sets}$ be a product-preserving functor, so we can identify $S(n) = S(\{0, 1, \dots, n-1\})$ with $S(1)^n$. The map $z: 0 \rightarrow 1$ gives elements $0 = T_z() \in S(1)$ and $1 = N_z() \in S(1)$, and the map $s: 2 \rightarrow 1$ gives binary operations $a + b = T_s(a, b)$ and $ab = N_s(a, b)$. Proposition 5.4 shows abstractly that this gives

a semiring structure. For a direct check of distributivity, we define maps

$$\begin{array}{ccccc} 3 & \xrightarrow{f} & 2 & \xrightarrow{g} & 1 \\ p' \uparrow & & & & \uparrow r' \\ 4 & \xrightarrow{q'} & & & 2 \end{array}$$

as follows:



The operation $N_g T_f$ is $(a_0, a_1, a_2) \mapsto a_0(a_1 + a_2)$, whereas the operation $T_{r'} N_{q'} R_{p'}$ is $(a_0, a_1, a_2) \mapsto a_0 a_1 + a_0 a_2$. Suppose instead that we define a bispan $\omega = \Delta(f, g) = (3 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} 1)$ as in Definition 5.5. From the definitions we have

$$B = \{(0, s) \mid s: 2 \rightarrow 3, \quad fs = 1_2\}.$$

If we define maps $s_0, s_1: 2 \rightarrow 3$ by

$$\begin{array}{ll} s_0(0) = 0 & s_0(1) = 1 \\ s_1(0) = 0 & s_1(1) = 2 \end{array}$$

then $B = \{(0, s_0), (0, s_1)\}$. We then have

$$A = 2 \times_1 B = \{(0, 0, s_0), (1, 0, s_0), (0, 0, s_1), (1, 0, s_1)\}.$$

It follows that we have a commutative diagram

$$\begin{array}{ccccccc} 3 & \xleftarrow{p'} & 4 & \xrightarrow{q'} & 2 & \xrightarrow{q'} & 1 \\ \parallel & & \downarrow \alpha & \simeq & \downarrow \beta & & \parallel \\ 3 & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & 1 \end{array}$$

where

$$\begin{array}{llll} \alpha(0) = (0, 0, s_0) & \alpha(1) = (1, 0, s_0) & \alpha(2) = (0, 0, s_1) & \alpha(3) = (1, 0, s_1) \\ \beta(0) = (0, s_0) & \beta(1) = (0, s_1) & & \end{array}$$

This gives $N_{r'} T_{q'} R_{p'} = N_r T_q R_p = N_g T_f$ as expected.

We now discuss composition of more general bispan.

DEFINITION 5.8. Consider a pair of bispan

$$\begin{array}{l} \omega_0 = (X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1) \in \mathcal{U}(X_0, X_1) \\ \omega_1 = (X_1 \xleftarrow{p_1} A_1 \xrightarrow{q_1} B_1 \xrightarrow{r_1} X_2) \in \mathcal{U}(X_1, X_2). \end{array}$$

We define a bispan

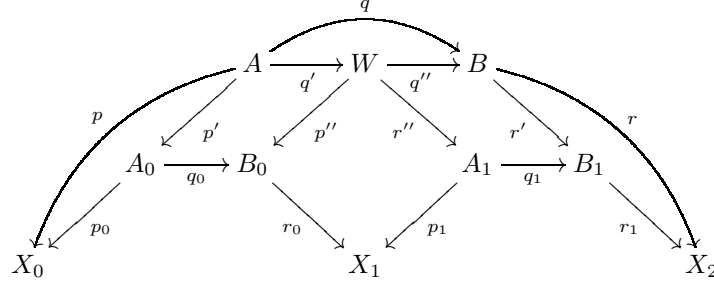
$$\omega_1 \circ \omega_0 = (X_0 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} X_2) \in \mathcal{U}(X_0, X_2)$$

as follows:

$$\begin{aligned}
A &= \{(a_0, a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0s = p_1, a_0 \in q_0^{-1}\{s(a_1)\}\} \\
B &= \{(b_1, s) \mid s: q_1^{-1}\{b_1\} \rightarrow B_0, r_0s = p_1\} \\
p(a_0, a_1, s) &= p_0(a_0) \\
q(a_0, a_1, s) &= (q_1(a_1), s) \\
r(b_1, s) &= r_1(b_1).
\end{aligned}$$

(Here it is implicit that $a_0 \in A_0$, $a_1 \in A_1$ and $b_1 \in B_1$.)

REMARK 5.9. It is sometimes useful to consider the enlarged diagram



where

$$\begin{aligned}
A &= \{(a_0, a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0s = p_1, a_0 \in q_0^{-1}\{s(a_1)\}\} \\
W &= \{(a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0s = p_1\} \\
B &= \{(b_1, s) \mid s: q_1^{-1}\{b_1\} \rightarrow B_0, r_0s = p_1\}
\end{aligned}$$

$$\begin{aligned}
p(a_0, a_1, s) &= p_0(a_0) & p'(a_0, a_1, s) &= a_0 & p''(a_1, s) &= s(a_1) \\
q(a_0, a_1, s) &= (q_1(a_1), s) & q'(a_0, a_1, s) &= (a_1, s) & q''(a_1, s) &= (q_1(a_1), s) \\
r(b_1, s) &= r_1(b_1) & r'(b_1, s) &= b_1 & r''(a_1, s) &= a_1
\end{aligned}$$

One can check that everything commutes and that the two rhombi are cartesian (but the middle square is not, in general).

PROPOSITION 5.10. *For any semiring R we have $f_{\omega_1 \circ \omega_0} = f_{\omega_1} f_{\omega_0}: R^{X_0} \rightarrow R^{X_2}$.*

PROOF. We first claim that in the expanded diagram we have

$$N_{q_1} R_{p_1} T_{r_0} = T_{r'} N_{q''} R_{p''}: R^{B_0} \rightarrow R^{B_1}.$$

Indeed, for any map $m: B_0 \rightarrow R$ and $b_1 \in B_1$, the left hand side gives

$$\prod_{a_1 \in q_1^{-1}\{b_1\}} \sum_{b_0 \in r_0^{-1}\{p_1(a_1)\}} m(b_0),$$

whereas the right hand side gives

$$\sum_{\substack{s: q_1^{-1}\{b_1\} \rightarrow B_0 \\ r_0s = p_1}} \prod_{a_1 \in q_1^{-1}\{b_1\}} m(s(a_1)).$$

To give a map $s: q_1^{-1}\{b_1\} \rightarrow B_0$ with $r_0s = p_1$ is the same as to pick out one summand in each of the factors on the left hand side. This means that the right hand side is just what we get by expanding out the left hand side, as required. We can now compose on the left with T_{r_1} and on the right with $N_{q_0} R_{p_0}$ to get

$$f_{\omega_1} f_{\omega_0} = T_{r_1} N_{q_1} R_{p_1} T_{r_0} N_{q_0} R_{p_0} = T_{r_1} T_{r'} N_{q''} R_{p''} N_{q_0} R_{p_0}: R^{X_0} \rightarrow R^{X_2}.$$

As the square (q', p', p'', q_0) is cartesian, we also have $R_{p''}N_{q_0} = N_{q'}R_{p'}$. Using this together with the functoriality of T , N and R we obtain

$$f_{\omega_1}f_{\omega_0} = T_{r_1}T_{r'}N_{q''}N_{q'}R_{p'}R_{p_0} = T_rN_qR_p = f_{\omega_1 \circ \omega_0}$$

as claimed. \square

Now suppose we have a third bispan ω_2 from X_2 to X_3 . It follows from the proposition that the bispans $\omega_2 \circ (\omega_1 \circ \omega_0)$ and $(\omega_2 \circ \omega_1) \circ \omega_0$ give the same natural semiring operation, so they are at least unnaturally isomorphic. It is reasonable to expect that there should be a natural isomorphism satisfying a pentagonal coherence identity. In order to prove this, we will introduce a single-step construction of multiple compositions. In more detail, consider a family of bispans

$$\omega_i = (X_i \xleftarrow{p_i} A_i \xrightarrow{q_i} B_i \xrightarrow{r_i} X_{i+1})$$

for $n \leq i < m$. We would like to build a composite bispan from X_n to X_m .

DEFINITION 5.11. First, for $n \leq i \leq j \leq m$ we put $A_i^j = \prod_{k=i}^{j-1} A_k$. In the case $i = j$ this should be interpreted as the one-point set: $A_i^i = \{1\}$. There are evident projections $\pi_i: A_i^j \rightarrow A_i$ and $\pi_{>i}: A_i^j \rightarrow A_{i+1}^j$.

CONSTRUCTION 5.12. Now consider a system of subsets $S_i \subseteq A_i^m$ (for $n \leq i \leq m$) together with maps $s_i: S_i \rightarrow B_{i-1}$ (for $n < i \leq m$). We say that the above data form a *term system* if

- (a) $S_m = \{1\}$
- (b) When $n \leq i < m$, the projection $\pi: A_i^m \rightarrow A_{i+1}^m$ has $\pi(S_i) \subseteq S_{i+1}$. Moreover, the square

$$\begin{array}{ccc} S_i & \xrightarrow{\pi_{>i}} & S_{i+1} \\ \pi_i \downarrow & & \downarrow s_{i+1} \\ A_i & \xrightarrow{q_i} & B_i \end{array}$$

is a pullback.

- (c) When $n < i < m$, the square

$$\begin{array}{ccc} B_{i-1} & \xleftarrow{s_i} & S_i \\ \tau_{i-1} \downarrow & & \downarrow \pi_i \\ X_i & \xleftarrow{p_i} & A_i \end{array}$$

commutes.

A *thread* for a term system as above is just an element of the set S_n . We write B_{nm} for the set of all term systems, and A_{nm} for the set of pairs consisting of a term system and a choice of thread. In other words, we have

$$A_{nm} = \{(\underline{a}, \underline{S}, \underline{s}) \mid (\underline{S}, \underline{s}) \in B_{nm}, \underline{a} \in S_n\}.$$

We define maps

$$X_n \xleftarrow{p_{nm}} A_{nm} \xrightarrow{q_{nm}} B_{nm} \xrightarrow{r_{nm}} X_m$$

as follows:

$$\begin{aligned} p_{nm}(\underline{a}, \underline{S}, \underline{s}) &= p_n(\pi_n(\underline{a})) \\ q_{nm}(\underline{a}, \underline{S}, \underline{s}) &= (\underline{S}, \underline{s}) \\ r_{nm}(\underline{S}, \underline{s}) &= r_{m-1}(s_m(1)). \end{aligned}$$

This gives a bispan ω_{nm} from X_n to X_m .

We now unpack this definition in the simplest cases.

PROPOSITION 5.13. *Suppose that $n = 0$ and $m = 1$, so our input data consists of a single bispan*

$$\omega_0 = (X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1).$$

Then the resulting bispan ω_{01} is naturally isomorphic to ω_0 .

PROOF. In this context a term system consists of sets S_0 and S_1 , together with a map $s_1: S_1 \rightarrow B_0$. Axiom (a) says that $S_1 = \{1\}$, so s_1 just gives a point $b_0 = s_1(1) \in B_0$. Axiom (b) therefore says that $S_0 = q_0^{-1}\{b_0\} \subseteq A_0$, and (c) is vacuously satisfied. We can therefore identify B_{01} with B_0 . More precisely, we have a bijection $\beta: B_0 \rightarrow B_{01}$ given by

$$\beta(b_0) = (q_0^{-1}\{b_0\}, \{1\}; 1 \mapsto b_0).$$

Similarly, there is a bijection $\alpha: A_0 \rightarrow A_{01}$ given by

$$\alpha(a_0) = (a_0; q_0^{-1}\{q_0(a_0)\}, \{1\}; 1 \mapsto q_0(a_0)).$$

It is clear from the definitions that the diagram

$$\begin{array}{ccccccc} X_0 & \xleftarrow{p_0} & A_0 & \xrightarrow{q_0} & B_0 & \xrightarrow{r_0} & X_1 \\ \parallel & & \downarrow \alpha & & \downarrow \beta & & \parallel \\ X_0 & \xleftarrow{p_{01}} & A_{01} & \xrightarrow{q_{01}} & B_{01} & \xrightarrow{r_{01}} & X_1 \end{array}$$

commutes, so we have the claimed isomorphism of bispan. \square

PROPOSITION 5.14. *Suppose that $n = 0$ and $m = 2$, so our input data consists of bispan*

$$\begin{aligned} \omega_0 &= (X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1) \in \mathcal{U}(X_0, X_1) \\ \omega_1 &= (X_1 \xleftarrow{p_1} A_1 \xrightarrow{q_1} B_1 \xrightarrow{r_1} X_2) \in \mathcal{U}(X_1, X_2). \end{aligned}$$

Then ω_{02} is naturally isomorphic to $\omega_1 \circ \omega_0$.

PROOF. We write $\omega_1 \circ \omega_0$ as

$$X_0 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} X_2$$

as in Definition 5.8.

In the construction of ω_{02} , a term system consists of sets $S_0 \subseteq A_0 \times A_1$ and $S_1 \subseteq A_1$ and $S_2 = \{1\}$, together with maps $s_1: S_1 \rightarrow B_0$ and $s_2: S_2 \rightarrow B_1$. These fit into a diagram

$$\begin{array}{ccccc} S_0 & \xrightarrow{\quad} & S_1 & \xrightarrow{\quad} & S_2 = \{1\} \\ & \searrow & & \searrow & \\ & & A_0 & \xrightarrow{\quad} & B_0 & & A_1 & \xrightarrow{\quad} & B_1 \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ X_0 & & & & X_1 & & & & X_2 \end{array}$$

in which the middle square is commutative and the other two quadrilaterals are cartesian. The map s_2 just gives a point $b_1 \in B_2$ and the cartesian property forces $S_1 = q_1^{-1}\{b_1\} \subseteq A_1$. Now s_1 is a map $q_1^{-1}\{b_1\} \rightarrow B_0$, and the map $r_0 s_1: q_1^{-1}\{b_1\} \rightarrow X_1$ must be the same as the restriction of p_1 to make the middle square commute. Once s_1 has been chosen, the cartesian property forces us to take

$$S_0 = \{(a_0, a_1) \in A_0 \times A_1 \mid a_1 \in q_1^{-1}\{b_1\}, a_0 \in q_0^{-1}\{s_1(a_1)\}\},$$

so there is no additional information in S_0 . This identifies the set B_{02} (of all term systems) with the set

$$B = \{(b_1, s) \mid s: q_1^{-1}\{b_1\} \rightarrow B_0, r_0 s = p_1\}.$$

Next, a point of A_{02} consists of a term system together with a thread (a_0, a_1) . Here b_1 must be equal to $q_1(a_1)$, so we need not record it as a separate piece of data. For (a_0, a_1) to be a thread we must have $a_0 \in q_0^{-1}\{s_1(a_1)\}$. This identifies A_{02} with the set

$$A = \{(a_0, a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0 s = p_1, a_0 \in q_0^{-1}\{s(a_1)\}\}.$$

We leave it to the reader to check that these identifications are compatible with the maps p, q and r , so they give an isomorphism of bispan. \square

Now suppose we have bispans $\omega_n, \dots, \omega_{m-1}$ as before, and an index k with $n < k < m$. By the procedure described above, we can construct a bispan ω_{nm} from X_n to X_m . Alternatively, we can construct ω_{nk} (from X_n to X_k) and ω_{km} (from X_k to X_m), and then construct $\omega_{km} \circ \omega_{nk}$.

PROPOSITION 5.15. *The bispan $\omega_{km} \circ \omega_{nk}$ is naturally isomorphic to ω_{nm} .*

PROOF. Let A'_{nm} and B'_{nm} denote the sets occurring in $\omega_{km} \circ \omega_{nk}$. A point of B'_{nm} consists of a term system $\beta = (S_k, \dots, S_m; s_{k+1}, \dots, s_m)$ together with a map $\sigma: q_{km}^{-1}\{\beta\} \rightarrow B_{nk}$ such that $r_{nk}\sigma = p_{kn}$. Here $q_{km}^{-1}\{\beta\}$ is easily identified with the set S_k , and after this identification p_{kn} becomes the map $p_k\pi_k: S_k \rightarrow X_k$. Now for $a \in S_k$ we have a point $\sigma(a) \in B_{nk}$. This will have the form

$$\sigma(a) = (\Sigma_n(a), \dots, \Sigma_k(a); \sigma_{n+1}(a), \dots, \sigma_k(a))$$

for some subsets $\Sigma_i(a) \subseteq A_i^k$ and some maps $\sigma_i(a): \Sigma_i(a) \rightarrow B_{i-1}$. Here $\Sigma_k(a) = \{1\}$ so we can define $s_k: S_k \rightarrow B_{k-1}$ by $s_k(a) = \sigma_k(a)(1)$. From the definitions we have $r_{nk}(\sigma(a)) = r_{k-1}(\sigma_k(a)(1)) = r_{k-1}(s_k(a))$, so the axiom $r_{nk}\sigma = p_{kn}$ becomes $r_{k-1}s_k = p_k\pi_k: S_k \rightarrow X_k$. Next, for $n \leq i < k$ we can split A_i^m as $A_i^k \times A_k^m$ and put

$$S_i = \{(a', a'') \in A_i^k \times A_k^m \mid a'' \in S_k \text{ and } a' \in \Sigma_i(a'')\}.$$

When $i > n$ we can also define $s_i: S_i \rightarrow B_{i-1}$ by

$$s_i(a', a'') = \sigma_i(a'')(a').$$

We claim that the list $\alpha = (S_n, \dots, S_m; s_{n+1}, \dots, s_m)$ is a term system for $(\omega_n, \dots, \omega_{m-1})$. Indeed, axiom (a) for α follows from axiom (a) for β . Similarly, axiom (b) for α follows from axiom (b) for β when $i \geq k$, and from axiom (b) for $\sigma(a'')$ when $i < k$. Axiom (c) works the same way except that the case where $i = k$ must be treated separately. That case says that the diagram

$$\begin{array}{ccc} B_{k-1} & \xleftarrow{s_k} & S_k \\ r_{k-1} \downarrow & & \downarrow \pi_k \\ X_k & \xleftarrow{p_k} & A_k \end{array}$$

must commute, and we observed above that this follows from our assumption that $r_{nk}\sigma = p_{kn}$. We thus have a point $\alpha \in B_{nm}$ as claimed. This was constructed in a natural way from β , so we can define a map $\phi: B'_{nm} \rightarrow B_{nm}$ by $\phi(\beta) = \alpha$. We leave it to the reader to check that this is a bijection.

Now consider instead the set A'_{nm} . The elements have the form $\alpha = (\alpha', \alpha'', \sigma)$, where

- α'' is an element of A_{km} , so it consists of a term system $\beta = (S_k, \dots, S_m; s_{k+1}, \dots, s_m)$ together with a thread $a'' \in S_k$;
- (β, σ) is a point in B'_{nm} , corresponding to a point

$$\phi(\beta, \sigma) = (S_n, \dots, S_m; s_{n+1}, \dots, s_m) \in B_{nm};$$

- α' is an element of A_{nk} with $q_{nk}(\alpha') = \sigma(\alpha'') \in B_{nk}$. This means that α' consists of a term system $(T_n, \dots, T_k; t_{n+1}, \dots, t_k)$ together with a thread $a' \in T_n$. The fact that $q_{nk}(\alpha') = \sigma(\alpha'')$ means that

$$T_i = \{u \in A_i^k \mid (u, a'') \in S_i\}$$

and $t_i(u) = s_i(u, a'')$. In particular, the sets T_i and maps t_j are determined by the other data that we have mentioned already. The thread $a' \in T_n$ can be combined with a'' to get a thread $a = (a', a'') \in S_n$ for $\phi(\beta, \sigma)$.

We now see that the construction

$$(\alpha', \alpha'', \sigma) \mapsto (\phi(\beta, \sigma), (a', a''))$$

gives a natural map $\psi: A'_{nm} \rightarrow A_{nm}$. We leave it to the reader to check that this is bijective and that the diagram

$$\begin{array}{ccccccc} X_n & \longleftarrow & A'_{nm} & \longrightarrow & B'_{nm} & \longrightarrow & X_m \\ \parallel & & \downarrow \psi & & \downarrow \phi & & \parallel \\ X_n & \longleftarrow & A_{nm} & \longrightarrow & B_{nm} & \longrightarrow & X_m \end{array}$$

commutes, so we have the required isomorphism of bispans. \square

Now suppose we have composable bispans ω_0 , ω_1 and ω_2 . We can use the proposition repeatedly to construct natural isomorphisms

$$\omega_2 \circ (\omega_1 \circ \omega_0) \simeq \omega_2 \circ \omega_{02} \simeq \omega_{03} \simeq \omega_{13} \circ \omega_0 \simeq (\omega_2 \circ \omega_1) \circ \omega_0.$$

By combining these, we obtain an associativity isomorphism

$$\alpha: \omega_2 \circ (\omega_1 \circ \omega_0) \rightarrow (\omega_2 \circ \omega_1) \circ \omega_0.$$

If we have another bispan ω_3 that can be composed with ω_2 , we get a pentagonal coherence diagram

$$\begin{array}{ccccc} \omega_3 \circ (\omega_2 \circ (\omega_1 \circ \omega_0)) & \xrightarrow{\alpha} & (\omega_3 \circ \omega_2) \circ (\omega_1 \circ \omega_0) & \xrightarrow{\alpha} & ((\omega_3 \circ \omega_2) \circ \omega_1) \circ \omega_0 \\ \downarrow 1 \circ \alpha & & & & \uparrow \alpha \circ 1 \\ \omega_3 \circ ((\omega_2 \circ \omega_1) \circ \omega_0) & \xrightarrow{\alpha} & (\omega_3 \circ (\omega_2 \circ \omega_1)) \circ \omega_0 & & \end{array}$$

By comparing everything with ω_{04} , one can check that this commutes. Similarly, one can check that bispans of the form

$$\eta_X = (X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{1} X)$$

act as identities for composition up to coherent natural isomorphism.

All this can probably be repackaged to give a quasicategory of bispans, which should be a quasicategorical Lawvere theory in the sense of Cranch [5]. However, we have not checked the details.

We have previously remarked that the morphism set $\overline{\mathcal{U}}(X, Y)$ has a canonical semiring structure. It is natural to expect that this should arise from a symmetric bimonoidal structure on the category $\mathcal{U}(X, Y)$. Such a structure can be given as follows: for bispans

$$\begin{aligned} \omega &= (X \xleftarrow{u} A \xrightarrow{v} B \xrightarrow{w} Y) \\ v &= (X \xleftarrow{p} C \xrightarrow{q} D \xrightarrow{r} Y) \end{aligned}$$

we put

$$\begin{aligned} \omega \oplus v &= (X \xleftarrow{(u,p)} A \amalg C \xrightarrow{v \amalg q} B \amalg D \xrightarrow{(w,r)} Y) \\ \omega \otimes v &= (X \leftarrow ((A \times_Y D) \amalg (B \times_Y C)) \rightarrow B \times_Y D \rightarrow Y). \end{aligned}$$

We will not make serious use of this so we leave further details to the reader.

DEFINITION 5.16. For any map $f: X \rightarrow Y$ of finite sets, we will write T_f , N_f and R_f for the evident bispans representing the corresponding semiring operations, namely

$$\begin{aligned} R_f &= (Y \xleftarrow{f} X \xrightarrow{1} X \xrightarrow{1} X) \in \mathcal{U}(Y, X) \\ N_f &= (X \xleftarrow{1} X \xrightarrow{f} Y \xrightarrow{1} Y) \in \mathcal{U}(X, Y) \\ T_f &= (X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{f} Y) \in \mathcal{U}(X, Y). \end{aligned}$$

PROPOSITION 5.17. (a) Any bispan $\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$ is naturally isomorphic to $T_r \circ N_q \circ R_p$.

(b) For all $X \xleftarrow{f} Y \xrightarrow{g} Z$ there are natural isomorphisms $T_{gf} \simeq T_g \circ T_f$ and $N_{gf} \simeq N_g \circ N_f$ and $R_{gf} \simeq R_f \circ R_g$. Moreover, T_1 , N_1 and R_1 are all equal to the identity bispan $\eta_X = (X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{1} X)$.

(c) There is also a natural isomorphism from $N_g \circ T_f$ to the distributor $\Delta(f, g)$.

(d) For any cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

there are natural isomorphisms $T_g \circ R_f \simeq R_k \circ T_h$ and $N_g \circ R_f \simeq R_k \circ N_h$.

(e) For any bijection $f: X \rightarrow Y$ there are natural isomorphisms $T_f \simeq N_f \simeq R_{f^{-1}}$.

PROOF. In each case we see by considering the induced semiring operations that there is at least an unnatural isomorphism between the relevant bispan, and we can produce a natural isomorphism by simply unwinding the definition of bispan composition. \square

PROPOSITION 5.18. For any finite set U , there is a product-preserving functor $P_U: \overline{\mathcal{U}} \rightarrow \overline{\mathcal{U}}$ given on objects by $P_U(X) = U \times X$, and on morphisms by

$$P_U[X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y] = [U \times X \xleftarrow{1 \times p} U \times A \xrightarrow{1 \times q} U \times B \xrightarrow{1 \times r} U \times Y].$$

PROOF. It is clear that this construction gives well-defined maps $\overline{\mathcal{U}}(X, Y) \rightarrow \overline{\mathcal{U}}(U \times X, U \times Y)$, and we just need to check that these are compatible with bispan composition. This follows directly from the definitions, as U is just carried through the various constructions and does not interact with the other data in any interesting way. As products in $\overline{\mathcal{U}}$ are given by disjoint unions, it is clear that they are preserved by P_U . \square

We now examine two special cases of the associativity isomorphism for bispan composition. These will turn out to be useful later.

PROPOSITION 5.19. For any maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ there is a functorially associated diagram

$$\begin{array}{ccccc} \tilde{A} & \xrightarrow{\alpha} & A & \xrightarrow{p} & W & \xrightarrow{f} & X \\ \downarrow i & & \downarrow q & & & & \downarrow g \\ \tilde{B} & \xrightarrow{\beta} & B & \xrightarrow{r} & & & Y \\ \downarrow j & & & & & & \downarrow h \\ \tilde{C} & \xrightarrow{k} & & & & & Z \end{array}$$

such that the square and two rectangles are cartesian, and

$$\begin{aligned} \Delta(f, g) &= (W \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y) \\ \Delta(f, hg) &= (W \xleftarrow{p\alpha} \tilde{A} \xrightarrow{j^i} \tilde{C} \xrightarrow{k} Z) \\ \Delta(r, h) &= (B \xleftarrow{\beta} \tilde{B} \xrightarrow{j} \tilde{C} \xrightarrow{k} Z). \end{aligned}$$

This will help us to analyse the operation $N_h N_g T_f$.

PROOF. We put

$$\begin{aligned}
A &= \{(x, s) \mid x \in X, s: g^{-1}\{g(x)\} \rightarrow W, fs = 1\} \\
B &= \{(y, s) \mid y \in X, s: g^{-1}\{y\} \rightarrow W, fs = 1\} \\
\tilde{A} &= \{(x, t) \mid x \in X, t: (hg)^{-1}\{hg(x)\} \rightarrow W, ft = 1\} \\
\tilde{B} &= \{(y, t) \mid y \in Y, t: (hg)^{-1}\{h(y)\} \rightarrow W, ft = 1\} \\
\tilde{C} &= \{(z, t) \mid z \in Z, t: (hg)^{-1}\{z\} \rightarrow W, ft = 1\} \\
\alpha(x, t) &= (x, t|_{g^{-1}\{g(x)\}}) \\
\beta(y, t) &= (y, t|_{g^{-1}\{y\}}) \\
i(x, t) &= (g(x), t) \\
j(y, t) &= (h(y), t) \\
k(z, t) &= z \\
p(x, s) &= s(x) \\
q(x, s) &= (g(x), s) \\
r(y, s) &= y.
\end{aligned}$$

It is straightforward to check that the diagram commutes, that the three regions are cartesian, and that $\Delta(f, g)$ and $\Delta(f, hg)$ are as described. The real point is to understand $\Delta(r, h)$. By definition, this has the form

$$B \xleftarrow{\beta^*} B^* \xrightarrow{j^*} C^* \xrightarrow{k^*} Z,$$

where

$$\begin{aligned}
C^* &= \{(z, u) \mid z \in Z, u: h^{-1}\{z\} \rightarrow B, ru = 1\} \\
B^* &= \{(y, u) \mid y \in Y, u: h^{-1}\{h(y)\} \rightarrow B, ru = 1\} \\
\beta^*(y, u) &= u(y) \\
j^*(y, u) &= (h(y), u) \\
k^*(z, u) &= z.
\end{aligned}$$

Consider a point $(z, u) \in C^*$. For $y \in h^{-1}\{z\}$ we then have $u(y) \in B$ with $r(u(y)) = y$. This means that $u(y)$ has the form (y, s_y) , where $s_y: g^{-1}\{y\} \rightarrow W$ with $fs_y = 1$. Now for any point $x \in (hg)^{-1}\{z\}$ we have $g(x) \in h^{-1}\{z\}$ and we can define $t(x) = s_{g(x)}(x) \in W$. This defines a map $t: (hg)^{-1}\{z\} \rightarrow W$ with $ft = 1$, so $(z, t) \in \tilde{C}$. This construction gives a map $\lambda: C^* \rightarrow \tilde{C}$, and it is not hard to check that this is bijective. From this we also get a canonical bijection $\mu: B^* = Y \times_Z C^* \rightarrow Y \times_Z \tilde{C} = \tilde{B}$. One can now check that the diagram

$$\begin{array}{ccccc}
B & \xleftarrow{\beta^*} & B^* & \xrightarrow{j^*} & C^* & \xrightarrow{k^*} & Z \\
\parallel & & \downarrow \mu \simeq & & \downarrow \lambda \simeq & & \parallel \\
B & \xleftarrow{\beta} & \tilde{B} & \xrightarrow{j} & \tilde{C} & \xrightarrow{k} & Z
\end{array}$$

commutes, giving the claimed identification of $\Delta(r, h)$. □

PROPOSITION 5.20. For any maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ there is a functorially associated diagram

$$\begin{array}{ccccc}
W & \xleftarrow{\tilde{p}} & \tilde{A} & \xrightarrow{\tilde{q}} & \tilde{B} & \xrightarrow{\tilde{r}} & Z \\
\parallel & & \downarrow i & & \downarrow k & & \parallel \\
W & \xleftarrow{p^*} & A^* & & & & \\
\downarrow f & & \downarrow j & & \downarrow & & \\
X & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Z
\end{array}$$

such that the bottom left square and the middle rectangle are cartesian, and

$$\begin{aligned}
\Delta(g, h) &= (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z) \\
\Delta(gf, h) &= (W \xleftarrow{\tilde{p}} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{\tilde{r}} Z) \\
\Delta(j, q) &= (A^* \xleftarrow{i} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{k} B).
\end{aligned}$$

This will help us to analyse the operation $N_h T_g T_f$.

PROOF. We put

$$\begin{aligned}
A &= \{(y, s) \mid y \in Y, s: h^{-1}\{h(y)\} \rightarrow X, gs = 1\} \\
B &= \{(z, s) \mid z \in Z, s: h^{-1}\{z\} \rightarrow X, gs = 1\} \\
\tilde{A} &= \{(y, t) \mid y \in Y, t: h^{-1}\{h(y)\} \rightarrow W, fgt = 1\} \\
\tilde{B} &= \{(z, t) \mid z \in Z, t: h^{-1}\{z\} \rightarrow W, fgt = 1\} \\
A^* &= \{(w, y, s) \mid w \in W, y \in Y, s: h^{-1}\{h(y)\} \rightarrow X, gs = 1, f(w) = s(y)\} \\
p(y, s) &= s(y) \\
q(y, s) &= (h(y), s) \\
r(y, s) &= y \\
\tilde{p}(y, t) &= t(y) \\
\tilde{q}(y, t) &= (h(y), t) \\
\tilde{r}(z, t) &= z \\
i(y, t) &= (t(y), y, gt) \\
j(w, y, s) &= (y, s) \\
k(z, t) &= (z, gt) \\
p^*(w, y, s) &= w.
\end{aligned}$$

It is straightforward to check that the diagram commutes, the bottom left square is cartesian, and that $\Delta(g, h)$ and $\Delta(gf, h)$ are as described. The real point is to understand $\Delta(j, q)$. By definition this has the form

$$A^* \xleftarrow{i'} A' \xrightarrow{q'} B' \xrightarrow{k'} B,$$

where

$$\begin{aligned}
B' &= \{(b, u) \mid b \in B, u: q^{-1}\{b\} \rightarrow A^*, ju = 1\} \\
A' &= A \times_B B'.
\end{aligned}$$

Consider a point $(b, u) \in B'$. The point $b \in B$ has the form (z, s) for some $z \in Z$ and $s: h^{-1}\{z\} \rightarrow X$ with $gs = 1$. The domain of u consists of the points (y, s) for $y \in h^{-1}\{z\}$, and the constraint $ju = 1$ means that $u(y, s) = (t(y), y, s)$ for some $t(y) \in W$. Moreover, we have $u(y, s) \in A^*$, and by inspecting the definition of A^* this gives $ft(y) = s(y)$. We thus have a map $t: h^{-1}\{z\} \rightarrow W$ with $ft = s$ and so $gft = gs = 1$. The construction $(b, u) \mapsto (z, t)$ now gives a map $B' \rightarrow \tilde{B}$, which is easily seen to be bijective. From this

we also get a canonical bijection $A' = A \times_B B' \rightarrow A \times_B \tilde{B} = \tilde{A}$. We leave it to the reader to check that these bijections are compatible with the maps in the distributor diagram, giving the claimed description of $\Delta(j, g)$. \square

PROPOSITION 5.21. *Suppose we have a diagram as follows in which both squares are pullbacks:*

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{\tilde{f}} & X & \xrightarrow{\tilde{g}} & Y \\ \downarrow i & & \downarrow j & & \downarrow k \\ W & \xrightarrow{f} & X & \xrightarrow{g} & Y. \end{array}$$

Then there is a commutative diagram

$$\begin{array}{ccccccc} \tilde{W} & \xleftarrow{\tilde{p}} & \tilde{A} & \xrightarrow{\tilde{q}} & \tilde{B} & \xrightarrow{\tilde{r}} & \tilde{Y} \\ \downarrow i & & \downarrow \alpha & & \downarrow \beta & & \downarrow k \\ W & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Y \end{array}$$

in which the top row is $\Delta(\tilde{f}, \tilde{g})$, the bottom row is $\Delta(f, g)$, and the middle and right squares are cartesian.

PROOF. We define the two rows by the standard distributor construction as in Definition 5.5. Consider a point $\tilde{b} = (\tilde{y}, \tilde{s}) \in \tilde{B}$. This means that $\tilde{s}: \tilde{g}^{-1}\{\tilde{y}\} \rightarrow \tilde{W}$ with $\tilde{f}\tilde{s} = 1$. As the square $(\tilde{X}, \tilde{Y}, X, Y)$ is cartesian, we see that j restricts to give a bijection $j_{\tilde{y}}: \tilde{g}^{-1}\{\tilde{y}\} \rightarrow g^{-1}\{k(\tilde{y})\}$. We can thus define $s: g^{-1}\{k(\tilde{y})\} \rightarrow W$ by $s = i\tilde{s}j_{\tilde{y}}^{-1}$. This satisfies $fs = fi\tilde{s}j_{\tilde{y}}^{-1} = j\tilde{f}\tilde{s}j_{\tilde{y}}^{-1} = jj_{\tilde{y}}^{-1} = 1$, so $(k(\tilde{y}), s) \in B$. We can thus define $\beta: \tilde{B} \rightarrow B$ by $\beta(\tilde{y}, \tilde{s}) = (k(\tilde{y}), s) = (k(\tilde{y}), i\tilde{s}j_{\tilde{y}}^{-1})$. Similarly, we define $\alpha: \tilde{A} \rightarrow A$ by $\alpha(\tilde{x}, \tilde{s}) = (j(\tilde{x}), i\tilde{s}j_{\tilde{y}}^{-1})$. It is easy to see that this gives a commutative diagram as claimed.

We now show that the right hand square is cartesian. Suppose we have a point $\tilde{y} \in \tilde{Y}$ and a point $b = (y, s) \in B$ with $r(b) = k(\tilde{y})$. By the definition of r we have $r(b) = y$, so the condition is that $k(\tilde{y}) = y$, so $j_{\tilde{y}}$ gives a bijection $\tilde{g}^{-1}\{\tilde{y}\} \rightarrow g^{-1}\{y\}$. Similarly, i restricts to give a bijection $i_{\tilde{y}}: (\tilde{g}\tilde{f})^{-1}\{\tilde{y}\} \rightarrow (gf)^{-1}\{y\}$. Next, as $(y, s) \in B$ we must have $fs = 1: g^{-1}\{y\} \rightarrow X$, which implies that s lands in $(gf)^{-1}\{y\}$. We can thus define $\tilde{s} = i_{\tilde{y}}^{-1}s j_{\tilde{y}}: \tilde{g}^{-1}\{\tilde{y}\} \rightarrow \tilde{W}$. We find that $\tilde{f}\tilde{s} = 1$, so we have an element $\tilde{b} = (\tilde{y}, \tilde{s}) \in \tilde{B}$. This is easily seen to be the unique element with $\tilde{r}(\tilde{b}) = \tilde{y}$ and $\beta(\tilde{b}) = b$, which gives the required pullback property. A similar argument shows that the middle square is also cartesian. \square

6. Tambara functors

Let \mathcal{U}_G be the bicategory that is the natural G -equivariant analogue of \mathcal{U} . Explicitly:

- The 0-cells are finite G -sets.
- The 1-cells from X to Y are diagrams $(X \leftarrow A \rightarrow B \rightarrow Y)$ of finite G -sets.
- The 2-cells from $(X \leftarrow A \rightarrow B \rightarrow Y)$ to $(X \leftarrow A' \rightarrow B' \rightarrow Y)$ consist of pairs (α, β) , where $\alpha: A \rightarrow A'$ and $\beta: B \rightarrow B'$ are isomorphisms of finite G -sets making the evident diagram commute.

Composition of 2-cells is the obvious thing, and composition of 1-cells is performed by calculating the composite in \mathcal{U} and giving it the evident G -action. In more detail, suppose we have equivariant bispan

$$\begin{aligned} \omega_0 &= (X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1) \in \mathcal{U}(X_0, X_1) \\ \omega_1 &= (X_1 \xleftarrow{p_1} A_1 \xrightarrow{q_1} B_1 \xrightarrow{r_1} X_2) \in \mathcal{U}(X_1, X_2). \end{aligned}$$

As before, we define a bispan

$$\omega_1 \circ \omega_0 = (X_0 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} X_2) \in \mathcal{U}(X_0, X_2)$$

as follows:

$$\begin{aligned}
A &= \{(a_0, a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0s = p_1, a_0 \in q_0^{-1}\{s(a_1)\}\} \\
B &= \{(b_1, s) \mid s: q_1^{-1}\{b_1\} \rightarrow B_0, r_0s = p_1\} \\
p(a_0, a_1, s) &= p_0(a_0) \\
q(a_0, a_1, s) &= (q_1(a_1), s) \\
r(b_1, s) &= r_1(b_1).
\end{aligned}$$

Note that in the definition of A , the map s is not required to have any kind of equivariance. We let G act on A by the rule

$$g \cdot (a_0, a_1, s) = (ga_0, ga_1, gsg^{-1}),$$

where gs denotes the composite

$$q_1^{-1}\{q_1(ga_1)\} \xrightarrow{g^{-1}} q_1^{-1}\{q_1(a_1)\} \xrightarrow{s} B_0 \xrightarrow{g} B_0.$$

We also let G act on B by a similar rule, and it is easy to check that the maps p , q and r respect these actions, so $\omega_1 \circ \omega_0$ is an equivariant bispan.

If $f: X \rightarrow Y$ is a map of finite G -sets, then the bispans T_f , N_f and R_f all have natural G -actions, so they can be regarded as 1-cells in \mathcal{U}_G . As the isomorphisms in Proposition 5.17 are natural, they are automatically equivariant. For ease of reference elsewhere, we record this formally:

- PROPOSITION 6.1. (a) Any G -equivariant bispan $\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$ is naturally isomorphic to $T_r \circ N_q \circ R_p$.
- (b) For all finite G -sets X , Y and Z , and all G -maps $X \xleftarrow{f} Y \xrightarrow{g} Z$, there are natural equivariant isomorphisms $T_{gf} \simeq T_g \circ T_f$ and $N_{gf} \simeq N_g \circ N_f$ and $R_{gf} \simeq R_f \circ R_g$. Moreover, T_1 , N_1 and R_1 are all equal to the identity bispan $\eta_X = (X \xleftarrow{1} X \xrightarrow{1} X \xrightarrow{1} X)$.
- (c) There is also a natural equivariant isomorphism from $N_g \circ T_f$ to the distributor $\Delta(f, g)$.
- (d) For any cartesian square

$$\begin{array}{ccc}
W & \xrightarrow{f} & X \\
g \downarrow & & \downarrow h \\
Y & \xrightarrow{k} & Z
\end{array}$$

(of finite G -sets and equivariant maps) there are natural equivariant isomorphisms $T_g \circ R_f \simeq R_k \circ T_h$ and $N_g \circ R_f = R_k \circ N_h$.

- (e) For any equivariant bijection $f: X \rightarrow Y$ there are natural equivariant isomorphisms $T_f \simeq N_f \simeq R_{f^{-1}}$.

Now let $\overline{\mathcal{U}}_G$ be the category whose objects are finite G -sets, and whose morphisms from X to Y are isomorphism classes of 1-cells in $\mathcal{U}_G(X, Y)$. One checks that $X \amalg Y$ is a categorical product of X and Y in $\overline{\mathcal{U}}_G$.

The key definition, taken from [22], is as follows.

DEFINITION 6.2. A *Tambara functor* for G is a product-preserving functor from $\overline{\mathcal{U}}_G$ to the category of sets. We write Tambara_G for the category of Tambara functors.

EXAMPLE 6.3. Let R be a semiring with an action of G , and put $cR(X) = \text{Map}_G(X, R)$ as before. For any bispan $\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$ we define $f_\omega: cR(X) \rightarrow cR(Y)$ by

$$f_\omega(s)(y) = \sum_{r(b)=y} \prod_{q(a)=b} s(p(a)).$$

This makes cR into a Tambara functor. Using the same unit and counit maps as in Proposition 3.16, we see that $c: \text{Semirings}_G \rightarrow \text{Tambara}_G$ is right adjoint to the functor $\omega: \text{Tambara}_G \rightarrow \text{Semirings}_G$ given by $\omega(S) = S(G)$.

EXAMPLE 6.4. Let $A(X)$ denote the set of isomorphism classes of finite G -sets over X , as in Example 3.3. We claim that this is the same as $\overline{\mathcal{U}}_G(\emptyset, X)$. Indeed, in any bispan $(\emptyset \xleftarrow{i} P \xrightarrow{j} Q \xrightarrow{k} X)$ the set P admits a map to \emptyset , which is only possible if $P = \emptyset$, in which case there is only one possibility for j , so everything is determined by $(Q \xrightarrow{k} X)$. The claim is clear from this. This means that A can be regarded as a functor, represented by \emptyset . One can check that the T , N and R operations can be described as follows.

(a) For $f: X \rightarrow Y$ and $[Q \xrightarrow{k} X] \in A(X)$ we have

$$T_f[Q \xrightarrow{k} X] = [Q \xrightarrow{fk} Y].$$

Next, if we put

$$S = \{(y, s) \mid y \in Y, s: f^{-1}\{y\} \rightarrow Q, ks = 1\}$$

and define $m: S \rightarrow Y$ by $m(y, s) = y$, then

$$N_f[Q \xrightarrow{k} X] = [S \xrightarrow{m} Y].$$

(b) For $e: W \rightarrow X$ we have

$$R_e[Q \xrightarrow{k} X] = [e^*Q \xrightarrow{\bar{k}} W],$$

where $e^*Q = \{(w, q) \in W \times Q \mid e(w) = k(q)\}$ and \bar{k} is the obvious projection.

In slightly different notation, if we write Q_x for the fibre $k^{-1}\{x\}$ and so on, we get

$$T_f[Q \xrightarrow{k} X]_y = \prod_{f(x)=y} Q_x$$

$$N_f[Q \xrightarrow{k} X]_y = \prod_{f(x)=y} Q_x$$

$$R_e[Q \xrightarrow{k} X]_w = Q_{e(w)}.$$

This is clearly analogous to Definition 5.3. We also observe that R_e comes from an evident functor

$$e^*: \{\text{finite } G\text{-sets over } X\} \rightarrow \{\text{finite } G\text{-sets over } W\},$$

and that T_e and N_e come from the left and right adjoints to this functor.

EXAMPLE 6.5. Let $\mathcal{R}(X)$ denote the category of G -equivariant complex vector bundles over X , and let $R(X)$ denote the set of isomorphism classes in $\mathcal{R}(X)$. (As X is just a finite G -set, the study of these vector bundles involves no topology or analysis, just combinatorics and representation theory.) For any G -map $f: X \rightarrow Y$ we have functors $T_f, N_f: \mathcal{R}(X) \rightarrow \mathcal{R}(Y)$ and $R_f: \mathcal{R}(Y) \rightarrow \mathcal{R}(X)$ given by

$$T_f(V)_y = \bigoplus_{f(x)=y} V_x$$

$$N_f(V)_y = \bigotimes_{f(x)=y} V_x$$

$$R_f(W)_x = W_{f(x)}.$$

One can check that this gives a Tambara functor.

We now want to make some remarks about comparison between different groups, so we will write $\mathcal{R}_G(X)$ rather than $\mathcal{R}(X)$. It is useful to note that $\mathcal{R}_G(G/H)$ is equivalent to the category \mathcal{R}_H of representations of H . Indeed, if V is an equivariant vector bundle over G/H then the fibre V_H at the identity coset has an action of H . On the other hand, if W is a representation of H then the set $G \times_H W$ is the total space of an equivariant vector bundle over G/H . These constructions are easily seen to be inverse to each other. We deduce that $\mathcal{R}_G(G/H)$ can be identified with the representation semiring R_H .

If $K \leq H \leq G$ then we have an obvious map $f: G/K \rightarrow G/H$, which gives maps

$$T_f: R_K \rightarrow R_H$$

$$N_f: R_K \rightarrow R_H$$

$$R_f: R_H \rightarrow R_K.$$

These are usually called *induction*, *tensor induction* and *restriction*, respectively.

