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FUNCTION SPACES AND DUALITY

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Introduction

This paper is devoted to a new approach to the duality in S-theory introduced by J. H. C. Whitehead and the author [10, 12, 14]. The duality as originally defined was based on imbedding X and its dual X' in a sphere S^n in such a way that each is an S-deformation retract of the complement of the other. If Y and Y' are similarly imbedded in S^n, there is a duality isomorphism D_n: $\{X, Y\} \approx \{Y', X'\}$ of the S-groups ($\{X, Y\}$ is, by definition, the limit with respect to k of the set of homotopy classes of maps $S^k X \to S^k Y$) having many of the properties one would expect (and hope for) in a duality. The construction of this isomorphism involves factoring a map into a composite of inclusion maps and retractions by deformation for each of which one knows what the dual map is, defining the dual of the map to be the composite of the duals of the factors, and then proving that the end result does not depend on the factorization and other choices involved in the construction. This method of defining the duality map is not explicit, and there seems to be no way of determining if a map $f: S^k X \to S^k Y$ represents an element of $\{X, Y\}$ corresponding (under D_n) to the element of $\{Y', X'\}$ represented by a map $f': S^{k'} Y' \to S^{k'} X'$ except to go back to the original construction of the duality.

The present paper presents a new treatment of this duality. It seems to be more natural and more general and gives more explicitly the relation between maps representing corresponding elements of the S-groups in question.

We start with the category of connected polyhedra having base points. If X and Y have base points x_0, y_0, respectively, we define $X \mathbin{⩕} Y$ to be the quotient of $X \times Y$ when $X \times y_0 \cup x_0 \times Y$ is collapsed to a single point. Then a duality map is a continuous map

$$u : X' \mathbin{⩕} X \longrightarrow S^n$$

for some n, such that the slant product $u^* s_n^* / z \in H^{n-q}(X)$ (s_n^* a generator of $H^n(S^n)$, $z \in H_q(X')$) induces an isomorphism[1]

$$\varphi_u : H_q(X') \longrightarrow H^{n-q}(X).$$

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[1] We shall always work with homology and cohomology modulo the point base. Thus, $H_q(X)$ is an abbreviation for $H_q(X, x_0)$ and $H^q(X)$ is an abbreviation for $H^q(X, x_0)$.

By (5.1) below, this is a generalization of the older concept of duality. u also induces a duality map

$$u_{p,q} : S^p X' \ast S^q X \longrightarrow S^{n+p+q}$$

for every $p \geqq 0$, $q \geqq 0$. Let $v : Y' \ast Y \to S^n$ be another duality map (for the same n). The main result, (5.9) below, asserts the existence of an isomorphism

$$D_n(u, v) : \{X, Y\} \approx \{Y', X'\}.$$

This isomorphism is characterized by the property (see (5.11) below) that for k, k' large enough if f: $S^k X \to S^k Y$, f': $S^{k'} Y' \to S^{k'} X'$ then $D_n\{f\} = \{f'\}$ if and only if the following diagram is homotopy commutative:

$$\begin{array}{ccc} S^{k'} Y' \ast S^{k'} X & \xrightarrow{1 \ast f} & S^{k'} Y' \ast S^k Y \\ {\scriptstyle f' \ast 1}\downarrow & & \downarrow{\scriptstyle v_{k',k}} \\ S^{k'} X' \ast S^k X & \xrightarrow{u_{k',k}} & S^{n+k+k'} \end{array}.$$

The fact that D_n is characterized by the homotopy commutativity of the above diagram makes possible a natural direct derivation of the main properties of the duality. This is done in §6.

The proof of the main theorem is given in §5. It is based on the concepts of spectrum and functional dual developed in the earlier sections. Spectra were introduced by Lima [6] in order to generalize S-theory. In S-theory one essentially replaces a space X by the sequence of spaces X, SX, $S^2X, \cdots$. By using spectra it is possible to extend this further by allowing sequences of spaces X_0, X_1, X_2, $\cdots$ together with maps $SX_n \to X_{n+1}$ for every n having certain convergence properties ((3.1), (3.2)). The definitions and basic properties of spectra are given in §3.

Given a polyhedron X there is associated a spectrum $\mathbf{F}(X)$ whose k^{th} space is the set of continuous maps $X \to S^k$ in the compact-open topology. In §4 the spectra $\mathbf{F}(X)$ are introduced and the "exponential law" for function spaces ((2.3) below) implies the existence of a duality isomorphism

$$D\colon \{Y, \mathbf{F}(X)\} \approx \{X, \mathbf{F}(Y)\}$$

as in (4.9) below. This duality underlies the duality $D_n(u, v)$ because if u: $X' \ast X \to S^n$ is a duality map there is a canonical equivalence of X' with the n^{th} suspension of $\mathbf{F}(X)$ (see (5.5) below).

Naturally the whole theory can be developed in relative form using carriers as in [14]. We have preferred to present only the absolute case in detail, but the final section indicates how the basic result appears in the relative form.

Most of the results are known [10, 12, 14], but (5.8) below is new. It implies that if $v\colon Y' \ast Y \to S^n$ is a duality map there is an isomorphism (see (6.9) below)

$$\Gamma\colon \{X, Z \ast Y'\} \approx \{X \ast Y, S^n Z\} .$$

In particular, let Y be a simply connected polyhedron with $H^i(Y) = 0$ except for $i = p$ (where $1 < p$, and we shall suppose $p \leqq n - 2$). Then we can take for Y' a simply connected polyhedron with $H_i(Y') = 0$ except for $i = n - p$. Let $G = H^p(Y) \approx H_{n-p}(Y')$. Then the isomorphism above implies the equivalence of two possible definitions of the S-group with coefficients in G, denoted by $\{X, Z; G\}$. We shall return to a consideration of these groups in a future publication.