EXAMPLE 6.6. In Section 13 we will show (following [22, Section 6]) that we can adjoin additive inverses to any semiring-valued Tambara functor, giving a ring-valued Tambara functor. We call this process *additive completion*.

EXAMPLE 6.7. Let E be a G -equivariant spectrum in the sense of stable homotopy theory. We remarked previously that the assignment $\pi_0^G(E)(X) = [\Sigma_G^\infty(X_+), E]^G$ gives a Mackey functor. We will show in Section 8 that when E has a strictly commutative product structure, then $\pi_0^G(E)$ is actually a Tambara functor. (This was also proved in [3], but we will explain an alternative approach that avoids many homotopical technicalities.) In this context we always have additive inverses, so $\pi_0^G(E)(X)$ is a ring rather than just a semiring. For the sphere spectrum it works out that $\pi_0^G(S)$ is the additive completion of the Burnside semiring Tambara functor as in Example 6.4. Similarly, for the complex K -theory spectrum KU it can be shown that $\pi_0^G(KU)$ is the additive completion of the representation semiring Tambara functor as in Example 6.5. Moreover, for any commutative ring A with an action of G one can define an equivariant Eilenberg-MacLane spectrum A with a strictly commutative product such that $\pi_0^G(HA)(X) = \text{Map}_G(X, A)$ as in Example 6.3. However, we will prove any of these identifications in the present memoir.

We now prove a small lemma which will be useful later, and which is a good exercise in handling the definitions.

LEMMA 6.8. Consider a map $g: X \rightarrow Y$ of finite G -sets, and note that this gives an equivariant splitting $Y = g(X) \amalg g(X)^c$ and thus $S(Y) = S(g(X)) \times S(g(X)^c)$ for any Tambara functor S . With respect to this decomposition we have $N_g(0) = (0, 1)$.

PROOF. Let f be the map $\emptyset \rightarrow X$. Recall that $S(\emptyset)$ consists of a single element which we can call a , and by definition the element $0 \in S(X)$ is $T_f(a)$, so $N_g(0) = N_g T_f(a)$. The distributor $\Delta(f, g)$ is easily identified as the diagram $(\emptyset \xleftarrow{1} \emptyset \xrightarrow{g} f(X)^c \xrightarrow{r} Y)$, where r is just the inclusion. This gives $N_g(0) = T_r N_q R_1(a) = T_r N_q(a) = T_r(1)$, and T_r is just the inclusion of the summand $S(f(X)^c)$ in $S(Y)$ so the claim follows. \square

PROPOSITION 6.9. The category $\overline{\mathcal{U}}_G$ is generated by the morphisms T_f , N_f and R_f , subject only to the following relations:

- (a) For any maps $X \xleftarrow{f} Y \xrightarrow{g} Z$ we have $T_{gf} = T_g T_f$ and $N_{gf} = N_g N_f$ and $R_{gf} = R_f R_g$. Moreover, T_1 , N_1 and R_1 are all equal to the identity.
- (b) If $\Delta(f, g) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$ then $N_g T_f = T_r N_q R_p$.
- (c) For any cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

we have $T_g R_f = R_k T_h$ and $N_g R_f = R_k N_h$.

PROOF. Let \mathcal{U}'_G be the category with the indicated generators and relations. In more detail, the objects are the finite G -sets, and the morphisms are the composable strings of T 's, N 's and R 's, modulo the smallest equivalence relation necessary to ensure that the given relations are satisfied and composition remains associative and unital. Proposition 6.1 tells us that the stated relations do in fact hold in $\overline{\mathcal{U}}_G$, so we have a functor $\pi: \mathcal{U}'_G \rightarrow \overline{\mathcal{U}}_G$ that is the identity on objects. Every morphism in $\overline{\mathcal{U}}_G(X, Y)$ can be written as $T_r N_q R_p$ for some p , q and r , so π is full. The real issue is to prove that π is faithful.

We first claim that every morphism in \mathcal{U}'_G can be expressed as $T_r N_q R_p$ for some p , q and r . It will clearly suffice to show that the class of morphisms of this form is closed under composition. The argument

can be presented schematically as follows:

$$\begin{aligned}
(TNR)(TNR) &= TN(RT)NR \stackrel{(c)}{=} TN(TR)NR \\
&= TNT(RN)R \stackrel{(c)}{=} TNT(NR)R \\
&= T(NT)NRR \stackrel{(b)}{=} T(TNR)NRR \\
&= TTN(RN)RR \stackrel{(c)}{=} TTN(NR)RR \\
&= (TT)(NN)(RRR) \stackrel{(a)}{=} TNR.
\end{aligned}$$

On the first line, we have a subword of the form $R_k T_h$, and relation (c) allows us to rewrite this in the form $T_g R_f$ for some other maps f and g . The other lines can be interpreted in a similar way.

Now suppose we have two morphisms $\xi, \xi' \in \mathcal{U}'_G(X, Y)$ with $\pi(\xi) = \pi(\xi')$. By the previous paragraph, we can choose bispans

$$\begin{aligned}
\omega &= (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y) \\
\omega' &= (X \xleftarrow{p'} A' \xrightarrow{q'} B' \xrightarrow{r'} Y)
\end{aligned}$$

such that $\xi = T_r N_q R_p$ and $\xi' = T_{r'} N_{q'} R_{p'}$. Now $\pi(\xi) = [\omega]$ and $\pi(\xi') = [\omega']$ so the assumption $\pi(\xi) = \pi(\xi')$ means that ω and ω' are isomorphic. Thus, there are equivariant bijections $\alpha: A \rightarrow A'$ and $\beta: B \rightarrow B'$ such that the diagram

$$\begin{array}{ccccc}
X & \longleftarrow & A & \longrightarrow & B & \longrightarrow & Y \\
\parallel & & \alpha \downarrow & & \beta \downarrow & & \parallel \\
X & \longleftarrow & A' & \longrightarrow & B' & \longrightarrow & Y
\end{array}$$

commutes. We can now apply (c) to the cartesian squares

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\alpha \downarrow & & \downarrow 1 \\
A' & \xrightarrow{1} & A'
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{\beta} & B' \\
\beta \downarrow & & \downarrow 1 \\
B' & \xrightarrow{1} & B'
\end{array}$$

to see that $T_\alpha = N_\alpha = R_\alpha^{-1}$ and $T_\beta = N_\beta = R_\beta^{-1}$. It follows easily from this that $T_r N_q R_p = T_{r'} N_{q'} R_{p'}$ in \mathcal{U}'_G , so $\xi = \xi'$ as required. \square

PROPOSITION 6.10. *For any finite set G -set U , there is a functor $P_U: \overline{\mathcal{U}}_G \rightarrow \overline{\mathcal{U}}_G$ given on objects by $P_U(X) = U \times X$, and on morphisms by*

$$P_U[X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y] = [U \times X \xleftarrow{1 \times p} U \times A \xrightarrow{1 \times q} U \times B \xrightarrow{1 \times r} U \times Y].$$

PROOF. Let G act in the obvious way on all ingredients in Proposition 5.18. \square

PROPOSITION 6.11. *Let S be a Tambara functor. Then each set $S(X)$ has a canonical structure as a semiring. Moreover, for every map $f: X \rightarrow Y$:*

- (a) *The map $R_f: S(Y) \rightarrow S(X)$ is a semiring homomorphism (which we use to regard $S(X)$ as an $S(Y)$ -module).*
- (b) *The map $T_f: S(X) \rightarrow S(Y)$ is a homomorphism of $S(Y)$ -modules (and in particular, respects addition).*
- (c) *The map $N_f: S(X) \rightarrow S(Y)$ sends 1 to 1 and respects multiplication.*

PROOF. We can identify $\overline{\mathcal{U}}$ with the (non-full) subcategory of $\overline{\mathcal{U}}_G$ where G acts trivially on everything, and this subcategory is closed under products. We thus have a product-preserving functor

$$\overline{\mathcal{U}} \xrightarrow{\text{inc}} \overline{\mathcal{U}}_G \xrightarrow{P_X} \overline{\mathcal{U}}_G \xrightarrow{S} \text{Sets}.$$

As in Proposition 5.4, the image of 1 under this functor is a semiring; but that image is just $S(X)$. By unwinding the definitions a little, we see that addition and multiplication are given by $a + b = T_s(a, b)$, and $ab = N_s(a, b)$, where $s: X \amalg X \rightarrow X$ is given by the identity on both copies of X . Similarly, the identity elements are $0 = T_z()$ and $1 = N_z()$, where $z: \emptyset \rightarrow X$ is the inclusion.

Now suppose we have a map $f: X \rightarrow Y$. This gives a commutative diagram

$$\begin{array}{ccccc} \emptyset & \xrightarrow{z} & X & \xleftarrow{s} & X \amalg X \\ \downarrow 1 & & \downarrow f & & \downarrow f \amalg f \\ \emptyset & \xrightarrow{z} & Y & \xleftarrow{s} & Y \amalg Y \end{array}$$

in which both squares are cartesian. By applying T to the diagram we see that T_f preserves addition (and zero). By applying N to the diagram we see that N_f preserves multiplication (and one). Using the cartesian property together with part (d) of Proposition 6.1 we see that R_f is a semiring homomorphism. All that is left is to check that T_f is a morphism of $S(Y)$ -modules, or more explicitly that $T_f(u R_f(v)) = T_f(u)v$ for all $u \in S(X)$ and $v \in S(Y)$. Consider the maps

$$\begin{array}{ccccc} X \amalg Y & \xrightarrow{f \amalg 1} & Y \amalg Y & \xrightarrow{s} & Y \\ \uparrow 1 \amalg f & & & & \uparrow f \\ X \amalg X & \xrightarrow{s} & & & X \end{array}$$

The operation $(u, v) \mapsto T_f(u)v$ is $N_s T_f \amalg 1$, whereas the operation $(u, v) \mapsto T_f(u R_f(v))$ is $T_f N_s R_f \amalg 1 \amalg f$. It will thus suffice to check that the bispan

$$\omega' = (X \amalg Y \xleftarrow{1 \amalg f} X \amalg X \xrightarrow{s} X \xrightarrow{f} Y)$$

is isomorphic to the distributor

$$\Delta(f \amalg 1, s) = (X \amalg Y \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$$

constructed in Definition 5.5. To analyse this, we write y_L for the copy of y in the left factor of $Y \amalg Y$, and y_R for the copy in the right factor, so $s^{-1}\{y\} = \{y_L, y_R\}$. This means that B is the set of pairs (y, t) , where $y \in Y$ and $t: \{y_L, y_R\} \rightarrow X \amalg Y$ with $(f \amalg 1) \circ t = 1$. This means that $t(y_R) = y_R$ and $t(y_L) = x_L$ for some $x \in X$ with $f(x) = y$. We see that everything is determined by x , so B can be identified with X . Next, A is the set of triples (w, y, t) , where (y, t) is as before and $w \in \{y_L, y_R\}$. The triples with $w = y_L$ form one copy of X , and the triples with $w = y_R$ form another copy of X , so we can identify A with $X \amalg X$. This means that ω is isomorphic to some bispan of the form

$$(X \amalg Y \leftarrow X \amalg X \rightarrow X \rightarrow Y);$$

we leave it to the reader to check that the maps are $1 \amalg f$, s and f . □

7. The group of order two

We now analyse how the theory works out for a group $G = \{1, \chi\}$ of order two, building on the corresponding result for Mackey functors in Theorem 4.9.

DEFINITION 7.1. A *Tambara pair* consists of rings A and B , together with a right action of G on A by semiring maps, a semiring map $\text{res}: B \rightarrow A^G$, and functions $\text{trace}, \text{norm}: A \rightarrow B$ satisfying

$$\begin{array}{ll} \text{trace}(0) = 0 & \text{norm}(1) = 1 \\ \text{trace}(a_0 + a_1) = \text{trace}(a_0) + \text{trace}(a_1) & \text{norm}(a_0 a_1) = \text{norm}(a_0) \text{norm}(a_1) \\ \text{trace}(\bar{a}) = \text{trace}(a) & \text{norm}(\bar{a}) = \text{norm}(a) \\ \text{res}(\text{trace}(a)) = a + \bar{a} & \text{res}(\text{norm}(a)) = a \bar{a} \\ \text{norm}(0) = 0 & \text{norm}(a_0 + a_1) = \text{norm}(a_0) + \text{norm}(a_1) + \text{trace}(a_0 \bar{a}_1) \\ \text{trace}(a \text{res}(b)) = \text{trace}(a)b. & \end{array}$$

We write TP for the category of Tambara pairs.

REMARK 7.2. Note that a Tambara pair can be regarded as a Mackey pair using addition and the trace map, and it can also be regarded as a Mackey pair in a different way using multiplication and the norm map. These structures encode all but the last three axioms for a Tambara pair.

CONSTRUCTION 7.3. Given a G -Tambara functor S , put $A = S(G/1) = S(G)$ and $B = S(G/G) = S(1)$ (so both of these are semirings). Next, as G is commutative we see that $\chi: G \rightarrow G$ is a G -map, with $\chi = \chi^{-1}$, so $T_\chi = N_\chi = R_\chi: A \rightarrow A$. We use this map to define an action of G on A by semiring maps.

The projection $\epsilon: G \rightarrow 1$ gives maps

$$\begin{aligned} \text{res} &= \epsilon^*: B \rightarrow A \\ \text{trace} &= T_\epsilon: A \rightarrow B \\ \text{norm} &= N_\epsilon: A \rightarrow B. \end{aligned}$$

We will prove that this gives an equivalence between Tambara functors and Tambara pairs. The first thing to check is that we at least have a functor.

PROPOSITION 7.4. *The above construction gives a faithful functor $F: \text{Tambara}_G \rightarrow \text{TP}$.*

PROOF. We first need to check that the construction gives a Tambara pair. In view of Remark 7.2 and Theorem 4.9, we need only consider the last three axioms. The identities $\text{norm}(0) = 0$ and $\text{trace}(a \text{res}(b)) = \text{trace}(a)b$ follows immediately from Lemma 6.8 and Proposition 6.1. Finally, we need to understand $\text{norm}(a_0 + a_1) = N_\epsilon T_s(a_0, a_1)$. By Proposition 5.6, the composite $N_\epsilon T_s$ can be written as $T_r N_q R_p$ for certain maps

$$G \amalg G \xleftarrow{p} E \xrightarrow{q} F \xrightarrow{r} 1.$$

As the target set is just a single point, the definitions can be simplified as follows:

$$\begin{aligned} F &= \{t: G \rightarrow G \amalg G \mid st = 1\} \\ E &= G \times F \\ p(g, t) &= t(g) \\ q(t, g) &= t \\ r(t) &= 0. \end{aligned}$$

We can name the elements of $G \amalg G$ in an obvious way as $\{1_L, \chi_L, 1_R, \chi_R\}$. The elements of F are $\{t_0, t_1, t_2, t_3\}$, where

$$\begin{array}{cccc} t_0(1) = 1_L & t_1(1) = 1_R & t_2(1) = 1_L & t_3(1) = 1_R \\ t_0(\chi) = \chi_L & t_1(\chi) = \chi_R & t_2(\chi) = \chi_R & t_3(\chi) = \chi_L. \end{array}$$

Note that t_0 and t_1 are equivariant, or in other words, fixed under the standard action of G by conjugation. On the other hand, we have $\overline{t_2} = t_3$ and $\overline{t_3} = t_2$. We now put $E_0 = \{t_0\} \simeq 1$ and $E_1 = \{t_1\} \simeq 1$ and $E_2 = \{t_2, t_3\} \simeq G$ and $F_i = G \times E_i$, so we have equivariant decompositions $E = E_0 \amalg E_1 \amalg E_2$ and $F = F_0 \amalg F_1 \amalg F_2$. It follows that $N_\epsilon T_s$ can be written as a sum of three terms, one for each bispan

$$\omega_i = (G \amalg G \xleftarrow{p_i} E_i \xrightarrow{q_i} F_i \xrightarrow{r_i} 1).$$

Now E_0 can be identified with G , and p_0 with the inclusion of the left factor in $G \amalg G$, and q_0 with ϵ . It follows that ω_0 gives the operation $(a_0, a_1) \mapsto \text{norm}(a_0)$. Similarly, ω_1 gives the operation $(a_0, a_1) \mapsto \text{norm}(a_1)$. For the last term, we define an equivariant bijection $m: E_2 \rightarrow G$ by $m(t_2) = 1$ and $m(t_3) = \chi$. One can check that $p_2: E_2 \rightarrow G \amalg G$ is also an equivariant bijection, and that the composite $m q_2 p_2^{-1}: G \amalg G \rightarrow G$ is $(1, \chi)$, so the associated operation $N_{m q_2 p_2^{-1}}: S(G) \times S(G) \rightarrow S(G)$ is $(a_0, a_1) \mapsto a_0 \overline{a_1}$. As m and p_2 are bijective we have $N_m = T_m$ and $N_{p_2^{-1}} = R_{p_2}$, and it is also clear that $\epsilon m = r_2: E_2 \rightarrow 1$. Putting this together we see that $T_{r_2} N_{q_2} R_{p_2}(a_0, a_1) = \text{trace}(a_0 \overline{a_1})$ as required.

It now follows that the pair $F(S) = (A, B)$ (with structure maps as above) is a Tambara pair. Now suppose we have another Tambara functor S' , with $F(S') = (A', B')$ say, and a morphism $\phi: S \rightarrow S'$ of Tambara functors. This has components $\phi_G: A = S(G) \rightarrow S'(G) = A'$ and $\phi_1: B = S(1) \rightarrow S'(1) = B'$,

and it is tautological that these commute with the all the Tambara pair structure maps. There is thus an evident way to make F into a functor $\text{Tambara}_G \rightarrow \text{TP}$. This is faithful by the same argument as for Mackey functors. \square

CONSTRUCTION 7.5. Let $P = (A, B)$ be a Tambara pair. For any finite G -set X , we put

$$EP(X) = \{(u, v) \in \text{Map}_G(X, A) \times \text{Map}(X^G, B) \mid u(x) = \text{res}(v(x)) \text{ for all } x \in X^G\},$$

as before. Now suppose we have a G -equivariant map $f: X \rightarrow Y$. Note that $f^{-1}(Y^G)$ will contain X^G , and possibly some free orbits as well. If we choose a point in each such free orbit, we get a decomposition $f^{-1}(Y^G) = X^G \amalg X_1 \amalg \overline{X_1}$ say.

- (a) We define $R_f: EP(Y) \rightarrow EP(X)$ by $R_f(m, n) = (m \circ f, n \circ f^G)$ (where $f^G: X^G \rightarrow Y^G$ is just the restriction of f).
- (b) We define $T_f: EP(X) \rightarrow EP(Y)$ by $T_f(u, v) = (m, n)$, where

$$\begin{aligned} m(y) &= \sum_{x \in f^{-1}\{y\}} u(x) \\ n(y) &= \sum_{x_0 \in X^G \cap f^{-1}\{y\}} v(x_0) + \sum_{x_1 \in X_1 \cap f^{-1}\{y\}} \text{trace}(u(x_1)). \end{aligned}$$

- (c) We define $N_f: EP(X) \rightarrow EP(Y)$ by $N_f(u, v) = (p, q)$, where

$$\begin{aligned} p(y) &= \prod_{x \in f^{-1}\{y\}} u(x) \\ q(y) &= \left(\prod_{x_0 \in X^G \cap f^{-1}\{y\}} v(x_0) \right) \left(\sum_{x_1 \in X_1 \cap f^{-1}\{y\}} \text{norm}(u(x_1)) \right). \end{aligned}$$

By reducing to the Mackey functor case, we see that these operators are well-defined.

PROPOSITION 7.6. *The above construction makes EP into a Tambara functor.*

PROOF. By applying Proposition 4.8 to the underlying additive and multiplicative Mackey pairs, we see that all composites of the operators T , N and R work correctly except possibly for composites of the form NT . Suppose we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ of finite G -sets, and we define

$$\Delta(f, g) = (X \xleftarrow{p} M \xrightarrow{q} N \xrightarrow{r} Z)$$

by the usual distribution rule:

$$\begin{aligned} N &= \{(z, s) \mid z \in Z, s: g^{-1}\{z\} \rightarrow X, fs = 1\} \\ M &= \{(y, s) \mid y \in Y, s: g^{-1}\{g(y)\} \rightarrow X, fs = 1\} \\ p(y, s) &= s(y) \\ q(y, s) &= (g(y), s) \\ r(z, s) &= z. \end{aligned}$$

We say that the chain $(X \xrightarrow{f} Y \xrightarrow{g} Z)$ is *distributable* if $N_g T_f = T_r N_q R_p: EP(X) \rightarrow EP(Z)$. We must show that this holds for all f and g .

Consider a pair $(u, v) \in EP(X)$, so $N_g T_f(u, v) = (d, e)$ and $T_r N_q R_p(u, v) = (d^*, e^*)$ say. Here $d: Z \rightarrow A$ is given by

$$d(z) = \prod_{y \in g^{-1}\{z\}} \sum_{x \in f^{-1}\{y\}} u(x),$$

and it is clear from the nonequivariant theory that this is the same as $d^*(z)$.

Now suppose that Z is a free G -set, so $Z^G = \emptyset$, so the set $\text{Map}(Z^G, B)$ in which e and e^* live has only one element, so $e = e^*$. It follows that in this case we have $N_g T_f = T_r N_q R_p$.

For a more general finite G -set Z , we note that everything happens independently over the different orbits in Z , so we can reduce to the cases $Z = G$ and $Z = 1$. If $Z = G$ then it is free, and this case has already been covered. We may thus assume that $Z = 1$, so the definitions simplify to

$$\begin{aligned} N &= \{s: Y \rightarrow X, fs = 1\} \\ M &= Y \times N \\ p(y, s) &= s(y) \\ q(y, s) &= s \\ r(s) &= 0. \end{aligned}$$

We now argue by induction on the number of orbits in Y . If Y is empty then the claim is easy. For the induction step, we may assume that $X \xrightarrow{f} Y \xrightarrow{g} 1$ is distributable, and we must show that the same holds for slightly larger chains. The first case to consider is a chain $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z$, where $Y_1 = Y \amalg 1$, and $X_1 = X \amalg X'$ for some X' , and $f_1 = f \amalg f'$ for the unique map $f': X' \rightarrow 1$. In this case the distributor is

$$X \amalg X' \xleftarrow{p_1} ((Y \times N \times X') \amalg (N \times X')) \xrightarrow{q_1} (N \times X') \xrightarrow{r_1} 1,$$

where

$$\begin{aligned} p_1(y, s, x') &= s(y) \in X \subseteq X \amalg X' \\ p_1(s, x') &= x' \in X' \subseteq X \amalg X' \\ q_1(y, s, x') &= (s, x') \\ q_1(s, x') &= (s, x'). \end{aligned}$$

Consider an element $(u_1, v_1) \in EP(X_1) = EP(X) \times EP(X')$, with components $(u, v) \in EP(X)$ and $(u', v') \in EP(X')$. By our inductive hypothesis we have

$$N_g T_f(u, v) = T_r N_q R_p(u, v) = e \in EP(1) \simeq B$$

say. Now split X' as $X'_0 \amalg (G \times X'_1)$, with G acting trivially on X'_0 and X'_1 . From the definitions, we have

$$N_{g_1} T_{f_1}(u_1, v_1) = \left(\sum_{x' \in X'_0} v'(x') + \sum_{x' \in X'_1} \text{trace}(u'(x')) \right) e.$$

We need to compare this with the element $e_1^* = T_{r_1} N_{q_1} R_{p_1}(u_1, v_1)$. To do this, we split N equivariantly as $N_0 \amalg (G \times N_1)$ with G acting trivially on N_0 and N_1 . This gives a splitting

$$N \times X' = N_0 \times X'_0 \amalg G \times ((N_1 \times X'_0) \amalg (N_0 \times X'_1) \amalg (G \times N_1 \times X'_1)).$$

Each orbit in $N \times X'$ contributes a term to e^* . These terms can be described as follows.

- (a) Consider an element $(s, x') \in N_0 \times X'_0$. As $s \in N_0 = N^G$, we see that it is an equivariant map $Y \rightarrow X$, so it gives an operator $R_s: EP(X) \rightarrow EP(Y)$. The element

$$N_g R_s(u, v) \in EP(1) \subseteq A \times B$$

will have the form $(\prod_y u(s(y)), w_s)$ for some $w_s \in B$ with $\text{res}(w_s) = \prod_y u(s(y))$. By unwinding the definitions, we see that the term in e^* corresponding to (s, x') is $w_s v'(x')$.

- (b) Each element $(s, x') \in N_1 \times X'_0$ contributes a term $\text{trace}\left(\left(\prod_{y \in Y} u(s(y))\right) u'(x')\right)$ to e^* . As $x' \in X'_0$ we have $u'(x') = \text{res}(v'(x'))$, and using the rule $\text{trace}(a \text{res}(b)) = \text{trace}(a)b$ we can rewrite this term as $\text{trace}\left(\prod_{y \in Y} u(s(y))\right) v'(x')$.
- (c) Each element $(s, x') \in N_0 \times X'_1$ contributes a term $\text{trace}\left(\left(\prod_{y \in Y} u(s(y))\right) u'(x')\right)$ to e^* . As in (a) we have $\prod_y u(s(y)) = \text{res}(w_s)$, so this term can be rewritten as $w_s \text{trace}(u'(x'))$.
- (d) Each element $(s, x') \in N_1 \times X'_1$ contributes two orbits in $N \times X'$, and so contributes two terms in e^* , namely $\text{trace}\left(\left(\prod_{y \in Y} u(s(y))\right) u'(x')\right)$ and $\text{trace}\left(\left(\prod_{y \in Y} u(s(y))\right) \overline{u'(x')}\right)$. Here $u'(x') + \overline{u'(x')} = \text{res}(\text{trace}(u'(x')))$ so the sum of these two terms is $\text{trace}\left(\prod_{y \in Y} u(s(y))\right) \text{trace}(u'(x'))$.

We also have

$$e = T_r N_q R_p(u, v) = \sum_{s \in N_0} w_s + \sum_{s \in N_1} \text{trace} \left(\prod_{y \in Y} u(s(y)) \right),$$

and it follows that

$$e^* = \left(\sum_{x' \in X'_0} v'(x') + \sum_{x' \in X'_1} \text{trace}(u'(x')) \right) e = N_{g_1} T_{f_1}(u_1, v_1)$$

as required.

The other case that we need to consider is a chain $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} 1$, where $Y_1 = Y \amalg G$, and $X_1 = X \amalg (G \times X')$ for some X' , and $f_1 = f \amalg \pi$, where $\pi: G \times X' \rightarrow G$ is the projection. In this case we have

$$EP(X_1) = EP(X) \times \text{Map}(X', A) \subseteq \text{Map}_G(X, A) \times \text{Map}((X')^G, B) \times \text{Map}(X', A),$$

and

$$N_{g_1} T_{f_1}(u, v, u') = \text{norm} \left(\sum_{x' \in X'} u'(x') \right) e$$

(where $e = N_g T_f(u, v) = T_r N_q R_p(u, v)$ as before). We need to compare this with the element $e^* = T_{r^*} N_{q^*} R_{p^*}(u, v)$ defined using the distributor

$$\Delta(f^*, g^*) = (X_1 \xleftarrow{p^*} Y_1 \times N^* \xrightarrow{q^*} N^* \xrightarrow{r^*} 1).$$

Here $N^* = \{s_1: Y_1 \rightarrow X_1 \mid f_1 s_1 = 1\}$. Given $s \in N$ and $x', x'' \in X'$ we define

$$s_1: Y_1 = Y \amalg \{1, \chi\} \rightarrow X_1 = X \amalg (G \times X')$$

by

$$\begin{aligned} s_1(y) &= s(y) \in X && \text{for } y \in Y \\ s_1(1) &= (1, x') \in \{1\} \times X' \\ s_1(\chi) &= (\chi, x'') \in \{\chi\} \times X' \end{aligned}$$

This construction identifies N^* with $N \times X' \times X'$. The resulting G -action is

$$\chi(s, x', x'') = (\chi s \chi, x'', x').$$

To classify the orbits, we first decompose N as $N_0 \amalg (G \times N_1)$ as before, and we choose a total order on X' .

- (a) For each $s \in N_0$ and $x' \in X'$ we have a fixed point (s, x', x') . If we put $w_s = N_g R_s(u, v) \in B$ as before, then the corresponding term in e^* is $\text{norm}(u(x'))w_s$.
- (b) For each $s \in N_1$ and $x', x'' \in X'$ we have a free orbit containing (s, x', x'') . The corresponding term in e^* is $\text{trace}(u'(x')\overline{u'(x'')}) \prod_y u(s(y))$.
- (c) For each $s \in N_0$ and $x', x'' \in X'$ with $x' < x''$ we have a free orbit containing (s, x', x'') . The corresponding term in e^* is $\text{trace}(u'(x')\overline{u'(x'')}) \prod_y u(s(y)) = \text{trace}(u'(x')\overline{u'(x'')})w_s$.

Recall that $\text{norm}(a_0 + a_1) = \text{norm}(a_0) + \text{norm}(a_1) + \text{trace}(a_0 \overline{a_1})$. It follows inductively from this that

$$\text{norm}\left(\sum_{i=0}^{n-1} a_i\right) = \sum_i \text{norm}(a_i) + \sum_{i < j} \text{trace}(a_i \overline{a_j})$$

We can use this to combine the terms of types (a) and (c); together they contribute $\text{norm}(z) \sum_{s \in N_0} w_s$, where $z = \sum_{x' \in X'} u'(x')$. Similarly, the sum of all terms of type (b) is

$$\text{trace} \left(z \overline{z} \sum_{s \in N_1} \prod_{y \in Y} u(s(y)) \right).$$

We can use the relation $z\bar{z} = \text{res}(\text{norm}(z))$ to rewrite this, and then combine with the (a) and (c) terms to get

$$e^* = \text{norm}(z) \left(\sum_{s \in N_0} w_s + \sum_{s \in N_1} \text{trace} \left(\prod_{y \in Y} u(s(y)) \right) \right) = \text{norm}(z)e = N_{g_1} T_{f_1}(u, v, u')$$

as required. \square

THEOREM 7.7. *The functor $F: \text{Tambara}_G \rightarrow \text{TP}$ is an equivalence, with inverse given by E .*

PROOF. There is an evident way to define E on morphisms, giving a functor $E: \text{TP} \rightarrow \text{Tambara}_G$. As in the Mackey functor case, there is a natural isomorphism $FE(A, B) \simeq (A, B)$ of Tambara pairs. Similarly, for any Tambara functor S there is a canonical system of bijections $\alpha_{S, X}: S(X) \rightarrow EFS(X)$ for all finite G -sets X , and these are evidently natural with respect to morphisms $S \rightarrow S'$ of Tambara functors. We still need to check, however, that for fixed S and varying X , the maps $\alpha_{S, X}$ give a morphism of Tambara functors. In other words, for any morphism $f: X \rightarrow Y$ of finite G -sets, we need to show that the diagrams

$$\begin{array}{ccc} S(X) & \xrightarrow{T_f} & S(Y) \\ \alpha \downarrow & & \downarrow \alpha \\ EFS(X) & \xrightarrow{T_f} & EFS(Y) \end{array} \quad \begin{array}{ccc} S(X) & \xrightarrow{N_f} & S(Y) \\ \alpha \downarrow & & \downarrow \alpha \\ EFS(X) & \xrightarrow{N_f} & EFS(Y) \end{array} \quad \begin{array}{ccc} S(X) & \xleftarrow{R_f} & S(Y) \\ \alpha \downarrow & & \downarrow \alpha \\ EFS(X) & \xleftarrow{R_f} & EFS(Y) \end{array}$$

commute. This is left as an exercise for the reader. \square

EXAMPLE 7.8. We will describe a Tambara pair T that does not arise from any of the standard constructions of Tambara functors that we have discussed previously. (It will be used as an interesting counterexample in Section 14.)

Let S denote the Tambara pair given by $A = B = \mathbb{Z}$ with $\bar{k} = k$ and $\text{trace}(k) = 2k$ and $\text{norm}(k) = k^2$ and $\text{res}(k) = k$. This corresponds to the Tambara functor $X \mapsto \text{Map}_G(X, \mathbb{Z})$ (with G acting trivially on \mathbb{Z}).

Next, let T denote the Tambara pair given as follows:

$$\begin{aligned} A &= \mathbb{Z}[\alpha]/\alpha^2 = \mathbb{Z} \oplus \mathbb{Z}\alpha \\ B &= \mathbb{Z}[\beta, \gamma]/(\beta^2, \beta\gamma, \gamma^2, 2\gamma) = \mathbb{Z} \oplus \mathbb{Z}\beta \oplus (\mathbb{Z}/2)\gamma \\ \overline{i + j\alpha} &= i + j\alpha \\ \text{res}(i + j\beta + k\gamma) &= i + 2j\alpha \\ \text{trace}(i + j\alpha) &= 2i + j\beta \\ \text{norm}(i + j\alpha) &= i^2 + ij\beta + j^2\gamma. \end{aligned}$$

Note here that $j^2\gamma = j\gamma$ because $2\gamma = 0$ and $j^2 - j$ is always even. Using this it is straightforward, if somewhat lengthy, to check all the axioms for a Tambara pair. There is an evident inclusion $\eta: S \rightarrow T$ and an augmentation $\epsilon: T \rightarrow S$ given by $\epsilon(i + j\alpha) = i$ and $\epsilon(i + j\beta + k\gamma) = i$; these are morphisms of Tambara pairs.

8. Tambara functors in stable homotopy theory

We next outline a construction showing that equivariant E_∞ ring spectra give rise to Tambara functors. Some examples were discussed in Example 6.7. Our construction is simpler than that of Brun [3] and covers the existing applications, although it is easy to imagine more advanced applications where something closer to Brun's approach might be needed.

A *Euclidean space* is a finite-dimensional vector space over \mathbb{R} equipped with an inner product. An *isometry* between Euclidean spaces is a linear map that preserves inner products (and so is injective, but not necessarily surjective). We write $\mathcal{L}(V, W)$ for the space of isometries from V to W ; this gives us a category \mathcal{L} . We also have a quotient category \mathcal{L}_0 , where the morphism set $\mathcal{L}_0(V, W)$ is the set of path-components in $\mathcal{L}(V, W)$. Note that

- (a) If $\dim(V) < \dim(W)$ then $\mathcal{L}(V, W)$ admits a transitive action of the connected group $SO(W)$, so $|\mathcal{L}_0(V, W)| = 1$.
- (b) If $\dim(V) = \dim(W) = n$ then $V \simeq W \simeq \mathbb{R}^n$ so $\mathcal{L}(V, W) \simeq O(n)$ and $|\mathcal{L}_0(V, W)| = 2$.
- (c) If $\dim(V) > \dim(W)$ then $\mathcal{L}_0(V, W) = \mathcal{L}(V, W) = \emptyset$.

This implies that the category \mathcal{L}_0 is filtered, and that the subcategory consisting of the objects \mathbb{R}^n and the standard inclusions between them is cofinal.

Next, there is a vector bundle over $\mathcal{L}(V, W)$ whose fibre at a map $\alpha: V \rightarrow W$ is the cokernel of α , or equivalently the orthogonal complement $W \ominus \alpha(V)$. We write cok for this bundle, and $\Theta(V, W) = \mathcal{L}(V, W)^{\text{cok}}$ for the associated Thom space. There are canonical isomorphisms $\text{cok}(\beta\alpha) \simeq \text{cok}(\beta) \oplus \text{cok}(\alpha)$, and using these we can define composition maps

$$\Theta(V, W) \wedge \Theta(U, V) \rightarrow \Theta(U, W),$$

giving a category Θ enriched in based spaces. There are also evident pairings

$$\Theta(V, W) \wedge \Theta(V', W') \rightarrow \Theta(V \oplus V', W \oplus W'),$$

using which we see that the direct sum gives a symmetric monoidal structure on Θ .

A *spectrum* is a functor E from Θ to the category Spaces_* of compactly generated, weak Hausdorff based spaces with the property that the maps

$$E: \Theta(V, W) \rightarrow \text{Spaces}_*(E(V), E(W))$$

are continuous and preserve basepoints. (It would be more standard to call E an *orthogonal prespectrum*, but we will just use the word *spectrum* for brevity.) A *ring spectrum* is a spectrum equipped with maps $\eta: S^0 \rightarrow E(0)$ and

$$\mu: E(V) \wedge E(W) \rightarrow E(V \oplus W)$$

that are commutative, associative, unital and compatible with the action of Θ . (Again, it would be more standard to say *strictly commutative orthogonal ring prespectrum*.)

Consider a spectrum E . For any Euclidean space V we have a one-point compactification S^V (which is homeomorphic to a sphere). We write $\Pi(V; E) = \text{Spaces}_*(S^V, E(V))$, and we let $\pi(V; E)$ be the set of path-components in $\Pi(V; E)$. This can be described as $\pi_d(E(V))$, where $d = \dim(V)$, so it will have a natural structure as an abelian group provided that $d \geq 2$.

We claim that the construction $V \mapsto \Pi(V; E)$ gives a functor $\mathcal{L} \rightarrow \text{Spaces}_*$. To see this, consider an isometry $\alpha: V \rightarrow W$, and a based map $f: S^V \rightarrow E(V)$. Any point $w \in W$ can be written uniquely as $\alpha(v) + u$ with $v \in V$ and $u \in \text{cok}_\alpha = W \ominus \alpha(V)$. This gives a pair $(\alpha, u) \in \Theta(V, W)$ and thus a map $(\alpha, \theta)_*: X(V) \rightarrow X(W)$. We define a map $\alpha_*(f): S^W \rightarrow E(W)$ by

$$\alpha_*(f)(\alpha(v) + u) = (\alpha, \theta)_*(f(v)).$$

We leave it the reader to check that this has the right continuity and functoriality properties. We now have a map

$$\mathcal{L}(V, W) \times \Pi(V; E) \rightarrow \Pi(W; E),$$

and we can pass to path components to get a map

$$\mathcal{L}_0(V, W) \times \pi(V; E) \rightarrow \pi(W; E),$$

giving a functor $\mathcal{L}_0 \rightarrow \text{Sets}$. We define $\pi_0(E)$ to be the colimit of this functor. By the cofinality statement that we mentioned before, $\pi_0(E)$ is also the colimit of the smaller diagram

$$\pi_{\mathbb{R}^0}(E(\mathbb{R}^0)) \rightarrow \pi_{\mathbb{R}^1}(E(\mathbb{R}^1)) \rightarrow \pi_{\mathbb{R}^2}(E(\mathbb{R}^2)) \rightarrow \pi_{\mathbb{R}^3}(E(\mathbb{R}^3)) \rightarrow \dots$$

However, our original description is better for analysing naturality questions.

It is known that our definition of π_0 is only the first glimpse of an elaborate homotopy theory of spectra involving spectrification functors, several different Quillen model structures and so on [14]. The associated homotopy category is equivalent to the stable category of spectra as originally introduced by Boardman. However, none of this is required for our purposes here; we can just work directly with π_0 .

Now let G be a finite group. A *G-Euclidean space* is a Euclidean space with isometric action of G . A *G-spectrum* is just a spectrum with an action of G .

Let E be a G -spectrum, and let V be a G -Euclidean space. Note that $\Theta(V, V)$ is just $\mathcal{L}(V, V)$ with an added basepoint. The composite

$$G \rightarrow \mathcal{L}(V, V) \xrightarrow{\text{inc}} \Theta(V, V) \xrightarrow{E} \text{Spaces}_*(E(V), E(V))$$

gives an action of G on $E(V)$. On the other hand, G is assumed to act on E , so it acts on $E(U)$ for every Euclidean space U , and this gives another action on $E(V)$. These actions commute, and so give an action of $G \times G$ on $E(V)$. Using the diagonal embedding $G \rightarrow G \times G$ we get a third action of G on $E(V)$. It is this third action that we will use unless otherwise specified. In other words, we let G act on $E(V)$ by combining the actions on all ingredients. This in turn gives an action of G on the space $\Pi(V; E)$, and we write $\Pi^G(V; E)$ for the subspace of G -fixed points. Equivalently, $\Pi^G(V; E)$ is the space of G -equivariant (continuous, based) maps from S^V to $E(V)$. We then define $\pi^G(V; E) = \pi_0(\Pi^G(V; E))$. As before, this gives a functor from the category $G\mathcal{L}_0$ to sets, where $G\mathcal{L}_0(V, W)$ is the set of path-components in the space $G\mathcal{L}(V, W)$ of equivariant isometries from V to W . We write $\pi_0^G(E)$ for the colimit of this functor.

Next, let S_1, \dots, S_r be a list containing precisely one representative of every isomorphism class of irreducible $\mathbb{R}[G]$ -modules. We can give S_i a G -invariant inner product (which will be unique up to a multiplicative constant) and thus regard it as a G -euclidean space. For an arbitrary G -Euclidean space V we put $Q_i(V) = \text{Hom}_G(S_i, V)$. Standard representation theory shows that $V \simeq \bigoplus_i Q_i(V) \otimes S_i$ and that the evident map

$$\text{Hom}_G(V, W) \rightarrow \prod_i \text{Hom}(Q_i(V), Q_i(W))$$

is an isomorphism. One can check that the spaces $Q_i(V)$ can be given inner products in a natural way, and that

$$G\mathcal{L}(V, W) \simeq \prod_i \mathcal{L}(Q_i(V), Q_i(W)).$$

Thus, the category $G\mathcal{L}$ is equivalent to the product of r copies of \mathcal{L} , and similarly for $G\mathcal{L}_0$. In particular, if $\dim(Q_i(W)) > \dim(Q_i(V))$ for all i (which will certainly hold if W contains a copy of $V \oplus \mathbb{R}[G]$) then $|G\mathcal{L}_0(V, W)| = 1$. It follows that $\pi_0^G(E)$ is also the colimit of the sequence

$$\pi^G(\mathbb{R}[G]^0; E) \rightarrow \pi^G(\mathbb{R}[G]^1; E) \rightarrow \pi^G(\mathbb{R}[G]^2; E) \rightarrow \pi^G(\mathbb{R}[G]^3; E) \rightarrow \dots$$

Now suppose we have a finite G -set X . We write $G\mathcal{L}[X]$ for the category of G -equivariant Euclidean vector bundles over X . Thus, for $V \in G\mathcal{L}[X]$ we have a Euclidean vector space V_x for each $x \in X$, together with isometric isomorphisms $g: V_x \rightarrow V_{gx}$ that compose in the obvious way. We now define $\Pi(V; E) = \prod_x \Pi(V_x; E)$, and note that this also has a natural G -action. We write $\Pi^G(V; E)$ for the subspace of fixed points, and $\pi^G(V; E)$ for the set of path-components in $\Pi^G(V; E)$. We then write $\pi_0^G(E)(X)$ for the colimit of the sets $\pi^G(V; E)$ over the category $G\mathcal{L}[X]_0$.

THEOREM 8.1. *For any G -spectrum E , the construction $X \mapsto \pi_0^G(E)(X)$ gives a Mackey functor in a natural way. Moreover, if E is a ring spectrum (in the strict sense that we mentioned previously) then this construction gives a Tambara functor.*

The rest of this section will form the proof.

CONSTRUCTION 8.2. Consider a morphism $f: X \rightarrow Y$ of finite G -sets and a G -spectrum E . We will define a map

$$R_f: \pi_0^G(E)(Y) \rightarrow \pi_0^G(E)(X).$$

Consider a G -Euclidean bundle W over Y , and a point $s \in \Pi^G(W; E)$, or in other words a system of based maps $s_y: S^{W_y} \rightarrow E(W_y)$ that are compatible with the G -action. For each $x \in X$ we let t_x denote the map

$$S^{(f^*W)_x} = S^{W_{f(x)}} \xrightarrow{s_{f(x)}} E(W_{f(x)}) = E((f^*W)_x).$$

This defines a point $t \in \Pi^G(f^*W; E)$, which depends naturally and continuously on all the ingredients, so we can define a map

$$R_f: \Pi^G(W; E) \rightarrow \Pi^G(f^*W; E)$$

by $s \mapsto t$. This in turn gives a map $R_f: \pi^G(W; E) \rightarrow \pi^G(f^*W; E)$. Next, by considering bundles with total space $\mathbb{R}[G]^N \times Y$ for large N we can see that the functor $f^*: \mathcal{GL}[Y]_0 \rightarrow \mathcal{GL}[X]_0$ is cofinal. Thus, after passing to the colimit over W we get a map

$$R_f: \pi_0^G(E)(Y) \rightarrow \pi_0^G(E)(X).$$

It is formal to check that this is contravariantly functorial.

CONSTRUCTION 8.3. Consider again a morphism $f: X \rightarrow Y$ of finite G -sets, and a ring G -spectrum E . We will define a map

$$N_f: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Y).$$

Consider a G -Euclidean bundle V over X , and put $(f_*V)_y = \bigoplus_{f(x)=y} V_x$. Now suppose we have a point $t \in \Pi^G(V; E)$, or in other words a system of based maps $t_x: S^{V_x} \rightarrow E(V_x)$ that are compatible with the G -action. For each $y \in Y$ we let s_y denote the composite

$$S^{(f_*V)_y} = \bigwedge_{x \in f^{-1}\{y\}} S^{V_x} \xrightarrow{\bigwedge_x t_x} \bigwedge_{x \in f^{-1}\{y\}} E(V_x) \xrightarrow{\mu} E \left(\bigoplus_{x \in f^{-1}\{y\}} V_x \right) = E((f_*V)_y).$$

This defines a point $s \in \Pi^G(f_*V; E)$, which depends naturally and continuously on all the ingredients, so we can define a map

$$N_f: \Pi^G(V; E) \rightarrow \Pi^G(f_*V; E)$$

by $t \mapsto s$. This in turn gives a map $N_f: \pi^G(V; E) \rightarrow \pi^G(f_*V; E)$.

We can now pass to the colimit over V to get a map $N_f: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Y)$. (In this case, the variances are the right way around so we get an induced map of colimits without needing to assume that f_* is cofinal. It will in fact be cofinal if f is surjective, but not otherwise.) It is formal to check that this construction is covariantly functorial.

PROPOSITION 8.4. *Suppose we have a cartesian square of finite G -sets as shown:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Then $R_h N_f = N_{f'} R_g: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Y')$.

PROOF. Let V be a G -Euclidean bundle over X . As the square is cartesian, we see that g induces a bijection $(f')^{-1}\{y'\} \rightarrow f^{-1}\{h(y')\}$ for all $y' \in Y'$. Using this we see that the bundle $W = h^*f_*V$ over Y' can also be identified with $(f')_*g^*V$. We claim that the following diagram commutes:

$$\begin{array}{ccc} \Pi^G(g^*V; E) & \xrightarrow{N_{f'}} & \Pi^G(W; E) \\ R_g \uparrow & & \uparrow R_h \\ \Pi^G(V; E) & \xrightarrow{N_f} & \Pi^G(f_*V; E). \end{array}$$

This is proved by simply working through the definitions; details are left to the reader. We can now pass to homotopy and take colimits to recover the original claim. \square

We now need to deal with transfers. The key ideas for our approach are due to Steiner [21], but we interpret them in a slightly different context.

DEFINITION 8.5. We write \mathcal{L}^\times for the category of Euclidean spaces and isometric isomorphisms, so $\mathcal{L}^\times(V, W) = \mathcal{L}(V, W)$ if $\dim(V) = \dim(W)$, and $\mathcal{L}^\times(V, W) = \emptyset$ otherwise.

DEFINITION 8.6. Let V be a Euclidean space. A *shrinking* of V is a family of maps $\alpha(t): V \rightarrow V$ (for $0 \leq t \leq 1$) such that

- (a) The combined map $[0, 1] \times V \rightarrow V$ (given by $(t, v) \mapsto \alpha(t)(v)$) is continuous.
(b) Each map $\alpha(t): V \rightarrow V$ is an open embedding, with

$$\|\alpha(t)(v) - \alpha(t)(v')\| \leq \|v - v'\|$$

for all $v, v' \in V$.

- (c) The map $\alpha(0): V \rightarrow V$ is the identity.

We call (b) the *shrinking condition*.

Given shrinkings α and β , the maps $\beta(t) \circ \alpha(t)$ form another shrinking. The space $\mathcal{S}(V)$ of shrinkings is thus a topological monoid in a natural way. There is also an evident equivariant analogue $G\mathcal{S}(V)$.

REMARK 8.7. The shrinking condition can often be checked differentially, as follows. Suppose we have a continuously differentiable map $\phi: V \rightarrow V$, so for each $v \in V$ we have a derivative $\phi'(v) \in \text{Hom}(V, V)$. The *operator norm* of a linear map $\lambda \in \text{Hom}(V, V)$ is the number

$$\|\lambda\|_{\text{op}} = \sup\{\|\lambda(w)\| \mid w \in W, \|w\| \leq 1\},$$

or equivalently, the largest eigenvalue of the self-adjoint operator $\lambda^\dagger \lambda$. Standard arguments show that $\|\phi(v) - \phi(w)\| \leq \|v - w\|$ for all v and w iff $\|\phi'(v)\|_{\text{op}} \leq 1$ for all v .

EXAMPLE 8.8. We can define $\theta \in \mathcal{S}(V)$ by $\theta(t)(v) = v/(1+t\|v\|)$. This has the property that $\theta(1)(V)$ is the open unit ball centred at the origin. More generally, given $a \in V$ and $r > 0$ we can define $\theta' \in \mathcal{S}(V)$ by

$$\theta'(t)(v) = tv + (1-t) \left(a + \frac{v}{1 + \|v\|/r} \right),$$

and then $\theta'(1)(V)$ is the open ball of radius r centred at a . In both cases one can check the shrinking condition by a differential calculation as in Remark 8.7; details are given in [21].

REMARK 8.9. Given any shrinking α , we can define shrinkings $H_u \alpha$ for $0 \leq u \leq 1$ by $(H_u \alpha)(t) = \alpha(1 - u + ut)$. This construction gives a contraction of the space $\mathcal{S}(V)$.

DEFINITION 8.10. Given an open embedding $f: V \rightarrow W$, we write $f^!$ for the map $S^W \rightarrow S^V$ obtained by the Pontrjagin-Thom construction, so $f^!(f(v)) = v$ and $f^!(w) = \infty$ if $w \notin f(V)$. Given a shrinking $\alpha \in \mathcal{S}(V)$ we write $\alpha^!$ for $\alpha(0)^!$ (which is homotopic to $\alpha(1)^! = 1^! = 1$). This gives a morphism $\mathcal{S}(V)^{\text{op}} \rightarrow \text{Spaces}_*(S^V, S^V)$ of topological monoids.

DEFINITION 8.11. Consider again a morphism $f: X \rightarrow Y$ of finite G -sets, equipped with a G -Euclidean bundle W over Y . A *T -lifting* of f is a system of shrinkings $\alpha_x \in \mathcal{S}(W_{f(x)})$ such that for each $y \in Y$, the resulting map

$$\alpha_{[y]} = (\alpha_x(0))_{x \in f^{-1}\{y\}} : \coprod_{x \in f^{-1}\{y\}} W_y \rightarrow W_y$$

is injective (and thus an open embedding).

LEMMA 8.12. *Let L be the space of T -liftings of f , and let L' be the space of G -equivariant injective maps $f': X \rightarrow \coprod_y W_y$ such that $f'(x) \in W_{f(x)}$ for all x . Then L is homotopy equivalent to L' . Moreover, if W contains a subbundle isomorphic to $\mathbb{C}[G] \times Y$, then L and L' are connected.*

PROOF. First, we can define $\pi: L \rightarrow L'$ by $\pi(\alpha)(x) = \alpha_x(0)(0)$. In the opposite direction, suppose we have a point $f' \in L'$. We define $r: Y \rightarrow (0, \infty]$ by

$$r(y) = \frac{1}{2} \inf\{\|f'(x) - f'(x')\| \mid x, x' \in f^{-1}\{y\}, x \neq x'\},$$

then we define

$$\alpha_x(t)(v) = tv + (1-t) \left(f'(x) + \frac{v}{1 + \|v\|/r(f(x))} \right).$$

This gives a T -lifting $\alpha \in L$. It depends continuously on f' , so we can define a map $\sigma: L' \rightarrow L$ by $\sigma(f') = \alpha$. By construction we have $\alpha_x(0)(0) = f'(x)$, so $\pi\sigma = 1: L' \rightarrow L'$. We claim that $\sigma\pi: L \rightarrow L$ is homotopic to the identity. The proof is essentially the same as that of the main theorem in [21], using the formulae

displayed in Figure 1 of that paper. The resulting homotopy involves no arbitrary choices, so it is not hard to check that it respects equivariance.

We next claim that L' (and therefore L) is path connected, provided that W contains a subbundle isomorphic to $\mathbb{C}[G] \times Y$. It is clear that everything works independently over the different orbits in Y , so we can reduce to the case $Y = G/H$. Next, we recall that the category of G -sets over G/H is equivalent to the category of H -sets, by the functor sending $(U \xrightarrow{p} G/H)$ to $p^{-1}\{H/H\}$. Similarly, the category of G -equivariant vector bundles over G/H is equivalent to the category of representations of H . Under this equivalence, the constant bundle $\mathbb{C}[G] \times G/H$ becomes a sum of $|G/H|$ copies of $\mathbb{C}[H]$, which certainly contains $\mathbb{C}[H]$. Using these observations we can reduce further to the case where Y is a single point, and W is a representation of G containing $\mathbb{C}[G]$.

Now choose a list H_1, \dots, H_r of subgroups of G containing precisely one subgroup in each conjugacy class. For each orbit in X we can choose a representative, whose stabiliser will be conjugate to one of the groups H_i . After changing the representative we can assume that the stabiliser is actually equal to H_i . Using this we obtain a splitting $X = \coprod_i G/H_i \times A_i$, where G acts trivially on A_i . This in turn gives

$$\mathrm{Map}_G(X, W) = \prod_i \mathrm{Map}(A_i, W^{H_i}).$$

Now put

$$U_i = \{w \in W \mid \mathrm{stab}_G(w) = H_i\} = W^{H_i} \setminus \bigcup_{K > H_i} W^K.$$

We find that

$$L' = \mathrm{Inj}_G(X, W) = \prod_i \mathrm{Inj}(A_i, U_i).$$

The assumption $W \geq \mathbb{C}[G]$ implies that W^{H_i} is a vector space of real dimension at least two, and that U_i is obtained from W^{H_i} by removing finitely many subspaces of real codimension at least two. Similarly, we find that $\mathrm{Map}(A_i, W^{H_i})$ has dimension at least two, and $\mathrm{Inj}(A_i, U_i)$ is obtained by removing finitely many subspaces of real codimension at least two. This implies that $\mathrm{Inj}(A_i, U_i)$ is path connected, and thus that L' is path connected. \square

CONSTRUCTION 8.13. Consider a morphism $f: X \rightarrow Y$ of finite G -sets, and an equivariant vector bundle W over Y . Let α be a T -lifting of f . We can apply the Pontrjagin-Thom construction to $\alpha_{[y]}$ to get a map

$$\alpha_{[y]}^!: S^{W_y} \rightarrow \bigvee_{x \in f^{-1}\{y\}} S^{W_x}.$$

Now suppose we have a point $t \in \Pi^G(f^*W; E)$, or in other words a system of based maps $t_x: S^{W_{f(x)}} \rightarrow E(W_{f(x)})$ that are compatible with the G -action. For each $y \in Y$ we let s_y denote the composite

$$S^{W_y} \xrightarrow{\alpha_{[y]}^!} \bigvee_{x \in f^{-1}\{y\}} S^{W_x} \xrightarrow{t_{[y]}} E(W_y),$$

where $t_{[y]}$ is given by t_x on the wedge summand indexed by x . This defines a point $s \in \Pi^G(W; E)$, which depends naturally and continuously on all the ingredients, so we can define a map

$$T_{f,\alpha}: \Pi^G(f^*W; E) \rightarrow \Pi^G(W; E)$$

by $t \mapsto s$. Note also that if we embed W in a larger bundle $W' = W \oplus A$ then the maps

$$\alpha'_x(t) = \alpha_x(t) \times 1: W'_{f(x)} = W_{f(x)} \times A_{f(x)} \rightarrow W_{f(x)} \times A_{f(x)} = W'_{f(x)}$$

also form a T -lifting of f , and the diagram

$$\begin{array}{ccc} \Pi^G(f^*W; E) & \xrightarrow{T_{f,\alpha}} & \Pi^G(W; E) \\ \downarrow & & \downarrow \\ \Pi^G(f^*W'; E) & \xrightarrow{T_{f,\alpha'}} & \Pi^G(W'; E) \end{array}$$

commutes.

Next, we can apply π_0 to the above map $T_{f,\alpha}$ to get a map $T_{f,\alpha}: \pi^G(f^*W; E) \rightarrow \pi^G(W; E)$. Provided that W contains $\mathbb{C}[G] \times Y$, the space of possible lifts α will be path-connected, so $T_{f,\alpha}: \pi^G(f^*W; E) \rightarrow \pi^G(W; E)$ will be independent of α , so we can denote it by T_f . Now note that the category of bundles W containing $\mathbb{C}[G] \times Y$ is cofinal in $G\mathcal{L}[Y]$, and the functor f^* from this subcategory to $G\mathcal{L}[X]$ is also cofinal. We can thus pass to the colimit to get a map $T_f: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Y)$.

PROPOSITION 8.14. *For any maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have*

$$T_{gf} = T_g T_f: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Z).$$

PROOF. Let U be a G -euclidean bundle over Z , giving a bundle $g^*(U)$ over Y and a bundle $(gf)^*(U)$ over X . Choose T -liftings α and β for f and g with respect to these bundles. Next, for $x \in X$ and $t \in [0, 1]$ let $\gamma_x(t)$ denote the composite

$$U_{gf(x)} \xrightarrow{\alpha_x(t)} U_{gf(x)} \xrightarrow{\beta_{f(x)}(t)} U_{gf(x)}.$$

These maps give a T -lifting for gf , and by inspection of the definitions we have

$$T_{gf,\gamma} = T_{g,\beta} T_{f,\alpha}: \Pi^G((gf)^*(U); E) \rightarrow \Pi^G(U; E).$$

The claim follows by taking π_0 and passing to colimits. \square

PROPOSITION 8.15. *Suppose we have a cartesian square of finite G -sets as shown:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Then $R_h T_f = T_{f'} R_g: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Y')$.

PROOF. Let W be a G -euclidean bundle over Y , and put $W' = h^*W$. Suppose we have a T -lifting α for f with respect to W . For $x' \in X'$ we put

$$\alpha'_{x'}(t) = \alpha_{g(x')}(t): ((f')^*W')_{x'} = W_{hf'(x')} = W_{fg(x')} \rightarrow ((f')^*W')_{x'}.$$

By inspection of the definitions we find that these maps give a T -lifting of f' , and that the diagram

$$\begin{array}{ccc} \Pi^G((f')^*W'; E) & \xrightarrow{T_{f',\alpha'}} & \Pi^G(W'; E) \\ \uparrow R_g & & \uparrow R_h \\ \Pi^G(f^*W; E) & \xrightarrow{T_{f,\alpha}} & \Pi^G(W; E) \end{array}$$

commutes. The claim follows by taking π_0 and passing to colimits. \square

PROPOSITION 8.16. *Suppose we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ of finite G -sets, with distributor*

$$\Delta(f, g) = (X \xrightarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y).$$

Then $N_g T_f = T_r N_q R_p: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Z)$.

PROOF. Consider a G -euclidean bundle W over Y . Note that for $(y, s) \in A$ we have $fp(y, s) = fs(y) = y$, so for $(z, s) \in B$ we have

$$(q_* p^* f^* W)_{(z,s)} = \bigoplus_{y \in g^{-1}\{z\}} (p^* f^* W)_{(y,s)} = \bigoplus_{y \in g^{-1}\{z\}} W_y = (r^* g_* W)_{(z,s)},$$

so we can identify $q_* p^* f^* W$ with $r^* g_* W$.

Now choose a T -lifting for f , consisting of maps $\alpha_x(t): W_{f(x)} \rightarrow W_{f(x)}$. Consider a point $(z, s) \in B$. For each $y \in g^{-1}\{z\}$ we have a point $s(y) \in X$ with $fs(y) = y$, and thus a shrunk isometry $\alpha_{s(y)}(t): W_y \rightarrow W_y$.

We can take the product over all $y \in g^{-1}\{z\}$ to get a map $\beta_{(z,s)}(t): (r^*g_*W)_y \rightarrow (r^*g_*W)_y$. These maps form a T -lifting of r . We now have maps

$$\Pi^G(f^*W; E) \xrightarrow{T_{f,\alpha}} \Pi^G(W; E) \xrightarrow{N_f} \Pi^G(g_*W; E)$$

and

$$\Pi^G(f^*W; E) \xrightarrow{R_p} \Pi^G(p^*f^*W; E) \xrightarrow{N_q} \Pi^G(q_*p^*f^*W; E) = \Pi^G(r^*g_*W; E) \xrightarrow{T_{r,\beta}} \Pi^G(g_*W; E).$$

We claim that the two composites are the same. To see this, consider an element $t \in \Pi^G(f^*W; E)$, consisting of maps $t_x: S^{W_{f(x)}} \rightarrow E(W_{f(x)})$ for all $x \in X$. The point $t' = T_{f,\alpha}(t) \in \Pi^G(W; E)$ consists of maps $t'_y: S^{W_y} \rightarrow E(W_y)$, where t'_y is the composite

$$S^{W_y} \xrightarrow{\alpha_{[y]}^1} \bigvee_{x \in f^{-1}\{y\}} S^{W_x} \xrightarrow{t_{[y]}} E(W_y).$$

Now the point $t'' = N_f t' \in \Pi^G(g_*W; E)$ consists of maps

$$t''_z: \bigwedge_{y \in g^{-1}\{z\}} S^{W_y} = S^{(g_*W)_z} \rightarrow E((g_*W)_z).$$

The map t''_z is obtained by smashing together the maps t'_y for all $y \in g^{-1}\{z\}$, then composing with the map

$$\bigwedge_{y \in g^{-1}\{z\}} E(W_y) \rightarrow E\left(\bigoplus_{y \in g^{-1}\{z\}} W_y\right)$$

given by the ring structure of the spectrum E . The smash product of the maps t'_y factors through the space

$$Q = \bigwedge_{y \in g^{-1}\{z\}} \bigvee_{x \in f^{-1}\{y\}} S^{W_x}.$$

As the smash product distributes over the wedge product, this can be rewritten as

$$Q = \bigvee_s \bigwedge_{y \in g^{-1}\{z\}} S^{W_y} = \bigvee_s \bigwedge_{y \in g^{-1}\{z\}} S^{W_{fs(y)}},$$

where s runs over the set of maps $f^{-1}\{y\} \rightarrow X$ with $fs = 1$. Using the fact that the Pontrjagin-Thom construction sends direct sums to smash products, we see that the map $S^{(r_*W)_z} = \bigwedge_{y \in g^{-1}\{z\}} S^{W_y} \rightarrow Q$ occurring in t''_z is just $\beta_{[z]}^!$. It is now straightforward to check that

$$N_g T_{f,\alpha} = T_{r,\beta} N_q R_p: \Pi^G(f^*W; E) \rightarrow \Pi^G(f_*W; E).$$

We now apply π_0 and pass to the colimit over W . The functor $f^*: G\mathcal{L}[Y] \rightarrow G\mathcal{L}[X]$ is cofinal, so $\lim_{\rightarrow W} \pi^G(f^*W; E) = \pi_0^G(E)(X)$. The functor $f_*: G\mathcal{L}[Y] \rightarrow G\mathcal{L}[Z]$ need not be cofinal, but we still have a comparison map $\lim_{\rightarrow W} \pi^G(f_*W; E) \rightarrow \pi_0^G(E)(Z)$. We deduce that $N_g T_f = T_r N_q R_p: \pi_0^G(E)(X) \rightarrow \pi_0^G(E)(Z)$ as claimed. \square

9. Tensor products of Tambara functors

Our main result is as follows:

PROPOSITION 9.1. *Let M and N be Tambara functors. Then there is a unique way to make the Mackey functor $M \boxtimes N$ into a Tambara functor such that $N_f(m \otimes n) = N_f(m) \otimes N_f(n)$ for all $f: X \rightarrow Y$ and all $(m, n) \in M(X) \times N(X)$. Moreover, with this structure, $M \boxtimes N$ is the coproduct of M and N in the category of Tambara functors.*

According to [17, Remark 1.9], this result also appears in an unpublished manuscript of Tambara.

The rest of this section constitutes the proof; the threads are gathered together in Corollary 9.7 and Lemma 9.8.

DEFINITION 9.2. Let M and N be Tambara functors, and let $X \xrightarrow{g} Y \xrightarrow{h} Z$ be equivariant maps of finite G -sets. We define

$$NT_{h,g}: M(X) \times N(X) \rightarrow (M \boxtimes N)(Z)$$

as follows. We first construct the distributor

$$\Delta(g, h) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$$

as in Definition 5.5, and then put

$$NT_{h,g}(m, n) = T_r(N_q R_p(m) \otimes N_q R_p(n)).$$

LEMMA 9.3. *Suppose we have maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ and elements $m' \in M(W)$ and $n \in N(X)$. Then*

$$NT_{h,g}(T_f(m'), n) = NT_{h,gf}(m', R_f(n)).$$

PROOF. We will use the diagram

$$\begin{array}{ccccccc} W & \xleftarrow{\tilde{p}} & \tilde{A} & \xrightarrow{\tilde{q}} & \tilde{B} & \xrightarrow{\tilde{r}} & Z \\ \parallel & & \downarrow i & & \downarrow k & & \parallel \\ W & \xleftarrow{p^*} & A^* & & & & \\ \downarrow f & & \downarrow j & & \downarrow & & \\ X & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Z \end{array}$$

constructed in Proposition 5.20. As $\Delta(g, h) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$ and the middle square is cartesian we have

$$NT_{h,g}(T_f(m'), n) = T_r(N_q R_p T_f(m') \otimes N_q R_p(n)) = T_r(N_q T_j R_{p^*}(m') \otimes N_q R_p(n)).$$

As $\Delta(j, q) = (A^* \xleftarrow{i} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{k} B)$, this can be rewritten as

$$NT_{h,g}(T_f(m'), n) = T_r(T_k N_{\tilde{q}} R_i R_{p^*}(m') \otimes N_q R_p(n)) = T_r(T_k N_{\tilde{q}} R_{\tilde{p}}(m') \otimes N_q R_p(n)).$$

Using Frobenius reciprocity with respect to k this becomes

$$NT_{h,g}(T_f(m'), n) = T_{rk}(N_{\tilde{q}} R_{\tilde{p}}(m') \otimes R_k N_q R_p(n)).$$

On the outside we have $rk = \tilde{r}$. For the second factor inside, we note that the middle rectangle is cartesian, so $R_k N_q = N_{\tilde{q}} R_{ji}$. Moreover, we have $pji = f\tilde{p}$, so $R_{ji} R_p = R_{\tilde{p}} R_f$. Putting this together we get

$$NT_{h,g}(T_f(m'), n) = T_{\tilde{r}}(N_{\tilde{q}} R_{\tilde{p}}(m') \otimes N_{\tilde{q}} R_{\tilde{p}} R_f(n)).$$

As

$$\begin{aligned} \Delta(gf, h) &= (W \xleftarrow{\tilde{p}} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{\tilde{r}} Z) \\ &= (W \xleftarrow{p^* i} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{rk} Z), \end{aligned}$$

this is the same as $NT_{h,gf}(m', R_f(n))$. \square

COROLLARY 9.4. *For any $h: Y \rightarrow Z$ there is a unique map $N_h: (M \boxtimes N)(Y) \rightarrow (M \boxtimes N)(Z)$ such that $N_h T_g(m \otimes n) = NT_{h,g}(m, n)$ for all $g: X \rightarrow Y$ and all $(m, n) \in M(X) \times N(X)$.*

PROOF. Lemma 9.3 gives an identity

$$NT_{h,g}(T_f(m'), n) = NT_{h,gf}(m', R_f(n)).$$

There is a similar identity

$$NT_{h,g}(m, T_f(n')) = NT_{h,gf}(R_f(m), n')$$

which can be proved in the same way or deduced using the twist map. The claim follows directly from these together with Proposition 3.14. \square

LEMMA 9.5. For any maps $X \xrightarrow{g} Y \xrightarrow{h} Z$ we have

$$N_h N_g = N_{hg}: (M \boxtimes N)(X) \rightarrow (M \boxtimes N)(Z).$$

PROOF. Consider another map $f: W \rightarrow X$ and elements $(m, n) \in M(W) \times N(W)$. We must show that $N_h N_g T_f(m \otimes n) = N_{hg} T_f(m \otimes n)$. We use the diagram

$$\begin{array}{ccccccc} \tilde{A} & \xrightarrow{\alpha} & A & \xrightarrow{p} & W & \xrightarrow{f} & X \\ \downarrow i & & \downarrow q & & & & \downarrow g \\ \tilde{B} & \xrightarrow{\beta} & B & \xrightarrow{r} & & & Y \\ \downarrow j & & & & & & \downarrow h \\ \tilde{C} & \xrightarrow{k} & & & & & Z \end{array}$$

constructed in Proposition 5.19. As $\Delta(f, g) = (W \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$, we have $N_g T_f(m \otimes n) = T_r(N_q R_p(m) \otimes N_q R_p(n))$. As $\Delta(r, h) = (B \xleftarrow{\beta} \tilde{B} \xrightarrow{j} \tilde{C} \xrightarrow{k} Z)$, we get

$$N_h N_g T_f(m \otimes n) = T_k(N_j R_\beta N_q R_p(m) \otimes N_j R_\beta N_q R_p(n)).$$

As the top left square is cartesian, we have $R_\beta N_q = N_j R_\alpha$, so we get

$$N_h N_g T_f(m \otimes n) = T_k(N_{ji} R_{p\alpha}(m) \otimes N_{ji} R_{p\alpha}(n)).$$

As $\Delta(f, hg) = (W \xleftarrow{p\alpha} \tilde{A} \xrightarrow{ji} \tilde{C} \xrightarrow{k} Z)$, this is the same as $N_{hg} T_f(m \otimes n)$, as required. \square

LEMMA 9.6. For any pullback square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{Y} \\ \downarrow j & & \downarrow k \\ X & \xrightarrow{g} & Y \end{array}$$

we have $R_k N_g = N_{\tilde{g}} R_j: (M \boxtimes N)(X) \rightarrow (M \boxtimes N)(\tilde{Y})$.

PROOF. Consider a map $f: W \rightarrow X$ and an element $u = T_f(m \otimes n) \in (M \boxtimes N)(X)$. We let \tilde{W} be the pullback of W , so we have a diagram

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{\tilde{f}} & X & \xrightarrow{\tilde{g}} & Y \\ \downarrow i & & \downarrow j & & \downarrow k \\ W & \xrightarrow{f} & X & \xrightarrow{g} & Y. \end{array}$$

in which both squares (and thus the full rectangle) are cartesian. As in Proposition 5.21 there is a commutative diagram

$$\begin{array}{ccccccc} \tilde{W} & \xleftarrow{\tilde{p}} & \tilde{A} & \xrightarrow{\tilde{q}} & \tilde{B} & \xrightarrow{\tilde{r}} & \tilde{Y} \\ \downarrow i & & \downarrow \alpha & & \downarrow \beta & & \downarrow k \\ W & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Y \end{array}$$

in which the middle and right squares are cartesian, the top row is $\Delta(\tilde{f}, \tilde{g})$, and the bottom row is $\Delta(f, g)$. We now have

$$\begin{aligned} N_g(u) &= T_r(N_q R_p(m) \otimes N_q R_p(n)) \\ R_k N_g(u) &= T_{\tilde{r}} R_\beta(N_q R_p(m) \otimes N_q R_p(n)) = T_{\tilde{r}}(R_\beta N_q R_p(m) \otimes R_\beta N_q R_p(n)). \end{aligned}$$

As the middle square is cartesian we have $R_\beta N_q = N_{\bar{q}} R_\alpha$. As $p\alpha = i\tilde{p}$ we have $R_\alpha R_p = R_{\bar{p}} R_i$. Putting this together, we get

$$R_k N_g(u) = T_{\bar{r}}(N_{\bar{q}} R_{\bar{p}} R_i(m) \otimes N_{\bar{q}} R_{\bar{p}} R_i(n)).$$

On the other hand, we have

$$\begin{aligned} R_j(u) &= T_{\bar{r}} R_i(m \otimes n) = T_{\bar{r}}(R_i(m) \otimes R_i(n)) \\ N_{\bar{g}} R_j(u) &= T_{\bar{r}}(N_{\bar{q}} R_{\bar{p}} R_i(m) \otimes N_{\bar{q}} R_{\bar{p}} R_i(n)), \end{aligned}$$

which is the same as $R_k N_g(u)$. \square

COROLLARY 9.7. *Our definition of N_h makes the Mackey functor $M \boxtimes N$ into a Tambara functor.*

PROOF. It will suffice to check that the relations in Proposition 6.9 are satisfied. Many of these involve only T and R so they are automatically satisfied, because $M \boxtimes N$ is a Mackey functor. Of the remaining relations, part (a) is covered by Lemma 9.5 (together with the easy fact that $N_1 = 1$). Part (b) is visibly built in to our definition of N_h . Part (c) is Lemma 9.6. \square

LEMMA 9.8. *There is a morphism $i: M \rightarrow M \boxtimes N$ of Tambara functors given by $i(m) = m \otimes 1$ for all $m \in M(X)$ (where 1 denotes the multiplicative identity element for the semiring structure on $N(X)$ discussed in Proposition 6.11). Similarly, there is a morphism $j: N \rightarrow M \boxtimes N$ given by $j(n) = 1 \otimes n$. Moreover, the diagram $M \xrightarrow{i} M \boxtimes N \xleftarrow{j} N$ is a coproduct.*

PROOF. Consider maps $W \xrightarrow{f} X \xrightarrow{g} Y$ and an element $m \in M(X)$. As $R_f: N(X) \rightarrow N(W)$ is a semiring map we have

$$R_f i(m) = R_f(m \otimes 1) = R_f(m) \otimes R_f(1) = R_f(m) \otimes 1 = i R_f(m).$$

Similarly, the map $N_g: N(X) \rightarrow N(Y)$ preserves 1 so we have

$$N_g i(m) = N_g(m \otimes 1) = N_g(m) \otimes N_g(1) = N_g(m) \otimes 1 = i N_g(m).$$

We can also use Frobenius reciprocity (Lemma 3.13) to get

$$T_g i(m) = T_g(m \otimes 1) = T_g(m \otimes R_g(1)) = T_g(m) \otimes 1 = i T_g(m).$$

This proves that i is a morphism of Tambara functors, and by symmetry the same is true of j .

Now suppose we have a Tambara functor S and morphisms $M \xrightarrow{d} S \xleftarrow{e} N$. We claim that for each X there is a unique map $r: (M \boxtimes N)(X) \rightarrow S(X)$ satisfying

$$r(T_q(m \otimes n)) = T_q(d(m)e(n))$$

for all $q: U \rightarrow X$ and $(m, n) \in M(U) \times N(U)$. In view of Proposition 3.14, it will suffice to show that

- (a) For all $U' \xrightarrow{r} U \xrightarrow{q} X$ and $(m', n) \in M(U') \times N(U)$ we have $T_{qr}(d(m')e(R_r(n))) = T_q(d(T_r(m'))e(n))$.
- (b) For all $U' \xrightarrow{r} U \xrightarrow{q} X$ and $(m, n') \in M(U) \times N(U')$ we have $T_{qr}(d(R_r(m))e(n')) = T_q(d(m)e(T_r(n')))$.

Part (a) can be rewritten as $T_{qr}(d(m')R_r(e(n))) = T_q(T_r(d(m'))e(n))$ and in this form it is a special case of Proposition 6.11(b). Part (b) is similar, so we have maps $r: (M \boxtimes N)(X) \rightarrow S(X)$ as claimed. It is clear by construction that $rT_f = T_f r$ for any $f: W \rightarrow X$. We claim that $rN_f = N_f r$ also holds. To see this, consider an element $z = T_q(m \otimes n) \in (M \boxtimes N)(W)$, and write the distributor $\Delta(q, f)$ as $(U \xleftarrow{p^*} A \xrightarrow{q^*} B \xrightarrow{r^*} X)$. We then have

$$\begin{aligned} N_f(z) &= T_{r^*}(N_{q^*} R_{p^*}(m) \otimes N_{q^*} R_{p^*}(n)) \\ rN_f(z) &= T_{r^*}(d(N_{q^*} R_{p^*}(m)) e(N_{q^*} R_{p^*}(n))) \\ &= T_{r^*}(N_{q^*} R_{p^*}(d(m)) N_{q^*} R_{p^*}(e(n))) \\ &= T_{r^*} N_{q^*} R_{p^*}(d(m) e(n)) \\ &= N_f T_q(d(m) e(n)) = N_f r(z), \end{aligned}$$

as claimed.

Now consider instead a map $g: V \rightarrow W$; we claim that $rR_g = R_g r: (M \boxtimes N)(W) \rightarrow S(V)$. To see this, we construct a pullback square as follows.

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{g}} & U \\ \tilde{q} \downarrow & & \downarrow q \\ V & \xrightarrow{g} & W \end{array}$$

For $z = T_q(m \otimes n)$ as before, we then have

$$\begin{aligned} R_g(z) &= R_g T_q(m \otimes n) = T_{\tilde{q}} R_{\tilde{g}}(m \otimes n) = T_{\tilde{q}}(R_{\tilde{g}}(m) \otimes R_{\tilde{g}}(n)) \\ rR_g(z) &= T_{\tilde{q}}(d(R_{\tilde{g}}(m)) e(R_{\tilde{g}}(n))) = T_{\tilde{q}}(R_{\tilde{g}}(d(m)) R_{\tilde{g}}(e(n))) \\ &= T_{\tilde{q}} R_{\tilde{g}}(d(m) e(n)) = R_g T_q(d(m) e(n)) = R_g r(z) \end{aligned}$$

as required. This proves that r is a morphism of Tambara functors. It is easily seen to be the unique one with $ri = d$ and $rj = e$, so the diagram $M \xrightarrow{i} M \boxtimes N \xleftarrow{j} N$ is a coproduct, as claimed. \square

10. Limits and colimits

In this section we will analyse limits and colimits in the categories Mackey_G and Tambara_G . We could do this by quoting general results about coloured Lawvere theories, but it is not hard to argue more directly.

PROPOSITION 10.1. *Let $S: \mathcal{J} \rightarrow \text{Tambara}_G$ be a diagram of Tambara functors, and put $S^*(X) = \lim_{\leftarrow j \in \mathcal{J}} S(j)(X)$. Then there is a canonical way to make S^* into a Tambara functor, and it becomes the limit of the diagram in Tambara_G . Moreover, the same applies for diagrams of Mackey functors.*

PROOF. It is standard that S^* is the limit of the diagram in the category of all functors from \overline{U}_G to sets. Moreover, as limits commute with products, we see that the map $S^*(X \amalg Y) \rightarrow S^*(X) \times S^*(Y)$ is a bijection for all X and Y in \overline{U}_G , so S^* is a Tambara functor. The rest is clear. \square

Filtered colimits are equally easy:

PROPOSITION 10.2. *Let $S: \mathcal{J} \rightarrow \text{Tambara}_G$ be a diagram of Tambara functors, where \mathcal{J} is a filtered category. Put $S^*(X) = \lim_{\rightarrow j \in \mathcal{J}} S(j)(X)$. Then there is a canonical way to make S^* into a Tambara functor, and it becomes the colimit of the diagram in Tambara_G . Moreover, the same applies for diagrams of Mackey functors.*

PROOF. It is standard that S^* is the colimit of the diagram in the category of all functors from \overline{U}_G to sets. Moreover, as filtered colimits in the category of sets commute with finite products, we see that the map $S^*(X \amalg Y) \rightarrow S^*(X) \times S^*(Y)$ is a bijection for all X and Y in \overline{U}_G , so S^* is a Tambara functor. The rest is clear. \square

COROLLARY 10.3. *The categories Mackey_G and Tambara_G have all set-indexed coproducts.*

PROOF. Finite coproducts exist in both categories: in Mackey_G they are the same as finite products, and in Tambara_G they are tensor products (by Proposition 9.1). Infinite coproducts can be expressed as filtered colimits of finite coproducts. \square

We can now generalise Definition 2.2:

DEFINITION 10.4.

A *congruence* on an algebraic structure P is a substructure of $P \times P$ that is also an equivalence relation. In more detail:

- (a) A congruence on a semigroup M_1 is a subsemigroup of $M_1 \times M_1$ that is also an equivalence relation.
- (b) A congruence on a semiring S_1 is a subsemiring of $S_1 \times S_1$ that is also an equivalence relation.
- (c) A congruence on a Mackey functor M is a sub-Mackey functor $E \leq M \times M$ such that the subset $E(X) \subseteq M(X) \times M(X)$ is an equivalence relation on $M(X)$ for all X .

- (d) A congruence on a Tambara functor S is a sub-Tambara functor $E \leq S \times S$ such that the subset $E(X) \subseteq S(X) \times S(X)$ is an equivalence relation on $S(X)$ for all X .

PROPOSITION 10.5.

- (a) For any morphism $\phi: S \rightarrow S'$ of Tambara functors there is a congruence $\text{eqker}(\phi)$ on S given by
$$\text{eqker}(\phi)(X) = \{(a, b) \in S(X) \times S(X) \mid \phi(a) = \phi(b) \in S'(X)\}.$$
- (b) For any congruence E on S there is a unique way to make the quotient sets $(S/E)(X) = S(X)/E(X)$ into a Tambara functor such that the projection $\pi: S \rightarrow S/E$ is a Tambara morphism.
- (c) If $E \leq \text{eqker}(\phi)$ then there is a unique Tambara morphism $\bar{\phi}: S/E \rightarrow S'$ with $\phi = \bar{\phi}\pi$, but if $E \not\leq \text{eqker}(\phi)$ then there is no such morphism.

Moreover, the corresponding statements also hold for Mackey functors.

PROOF. Most of this is clear but we offer a few pointers.

- (a) If ω is a bispan from X to Y and $(a, b) \in \text{eqker}(\phi)(X)$ then

$$\phi(f_\omega(a)) = f_\omega(\phi(a)) = f_\omega(\phi(b)) = \phi(f_\omega(b))$$

so $f_\omega(a, b) \in \text{eqker}(\phi)(Y)$. Moreover, it is clear from the diagram

$$\begin{array}{ccc} S(X \amalg Y) & \xrightarrow{\phi} & S'(X \amalg Y) \\ (R_i, R_j) \downarrow \simeq & & \simeq \downarrow (R_i, R_j) \\ S(X) \times S(Y) & \xrightarrow{\phi \times \phi} & S'(X) \times S'(Y) \end{array}$$

that $\text{eqker}(\phi)(X \amalg Y) = \text{eqker}(\phi)(X) \times \text{eqker}(\phi)(Y)$. This proves that $\text{eqker}(\phi)$ is a sub-Tambara functor of $S \times S$. It is clear that it is also an equivalence relation.

- (b) Consider a bispan ω from X to Y . Any element $a \in (S/E)(X)$ can be written as $\pi(a_0)$ for some $a_0 \in S(X)$. If $a = \pi(a_0) = \pi(a_1)$ then $(a_0, a_1) \in E(X)$ but E is a sub-Tambara functor of $S \times S$ so $(f_\omega(a_0), f_\omega(a_1)) \in E(Y)$ so $\pi(f_\omega(a_0)) = \pi(f_\omega(a_1))$. We thus have a well-defined operation $f_\omega: (S/E)(X) \rightarrow (S/E)(Y)$ given by $\pi(a_0) \mapsto \pi(f_\omega(a_0))$. Moreover, for any finite G -sets X and Y we have $S(X \amalg Y) = S(X) \times S(Y)$ and $E(X \amalg Y) = E(X) \times E(Y)$; it follows directly from this that $(S/E)(X \amalg Y) = (S/E)(X) \times (S/E)(Y)$. The rest is now clear.
- (c) Left to the reader. □

PROPOSITION 10.6. *The category of Tambara functors has coequalisers, as does the category of Mackey functors.*

PROOF. Consider a pair of Tambara morphisms $\phi, \psi: S' \rightarrow S$. Let \mathcal{E} be the collection of all congruences $E \leq S \times S$ such that for all finite G -sets X and all elements $a' \in S'(X)$ we have $(\phi(x), \psi(x)) \in E(X)$. (Note that any congruence is determined by the subsets $E(G/H) \subseteq S(G/H) \times S(G/H)$ for all $H \leq G$, so \mathcal{E} is a set rather than a proper class.) Now put

$$E_1(X) = \{(a, b) \in S(X) \times S(X) \mid (a, b) \in E(X) \text{ for all } E \in \mathcal{E}\}.$$

One can check that this is itself a congruence with $(\phi(a'), \psi(a')) \in E_1(X)$ for all $a' \in S'(X)$, so E_1 is the smallest element of \mathcal{E} . Using Proposition 10.5 we see that the projection $\pi: S \rightarrow S/E_1$ is a coequaliser for ϕ and ψ . The proof for Mackey functors is the same. □

COROLLARY 10.7. *The category of Tambara functors has colimits for all set-indexed diagrams, as does the category of Mackey functors.*

PROOF. For any diagram $S: \mathcal{J} \rightarrow \text{Tambara}_G$, there is a well-known way to construct the colimit as the coequaliser of a pair of maps

$$\coprod_{u: i \rightarrow j} S(i) \rightrightarrows \coprod_k S(k).$$

The proof for Mackey functors is the same. □

PROPOSITION 10.8. *Let $\phi, \psi: S' \rightarrow S$ be morphisms of Tambara functors, with coequaliser $\pi: S \rightarrow Q$. Suppose that there is a Tambara morphism $\sigma: S \rightarrow S'$ with $\phi\sigma = \psi\sigma = 1$ (or in other words, that we have a reflexive coequaliser diagram). Then for each finite G -set X , the map $\pi: S(X) \rightarrow Q(X)$ is a coequaliser (in the category of sets) for the maps $\phi, \psi: S'(X) \rightarrow S(X)$.*

PROOF. For each finite G -set X we put

$$F(X) = \text{img}((\phi, \psi): S'(X) \rightarrow S(X) \times S(X))$$

$$E(X) = \text{the smallest equivalence relation on } S(X) \text{ containing } F(X) .$$

We claim that E is actually a sub-Tambara functor of $S \times S$. Assuming this, we see that it is actually a congruence on S , and thus that it is the smallest congruence containing F , so $Q = S/E$. It is clear by construction that $S(X)/E(X)$ is the coequaliser of the maps $\phi, \psi: S'(X) \rightarrow S(X)$, so the proposition will follow.

To see that E is a sub-Tambara functor, we first consider a bispan ω from X to Y . Put

$$E^*(X) = \{(a, b) \in S(X) \times S(X) \mid (f_\omega(a), f_\omega(b)) \in E(Y)\}.$$

As $E(Y)$ is an equivalence relation, it follows easily that $E^*(X)$ is an equivalence relation. Moreover, as F is a sub-Tambara functor we have $(f_\omega \times f_\omega)(F(X)) \subseteq F(Y) \subseteq E(Y)$, so $F(X) \subseteq E^*(X)$. Using the minimality property of $E(X)$ we deduce that $E(X) \subseteq E^*(X)$, which means that $(f_\omega \times f_\omega)(E(X)) \subseteq E(Y)$. This proves that E is a subfunctor of $S \times S$.

Now let X and Y be any two finite G -sets. By applying the above to the bispans $R_i \in \overline{\mathcal{U}}_G(X \amalg Y, X)$ and $R_j \in \overline{\mathcal{U}}_G(X \amalg Y, Y)$ we see that $E(X \amalg Y) \subseteq E(X) \times E(Y)$. We would like to prove that in fact $E(X \amalg Y) = E(X) \times E(Y)$. To see this, consider an element $a \in S(X)$ and put

$$E[a](Y) = \{(b_0, b_1) \in S(Y) \times S(Y) \mid ((a, b_0), (a, b_1)) \in E(X \amalg Y)\}.$$

Note that for any $a' \in S'(Y)$ we have a point $(\sigma(a), a') \in S'(X \amalg Y)$ with $\phi(\sigma(a), a') = (a, \phi(a'))$ and $\psi(\sigma(a), a') = (a, \psi(a'))$ so

$$((a, \phi(a')), (a, \psi(a'))) \in F(X \amalg Y).$$

This shows that $F(Y) \subseteq E[a](Y)$, and $E[a](Y)$ is easily seen to be an equivalence relation, so $E(Y) \subseteq E[a](Y)$. In other words, whenever $a \in S(X)$ and $(b_0, b_1) \in E(Y)$ we have $((a, b_0), (a, b_1)) \in E(X \amalg Y)$. By a similar argument, whenever $(a_0, a_1) \in E(X)$ and $b \in S(Y)$ we have $((a_0, b), (a_1, b)) \in E(X \amalg Y)$. As $E(X \amalg Y)$ is a transitive relation, we deduce that for all $(a_0, a_1) \in E(X)$ and all $(b_0, b_1) \in E(Y)$ we have $((a_0, b_0), (a_1, b_1)) \in E(X \amalg Y)$. Thus, we have $E(X) \times E(Y) \subseteq E(X \amalg Y)$ as required. This completes the proof that E is a sub-Tambara functor of $S \times S$, and the proposition follows from this as we explained earlier. \square

Congruences in an additively complete Tambara functor biject with Tambara ideals, which have been studied in detail by Nakaoka [18]. We recall the main definition:

DEFINITION 10.9. Let S be an additively complete Tambara functor. An *ideal* in S is a collection of ideals $I(X) \leq S(X)$ such that for every $f: X \rightarrow Y$ we have

$$\begin{aligned} R_f(I(Y)) &\subseteq I(X) \\ T_f(I(X)) &\subseteq I(Y) \\ N_f(I(X)) &\subseteq I(Y) + N_f(0). \end{aligned}$$

Recall here that $N_f(0)$ is $(0, 1)$ with respect to the splitting

$$S(Y) = S(f(X)) \times S(Y \setminus f(X)),$$

as was proved in Lemma 6.8.

PROPOSITION 10.10. *Let S be an additively complete Tambara functor. Then for any congruence E on S we can define a Tambara ideal I_E by*

$$I_E(X) = \{a \in S(X) \mid (a, 0) \in E(X)\}.$$

Moreover, the construction $E \mapsto I_E$ gives a bijection between congruences and ideals.

This is more or less clear from the results of [18], but we will spell it out.

PROOF. Let π denote the usual quotient morphism from S to S/E . As $S(X)$ has additive inverses, the same is true of the quotient $(S/E)(X)$, so $(S/E)(X)$ is also a ring rather than just a semiring. Moreover, $I_E(X)$ is the kernel of the map $\pi: S(X) \rightarrow (S/E)(X)$. As R_f and T_f are additive homomorphisms that commute with π , it is clear that $R_f(I_E(Y)) \subseteq I_E(X)$ and $T_f(I_E(X)) \subseteq I_E(Y)$. Now suppose we have an element $a \in I_E(X)$. Then $(a, 0) \in E(X)$ but E is a sub-Tambara functor of $S \times S$ so $(N_f(a), N_f(0)) \in E(Y)$. Also $E(Y)$ is an equivalence relation, so it contains $(-N_f(0), -N_f(0))$, and it is closed under addition, so $(N_f(a) - N_f(0), 0) \in E(Y)$. This means that $N_f(a) - N_f(0) \in I_E(Y)$, so $N_f(I_E(X)) \subseteq I_E(Y) + N_f(0)$ as required.

Now let J be an arbitrary Tambara ideal, and put

$$F_J(X) = \{(a, b) \in S(X) \mid a - b \in J(X)\}.$$

It is clear that this is a Mackey functor congruence; we claim that it is also closed under norm maps. Suppose we have $(a, b) \in F_J(X)$, so $a = b + c$ for some $c \in J(X)$. The pair (b, c) gives an element of the semiring

$$S(X \times \{0, 1\}) \simeq S(X \amalg X) \simeq S(X) \times S(X).$$

The distributor for

$$X \times \{0, 1\} \xrightarrow{\text{proj}} X \xrightarrow{f} Y$$

is easily identified with the diagram

$$X \times \{0, 1\} \xleftarrow{(p_0, p_1)} A_0 \amalg A_1 \xrightarrow{(q_0, q_1)} B \xrightarrow{r} Y,$$

where

$$\begin{aligned} A_0 &= \{(x, C) \mid x \in X, C \subseteq f^{-1}\{f(x)\}, x \notin C\} \\ A_1 &= \{(x, C) \mid x \in X, C \subseteq f^{-1}\{f(x)\}, x \in C\} \\ B &= \{(y, C) \mid y \in Y, C \subseteq f^{-1}\{y\}\} \\ p_i(x, C) &= (x, i) \\ q_i(x, C) &= (f(x), C) \\ r(y, C) &= y. \end{aligned}$$

It follows that if we put

$$\begin{aligned} b' &= R_{p_0}(b) \in S(A_0) & b'' &= N_{q_0}(b') \in S(B) \\ c' &= R_{p_1}(c) \in J(A_1) & c'' &= N_{q_1}(c') \in J(B) + N_{q_1}(0) \end{aligned}$$

then

$$N_f(a) = N_f(b + c) = N_f T_{\text{proj}}(b, c) = T_r(b'' c'').$$

Now split B as $B_0 \amalg B_1$, where

$$\begin{aligned} B_0 &= \{(y, \emptyset) \mid y \in Y\} \simeq Y \\ B_1 &= \{(y, C) \in B \mid C \neq \emptyset\} = q_1(A_1), \end{aligned}$$

and put $r_i = r|_{B_i}$. We write $b'' = (b''_0, b''_1)$ and $c'' = (c''_0, c''_1)$ with respect to the splitting $S(B) = S(B_0) \times S(B_1)$. As $c'' \in J(B) + N_{q_1}(0)$ and $B_1 = q_1(A_1)$ we have $c''_0 \in 1 + J(B_0)$ and $c''_1 \in J(B_1)$, so

$$N_f(a) = T_{r_0}(b''_0 c''_0) + T_{r_1}(b''_1 c''_1) \in J(Y) + T_{r_0}(b''_0).$$

Now r_0 is an isomorphism, and the preimage of B_0 in $A_0 \amalg A_1$ is $\{(x, \emptyset) \mid x \in X\} \subseteq A_0$, and this is mapped isomorphically by p_0 to X . Using this it is not hard to see that $T_{r_0}(b''_0) = N_f(b)$. We thus have $N_f(a) \in N_f(b) + J(Y)$, so $N_f(a, b) \in F_J(Y)$ as required.