1. Preliminaries

We shall be concerned exclusively with topological spaces X with base points x_0. By a *map* $f\colon X \to Y$ we shall mean a continuous function from X to Y preserving base points (so $fx_0 = y_0$). A *homotopy* between two maps will mean a homotopy relative to the base points. By a *polyhedron* we mean a finite CW-complex [17] with a vertex as base point.

We use the notation $[X, Y]$ to denote the set of homotopy classes of maps $X \to Y$. If $f\colon X \to Y$ then $[f]$ denotes the homotopy class of f. If $g\colon X \to X'$, $h\colon Y \to Y'$ we let $g^\#\colon [X', Y] \to [X, Y]$, $h_\#\colon [X, Y] \to [X, Y']$ denote the induced maps defined by $g^\#[f] = [fg]$, $h_\#[f] = [hf]$.

If X and Y are spaces the *sum* $X \vee Y$ will denote the subset $X \times y_0 \cup x_0 \times Y$ of $X \times Y$ with (x_0, y_0) as base point. It is the union of disjoint copies of X and Y in which the base points have been identified. It is easy to verify that, up to canonical homeomorphism, the operation of forming the sum is commutative and associative. If A is a closed subset of X containing x_0 we define X/A to be the quotient space obtained by identifying A to a single point (to be used as base point for X/A). We define the *reduced product* $X \ast Y$ to be the quotient space $X \times Y/X \vee Y$. If $x \in X$, $y \in Y$, then $x \ast y$ will denote the point of $X \ast Y$ obtained from (x, y) by the collapsing map $X \times Y \to X \ast Y$. Thus $x \ast y_0 = x_0 \ast y_0 = x_0 \ast y$ for all $x \in X$, $y \in Y$. If $f\colon X \to X'$, $g\colon Y \to Y'$ we use $f \vee g\colon X \vee Y \to X' \vee Y'$, $f \ast g\colon X \ast Y \to X' \ast Y$ for the corresponding maps.

The map $x \ast y \to y \ast x$ is a canonical homeomorphism of $X \ast Y$ onto $Y \ast X$. The canonical map $(x \ast y) \ast z \to x \ast (y \ast z)$ is not, in general, a homeomorphism of $(X \ast Y) \ast Z$ onto $X \ast (Y \ast Z)$; however, if two of X, Y, Z are compact Hausdorff spaces, it follows from [3; pp. 220–225] that it is a homeomorphism (because $(X \ast Y) \ast Z$ and $X \ast (Y \ast Z)$ are both quotient spaces of $X \times Y \times Z$). There is also a canonical homeomorphism of $X \divideontimes Y \vee X' \divideontimes Y$ onto $(X \vee X') \divideontimes Y$ induced by the inclusions $X \divideontimes Y \subset (X \vee X') \divideontimes Y$, $X' \divideontimes Y \subset (X \vee X') \divideontimes Y$. In the future we shall use a double headed arrow $\longleftrightarrow$ to denote a natural homeomorphism resulting from the commutative, associative, and distributive properties of the sum and reduced product. For example, $(X \divideontimes Y) \divideontimes Z \longleftrightarrow Y \divideontimes (Z \divideontimes X)$ denotes the map $(x \divideontimes y) \divideontimes z \to y \divideontimes (z \divideontimes x)$, which is a homeomorphism if two of X, Y, Z are compact Hausdorff.

If X and Y are CW-complexes with vertices as base points, then $X \vee Y$ is a CW-complex whose set of cells is the union of the sets of cells of X and of Y with the base vertices identified. If at least one of X, Y is locally finite, then $X \divideontimes Y$ is a CW-complex with cells $e \divideontimes e'$ where e, e' are cells of X, Y, respectively, different from the base vertices together with one more cell, the base vertex $x_0 \divideontimes y_0$ of $X \divideontimes Y$.

In the sequel we shall be mainly interested in spaces in which the base point is smoothly imbedded in a suitable sense. Following Puppe [9] we say x_0 is a *non-degenerate base point* of X if there exists a neighborhood U of x_0 and continuous maps D: $U \times I \to X$, u: $X \to I$ such that:

(1) $D(x, 0) = x$, $D(x_0, t) = x_0$, $D(x, 1) = x_0$

(2) $u(x_0) = 1$, $u(X - U) = 0$.

Puppe proved the following properties:

(1.1) There is no loss of generality in the definition if the neighborhood U is assumed to be closed.

(1.2) Any point of a CW-complex is a non-degenerate base point.

(1.3) If X and Y have non-degenerate base points, so do $X \vee Y$ and $X \divideontimes Y$.

We shall need the following additional properties:

LEMMA (1.4). *If x_0 is a non-degenerate base point of X there exists a homotopy h: $X \times I \to X$ and neighborhoods W, V of x_0 such that $\overline{W} \subset$ interior V and*

(1.5) $$h(x, 0) = x, \quad h(x_0, t) = x_0, \quad h(V \times 1) = x_0 .$$

PROOF. By (1.1) we choose a closed neighborhood U of x_0 and maps D, u satisfying (1), (2) above. Then define

$$h(x, t) = \begin{cases} x & \text{if } x \notin U \\ D(x, 2u(x)t) & \text{if } x \in U,\ u(x) \leqq 1/2 \\ D(x, t) & \text{if } x \in U,\ u(x) \leqq 1/2 . \end{cases}$$

Let $V = \{x \mid u(x) > 1/2\}$, $W = \{x \mid u(x) > 3/4\}$. Then all the conditions are satisfied for these choices of h, V, W.

Lemma (1.6). *If X and Y have non-degenerate base points, then the collapsing map*

$$k\colon (X\times Y, X\vee Y)\to (X\divideontimes Y, x_0\divideontimes y_0)$$

induces isomorphisms of the corresponding homology and cohomology groups[2].