It is clear that the constructions $E \mapsto I_E$ and $J \mapsto F_J$ are inverse to each other, so we have a bijection between congruences and ideals. \square

11. Semigroup semirings

For any set M , we have a free semigroup $\mathbb{N}[M]$ with one basis element (denoted $[m]$) for each element $m \in M$. If M is itself a semigroup, we can introduce a product on the set $\mathbb{N}[M]$ by the usual rule

$$\left(\sum_i a_i [m_i]\right) \left(\sum_j b_j [n_j]\right) = \sum_{i,j} a_i b_j [m_i + n_j]$$

(so $[m][n] = [m + n]$). This makes $\mathbb{N}[M]$ into a semiring. Conversely, if S is a semiring, we write US for S considered as a semigroup under multiplication. It is easy to see that there is an adjunction

$$\text{Semirings}(\mathbb{N}[M], S) \simeq \text{Semigroups}(M, US).$$

Our object in this section is to set up an analogous theory for Mackey functors and Tambara functors, which reduces to the above in the case where the group of equivariance is trivial. This construction was also given by Nakaoka [20].

DEFINITION 11.1. We define $U: \text{Tambara}_G \rightarrow \text{Mackey}_G$ as follows. For a Tambara functor S , the corresponding Mackey functor US is given on objects by $US(X) = S(X)$. For any map $f: X \rightarrow Y$, the operator $R_f^{US}: US(Y) \rightarrow US(X)$ is just the same as $R_f^S: S(Y) \rightarrow S(X)$, and the operator $T_f^{US}: US(X) \rightarrow US(Y)$ is the same as $N_f^S: S(X) \rightarrow S(Y)$. We call US the *underlying multiplicative Mackey functor* of S .

Our problem is to understand the left adjoint to U .

DEFINITION 11.2. Let M be a Mackey functor. For any finite G -set X , we define a groupoid $\mathcal{A}[M](X)$ as follows. The objects are triples (U, u, m) , where U is a finite G -set and $u: U \rightarrow X$ is an equivariant map and $m \in M(U)$. The morphisms from (U, u, m) to (U', u', m') are the equivariant bijections $p: U \rightarrow U'$ for which $u'p = u$ and $R_p(u') = u$ (or equivalently $u' = T_p(u)$). We write $A[M](X)$ for the set of isomorphism classes in this groupoid. We also write $[U, u, m]$ for the isomorphism class of (U, u, m) .

DEFINITION 11.3. Suppose we have an object $(U, u, m) \in \mathcal{A}[M](X)$.

- (a) For any map $g: X \rightarrow Y$, we define $T_g(U, u, m) = (U, gu, m) \in \mathcal{A}[M](Y)$. There is an evident way to define an action on morphisms so that this becomes a functor $T_f: \mathcal{A}[M](X) \rightarrow \mathcal{A}[M](Y)$. It therefore induces an operation $T_f: A[M](X) \rightarrow A[M](Y)$.
- (b) We also define $N_g(U, u, m) \in \mathcal{A}[M](Y)$ as follows: we form the distributor $\Delta(u, g) = (U \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$, then put $N_g(U, u, m) = (B, r, T_q R_p(m))$. This again induces an operation $N_f: A[M](X) \rightarrow A[M](Y)$.
- (c) Suppose instead we have a map $f: W \rightarrow X$. We form the pullback square

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{u}} & W \\ \tilde{f} \downarrow & & \downarrow f \\ U & \xrightarrow{u} & X \end{array}$$

and then put $R_f(U, u, m) = (\tilde{U}, \tilde{u}, R_{\tilde{f}}(m))$. This gives an operation $R_f: A[M](X) \rightarrow A[M](W)$.

PROPOSITION 11.4. *The above definitions make $A[M]$ into a Tambara functor. Moreover, this construction gives a functor $A[-]: \text{Mackey}_G \rightarrow \text{Tambara}_G$, which is left adjoint to U .*

The rest of this section will constitute the proof. □

LEMMA 11.5. *For any $X \xrightarrow{g} Y \xrightarrow{h} Z$ we have*

$$\begin{aligned} R_{hg} &= R_g R_h: A[M](Z) \rightarrow A[M](X) \\ T_{hg} &= T_h T_g: A[M](X) \rightarrow A[M](Z). \end{aligned}$$

PROOF. Straightforward. □

LEMMA 11.6. *For any $X \xrightarrow{g} Y \xrightarrow{h} Z$ we have $N_{hg} = N_h N_g: A[M](X) \rightarrow A[M](Z)$.*

PROOF. Consider an object $(W, f, m) \in \mathcal{A}[M](X)$. We have a chain of maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ which we use to build a diagram

$$\begin{array}{ccccc}
\tilde{A} & \xrightarrow{\alpha} & A & \xrightarrow{p} & W & \xrightarrow{f} & X \\
\downarrow i & & \downarrow q & & & & \downarrow g \\
\tilde{B} & \xrightarrow{\beta} & B & \xrightarrow{r} & & & Y \\
\downarrow j & & & & & & \downarrow h \\
\tilde{C} & \xrightarrow{k} & & & & & Z
\end{array}$$

as in Proposition 5.19. Now $\Delta(f, g) = (W \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$, so $N_g[W, f, m] = [B, r, T_q R_p(m)]$. To apply N_h to this, we use the distributor $\Delta(r, h) = (B \xleftarrow{\beta} \tilde{B} \xrightarrow{j} \tilde{C} \xrightarrow{k} Z)$, giving $N_h N_g[W, f, m] = [\tilde{C}, k, T_j R_\beta T_q R_p(m)]$. As the top left square is cartesian we have $R_\beta T_q = T_i R_\alpha$, so $N_h N_g[W, f, m] = [\tilde{C}, k, T_{ji} R_{p\alpha}(m)]$. We also know that $\Delta(f, hg) = (W \xleftarrow{p\alpha} \tilde{A} \xrightarrow{ji} \tilde{C} \xrightarrow{k} Z)$, so this is the same as $N_{hg}[W, f, m]$, as required. \square

LEMMA 11.7. *Suppose we have maps $X \xrightarrow{g} Y \xrightarrow{h} Z$ with distributor $\Delta(g, h) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$. Then $N_h T_g = T_r N_q R_p: \mathcal{A}[M](X) \rightarrow \mathcal{A}[M](Z)$.*

PROOF. Consider an object $(W, f, m) \in \mathcal{A}[M](X)$. We use the diagram

$$\begin{array}{ccccccc}
W & \xleftarrow{\tilde{p}} & \tilde{A} & \xrightarrow{\tilde{q}} & \tilde{B} & \xrightarrow{\tilde{r}} & Z \\
\parallel & & \downarrow i & & \downarrow k & & \parallel \\
W & \xleftarrow{p^*} & A^* & & & & \\
\downarrow f & & \downarrow j & & & & \\
X & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Z
\end{array}$$

constructed in Proposition 5.20. We have $T_g(W, f, m) = (W, gf, m)$ and $\Delta(gf, h) = (W \xleftarrow{\tilde{p}} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{\tilde{r}} Z)$ so $N_h T_g(W, f, m) = (\tilde{B}, \tilde{r}, N_{\tilde{q}} R_{\tilde{p}}(m))$. On the other hand, as the bottom left square is cartesian, we have $R_p(W, f, m) = (A^*, j, R_{p^*}(m))$. As $\Delta(j, q) = (A^* \xleftarrow{i} \tilde{A} \xrightarrow{\tilde{q}} \tilde{B} \xrightarrow{k} B)$, this gives

$$N_q R_p(W, f, m) = (\tilde{B}, k, N_{\tilde{q}} R_i R_{p^*}(m)) = (\tilde{B}, k, N_{\tilde{q}} R_{\tilde{p}}(m)).$$

We now apply T_r , noting that $rk = \tilde{r}$, to get

$$T_r N_q R_p(W, f, m) = (\tilde{B}, \tilde{r}, N_{\tilde{q}} R_{\tilde{p}}(m)) = N_h T_g(W, f, m)$$

as required. \square

LEMMA 11.8. *Suppose we have a pullback square*

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & \tilde{Y} \\
j \downarrow & & \downarrow k \\
X & \xrightarrow{g} & Y
\end{array}$$

Then $R_k T_g = T_{\tilde{g}} R_j$ and $R_k N_g = N_{\tilde{g}} R_j$ as operators $\mathcal{A}[M](X) \rightarrow \mathcal{A}[M](\tilde{Y})$.

PROOF. Consider an object $z = (W, f, m) \in A[M](X)$. First note that $T_g(z) = (W, gf, m)$. Let \widetilde{W} be the pullback of \widetilde{W} along f , so we have a diagram

$$\begin{array}{ccccc} \widetilde{W} & \xrightarrow{\tilde{f}} & \widetilde{X} & \xrightarrow{\tilde{g}} & \widetilde{Y} \\ \downarrow i & & \downarrow i & & \downarrow j \\ W & \xrightarrow{f} & X & \xrightarrow{g} & Y \end{array}$$

in which both squares are cartesian. It follows that the outer rectangle is also cartesian, so $R_k T_g(z) = R_k(W, gf, m) = (\widetilde{W}, \tilde{g}\tilde{f}, R_i(m))$. On the other hand, we have $R_j(z) = (\widetilde{W}, \tilde{f}, R_i(m))$, so $T_{\tilde{g}} R_j(z) = (\widetilde{W}, \tilde{g}\tilde{f}, R_i(m))$, which is the same as $R_k T_g(z)$, as required.

Now consider the diagram

$$\begin{array}{ccccccc} \widetilde{W} & \xleftarrow{\tilde{p}} & \widetilde{A} & \xrightarrow{\tilde{q}} & \widetilde{B} & \xrightarrow{\tilde{r}} & \widetilde{Y} \\ \downarrow i & & \downarrow \alpha & & \downarrow \beta & & \downarrow k \\ W & \xleftarrow{p} & A & \xrightarrow{q} & B & \xrightarrow{r} & Y \end{array}$$

constructed in Proposition 5.21. The bottom row is $\Delta(f, g)$, so we have $N_g(z) = (B, r, T_q R_p(m))$. As the right hand square is cartesian, this gives $R_k N_g(z) = (\widetilde{B}, \tilde{r}, R_\beta T_q R_p(m))$. As the middle square is also cartesian we have $R_\beta T_q = T_{\tilde{q}} R_\alpha: M(A) \rightarrow M(\widetilde{B})$. As $p\alpha = i\tilde{p}$ we also have $R_\alpha R_p = R_{\tilde{p}} R_i: M(W) \rightarrow M(\widetilde{A})$. Putting this together, we get $R_k N_g(z) = (\widetilde{B}, \tilde{r}, T_{\tilde{q}} R_{\tilde{p}} R_i(m))$. On the other hand, we have $R_j(z) = (\widetilde{W}, \tilde{f}, R_i(m))$ and $\Delta(\tilde{f}, \tilde{g})$ is the top row of the above diagram so we also have $N_{\tilde{g}} R_j(z) = (\widetilde{B}, \tilde{r}, T_{\tilde{q}} R_{\tilde{p}} R_i(m))$ as required. \square

COROLLARY 11.9. *The above definitions make $A[M]$ into a Tambara functor.*

PROOF. It will suffice to check the conditions in Proposition 6.9. Part (a) is covered by Lemmas 11.5 and 11.6. Part (b) is Lemma 11.7, and part (c) is Lemma 11.8. \square

DEFINITION 11.10. For any Mackey functor M , we define $\eta: M(X) \rightarrow UA[M](X)$ by $\eta(m) = [X, 1, m]$. For any Tambara functor S , we define $\epsilon: A[US](X) \rightarrow S(X)$ by $\epsilon[W, f, m] = T_f(m)$.

PROPOSITION 11.11. *The above definitions give natural maps that satisfy the triangular identities and so display $A[-]: \text{Mackey}_G \rightarrow \text{Tambara}_G$ as left adjoint to $U: \text{Tambara}_G \rightarrow \text{Mackey}_G$.*

PROOF. We first claim that η gives a morphism of Mackey functors. For any map $g: X \rightarrow Y$, we must show that the diagrams

$$\begin{array}{ccc} M(X) & \xrightarrow{T_g^M} & M(Y) \\ \eta \downarrow & & \downarrow \eta \\ UA[M](X) & \xrightarrow{T_g^{UA[M]}} & UA[M](Y) \end{array} \quad \begin{array}{ccc} M(Y) & \xrightarrow{R_g^M} & M(X) \\ \eta \downarrow & & \downarrow \eta \\ UA[M](Y) & \xrightarrow{R_g^{UA[M]}} & UA[M](X) \end{array}$$

commute. By definition, these are the same as

$$\begin{array}{ccc} M(X) & \xrightarrow{T_g^M} & M(Y) \\ \eta \downarrow & & \downarrow \eta \\ A[M](X) & \xrightarrow{N_g^{A[M]}} & A[M](Y) \end{array} \quad \begin{array}{ccc} M(Y) & \xrightarrow{R_g^M} & M(X) \\ \eta \downarrow & & \downarrow \eta \\ A[M](Y) & \xrightarrow{R_g^{A[M]}} & A[M](X) \end{array}$$

Consider an element $m \in M(X)$. The distributor $\Delta(1, g)$ is just $(X \xleftarrow{1} X \xrightarrow{g} Y \xrightarrow{1} Y)$, so

$$N_g \eta(m) = N_g[X, 1, m] = [Y, N_g R_1(m)] = \eta(N_g(m)),$$

which proves that the left square commutes. Consider instead an element $n \in M(Y)$. As the square

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ g \downarrow & & \downarrow g \\ Y & \xrightarrow{1} & Y \end{array}$$

is cartesian, we have

$$R_g \eta(n) = R_g[Y, 1, n] = [X, 1, R_g(n)] = \eta R_g(n)$$

as required. It is also clear that η is natural with respect to morphisms $M \rightarrow M'$ of Mackey functors.

We next claim that ϵ gives a morphism of Tambara functors. For any map $g: X \rightarrow Y$, we must show that the diagrams

$$\begin{array}{ccc} A[US](X) \xrightarrow{T_g^{A[US]}} A[US](Y) & A[US](X) \xrightarrow{N_g^{A[US]}} A[US](Y) & A[US](Y) \xrightarrow{R_g^{A[US]}} A[US](X) \\ \epsilon \downarrow & \epsilon \downarrow & \epsilon \downarrow \\ S(X) \xrightarrow{T_g^S} S(Y) & S(X) \xrightarrow{N_g^S} S(Y) & S(Y) \xrightarrow{R_g^S} S(X) \end{array}$$

commute. Consider an element $a = [W, f, m] \in A[US](X)$. We then have

$$\epsilon(T_g(a)) = \epsilon[W, gf, m] = T_{gf}(m) = T_g(T_f(m)) = T_g\epsilon(a),$$

which proves that the first diagram commutes. Now consider the distributor $\Delta(f, g) = (W \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$. We have

$$\epsilon(N_g(a)) = \epsilon[B, r, N_q R_p(m)] = T_r N_q R_p(m) = N_g T_f(m) = N_g \epsilon(a),$$

which proves that the middle diagram commutes.

Now consider instead an element $b = [V, e, n] \in A[US](Y)$. We form a pullback square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ f \downarrow & & \downarrow e \\ X & \xrightarrow{g} & Y \end{array}$$

so that $R_g(b) = [W, f, R_j(n)]$. This gives

$$\epsilon R_g(b) = T_f R_j(n) = R_g T_e(n) = R_g \epsilon(b),$$

which proves that the right hand diagram commutes. It is also clear that ϵ is natural with respect to morphisms $S \rightarrow S'$ of Tambara functors.

This just leaves the triangular identities. The first of these says that the composite

$$US(X) \xrightarrow{\eta} UA[US](X) \xrightarrow{U\epsilon} US(X)$$

is the identity. For any $s \in US(X) = S(X)$ we have $\epsilon(\eta(s)) = \epsilon[X, 1, s] = T_1^{US}(s) = s$ as required. The second one says that the composite

$$A[M](X) \xrightarrow{A[\eta]} A[UA[M]](X) \xrightarrow{\epsilon} A[M](X)$$

is also the identity. Consider an element $a = [W, f, m] \in A[M](X)$. We then have $A[\eta](a) = [W, f, \eta(m)] = [W, f, [W, 1, m]]$, so

$$\epsilon(A[\eta](a)) = T_f^{A[M]}[W, 1, m] = [W, f, m] = a,$$

as required. \square

12. Green semirings

DEFINITION 12.1. A *Green semiring* is a Mackey functor S equipped with a morphism $\mu: S \boxtimes S \rightarrow S$ that is commutative, associative and unital in the obvious sense.

We will not make much use of Green semirings. Our purpose in this section is simply to explain some nonobvious ways to understand their relationship with Tambara functors.

PROPOSITION 12.2. *Let S be a Mackey functor. To make S into a Green semiring is the same as to give each set $S(X)$ a semiring structure (extending the usual semigroup structure) in such a way that*

- (a) *For every map $f: X \rightarrow Y$ of finite G -sets, the resulting map $R_f: S(Y) \rightarrow S(X)$ is a homomorphism of semirings.*
- (b) *Moreover, for any $a \in S(X)$ and $b \in S(Y)$ we have $T_f(a R_f(b)) = T_f(a) b$ in $S(Y)$.*

PROOF. This follows easily from Proposition 3.14. □

DEFINITION 12.3. We say that an equivariant map $f: U \rightarrow V$ is *green* if for all $u \in U$, the image $f(u) \in V$ has the same isotropy group as u . We say that a bispan $\omega = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Y)$ is *green* if the map q is green.

Here of course it is automatic that the isotropy group of $f(u)$ is at least as large as that of u ; the force of the condition is that it can be no larger.

PROPOSITION 12.4. *The composite of any two composable green bispans is again green, and all identity bispans are green, so the green bispans give a subcategory $\overline{\mathcal{G}}_G \subseteq \overline{\mathcal{U}}_G$ (containing all the objects but not all the morphisms). Moreover, this subcategory is closed under products.*

PROOF. Consider a pair of green bispans

$$\begin{aligned}\omega_0 &= (X_0 \xleftarrow{p_0} A_0 \xrightarrow{q_0} B_0 \xrightarrow{r_0} X_1) \in \mathcal{U}(X_0, X_1) \\ \omega_1 &= (X_1 \xleftarrow{p_1} A_1 \xrightarrow{q_1} B_1 \xrightarrow{r_1} X_2) \in \mathcal{U}(X_1, X_2).\end{aligned}$$

The composite is the bispan

$$\omega = (X_0 \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} X_2) \in \mathcal{U}(X_0, X_2)$$

given by

$$\begin{aligned}A &= \{(a_0, a_1, s) \mid s: q_1^{-1}\{q_1(a_1)\} \rightarrow B_0, r_0 s = p_1, a_0 \in q_0^{-1}\{s(a_1)\}\} \\ B &= \{(b_1, s) \mid s: q_1^{-1}\{b_1\} \rightarrow B_0, r_0 s = p_1\} \\ p(a_0, a_1, s) &= p_0(a_0) \\ q(a_0, a_1, s) &= (q_1(a_1), s) \\ r(b_1, s) &= r_1(b_1).\end{aligned}$$

Consider a point $a = (a_0, a_1, s) \in A$ and an element $g \in G$ that satisfies $g.q(a) = q(a)$; we must show that $g.a = a$. The condition $g.q(a) = q(a)$ means that $g.q_1(a_1) = q_1(a_1)$ (so g preserves the fibre $F = q_1^{-1}\{q_1(a_1)\}$) and that the map $s: F \rightarrow B_0$ satisfies $g \circ s = s \circ g$. As ω_1 is green and $g.q_1(a_1) = q_1(a_1)$ we must have $g.a_1 = a_1$. As s is g -equivariant this gives $g.s(a_1) = s(a_1)$. As $a \in A$ we also have $q_0(a_0) = s(a_1)$, so $g.q_0(a_0) = q_0(a_0)$. As ω_0 is green we can conclude that $g.a_0 = a_0$. As a_0 and a_1 are fixed by g and s is g -equivariant we see that the triple $a = (a_0, a_1, s)$ has $g.a = a$, as required. This shows that composites of green bispans are green, and the corresponding fact for identities is trivial.

Now consider a pair of objects $Y, Z \in \overline{\mathcal{U}}_G$. We have inclusions $Y \xrightarrow{i} Y \amalg Z \xleftarrow{j} Z$, and we have seen that the resulting diagram $Y \xleftarrow{R_i} Y \amalg Z \xrightarrow{R_j} Z$ is a product diagram in $\overline{\mathcal{U}}_G$. Using the proof of this we see that a bispan $\alpha \in \mathcal{U}_G(U, Y \amalg Z)$ is green iff the composites $R_i \circ \alpha$ and $R_j \circ \alpha$ are green. The bispans R_i and R_j themselves are also clearly green. It follows that our diagram is still a product diagram in $\overline{\mathcal{G}}_G$. □

DEFINITION 12.5. A *Green functor* is a product-preserving functor $\overline{\mathcal{G}}_G \rightarrow \text{Sets}$. Any Green functor has operators T_f, N_f and R_f just as for Tambara functors, except that N_f is only defined when f is green. We write Green_G for the category of Green functors, and Green'_G for the category of Green semirings.

The rest of this section is devoted to constructing an equivalence $\text{Green}_G \simeq \text{Green}'_G$.

LEMMA 12.6. *Any projection map $\pi: I \times V \rightarrow V$ (where G acts trivially on I) is green. Conversely, if $p: U \rightarrow V$ is green and V is a G -orbit then p can be written as a composite $U \xrightarrow{m} I \times V \xrightarrow{\pi} V$, where I is again G -fixed and m is an equivariant bijection.*

PROOF. The first claim is clear. For the second claim, choose a point $v_1 \in V$, and let H be the isotropy group of v_1 . Put $I = p^{-1}\{v_1\} \subseteq U$. For each $u \in U$ we claim that the orbit Gu meets I in a single point. Indeed, as V is an orbit, we must have $v_1 = g.p(u)$ for some g , which means that $g.u \in I$. If we also have $g'.u \in I$ then $g'g^{-1}.v_1 = g'g^{-1}.p(gu) = p(g'u) = v_1$, so $g'g^{-1} \in H$. However, the green condition means that H also stabilises every point in I , so $g'g^{-1}.gu = gu$, or in other words $g'u = gu$ as required. We can thus define $m_1(u)$ to be the unique point in $Gu \cap I$; this gives a map $m_1: U \rightarrow I$ which is equivariant if we give I the trivial action. We now put $m(u) = (m_1(u), p(u)) \in I \times V$ (so $\pi m = p$). Suppose that $m(u) = m(u')$. As $m_1(u) = m_1(u')$ we see that u and u' lie in the same orbit, say $u' = g.u$. As $p(u') = p(u)$ we see that g stabilises $p(u)$ and thus (by the green condition) stabilises u , so $u' = u$. This shows that m is injective. Now consider an arbitrary point $(u, v) \in I \times V$. As V is an orbit there exists g with $g.v_1 = v$, and it follows that $m(g.u) = (u, v)$, proving that m is surjective. \square

LEMMA 12.7. *Let S be a Green semiring, and let $f: X \rightarrow Y$ be a green map. Then there is a unique function $N_f: S(X) \rightarrow S(Y)$ with the following property: for every transitive G -set U and every G -map $j: U \rightarrow Y$ and every $m \in S(X)$, we have*

$$R_j N_f(m) = \prod_{i: U \rightarrow X, fi=j} R_i(m) \in S(U).$$

PROOF. We can write $Y = U_1 \amalg \cdots \amalg U_r$, where the sets U_k are the orbits in Y , and let $j_k: U_k \rightarrow Y$ be the inclusion. As S is product-preserving we find that the maps $R_{j_k}: S(Y) \rightarrow S(U_k)$ give a bijection $\lambda: S(Y) \rightarrow \prod_k S(U_k)$. Define $\nu_k: S(X) \rightarrow S(U_k)$ by $\nu_k(m) = \prod_i R_i(m)$, where i runs over equivariant maps $U_k \rightarrow X$ with $fi = j_k$. These maps combine to give a map $\nu: S(X) \rightarrow \prod_k S(U_k)$, and we put $N_f = \lambda^{-1}\nu: S(X) \rightarrow S(Y)$. It is clear that this is the only map that can possibly have the stated property.

Now let U be an arbitrary transitive G -set, and let $j: U \rightarrow Y$ be a G -map. Then we must have $j = j_k p$ for some k and some surjective G -map $p: U \rightarrow U_k$. Let I_k be the set of G -maps $i': U_k \rightarrow X$ with $f i' = j_k$, and let I be the set of G -maps $i: U \rightarrow X$ with $fi = j$. We have a map $p^*: I_k \rightarrow I$ given by $p^*(i') = i'p$, and using the fact that f is green we see that this is bijective. It follows that

$$R_j N_f(m) = R_p R_{j_k} N_f(m) = R_p \prod_{i' \in I_k} R_{i'}(m) = \prod_{i' \in I_k} R_{i'p}(m) = \prod_{i \in I} R_i(m)$$

as required. \square

LEMMA 12.8. *For a projection map $\pi: I \times Y \rightarrow Y$ we have $N_\pi(m) = \prod_{i \in I} R_{\sigma_i}(m)$, where $\sigma_i: Y \rightarrow I \times Y$ is given by $\sigma_i(y) = (i, y)$.*

PROOF. Straightforward. \square

LEMMA 12.9. *The maps N_f as in Lemma 12.7 make S into a Green functor.*

PROOF. By the evident analog of Proposition 6.9, it will suffice to check the following:

- (a) For all finite G -sets X, Y and Z , and all G -maps $X \xleftarrow{f} Y \xrightarrow{g} Z$, we have $R_{gf} = R_f R_g: S(Z) \rightarrow S(X)$ and $T_{gf} = T_g T_f: S(X) \rightarrow S(Z)$. Moreover, if f and g are green, we also have $N_{gf} = N_g N_f: S(X) \rightarrow S(Z)$.
- (b) If we have G -maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ with distributor $\Delta(f, g) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z)$ (where g is green, and q is therefore also green) then $N_g T_f = T_r N_q R_p: S(X) \rightarrow S(Z)$.
- (c) For any cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

(of finite G -sets and equivariant maps) we have $T_f R_g = R_h T_k: S(X) \rightarrow S(Y)$. If f and h are green, we also have $N_f R_g = R_h N_k: S(X) \rightarrow S(Y)$.

In part (a), the equations $R_{gf} = R_f R_g$ and $T_{gf} = T_g T_f$ hold because S is assumed to be a Mackey functor. It is also clear that the composite $N_g N_f$ has the property that defines N_{gf} .

Now consider a square as in (c). The equation $T_f R_g = R_h T_k$ holds because S is assumed to be a Mackey functor. Suppose that f (and therefore also h) is green, and consider an element $m \in S(Y)$. We need to show that $N_f R_g(m) = R_h N_k(m) \in S(X)$, and it will suffice to show that $R_j N_f R_g(m) = R_j R_h N_k(m)$ for all G -orbits U and all injective G -maps $j: U \rightarrow X$. On the left hand side, we have $R_j N_f R_g(m) = \prod_{i \in I} R_i R_g(m) = \prod_{i \in I} R_{gi}(m)$, where I is the set of G -maps $i: U \rightarrow W$ with $fi = j$. On the right hand side, we have

$$R_j R_h N_k(m) = R_{hj} N_k(m) = \prod_{i' \in I'} R_{i'}(m),$$

where I' is the set of maps $i': U \rightarrow Y$ that satisfy $ki' = hj$. As the square is cartesian, we see that the map $i \mapsto gi$ gives a bijection $I \rightarrow I'$, so $R_j N_f R_g(m) = R_j R_h N_k(m)$ as claimed.

We now turn to (b). It is clear that everything works independently over the different orbits of Z , so we may assume that Z itself is a single orbit. Using Lemma 12.6 we can then reduce to the case where $Y = I \times Z = \prod_{i \in I} Z$ and $g: I \times Z \rightarrow Z$ is just the projection. We can then split the diagram $(X \xrightarrow{f} I \times Y)$ as a disjoint union of diagrams $(X_i \xrightarrow{f_i} Z)$. An element $m \in S(X)$ corresponds to a system of elements $m_i \in S(X_i)$, and using Lemma 12.8 we find that $N_g T_f(m) = \prod_i T_{f_i}(m_i)$. Next, by inspecting the definitions we find that the distributor $\Delta(f, g)$ has the form $(X \xleftarrow{p} I \times B \xrightarrow{\pi} B \xrightarrow{r} Z)$, where

$$\begin{aligned} B &= \{(z, u) \mid z \in Z, u \in \prod_i f_i^{-1}\{z\} \subseteq \prod_i X_i\} \\ p(i, z, u) &= u_i \\ r(z, u) &= z. \end{aligned}$$

Using this together with Lemma 12.8 again we get

$$T_r N_\pi R_p(m) = T_r(\prod_i R_{p_i}(m_i)),$$

where $p_i(z, u) = p(i, z, u) = u_i$.

It will now be convenient to identify I with $\{1, \dots, n\}$ say. Put

$$\begin{aligned} B' &= \{(z, u) \mid z \in Z, u \in \prod_{i < n} f_i^{-1}\{z\} \subseteq \prod_i X_i\} \\ p'_i(z, u) &= u_i \quad (\text{for } i < n) \\ r'(z, u) &= z \end{aligned}$$

and let $t: B \rightarrow B'$ be the evident truncation map. We then have a cartesian square as follows:

$$\begin{array}{ccc} B & \xrightarrow{p_n} & X_n \\ \downarrow t & \searrow r & \downarrow f_n \\ B' & \xrightarrow{r'} & Z \end{array}$$

Using Frobenius reciprocity twice, and the fact that $p_i = p'_i t$ for $i < n$, and the cartesian property of the square, we obtain

$$\begin{aligned}
T_r \left(\prod_i R_{p_i}(m_i) \right) &= T_{r'} T_t \left(R_{p_n}(m_n) R_t \prod_{i < n} R_{p'_i}(m_i) \right) \\
&= T_{r'} \left(T_t R_{p_n}(m_n) \prod_{i < n} R_{p'_i}(m_i) \right) \\
&= T_{r'} \left(R_{r'} T_{f_n}(m_n) \prod_{i < n} R_{p'_i}(m_i) \right) \\
&= T_{f_n}(m_n) T_{r'} \left(\prod_{i < n} R_{p'_i}(m_i) \right).
\end{aligned}$$

By an evident inductive extension of this, we get

$$T_r \left(\prod_i R_{p_i}(m_i) \right) = \prod_i T_{f_i}(m_i)$$

or in other words $T_r N_\pi R_p(m) = N_g T_f(m)$ as claimed. \square

LEMMA 12.10. *Let S be a Green functor. Then the underlying Mackey functor has a canonical structure as a Green semiring.*

PROOF. We first claim that each set $S(X)$ has a canonical structure as a semiring. Moreover, for every map $f: X \rightarrow Y$:

- (a) The map $R_f: S(Y) \rightarrow S(X)$ is a semiring homomorphism (which we use to regard $S(X)$ as an $S(Y)$ -module).
- (b) The map $T_f: S(X) \rightarrow S(Y)$ is a homomorphism of $S(Y)$ -modules (and in particular, respects addition).
- (c) If f is green then the map $N_f: S(X) \rightarrow S(Y)$ sends 1 to 1 and respects multiplication.

This is proved in the same way as Proposition 6.11; we just need to observe that the bispans used to define the semiring structure are all green.

In view of Proposition 12.2, this gives the required Green semiring structure. \square

PROPOSITION 12.11. *The categories Green_G and Green'_G are equivalent.*

PROOF. If S is a Green semiring, we write $\Phi(S)$ for the same Mackey functor equipped with norm maps N_f as given by Lemma 12.7. Using Lemma 12.9 we see that this gives a functor $\Phi: \text{Green}'_G \rightarrow \text{Green}_G$. Similarly, Lemma 12.10 defines a functor $\Psi: \text{Green}_G \rightarrow \text{Green}'_G$. It is easy to see that these are inverse to each other. \square

13. Additive completion

Given a commutative semigroup A , there is a universal example of a commutative group A^+ equipped with a homomorphism $A \rightarrow A^+$; we call this the *additive completion* of A . In this section we explain and extend Tambara's proof [22, Theorem 6.1] of the following result:

THEOREM 13.1. *If M is a Mackey functor then there is another Mackey functor M^+ with $M^+(X) = M(X)^+$ for all X . If S is a Tambara functor then there is another Tambara functor S^+ with $S^+(X) = S(X)^+$ for all X .*

The Mackey functor case is straightforward, because Mackey functors can be regarded as additive functors from $\overline{\mathcal{A}}_G$ to Semigroups, and additive completion gives an additive functor $\text{Semigroups} \rightarrow \text{Ab}$. The real problem is to handle the nonlinearity of the norm operations for a Tambara functor. The rest of this section will constitute the proof.

We will use the following construction for A^+ :

DEFINITION 13.2. Let A be a semigroup. We put

$$\widehat{A} = A^2 = \{(a_+, a_-) \mid a_+, a_- \in A\},$$

and regard this as a semigroup using the obvious coordinatewise addition. We then put

$$E_A = \{((a_+, a_-), (b_+, b_-)) \in \widehat{A} \times \widehat{A} \mid a_+ + b_- + x = a_- + b_+ + x \text{ for some } x \in A\}$$

and check that this is a congruence on \widehat{A} . It follows that the quotient set \widehat{A}/E_A inherits a semigroup structure; we write A^+ for this semigroup.

PROPOSITION 13.3. *Let S be a semiring. Then the rule*

$$(a_+, a_-).(b_+, b_-) = (a_+b_+ + a_-b_-, a_+b_- + a_-b_+)$$

makes \widehat{S} into a semiring (with identity element $(1, 0)$). Moreover, E_S is a subsemiring of $\widehat{S} \times \widehat{S}$.

PROOF. The semiring axioms for \widehat{S} are straightforward and are left to the reader. (Essentially, we have defined \widehat{S} to be the group semiring $S[C_2]$.) It is also clear that E_S contains the additive and multiplicative identity elements and is closed under addition. The real issue is to prove that E_S is closed under multiplication. Consider elements $s_i = ((a_{i+}, a_{i-}), (b_{i+}, b_{i-})) \in E_S$ for $i = 0, 1$. This means that there are elements $x_i \in S$ such that $P_i = Q_i$, where

$$\begin{aligned} P_i &= a_{i+} + b_{i-} + x_i \\ Q_i &= a_{i-} + b_{i+} + x_i. \end{aligned}$$

We now have $s_0s_1 = ((a_+, a_-), (b_+, b_-))$, where

$$\begin{aligned} a_+ &= a_{0+}a_{1+} + a_{0-}a_{1-} & a_- &= a_{0+}a_{1-} + a_{0-}a_{1+} \\ b_+ &= b_{0+}b_{1+} + b_{0-}b_{1-} & b_- &= b_{0+}b_{1-} + b_{0-}b_{1+}. \end{aligned}$$

Put

$$\begin{aligned} x &= a_{1+}x_0 + a_{0+}x_1 + x_0b_{1-} + x_1b_{0-} + 2x_0x_1 + a_{1+}b_{0-} + a_{0+}b_{1-} + b_{0-}b_{1-} \\ u_0 &= a_{0-}a_{1-} + x_0x_1 + b_{0+}b_{1-} + b_{1+}b_{0-} \\ u_1 &= b_{0+}b_{1-} + b_{1+}b_{0-} + b_{0+}b_{1+} + b_{0+}x_1 + b_{1+}x_0 + 2x_0x_1 \\ u_2 &= b_{0+}b_{1-} + b_{1+}b_{0-} + b_{0+}b_{1+} + a_{0+}a_{1-} + a_{1+}a_{0-} + a_{0-}b_{1-} + a_{1-}b_{0-} \\ u_3 &= b_{0+}b_{1+} + a_{0+}a_{1-} + a_{1+}a_{0-} + a_{0+}x_1 + a_{1+}x_0 + 2x_0x_1. \end{aligned}$$

By direct expansion one can check that

$$\begin{aligned} a_+ + b_- + x &= P_0P_1 + u_0 \\ Q_0Q_1 + u_0 &= a_{0-}Q_1 + a_{1-}Q_0 + u_1 \\ a_{0-}P_1 + a_{1-}P_0 + u_1 &= x_0Q_1 + x_1Q_0 + u_2 \\ x_0P_1 + x_1P_0 + u_2 &= b_{0-}Q_1 + b_{1-}Q_0 + u_3 \\ b_{0-}P_1 + b_{1-}P_0 + u_3 &= a_- + b_+ + x. \end{aligned}$$

We are also given that $P_i = Q_i$, so it follows that $a_+ + b_- + x = a_- + b_+ + x$, which proves that $s_0s_1 \in E_S$ as required. \square

COROLLARY 13.4. *The set $S^+ = \widehat{S}/E_S$ has a unique semiring structure for which the quotient map $\widehat{S} \rightarrow S^+$ is a semiring homomorphism.*

PROOF. This follows formally from the fact that E_S is both an equivalence relation and a subsemiring in $\widehat{S} \times \widehat{S}$. \square

PROPOSITION 13.5. *If S is a Green functor, then \widehat{S} and S^+ also have canonical structures as Green functors.*

PROOF. We will actually work with Green semiring structures, which are equivalent to Green functor structures as explained in Section 12. It is clear that Proposition 13.3 gives the required semiring structures on $\widehat{S}(X)$ and $S^+(X)$ and that the maps R_f are semiring morphisms. The key issue is to check the Frobenius reciprocity formula $T_f(a)b = T_f(aR_f(b))$ for $f: X \rightarrow Y$ and $a \in \widehat{S}(X)$ and $b \in \widehat{S}(Y)$. Here we have $a = (a_+, a_-)$ for some $a_+, a_- \in S(X)$ and similarly $b = (b_+, b_-)$ for some $b_+, b_- \in S(Y)$. We thus have

$$\begin{aligned} T_f(a)b &= (T_f(a_+), T_f(a_-)) \cdot (b_+, b_-) \\ &= (T_f(a_+)b_+ + T_f(a_-)b_-, T_f(a_+)b_- + T_f(a_-)b_+) \\ &= (T_f(a_+R_f(b_+) + a_-R_f(b_-)), T_f(a_+R_f(b_-) + a_-R_f(b_+))) \\ &= T_f(aR_f(b)) \end{aligned}$$

as required. The quotient map $\widehat{S}(X) \rightarrow S^+(X)$ is a surjective semiring morphism that commutes with maps of the form R_f and T_f . The reciprocity formula for S^+ therefore follows from the formula for \widehat{S} . \square

We next want to define norm maps for S^+ (assuming that S itself is a Tambara functor). A key problem is to define $N_f(-1) \in S^+(Y)$ for every $f: X \rightarrow Y$. If $|f^{-1}\{y\}| = d$ for all y then it is easy to think that $N_f(-1)$ should be $(-1)^d$, and if we assume this then it is not hard to determine how N_f should behave in general. Unfortunately, this approach does not lead to well-defined operations satisfying the Tambara axioms. To see this, consider the case where $|G| = 2$ and f is the unique map $G \rightarrow 1$ (so $d = 2$). As in Section 7 we need to have

$$0 = N_f(0) = N_f(1 + (-1)) = N_f(1) + T_f(-1) + N_f(-1) = 1 - T_f(1) + N_f(-1),$$

so $N_f(-1) = T_f(1) - 1 \in S^+(1)$. If S is the Burnside semiring Tambara functor and $t = T_f(1) \in S(1)$ then $S(1) = \mathbb{N}\{1, t\}$ (with $t^2 = 2t$) and $S^+(1) = \mathbb{Z}\{1, t\}$ so the element $N_f(-1) = t - 1$ is definitely different from $(-1)^d = 1$. Because of this it is hard to give an explicit formula for norm maps on S^+ . Instead, Tambara used an indirect approach which we now explain.

Consider a map $f: X \rightarrow Y$ of finite G -sets, and put

$$\begin{aligned} V(f) &= \{(y, C) \mid y \in Y, C \subseteq f^{-1}\{y\}\} \\ V_n(f) &= \{(y, C) \in V(f) \mid |C| = n\}. \end{aligned}$$

Given a Green functor S , we will define a ‘‘convolution product’’ (different from the standard product) on $S(V(f))$. We start with the nonequivariant case for motivation.

Suppose we have a semiring S_1 and we put $S(X) = \text{Map}(X, S_1)$ for all X . We can define the convolution product on $S(V(f))$ by

$$(a_1 \vee a_2)(y, C) = \sum_{C=C_1 \amalg C_2} a_1(y, C_1) a_2(y, C_2).$$

This is easily seen to make $S(V(f))$ into a commutative semiring, with identity element given by

$$e(y, C) = \begin{cases} 1 & \text{if } C = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $S(V(f))$ splits as the product of pieces $S(V_n(f))$ (which are zero for n sufficiently large) and $S(V_n(f)) \vee S(V_m(f)) \subseteq S(V_{n+m}(f))$ so in fact we have a graded semiring. We can define $\chi: S(X) \rightarrow S(V(f))$ by

$$\chi(a)(y, C) = \prod_{x \in C} a(x)$$

and it is easy to see that $\chi(0) = e$ and $\chi(a + b) = \chi(a)\chi(b)$. There is an inclusion $j: Y \rightarrow V(f)$ given by $j(y) = (y, f^{-1}\{y\})$ and we have

$$R_j \chi(a)(y) = \chi(a)(y, f^{-1}\{y\}) = \prod_{f(x)=y} a(x) = (N_f a)(y).$$

This shows that χ determines N_f .

We now give the corresponding definitions for Green functors and Tambara functors.

DEFINITION 13.6. For $n, m \geq 0$ we put

$$V^m(f) = \{(y, C_1, \dots, C_m) \mid y \in Y, C_i \subseteq f^{-1}\{y\}, C_i \cap C_j = \emptyset \text{ for } i \neq j\}$$

$$V_n^m(f) = \{(y, C_1, \dots, C_m) \in V^m(f) \mid \sum_i |C_i| = n\}.$$

We define maps $p_1, p_2, p_{12}: V^2(f) \rightarrow V(f)$ by

$$\begin{aligned} p_1(y, C_1, C_2) &= (y, C_1) \\ p_2(y, C_1, C_2) &= (y, C_2) \\ p_{12}(y, C_1, C_2) &= (y, C_1 \cup C_2). \end{aligned}$$

Now let S be a Green functor. For $a_1, a_2 \in S(V(f))$ we define

$$a_1 \vee a_2 = T_{p_{12}}(R_{p_1}(a_1) R_{p_2}(a_2)) \in S(V(f)).$$

We also write $e = T_k(1)$, where $k: Y \rightarrow V(f)$ is given by $k(y) = (y, \emptyset)$ (so $k: Y \simeq V_0(f)$).

PROPOSITION 13.7. *The above product makes $S(V(f))$ into a commutative graded semiring.*

PROOF. We will prove associativity and leave the rest to the reader. Just by expanding the definitions, we have

$$\begin{aligned} (a_1 \vee a_2) \vee a_3 &= T_{p_{12}}(R_{p_1}(a_1 \vee a_2) R_{p_2}(a_3)) \\ &= T_{p_{12}}(R_{p_1} T_{p_{12}}(R_{p_1}(a_1) R_{p_2}(a_2)) R_{p_2}(a_3)). \end{aligned}$$

Next, we define maps as follows:

$$\begin{array}{ll} q_i: V^3 \rightarrow V & q_i(y, C_1, C_2, C_3) = (y, C_i) \\ q_{12}: V^3 \rightarrow V & q_{1,2}(y, C_1, C_2, C_3) = (y, C_1 \cup C_2) \\ q_{123}: V^3 \rightarrow V & q_{123}(y, C_1, C_2, C_3) = (y, C_1 \cup C_2 \cup C_3) \\ q_{1,2}: V^3 \rightarrow V^2 & q_{1,2}(y, C_1, C_2, C_3) = (y, C_1, C_2) \\ q_{12,3}: V^3 \rightarrow V^2 & q_{12,3}(y, C_1, C_2, C_3) = (y, C_1 \cup C_2, C_3). \end{array}$$

These satisfy

$$\begin{array}{lll} p_1 q_{1,2} = q_1 & p_2 q_{1,2} = q_2 & p_{12} q_{1,2} = q_{12} \\ p_1 q_{12,3} = q_{12} & p_2 q_{12,3} = q_3 & p_{12} q_{12,3} = q_{123}, \end{array}$$

and one can check that the square

$$\begin{array}{ccc} V^3 & \xrightarrow{q_{1,2}} & V^2 \\ q_{12,3} \downarrow & & \downarrow p_{12} \\ V^2 & \xrightarrow{p_1} & V \end{array}$$

is cartesian. This means that $R_{p_1} T_{p_{12}} = T_{q_{12,3}} R_{q_{1,2}}$. Using this together with Frobenius reciprocity for $q_{12,3}$ we get

$$\begin{aligned} (a_1 \vee a_2) \vee a_3 &= T_{p_{12}}(T_{q_{12,3}} R_{q_{1,2}}(R_{p_1}(a_1) R_{p_2}(a_2)) R_{p_2}(a_3)) \\ &= T_{p_{12} q_{12,3}}(R_{q_{1,2}} R_{p_1}(a_1) R_{q_{1,2}} R_{p_2}(a_2) R_{q_{12,3}} R_{p_2}(a_3)) \\ &= T_{q_{123}}(R_{q_1}(a_1) R_{q_2}(a_2) R_{q_3}(a_3)). \end{aligned}$$

A similar argument gives the same description for $a_1 \vee (a_2 \vee a_3)$. Note that we have only used ordinary products and not norms, so we only need a Green functor, not a Tambara functor. \square

DEFINITION 13.8. We put

$$U(f) = \{(x, C) \mid x \in X, x \in C \subseteq f^{-1}\{f(x)\}\},$$

and we define maps $X \xleftarrow{r} U(f) \xrightarrow{t} V(f)$ by $r(x, C) = x$ and $t(x, C) = (f(x), C)$. We also define $j: Y \rightarrow V(f)$ by $j(y) = (y, f^{-1}\{y\})$. For any Tambara functor S , we then define $\chi = N_t R_r: S(X) \rightarrow S(V(f))$.

PROPOSITION 13.9. *The map χ satisfies $\chi(0) = e$ and $\chi(a_1 + a_2) = \chi(a_1) \vee \chi(a_2)$.*

PROOF. First, we have $\chi(0) = N_t(0)$. We observe that $\text{img}(t)^c = V_0(f)$, and using Lemma 6.8 it follows that $\chi(0) = e$.

Now suppose we have elements $a_1, a_2 \in S(X)$ and we put $b_i = R_r(a_i)$ and $c_i = N_t(b_i) = \chi(a_i)$. We must show that $\chi(a_1 + a_2) = c_1 \vee c_2$. The elements b_i combine to give an element $b \in S(U(f) \times \{1, 2\})$ and the projection $s: U(f) \times \{1, 2\} \rightarrow U(f)$ has $T_s(b) = b_1 + b_2$. We therefore need to understand the distributor $\Delta(s, t)$. For any point $(y, C) \in V(f)$ we have $t^{-1}\{(y, C)\} \simeq C$. Moreover, to give a map $m: t^{-1}\{(y, C)\} \rightarrow U(f) \times \{1, 2\}$ with $sm = 1$ is the same as to give an arbitrary map $C \rightarrow \{1, 2\}$, or equivalently a splitting of C as a disjoint union of subsets C_1 and C_2 . Using this we can identify $\Delta(s, t)$ with the diagram

$$U(f) \times \{1, 2\} \xleftarrow{u} U^2(f) \xrightarrow{t_2} V^2(f) \xrightarrow{p_{12}} V(f)$$

where

$$\begin{aligned} U^2(f) &= \{(x, C_1, C_2) \mid x \in C_1 \amalg C_2 \subseteq f^{-1}\{f(x)\}\} \\ u(x, C_1, C_2) &= \begin{cases} (x, C_1 \cup C_2, 1) & \text{if } x \in C_1 \\ (x, C_1 \cup C_2, 2) & \text{if } x \in C_2. \end{cases} \\ t_2(x, C_1, C_2) &= (f(x), C_1, C_2). \end{aligned}$$

We conclude that

$$\chi(a_1 + a_2) = N_t T_s(b) = T_{p_{12}} N_{t_2} R_u(b) = T_{p_{12}} N_{t_2} R_{(r \times 1) \circ u}(a).$$

On the other hand, we have

$$\chi(a_1) \vee \chi(a_2) = T_{p_{12}}(R_{p_1}(\chi(a_1))R_{p_2}(\chi(a_2))).$$

Here $R_{p_1}(\chi(a_1)) = R_{p_1} N_t R_r(a_1)$. We have a commutative diagram

$$\begin{array}{ccccc} U(f) & \xleftarrow{u_1} & U^{2,1}(f) & \xrightarrow{t_{2,1}} & V^2(f) \\ r \downarrow & & v_1 \downarrow & & p_1 \downarrow \\ X & \xleftarrow{r} & U(f) & \xrightarrow{t} & V(f), \end{array}$$

where

$$\begin{aligned} U^{2,1}(f) &= \{(x, C_1, C_2) \in U^2 \mid x \in C_1\} \\ v_1(x, C_1, C_2) &= (x, C_1) \\ t_{2,1} &= t_2|_{U^{2,1}(f)} \\ u_1 &= u|_{U^{2,1}(f)}. \end{aligned}$$

The right hand square is cartesian, giving

$$R_{p_1}(\chi(a_1)) = R_{p_1} N_t R_r(a_1) = N_{t_{2,1}} R_{v_1} R_r(a_1) = N_{t_{2,1}} R_{r u_1}(a_2).$$

After obtaining a similar expression for $R_{p_2}(\chi(a_2))$ and noting that $U^2(f) = U^{2,1}(f) \amalg U^{2,2}(f)$ we find that $R_{p_1}(\chi(a_1))R_{p_2}(\chi(a_2)) = N_{t_2} R_{(r \times 1) \circ u}(a)$ so $\chi(a_1) \vee \chi(a_2) = \chi(a_1 + a_2)$ as claimed. \square

PROPOSITION 13.10. *The map χ also satisfies $R_j \chi(a) = N_f(a)$.*

PROOF. We have a commutative diagram

$$\begin{array}{ccc} & X & \xrightarrow{f} Y \\ & \downarrow i & \downarrow j \\ X & \xleftarrow{r} U(f) & \xrightarrow{t} V(f) \end{array}$$

where $i(x) = (x, f^{-1}\{f(x)\})$. One can check that the square is cartesian, so $R_j \chi = R_j N_t R_r = N_f R_i R_r = N_f$. \square

DEFINITION 13.11. As before we define $k: Y \rightarrow V(f)$ by $k(y) = (y, \emptyset)$, and we put

$$GS(V(f)) = \{a \in S(V(f)) \mid R_k(a) = 1\}.$$

PROPOSITION 13.12. For any Green functor S , the map $R_k: S(V(f)) \rightarrow S(Y)$ satisfies $R_k(e) = 1$ and $R_k(b_1 \vee b_2) = R_k(b_1) R_k(b_2)$. If S is a Tambara functor then we also have $R_k(\chi(a)) = 1$.

PROOF. Define $k^2: Y \rightarrow V^2(f)$ by $k^2(y) = (y, \emptyset, \emptyset)$. We then have commutative diagrams as follows, in which the squares are cartesian:

$$\begin{array}{ccc} \begin{array}{ccc} Y & \xrightarrow{1} & Y \\ \downarrow 1 & & \downarrow k \\ Y & \xrightarrow{k} & V(f) \end{array} & \begin{array}{ccc} & Y & \xrightarrow{1} & Y \\ & \swarrow k & \downarrow k^2 & \downarrow k \\ V(f) & \xleftarrow{p_i} & V^2(f) & \xrightarrow{p_{12}} & V(f) \end{array} & \begin{array}{ccc} & \emptyset & \longrightarrow & Y \\ & \swarrow & \downarrow & \downarrow k \\ X & \xleftarrow{r} & U(f) & \xrightarrow{t} & V(f) \end{array} \end{array}$$

The first diagram gives $R_k(e) = R_k T_k(1) = 1$. The second diagram gives

$$R_k(b_1 \vee b_2) = R_k T_{p_{12}}(R_{p_1}(b_1) R_{p_2}(b_2)) = R_{k^2}(R_{p_1}(b_1) R_{p_2}(b_2)) = R_k(b_1) R_k(b_2).$$

The third diagram gives

$$R_k \chi(a) = R_k N_t R_r(a) = N_{\emptyset \rightarrow Y} R_{\emptyset \rightarrow X}(a) = N_{\emptyset \rightarrow Y}(1) = 1.$$

□

PROPOSITION 13.13. The set $GS(V(f))$ is a submonoid of $S(V(f))$ (under the convolution product). If S is additively complete, then $GS(V(f))$ is an abelian group under convolution.

PROOF. From the facts that $R_k(e) = 1$ and $R_k(b_1 \vee b_2) = R_k(b_1) R_k(b_2)$ it is clear that $GS(V(f))$ is a submonoid. Now suppose that S is additively complete and $a \in GS(V(f))$. Put $N = \max\{|f^{-1}\{y\}| \mid y \in Y\}$, so the standard grading on $S(V(f))$ is zero in degrees larger than N . We can thus write $a = \sum_{i=0}^N a_i$ with a_i in degree i . As $R_k(a) = 1$ and k gives a bijection $Y \rightarrow V_0(f)$ we see that $a_0 = e$. Put $b = \sum_{i=1}^N a_i$ so $a = e + b$. Using the grading we see that $b^{N+1} = 0$, so the element $c = \sum_{i=0}^n (-b)^i$ gives an inverse for a . □

DEFINITION 13.14. Let S be a Tambara functor, and let $\eta: S \rightarrow S^+$ be the evident morphism of Green functors. As $\eta\chi: S(X) \rightarrow GS^+(V(f))$ is a homomorphism from a semigroup to a group, the universal property of $S^+(X)$ gives us a homomorphism $S^+(X) \rightarrow GS^+(V(f))$ making the left square below commute.

$$\begin{array}{ccccccc} S(X) & \xrightarrow{\chi} & GS(V(f)) & \xrightarrow{\text{inc}} & S(V(f)) & \xrightarrow{R_j} & S(Y) \\ \eta \downarrow & & \eta \downarrow & & \downarrow \eta & & \downarrow \eta \\ S^+(X) & \xrightarrow{\chi^+} & GS^+(V(f)) & \xrightarrow{\text{inc}} & S^+(V(f)) & \xrightarrow{R_j} & S^+(Y) \end{array}$$

The other two squares commute automatically. We also recall that the top composite $S(X) \rightarrow S(Y)$ is N_f , and we define $N_f: S^+(X) \rightarrow S^+(Y)$ to be the bottom composite.

In order to establish the properties of these norm maps, we need to recall the theory of polynomial maps between abelian groups [7, 8].

DEFINITION 13.15. Let A and B be abelian groups. We will write M for the set of all functions (not necessarily homomorphisms) from A to B .

- (a) We define $\epsilon: M \rightarrow B$ by $\epsilon(f) = f(0)$.
- (b) For $a \in A$ we define $\delta[a]: M \rightarrow M$ by $(\delta[a]f)(x) = f(a+x) - f(x)$. It is easy to see that $\delta[a]\delta[a'] = \delta[a']\delta[a]$ for all $a, a' \in A$.
- (c) Given finite sets $J \subseteq I$ and a map $a: I \rightarrow A$ we also write $\delta[J, a] = \prod_{j \in J} \delta[a(j)]: M \rightarrow M$.
- (d) We say that $f \in M$ is *polynomial of degree at most n* if $\delta[I, a]f = 0$ whenever $|I| > n$. We say that $f \in M$ is *polynomial* if this condition is satisfied for some n .

REMARK 13.16. It is easy to see that f is polynomial of degree at most 0 iff it is constant, and polynomial of degree at most 1 iff it is a constant plus a homomorphism.

REMARK 13.17. If we put $\sigma(J, a) = \sum_{j \in J} a(j)$ we find that

$$(\delta[I, a]f)(x) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} f(\sigma(J, a) + x).$$

REMARK 13.18. If $a(i) = 0$ for some $i \in I$, it is clear that $\delta[I, a]f = 0$ for all f .

LEMMA 13.19. For any $f \in M$ and $a: I \rightarrow A$ we have

$$f(a + \sigma(I, a)) = \sum_{J \subseteq I} (\delta[J, a]f)(x).$$

PROOF. By definition we have

$$\begin{aligned} \sum_{J \subseteq I} (\delta[J, a]f)(x) &= \sum_{K \subseteq J \subseteq I} (-1)^{|J \setminus K|} f(\sigma(K, a) + x) \\ &= \sum_{K \subseteq I} \left(f(\sigma(K, a) + x) \sum_{K \subseteq J \subseteq I} (-1)^{|J \setminus K|} \right). \end{aligned}$$

It is straightforward to check that the inner sum is 1 when $K = I$ and 0 otherwise, and the claim follows. \square

PROPOSITION 13.20. Suppose we have maps $A \xrightarrow{f} B \xrightarrow{g} C$ where f is polynomial of degree at most n , and g is polynomial of degree at most m . Then gf is polynomial of degree at most nm .

PROOF. Consider a map $a: I \rightarrow A$, and an element $x \in A$. Let PI be the set of subsets of I , and for $J \in PI$ put

$$b(J) = (\delta[J, a]f)(x).$$

Now

$$(\delta[I, a]gf)(x) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} g(f(\sigma(J, a) + x)),$$

but Lemma 13.19 gives

$$f(\sigma(J, a) + x) = \sum_{K \subseteq J} (\delta[K, a]f)(x) = \sum_{K \in PJ} b(K).$$

This in turn gives

$$g(f(\sigma(J, a) + x)) = g(\sigma(PJ, b) + 0) = \sum_{T \subseteq PJ} (\delta[T, b]g)(0),$$

so

$$(\delta[I, a]gf)(x) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{T \subseteq PJ} (\delta[T, b]g)(0) = \sum_{T \subseteq PI} (\delta[T, b]g)(0) \sum_{\bigcup T \subseteq J \subseteq I} (-1)^{|I \setminus J|}.$$

Here it is easy to see that the inner sum is 1 if $\bigcup T = I$, and 0 otherwise. We conclude that

$$(\delta[I, a]gf)(x) = \sum_{T \subseteq PI, \bigcup T = I} (\delta[T, b]g)(0).$$

For the term $(\delta[T, b]g)(0)$ to be nonzero we must have $|T| \leq m$. Moreover, by Remark 13.18 we must also have $b(K) \neq 0$ for all $K \in T$, which forces $|K| \leq n$. Together these imply that $|\bigcup T| \leq nm$, but $\bigcup T = I$ so $|I| \leq nm$. It follows that gf is polynomial of degree at most nm , as claimed. \square

PROPOSITION 13.21. Let $f, g: A \rightarrow B$ be polynomial maps. Let A_0 be a subsemigroup of A that generates A as a group, and suppose that $f|_{A_0} = g|_{A_0}$. Then $f = g$.

PROOF. The difference $h = f - g$ has $h(A_0) = 0$, and it is clearly polynomial of some degree d say. If $d = 0$ then h is constant but $h(A_0) = 0$ so $h = 0$. Now suppose that $d > 0$. Any element $a \in A$ can be written as $a_+ - a_-$ with $a_+, a_- \in A_0$. Put $k = \delta[a_-]h$, and note that this is polynomial of degree at most $d - 1$. For $x \in A_0$ we have $a_- + x \in A_0$ and $k(x) = h(a_- + x) - h(x) = 0 - 0 = 0$. By induction it follows that $k = 0$, so $h(x) = h(a_- + x)$ for all x . Now take $x = a$ to get $h(a) = h(a_+) \in h(A_0) = 0$. We conclude that $h = 0$ as required. \square

LEMMA 13.22. *Let $f: X \rightarrow Y$ be a map of finite G -sets, and let d be the maximum of the numbers $|f^{-1}\{y\}|$ for $y \in Y$. Then the map $N_f: S^+(X) \rightarrow S^+(Y)$ is polynomial of degree at most d .*

PROOF. We have $N_f = R_j \chi^+$, where R_j is a homomorphism and so is polynomial of degree at most one. In view of Proposition 13.20, it will suffice to prove that χ^+ is polynomial of degree at most d . For this proof we will just write uv for the convolution product $u \vee v$. We also note that when $k > d$ we have $V_k(f) = \emptyset$ and so $S^+(V(f))_k = 0$. It follows that the ideal $K = \sum_{k>0} S^+(V(f))_k$ satisfies $K^{d+1} = 0$. We have seen that the image of χ^+ is contained in $GS^+(V(f))$ so we can write $\chi^+(a) = 1 + \chi_0(a)$ for some map $\chi_0: S(X) \rightarrow K$. We can rearrange the relation $\chi^+(a+x) = \chi^+(a)\chi^+(x)$ to get $\delta[a]\chi^+ = \chi_0(a)\chi^+$. It follows inductively that for any map $a: I \rightarrow S^+(X)$ we have

$$\delta[I, a]\chi^+ = \left(\prod_{i \in I} \chi_0(a(i)) \right) \chi^+.$$

Using $K^{d+1} = 0$ we deduce that $\delta[I, a]\chi^+ = 0$ when $|I| > d$, as required. \square

PROPOSITION 13.23. *The maps $N_f: S^+(X) \rightarrow S^+(Y)$ make S^+ into a Tambara functor. Moreover, if T is any additively complete Tambara functor and $\phi: S \rightarrow T$ is a morphism of Tambara functors then there is a unique morphism $\phi^+: S^+ \rightarrow T$ of Tambara functors with $\phi^+ \eta = \phi$.*

PROOF. Suppose we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. We must show that $N_{gf} = N_g N_f: S^+(X) \rightarrow S^+(Z)$. By construction, the following diagrams commute:

$$\begin{array}{ccccc} S(X) & \xrightarrow{N_f} & S(Y) & \xrightarrow{N_g} & S(Z) & & S(X) & \xrightarrow{N_{gf}} & S(Z) \\ \eta \downarrow & & \eta \downarrow & & \eta \downarrow & & \eta \downarrow & & \eta \downarrow \\ S^+(X) & \xrightarrow{N_f} & S^+(Y) & \xrightarrow{N_g} & S^+(Z) & & S^+(X) & \xrightarrow{N_{gf}} & S^+(Z). \end{array}$$

We also know that $N_{gf} = N_g N_f$ on $S(X)$, and it follows that $N_{gf} = N_g N_f$ on the image of $\eta: S(X) \rightarrow S^+(X)$. Moreover, both N_{gf} and $N_g N_f$ are algebraic, so it follows from Proposition 13.21 that $N_{gf} = N_g N_f$ on $S^+(X)$. All the other Tambara functor axioms can be verified in the same way.

Now suppose we have an additively complete Tambara functor T and a morphism $\phi: S \rightarrow T$ of Tambara functors. It is clear that there is a unique morphism $\phi^+: S^+ \rightarrow T$ of Mackey functors with $\phi^+ \eta = \phi$, and we just need to check that this is compatible with norm maps. In more detail, given $f: X \rightarrow Y$ we must show that for any $f: X \rightarrow Y$ we have $N_f \phi^+ = \phi^+ N_f: S^+(X) \rightarrow T(Y)$. Here both $N_f \phi^+$ and $\phi^+ N_f$ are algebraic, so it will suffice to check that they agree on $\eta(S(X))$, but that is clear by naturality. \square

14. Modules over Tambara functors

In this section we study two different possible definitions for modules over a Tambara functor.

DEFINITION 14.1. Let S be a Tambara functor. As in the Section 12 we see that there are canonical maps $A \xrightarrow{\eta} S \xleftarrow{\mu} S \boxtimes S$ making S into a Green ring. By a *naive S -module* we mean a Mackey functor M equipped with a map $\nu: S \boxtimes M \rightarrow M$ making the obvious diagram commute:

$$\begin{array}{ccc} S \boxtimes S \boxtimes M & \xrightarrow{\mu \boxtimes 1} & S \boxtimes M \xleftarrow{\eta \boxtimes 1} M \\ \downarrow 1 \boxtimes \nu & & \downarrow \nu \swarrow 1 \\ S \boxtimes M & \xrightarrow{\nu} & M \end{array}$$

We write NMod_S for the category of naive S -modules.

REMARK 14.2. In view of Proposition 3.14, to give a naive S -module structure on M is the same as to make each $M(X)$ into an $S(X)$ -module in such a way that

- (a) For all $f: X \rightarrow Y$ and $b \in S(Y)$ and $n \in M(Y)$ we have $R_f(bn) = R_f(b) R_f(n)$.
- (b) For all $f: X \rightarrow Y$ and $b \in S(Y)$ and $m \in M(X)$ we have $bT_f(m) = T_f(R_f(b) m)$.

In particular, if S is additively complete, then multiplication by $-1 \in S(X)$ provides additive inverses in $M(X)$, so $M(X)$ is also additively complete. Given this, one can check that NMod_S is an abelian category.

As the category NMod_S depends only on the underlying Green ring, it is natural to ask whether there is a more refined category that somehow takes account of the norm maps. We can define such a category following an idea of Waldhausen in the context of stable homotopy theory, which takes a foundational result in the André-Quillen homology theory for commutative rings and turns it into a definition. This will of course mean that our definition is a good basis for an André-Quillen homology theory for Tambara functors. One can thus expect it to be useful when studying topological André-Quillen homology for equivariant ring spectra, which is important for a number of applications.

DEFINITION 14.3. Let S be a Tambara functor. We write AugAlg_S for the category of augmented S -algebras, or in other words Tambara functors T equipped with morphisms $S \xrightarrow{\eta} T \xrightarrow{\epsilon} S$ such that $\epsilon\eta = 1$. This category has finite products, given by

$$(T \times_S T')(X) = \{(a, a') \in T(X) \times T'(X) \mid \epsilon(a) = \epsilon(a') \in S(X)\},$$

so it is meaningful to talk about semigroup objects in AugAlg_S . (As always, our semigroups are assumed to be commutative.) We call these S -modules, and we write Mod_S for the category of such objects.

REMARK 14.4. In any category \mathcal{C} with finite products, one can check that the category $\text{Semigroups}(\mathcal{C})$ of semigroup objects in \mathcal{C} is semiadditive, with finite products (and therefore finite coproducts) given by products in the underlying category \mathcal{C} . Thus, Mod_S is always a semiadditive category.

DEFINITION 14.5. For $T \in \text{Mod}_S$, we put

$$(\Lambda T)(X) = \{a \in T(X) \mid \epsilon(a) = 0 \in S(X)\}.$$

It is clear this is a sub-Mackey functor of T and that the product map $T \boxtimes T \rightarrow T$ restricts to give a product $S \boxtimes \Lambda T \rightarrow \Lambda T$ making ΛT a naive S -module. We thus have a functor $\Lambda: \text{Mod}_S \rightarrow \text{NMod}_S$.

PROPOSITION 14.6. Let S be an additively complete Tambara functor.

- (a) Any $T \in \text{Mod}_S$ has a natural splitting $T = S \oplus \Lambda T$ as Mackey functors.
- (b) With respect to the splitting in (a), the semigroup structure map $\sigma: T(X) \times_{S(X)} T(X) \rightarrow T(X)$ is given by $\sigma(a, u, v) = (a, u + v)$.
- (c) The map $\chi(a, u) = (a, -u)$ is a morphism of augmented S -algebras, so T is actually a group object (not just a semigroup object) in AugAlg_S .
- (d) For any morphism $\phi: T \rightarrow T'$ in Mod_S , the Mackey functors $S \oplus \ker(\Lambda\phi)$, $S \oplus \text{img}(\Lambda\phi)$ and $S \oplus \text{cok}(\Lambda\phi)$ have unique structures as S -modules such that the evident Mackey morphisms

$$S \oplus \ker(\Lambda\phi) \rightarrow T \rightarrow S \oplus \text{img}(\Lambda\phi) \rightarrow T' \rightarrow S \oplus \text{cok}(\Lambda\phi)$$

are S -module morphisms.

- (e) Mod_S is an abelian category.

PROOF. (a) Any $a \in T(X)$ can be written as $\eta(\epsilon(a)) + (a - \eta(\epsilon(a)))$ with $a - \eta(\epsilon(a)) \in \Lambda T(X)$. The claim is clear from this.

- (b) Because σ is supposed to give a semigroup structure, there must be a morphism $\zeta: S \rightarrow T$ in AugAlg_S (to provide the zero) such that the diagram

$$\begin{array}{ccccc} T & \xrightarrow{(\zeta, 1)} & T \times_S T & \xleftarrow{(1, \zeta)} & T \\ & \searrow 1 & \downarrow \sigma & \swarrow 1 & \\ & & T & & \end{array}$$

commutes. Now η is the only morphism from S to T in AugAlg_S , so we must have $\zeta = \eta$. Given this, and using the splitting from (a), commutativity of the diagram means that $\sigma(a, 0, v) = (a, v)$ and $\sigma(a, u, 0) = (a, u)$. In particular, we have $\sigma(a, 0, 0) = (a, 0)$ and $\sigma(0, u, 0) = (0, u)$ and $\sigma(0, 0, v) = (0, v)$. Moreover, σ arises from a morphism $T \times_S T \rightarrow T$ of Mackey functors, so it preserves addition. Claim (b) clearly follows.

- (c) We are given that σ is a Tambara morphism, and it follows that we can define a Tambara morphism $\theta: T \times_S T \rightarrow T \times_S T$ by $\theta(s, t) = (\sigma(s, t), t)$. Now (b) tells us that $\theta(a, u, v) = (a, u + v, v)$, so the component

$$\theta_X: (T \times_S T)(X) \rightarrow (T \times_S T)(X)$$

is bijective for all X , with inverse $(a, u, v) \mapsto (a, u - v, v)$. It is standard that if all components of a natural transformation are isomorphisms, then their inverses form a natural transformation inverse to the original one. Thus θ is invertible, with $\theta^{-1}(a, u, v) = (a, u - v, v)$. The map χ is $\theta^{-1} \circ (\eta, 1)$, so it is a morphism of Tambara functors. It is clearly also compatible with augmentation and satisfies $\sigma \circ (1, \chi) = \eta\epsilon$, so it provides inverses for the semigroup structure on T .

- (d) It is clear that $S \oplus \text{img}(\Lambda\phi)$ is the image of the Tambara morphism ϕ and so has a unique Tambara structure compatible with the evident morphisms

$$T \rightarrow S \oplus \text{img}(\Lambda\phi) \rightarrow T'$$

It is also clear that this is compatible with the augmentations. Moreover, we see from (b) that this is also compatible with the relevant semigroup structures. Next, $S \oplus \ker(\Lambda\phi)$ can be expressed as the equaliser of ϕ and the composite $T \xrightarrow{\epsilon} S \xrightarrow{\eta'} T'$, both of which are Tambara morphisms. All claims about $S \oplus \ker(\Lambda\phi)$ follow from this description. Next, we claim that the functor $J = \text{img}(\Lambda\phi) \leq \Lambda T' \leq T'$ is a Tambara ideal, as in Definition 10.9. Indeed, it is clear that J is a sub-Mackey functor, and that $J(X)$ is an ideal in $T(X)$ for all X . Now suppose we have a map $f: X \rightarrow Y$ of finite G -sets and an element $b \in \text{img}(\Lambda\phi)(X)$, so $b = \phi(a)$ for some $a \in T(X)$ with $\epsilon(a) = 0$. As ϕ is natural we have $N_f^{T'}(b) = \phi(N_f^T(a))$ and $N_f^{T'}(0) = \phi(N_f^T(0))$, so

$$N_f^{T'}(b) = \phi(N_f^T(a) - N_f^T(0)) + N_f^{T'}(0).$$

Because $\epsilon(a) = 0$ we also have

$$\epsilon(N_f^T(a) - N_f^T(0)) = N_f^S(\epsilon(a)) - N_f^S(\epsilon(0)) = 0,$$

so $N_f^T(a) - N_f^T(0) \in \Lambda(T)(X)$, so $N_f^{T'}(b) \in J(Y) + N_f^{T'}(0)$ as required. It follows that the quotient $T'/\text{img}(\Lambda\phi) = S \oplus \text{cok}(\Lambda\phi)$ has a natural structure as a Tambara functor. Using (a) and (b) we again see that everything is compatible with augmentations and semigroup structures.

- (e) This is clear from (a) to (d). □

Although the structure of genuine modules is more subtle than that of naive modules, it turns out that every naive module can be given a canonical genuine module structure. However, not every genuine module arises from this construction. We can state this more precisely as follows:

PROPOSITION 14.7. *There is a functor $\Pi: \text{NMod}_S \rightarrow \text{Mod}_S$ and a natural isomorphism $\Lambda\Pi(M) \simeq M$ in NMod_S . There is also a natural morphism $\phi: \Pi\Lambda(T) \rightarrow T$ of Mackey functors for all $T \in \text{Mod}_S$, which is an isomorphism if S is additively complete. However, ϕ need not be a morphism of Tambara functors.*

The rest of this section will constitute the proof.

We first explain how the theorem generalises a straightforward and well-known fact. Let S be a commutative semiring, and let M be an S -module. We can then define a semiring structure on the group $\Pi M = S \oplus M$ by $(s, m) \cdot (s', m') = (ss', sm' + s'm)$. We have ring homomorphisms $S \xrightarrow{\eta} \Pi M \xrightarrow{\epsilon} S$ given by $\eta(s) = (s, 0)$ and $\epsilon(s, m) = s$, so ΠM is an augmented S -algebra. The copy of M inside ΠM is an ideal satisfying $M^2 = 0$. The categorical product of augmented S -algebras has the property that

$$(S \oplus I) \times_S (S \oplus J) = S \oplus I \oplus J.$$

Using this, we find that the map $(s, m, m') \mapsto (s, m + m')$ can be regarded as an augmented algebra homomorphism $\Pi M \times_S \Pi M \rightarrow \Pi M$, making ΠM a commutative semigroup object in AugAlg_S . We thus have a functor $\Pi: \text{NMod}_S \rightarrow \text{Mod}_S$, and it is clear that $\Lambda\Pi M = M$. In the opposite direction, suppose we have an object $T \in \text{Mod}_S$. We can then define $\phi: \Pi\Lambda T = S \times \Lambda T \rightarrow T$ by $\phi(s, m) = s + m$. This clearly preserves addition. Moreover, T is assumed to be a semigroup object, with addition given by a semiring homomorphism $\sigma: T \times_S T \rightarrow T$ say. Given $m_0, m_1 \in \Lambda T$ we have elements $n_0 = (m_0, 0)$ and $n_1 = (0, m_1)$

in $T \times_S T$ with $n_0 n_1 = 0$. It is formal that $\sigma(n_i) = m_i$ and σ preserves products so $m_0 m_1 = 0$. Using this we see that ϕ also preserves multiplication. If S (and therefore T) is additively complete, we have an inverse given by $\phi^{-1}(t) = (\epsilon(t), t - \epsilon(t))$.

To see what can go wrong when S is not additively complete, consider the sets

$$\begin{aligned} T &= \mathbb{N}[x]/x^2 = \{n + mx \mid n, m \in \mathbb{N}\} \\ T' &= \{n + mx \in T \mid n = m = 0 \text{ or } n > 0\} \\ T'' &= T / ((n + mx) \sim n \text{ whenever } n > 0). \end{aligned}$$

We can regard T as an \mathbb{N} -algebra with augmentation $\epsilon(n + mx) = n$. In fact, it is a semigroup object in $\text{AugAlg}_{\mathbb{N}}$ with addition map $\sigma: T \times_{\mathbb{N}} T \rightarrow T$ given by $\sigma(n + mx, n + kx) = n + (m + k)x$. One can check that T' is a subobject of T and T'' is a quotient object. We have $\Lambda T' = 0$ so the map $\Pi \Lambda T' \rightarrow T'$ is not surjective. On the other hand, we have $\Lambda T'' = \mathbb{N}x$ and $\Pi \Lambda T'' = T$ so the map $\Pi \Lambda T'' \rightarrow T''$ is not injective.

We now start to define a functor Π in the Tambara context.

DEFINITION 14.8. Fix a Tambara functor S . For any naive S -module M we put $\Pi M = S \oplus M$, which is a Mackey functor in an obvious way. We define maps $S \xrightarrow{\eta} \Pi M \xrightarrow{\epsilon} S$ by $\eta(a) = (a, 0)$ and $\epsilon(a, m) = a$. We also define $\sigma: \Pi M \times_S \Pi M \rightarrow \Pi M$ by $\sigma(a, m, m') = (a, m + m')$.

Now consider a map $f: X \rightarrow Y$ of finite G -sets. We put

$$F(f) = \{(x, x') \in X^2 \mid x \neq x', f(x) = f(x')\},$$

and we let $\pi, \pi': F(f) \rightarrow X$ be the obvious projections. We then define $N_f: (\Pi M)(X) \rightarrow (\Pi M)(Y)$ by

$$N_f(a, m) = (N_f(a), T_f((N_{\pi} R_{\pi'}(a)).m))$$

To see that this definition is reasonable, consider the case where G is the trivial group, so we have a semiring S_1 and an S_1 -module M_1 such that $S(X) = \text{Map}(X, S_1)$ and $M(X) = \text{Map}(X, M_1)$ for all X . From these data we can construct the augmented S_1 -algebra ΠM_1 and then the Tambara functor $T(X) = \text{Map}(X, \Pi M_1)$. Alternatively, we can define a Mackey functor ΠM and construct norm maps as in Definition 14.8. There is an obvious way to identify T with ΠM as Mackey functors, and Definition 14.8 is designed to ensure that this identification is compatible with norm maps. To see this, consider a map $f: X \rightarrow Y$ and maps $a: X \rightarrow S_1$ and $m: X \rightarrow M_1$, and put $(b, n) = N_f^T(s, m) \in T(Y)$. It will be notationally convenient to think of S_1 and M_1 as subsets of ΠM_1 so (a, m) can be written as $a + m$. By definition we have

$$b(y) + n(y) = \prod_{f(x)=y} (a(x) + m(x)).$$

Recall that the product in ΠM_1 of any two elements of M_1 is zero. Thus, in expanding out the above product, we need only consider the monomial $\prod_{f(x)=y} a(x)$ (which is $N_f^S(a)(y)$) and the monomials that involve a single factor $m(x)$. This observation gives

$$\begin{aligned} n(y) &= \sum_{f(x)=y} m(x) \prod_{f(x')=y, x' \neq x} a(x') \\ &= \sum_{f(x)=y} m(x) \prod_{x' \mid (x, x') \in F(f)} (R_{\pi'} a)(x, x') = (T_f(m N_{\pi} R_{\pi'}(a)))(y) \end{aligned}$$

as required.

PROPOSITION 14.9. *The above definition makes ΠM into an object of Mod_S .*

The proof will be divided into several lemmas.

LEMMA 14.10. *For any maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $N_{gf} = N_g N_f: (\Pi M)(X) \rightarrow (\Pi M)(Y)$.*

PROOF. For any map $u: A \rightarrow B$ we will write π_u and π'_u for the two projections $F(u) \rightarrow A$ that were previously called π and π' . We also note that $F(f) \subseteq F(gf) \subseteq X^2$, and we introduce the sets

$$\begin{aligned} P &= \{(y, x') \in Y \times X \mid y \neq f(x'), g(y) = gf(x')\} \\ Q &= \{(x, y') \in X \times Y \mid f(x) \neq y', gf(x) = g(y')\} \\ F^*(gf) &= F(gf) \setminus F(f) \\ &= \{(x, x') \in X \times X \mid f(x) \neq f(x'), gf(x) = gf(x')\}. \end{aligned}$$

We let θ and θ' denote the two projections $F^*(gf) \rightarrow X$, so we have a commutative diagram

$$\begin{array}{ccccc} & & \theta & & \\ & & \curvearrowright & & \\ F^*(gf) & \xrightarrow{1 \times f} & Q & \xrightarrow{\quad} & X \\ \downarrow f \times 1 & & \downarrow f \times 1 & & \downarrow f \\ \theta' \downarrow & P & \xrightarrow{1 \times f} & F(g) & \xrightarrow{\pi_g} & Y \\ & \downarrow & & \downarrow \pi'_g & & \\ & X & \xrightarrow{\quad} & Y & & \end{array}$$

in which the three squares are cartesian.

Consider an element $(a, m) \in (\text{IIM})(X)$. Put

$$\begin{aligned} u &= N_{\pi_f} R_{\pi'_f}(a) \in S(X) \\ v &= N_{\pi_g} R_{\pi'_g} N_f(a) \in S(Y), \end{aligned}$$

so

$$\begin{aligned} N_f(a, m) &= (N_f(a), T_f(mu)) \\ N_g N_f(a, m) &= (N_{gf}(a), T_g(T_f(mu)v)). \end{aligned}$$

Recall that $T_f(mu)v = T_f(muR_f(v))$. Here $R_f(v) = R_f N_{\pi_g} R_{\pi'_g} N_f(a)$, and using the cartesian properties of our diagram we see that this is the same as $N_\theta R_{\theta'}(a)$. We now have

$$uR_f(v) = N_{\pi_f} R_{\pi'_f}(a) N_\theta R_{\theta'}(a).$$

Using the splitting $F(gf) = F(f) \amalg F^*(gf)$ we see that this is the same as $N_{\pi_{gf}} R_{\pi'_{gf}}(a)$. Putting this together, we obtain

$$N_g N_f(a, m) = \left(N_{gf}(a), T_{gf}(m N_{\pi_{gf}} R_{\pi'_{gf}}(a)) \right),$$

which is by definition $N_{gf}(a, m)$. □

LEMMA 14.11. *For any cartesian square*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ g \downarrow & & \downarrow h \\ Y & \xrightarrow{k} & Z \end{array}$$

we have $R_k N_h = N_g R_f: (\text{IIM})(Y) \rightarrow (\text{IIM})(X)$.

PROOF. By definition we have

$$\begin{aligned} R_k N_h(a, m) &= (R_k N_h(a), R_k T_h(m N_{\pi_h} R_{\pi'_h}(a))) \\ &= (N_g R_f(a), T_g R_f(m N_{\pi_h} R_{\pi'_h}(a))) \\ &= (N_g R_f(a), T_g(R_f(m) R_f N_{\pi_h} R_{\pi'_h}(a))) \\ N_g R_f(a, m) &= (N_g R_f(a), T_g(R_f(m) N_{\pi_g} R_{\pi'_g} R_f(a))). \end{aligned}$$

It will therefore suffice to prove that

$$R_f N_{\pi_h} R_{\pi'_h}(a) = N_{\pi_g} R_{\pi'_g} R_f(a).$$

Next, consider a point $(w, w') \in F(g)$, so $w \neq w'$ but $g(w) = g(w')$. Using the cartesian property of the given square, we see that $f(w) \neq f(w')$ but $hf(w) = hf(w')$, so $(f(w), f(w')) \in F(h)$. We can thus define $f_2: F(g) \rightarrow F(h)$ by $f_2(w, w') = (f(w), f(w'))$, and this gives a commutative diagram as follows:

$$\begin{array}{ccccc} W & \xleftarrow{\pi'_g} & F(g) & \xrightarrow{\pi_g} & W \\ \downarrow f & & \downarrow f_2 & & \downarrow f \\ X & \xleftarrow{\pi'_h} & F(h) & \xrightarrow{\pi_h} & X \end{array}$$

It is straightforward to check that the squares are in fact cartesian. Using the cartesian property of the right square and the commutativity of the left square we get

$$R_f N_{\pi_h} R_{\pi'_h} = N_{\pi_g} R_{f_2} R_{\pi'_h} = N_{\pi_g} R_{\pi'_g} R_f$$

as required. \square

LEMMA 14.12. *Suppose we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ with distributor*

$$\Delta(f, g) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z).$$

Then $N_g T_f = T_r N_q R_p: (\Pi M)(X) \rightarrow (\Pi M)(Z)$.

PROOF. Consider a point $(a, m) \in (\Pi M)(X)$ and put $b = N_g T_f(a) = T_r N_q R_p(a) \in S(Z)$. We have $N_g T_f(a, m) = (b, n)$ and $T_r N_q R_p(a, m) = (b, n')$ for certain elements $n, n' \in M(Z)$ and we need to show that $n = n'$. Note that

$$\begin{aligned} T_r N_q R_p(a, m) &= T_r N_q(R_p(a), R_p(m)) = T_r(N_q R_p(a), T_q(R_p(m) N_{\pi_q} R_{\pi'_q} R_p(a))) \\ &= (b, T_{rq}(R_p(m) N_{\pi_q} R_{\pi'_q}(a))), \end{aligned}$$

so $n' = T_{rq}(R_p(m) N_{\pi_q} R_{\pi'_q}(a))$. On the other hand, one can see from the definitions that $rq = gfp: A \rightarrow Z$. Using this together with Frobenius reciprocity for p we get

$$n' = T_{gf} T_p(R_p(m) N_{\pi_q} R_{\pi'_q}(a)) = T_{gf}(m T_p N_{\pi_q} R_{\pi'_q}(a)).$$

We write $c' = T_p N_{\pi_q} R_{\pi'_q}(a) \in S(X)$ so that $n' = T_{gf}(m c')$.

Next, using the definitions and Frobenius reciprocity for f we have

$$\begin{aligned} N_g T_f(a, m) &= N_g(T_f(a), T_f(m)) = (b, T_g(T_f(m) N_{\pi_g} R_{\pi'_g} T_f(a))) \\ &= (b, T_{gf}(m R_f N_{\pi_g} R_{\pi'_g} T_f(a))). \end{aligned}$$

Thus, if we put $c = R_f N_{\pi_g} R_{\pi'_g} T_f(a) \in S(X)$ we have $n = T_{gf}(m c)$. It will thus suffice to prove that $c = c'$.

We now define a cartesian square

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \downarrow \psi & & \downarrow f \\ F(g) & \xrightarrow{\pi'_g} & Y \end{array}$$

by

$$\begin{aligned} P &= \{(y, x') \in Y \times X \mid y \neq f(x'), g(y) = gf(x')\} \\ \phi(y, x') &= x' \\ \psi(y, x') &= (y, f(x')). \end{aligned}$$

This gives $R_{\pi'_g} T_f = T_\psi R_\phi$ so $c = R_f N_{\pi_g} T_\psi R_\phi(a)$.

We next want to construct the distributor $\Delta(\psi, \pi_g)$. To describe this, we put $G_y = g^{-1}\{g(y)\} \setminus \{y\}$. The distributor involves the fibres $\pi_g^{-1}\{y\} \subseteq F(g)$, but π'_g gives a natural bijection $\pi_g^{-1}\{y\} \rightarrow G_y$ and it will be convenient to use G_y instead. We can now identify $\Delta(\psi, \pi_g)$ with the diagram

$$P \xleftarrow{p^*} A^* \xrightarrow{q^*} B^* \xrightarrow{r^*} Y,$$

where

$$\begin{aligned} B^* &= \{(y, s) \mid y \in Y, s: G_y \rightarrow X, fs = 1\} \\ A^* &= \{(y, y', s) \mid y \in Y, s: G_y \rightarrow X, fs = 1, y' \in G_y\} \\ p^*(y, y', s) &= (y, s(y')) \\ q^*(y, y', s) &= (y, s) \\ r^*(y, s) &= y. \end{aligned}$$

This gives

$$c = R_f T_{r^*} N_{q^*} R_{\phi p^*}(a).$$

We now construct another diagram

$$\begin{array}{ccccc} F(q) & \xrightarrow{\pi_q} & A & \xrightarrow{p} & X \\ & \swarrow p\pi'_q & \downarrow \sigma & & \downarrow f \\ X & \xleftarrow{\phi p^*} & A^* & \xrightarrow{q^*} & B^* & \xrightarrow{r^*} & Y \end{array}$$

Recall that

$$A = \{(y, t) \mid y \in Y, t: g^{-1}\{g(y)\} \rightarrow X, ft = 1\}$$

and $p(y, t) = t(y)$. We define $\rho(y, t) = (y, t|_{G_y})$; this gives a map $\rho: A \rightarrow B^*$ making the right square commute. Next, recall that

$$B = \{(z, t) \mid z \in Z, t: g^{-1}\{z\} \rightarrow X, ft = 1\}$$

and $q(y, t) = (g(y), t)$. Using this we get

$$F(q) = \{(y, y', t) \mid (y, y') \in F(g), t: g^{-1}\{g(y)\} = g^{-1}\{g(y')\} \rightarrow X, ft = 1\},$$

so we can define $\sigma: F(q) \rightarrow A^*$ by $\sigma(y, y', t) = (y, y', t|_{G_y})$. This makes the middle square and the left hand triangle commute. One can check that the two squares are cartesian, and it follows that

$$R_f T_{r^*} N_{q^*} R_{\phi p^*} = T_p R_\rho N_{q^*} R_{\phi p^*} = T_p N_{\pi_q} R_\sigma R_{\phi p^*} = T_p N_{\pi_q} R_{p\pi'_q}.$$

This implies that $c = c'$, as required. \square

PROOF OF PROPOSITION 14.9. We first show that ΠS is a Tambara functor. It will suffice to check the conditions in Proposition 6.9. The nonobvious part of condition (a) is covered by Lemma 14.10. Condition (b) is Lemma 14.12, and the nonobvious part of (c) is Lemma 14.11.

Next, it is clear by inspection that the maps $S \xrightarrow{\eta} \Pi M \xrightarrow{\epsilon} S$ and $\Pi M \times_S \Pi M \xrightarrow{\sigma} \Pi M$ preserve norm maps as well as being Mackey morphisms, so they make ΠM into a semigroup object in AugAlg_S , or in other words, an object of Mod_S . \square

PROOF OF PROPOSITION 14.7. We now have a functor $\Pi: \text{NMod}_S \rightarrow \text{Mod}_S$, and it is clear that $\Pi \Pi = 1$. We can define $\phi: \Pi \Lambda T \rightarrow T$ by $\phi(a, m) = a + m$, and it is clear that this is a morphism of Mackey functors. In fact, the same argument that we used in the semiring case shows that ϕ preserves multiplication, so it is a morphism of Green functors. In the additively complete case we again have an inverse map $\phi^{-1}(t) = (\epsilon(t), t - \epsilon(t))$. Example 14.13 will exhibit a case where ϕ does not preserve more general norms. \square

EXAMPLE 14.13. We will work in the category TP of Tambara pairs as described in Section 7. We proved there that TP is equivalent to the category of Tambara functors for the group of order 2.