Proof. By (1.4) we find $h\colon X\times I\to X$ and neighborhoods W, V of x_0 in X satisfying (1.5) and similarly $h'\colon Y\times I\to Y$ and neighborhoods W', V' of y_0 in Y also satisfying (1.5). Define

$$\bar{h}\colon (X\times Y)\times I\to X\times Y,\quad \bar{\bar{h}}\colon (X\divideontimes Y)\times I\to X\divideontimes Y$$

by $\bar{h}((x, y), t) = (h(x, t), h'(y, t))$, $\bar{\bar{h}}(x\divideontimes y, t) = h(x, t)\divideontimes h'(y, t)$. Let $\bar{f}\colon (X\times Y, X\times V'\cup V\times Y)\to (X\times Y, X\vee Y)$ and

$$\bar{\bar{f}}\colon (X\divideontimes Y, X\divideontimes V'\cup V\divideontimes Y)\to (X\divideontimes Y, x_0\divideontimes y_0)$$

be defined by $\bar{f}(x, y) = \bar{h}((x, y), 1)$, $\bar{\bar{f}}(x\divideontimes y) = \bar{\bar{h}}(x\divideontimes y, 1)$. Let

$$k'\colon (X\times Y, X\times V'\cup V\times Y)\to (X\divideontimes Y, X\divideontimes V'\cup V\divideontimes Y)$$

be the collapsing map and let

$$i\colon (X\times Y, X\vee Y)\subset (X\times Y, X\times V'\cup V\times Y),$$
$$j\colon (X\divideontimes Y, x_0\divideontimes y_0)\subset (X\divideontimes Y, X\divideontimes V'\cup V\divideontimes Y)$$

be inclusion maps. Then we have the commutative diagram

$$\begin{array}{ccc}
H_q(X\times Y, X\vee Y) & \xrightarrow{k_*} & H_q(X\divideontimes Y, x_0\divideontimes y_0)\\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*}\\
H_q(X\times Y, X\times V'\cup V\times Y) & \xrightarrow{k'_*} & H_q(X\divideontimes Y, X\divideontimes V'\cup V\divideontimes Y)\\
\downarrow{\scriptstyle \bar{f}_*} & & \downarrow{\scriptstyle \bar{\bar{f}}_*}\\
H_q(X\times Y, X\vee Y) & \xrightarrow{k_*} & H_q(X\divideontimes Y, x_0\divideontimes y_0)\,.
\end{array}$$

In this diagram $\bar{f}_* i_*$ is just the homomorphism induced by $\bar{h}|(X\times Y)\times 1$ regarded as a map of $(X\times Y, X\vee Y)$ into itself. Since $\bar{h}$ is a homotopy between the identity map of $(X\times Y, X\vee Y)$ and this map (note that

[2] Here and later when we do not specify a homology (or cohomology) theory we mean any one satisfying the Eilenberg-Steenrod axioms [4]. We assume that all pairs for which we have to consider homology groups are admissible for the theory. Though some of the results are valid more generally we shall always assume the coefficient group of the theory to be the integers.

$\bar{h}((X \vee Y) \times I) \subset X \vee Y)$, it follows that $\bar{f}_* i_*$ is the identity map. Similarly $\bar{\bar{f}}_* j_*$ is the identity map of $H_q(X \divideontimes Y, x_0 \divideontimes y_0)$.

Now $X \times W' \cup W \times Y$ is an open subset of $X \times Y$ whose closure, $X \times \overline{W}' \cup \overline{W} \times Y$, is contained in the interior of $X \times V' \cup V \times Y$. Similarly $X \divideontimes W' \cup W \divideontimes X$ is an open subset of $X \divideontimes Y$ whose closure is contained in the interior of $X \divideontimes V' \cup V \divideontimes Y$. In the commutative diagram (where k'' is the appropriate collapsing map, j', j'' are excisions, and $A = X \times W' \cup W \times Y$, $B = X \times V' \cup V \times Y$, $C = X \divideontimes W' \cup W \divideontimes Y$, $D = X \divideontimes V' \cup V \divideontimes Y$)

$$\begin{array}{ccc} H_q(X \times Y - A, B - A) & \xrightarrow{j'_*} & H_q(X \times Y, B) \\ {\scriptstyle k''_*}\big\downarrow & & \big\downarrow{\scriptstyle k'_*} \\ H_q(X \divideontimes Y - C, D - C) & \xrightarrow{j''_*} & H_q(X \divideontimes Y, D) \end{array},$$

it follows that j'_*, j''_* are isomorphisms. Since k'' is a homeomorphism, k''_* is also an isomorphism so, by the commutativity of the diagram, k'_* is an isomorphism.

Returning to the larger diagram considered earlier we have shown that the two vertical composites are identities and the middle horizontal map k'_* is an isomorphism. It follows purely formally from these properties and the commutativity of the diagram that k_* is also an isomorphism.

A similar argument applies for cohomology giving the result.

It follows from (1.6) and the Künneth theorem that if X and Y have non-degenerate base points we have homomorphisms

$$H_q(X) \otimes H_q(Y) \longrightarrow H_{p+q}(X \divideontimes Y)$$
$$H^p(X) \otimes H^q(Y) \longrightarrow H^{p+q}(X \divideontimes Y)\ .$$

If $z \in H_p(X)$, $z' \in H_q(Y)$, we let $z \divideontimes z'$ denote the corresponding element of $H_{p+q}(X \divideontimes Y)$. Similarly if $u \in H^p(X)$, $u' \in H^q(Y)$, $u \divideontimes u'$ will denote the corresponding element of $H^{p+q}(X \divideontimes Y)$. If we let $\langle u, z \rangle$ denote the value of the cohomology class u on the homology class z, then we see that

$$\langle u \divideontimes u', z \divideontimes z' \rangle = \langle u, z \rangle \cdot \langle u', z' \rangle\ .$$

Let I denote the unit interval with 0 as base point. Then the *cone* TX over X is defined to be $X \divideontimes I$, and X is imbedded in TX by the map $x \to x \divideontimes 1$. Let S^1 denote I with 0 and 1 identified to a single point, denoted by 0 and used as base point for S^1. Then the *suspension* SX is defined to be $X \divideontimes S^1$. By iteration we define

$$S^p X = S(S^{p-1} X) \qquad \text{for } p > 1\ .$$

There are canonical homeomorphisms of STX with TSX and, inductively, of S^pTX with TS^pX by means of which we shall identify these spaces.

If $f: X \to Y$ we define $Tf: TX \to TY$, $S^pf: S^pX \to S^pY$ to be the induced maps (e.g., Tf is defined by $Tf(x \divideontimes t) = fx \divideontimes t$ for $x \in X, t \in I$). In this way T and S are functors. It is clear that if X or Y is compact there are canonical homeomorphisms

$$S(X \divideontimes Y) \longleftrightarrow SX \divideontimes Y \longleftrightarrow X \divideontimes SY .$$

There is also a canonical homeomorphism

$$TX/X \approx SX .$$

LEMMA (1.7). *If X has a non-degenerate base point the collapsing map k: $(TX, X) \to (SX, x_0)$ induces isomorphisms of all the homology and cohomology groups.*

PROOF. Let $h: X \times I \to X$ and W, V satisfy (1.5). Define $\bar{h}: TX \times I \to TX$ by

$$\bar{h}((x \divideontimes t), s) = \begin{cases} h(x, s) \divideontimes (t + st) & \text{if } 0 \leqq t \leqq 1/2 \\ h(x, s) \divideontimes (s + t - st) & \text{if } 1/2 \leqq t \leqq 1 . \end{cases}$$

Then letting $A = V \divideontimes I \cup X \divideontimes [1/2, 1] \subset TX$ we see that

$$\bar{h}(x \divideontimes t, 0) = x \divideontimes t, \quad \bar{h}(X \times I) \subset X, \quad \bar{h}(A \times 1) \subset X .$$

Let $\bar{f}$: $(TX, A) \to (TX, X)$ be the map defined by $\bar{h} | TX \times \mathrm{I}$ and let i: $(TX, X) \subset (TX, A)$. Then $\bar{h}$ is a homotopy between the identity map of (TX, X) and $\bar{f}i$ so the composite

$$H_q(TX, X) \xrightarrow{i_*} H_q(TX, A) \xrightarrow{\bar{f}_*} H_q(TX, X)$$

is the identity.

Passing to the quotient by X, the map $\bar{h}$ defines a map $\bar{\bar{h}}$: $(TX/X) \times I \to TX/X$ and $\bar{f}$ defines a map $\bar{\bar{f}}$: $(TX/X, A/X) \to (TX/X, x_0)$ such that, if j: $(TX/X, x_0) \subset (TX/X, A/X)$, then $\bar{\bar{h}}$ is a homotopy between the identity map of $(TX/X, x_0)$ and $\bar{\bar{f}}j$. Then we have the commutative diagram (where the vertical maps are induced by appropriate collapsing maps)

$$\begin{array}{ccccc} H_q(TX, X) & \xrightarrow{i_*} & H_q(TX, A) & \xrightarrow{\bar{f}_*} & H_q(TX, X) \\ \downarrow k_* & & \downarrow k'_* & & \downarrow k_* \\ H_q(TX/X, x_0) & \xrightarrow{j_*} & H_q(TX/X, A/X) & \xrightarrow{\bar{\bar{f}}_*} & H_q(TX/X, x_0) \end{array} ,$$

and the composite across each row is the identity map. We can excise

$W \ltimes I \cup X \ltimes (3/4, 1]$ from the pair (TX, A) and $(W \ltimes I \cup X \ltimes (3/4, 1])/X$ from the pair $(TX/X, A/X)$ to obtain identical pairs. Therefore, as in (1.6) k'_* is an isomorphism so it follows (again as in (1.6)) that k_* is an isomorphism which completes the proof.

Since TX is contractible to x_0 the map

$$\partial\colon H_{q+1}(TX, X) \longrightarrow H_q(X)$$

is an isomorphism[1]. If X has a non-degenerate base point, it follows from (1.7) that we have an isomorphism

$$k_*\colon H_{q+1}(TX, X) \longrightarrow H_{q+1}(SX) .$$

The composite $k_*\partial^{-1}$ will be denoted by

$$S\colon H_q(X) \longrightarrow H_{q+1}(SX)$$

and is an isomorphism if X has a non-degenerate base point. Similarly we let

$$S\colon H^q(X) \longrightarrow H^{q+1}(SX)$$

denote the composite

$$H^q(X) \xrightarrow{\delta} H^{q+1}(TX, X) \xrightarrow{k^{*-1}} H^{q+1}(SX)$$

defined when X has a non-degenerate base point. Then we see that if $z \in H_q(X)$, $u \in H^q(X)$,

$$\langle Su, Sz\rangle = \langle u, z\rangle .$$

We define S^0 to be the two point space consisting of 0 and 1 with 0 as base point. For $n \geqq 1$ let S^n be defined inductively by $S^n = S(S^{n-1})$ (for $n = 1$ we have $S(S^0) = S^0 \ltimes S^1$, which is homeomorphic to S^1 by the map $1 \ltimes t \to t$, so this notation is consistent with the earlier definition of S^1). Since S^n is a CW-complex it has a non-degenerate base point so

$$S\colon H_n(S^n) \approx H_{n+1}(S^{n+1}) \quad\text{and}\quad S\colon H^n(S^n) \approx H^{n+1}(S^{n+1}) .$$

We let $s_0 \in H_0(S^0)$ be the integral homology class which is represented by the point 1 with value 1 and the point 0 with value 0. We define $s_n \in H_n(S^n)$ inductively by

$$s_n = S(s_{n-1}) \qquad\qquad \text{for } n \geqq 1 .$$

We also define $s_n^* \in H^n(S^n)$ by the condition $\langle s_n^*, s_n\rangle = 1$. Then $Ss_{n-1}^* = s_n^*$ for $n \geqq 1$.