Let S and T be the Tambara pairs described in Example 7.8. Put $T_2 = T \times_S T$, which consists of the semirings

$$\begin{aligned} A_2 &= A \times_{\mathbb{Z}} A = \mathbb{Z}\{1, \alpha_0, \alpha_1\} \\ B_2 &= B \times_{\mathbb{Z}} B = \mathbb{Z}\{1, \beta_0, \beta_1\} \oplus (\mathbb{Z}/2)\{\gamma_0, \gamma_1\}. \end{aligned}$$

We define $\sigma: S_2 \rightarrow S$ by

$$\begin{aligned} \sigma(i + j_0\alpha_0 + j_1\alpha_1) &= i + (j_0 + j_1)\alpha \\ \sigma(i + j_0\beta_0 + j_1\beta_1 + k_0\gamma_0 + k_1\gamma_1) &= i + (j_0 + j_1)\beta + (k_0 + k_1)\gamma. \end{aligned}$$

One can check that this is a morphism of Tambara pairs. For example, we will show that σ commutes with the norm map. Note that T_2 is defined as a sub-Tambara-pair of $T \times T$, which gives the rule

$$\text{norm}(i + j_0\alpha_0 + j_1\alpha_1) = i^2 + ij_0\beta_0 + ij_1\beta_1 + j_0^2\gamma_0 + j_1^2\gamma_1,$$

so

$$\sigma(\text{norm}(i + j_0\alpha_0 + j_1\alpha_1)) = i^2 + (ij_0 + ij_1)\beta + (j_0^2 + j_1^2)\gamma.$$

On the other hand, we have

$$\text{norm}(\sigma(i + j_0\alpha_0 + j_1\alpha_1)) = \text{norm}(i + (j_0 + j_1)\alpha) = i^2 + (ij_0 + ij_1)\beta + (j_0^2 + j_1^2 + 2j_0j_1)\gamma.$$

These are the same because $2\gamma = 0$. It is clear that σ is commutative, associative and unital, so we have given T the structure of a commutative group object in AugAlg_S , so $T \in \text{Mod}_S$. We can use ϕ to identify $\Pi\Lambda T$ with T as Mackey pairs. However, we claim that this does not respect norm maps. Indeed, in T we have $\text{norm}(\alpha) = \gamma$ by definition. Let ϵ be the map $G \rightarrow 1$, so the value of $\text{norm}(\alpha)$ in $\Pi\Lambda T$ is found by calculating $N_\epsilon(0, \alpha)$ as in Definition 14.8. This gives

$$N_\epsilon(0, \alpha) = (N_\epsilon(0), T_\epsilon(\alpha N_{\pi_\epsilon} R_{\pi'_\epsilon}(0))) = (N_\epsilon(0), T_\epsilon(\alpha N_{\pi_\epsilon}(0))).$$

Here the maps $\epsilon: G \rightarrow 1$ and $\pi_\epsilon: F(\epsilon) \rightarrow G$ are both surjective, so Lemma 6.8 tells us us that $N_\epsilon(0) = 0$ and $N_{\pi_\epsilon}(0) = 0$. It follows that $\text{norm}(\alpha) = 0$ in $\Pi\Lambda T$, so ϕ is not a Tambara morphism.

15. Coefficient systems

In this section we introduce the category CSys_G of coefficient systems, and the related category MCSys_G of multiplicative coefficient systems. These will be linked by adjunctions to Mackey_G and Tambara_G . Later we will define \mathbb{Q} -linear analogues denoted by $\mathbb{Q}\text{CSys}_G$ and so on, and show that our adjunctions restrict to give equivalences $\mathbb{Q}\text{Mackey}_G \simeq \mathbb{Q}\text{CSys}_G$ and $\mathbb{Q}\text{Tambara}_G \simeq \mathbb{Q}\text{MCSys}_G$.

DEFINITION 15.1. Let Orb_G^\times denote the category of transitive G -sets and equivariant isomorphisms between them. A *coefficient system* will mean a functor from Orb_G^\times to the category of semigroups. We write CSys_G for the category of coefficient systems.

REMARK 15.2. As all morphisms in Orb_G^\times are invertible, covariant functors can be converted to contravariant functors and vice-versa. We prefer to use covariant functors here to maximise compatibility with our later discussion of rational Tambara functors. Given a coefficient system N and an isomorphism $f: U \rightarrow V$ of transitive G -sets, we will write f_* for the resulting map $N(U) \rightarrow N(V)$.

DEFINITION 15.3. For any subgroup $H \leq G$ and any $g \in G$ we define $\rho(g): G/H \rightarrow G/gHg^{-1}$ by

$$\rho(g)(xH) = xHg^{-1} = xg^{-1}gHg^{-1}.$$

Now let N and N' be coefficient systems. For any map $u: N(G/H) \rightarrow N'(G/H)$ we write $g.u$ for the composite

$$N(G/gHg^{-1}) \xrightarrow{\rho(g^{-1})_*} N(G/H) \xrightarrow{u} N'(G/H) \xrightarrow{\rho(g)_*} N'(G/gHg^{-1}).$$

This defines a map

$$g: \text{Map}(N(G/H), N'(G/H)) \rightarrow \text{Map}(N(G/gHg^{-1}), N'(G/gHg^{-1})).$$

We give the set $\prod_{K \leq G} \text{Map}(N(G/K), N'(G/K))$ the unique G -action such that the diagrams

$$\begin{array}{ccc} \prod_{K \leq G} \text{Map}(N(G/H), N'(G/H)) & \xrightarrow{g} & \prod_{K \leq G} \text{Map}(N(G/H), N'(G/H)) \\ \pi_H \downarrow & & \downarrow \pi_{gHg^{-1}} \\ \text{Map}(N(G/H), N'(G/H)) & \xrightarrow{g} & \text{Map}(N(G/gHg^{-1}), N'(G/gHg^{-1})) \end{array}$$

commute.

PROPOSITION 15.4. *Let N and N' be coefficient systems. Then there is a natural isomorphism*

$$\text{CSys}_G(N, N') = \left[\prod_{K \leq G} \text{Map}(N(G/K), N'(G/K)) \right]^G$$

PROOF. Consider a morphism $u: N \rightarrow N'$ of coefficient systems. For each subgroup $H \leq G$ we have a component $u_H \in \text{Map}(N(G/H), N'(G/H))$. Together these give an element $\theta(u) \in \prod_H \text{Map}(N(G/H), N'(G/H))$. This determines u , because every orbit is isomorphic to G/H for some H . Every morphism $G/H \rightarrow G/K$ has the form $\rho(g)$ for some g with $gHg^{-1} = K$. Naturality with respect to such a morphism is equivalent to the identity $u_{gHg^{-1}} = g \cdot u_H$, and this holds for all g and H iff $\theta(u)$ is a G -fixed point in $\prod_H \text{Map}(N(G/H), N'(G/H))$. The claim is clear from this. \square

At the expense of some arbitrary choices, we can cut this description down further, as follows.

DEFINITION 15.5. A *subgroup system* for G is a list (H_1, \dots, H_r) of subgroups such that

- (a) Each subgroup of G is conjugate to H_i for precisely one value of i .
- (b) We have $|H_i| \leq |H_j|$ whenever $i \leq j$.

It is clear that we must have $H_1 = 1$ and $H_r = G$. We let N_i denote the normaliser in G of H_i , and put $W_i = N_i/H_i$.

Given a subgroup system as above, we note that any transitive G -set is isomorphic to G/H_i for a unique value of i , and that the automorphism group of G/H_i is W_i . It follows that if we regard W_i as a groupoid with one object, then Orb_G^\times is equivalent to the coproduct $\coprod_i W_i$. Using this we get an equivalence

$$\text{CSys}_G \simeq \prod_{i=1}^r \text{Mod}_{\mathbb{N}[W_i]}.$$

We will exhibit an adjunction between functors $q: \text{Mackey}_G \rightarrow \text{CSys}_G$ and $r: \text{CSys}_G \rightarrow \text{Mackey}_G$, and show that this restricts to an equivalence between the corresponding rational categories.

DEFINITION 15.6. Given $M \in \text{Mackey}_G$ and $U \in \text{Orb}_G^\times$ we let $(qM)(U)$ be the quotient of $M(U)$ by the sum of all subsemigroups $T_u M(U')$, where $u: U' \rightarrow U$ is a map of transitive G -sets that is not an isomorphism. This is clearly functorial for isomorphisms $U_0 \rightarrow U_1$, and for arbitrary morphisms $M_0 \rightarrow M_1$ of Mackey functors. It therefore gives a functor $q: \text{Mackey}_G \rightarrow \text{CSys}_G$.

REMARK 15.7. In slightly different language, $(qM)(G/H)$ is $M(G/H)$ modulo transfers from $M(G/K)$ for proper subgroups $K < H$.

DEFINITION 15.8. We define a functor $\Phi: (\overline{\mathcal{A}}_G)^{\text{op}} \rightarrow \text{CSys}_G$ as follows. On objects, we define $\Phi(X)(T) = \mathbb{N}\{\text{Map}_G(T, X)\}$ (so $\Phi(X)(G/H) = \mathbb{Z}\{X^H\}$). We write $[u]$ for the basis element corresponding to a G -map $u: T \rightarrow X$.

A morphism $X \rightarrow Y$ in \mathcal{A}_G is represented by a diagram $X \xleftarrow{p} A \xrightarrow{q} Y$ of finite G -sets. The corresponding map

$$\alpha: \mathbb{N}\{\text{Map}_G(T, Y)\} \rightarrow \mathbb{N}\{\text{Map}_G(T, X)\}$$

sends $[T \xrightarrow{v} Y]$ to the sum of the elements $[T \xrightarrow{a} A \xrightarrow{p} X]$ for all G -maps $T \xrightarrow{a} A$ such that $qa = v$.

Alternatively, the functor $\text{Map}_G(T, -)$ (from finite G -sets to finite sets) preserves finite limits and so induces a functor $\overline{\mathcal{A}}_G \rightarrow \overline{\mathcal{A}}_1$. However, there is an evident isomorphism $\overline{\mathcal{A}}_G \simeq (\overline{\mathcal{A}}_G)^{\text{op}}$, and $\overline{\mathcal{A}}_1$ is canonically

equivalent to the category of finitely generated free abelian semigroups, so we get a functor $(\mathcal{A}_G)^{\text{op}} \rightarrow \text{Semigroups}$, which we denote by ϕ^T . We then have $\Phi(X)(T) = \phi^T(X)$.

DEFINITION 15.9. We now define $r: \text{CSys}_G \rightarrow \text{Mackey}_G$ by

$$r(N)(X) = \text{CSys}_G(\Phi(X), N) = \left[\prod_{H \leq G} \text{Map}(X^H, N(G/H)) \right]^G.$$

REMARK 15.10. One checks that $\Phi(G)(G) = \mathbb{N}[G]$ but $\Phi(G)(T) = 0$ if $T \neq G$. It follows that $r(N)(G) = N(G)$.

We can rewrite the definition in a useful way when X is an orbit.

PROPOSITION 15.11. *There is a natural isomorphism*

$$r(N)(G/H) = \left[\prod_{K \leq H} N(G/K) \right]^H.$$

If K_1, \dots, K_r is a subgroup system for H , this can be written as

$$r(N)(G/H) = \prod_{i=1}^r N(G/K_i)^{W_H K_i}.$$

PROOF. From the definitions, we have

$$r(N)(G/H) = \left[\prod_{K \leq G} \text{Map}((G/H)^K, N(G/K)) \right]^G.$$

Suppose we have $n \in r(N)(G/H)$. For each $K \leq H$, the basepoint $H \in G/H$ is fixed by K , so we have an element $n_K(H) \in N(G/K)$. We can thus define a map

$$\phi: r(N)(G/H) \rightarrow \prod_{K \leq H} N(G/K)$$

by $\phi(n)_K = n_K(H)$. As n is G -invariant, it follows that $\phi(n)$ is H -invariant.

Now suppose instead we have an element $m \in \left[\prod_{K \leq H} N(G/K) \right]^H$. Consider a subgroup $L \leq G$, and an element $xH \in (G/H)^L$. This means that $LxH = xH$ and so $x^{-1}Lx \leq H$. We thus have an element $m_{x^{-1}Lx} \in N(G/x^{-1}Lx)$ and a map $\rho(x): G/x^{-1}Lx \rightarrow G/L$ giving an element $\rho(x)_*(m_{x^{-1}Lx}) \in N(G/L)$. Using the fact that m is H -invariant, we see that this element does not depend on the choice of element x representing the coset xH . We can thus define $\psi(m) \in \prod_L \text{Map}((G/H)^L, N(G/L))$ by $\psi(m)_L(xH) = \rho(x)_*(m_{x^{-1}Lx})$. One can check that this is invariant under the action of G , so we have defined a map

$$\psi: \left[\prod_{K \leq H} N(G/K) \right]^H \rightarrow r(N)(G/H).$$

We leave it to the reader to show that this is inverse to ϕ . \square

REMARK 15.12. As yet another way to formulate the definition, we introduce the category $(\text{Orb}_G^\times \downarrow X)$. The objects are diagrams $(U \xrightarrow{x} X)$, where U is a G -orbit and x is a G -map. The morphisms from $(U \xrightarrow{x} X)$ to $(U' \xrightarrow{x'} X)$ are the G -isomorphisms $p: U \rightarrow U'$ with $x'p = x$. Any coefficient system N gives a functor $(U \xrightarrow{x} X) \mapsto N(U)$ from this category to Semigroups, and $rN(X)$ is easily seen to be the inverse limit of this functor. We will use this interpretation without further comment when convenient.

One can check that in this picture the operations associated to a map $f: X \rightarrow Y$ are

$$\begin{aligned}(R_f n)(U \xrightarrow{x} X) &= n(U \xrightarrow{fx} Y) \\ (T_f m)(U \xrightarrow{y} Y) &= \sum_{x \in \text{Map}_G(U, X), fx=y} m(U \xrightarrow{x} X).\end{aligned}$$

PROPOSITION 15.13. *There is a natural map $\eta: M \rightarrow rqM$ of Mackey functors given by*

$$(\eta m)(U \xrightarrow{x} X) = \pi R_x(u) \in (qM)(U)$$

(where π is the quotient map $M(U) \rightarrow qM(U)$).

The proof will be given after some preliminaries.

DEFINITION 15.14. For any equivariant map $f: X \rightarrow Y$, we put

$$\text{Sec}(f) = \{s \in \text{Map}_G(Y, X) \mid fs = 1_Y\} = \{\text{equivariant sections of } f\}.$$

If Y is a single G -orbit, we also put

$$\text{Sec}'(f) = \{\text{orbits } U \subseteq X \mid f: U \rightarrow Y \text{ is an isomorphism.}\}$$

LEMMA 15.15. (a) *Let $f: U \rightarrow V$ be a G -map between transitive G -sets. Then f is automatically surjective (and so $|U| \geq |V|$). Moreover, f is an isomorphism iff it is injective iff it is a split epimorphism iff $|U| = |V|$.*

(b) *More generally, let $f: X \rightarrow V$ be a map of finite G -sets where V is transitive. We then have*

$$\text{Sec}'(f) = \{\text{orbits } U \subseteq X \mid |U| = |V|\}.$$

Moreover, there is a bijection $\text{Sec}(f) \simeq \text{Sec}'(f)$ given by $s \mapsto s(V)$ and $U \mapsto (f|_U)^{-1}$.

(c) *Suppose we have maps $V \xrightarrow{y} Y \xleftarrow{f} X$ of finite G -sets, where V is transitive. Form a pullback square*

$$\begin{array}{ccc} U & \xrightarrow{x'} & X \\ e \downarrow & & \downarrow f \\ V & \xrightarrow{y} & Y, \end{array}$$

and let U_1, \dots, U_n be the G -orbits in V . Let $e_i: U_i \rightarrow V$ and $x'_i: U_i \rightarrow X$ be the restrictions of e and x' . We may assume that the U_i are numbered so that e_1, \dots, e_p are isomorphisms (for some p , possibly $p = 0$) and e_{p+1}, \dots, e_n are not isomorphisms. For $i \leq p$ put $x_i = x'_i e_i^{-1}: V \rightarrow X$. Then the maps x_1, \dots, x_p are all different, and this is a complete list of all G -maps $x: V \rightarrow X$ with $fx = y$.

PROOF. (a) It is clear that $f(U)$ is a nonempty G -invariant subset of the transitive G -set V , so $f(U) = V$. If f is a split epimorphism of G -sets then the splitting map $g: V \rightarrow U$ must also be surjective by the same logic, and it follows easily that g is an inverse for f . The rest is clear.

(b) If $s \in \text{Sec}(f)$ is a section then $s(V) \subseteq X$ is an orbit and $f: s(V) \rightarrow V$ is a split epimorphism of G -sets, hence an isomorphism by (a). Everything else is clear from this.

(c) The G -maps $x: V \rightarrow X$ lifting y biject naturally with the sections of the map $e: U \rightarrow V$. Given this, claim (c) follows from (b). □

PROOF OF PROPOSITION 15.13. Given maps $W \xrightarrow{g} X \xrightarrow{f} Y$ of finite G -sets, we must show that the diagram

$$\begin{array}{ccccc} M(W) & \xleftarrow{R_g} & M(X) & \xrightarrow{T_f} & M(Y) \\ \eta \downarrow & & \downarrow \eta & & \downarrow \eta \\ rqM(W) & \xleftarrow{R_g} & rqM(X) & \xrightarrow{T_f} & rqM(Y) \end{array}$$

commutes. For the left square, we have

$$\begin{aligned} (R_g \eta m)(U \xrightarrow{w} W) &= (\eta m)(U \xrightarrow{gw} X) = \pi R_{gw}(m) = \pi R_w(R_g m) \\ &= (\eta R_g m)(U \xrightarrow{w} W) \end{aligned}$$

as required.

For the right hand side, we have $(\eta T_f m)(U \xrightarrow{y} Y) = \pi R_y T_f(m)$. If we form a pullback as in Lemma 15.15 we find that

$$\pi R_y T_f = \pi T_e R_v = \sum_{i=1}^n \pi T_{e_i} R_{v_i}.$$

If $i > p$ then $e_i: V_i \rightarrow U$ is not an isomorphism so $\pi T_{e_i} = 0$. If $i \leq p$ then e_i is an isomorphism so $T_{e_i} = R_{e_i^{-1}}$ so $\pi T_{e_i} R_{v_i} = \pi R_{v_i e_i^{-1}} = \pi R_{x_i}$. It follows that

$$(\eta T_f m)(U \xrightarrow{y} Y) = \pi R_y T_f(m) = \sum_{i=1}^p \pi R_{x_i}(m) = \sum_{i=1}^p (\eta m)(U \xrightarrow{x_i} X) = (T_f \eta m)(U \xrightarrow{y} Y)$$

as required. This proves that $\eta: M \rightarrow r q M$ is a morphism of Mackey functors. Naturality in M is clear. \square

PROPOSITION 15.16. *There is a natural map $\epsilon: q r N \rightarrow N$ of coefficient systems given by $\epsilon \pi(n) = n(U \xrightarrow{1} U) \in N(U)$ (for all transitive G -sets U and elements $n \in (rN)(U)$).*

PROOF. Suppose we have an element $n \in (rN)(U)$, so we have $n(U' \xrightarrow{t} U) \in N(U')$ for all transitive U' and all G -maps $t: U' \rightarrow U$. In particular, we can put $\epsilon' n = n(U \xrightarrow{1} U) \in N(U)$. This defines a map $\epsilon': (rN)(U) \rightarrow N(U)$. Suppose that $n = T_f(p)$ for some map $f: V \rightarrow U$ (where V is transitive and f is not invertible) and $p \in (rN)(V)$. Then the element $\epsilon' n = (T_f p)(U \xrightarrow{1} U)$ is by definition a sum of terms indexed the equivariant sections of f , but there are no sections, so $\epsilon' n = 0$. It follows that $\epsilon': rN(U) \rightarrow N(U)$ induces a map $\epsilon: q r N(U) \rightarrow N(U)$. This is easily seen to be natural for isomorphisms $U_0 \rightarrow U_1$ of transitive G -sets, so we have a morphism $\epsilon: q r N \rightarrow N$ of coefficient systems. Naturality in N is also clear. \square

PROPOSITION 15.17. *The maps η and ϵ are the unit and counit of an adjunction between q and r .*

PROOF. The map η gives rise to a natural map

$$\lambda: \text{CSys}_G(qM, N) \rightarrow \text{Mackey}_G(M, rN)$$

by $\lambda(\alpha) = (M \xrightarrow{\eta} r q M \xrightarrow{r(\alpha)} rN)$. Suppose we have a map $x: U \rightarrow X$ of finite G -sets with U transitive, and an element $m \in M(X)$. We then have $\lambda(\alpha)(m) \in (rN)(X)$ and thus $\lambda(\alpha)(m)(U \xrightarrow{x} X) \in N(U)$. On the other hand, we have $R_x(m) \in M(U)$ and so $\pi(R_x(m)) \in qM(U)$ and $\alpha(\pi(R_x(m))) \in N(U)$. By inspecting the definitions, we obtain the formula

$$\lambda(\alpha)(m)(U \xrightarrow{x} X) = \alpha(\pi(R_x(m))).$$

Similarly, the map ϵ gives rise to a natural map

$$\rho: \text{Mackey}_G(M, rN) \rightarrow \text{CSys}_G(qM, N)$$

by $\rho(\beta) = (qM \xrightarrow{q(\beta)} q r N \xrightarrow{\epsilon} N)$. For any transitive G -set U and $m \in M(U)$ we have $\beta(m) \in (rN)(U)$ and so $\beta(m)(U \xrightarrow{1} U) \in N(U)$. By inspecting the definitions, we obtain the formula

$$\rho(\beta)(\pi(m)) = \beta(m)(U \xrightarrow{1} U).$$

We need to show that λ and ρ are inverse to each other. If we start with $\alpha: qM \rightarrow N$ we have

$$\rho(\lambda(\alpha))(\pi(m)) = \lambda(\alpha)(m)(U \xrightarrow{1} U) = \alpha(\pi(R_1(m))) = \alpha(\pi(m)),$$

so $\rho(\lambda(\alpha)) = \alpha$. If instead we start with $\beta: M \rightarrow rN$ we have

$$\lambda(\rho(\beta))(m)(U \xrightarrow{x} X) = \rho(\beta)(\pi(R_x(m))) = \beta(R_x(m))(U \xrightarrow{1} U).$$

Now, β is assumed to be a morphism $M \rightarrow rN$ of Mackey functors so $\beta R_x = R_x \beta$. The map $R_x: (rN)(X) \rightarrow (rN)(U)$ is defined by

$$(R_x n)(T \xrightarrow{t} U) = n(T \xrightarrow{xt} X).$$

In particular, we have $(R_x n)(U \xrightarrow{1} U) = n(U \xrightarrow{x} X)$. Putting this together, we get

$$\lambda(\rho(\beta))(m)(U \xrightarrow{x} X) = R_x(\beta(m))(U \xrightarrow{1} U) = \beta(m)(U \xrightarrow{x} X).$$

This holds for all X, U, x and m , so $\lambda(\rho(\beta)) = \beta$ as required. \square

PROPOSITION 15.18. *There is a natural map $\sigma: N \rightarrow qrN$ (for $N \in \text{CSys}_G$) such that $\epsilon\sigma = 1$.*

PROOF. Consider an object $T \in \text{Orb}_G^\times$ and an element $u \in N(T)$. Given an orbit U and a G -map $p: U \rightarrow T$ we define $\sigma_0(u)(U, p) = p_*^{-1}(u)$ if p is an isomorphism, and $\sigma_0(u)(U, p) = 0$ otherwise. This gives a system of maps $\sigma_0(u): \text{Map}_G(U, T) \rightarrow N(U)$ that are natural for isomorphisms of U , or in other words an element $\sigma_0(u) \in (rN)(T)$. We let $\sigma(u)$ denote the image of $\sigma_0(u)$ in $(qrN)(T)$. This defines a natural map $\sigma: N \rightarrow qrN$, and it is clear from the definitions that $\epsilon\sigma = 1$. \square

Recall that Mackey_G is a symmetric monoidal category under the box product operation \boxtimes , with the Burnside semiring Mackey functor A acting as the unit. There is also a simpler symmetric monoidal structure on CSys_G given by $(N \otimes N')(U) = N(U) \otimes N'(U)$. For this, the unit is the constant functor $c\mathbb{N}$ given by $c\mathbb{N}(U) = \mathbb{N}$ for all U .

PROPOSITION 15.19. *There are natural isomorphisms $qA \rightarrow c\mathbb{N}$ and $q(M \boxtimes M') \rightarrow q(M) \otimes q(N')$ making q a symmetric monoidal functor.*

PROOF. Let T be a transitive G -set. Recall that $A(T)$ is the set of isomorphism classes of finite G -sets equipped with a map to U . We define $\alpha': A(T) \rightarrow \mathbb{N}$ by

$$\alpha'[W \xrightarrow{t} T] = |\text{Sec}(t)| = |\text{Sec}'(t)|.$$

It is easy to see that this is a homomorphism. It has $\alpha'[T \xrightarrow{1} T] = 1$, so it is surjective.

Recall also that for $f: U \rightarrow T$ and $[W \xrightarrow{u} U] \in A(U)$ we have $T_f[W \xrightarrow{u} U] = [W \xrightarrow{fu} T]$, and note that if u has no sections then fu has no sections. From this it is easy to see that α' induces an isomorphism $qA(T) \rightarrow \mathbb{N} = c\mathbb{N}(T)$. This is clearly natural and so gives an isomorphism $qA \rightarrow c\mathbb{N}$.

Now suppose we have two Mackey functors M and M' . Given a map $u: X \rightarrow U$ and elements $m \in M(X)$ and $m' \in M'(X)$ we define

$$\beta'(u, m, m') = \sum_{s \in \text{Sec}(u)} \pi(R_s(m)) \otimes \pi(R_s(m')) \in (qM)(U) \otimes (qM')(U) = (qM \otimes qM')(U).$$

Now suppose we have a map $p: W \rightarrow X$ and $m \in M(X)$ and $n' \in M'(W)$. We claim that $\beta'(up, R_p(m), n') = \beta'(u, m, T_p(n'))$. Indeed, we have

$$\beta'(up, R_p(m), n') = \sum_{t \in \text{Sec}(up)} \pi R_{pt}(m) \otimes \pi R_t(n') = \sum_{s \in \text{Sec}(u)} \pi R_s(m) \otimes \left(\sum_{\substack{t: T \rightarrow W \\ pt = s}} \pi R_t(n') \right)$$

$$\beta'(u, m, T_p(n')) = \sum_{s \in \text{Sec}(u)} \pi R_s(m) \otimes \pi R_s T_p(n').$$

For $s \in \text{Sec}(u)$ we can form a pullback square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{s}} & W \\ \bar{p} \downarrow & & \downarrow p \\ T & \xrightarrow{s} & X. \end{array}$$

We then have $\pi R_s T_p = \pi T_{\tilde{p}} R_{\tilde{s}}$. We can write this as a sum over the orbits $V \subseteq \tilde{T}$. If the map $\tilde{p}|_V: V \rightarrow T$ is not an isomorphism, then the corresponding contribution is killed by π . Lemma 15.15(c) tells us that the remaining orbits biject with G -maps $t: T \rightarrow W$ satisfying $pt = s$. The identity $\beta'(up, R_p(m), n') = \beta'(u, m, T_p(n'))$ now follows easily. By a similar argument, for any $n \in M(W)$ and $m' \in M'(X)$ we have $\beta'(up, n, R_p(m')) = \beta'(u, T_p(n), m')$. It now follows that there is a unique map $\beta'': (M \boxtimes M')(U) \rightarrow (qM \otimes qM')(U)$ satisfying $\beta''(T_u(m \otimes m')) = \beta'(u, m, m')$ for all (u, m, m') as above. It is clear by construction that $\beta'' T_p = 0$ for all non-split maps $V \xrightarrow{p} U$ of orbits, so there is an induced map $\beta: q(M \boxtimes M')(U) \rightarrow (qM \otimes qM')(U)$.

In the opposite direction, suppose we have $m \in M(U)$ and $m' \in M'(U)$, giving $\pi(m \otimes m') \in q(M \boxtimes M')(U)$. If m has the form $T_p(n)$ for some non-split map $V \xrightarrow{p} U$ of orbits, then we have $\pi(m \otimes m') = \pi T_p(n \otimes R_p(m')) = 0$. Using this and its symmetrical counterpart, we see that there is a well-defined map $\gamma: qM(U) \otimes qM'(U) \rightarrow q(M \boxtimes M')(U)$ such that $\gamma(\pi(m) \otimes \pi(m')) = \pi(m \otimes m')$. We leave the reader to check that this is inverse to β . \square

DEFINITION 15.20. A *multiplicative coefficient system* is a covariant functor $\text{Orb}_G \rightarrow \text{Rings}$, where Orb_G is the category of transitive G -sets and equivariant maps. We write MCSys_G for the category of such objects. Given $R \in \text{MCSys}_G$ and a morphism $f: U \rightarrow T$ in Orb_G^\times we write N_f for the induced map $R(U) \rightarrow R(T)$. If f is an isomorphism we may use the alternative notation f_* for N_f . By this rule, we can regard R as a covariant functor $\text{Orb}_G^\times \rightarrow \text{Ab}$, or in other words, a coefficient system.

PROPOSITION 15.21. *If S is a G -Tambara functor, then qS is naturally a multiplicative coefficient system.*

The proof will be given after the following lemma.

LEMMA 15.22. *Let $X \xrightarrow{g} T \xrightarrow{f} U$ be maps of finite G -sets, where T and U are orbits. Let S be a Tambara functor, and let $\pi: S(U) \rightarrow qS(U)$ be the quotient map. Then*

$$\pi N_f T_g = \sum_{k \in \text{Sec}(g)} \pi N_f R_k: S(X) \rightarrow qS(U).$$

PROOF. Let $X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} U$ be the distributor $\Delta(g, f)$, so $\pi N_f T_g = \pi T_r N_q R_p: S(X) \rightarrow qS(U)$. Now write B as a disjoint union of orbits, say $B = \coprod_i B_i$, and let r_i be the restriction of r to B_i . If r_i is not an isomorphism then $\pi T_{r_i} = 0$ by the definition of q . If r_i is an isomorphism then the map $m_i = (U \xrightarrow{r_i^{-1}} B_i \rightarrow B)$ is an equivariant section of r , with $T_{r_i} = R_{m_i}$. Using this we see that $\pi T_r = \sum_{m \in \text{Sec}(r)} \pi R_m$, and so $\pi T_r N_q R_p = \sum_{m \in \text{Sec}(r)} \pi R_m N_q R_p$.

Now consider a section $k \in \text{Sec}(g)$. Recall that

$$B = \{(u, s) \mid u \in U, s: f^{-1}\{u\} \rightarrow X, gs = 1\}.$$

For any $u \in B$ we put $k'(u) = (u, k|_{f^{-1}\{u\}}) \in B$, so $rk'(u) = u$. It is not hard to check that the construction $k \mapsto k'$ gives a bijection $\text{Sec}(g) \rightarrow \text{Sec}(r)$, so $\pi T_r N_q R_p = \sum_{k \in \text{Sec}(g)} \pi R_{k'} N_q R_p$.

Next, define $k'': T \rightarrow A$ by $k''(t) = (f(t), k|_{f^{-1}\{f(t)\}})$. We then have a commutative diagram as follows, in which the square is cartesian:

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ k \swarrow & & \downarrow k' \\ X & \xleftarrow{p} & A \xrightarrow{q} B \end{array}$$

This gives

$$R_{k'} N_q R_p = N_f R_{k''} R_p = N_f R_{pk''} = N_f R_k$$

as required. \square

PROOF OF PROPOSITION 15.21. First, for any non-isomorphic map $p: V \rightarrow U$ of G -orbits, the Frobenius reciprocity formula $T_p(a)b = T_p(a R_p(b))$ shows that the image of T_p is an ideal in $S(U)$. It follows easily that qS has a unique semiring structure for which the projection $\pi: S(U) \rightarrow qS(U)$ is a homomorphism.

Next, consider a map $f: T \rightarrow U$ of G -orbits. We first apply Lemma 15.22 to the fold map $T \amalg T \rightarrow T$, noting that this has only the two obvious sections; the conclusion is that $\pi N_f(a_0 + a_1) = \pi N_f(a_0) + \pi N_f(a_1)$ for all $a_0, a_1 \in S(T)$, so $\pi N_f: S(T) \rightarrow qS(U)$ is an additive homomorphism. Now apply the lemma instead to a map $g: X \rightarrow T$, where X is an orbit and g is not an isomorphism. We recall from Lemma 15.15 that $\text{Sec}(g) = \emptyset$, so $\pi N_f T_g = 0$. It follows that there is a unique additive homomorphism $N_f: qS(T) \rightarrow qS(U)$ such that the square

$$\begin{array}{ccc} S(X) & \xrightarrow{N_f} & S(Y) \\ \pi \downarrow & & \downarrow \pi \\ qS(X) & \xrightarrow{N_f} & qS(Y) \end{array}$$

commutes. As $N_f: S(X) \rightarrow S(Y)$ preserves products, the same is true of the induced map $qS(X) \rightarrow qS(Y)$. \square

PROPOSITION 15.23. *If R is a multiplicative coefficient system, then rR is naturally a Tambara functor.*

PROOF. Consider a map $f: X \rightarrow Y$; we need to define $N_f: (rR)(X) \rightarrow (rR)(Y)$, and check that it is compatible with all other structure. Consider $m \in (rR)(X)$, so we have an element $m(T \xrightarrow{x} X) \in R(T)$ for each transitive G -set T and each G -map $x: T \rightarrow X$. Consider instead a map $y: T \rightarrow Y$, and form the pullback square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{u} & X \\ v \downarrow & & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

Each orbit $U \subseteq \tilde{T}$ comes equipped with maps $T \xleftarrow{v} U \xrightarrow{u} X$ so we have $m(U \xrightarrow{u} X) \in R(U)$ and $N_v m(U \xrightarrow{u} X) \in R(T)$. Let $p \in R(T)$ be the product of all these terms, as U runs over the orbits in \tilde{T} . This is completely natural, so we can define $(N_f m)(T \xrightarrow{y} Y) = p$ and this defines a map $N_f: (rR)(X) \rightarrow (rR)(Y)$ as required. It is clear that this depends functorially on f .

We next check the Mackey property. Consider a pullback square

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{k} & Y' \end{array},$$

and an element $m \in (rR)(X')$. We must show that

$$(N_f R_j(m))(T \xrightarrow{y} Y) = (R_k N_{f'}(m))(T \xrightarrow{y} Y)$$

for all G -orbits T and all G -maps $T \xrightarrow{y} Y$. We define \tilde{T} as before, giving a diagram

$$\begin{array}{ccccc} \tilde{T} & \xrightarrow{u} & X & \xrightarrow{j} & X' \\ v \downarrow & & \downarrow f & & \downarrow f' \\ T & \xrightarrow{y} & Y & \xrightarrow{k} & Y' \end{array}$$

in which both squares are cartesian. This gives

$$(N_f R_j(m))(T \xrightarrow{y} Y) = \prod_{U \in \text{orb}(\tilde{T})} N_{U \rightarrow T}(R_j m)(U \xrightarrow{u} X) = \prod_{U \in \text{orb}(\tilde{T})} N_{U \rightarrow T} m(U \xrightarrow{j u} X').$$

On the other hand, we have

$$(R_k N_{f'}(m))(T \xrightarrow{y} Y) = (N_{f'} m)(T \xrightarrow{k y} Y').$$

As both squares are cartesian we see that the full rectangle is also cartesian and so can be used to compute $N_{f'}$, giving the same answer as before.

Finally, consider a pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ with distributor

$$\Delta(f, g) = (X \xleftarrow{p} A \xrightarrow{q} B \xrightarrow{r} Z).$$

Consider an element $m \in (rR)(X)$, a G -orbit T and a G -map $T \xrightarrow{z} Z$. We must show that

$$(N_g T_f m)(T \xrightarrow{z} Z) = (T_r N_q R_p m)(T \xrightarrow{z} Z).$$

For the left hand side, we form the pullback square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{u} & Y \\ v \downarrow & & \downarrow g \\ T & \xrightarrow{z} & Z. \end{array}$$

For each orbit $U \subseteq \tilde{T}$ we let L_U denote the the set of G -maps $s: U \rightarrow X$ with $fs = u|_U$. We then put $L = \prod_U L_U$, which can be identified with the set of G -maps $t: \tilde{T} \rightarrow X$ with $ft = U$. From the definitions we have

$$(N_g T_f m)(T \xrightarrow{z} Z) = \prod_U N_{v|_U}((T_f m)(U \xrightarrow{u} Y)) = \prod_U N_{v|_U} \left(\sum_{s \in L_U} m(U \xrightarrow{s} X) \right).$$

We can next use the splitting $L = \prod_U L_U$ to expand out the product, and recall that $N_{v|_U}: R(U) \rightarrow R(T)$ is a ring map, to get

$$(N_g T_f m)(T \xrightarrow{z} Z) = \sum_{t \in L} \prod_U N_{v|_U}(m(U \xrightarrow{t|_U} X)).$$

Now consider instead the right hand side. Put $m' = R_p m$ and $m'' = N_q m'$ so $T_r N_q R_p m = T_r m''$. By definition, $(T_r m'')(T \xrightarrow{z} Z)$ can be written as a sum over G -maps $T \rightarrow B$ lifting z . For any $t \in L$ we can define $t': T \rightarrow B$ by

$$t'(a) = \left(z(a), g^{-1}\{z(a)\} \xrightarrow{u^{-1}} v^{-1}\{a\} \xrightarrow{t} X \right).$$

One can check that $rt' = z$, and that any G -map with $rt' = z$ arises in this way. We therefore have

$$(T_r m'')(T \xrightarrow{z} Z) = \sum_{t \in L} m''(T \xrightarrow{t'} B) = \sum_{t \in L} (N_q m')(T \xrightarrow{t'} B).$$

Next, we have a cartesian square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{t''} & A \\ v \downarrow & & \downarrow q \\ T & \xrightarrow{t'} & B \end{array}$$

where

$$t''(a) = \left(u(a), g^{-1}\{g(u(a))\} = g^{-1}\{z(v(a))\} \xrightarrow{u^{-1}} v^{-1}\{v(a)\} \xrightarrow{t} X \right).$$

This gives

$$(N_q m')(T \xrightarrow{t'} B) = \prod_U N_{v|_U}(m'(U \xrightarrow{t''} A)).$$

Moreover, we have $m' = R_p m$ so $m'(U \xrightarrow{t''} A) = m(U \xrightarrow{pt''} X)$. It is clear from the definitions that $pt'' = t$. After unwinding all this we get

$$(T_r N_q R_p m)(T \xrightarrow{z} Z) = \sum_{t \in L} \prod_U N_{v|_U}(m(U \xrightarrow{t} X)),$$

which is the same as $(N_g T_f m)(T \xrightarrow{z} Z)$, as required. \square

REMARK 15.24. The Tambara structure on rR makes $(rR)(X)$ into a semiring, with multiplication on $(rR)(X)$ given by N_s , where $s: X \amalg X \rightarrow X$ is the fold map. It is not hard to check that the rule is just the obvious one:

$$(mm')(T \xrightarrow{x} X) = m(T \xrightarrow{x} X) m'(T \xrightarrow{x} X) \in R(T).$$

PROPOSITION 15.25. *The functor $q: \text{Tambara}_G \rightarrow \text{MCSys}_G$ is left adjoint to $r: \text{MCSys}_G \rightarrow \text{Tambara}_G$.*

PROOF. First, consider a Tambara functor S , and a map $f: X \rightarrow Y$ of finite G -sets. We claim that the square

$$\begin{array}{ccc} S(X) & \xrightarrow{N_f} & S(Y) \\ \eta \downarrow & & \downarrow \eta \\ (rqS)(X) & \xrightarrow{N_f} & (rqS)(Y) \end{array}$$

commutes. To see this, consider an element $m \in S(X)$, a G -orbit T and a G -map $y: T \rightarrow Y$. We must show that

$$(\eta N_f m)(T \xrightarrow{y} Y) = (N_f \eta m)(T \xrightarrow{y} Y) \in (qS)(T).$$

The left hand side is by definition $\pi(R_y N_f m)$. We form the pullback square

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{u} & X \\ v \downarrow & & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

so the Mackey property gives $R_y N_f = N_v R_u$. We split \tilde{T} as a disjoint union of orbits U and note that

$$N_v R_u(m) = \coprod_U N_{v|_U} R_{v|_U} m.$$

The norm maps for qS satisfy $\pi N = N \pi$ by construction, so the left hand side of our equation is $\prod_U N_{v|_U} \pi(R_{v|_U} m)$. The right hand side is

$$(N_f \eta m)(T \xrightarrow{y} Y) = \prod_U N_{v|_U} ((\eta m)(U \xrightarrow{u|_U} X)) = \prod_U N_{v|_U} \pi(R_{v|_U} m),$$

which is the same as the left hand side. We have seen previously that η is a morphism of Mackey functors, so it commutes with the T and R operators as well as the norms, so it is a Tambara morphism.

Now consider a multiplicative coefficient system R , and a map $f: T \rightarrow U$ of G -orbits. We claim that the square

$$\begin{array}{ccc} (qrR)(T) & \xrightarrow{N_f} & (qrR)(U) \\ \epsilon \downarrow & & \downarrow \epsilon \\ R(T) & \xrightarrow{N_f} & R(U) \end{array}$$

commutes. To see this, consider an element $m \in (rR)(T)$. We then have $\epsilon N_f(\pi(m)) = (N_f(\pi(m)))(U \xrightarrow{1} U)$. This is by definition a product of factors indexed by the orbits in the pullback of $U \xrightarrow{1} U$ along $f: T \rightarrow U$. Of course this pullback is just T , so there is only one orbit, and the corresponding factor is $N_f m$ as required. This means that ϵ is a morphism of multiplicative coefficient systems.

We previously proved triangular identities for η and ϵ . We can now reinterpret these as identities in Tambara_G and MCSys_G rather than Mackey_G and CSys_G , and they show that we still have an adjunction in this richer context. \square

Recall that in Section 3 we defined a functor $\omega: \text{Mackey}_G \rightarrow \text{Semigroups}_G$ by $\omega(M) = M(G)$, and studied its left and right adjoints $d, c: \text{Semigroups}_G \rightarrow \text{Mackey}_G$. In Example 6.3 we remarked that we can define $\omega: \text{Tambara}_G \rightarrow \text{Semirings}_G$ and its right adjoint $c: \text{Tambara}_G \rightarrow \text{Semirings}_G$ in the same way. However, in this context it is much harder to understand the left adjoint to ω . This will be the subject of Section 19, where (following Brun [4]) we build a connection with Witt rings in the sense of Dress and Siebeneicher [8]. For the moment we just discuss a parallel adjunction in terms of coefficient systems, which is easier.

DEFINITION 15.26. We define $\omega': \text{MCSys} \rightarrow \text{Semirings}_G$ by $\omega'(R) = R(G)$.

REMARK 15.27. Recall that for any Tambara functor S we have $(qS)(G) = S(G)$. On the other hand, using Remark 15.10 we see that $(rR)(G) = R(G)$ for any multiplicative coefficient system R . This means that the diagram

$$\begin{array}{ccccc} \text{Tambara}_G & \xrightarrow{q} & \text{MCSys}_G & \xrightarrow{r} & \text{Tambara}_G \\ \omega \downarrow & & \downarrow \omega' & & \downarrow \omega \\ \text{Semirings}_G & \xlongequal{\quad} & \text{Semirings}_G & \xlongequal{\quad} & \text{Semirings}_G \end{array}$$

commutes up to natural isomorphism.

DEFINITION 15.28. Consider an object $A \in \text{Semirings}_G$. For any G -orbit T , let $L'A(T)$ be the commutative semiring generated by symbols $i_t(a)$ (for $t \in T$ and $a \in A$) modulo relations

$$\begin{aligned} i_t(0) &= 0 & i_t(1) &= 1 \\ i_t(a+b) &= i_t(a) + i_t(b) & i_t(ab) &= i_t(a)i_t(b) \\ i_{gt}(a) &= i_t(g^{-1}a). \end{aligned}$$

Next, for any morphism $u: T \rightarrow U$ of orbits, there is an induced semiring map $N_u: L'A(T) \rightarrow L'A(U)$ given by $N_u i_t(a) = i_{u(t)}(a)$. This makes $L'A$ into a multiplicative coefficient system.

REMARK 15.29. It is straightforward to check the universal property

$$\text{Semirings}(L'A(T), B) = \text{Map}_G(T, \text{Semirings}(A, B)) = \text{Semirings}_G(A, \text{Map}(T, B))$$

for all semirings B . Note that neither $L'A(T)$ nor B has a G -action here, so the G -actions on $\text{Semirings}(A, B)$ and $\text{Map}(T, B)$ are purely determined by the G -actions on A and T .

We can give an alternative description of L' as follows.

DEFINITION 15.30. Let A be a semiring with an action of G . We let $A_{[G]}$ denote the quotient of A by the smallest semiring congruence containing (a, ga) for all $a \in A$ and $g \in G$. We call this the *coinvariant semiring* for A .

REMARK 15.31. $A_{[G]}$ is in general a proper quotient of the coinvariant semigroup A_G as in Definition 2.7. For example, if G is the group of order two acting by conjugation on \mathbb{C} , then $\mathbb{C}_G = \mathbb{R}$ but $\mathbb{C}_{[G]} = 0$.

REMARK 15.32. It is now not hard to see that $(L'A)(G/K) = A_{[K]}$. It follows from Proposition 15.11 that

$$(rL'A)(G/H) = \left[\prod_{K \leq H} A_{[K]} \right]^H.$$

PROPOSITION 15.33. *The functor $L': \text{Semirings}_G \rightarrow \text{MCSys}_G$ is left adjoint to $\omega': \text{MCSys}_G \rightarrow \text{Semirings}_G$.*

PROOF. For any $A \in \text{Semirings}_G$ we can define

$$\eta: A \rightarrow \omega' L'A = L'A(G)$$

by $\eta(a) = i_1(a)$. This is a semiring map and satisfies

$$g\eta(a) = N_{\rho(g)}(i_1(a)) = i_{\rho(g)(1)}(a) = i_{g^{-1}}(a) = i_1(ga) = \eta(ga),$$

so it is G -equivariant.

Now consider a multiplicative coefficient system R . Let T be a G -orbit. For any $t \in T$ we have a G -map $\hat{t}: G \rightarrow T$ given by $\hat{t}(g) = gt$. We let $\epsilon: L'(R(G))(T) \rightarrow R(T)$ be the unique ring map such that $\epsilon(i_t(a)) = N_{\hat{t}}(a)$ for all $a \in R(G)$. This defines a natural map $\epsilon: L'\omega' \rightarrow 1$.

We leave it to the reader to check the triangle identities, giving an adjunction as claimed. \square

16. Filtrations

DEFINITION 16.1. (a) We say that a G -set X is k -free if all stabiliser groups of points in X have order at most k .

(b) For any G -orbit U , the *coorder* of U is $|G|/|U|$ (which is the order of the stabiliser group of any point in U).

(c) We say that a coefficient system N is k -pure if $N(U) = 0$ whenever $\text{coord}(U) \neq k$.

(d) We say that a Mackey functor M is k -pure if the coefficient system qM is k -pure.

(e) For any coefficient system N , we define subsystems $F_k N$ by

$$F_k N(U) = \begin{cases} N(U) & \text{if } \text{coord}(U) \leq k \\ 0 & \text{otherwise.} \end{cases}$$

(f) For any Mackey functor M and any G -set X , we let $F_k M(X)$ denote the set of elements $m \in M(X)$ that can be written in the form $m = T_p n$ for some k -free G -set W , some G -map $p: W \rightarrow X$, and some element $n \in M(W)$.