Inductively, we define natural homeomorphisms $S^pX \longleftrightarrow X \ltimes S^p$ for $p \geqq 0$ in such a way that we have commutative diagrams

$$\begin{array}{ccc} S^{p+1}X = S^pX \divideontimes S^1 & \longleftrightarrow & (X \divideontimes S^p) \divideontimes S^1 \\ \updownarrow & & \updownarrow \\ X \divideontimes S^{p+1} & = & X \divideontimes (S^p \divideontimes S^1)\,. \end{array}$$

It follows that we have natural homeomorphisms

$$S^p(S^qX) \longleftrightarrow S^{p+q}X$$

and that we have a commutative diagram

$$\begin{array}{ccc} S^p(S^qX) & \longleftrightarrow & S^q(S^pX) \\ \updownarrow & & \updownarrow \\ S^{p+q}X & \longleftrightarrow & S^{p+q}X \\ \updownarrow & & \updownarrow \\ S \divideontimes S^{p+q} & \xrightarrow{\rho} & X \divideontimes S^{p+q} \end{array}$$

where $\rho = 1 \divideontimes \varepsilon$ where $1\colon X \subset X$ and $\varepsilon\colon S^{p+q} \to S^{p+q}$ has degree $(-1)^{pq}$. We shall have occasion later to use these natural homeomorphisms and shall have to know the degrees of the maps of the spheres involved. Given a space of the form $X \divideontimes S^n$ we let $-1\colon X \divideontimes S^n \to X \divideontimes S^n$ denote a map of the form $1 \divideontimes \varepsilon$ where $\varepsilon\colon S^n \to S^n$ has degree -1. Two such maps are homotopic, and when we refer to such a map the homotopy type of the map will be the only thing of importance in the discussion.

LEMMA (1.8). *Under the homeomorphism $S^pX \longleftrightarrow X \divideontimes S^p$ if $z \in H_q(X)$ then $S^p(z)$ corresponds to $(-1)^{pq}z \divideontimes s_p$.*

PROOF. It clearly suffices to verify the lemma for $p = 1$ and use induction on p. Let σ denote the 1-cell of I oriented so that $\partial\sigma = 1 - 0$. Then $\sigma \in H_1(I, 0 \cup 1)$ and $(-1)^q z \divideontimes \sigma \in H_{q+1}(TX, X)$ is such that $\partial((-1)^q z \divideontimes \sigma) = z$. Since the natural map $I \to S^1$ sends σ into a representative of s_1, we have

$$Sz = k_*((-1)^q z \divideontimes \sigma) = (-1)^q z \divideontimes s_1\,.$$

2. Function spaces

If X and Y are topological spaces with base points, we let $F(X, Y)$ denote the space of maps $X \to Y$ (sending x_0 into y_0) topologized by the compact-open topology and with the constant map $\omega_0\colon X \to y_0$ as base point. This topology has the following properties [2, 5, 8, 15]:

(2.1) If X is locally compact Hausdorff, the evaluation map

$$E\colon F(X, Y) \divideontimes X \to Y$$

defined by $E(f \divideontimes x) = f(x)$ is continuous.

(2.2) For locally compact Hausdorff X and arbitrary Z a map $g: Z \to F(X, Y)$ is continuous if and only if $g' = E \circ (g ⋇ 1): Z ⋇ X \to Y$ is continuous (where $1: X \subset X$).

(2.3) For locally compact Hausdorff X and arbitrary Z the map $g \to g'$ of (2.2) is a homeomorphism of $F(Z, F(X, Y))$ onto $F(Z ⋇ X, Y)$.

We shall also need the following.

LEMMA (2.4). *If X is compact and Y has a non-degenerate base point then $F(X, Y)$ also has a non-degenerate base point.*

PROOF. Let U be a neighborhood of y_0 and let maps $D: U \times I \to Y$, $u: Y \to I$ be given satisfying the conditions (1), (2) guaranteed by the non-degeneracy of y_0 in Y. Let $U' = \{f \in F(X, Y) | fX \subset U\}$. Then $U' \supset \{f | fX \subset \text{interior } U\}$, which is an open set containing the constant map ω_0, so U' is a neighborhood of ω_0. Define $D': U' \times I \to F(X, Y)$ by $D'(f, t)(x) = D(f(x), t)$. We show D' is continuous. If (C, V) denotes the set $\{f \in F(X, Y) | fC \subset V\}$ where C is compact in X and V is open in Y, then (C, V) forms a sub-base for the topology on $F(X, Y)$. If $D'(f_0, t_0) \in (C, V)$ then $D(f_0(C), t_0) \subset V$ so there is an open neighborhood W of f_0C and an open neighborhood N of t_0 with $D(W \times N) \subset V$. Then $D'((C, W) \times N) \subset (C, V)$ proving D' is continuous. Furthermore,

$$\begin{array}{llll} D'(f, 0)(x) = D(fx, 0) = fx & \text{so } D'(f, 0) = f \\ D'(\omega_0, t)(x) = D(y_0, t) = y_0 & \text{so } D'(\omega_0, t) = \omega_0 \\ D'(f, 1)(x) = D(f(x), 1) = y_0 & \text{so } D'(U' \times 1) = \omega_0 , \end{array}$$

and D' has all the requistite properties.