LEMMA 16.2. *If Y is k -free and there exists a G -map $f: X \rightarrow Y$ then X is also k -free.*

PROOF. This is clear because $\text{stab}_G(x) \leq \text{stab}_G(f(x))$. \square

PROPOSITION 16.3. *$F_k M$ is a sub-Mackey functor of M , which satisfies $F_k M(X) = M(X)$ whenever X is k -free. Moreover, we have $qF_k M = F_k qM$.*

PROOF. It is clear from the definitions that $(F_k M)(X_0 \amalg X_1) = (F_k M)(X_0) \times (F_k M)(X_1)$. It is also clear by construction that for any map $f: X \rightarrow Y$, we have $T_f F_k M(X) \subseteq F_k M(Y)$. Now consider an element $m \in F_k M(Y)$. By definition there exists a map $p: W \rightarrow Y$ and an element $n \in M(W)$ with $m = T_p n$. Now form a pullback square

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & W \\ q \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y. \end{array}$$

As V admits a map to W , it must be k -free. We also have $R_p m = T_q R_{\tilde{f}} n$, so $R_f m \in F_k M(X)$. This shows that $F_k M$ is indeed a sub-Mackey functor. If X is k -free then we can write every element $m \in M(X)$ as $T_1 m$, showing that $F_k M(X) = M(X)$.

Now let U be a G -orbit. If $\text{coord}(U) \leq k$ then it is clear that $F_k M(U) = M(U)$, and more generally $F_k M(V) = M(V)$ for all G -orbits that admit a map to U . Thus, all ingredients in the definition of $qF_k M(U)$ are the same as the corresponding ingredients for $qM(U)$, so $qF_k M(U) = qM(U)$. Suppose instead that $\text{coord}(U) > k$. Every element $m \in F_k M(U)$ can be written as $T_p n$ for some k -free G -set Y and some $n \in M(Y)$. Note that $M(Y) = F_k M(Y)$, and that no orbit in Y can map isomorphically to U . It follows that m becomes zero in $qF_k M(U)$, but m was arbitrary so $qF_k M(U) = 0$. This shows that $qF_k M = F_k qM$. \square

PROPOSITION 16.4. *Let M be a k -pure Mackey functor.*

(a) *Then for orbits T with $\text{coord}(T) < k$ we have $M(T) = qM(T) = 0$.*

(b) *For orbits T with $\text{coord}(T) = k$ we have $qM(T) = M(T)$.*

(c) *For $(k-1)$ -free G -sets X we have $M(X) = 0$.*

(d) *For k -free G -sets X , the map $\eta: M(X) \rightarrow rqM(X)$ is an isomorphism.*

(e) *If $qM(T) = 0$ whenever $\text{coord}(T) = k$, then $M = 0$.*

PROOF. First, k -purity of M means by definition that $qM(T) = 0$ whenever $\text{coord}(T) \neq k$.

Next, recall that $qM(T)$ is the quotient of $M(T)$ by the sum of the images of transfers from orbits with strictly larger coorder. Thus, if $M(U) = 0$ such orbits (which is vacuously true if $T = G$), then $qM(T) = M(T)$. In particular, if $qM(T) = 0$, then $M(T) = 0$. Claims (a) and (b) follow easily by induction on the coorder. The same argument also proves (e).

Claim (c) follows from (a) by decomposing X into orbits. Similarly, it will be enough to prove (d) when $X = G/H$ with $|H| = k$. Then Proposition 15.11 gives

$$rqM(X) = \left[\prod_{K \leq H} qM(G/K) \right]^H.$$

If K is a proper subgroup of H then $qM(G/K) = 0$, but in the case $K = H$ we have $qM(G/H) = M(G/H)$. Thus, the above product is just $M(G/H)$, and H acts trivially on this, so $rqM(G/H) = M(G/H)$ as required. (We leave the reader to check that the isomorphism $M(G/H) \rightarrow rqM(G/H)$ implicit in this argument is just the same as η .) \square

PROPOSITION 16.5. *Let M be a k -pure, additively complete Mackey functor. Then $F_j M = 0$ for $j < k$, and $F_j M = M$ for $j \geq k$.*

PROOF. First suppose that $j < k$. We then have $M(Y) = 0$ for all j -free sets Y , and it follows immediately that $F_j M = 0$. Next, as q has a right adjoint, it preserves colimits, so $q(M/F_k M) = q(M)/q(F_k M)$. Now $q(F_k M) = F_k q(M)$ by Proposition 16.3, and purity implies that $F_k q(M) = q(M)$, so we have $q(M/F_k M) = 0$. Proposition 16.4 therefore tells us that $M/F_k M = 0$. As M is additively complete, pathologies like Remark 2.9 cannot occur, and we deduce that $F_k M = M$ as claimed. If $k \leq j \leq |G|$ then it is clear that $F_k M \leq F_j M$, and so $F_j M = M$ as well. \square

LEMMA 16.6. *Let M be a semigroup, and let N be a subsemigroup that is additively complete. Then the set*

$$E_N = \{(m_0, m_1) \in M^2 \mid m_1 = m_0 + n \text{ for some } n \in N\}$$

is the smallest congruence on M containing $0 \times N$, so $M/N = M/E_N$. It follows that $M/N = 0$ iff $M = N$, and that M/N is additively complete iff M is additively complete.

PROOF. Straightforward. \square

PROPOSITION 16.7. *Let M be a Mackey functor such that the coefficient system qM is additively complete. Then M is also additively complete, as are the subobjects $F_k M$ and the quotient objects $F_k M/F_{k-1} M$.*

PROOF. Consider an orbit G/H . We may assume by induction that all the semigroups $M(G/K)$ with $|K| < |H|$ are additively complete. Thus, if we let N denote the sum of the images of the maps $T_p: M(G/K) \rightarrow M(G/H)$, then N is additively complete. By assumption the quotient $qM(G/H) = M(G/H)/N$ is additively complete, and it follows from the Lemma that $M(G/H)$ is additively complete. Any finite G -set X can be written as a disjoint union of orbits, so $M(X)$ is additively complete as claimed.

Next, recall that $qF_k M = F_k qM$. From this it is clear that $qF_k M$ is also additively complete, so $F_k M$ is additively complete. It is clear that any quotient of additively complete Mackey functors is again additively complete. In particular, this applies to $F_k M/F_{k-1} M$. \square

17. Rational Mackey functors and Tambara functors

DEFINITION 17.1. We say that a semigroup A is *rational* if every element has an additive inverse, and the map $n.1_A: A \rightarrow A$ is an isomorphism for all positive integers n . (This means that A has a unique structure as a vector space over \mathbb{Q} extending the given addition law.) We say that a Mackey functor M is rational if $M(X)$ is rational for all X , and similarly for Green functors, Tambara functors and (multiplicative) coefficient systems. We write $\mathbb{Q}\text{Mackey}_G$ for the category of rational Mackey functors, and similarly for $\mathbb{Q}\text{Green}_G$, $\mathbb{Q}\text{Tambara}_G$, $\mathbb{Q}\text{CSys}_G$ and $\mathbb{Q}\text{MCSys}_G$.

THEOREM 17.2. *The functors q and r restrict to give an equivalence $\mathbb{Q}\text{Mackey}_G \simeq \mathbb{Q}\text{CSys}_G$, and also an equivalence $\mathbb{Q}\text{Tambara}_G \simeq \mathbb{Q}\text{MCSys}_G$.*

The proof will follow after some preparatory results.

DEFINITION 17.3. Let k be a divisor of $|G|$. We put

$$C_k = \{A \subseteq G \mid A = Hx \text{ for some } x \in G \text{ and } H \leq G \text{ with } |H| = k\}.$$

After noting that $gHx = (gHg^{-1})gx$, we see that C_k has a natural G -action by multiplication on the left.

LEMMA 17.4. *All orbits in C_k have coorder k . Moreover, for every orbit U of coorder k we have $|\text{Map}_G(U, C_k)| = |G|/k = |U|$.*

PROOF. It is clear that $\text{stab}_G(Hx) = H$. Thus, for $K \leq G$ with $|K| = k$ we have

$$|\text{Map}_G(G/K, C_k)| = |(C_k)^K| = |\{Kx \mid x \in G\}| = |G|/k.$$

□

PROPOSITION 17.5. *Let N be a rational coefficient system. Then the map $\epsilon: qrN \rightarrow N$ is an isomorphism.*

PROOF. It is clear that N is a direct sum of pure coefficient systems, so we may assume that N itself is k -pure for some k dividing $|G|$. Given this, Proposition 15.11 can be rewritten as

$$rN(G/H) = \left[\prod_{K \leq H, |K|=k} N(G/K) \right]^H.$$

If $|H| < k$ then the product has no terms, so $rN(G/H) = 0$, so certainly $qrN(G/H) = 0 = N(G/H)$. If $|H| = k$ then the product is just $N(G/H)$ with H acting trivially, so $rN(G/H) = N(G/H)$. Moreover, the previous case shows that there is nothing to kill to form $qrN(G/H)$, so $qrN(G/H) = N(G/H)$ as well. We now see that the map $\epsilon: qrN(U) \rightarrow N(U)$ is an isomorphism whenever $\text{coord}(U) \leq k$.

Now suppose instead that $\text{coord}(U) > k$. We then have $N(U) = 0$, so we must show that $qrN(U) = 0$. Let $p: C_k \times U \rightarrow U$ be the projection. Consider an element $m \in rN(U)$, given by a natural system of elements $m(V \xrightarrow{u} U) \in N(V)$ for all G -orbits V (wlog with $\text{coord}(V) = k$) and all G -maps $u: V \rightarrow U$. From the definitions we have

$$\begin{aligned} (R_p m)(V \xrightarrow{(c,u)} C_k \times U) &= m(V \xrightarrow{u} U) \\ (T_p R_p m)(V \xrightarrow{u} U) &= \sum_{c \in \text{Map}_G(U, C_k)} (R_p m)(V \xrightarrow{(c,u)} C_k \times U) \\ &= |\text{Map}_G(U, C_k)| m(V \xrightarrow{u} U) \\ &= (|G|/k) m(V \xrightarrow{u} U), \end{aligned}$$

so $T_p R_p m = (|G|/k)m$. Now let $\pi: rN(U) \rightarrow qrN(U)$ be the projection as usual. Note that C_k is k -free so the same is true of $C_k \times U$, but $\text{coord}(U) > k$; it follows that $\pi T_p = 0$. We therefore have $(|G|/k)\pi(m) = 0 \in qrN(U)$, but N is rational so $\pi(m) = 0$ as required. □

COROLLARY 17.6. *Let M be a rational Mackey functor. Then the map $\eta: M \rightarrow rqM$ is surjective.*

PROOF. The triangular identity for the (q, r) adjunction says that the composite

$$qM \xrightarrow{q\eta_M} qrqM \xrightarrow{\epsilon_{qM}} qM$$

is the identity. Using the proposition we see that ϵ_{qM} is an isomorphism, so $q\eta_M$ is also an isomorphism. Next, we note that q has a right adjoint, so it preserves colimits, so $q(\text{cok}(\eta_M)) = \text{cok}(q\eta_M) = 0$. Proposition 16.4(e) now tells us that $\text{cok}(\eta_M) = 0$. As everything is rational and therefore additively complete, we can deduce that η_M is surjective. □

PROPOSITION 17.7. *Let k be a divisor of $|G|$, and let M be a k -pure rational Mackey functor. Then the map $\eta: M \rightarrow rqM$ is an isomorphism.*

PROOF. As in Remark 15.12, we can think of $rqM(X)$ as the inverse limit of the functor $(\text{Orb}_G^\times \downarrow X) \rightarrow \text{Semigroups}$ given by $(U \xrightarrow{x} X) \mapsto qM(U)$. As qM is k -pure we have $qM(U) = 0$ unless U has coorder k , and in that case Proposition 16.4 tells us that $qM(U) = M(U)$. We can thus let \mathcal{P} be the full subcategory of $(\text{Orb}_G^\times \downarrow X)$ containing the diagrams $(U \xrightarrow{x} X)$ where U has coorder k , and we find that $rqM(X)$ is the limit of the functor $\mathcal{P} \rightarrow \text{Semigroups}$ given by $(U \xrightarrow{x} X) \mapsto M(U)$. For any element m of this inverse limit, we define $\psi(m) \in M(X)$ by

$$\psi(m) = \sum_{U \in \text{orb}(C_k)} \sum_{x \in \text{Map}_G(U, X)} T_x(m(U \xrightarrow{x} X)).$$

We claim that this defines a morphism $\psi: rqM \rightarrow M$ of Mackey functors. To see this, consider a map $f: X \rightarrow Y$ of finite G -sets, and the resulting diagram

$$\begin{array}{ccccc} rqM(X) & \xrightarrow{T_f} & rqM(Y) & \xrightarrow{R_f} & rqM(X) \\ \psi \downarrow & & \psi \downarrow & & \psi \downarrow \\ M(X) & \xrightarrow{T_f} & M(Y) & \xrightarrow{R_f} & M(X). \end{array}$$

For the left square, we recall that transfers in rqM are defined by

$$(T_f m)(U \xrightarrow{y} Y) = \sum_{x \in \text{Map}_G(U, X), fx=y} m(U \xrightarrow{x} X),$$

so

$$\begin{aligned} \psi T_f m &= \sum_U \sum_{y \in \text{Map}_G(U, Y)} T_y((T_f m)(U \xrightarrow{y} Y)) = \sum_U \sum_{y \in \text{Map}_G(U, Y)} \sum_{x \in \text{Map}_G(U, X), fx=y} T_y(m(U \xrightarrow{x} X)) \\ &= \sum_U \sum_{x \in \text{Map}_G(U, X)} T_{fx}(m(U \xrightarrow{x} X)) = T_f \left(\sum_U \sum_{x \in \text{Map}_G(U, X)} T_x(m(U \xrightarrow{x} X)) \right) \\ &= T_f \psi(m). \end{aligned}$$

For the right square, consider an element $n \in rqM(Y)$. We then have

$$R_f \psi(n) = \sum_U \sum_{y \in \text{Map}_G(U, Y)} (R_f T_y n)(U \xrightarrow{y} Y)$$

To analyse this, we form a pullback square

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{i} & U \\ j \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y. \end{array}$$

This gives $R_f T_y = T_i R_j$, and this can be written as a sum over orbits $V \subseteq \tilde{U}$. As V admits a map to U , it must have coorder at most k . If the coorder is strictly less than k then $M(V) = 0$ and so V does not contribute to $T_i R_j n$. The orbits of coorder k map isomorphically to U and so biject with G -maps $x: U \rightarrow X$ satisfying $fx = y$. We deduce that

$$\begin{aligned} R_f \psi(n) &= \sum_U \sum_{y \in \text{Map}_G(U, Y)} \sum_{x \in \text{Map}_G(U, X), fx=y} T_x(m(U \xrightarrow{y} Y)) \\ &= \sum_U \sum_{x \in \text{Map}_G(U, X)} T_x(m(U \xrightarrow{fx} Y)) = \psi R_f(m). \end{aligned}$$

Next, we claim that $\psi \eta(m) = (|G|/k)m$ for all G -sets X and elements $m \in M(X)$. Indeed, Proposition 16.5 tells us that $M = F_k M$, so every element of $M(X)$ has the form $T_p n$ for some k -free set W and some G -map $p: W \rightarrow X$. Using this we can reduce to the case where X is a k -free orbit. If the coorder of X is strictly

less than k , then $M(X) = 0$ and there is nothing to prove. We therefore assume that $\text{coord}(X) = k$. We now have $\eta(m)(U \xrightarrow{x} X) = \pi(R_x m) \in qM(U)$, and the π makes no difference if $\text{coord}(U) = k$. We therefore have

$$\psi\eta(m) = \sum_{U \in \text{orb}(C_k)} \sum_{x \in \text{Map}_G(U, X)} T_x R_x m.$$

Here both U and X are orbits of coorder k , so any G -map $x: U \rightarrow X$ is an isomorphism, so $T_x R_x$ is the identity. We also see that the inversion map $x \mapsto x^{-1}$ gives a bijection $\text{Map}_G(U, X) \rightarrow \text{Map}_G(X, U)$, so

$$\sum_{U \in \text{orb}(C_k)} |\text{Map}_G(U, X)| = \left| \prod_{U \in \text{orb}(C_k)} \text{Map}_G(X, U) \right| = |\text{Map}_G(X, C_k)| = |G|/k.$$

This gives $\psi\eta(m) = (|G|/k)m$ as claimed. As M is rational, we conclude that $\psi\eta$ is an isomorphism, so η is injective and ψ is surjective. We saw in Corollary 17.6 that η is also surjective, so it is an isomorphism. \square

PROOF OF THEOREM 17.2. We need to show that the unit map $\eta: M \rightarrow rqM$ and the counit map $\epsilon: qrN \rightarrow N$ are isomorphisms when N and M are rational. (This will prove both the Mackey functor statement and the Tambara functor statement.)

The counit is covered by Proposition 17.5. For the unit, it will be enough to prove by induction that the maps $\eta: F_k M \rightarrow rqF_k M$ are isomorphisms for all k . Put $Q_k M = F_k M / F_{k-1} M$, and consider the commutative diagram

$$\begin{array}{ccccc} F_{k-1}M & \xrightarrow{\quad} & F_k M & \longrightarrow & Q_k M \\ \eta \downarrow & & \eta \downarrow & & \downarrow \eta \\ rqF_{k-1}M & \xrightarrow{\quad} & rqF_k M & \longrightarrow & rqQ_k M. \end{array}$$

The top row is short exact by definition. Proposition 16.3 tells us that $qF_j = F_j q$, and q preserves colimits so $qQ_k M = (qF_k M) / (qF_{k-1} M) = (F_k qM) / (F_{k-1} qM)$. It follows that $Q_k M$ is k -pure, and that the sequence $qF_{k-1} M \rightarrow qF_k M \rightarrow qQ_k M$ is a split short exact sequence of coefficient systems. As any additive functor preserves split short exact sequences, we see that the bottom row of the above diagram is again short exact. The right hand vertical map is an isomorphism by Proposition 17.7, and the left hand map can be assumed to be an isomorphism by induction, so the middle map is an isomorphism by the Five Lemma. \square

EXAMPLE 17.8. Consider the rational Burnside ring Tambara functor

$$\mathbb{Q} \otimes A(X) = \mathbb{Q} \otimes \overline{\mathcal{A}}_G(1, X) \simeq \mathbb{Q} \otimes \overline{\mathcal{U}}_G(\emptyset, X).$$

We saw in Proposition 15.19 that $qA = c\mathbb{N}$. In the same way, we can check that $q(\mathbb{Q} \otimes A) = c\mathbb{Q}$, where $c\mathbb{Q}$ denotes the constant functor $c\mathbb{Q}: \text{Orb}_G^\times \rightarrow \text{Rings with value } \mathbb{Q}$. It follows that $\mathbb{Q} \otimes A \simeq rc\mathbb{Q}$, and so

$$\mathbb{Q} \otimes A(X) = \left[\prod_{H \leq G} \text{Map}(X^H, \mathbb{Q}) \right]^G.$$

We next want to discuss the rational representation ring. For this, we first need to investigate naturality properties of the construction sending G/H to the centre of H .

CONSTRUCTION 17.9. For any G -orbit T we have a translation category $\text{Trans}(G, T)$ and a functor

$$F_T: \text{Trans}(G, T) \rightarrow \text{Groups}$$

given by

$$F_T(x) = \text{stab}_G(x) = \{g \in G \mid gx = x\}.$$

We write Z_T for the inverse limit of F_T . One checks that $Z_{G/H}$ is the centre of H . In particular, if H is abelian then $Z_{G/H} = H$.

Now suppose we have a map $q: U \rightarrow T$ and thus $q_*: \text{Trans}(G, U) \rightarrow \text{Trans}(G, T)$. There is a natural monomorphism $F_U \rightarrow F_T \circ q_*$, and by general nonsense this gives maps

$$Z_U \rightarrow \varprojlim (F_T \circ q_*) \leftarrow Z_T.$$

We claim that if T (and thus U) has abelian isotropy then the first of these maps is injective and the second is an isomorphism, so we have a natural inclusion $Z_U \rightarrow Z_T$. Indeed, if q is just the projection $G/H \rightarrow G/K$ (with $H \leq K$), then the above maps are easily identified with the maps

$$ZH \rightarrow Z_K H \leftarrow ZK,$$

and the claim about the abelian case follows immediately.

One can now check that this construction gives a functor

$$\{G\text{-orbits with abelian isotropy}\} \rightarrow \{\text{abelian groups}\}.$$

EXAMPLE 17.10. Let R be the representation semiring Tambara functor, as in Example 6.5. Recall that $R(G/H)$ is just the semiring of isomorphism classes of complex representations of H . Consider the transfer maps $\mathbb{Q} \otimes R(G/C) \rightarrow \mathbb{Q} \otimes R(G/H)$ for cyclic subgroups $C \leq H$. Artin's induction theorem says that the sum of the images of these maps is all of $\mathbb{Q} \otimes R(G/H)$. It follows that the corresponding multiplicative coefficient system has $q(\mathbb{Q} \otimes R)(T) = 0$ unless T has cyclic isotropy groups. If C is cyclic of order n one can check that $q(\mathbb{Q} \otimes R)(G/C)$ is isomorphic to the cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$.

We can make this more functorial as follows. First, put $C^* = \text{Hom}(C, S^1)$, which is again cyclic of order n . One can check that $\mathbb{Q}[C^*]$ has a unique maximal ideal \mathfrak{m}_C such that the natural map from C^* to the field $K(C) = \mathbb{Q}[C^*]/\mathfrak{m}$ is injective. Next, suppose we have an injective homomorphism $i: C \rightarrow D$ of abelian groups, with $|D|/|C| = m$ say. Given a character $\alpha \in C^*$ we can choose $\beta \in D^*$ with $i^*(\beta) = \alpha$, and it is not hard to see that β^m is independent of the choice of β , so we can define $i_\bullet(\alpha) = \beta^m$. This is an injective homomorphism of cyclic groups, and it follows that it induces a homomorphism $i_\bullet: K(C) \rightarrow K(D)$. One can check that for any G -orbit T with cyclic isotropy, there is a natural isomorphism $q(\mathbb{Q} \otimes R)(T) \simeq K(Z_T)$.

EXAMPLE 17.11. Put $R(T) = \text{Map}(T, \mathbb{Q})$ when T is a free orbit, and $R(T) = 0$ if T is not free. For any map $f: U \rightarrow T$, either T is free (in which case f is an isomorphism and we put $R_f = (f^{-1})^*: \text{Map}(U, \mathbb{Q}) \rightarrow \text{Map}(T, \mathbb{Q})$) or T is not free (in which case $R_f = 0$). This gives a functor $\text{Orb}_G^\times \rightarrow \text{Alg}_\mathbb{Q}$, and the associated Tambara functor is just $S(X) = \text{Map}(X, \mathbb{Q})$.

PROPOSITION 17.12. *The restricted functor $\omega: \mathbb{Q}\text{Tambara}_G \rightarrow \mathbb{Q}\text{Semirings}_G$ is right adjoint to the functor rL' (where L' is as in Definition 15.28).*

PROOF. This is clear from Propositions 15.25 and 15.33. □

18. Change of groups

In this section we will investigate various functors relating the categories $\overline{\mathcal{U}}_G$, Tambara_G and MCSys_G for different groups G .

Let H be a subgroup of G . There is then an evident forgetful functor $\text{res}: \overline{\mathcal{U}}_G \rightarrow \overline{\mathcal{U}}_H$. We can also define a functor from finite H -sets to finite G -sets by $\text{ind}(Y) = G \times_H Y$. If T is a set of coset representatives then the underlying set of $\text{ind}(Y)$ is just $T \times Y$. Using this, we see that ind preserves all the constructions used to define composition in $\overline{\mathcal{U}}_H$, so we get a functor $\text{ind}: \overline{\mathcal{U}}_H \rightarrow \overline{\mathcal{U}}_G$.

PROPOSITION 18.1. *The functor $\text{res}: \overline{\mathcal{U}}_G \rightarrow \overline{\mathcal{U}}_H$ is left adjoint to $\text{ind}: \overline{\mathcal{U}}_H \rightarrow \overline{\mathcal{U}}_G$. Moreover, both these functors preserve categorical products.*

PROOF. First, consider a G -set W equipped with a map $f: W \rightarrow \text{ind}(Y) = G \times_H Y$. Note that $\text{ind}(Y)$ contains $H \times_H Y = Y$, and put $W_0 = f^{-1}(W)$, which is an H -set. There is a unique G -map $G \times_H W_0 \rightarrow W$ extending the identity on W_0 , and one checks that this is an isomorphism. It follows that the category of G -sets over $\text{ind}(Y)$ is equivalent to the category of H -sets over Y .

Any morphism in $\overline{\mathcal{U}}_H(\text{res}(X), Y)$ is represented by a diagram

$$P_0 = (X \xleftarrow{f_0} A_0 \xrightarrow{g_0} B_0 \xrightarrow{h_0} Y)$$

of finite H -sets. Let f be the unique G -equivariant extension of f_0 over $\text{ind}(A_0)$, so we have a diagram

$$P = (X \xleftarrow{f} \text{ind}(A_0) \xrightarrow{\text{ind}(g_0)} \text{ind}(B_0) \xrightarrow{\text{ind}(h_0)} \text{ind}(Y)),$$

representing a morphism in $\overline{\mathcal{U}}_G(X, \text{ind}(Y))$. Using our first paragraph, we see that this procedure gives a bijection $\overline{\mathcal{U}}_H(\text{res}(X), Y) = \overline{\mathcal{U}}_G(X, \text{ind}(Y))$.

Products in $\overline{\mathcal{U}}_G$ and $\overline{\mathcal{U}}_H$ are given by disjoint union, and it is clear that both ind and res preserve disjoint unions. \square

DEFINITION 18.2. We define $\text{coind}: \text{Tambara}_H \rightarrow \text{Tambara}_G$ and $\text{res}: \text{Tambara}_G \rightarrow \text{Tambara}_H$ by $\text{coind}(S) = S \circ \text{res}$ and $\text{res}(S') = S' \circ \text{ind}$.

PROPOSITION 18.3. *The functor $\text{coind}: \text{Tambara}_H \rightarrow \text{Tambara}_G$ is right adjoint to $\text{res}: \text{Tambara}_G \rightarrow \text{Tambara}_H$.*

PROOF. This follows formally from Proposition 18.1, using the (co)unit maps and triangular identities. \square

19. Witt vectors

We now discuss the relationship between Tambara functors and the generalised Witt rings of Dress and Siebeneicher [8]. Similar results have been obtained by Brun [4], Elliott [9] and Nakaoka [20].

PROPOSITION 19.1. *The functor $\omega: \text{Tambara}_G \rightarrow \text{Semirings}_G$ (given by $\omega(S) = S(G)$) has a left adjoint $L: \text{Semirings}_G \rightarrow \text{Tambara}_G$.*

PROOF. Put $P(X) = \overline{\mathcal{U}}_G(G, X)$, so P is a Tambara functor and the Yoneda lemma gives $\text{Tambara}_G(P, S) \simeq S(G)$. For any $A \in \text{Semirings}_G$, we let L_0A denote the coproduct of copies of P indexed by A , so $\text{Tambara}_G(L_0A, S) = \text{Map}(A, S(G))$. Similarly, we let L_1A denote the coproduct of copies of P indexed by the set

$$Q = 1 \amalg 1 \amalg (A \times A) \amalg (A \times A) \amalg (G \times A),$$

so

$$\begin{aligned} \text{Tambara}_G(L_1A, S) &= \text{Map}(Q, S(G)) \\ &= S(G) \times S(G) \times \text{Map}(A \times A, S(G)) \times \text{Map}(A \times A, S(G)) \times \text{Map}(G \times A, S(G)). \end{aligned}$$

Given a map $f \in \text{Map}(A, S(G))$ we define $\phi^*(f) \in \text{Map}(Q, S(G))$ by

$$\phi^*(f) = (f(0), f(1), (a, b) \mapsto f(a+b), (a, b) \mapsto f(ab), (g, a) \mapsto f(ga)).$$

We also define $\psi^*(f) \in \text{Map}(Q, S(G))$ by

$$\psi^*(f) = (0, 1, (a, b) \mapsto f(a) + f(b), (a, b) \mapsto f(a)f(b), (g, a) \mapsto gf(a)).$$

It is clear that the equaliser of the maps

$$\phi^*, \psi^*: \text{Map}(A, S(G)) \rightarrow \text{Map}(Q, S(G))$$

is $\text{Semirings}_G(A, S(G)) = \text{Semirings}_G(A, \omega(S))$. Moreover, the Yoneda lemma implies that ϕ^* and ψ^* arise from morphisms $\phi, \psi: L_1A \rightarrow L_0A$ in Tambara_G . If we let LA denote the coequaliser of these maps (which exists by Proposition 10.6), we obtain an isomorphism $\text{Tambara}_G(LA, S) \simeq \text{Semirings}_G(A, \omega(S))$, which is natural in S . It is now standard that there is a canonical way to define L on morphisms, giving a functor $\text{Semirings}_G \rightarrow \text{Tambara}_G$ that is left adjoint to ω . \square

DEFINITION 19.2. For any semiring A , we define a semiring W_GA as follows: we give A the trivial G -action, then apply the functor $L: \text{Semirings}_G \rightarrow \text{Tambara}_G$, then evaluate on the singleton G -set to get $W_GA = (LA)(1)$. We call this the semiring of G -Witt vectors for A .

When A is additively complete, we will show that $W_G(A)$ is the same as the ring defined by Dress and Siebeneicher [8]. In particular, if G is cyclic of prime-power order, we recover the usual p -typical Witt vectors.

One could think about generalising the definition to cover the case where A has nontrivial G -action. Some results in this context have been stated by Brun, but they appear to contain some inaccuracies. We will therefore ignore this possible generalisation: for the rest of this section we take A to be a semiring equipped with the trivial G -action.

We suspect that the proper context for our results is really a theory of global Tambara functors, similar to Webb's theory of global Mackey functors [23]. However, we will leave that for future work.

We now formulate our comparison with the theory of Dress and Siebeneicher in more detail.

DEFINITION 19.3. We write $\text{Sub}(G)$ for the set of subgroups of G , and $\text{sub}(G)$ for the quotient set of conjugacy classes of subgroups. We define a map

$$\gamma: \text{Map}(\text{sub}(G), A) \rightarrow \text{Map}(\text{sub}(G), A)$$

(called the *ghost map*) by

$$\gamma(a)([H]) = \sum_{[K]} |(G/H)^K| a([K])^{|H|/|K|}.$$

(Note here that if $(G/H)^K$ is nonempty then K is conjugate to a subgroup of H so $|H|/|K|$ is a positive integer.)

THEOREM 19.4. *Let A be an additively complete semiring. Then $W_G A$ is also additively complete. Moreover, there is a natural bijection $\tau: \text{Map}(\text{sub}(G), A) \rightarrow W_G A$ and a natural ring map $\phi: W_G A \rightarrow \text{Map}(\text{sub}(G), A)$ such that $\phi\tau = \gamma$.*

This is essentially equivalent to the main result of Brun in [4], but with our richer theory of Tambara functors in hand, we can give a somewhat different perspective on the proof.

PROOF. Combine Propositions 19.18 and 19.21 below. \square

Dress and Siebeneicher proved that there is a unique natural ring structure on $\text{Map}(\text{sub}(G), A)$ such that γ is a ring homomorphism. We can use τ to transport the ring structure on $W_G A$ to $\text{Map}(\text{sub}(G), A)$, and by the uniqueness clause, this must give the ring structure discussed by Dress and Siebeneicher. They define $W_G A$ to be $\text{Map}(\text{sub}(G), A)$ with this structure, so τ gives an isomorphism from their $W_G A$ to our $W_G A$.

We now start work on the proof of Theorem 19.4.

First, it turns out that the functor L can be recovered from the functors W_H for subgroups H of G , as explained in Corollary 19.6 below.

LEMMA 19.5. *The following diagram commutes up to natural isomorphism:*

$$\begin{array}{ccc} \text{Semirings}_G & \xrightarrow{L} & \text{Tambara}_G \\ \text{res} \downarrow & & \downarrow \text{res} \\ \text{Semirings}_H & \xrightarrow{L} & \text{Tambara}_H \end{array}$$

PROOF. It will suffice to show that the followings diagram of right adjoints is commutative:

$$\begin{array}{ccc} \text{Semirings}_G & \xleftarrow{\omega} & \text{Tambara}_G \\ \text{coind} \uparrow & & \uparrow \text{coind} \\ \text{Semirings}_H & \xleftarrow{\omega} & \text{Tambara}_H \end{array}$$

Here $\text{coind}: \text{Semirings}_H \rightarrow \text{Semirings}_G$ is given by $\text{coind}(A) = \text{Map}_H(G, A)$.

If $S \in \text{Tambara}_H$, then $\omega \text{coind}(S) = \text{coind}(S)(G) = S(\text{res}(G))$, whereas $\text{coind} \omega(S) = \text{Map}_H(G, S(H))$. Given $g \in G$ we define $i_g: H \rightarrow G$ by $i_g(h) = hg$. This is an H -map and so gives $i_g^*: S(G) \rightarrow S(H)$. Putting these maps together for all g , we get a map $\xi: S(G) \rightarrow \text{Map}_H(G, S(H))$. If we choose a set T of coset representatives, we can identify ξ with the standard map $S(\coprod_{t \in T} H) \rightarrow \prod_{t \in T} S(H)$, which is an isomorphism, as required. \square

COROLLARY 19.6. *For any semiring A and any $H \leq G$ we have $(LA)(G/H) = W_H(A)$.*

PROOF. We will write L_G and L_H for the functors L defined using G and H respectively. The lemma gives

$$(L_G A)(G/H) = (L_G A)(\text{ind}(1)) = (\text{res } L_G A)(1) = (L_H A)(1) = W_H(A).$$

□

REMARK 19.7. One can check directly that for the trivial group 1 we have $W_1 A = A$. Thus, for general G we have $\omega L A = (L A)(G/1) = W_1 A = A$. More precisely, after chasing through the definitions we see that the unit map $A \rightarrow \omega L A = (L A)(G)$ for the (L, ω) adjunction is an isomorphism.

PROPOSITION 19.8. *There are natural maps $\nu: A \rightarrow LA(U)$ for all orbits U , with the following properties:*

- (a) $\nu: A \rightarrow LA(G) = \omega L A$ is the unit map for the (L, ω) adjunction.
- (b) For any map $f: U \rightarrow V$ of orbits, we have $N_f(\nu_U(a)) = \nu_V(a)$.
- (c) $\nu(1) = 1$, and $\nu(ab) = \nu(a)\nu(b)$.
- (d) $\nu(0) = 0$.

(However, ν does not respect addition.)

PROOF. We define $\nu_G: A \rightarrow LA(G)$ to be the unit so that (a) holds. Given any orbit U we choose a point $u \in U$. This gives a G -map $\hat{u}: G \rightarrow U$ by $\hat{u}(g) = gu$, and this gives a map $N_{\hat{u}}: LA(G) \rightarrow LA(U)$. If we use a different point $v \in U$ then $v = xu$. We then have $\hat{v} = \hat{u} \circ \rho(x)$ (where $\rho(x)(g) = gx^{-1}$), but the action of G on $LA(G) \simeq A$ via ρ is trivial by assumption, so $N_{\hat{u}} = N_{\hat{v}}$. We can thus put $\nu_U = T_{\hat{u}} \circ \nu_G$ (for any choice of u). All the claimed properties are clear. □

PROPOSITION 19.9. *The multiplicative coefficient system $L'A$ is just the constant functor $\text{Orb}_G \rightarrow \text{Semirings}$ with value A .*

PROOF. Recall from Definition 15.28 that $L'A(U)$ is generated by symbols $i_u(a)$ (for $u \in U$ and $a \in A$) modulo relations

$$\begin{aligned} i_t(0) &= 0 & i_t(1) &= 1 \\ i_t(a+b) &= i_t(a) + i_t(b) & i_t(ab) &= i_t(a)i_t(b) \\ i_{gt}(a) &= i_t(g^{-1}a). \end{aligned}$$

As G acts transitively on U and trivially on A , the last equation tells us that all the maps i_u are the same, so we can just call them i ; and the map $i: A \rightarrow L'A(U)$ is an isomorphism. The norm maps satisfy $N_f i_u = i_{fu}$ by definition, and this means that we can identify N_f with 1_A . □

COROLLARY 19.10. *There is a natural isomorphism*

$$rL'A(X) = \text{Map}(\pi_0(\text{Orb}_G^\times \downarrow X), A).$$

PROOF. See Remark 15.12. □

PROPOSITION 19.11. *Let A be a semiring with trivial G -action. Then for all orbits U , the composite $\bar{\nu} = (A \xrightarrow{\nu} LA(U) \xrightarrow{\pi} qLA(U))$ is an isomorphism. Moreover, for any map $f: U \rightarrow V$ of orbits, we have $N_f \bar{\nu}_U = \bar{\nu}_V$. These maps therefore assemble to give an isomorphism $\bar{\nu}: L'A \rightarrow qLA$ of multiplicative coefficient systems.*

PROOF. As q is left adjoint to r and L is left adjoint to ω , we see that qL is left adjoint to ωr , which is the same as ω' by Remark 15.27. As ω' is right adjoint to L' , we have $qL = L'$. By unwinding the definitions we see that this map is the same as $\bar{\nu}$. □

COROLLARY 19.12. *If A is additively complete, then so is LA . Moreover, the filtration layers $F_k LA$ and their quotients are also additively complete.*

PROOF. It is clear from the Proposition that qLA is additively complete, and the rest follows from Proposition 16.7. □

DEFINITION 19.13. We define a Tambara morphism $\phi: LA \rightarrow rL'A$ by

$$\phi = (LA \xrightarrow{\eta} rqlA \xrightarrow{r(\overline{\nu}^{-1})} rL'A).$$

Equivalently, ϕ corresponds to $\overline{\nu}^{-1}$ with respect to the (q, r) adjunction. By evaluating at G/G and using Proposition 15.11 to analyse rL' we get a map

$$\phi: W_G A = LA(G/G) \rightarrow rL'A(G/G) = \text{Map}(\text{sub}(G), A).$$

REMARK 19.14. If A is a \mathbb{Q} -algebra, we see from Proposition 17.12 and Remark 15.32 that $\phi: LA \rightarrow rL'A$ is an isomorphism.

LEMMA 19.15. *For any diagram $(U \xrightarrow{x} X)$ in $(\text{Orb}_G^\times \downarrow X)$, the function $T_x \nu_U: A \rightarrow LA(X)$ depends only on the isomorphism class of $(U \xrightarrow{x} X)$ in $(\text{Orb}_G^\times \downarrow X)$.*

PROOF. If $(V \xrightarrow{y} X)$ is isomorphic to $(U \xrightarrow{x} X)$ then there is an isomorphism $f: U \rightarrow V$ with $yf = x$. This gives $T_x \nu_U = T_y T_f \nu_U$. However, as f is an isomorphism we have $T_f = R_f^{-1} = N_f$, and we have seen that $N_f \nu_U = \nu_V$, so $T_x \nu_U = T_y \nu_V$ as required. \square

DEFINITION 19.16. We define maps $\tau: rL'A(X) \rightarrow LA(X)$ as follows. Choose a list of diagrams $(U_i \xrightarrow{x_i} X)$ (for $1 \leq i \leq m$ say) containing precisely one representative of each isomorphism class in $(\text{Orb}_G^\times \downarrow X)$ and put

$$\tau(a) = \sum_{i=1}^m T_{x_i} \nu_{U_i}(a).$$

It follows from Lemma 19.15 that this does not depend on the choice of representatives.

REMARK 19.17. The map τ does not respect addition or multiplication, so it does not give a morphism of Mackey functors or Tambara functors. One can check that it is compatible with R_f and T_f when f is injective, but not otherwise.

PROPOSITION 19.18. *The composite*

$$\text{Map}(\text{sub}(G), A) = rL'A(1) \xrightarrow{\tau} LA(1) = W_G(A) \xrightarrow{\phi} rL'A(1) = \text{Map}(\text{sub}(G), A)$$

is the ghost map γ .

PROOF. First, we have already seen how to identify $LA(G)$ and $rL'A(G)$ with A , and we leave the reader to check that under these identifications the maps $\tau: rL'A(G) \rightarrow LA(G)$ and $\phi: LA(G) \rightarrow rL'A(G)$ are just the identity.

Now consider a subgroup $H \leq G$, and let p and q denote the obvious maps $G \xrightarrow{p} G/H \xrightarrow{q} 1$. Consider an element $a \in LA(G) = A$. As ϕ is a Tambara morphism we have

$$\phi(T_q \nu(a)) = \phi(T_q N_p(a)) = T_q N_p(\phi(a)) = T_q N_p(a).$$

The last expression involves the operators T_q and N_p on $rL'A$, which are determined by Remark 15.12 and Proposition 15.23. First, for any morphism $G/K \xrightarrow{t} G/H$, the element $(N_p a)(G/K \xrightarrow{t} G/H)$ is by definition a product indexed by the orbits $U \subseteq G \times_{G/H} (G/K)$. Each such orbit admits a projection map to G and so must be isomorphic to G . It follows that the term corresponding to each orbit is just a . By counting the number of points in $G \times_{G/H} (G/K)$ we also see that the number of orbits is $|H|/|K|$. We conclude that $(N_p a)(G/K \xrightarrow{t} G/H) = a^{|H|/|K|}$. Moreover, $(T_q N_p a)(G/K \rightarrow 1)$ is by definition the sum of the above elements for all G -maps $t: G/K \rightarrow G/H$. We now have

$$\phi(T_q \nu(a))(G/K \rightarrow 1) = |(G/H)^K| a^{|H|/|K|}.$$

Next, we note that there is a bijection $\text{sub}(G) \rightarrow \pi_0(\text{Orb}_G^\times)$ given by $H \mapsto G/H$. We choose a list of subgroups H_1, \dots, H_m containing precisely one representative of each conjugacy class, and we make some slight notational changes corresponding to the above bijection, giving

$$\phi(\tau(a))(H_j) = \sum_i |(G/H_i)^{H_j}| a^{H_i/|H_j|}.$$

This is the same as the ghost map, as claimed. \square

COROLLARY 19.19. *Suppose that A is additively complete and torsion-free. Then the maps $\tau : rL'A(X) \rightarrow LA(X)$ are injective for all X .*

PROOF. All the relevant functors convert disjoint unions to products, so we can reduce to the case $X = G/H$. Here $LA(G/H) = W_H A$ and similarly $rL'A(G/H) = \text{Map}(\text{sub}(H), A)$. One can check that these identifications are compatible with τ , so we can reduce further to the case $X = G/G = 1$ which appears in the Proposition. There we have $\phi\tau = \gamma$, so it will suffice to check that γ is injective. Choose a subgroup system H_1, \dots, H_r (as in Definition 15.5). Suppose we have elements $a, b \in \text{Map}(\text{sub}(G), A)$ with $\gamma(a) = \gamma(b)$. Fix $k \leq |G|$, and assume inductively that $a_i = b_i$ for all $i > k$. After cancelling the terms corresponding to H_i for $i > k$, we get

$$\sum_{i \leq k} |(G/H_i)^{H_j}| a(H_i)^{|H_i|/|H_j|} = \sum_{i \leq k} |(G/H_i)^{H_j}| b(H_i)^{|H_i|/|H_j|}$$

for all j . Take $j = k$, and note that the definition of a subgroup system implies that $(G/H_i)^{H_k} = \emptyset$ for $i < k$. Thus, we only have the term for $i = k$, giving $|(G/H_k)^{H_k}| a(H_k) = |(G/H_k)^{H_k}| b(H_k)$. As $|(G/H_k)^{H_k}| > 0$ and A is torsion-free we deduce that $a(H_k) = b(H_k)$. The claim now follows by decreasing induction on k . \square

PROPOSITION 19.20. *Suppose that A is additively complete. Then the maps $\tau : rL'A(X) \rightarrow LA(X)$ are surjective for all X .*

PROOF. Firstly, all semigroups that we consider will be additively complete by Corollary 19.12, so we can use standard methods with kernels and cokernels and so on.

We filter LA by sub-Mackey functors $F_k LA$ as in Section 16, and put $Q_k LA = F_k LA / F_{k-1} LA$. Next, let $\mathcal{O}_k(X)$ be the category of orbits of coorder k over X , so $(\text{Orb}_G^\times \downarrow X) = \coprod_k \mathcal{O}_k(X)$. Put $P_k(X) = \text{Map}(\pi_0(\mathcal{O}_k(X)), A)$, so $rL'A(X) = \prod_k P_k(X)$. Let τ_k be the restriction of τ to $P_k(X)$. It will be enough to prove that $F_k LA(X) = F_{k-1} LA(X) + \text{img}(\tau_k)$, or equivalently that the composite

$$\sigma_k = (P_k(X) \xrightarrow{\tau_k} F_k LA(X) \rightarrow Q_k LA(X))$$

is surjective. Now choose a system of representatives $(U_i \xrightarrow{x_i} X)$ for the isomorphism classes in $\mathcal{O}_k(X)$. It is straightforward to check that for any Mackey functor M we have $F_k M(U_i) = M(U_i)$ and $Q_k M(U_i) = qM(U_i)$. Moreover, we have $F_k M(X) = F_{k-1} M(X) + \sum_i T_{x_i} M(U_i)$, so the maps T_{x_i} induce an epimorphism $\sigma'_k : \prod_i qM(U_i) \rightarrow Q_k M(X)$. In the case $M = LA$ we have $qM(U_i) = A$. After recalling that the composite $\tau = \pi\nu$ respects addition, we find that $\sigma_k = \sigma'_k$, so σ_k is surjective as required. \square

PROPOSITION 19.21. *If A is additively complete, then $\tau : rL'A(X) \rightarrow LA(X)$ is bijective for all X .*

PROOF. This is clear from Corollary 19.19 and Proposition 19.20 if A is torsion-free.

Now suppose we have a coequaliser diagram

$$A_2 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} A_1 \xrightarrow{\gamma} A_0$$

in the category of rings. Suppose also that this is a reflexive coequaliser, which means that there is a ring map $\sigma : A_1 \rightarrow A_2$ with $\alpha\sigma = \beta\sigma = 1$. Suppose that the result holds for A_1 and A_2 ; we claim that it also holds for A_0 . To see this, consider the diagram

$$\begin{array}{ccccc} rL'A_2(X) & \rightrightarrows & rL'A_1(X) & \longrightarrow & rL'A_0(X) \\ \tau \downarrow & & \tau \downarrow & & \downarrow \tau \\ LA_2(X) & \rightrightarrows & LA_1(X) & \longrightarrow & LA_0(X). \end{array}$$

As L has a right adjoint, it preserves reflexive coequalisers, and Proposition 10.8 tells us that reflexive coequalisers can be computed separately for each X , so the bottom row above is a reflexive coequaliser in the category of sets. From the explicit description in Corollary 19.10 we see that the top row is also a reflexive coequaliser. The first two vertical maps are bijective by assumption, and it follows that the same is true of the third.