Define $u': F(X, Y) \to I$ by $u' f = \inf_x uf(x)$. Then $u'(\omega_0) = 1$, $u'(f) = 0$ if $fX \not\subset U$, so u' has also the requisite properties and U', D', u' show that x_0 is a non-degenerate base point of $F(X, Y)$.

There are natural maps $\lambda: SF(X, Y) \to F(X, SY)$, $\mu: F(X, Y) \to F(SX, SY)$ defined by

$$(\lambda(f ⋇ t))(x) = fx ⋇ t, \quad \mu f(x ⋇ t) = fx ⋇ t$$

for $f \in F(X, Y)$, $x \in X$, $t \in S^1$. If X is compact and $E: F(X, Y) ⋇ X \to Y$, $E': F(X, SY) ⋇ X \to SY$, $E'': F(SX, SY) ⋇ SX \to SY$ denote the appropriate evaluation maps, we have the commutative diagrams

$$(2.5) \qquad \begin{array}{ccc} SF(X, Y) ⋇ X & \longleftrightarrow & S(F(X, Y) ⋇ X) \\ \downarrow{\scriptstyle \lambda ⋇ 1} & & \downarrow{\scriptstyle SE} \\ F(X, SY) ⋇ X & \xrightarrow{E'} & SY \end{array}$$

$$\begin{array}{ccc} F(X, Y) \mathbin{\#} SX & \longleftrightarrow & S(F(X, Y) \mathbin{\#} X) \\ \mu \mathbin{\#} 1 \downarrow & & \downarrow SE \\ F(SX, SY) \mathbin{\#} SX & \xrightarrow{E''} & SY \end{array} \tag{2.6}$$

$$\begin{array}{ccccc} SF(X, Y) & \xrightarrow{\lambda} & F(X, SY) & \xrightarrow{\mu} & F(SX, S(SY)) \\ S\mu \downarrow & & & & \nearrow \rho \\ SF(SX, SY) & \xrightarrow{\lambda} & F(SX, S(SY)) , & & \end{array} \tag{2.7}$$

where ρ is induced by the map $(y \mathbin{\#} t) \mathbin{\#} t' \to (y \mathbin{\#} t') \mathbin{\#} t$ of $S(SY)$ into itself.

Assume X is a connected polyhedron of dimension $\leqq m$. Assume $n > m$ and let E: $F(X, S^n) \mathbin{\#} X \to S^n$ be the evaluation map and $s_n^* \in H^n(S^n)$ be the standard generator. Then $E^*s_n^* \in H^n(F(X, S^n) \mathbin{\#} X)$ and if $z \in H_q(F(X, S^n))$ then the slant product [7, 11] $E^*s_n^*/z \in H^{n-q}(X)$ is defined. If we define

$$\varphi\colon H_q(F(X, S^n)) \longrightarrow H^{n-q}(X)$$

by $\varphi(z) = E^*s_n^*/z$, then Moore [7; Theorem 3] has shown:

(2.8) *φ is an isomorphism of the reduced groups for $q < 2(n-m)$.*

Let f: $X \to X'$ and define $\bar{f}$: $F(X', S^n) \to F(X, S^n)$ by $\bar{f}(\omega) = \omega f$ for $\omega \in F(X', S^n)$. Then we have a commutative diagram

$$\begin{array}{ccc} F(X', S^n) \mathbin{\#} X & \xrightarrow{1 \mathbin{\#} f} & F(X', S^n) \mathbin{\#} X' \\ \bar{f} \mathbin{\#} 1 \downarrow & & \downarrow E' \\ F(X, S^n) \mathbin{\#} X & \xrightarrow{E} & S^n \end{array} \tag{2.9}$$

This together with naturality properties of the slant product [11; (11.1)] implies the commutativity of

$$\begin{array}{ccc} H_q(F(X', S^n)) & \xrightarrow{\varphi} & H^{n-q}(X') \\ \bar{f}_* \downarrow & & \downarrow f^* \\ H_q(F(X, S^n)) & \xrightarrow{\varphi} & H^{n-q}(X) . \end{array} \tag{2.10}$$

The slant product has the following easily verified property. Let $w \in H^r(X)$, $w' \in H^n(Y \mathbin{\#} Z)$, $z \in H_r(X)$, $z' \in H_q(Y)$, then $w \mathbin{\#} w' \in H^{r+n}(X \mathbin{\#} (Y \mathbin{\#} Z))$, $z \mathbin{\#} z' \in H_{r+q}(X \mathbin{\#} Y)$ and

$$w \mathbin{\#} w'/z \mathbin{\#} z' = \langle w, z\rangle w'/z' . \tag{2.11}$$

(2.11) together with [11; (11.1)], (1.8) and the commutativity of (2.5) gives the commutativity of

$$\begin{array}{ccc} H_q(F(X, S^n)) & \xrightarrow{S} & H_{q+1}(SF(X, S^n)) \\ \varphi\downarrow & & \downarrow\lambda_* \\ H^{n-q}(X) & \xleftarrow{\varphi} & H_{q+1}(F(X, S^{n+1})) \end{array} , \tag{2.12}$$

and (2.11), [11; (11.1)], (1.8) and the commutativity of (2.6) give the commutativity of

$$\begin{array}{ccc} H_q(F(X, S^n)) & \xrightarrow{\mu_*} & H_q(F(SX, S^{n+1})) \\ \varphi\downarrow & & \downarrow(-1)^q\varphi \\ H^{n-q}(X) & \xrightarrow{S} & H^{n-q+1}(SX) \end{array} . \tag{2.13}$$

3. Direct spectra

By a (*direct*) *spectrum* $\mathbf{X}=(X_k, \rho_k)$ we shall mean a sequence of topological spaces (with non-degenerate base points) X_k for $k = 0, 1, \cdots$ and continuous mappings ρ_k: $SX_k \to X_{k+1}$ for $k \geqq 0$ such that[3]:

(3.1) There exists an integer Q (positive or negative) such that $\pi_{q+k}(X_k) = 0$ for all $k \geqq 0$ and all $q \leqq Q$.

(3.2) For any (positive or negative) integer q there exists an integer N_q such that for $k \geqq N_q$

$$\rho_{k*}: H_{q+k+1}(SX_k) \approx H_{q+k+1}(X_{k+1}) .$$

For any positive or negative integer q the groups $H_{q+k}(X_k)$ together with the homomorphisms

$$H_{q+k}(X_k) \xrightarrow{S} H_{q+k+1}(SX_k) \xrightarrow{\rho_{k*}} H_{q+k+1}(X_{k+1})$$

form a direct system of groups. We define $H_q(\mathbf{X})$ to be the limit of this sequence. It follows from (3.2) above that $H_q(\mathbf{X}) \approx H_{q+k}(X_k)$ for $k \geqq N_q$. It follows from (3.1) that $H_{q+k}(X_k) = 0$ for $q \leqq Q$ so $H_q(\mathbf{X}) = 0$ for $q \leqq Q$. Then (3.2) implies that for given q

$$\rho_{k*}S: H_{j+k}(X_k) \approx H_{j+k+1}(X_{k+1})$$

for all $j \leqq q$ and all sufficiently large k (merely choose k larger than N_{Q+1}, $N_{Q+2}, \cdots, N_q$).

If X is a space with a non-degenerate base point, there is a spectrum $\mathrm{S}(X)$ consisting of the sequence S^kX for $k \geqq 0$ and the identity maps ρ_k: $S(S^kX) \subset S^{k+1}X$.

[3] This concept of direct spectrum is a slight modification of the one used by Lima [6].

To obtain a more interesting example of a spectrum let X be a connected polyhedron and let Y be a space with a non-degenerate base point. Let $\mathbf{F}(X, Y)$ denote the sequence of spaces $F(X, S^kY)$ for $k \geqq 0$ and the sequence of maps ρ_k: $SF(X, S^kY) \to F(X, S^{k+1}Y)$ each defined to be the map λ of §2. We show $\mathbf{F}(X, Y)$ is a spectrum. By (2.3) we have

$$[S^q, F(X, S^kY)] \approx [S^q \divideontimes X, S^kY] ,$$

and we know $[S^q \divideontimes X, S^kY] = 0$ if $k > q + \dim X$. Hence $\pi_q(F(X,S^kY)) = 0$ for $q < k - \dim X$ so (3.1) is satisfied by taking $Q = - \dim X - 1$.

To show that (3.2) is satisfied we need some preparation. Let

$$E_k: F(X, S^kY) \divideontimes X \to S^kY$$

be the evaluation map. If $f: S^j \to F(X, S^kY)$ represents an element $[f] \in [S^j, F(X, S^kY)]$ we define $\psi_k[f] \in [S^jX, S^kY]$ by $\psi_k[f] = [g]$ where g is the composite

$$S^jX \longleftrightarrow S^j \divideontimes X \xrightarrow{f \divideontimes 1} F(X, S^kY) \divideontimes X \xrightarrow{E_k} S^kY .$$

It follows from (2.3) that ψ_k is well defined and is a $1-1$ correspondence

$$\psi_k: [S^j, F(X, S^kY)] \approx [S^jX, S^kY] .$$

Consider the diagram

$$(3.3) \qquad \begin{array}{ccc} [S^j, F(X, S^kY)] & \xrightarrow{\psi_k} & [S^jX, S^kY] \\ \downarrow S & & \\ [S^{j+1}, SF(X, S^kY)] & & \downarrow S \\ \downarrow \lambda_\# & & \\ [S^{j+1}, F(X, S^{k+1}Y)] & \xrightarrow{\psi_{k+1}} & [S^{j+1}X, S^{k+1}Y] . \end{array}$$

This diagram is commutative in view of the commutativity of (2.5) and the definitions of ψ_k, ψ_{k+1}.

In (3.3) the maps ψ_k, ψ_{k+1} are $1-1$ by (2.3). By the suspension theorem [13; (7.2)] the right hand vertical map is a $1-1$ correspondence if $j \leqq 2k - 2 - \dim X$ (because S^kY is $(k-1)$-connected) and the left hand vertical map S is a $1-1$ correspondence if $j \leqq 2k - 2 - \dim X$ (because $F(X, S^kY)$ is $(k - \dim X - 1)$-connected). Therefore, the commutativity of (3.3) implies that $\lambda_\#$ is an isomorphism

$$\lambda_\#: \pi_{j+1}(SF(X, S^kY)) \approx \pi_{j+1}(F(X, S^{k+1}Y)) \qquad \text{for } j \leqq 2k - 2 - 2\dim X .$$

It follows [16] that for $j \leqq k - 3 - 2\dim X$ we have isomorphisms

$$\lambda_*: H_{k+j+1}(SF(X, S^k Y)) \approx H_{k+j+1}(F(X, S^{k+1} Y)) .$$

Hence, (3.2) is satisfied by $N_q = q + 3 + 2 \dim X$ so $\mathbf{F}(X, Y)$ is a spectrum.