Now let A_0 be any ring. Let A_1 be a polynomial algebra over \mathbb{Z} with one generator for each element of A_0 ; then A_1 is torsion-free, and there is an evident surjective homomorphism $\gamma: A_1 \rightarrow A_0$. Put

$$A_2 = \{(a, a') \in A_1 \times A_1 \mid \gamma(a) = \gamma(a')\}.$$

This is clearly also torsion-free. Let $\alpha, \beta: A_2 \rightarrow A_1$ be the two projections, and define $\sigma: A_1 \rightarrow A_2$ by $\sigma(a) = (a, a)$. This gives a reflexive coequaliser diagram, and we deduce that $\tau: rL'A(X) \rightarrow LA(X)$ is bijective as claimed. \square

We next discuss how our approach relates to that of Elliott [10] (but we will gloss over certain issues related to additive completion). For any semigroup M we have a semiring $\mathbb{N}[M]$ (and a ring $\mathbb{Z}[M]$) to which we can apply the above discussion to define a semiring $W_G(\mathbb{N}[M])$ and a ring $W_G(\mathbb{Z}[M])$. Elliott mostly considers this case, and deduces results for more general rings by noting that they can be expressed as quotients of polynomial rings. With hindsight we can see that he is really working with Tambara functors, and most of the numerous maps that he constructs have natural interpretations in that theory. One key point is as follows: we can construct a ‘‘coconstant’’ Mackey functor dM as in Definition 3.17, and then do the Tambara analogue of the semigroup-semiring construction to get a Tambara functor $A[dM]$ as in Section 11. We then have

$$\begin{aligned} \text{Tambara}_G(A[dM], S) &= \text{Mackey}_G(dM, US) = \text{Semigroups}_G(M, US(G)) = \text{Semirings}_G(\mathbb{N}[M], S(G)) \\ &= \text{Tambara}_G(L\mathbb{N}[M], S), \end{aligned}$$

so $L\mathbb{N}[M] \simeq A[dM]$. In this case we can define maps

$$\nu'_U: \mathbb{N}[M] \rightarrow L\mathbb{N}[M](U) = A[dM](U)$$

(for G -orbits U) as follows. First, recall that $dM(U) = \text{Map}(U, M)_G$, and as G acts trivially on M , this is easily identified with M . Next, every element of $A[dM](U)$ is represented by a pair $(X \xrightarrow{t} U, m)$, where $m \in dM(X)$. Thus, for any $m \in M$ the pair $(U \xrightarrow{1} U, m)$ represents an element $\lambda(m) \in A[dM](U)$. From the definitions we see that $\lambda(m_0 + m_1) = \lambda(m_0)\lambda(m_1)$. We can thus define a semiring map $\nu'_U: \mathbb{N}[M] \rightarrow A[dM](U)$ by $\nu'_U(\sum_i n_i [m_i]) = \sum_i n_i \lambda(m_i)$.

These maps ν'_U have similar properties to the maps ν_U considered earlier, and we can use them in the same way to define maps $\tau': rL'\mathbb{N}[M](X) \rightarrow L\mathbb{N}[M](X)$ analogous to τ . In more detail, we choose a list of diagrams $(U_i \xrightarrow{x_i} X)$ (for $1 \leq i \leq m$ say) containing precisely one representative of each isomorphism class in $(\text{Orb}_G^\times \downarrow X)$ and put

$$\tau'(a) = \sum_{i=1}^m T_{x_i} \nu'_{U_i}(a).$$

This map τ' is easier to use than τ because it is additive, and it is also a bijection. This can be proved along the same lines used for τ , or one can just exhibit an inverse as follows.

DEFINITION 19.22. For any orbit U and any element $[W \xrightarrow{u} U, m] \in A[dM](U)$ (so $m \in dM(W)$) we define

$$\beta_0[W \xrightarrow{u} U, M] = \sum_{w \in \text{Sec}(u)} R_w(m) \in \mathbb{N}[dM(U)] = \mathbb{N}[M].$$

For any G -set X and any element $a \in A[dM](X)$ we define

$$\beta'(a) \in rL'\mathbb{N}[M](X) = \text{Map}(\pi_0(\text{Orb}_G^\times \downarrow X), \mathbb{N}[M])$$

by $\beta'(a)(U \xrightarrow{x} X) = \beta_0(R_x(a))$.

It is not hard to check directly that β' is inverse to τ' . We also have a map $\gamma' = \phi\tau'$ analogous to the ghost map γ . To describe this, we first define maps $a \mapsto a^{(k)}$ on $\mathbb{N}[M]$ (for $k \in \mathbb{N}$) by

$$\left(\sum_i n_i [m_i] \right)^{\langle k \rangle} = \sum_i n_i [km_i].$$

This is similar to the usual power map $a \mapsto a^k$, because $a^k = a^{\langle k \rangle}$ whenever $a = [m]$ for some m , and also $a^p = a^{\langle p \rangle} \pmod{p}$ whenever p is prime. One can check that

$$\gamma'(a)([H]) = \sum_{[K]} |(G/H)^K| a([K])^{\langle |H|/|K| \rangle}.$$

A key part of Elliott's argument is to show that ϕ and ϕ' have the same image.

A. Semiadditive categories

In this appendix we set up a theory of semiadditive categories. This is largely the same as the more familiar theory of additive categories. However, there is one technical point that creates a need for a detailed account. Given a small semiadditive category \mathcal{A} , we write $\text{Mod}(\mathcal{A})$ for the category of product-preserving functors $M: \mathcal{A} \rightarrow \text{Sets}$. We will show that every such M can also be regarded as an additive functor $\mathcal{A} \rightarrow \text{Semigroups}$. Now suppose that \mathcal{A} has a biadditive tensor product making it a symmetric monoidal category. We would like to use the well-known construction of Day [6] to obtain from this a biadditive tensor product on $\text{Mod}(\mathcal{A})$. However, there are various subtleties about whether we require additivity at various stages in the construction, and whether we use Sets or Semigroups as the ambient category. For our applications to Tambara functors, the key point is that we can work with apparently non-additive constructions and still get a biadditive functor $\boxtimes: \text{Mod}(\mathcal{A}) \times \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{A})$. This will be explained in more detail below.

A.1. Basics.

DEFINITION A.1. Let \mathcal{A} be a category. We say that \mathcal{A} is *semiadditive* if

- (a) It has finite products (and in particular, a terminal object, denoted by 0).
- (b) The terminal object is also initial (so for each a and b we have unique maps $a \rightarrow 0 \rightarrow b$, whose composite we call the zero map).
- (c) Let $a_1 \xleftarrow{p_1} a \xrightarrow{p_2} a_2$ be a product diagram, and let $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ be the unique maps such that

$$p_1 i_1 = 1 \quad p_2 i_1 = 0 \quad p_1 i_2 = 0 \quad p_2 i_2 = 1.$$

Then the diagram $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ is a coproduct diagram.

REMARK A.2. By an evident inductive extension of (c), any product of finitely many factors is also a coproduct. (The case of no factors is (b), and the case of one factor is trivial.) In more detail, given objects a_1, \dots, a_n there is an object $a = \bigoplus_t a_t$ and morphisms

$$a_t \xrightarrow{i_t} a \xrightarrow{p_u} a_u$$

such that $p_t i_t = 1$, and $p_u i_t = 0$ for all $u \neq t$, and the maps i_t give a coproduct diagram, and the maps p_u give a product diagram. For any object x and system of maps $f_u: x \rightarrow a_u$, we write $[f_1, \dots, f_n]$ for the unique map $x \rightarrow a$ such that $p_u \circ [f_1, \dots, f_n] = f_u$ for all u . Dually, given maps $g_t: a_t \rightarrow y$, we write $\langle g_1, \dots, g_n \rangle$ the unique map $a \rightarrow y$ with $\langle g_1, \dots, g_n \rangle \circ j_t = g_t$ for all t . Note that in the case $n = 2$ we have

$$\begin{aligned} i_1 &= [1, 0]: a_1 \rightarrow a_1 \oplus a_2 & i_2 &= [0, 1]: a_2 \rightarrow a_1 \oplus a_2 \\ p_1 &= \langle 1, 0 \rangle: a_1 \oplus a_2 \rightarrow a_1 & p_2 &= \langle 0, 1 \rangle: a_1 \oplus a_2 \rightarrow a_2. \end{aligned}$$

We now resolve an apparent asymmetry in the definition.

LEMMA A.3. Let \mathcal{A} be a semiadditive category. Let $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ be a coproduct diagram, and let $a_1 \xleftarrow{p_1} a \xrightarrow{p_2} a_2$ be the unique maps such that

$$p_1 i_1 = 1 \quad p_2 i_1 = 0 \quad p_1 i_2 = 0 \quad p_2 i_2 = 1.$$

Then the diagram $a_1 \xleftarrow{p_1} a \xrightarrow{p_2} a_2$ is a product diagram.

PROOF. By axiom (a), there exists a product diagram $a_1 \xleftarrow{p'_1} a' \xrightarrow{p'_2} a_2$. From this we can build a coproduct diagram $a_1 \xrightarrow{i'_1} a' \xleftarrow{i'_2} a_2$ as in axiom (c). By the standard uniqueness property of coproducts, there is a unique map $f: a \rightarrow a'$ with $i'_t f = i_t$ for $t = 1, 2$, and this map f is an isomorphism. It follows easily that the diagram $a_1 \xleftarrow{p'_1 f} a \xrightarrow{p'_2 f} a_2$ is a product diagram, so it will suffice to prove that $p'_1 f = p_1: a \rightarrow a_1$ and $p'_2 f = p_2$. Note that $p'_1 f i_1 = p'_1 i'_1 = 1 = p_1 i_1$ and $p'_1 f i_2 = p'_1 i'_2 = 0 = p_1 i_2$. As $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ is a coproduct diagram, it follows that $p'_1 f = p_1$, and a symmetrical argument shows that $p'_2 f = p_2$, as required. \square

DEFINITION A.4. A *preadditive category* is a category \mathcal{B} with a given semigroup structure on each morphism set $\mathcal{B}(b_0, b_1)$, with the property that each composition map

$$\mathcal{B}(b_1, b_2) \times \mathcal{B}(b_0, b_1) \rightarrow \mathcal{B}(b_0, b_2)$$

is bilinear. Now let $F: \mathcal{B} \rightarrow \mathcal{C}$ be a functor between preadditive categories. We say that F is a *preadditive functor* if each map

$$F: \mathcal{B}(b_0, b_1) \rightarrow \mathcal{C}(Fb_0, Fb_1)$$

is a semigroup homomorphism.

PROPOSITION A.5. *Let \mathcal{A} be a semiadditive category. Then \mathcal{A} has a canonical structure as a preadditive category.*

PROOF. Given maps $g_1, g_2: b \rightarrow c$ we define

$$g_1 + g_2 = (b \xrightarrow{[1,1]} b \oplus b \xrightarrow{\langle g_1, g_2 \rangle} c).$$

In the case $g_2 = 0$ we observe that $g_1 p_1: b \oplus b \rightarrow c$ has the defining property of $\langle g_1, 0 \rangle$; it follows easily that $g_1 + 0 = g_1$. Next, put $\sigma = [p_2, p_1]: b \oplus b \rightarrow b \oplus b$. One can check that this is the same as $\langle i_2, i_1 \rangle$, and that $\sigma \circ [1, 1] = [1, 1]$ and $\langle g_1, g_2 \rangle \circ \sigma = \langle g_2, g_1 \rangle$. Using this we see that $g_2 + g_1 = g_1 + g_2$. Finally, if g_3 is a third morphism from b to c then it is not hard to check that $g_1 + (g_2 + g_3)$ and $(g_1 + g_2) + g_3$ are both equal to the composite

$$b \xrightarrow{[1,1,1]} b \oplus b \oplus b \xrightarrow{\langle g_1, g_2, g_3 \rangle} c.$$

It follows that $\mathcal{A}(b, c)$ is a semigroup.

Now consider a map $h: c \rightarrow d$. It is clear from the definitions that $h \circ \langle g_1, g_2 \rangle = \langle hg_1, hg_2 \rangle$ and thus that $h \circ (g_1 + g_2) = (hg_1) + (hg_2)$. Consider instead a map $f: a \rightarrow b$. This of course induces a map

$$f \oplus f = [fp_1, fp_2]: a \oplus a \rightarrow b \oplus b.$$

By considering the diagram

$$\begin{array}{ccccc} a & \xrightarrow{[1,1]} & a \oplus a & \xrightarrow{\langle g_1 f, g_2 f \rangle} & c \\ f \downarrow & & \downarrow f \oplus f & & \parallel \\ b & \xrightarrow{[1,1]} & b \oplus b & \xrightarrow{\langle g_1, g_2 \rangle} & c \end{array}$$

we see that $(g_1 + g_2)f = (g_1 f) + (g_2 f)$. This proves that composition is bilinear. \square

PROPOSITION A.6. *Let \mathcal{A} be a semiadditive category. Then in the context of axiom (c) or Lemma A.3 we always have*

$$i_1 p_1 + i_2 p_2 = 1: a \rightarrow a.$$

PROOF. Suppose that the maps i_t and p_t are as in axiom (c) or Lemma A.3. We then have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1 i_1 p_1 + p_1 i_2 p_2 = 1 \circ p_1 + 0 \circ p_2 = p_1 = p_1 \circ 1,$$

and similarly $p_2 \circ (i_1 p_1 + i_2 p_2) = p_2 \circ 1$. As $a_1 \xleftarrow{p_1} a \xrightarrow{p_2} a_2$ is a product diagram, this means that $i_1 p_1 + i_2 p_2 = 1$. \square

PROPOSITION A.7. *Let \mathcal{A} be a preadditive category.*

(a) *An object $a \in \mathcal{A}$ is initial iff it is terminal iff the semigroup $\mathcal{A}(a, a)$ is trivial.*

(b) Suppose we have a diagram

$$a_1 \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{array} x \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{array} a_2$$

with

$$p_1 i_1 = 1 \quad p_2 i_1 = 0 \quad p_1 i_2 = 0 \quad p_2 i_2 = 1 \quad i_1 p_1 + i_2 p_2 = 1.$$

Then the diagram $a_1 \xrightarrow{i_1} x \xleftarrow{i_2} a_2$ is a coproduct, and the diagram $a_1 \xleftarrow{p_1} x \xrightarrow{p_2} a_2$ is a product.

PROOF. (a) If a is either initial or terminal then $\mathcal{A}(a, a)$ is a singleton and so is trivial as a semigroup. Conversely, if $\mathcal{A}(a, a)$ is trivial then in particular the identity morphism is the same as the zero element. Now for any $f: x \rightarrow a$ we have $f = 1_a \circ f = 0 \circ f$, and because composition is biadditive this is the zero element of $\mathcal{A}(x, a)$. This proves that a is terminal, and a dual argument shows that it is also initial.

(b) Suppose we have maps $a_1 \xrightarrow{f_1} u \xleftarrow{f_2} a_2$. We put $f = f_1 p_1 + f_2 p_2: x \rightarrow u$. This has

$$f i_1 = f_1 p_1 i_1 + f_2 p_2 i_1 = f_1 \circ 1 + f_2 \circ 0 = f_1,$$

and similarly $f i_2 = f_2$. Let $f^*: x \rightarrow u$ be any map with $f^* i_t = f_t$ for $t = 0, 1$. We then have

$$f^* = f^* \circ 1_x = f^* \circ (i_1 p_1 + i_2 p_2) = f^* i_1 p_1 + f^* i_2 p_2 = f_1 p_1 + f_2 p_2 = f.$$

This proves that the diagram $a_1 \xrightarrow{i_1} x \xleftarrow{i_2} a_2$ is a coproduct. Dually, given maps $a_1 \xleftarrow{g_1} v \xrightarrow{g_2} a_2$ we find that the map $g = i_1 g_1 + i_2 g_2: v \rightarrow x$ is the unique one satisfying $p_t g = g_t$ for $t = 0, 1$ so the diagram $a_1 \xleftarrow{p_1} x \xrightarrow{p_2} a_2$ is a product. □

COROLLARY A.8. Let \mathcal{A} be a preadditive category. Suppose that there is an object satisfying part (a) of the Proposition, and that for all objects a_1 and a_2 there is a diagram as in part (b) of the Proposition. Then \mathcal{A} is semiadditive.

PROOF. Follows directly from the Proposition. □

PROPOSITION A.9. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between semiadditive categories. Then the following are equivalent:

- (a) F sends finite coproduct diagrams to finite coproduct diagrams.
- (b) F sends finite product diagrams to finite product diagrams.
- (c) F is a preadditive functor.

(We say that F is semiadditive if these conditions hold.)

PROOF. We first claim that any of the three conditions implies that F sends the zero object to the zero object (and thus sends zero morphisms to zero morphisms). In case (a) this holds because 0 is the coproduct of the empty family, and in case (b) because 0 is the product of the empty family. In case (c) we note that $1 = 0$ as endomorphisms of 0 . Any functor preserves identity maps, and F preserves zero maps by hypothesis, so $1 = 0$ as endomorphisms of $F0$. This means that the unique maps $0 \rightarrow F0 \rightarrow 0$ are inverse to each other, so $F0 \simeq 0$ as claimed.

Now suppose that (a) holds. Consider a product diagram $a_1 \xleftarrow{p_1} a \xrightarrow{p_2} a_2$. Let $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ be the unique maps such that

$$p_1 i_1 = 1 \quad p_2 i_1 = 0 \quad p_1 i_2 = 0 \quad p_2 i_2 = 1,$$

so the diagram $a_1 \xrightarrow{i_1} a \xleftarrow{i_2} a_2$ is a coproduct diagram by axiom (c) in Definition A.1. By assumption, the diagram $F a_1 \xrightarrow{F i_1} F a \xleftarrow{F i_2} F a_2$ is a coproduct diagram in \mathcal{B} . We can apply Lemma A.3 to this diagram to see that $F a_1 \xleftarrow{F p_1} F a \xrightarrow{F p_2} F a_2$ is a product diagram. This shows that (a) implies (b), and a dual argument shows that (b) implies (a). If these conditions hold, then F respects all the structure used to define addition,

so we see that (c) also holds. Conversely, suppose that (c) holds. Any binary product or coproduct diagram in \mathcal{A} gives rise to a diagram

$$a_1 \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{array} x \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{array} a_2$$

with

$$p_1 i_1 = 1 \quad p_2 i_1 = 0 \quad p_1 i_2 = 0 \quad p_2 i_2 = 1 \quad i_1 p_1 + i_2 p_2 = 1$$

as in Proposition A.7. Applying F gives a diagram in \mathcal{B} with the same properties, which is therefore both a product and a coproduct. This shows that (c) implies (a) and (b). \square

A.2. Modules.

DEFINITION A.10. Let \mathcal{A} be a small semiadditive category. An \mathcal{A} -module is a product-preserving functor $M: \mathcal{A} \rightarrow \text{Sets}$. We write $\mathcal{M}(\mathcal{A})$ for the category of \mathcal{A} -modules. If $f: a \rightarrow b$ is a morphism in \mathcal{A} and M is an \mathcal{A} -module, we write f_* for the induced map $M(a) \rightarrow M(b)$.

We next show that if M is a \mathcal{A} -module, then the sets $M(a)$ have canonical structures as semigroups. The cleanest way to formulate this is as follows.

DEFINITION A.11. Let \mathcal{A} be a small preadditive category. We write $\mathcal{M}'(\mathcal{A})$ for the category of preadditive functors from $M: \mathcal{A} \rightarrow \text{Semigroups}$.

PROPOSITION A.12. *The forgetful functor $U: \text{Semigroups} \rightarrow \text{Sets}$ induces an isomorphism of categories $U_*: \mathcal{M}'(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{A})$.*

PROOF. First, recall that products of semigroups are just constructed by giving the product set a suitable semigroup structure, which means that U preserves products, so U_* gives a functor from $\mathcal{M}'(\mathcal{A})$ to $\mathcal{M}(\mathcal{A})$ as indicated.

Now suppose we have $M \in \mathcal{M}(\mathcal{A})$. Consider an object $a \in \mathcal{A}$. As M preserves products we see that $M(0)$ is a singleton. We also have a unique map $0: 0 \rightarrow a$, giving a map $0_*: M(0) \rightarrow M(a)$; we again write 0 for the unique element in the image of this map.

Now construct a biproduct diagram

$$a \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{array} a \oplus a \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{array} a$$

as before. Put $s = p_1 + p_2 = \langle 1, 1 \rangle: a \oplus a \rightarrow a$. By hypothesis, the map $((p_1)_*, (p_2)_*): M(a \oplus a) \rightarrow M(a)^2$ is a bijection. Given $m_1, m_2 \in M(a)$ we let $m \in M(a \oplus a)$ be the unique element with $(p_t)_*(m) = m_t$ for $t = 1, 2$, then we define $m_1 + m_2 = s_*(m) \in M(a)$. By an argument similar to that for Proposition A.5, this gives an semigroup structure on $M(a)$. Now let $f: a \rightarrow b$ be a morphism in \mathcal{A} . By considering the diagram

$$\begin{array}{ccccc} M(a) \times M(a) & \xleftarrow[\cong]{((p_1)_*, (p_2)_*)} & M(a \oplus a) & \xrightarrow{s_*} & M(a) \\ f_* \times f_* \downarrow & & \downarrow (f \oplus f)_* & & \downarrow f_* \\ M(b) \times M(b) & \xleftarrow[\cong]{((p_1)_*, (p_2)_*)} & M(b \oplus b) & \xrightarrow{s_*} & M(b) \end{array}$$

we see that $f_*(m_1 + m_2) = f_*(m_1) + f_*(m_2)$, so f_* is a homomorphism. (As we are dealing with semigroups rather than groups we need to check separately that $f_*(0) = 0$, but this is easy.) We can thus use these structures to regard M as a functor $\mathcal{A} \rightarrow \text{Semigroups}$, which preserves products and so is semiadditive. In other words, we have an object in $\mathcal{M}'(\mathcal{A})$, which we call FM .

Now suppose we have a natural transformation $\alpha: M \rightarrow N$ between product-preserving functors $\mathcal{A} \rightarrow \text{Sets}$. By considering the diagram

$$\begin{array}{ccccc}
M(a) \times M(a) & \xleftarrow[\cong]{((p_1)_*, (p_2)_*)} & M(a \oplus a) & \xrightarrow{s_*} & M(a) \\
\alpha_a \times \alpha_a \downarrow & & \downarrow \alpha_{a \oplus a} & & \downarrow \alpha_a \\
N(a) \times N(a) & \xleftarrow[\cong]{((p_1)_*, (p_2)_*)} & N(a \oplus a) & \xrightarrow{s_*} & N(a)
\end{array}$$

we see that $\alpha_a(m_1 + m_2) = \alpha_a(m_1) + \alpha_a(m_2)$. We also have $\alpha_a(0) = 0$, so α_a is a homomorphism. Using this, we see that our construction actually gives a functor $F: \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}'(\mathcal{A})$. It is clear by construction that the composite $U_* \circ F: \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{A})$ is equal (not just isomorphic) to the identity.

In the opposite direction, suppose we have an object $M' \in \mathcal{M}'(\mathcal{A})$. Consider an object $a \in \mathcal{A}$ and elements $m_1, m_2 \in M'(a)$. As $M'(a \oplus a)$ has a specified semigroup structure, it is meaningful to define $m = (i_1)_*(m_1) + (i_2)_*(m_2) \in M'(a \oplus a)$. As M' is a functor with values in Semigroups we see that $(p_t)_*: M'(a \oplus a) \rightarrow M'(a)$ is a homomorphism, and using this it follows that $(p_t)_*(m) = m_t$, so we have the same element m as discussed previously. Similarly, the map $s_*: M'(a \oplus a) \rightarrow M'(a)$ is a homomorphism, and $s_1 = s_2 = 1$. It follows that $s_*(m) = m_1 + m_2$. This means that the originally given semigroup structure on $M'(a)$ is the same as the one constructed by the functor F , so $M' = FU_*(M')$. Thus, the functors F and U_* are mutually inverse. \square

PROPOSITION A.13. *Let \mathcal{A} be a small semiadditive category, and let \mathcal{A}_0 be full subcategory such that every object in \mathcal{A} can be expressed as a coproduct of some finite family of objects in \mathcal{A}_0 . Then the restriction functor $\text{res}: \mathcal{M}'(\mathcal{A}) \rightarrow \mathcal{M}'(\mathcal{A}_0)$ is an equivalence of categories.*

PROOF. We first claim that res is faithful. Consider a pair of morphisms $\alpha, \beta: M \rightarrow N$ in $\mathcal{M}'(\mathcal{A})$ with $\text{res}(\alpha) = \text{res}(\beta)$, and an object $a \in \mathcal{A}$; we must show that $\alpha_a = \beta_a: M(a) \rightarrow N(a)$. By hypothesis we can find objects $a_1, \dots, a_n \in \mathcal{A}_0$ and morphisms $a_t \xrightarrow{i_t} a \xrightarrow{p_t} a_t$ as in Remark A.2, so $p_t i_t = 1$ and $p_u i_t = 0$ for $u \neq t$ and $\sum_t i_t p_t = 1$. We have seen that N preserves products, and the maps p_t give a product diagram, so it will suffice to show that $N(p_t) \circ \alpha_a = N(p_t) \circ \beta_a$ for all t . By naturality we have $N(p_t) \circ \alpha_a = \alpha_{a_t} \circ M(p_t)$ and similarly for β . However, as $a_t \in \mathcal{A}_0$ and $\text{res}(\alpha) = \text{res}(\beta)$ we have $\alpha_{a_t} = \beta_{a_t}$, and the claim follows.

We next claim that res is full. To see this, consider a morphism $\alpha_0: \text{res}(M) \rightarrow \text{res}(N)$. Fix an object $a \in \mathcal{A}$. We claim that there is a unique map $\alpha_a: M(a) \rightarrow N(a)$ such that for all $a' \in \mathcal{A}_0$ and $f: a \rightarrow a'$, the diagram

$$\begin{array}{ccc}
M(a) & \xrightarrow{\alpha_a} & N(a) \\
M(f) \downarrow & & \downarrow N(f) \\
M(a') & \xrightarrow{(\alpha_0)_{a'}} & N(a')
\end{array}$$

commutes. To see this, we choose objects $a_t \in \mathcal{A}_0$ and morphisms i_t and p_t as before. As the maps $N(p_t)$ give a product diagram, we see that there is a unique map $\alpha_a: M(a) \rightarrow N(a)$ with $N(p_t) \circ \alpha_a = (\alpha_0)_{a_t} \circ M(p_t)$ for all t . Now consider an arbitrary object $a' \in \mathcal{A}_0$ and a morphism $f: a \rightarrow a'$. Recall that $1_a = \sum_t i_t p_t$, so $f = \sum_t (f i_t) p_t$. Now $f i_t: a_t \rightarrow a'$ is a morphism in \mathcal{A}_0 , so $(\alpha_0)_{a'} M(f i_t) = N(f i_t) (\alpha_0)_{a_t}$. Using this we get

$$\begin{aligned}
(\alpha_0)_{a'} M(f) &= \sum_t (\alpha_0)_{a'} \circ M(f i_t) \circ M(p_t) = \sum_t N(f i_t) \circ (\alpha_0)_{a_t} \circ M(p_t) \\
&= \sum_t N(f i_t) \circ N(p_t) \circ \alpha_a = \sum_t N(f i_t p_t) \circ \alpha_a = N(f) \circ \alpha_a,
\end{aligned}$$

so α_a has the claimed property. Uniqueness is clear, and naturality follows from uniqueness. We thus have a morphism $\alpha: M \rightarrow N$ with $\text{res}(\alpha) = \alpha_0$ as required.

Now consider an object $M_0 \in \mathcal{M}'(\mathcal{A}_0)$. For any $a \in \mathcal{A}$ we have a preadditive functor $K_a: \mathcal{A}_0 \rightarrow \text{Semigroups}$ given by $K_a(a') = \mathcal{A}(a, a')$. This gives an object $K_a \in \mathcal{M}'(\mathcal{A}_0)$, and we define

$$M(a) = \mathcal{M}'(\mathcal{A}_0)(K_a, M_0),$$

and note that this has a semigroup structure under pointwise addition. Now K_a is a preadditive contravariant functor of a , so M is a preadditive covariant functor, or in other words $M \in \mathcal{M}'(\mathcal{A})$. It follows from the Yoneda Lemma that $\text{res}(M) \simeq M_0$, so res is also essentially surjective. \square

A.3. Tensor products. Let \mathcal{A} be a small semiadditive category, and suppose that \mathcal{A} has a symmetric monoidal structure given by a functor $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ with a unit object $1 \in \mathcal{A}$. For typographical convenience, we will also use the symbol μ for the functor $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. We will assume that this functor is biadditive, which means that for $f_1, f_2: a \rightarrow b$ and $g_1, g_2: c \rightarrow d$ we have

$$(f_1 + f_2) \otimes (g_1 + g_2) = f_1 \otimes g_1 + f_1 \otimes g_2 + f_2 \otimes g_1 + f_2 \otimes g_2: a \otimes c \rightarrow b \otimes d,$$

and also $f \otimes 0 = 0$ and $0 \otimes g = 0$.

DEFINITION A.14. Let M and N be \mathcal{A} -modules. We define $M * N: \mathcal{A} \times \mathcal{A} \rightarrow \text{Sets}$ by $(M * N)(a, b) = M(a) \times N(b)$. We then put $M \boxtimes N = \lim_{\rightarrow \mu} (M * N)$ (the left Kan extension of $M * N$ along μ), so $M \boxtimes N: \mathcal{A} \rightarrow \text{Sets}$.

We can make this more explicit by recalling the standard construction of Kan extensions as colimits over comma categories. Write $Q = M \boxtimes N$ for brevity. Consider a morphism $f: u \otimes v \rightarrow a$ in \mathcal{A} together with elements $m \in M(u)$ and $n \in N(v)$. These data give an element $\theta(f, m, n) \in Q(a)$, and all elements of $Q(a)$ arise in this way. Now suppose we have maps $r: u' \rightarrow u$ and $s: v' \rightarrow v$, and elements $m' \in M(u')$ and $n' \in N(v')$, such that $r_*(m') = m$ and $s_*(n') = n$. In this context we have

$$\theta(f, m, n) = \theta(f, r_*(m'), s_*(n')) = \theta(f \circ (r \otimes s), m', n') \in Q(a).$$

Moreover, all identities between elements of $Q(a)$ are consequences of identities of this type. For any map $g: a \rightarrow b$, the induced map $g_*: Q(a) \rightarrow Q(b)$ is just $g_*(\theta(f, m, n)) = \theta(gf, m, n)$.

PROPOSITION A.15. *The functor $M \boxtimes N: \mathcal{A} \rightarrow \text{Sets}$ preserves products (and so is an \mathcal{A} -module).*

PROOF. Consider a finite family of objects $(a_t)_{t \in T}$. Put $a = \bigoplus_t a_t$ and let $a_t \xrightarrow{i_t} a \xrightarrow{p_t} a_t$ be the usual maps. Let $\phi: Q(a) \rightarrow \prod_t Q(a_t)$ be the map whose t 'th component is $(p_t)_*: Q(a) \rightarrow Q(a_t)$. The claim is that ϕ is a bijection.

Suppose we have maps $f_t: u_t \otimes v_t \rightarrow a_t$ and elements $m_t \in M(u_t)$ and $n_t \in N(v_t)$ giving elements $q_t = \theta(f_t, m_t, n_t) \in Q(a_t)$. Put $u = \bigoplus_t u_t$ and $v = \bigoplus_t v_t$. By the product property of a , there is a unique map $f: u \otimes v \rightarrow a$ such that the diagram

$$\begin{array}{ccc} u \otimes v & \xrightarrow{f} & a \\ p_t \otimes p_t \downarrow & & \downarrow p_t \\ u_t \otimes v_t & \xrightarrow{f_t} & a_t \end{array}$$

commutes for all t . As the functor M preserves products, there is a unique element $m \in M(u)$ with $(p_t)_*(m) = m_t \in M(u_t)$ for all t . Similarly, there is a unique element $n \in N(v)$ with $(p_t)_*(n) = n_t \in N(v_t)$ for all t . Now put $q = \theta(f, m, n) \in Q(a)$. We have

$$(p_t)_*(q) = \theta(p_t f, m, n) = \theta(f_t \circ (p_t \otimes p_t), m, n) = \theta(f_t, (p_t)_*(m), (p_t)_*(n)) = \theta(f_t, m_t, n_t) = q_t,$$

so $\phi(q) = (q_t)_{t \in T}$. This proves that ϕ is surjective.

The element q here depends on the maps f_t and the elements m_t and n_t , so we can define

$$\psi_0((f_t)_{t \in T}, (m_t)_{t \in T}, (n_t)_{t \in T}) = q.$$

We next claim that this map ψ_0 is compatible with the defining relations for Q , so it induces a map $\psi: \prod_t Q(a_t) \rightarrow Q(a)$ with $\psi((q_t)_{t \in T}) = q$. To see this, suppose we have maps $r_t: u'_t \rightarrow u_t$ and $s_t: v'_t \rightarrow v_t$, together with elements $m'_t \in M(u'_t)$ and $n'_t \in N(v'_t)$ satisfying $(r_t)_*(m'_t) = m_t$ and $(s_t)_*(n'_t) = n_t$. This means that $q_t = \theta(f_t \circ (r_t \otimes s_t), m'_t, n'_t)$. Put

$$q' = \psi_0((f_t \circ (r_t \otimes s_t))_{t \in T}, (m'_t)_{t \in T}, (n'_t)_{t \in T});$$

we need to show that $q' = q$. For this, we put $u' = \bigoplus_t u'_t$ and $v' = \bigoplus_t v'_t$, and let m' and n' be the evident elements of $M(u')$ and $N(v')$. By definition, we have $q' = \theta(g, m', n')$, where $g: u' \otimes v' \rightarrow a$ is the unique map such that $p_t \circ g = f_t \circ (r_t \otimes s_t) \circ (p_t \otimes p_t)$ for all t . By inspecting the diagram

$$\begin{array}{ccccc} u' \otimes v' & \xrightarrow{r \otimes s} & u \otimes v & \xrightarrow{f} & a \\ p_t \otimes p_t \downarrow & & p_t \otimes p_t \downarrow & & p_t \downarrow \\ u'_t \otimes v'_t & \xrightarrow{r_t \otimes s_t} & u_t \otimes v_t & \xrightarrow{f_t} & a_t \end{array}$$

we see that $g = f \circ (r \otimes s)$, so $q' = \theta(f \circ (r \otimes s), m', n')$. By the defining relations for $Q(a)$, this is the same as $\theta(f, r_*(m'), s_*(n'))$. It is straightforward to check that $r_*(m') = m$ and $s_*(n') = n$, so $q' = q$ as claimed. It follows easily that there is a well-defined map $\psi: \prod_t Q(a_t) \rightarrow Q(a)$ as described previously, and that $\phi\psi = 1: Q(a) \rightarrow Q(a)$.

In the opposite direction, consider an arbitrary element $q \in Q(a)$, where $a = \bigoplus_t a_t$ as before. We can represent q as $\theta(f, m, n)$ for some morphism $f: u \otimes v \rightarrow a$ and elements $m \in M(u)$ and $n \in N(v)$. This means that $\phi(q) = (\theta(\pi_t f, m, n))_{t \in T}$. Put $u' = \bigoplus_{t \in T} u$ and $v' = \bigoplus_{t \in T} v$. Let $m' \in M(u')$ be the unique element such that $(p_t)_*(u') = u$ for all t , and similarly for $n' \in N(v')$. By the product property of a , there is a unique map $g: u' \otimes v' \rightarrow a$ such that the following diagram commutes for all t :

$$\begin{array}{ccc} u' \otimes v' & \xrightarrow{g} & a \\ p_t \otimes p_t \downarrow & & p_t \downarrow \\ u \otimes v & \xrightarrow{f} & a \xrightarrow{p_t} a_t. \end{array}$$

From the definitions, we see that $\psi\phi(q) = \theta(g, m', n')$. Now let $d: u \rightarrow u'$ be the diagonal map, so $p_t d = 1: u \rightarrow u$ for all t . Similarly, let $e: v \rightarrow v'$ be the diagonal map. One can then check that $m' = d_*(m)$ and $n' = e_*(n)$, and also that $g \circ (d \otimes e) = f$. From the defining relations for Q we therefore have

$$\psi\phi(q) = \theta(g, d_*(m), e_*(n)) = \theta(g \circ (d \otimes e), m, n) = \theta(f, m, n) = q.$$

Thus, ψ is the required inverse for ϕ . □

CONSTRUCTION A.16. Let N be an \mathcal{A} -module, and u an object of \mathcal{A} . We write $T_u N$ for the composite functor

$$\mathcal{A} \xrightarrow{u \otimes (-)} \mathcal{A} \xrightarrow{N} \mathbf{Sets}.$$

As both $u \otimes (-)$ and N preserve products, the same is true of $T_u N$, so $T_u N$ is again an \mathcal{A} -module. It is formal to check that the assignment $u \mapsto T_u N$ gives an additive (and therefore product-preserving) functor $\mathcal{A} \rightarrow \mathbf{Mod}(\mathcal{A})$. Now let M be another \mathcal{A} -module. We define $\underline{\mathbf{Hom}}(M, N): \mathcal{A} \rightarrow \mathbf{Sets}$ by

$$\underline{\mathbf{Hom}}(M, N)(u) = \mathbf{Mod}(\mathcal{A})(M, T_u N).$$

It is again easy to see that this preserves products, so it is itself an \mathcal{A} -module.

PROPOSITION A.17. *For all \mathcal{A} -modules L, M and N there are natural isomorphisms*

$$\mathbf{Mod}(\mathcal{A})(L, \underline{\mathbf{Hom}}(M, N)) \simeq \mathbf{Mod}(\mathcal{A})(L \boxtimes M, N).$$

PROOF. A morphism $\alpha: L \rightarrow \underline{\mathbf{Hom}}(M, N)$ consists of maps $\alpha_u: L(u) \rightarrow \mathbf{Mod}(\mathcal{A})(M, T_u N)$ for all u that are natural in u . Thus, for each $r \in L(u)$ we have $\alpha_u(r): M \rightarrow T_u N$, consisting of maps

$$\alpha_u(r)_v: M(v) \rightarrow T_u N(v) = N(u \otimes v)$$

that are natural in v . Recall that we have functors $L * M, \mu^* N: \mathcal{A} \times \mathcal{A} \rightarrow \mathbf{Sets}$ given by $(L * M)(u, v) = L(u) \times N(v)$ and $(\mu^* N)(u, v) = N(u \otimes v)$. We define $\alpha^\#: L * M \rightarrow \mu^* N$ by

$$\alpha_{u,v}^\#(r, s) = \alpha_u(r)_v(s).$$

It is straightforward to check that this construction is bijective. On the other hand, by the definition of Kan extensions, we see that natural maps $L * M \rightarrow \mu^* N$ biject with morphisms $L \boxtimes M \rightarrow N$. □

PROPOSITION A.18. *The functor \boxtimes gives a symmetric monoidal structure on $\text{Mod}(\mathcal{A})$. The unit object is the functor $I(a) = \mathcal{A}(1, a)$, where 1 is the unit object in \mathcal{A} .*

PROOF. First, let $\tau: a \otimes b \rightarrow b \otimes a$ be the symmetry isomorphism for \mathcal{A} . For modules M, N we then have an obvious isomorphism $(M * N) \circ \tau \simeq N * M$, and using standard properties of Kan extensions this gives an isomorphism $M \boxtimes N \simeq N \boxtimes M$.

Next, the Yoneda lemma gives

$$\underline{\text{Hom}}(I, N)(u) = \text{Mod}(\mathcal{A})(I, T_u N) = T_u N(1) = N(u \otimes 1) = N(u)$$

for all u , so $\underline{\text{Hom}}(I, N) = N$. Now take $M = I$ in Proposition A.17 to get $\text{Mod}(\mathcal{A})(L, N) \simeq \text{Mod}(\mathcal{A})(L \boxtimes I, N)$. This is natural in L and N and so gives $L \boxtimes I \simeq L$.

Now suppose we have three modules L, M and N . We would like to give a natural isomorphism $(L \boxtimes M) \boxtimes N \simeq L \boxtimes (M \boxtimes N)$. For any module P , we let $T(P)$ denote the set of natural maps

$$L(u) \times M(v) \times N(w) \rightarrow P(u \otimes v \otimes w)$$

(of functors $\mathcal{A}^3 \rightarrow \text{Sets}$). It will suffice to show that the functor T is represented by both $(L \boxtimes M) \boxtimes N$ and $L \boxtimes (M \boxtimes N)$. Proposition A.17 gives

$$\text{Mod}(\mathcal{A})((L \boxtimes M) \boxtimes N, P) \simeq \text{Mod}(\mathcal{A})(L \boxtimes M, \underline{\text{Hom}}(N, P)) \simeq [\mathcal{A}^2, \text{Sets}](L * M, \mu^* \underline{\text{Hom}}(N, P)).$$

This is the set of maps $L(u) \times M(v) \rightarrow \underline{\text{Hom}}(N, P)(u \otimes v)$ that are natural in u and v . After filling in the definition of $\underline{\text{Hom}}$ this becomes the set of maps $L(u) \times M(v) \times N(w) \rightarrow P(u \otimes v \otimes w)$ that are natural in all variables, or in other words $T(P)$, as required. A similar argument covers the case of $L \boxtimes (M \boxtimes N)$. We leave further questions of naturality and coherence to the reader. \square

PROPOSITION A.19. *Let $H_a: \mathcal{A} \rightarrow \text{Sets}$ be the functor represented by a . Then H_a preserves products, so it can be regarded as an \mathcal{A} -module. Moreover, there are natural isomorphisms $H_a \boxtimes H_b \simeq H_{a \otimes b}$ and $\underline{\text{Hom}}(H_a, M) \simeq T_a M$ for all M .*

PROOF. It is tautological that H_a preserves products. There are natural identifications

$$\underline{\text{Hom}}(H_a, M)(b) = \text{Mod}(\mathcal{A})(H_a, T_b M) = (T_b M)(a) = M(a \otimes b) = (T_a M)(b),$$

which gives $\underline{\text{Hom}}(H_a, M) = T_a M$. This in turn gives natural isomorphisms

$$\begin{aligned} \text{Mod}(\mathcal{A})(H_a \otimes H_b, M) &= \text{Mod}(\mathcal{A})(H_a, \underline{\text{Hom}}(H_b, M)) = \text{Mod}(\mathcal{A})(H_a, T_b M) \\ &= (T_b M)(a) = M(a \otimes b) = \text{Mod}(\mathcal{A})(H_{a \otimes b}, M). \end{aligned}$$

By the Yoneda Lemma, this gives $H_a \otimes H_b \simeq H_{a \otimes b}$. \square

Bibliography

- [1] Francis Borceux, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994. Categories and structures. MR1313497 (96g:18001b)
- [2] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin, 1973. MR0420609 (54 #8623a)
- [3] M. Brun, *Witt vectors and equivariant ring spectra applied to cobordism*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 351–385, DOI 10.1112/plms/pdl010. MR2308231 (2008a:55008)
- [4] Morten Brun, *Witt vectors and Tambara functors*, Adv. Math. **193** (2005), no. 2, 233–256. MR2136887 (2005m:13034)
- [5] James Cranch, *Algebraic theories and $(\infty, 1)$ -categories*, available at [arXiv:1011.3243v1](https://arxiv.org/abs/1011.3243v1)[math.AT].
- [6] Brian Day, *On closed categories of functors*, Reports of the Midwest Category Seminar, IV, Lecture Notes in Mathematics, Vol. 137, Springer, Berlin, 1970, pp. 1–38. MR0272852 (42 #7733)
- [7] Andreas Dress, *Operations in representation rings*, Representation theory of finite groups and related topics (Proc. Sympos. Pure Math., Vol. XXI, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 39–45. MR0325740 (48 #4086)
- [8] Andreas W. M. Dress and Christian Siebeneicher, *The Burnside ring of profinite groups and the Witt vector construction*, Adv. in Math. **70** (1988), no. 1, 87–132. MR947758 (89m:20025)
- [9] Jesse Elliott, *Binomial rings, integer-valued polynomials, and λ -rings*, J. Pure Appl. Algebra **207** (2006), no. 1, 165–185. MR2244389 (2007f:13031)
- [10] ———, *Constructing Witt-Burnside rings*, Adv. Math. **203** (2006), no. 1, 319–363. MR2231048
- [11] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, DOI 10.1016/S0022-4049(02)00135-4. Special volume celebrating the 70th birthday of Professor Max Kelly. MR1935979 (2003h:55026)
- [12] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001)
- [13] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, 1971.
- [14] Michael A. Mandell and J. Peter May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), 108.
- [15] Hiroyuki Nakaoka, *Biset transformations of Tambara functors*, available at [arXiv:1105.0714](https://arxiv.org/abs/1105.0714).
- [16] ———, *On the fractions of semi-Mackey and Tambara functors*, available at [arXiv:1103.3991](https://arxiv.org/abs/1103.3991).
- [17] ———, *A generalization of The Dress construction for a Tambara functor, and polynomial Tambara functors*, available at [arXiv:1012.1911](https://arxiv.org/abs/1012.1911).
- [18] ———, *Ideals of Tambara functors*, available at [arXiv:1101.5982](https://arxiv.org/abs/1101.5982).
- [19] ———, *Tambara functors on profinite groups and generalized Burnside functors*, Comm. Algebra **37** (2009), no. 9, 3095–3151, DOI 10.1080/00927870902747605. MR2554193 (2011b:19002)
- [20] ———, *Tambaraization of a Mackey functor and its application to the Witt-Burnside construction*, Adv. Math. **227** (2011), no. 5, 2107–2143, DOI 10.1016/j.aim.2011.04.015. MR2803797
- [21] R. Steiner, *A canonical operad pair*, Mathematical Proceedings of the Cambridge Philosophical Society **86** (1979), 443–449.
- [22] D. Tambara, *On multiplicative transfer*, Comm. Algebra **21** (1993), no. 4, 1393–1420. MR1209937 (94a:19002)
- [23] Peter Webb, *Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces*, J. Pure Appl. Algebra **88** (1993), no. 1-3, 265–304, DOI 10.1016/0022-4049(93)90030-W. MR1233331 (94f:20101)
- [24] nLab contributors, *Lawvere theory*, available at <http://ncatlab.org/nlab/show/Lawvere+theory>.
- [25] ———, *Bicategory*, available at <http://ncatlab.org/nlab/show/bicategory>.