If $\mathbf{X} = (X_k, \rho_k)$ is any spectrum, we define its *suspension* $S\mathbf{X}$ to be the spectrum consisting of the sequence of spaces whose k^{th} term is X_{k+1} and the sequence of maps whose k^{th} term is the map $\rho_{k+1}: SX_{k+1} \to X_{k+2}$. Inductively, we define

$$S^p\mathbf{X} = S(S^{p-1}\mathbf{X}) \qquad \text{for } p > 1.$$

There are isomorphisms

$$S: H_q(\mathbf{X}) \approx H_{q+1}(S\mathbf{X})$$

defined by passing to the limit with the homomorphisms

$$H_{k+q}(X_k) \xrightarrow{S} H_{k+q+1}(SX_k) \xrightarrow{\rho_{k*}} H_{k+q+1}(X_{k+1}) ,$$

where the first map is always an isomorphism (because X_k has a non-degenerate base point), and the second map is an isomorphism for k large enough by (3.2).

Let Y be a polyhedron and let $\mathbf{X} = (X_k, \rho_k)$ be a spectrum. We define $\{Y, \mathbf{X}\}$ to be the direct limit of the groups[4] $[S^k Y, X_k]$ relative to the homomorphisms

$$[S^k Y, X_k] \xrightarrow{S} [S^{k+1} Y, SX_k] \xrightarrow{\rho_{k\#}} [S^{k+1} Y, X_{k+1}] .$$

Since X_k is $(Q + k)$-connected by (3.1) and $\dim S^k Y = k + \dim Y$, it follows [13; (7.2)] that for $k \geqq \dim Y - 2Q$ we have the isomorphism

$$S: [S^k Y, X_k] \approx [S^{k+1} Y, SX_k] .$$

We also know that for sufficiently large k the map ρ_k induces isomorphisms

$$\rho_{k*}: H_{j+k+1}(SX_k) \approx H_{j+k+1}(X_{k+1}) \quad \text{for } j \leqq \dim Y + 1 .$$

For sufficiently large k, SX_k and X_{k+1} are simply-connected (by 3.1) so the above condition implies

$$\rho_{k\#}: \pi_{j+k+1}(SX_k) \approx \pi_{j+k+1}(X_{k+1}) \qquad \text{for } j \leqq \dim Y ,$$

which, in turn, implies that we have a 1 – 1 correspondence

$$\rho_{k\#}: [Z, SX_k] \approx [Z, X_{k+1}]$$

for any polyhedron Z with $\dim Z \leqq k + \dim Y + 1$. In particular,

$$\rho_{k\#}: [S^{k+1} Y, SX_k] \approx [S^{k+1} Y, X_{k+1}]$$

[4] These are track groups [1] which are defined for $k \geqq 1$ and are abelian for $k \geqq 2$.

for k large enough. Combining the above isomorphisms we see that $\{Y, \mathbf{X}\} \approx [S^k Y, X_k]$ for k large enough.

For any (positive or negative) integer p we define

$$\{Y, \mathbf{X}\}_p = \begin{cases} \{S^p Y, \mathbf{X}\} & \text{if } p \geqq 0 \\ \{Y, S^{-p}\mathbf{X}\} & \text{if } p \leqq 0 . \end{cases}$$

These groups are attained by $[S^{k+p} Y, X_k]$ for k large enough. We also see that $\{Y, S\mathbf{X}\}_p = \{Y, \mathbf{X}\}_{p-1}$ and $\{SY, \mathbf{X}\}_p = \{Y, \mathbf{X}\}_{p+1}$. If $f\colon S^{p+k} Y \to X_k$ we let $\{f\} \in \{Y, \mathbf{X}\}_p$ denote the element determined by f. If X is a space with a non-degenerate base point, we have already defined the spectrum $\mathbf{S}(X)$ and we define $\{Y, X\}_p$ by

$$\{Y, X\}_p = \{Y, \mathbf{S}(X)\}_p .$$

This is the same as the S-group defined in [12; p. 66].

THEOREM (3.4). *Let X and Y be polyhedra and let Z be a space with a non-degenerate base point. There is an isomorphism*

$$\Lambda\colon \{X, \mathbf{F}(Y, Z)\} \approx \{X \mathbin{\#} Y, Z\}$$

such that if $f\colon S^k X \to F(Y, S^k Z)$ represents $\{f\} \in \{X, \mathbf{F}(Y, Z)\}$ then $\Lambda\{f\}$ is represented by the composite

$$S^k(X \mathbin{\#} Y) \longleftrightarrow S^k X \mathbin{\#} Y \xrightarrow{f \mathbin{\#} 1} F(Y, S^k Z) \mathbin{\#} Y \xrightarrow{E} S^k Z .$$

PROOF. It follows from (2.3) that the map sending f into the above composite, call it $\bar{f}$, is a homeomorphism

$$F(S^k X, F(Y, S^k Z)) \approx F(S^k(X \mathbin{\#} Y), S^k Z)$$

so induces a $1-1$ correspondence

$$\Lambda_k\colon [S^k X, F(Y, S^k Z)] \approx [S^k(X \mathbin{\#} Y), S^k Z] .$$

We show this correspondence is homomorphic for $k \geqq 2$. Let $f, g\colon S^k X \to F(Y, S^k Z)$. We can find closed subsets $A, B \subset S^k X$ such that $x_0 \in A \cap B$, $A \cup B = S^k X$, $f \simeq f_1$, $g \simeq g_1$ where $f_1 | A = \omega_0$, $g_1 | B = \omega_0$ (where ω_0 is the constant map $Y \to z_0$). Define $h\colon S^k X \to F(Y, S^k Z)$ by $h | A = g_1 | A$, $h | B = f_1 | B$. It follows from basic properties of the track addition [1] that $[f] + [g] = [h]$. Let $\bar{A}, \bar{B} \subset S^k(X \mathbin{\#} Y)$ correspond to $A \mathbin{\#} Y$, $B \mathbin{\#} Y$, respectively, under the homeomorphism $S^k(X \mathbin{\#} Y) \longleftrightarrow S^k X \mathbin{\#} Y$. Then $x_0 \mathbin{\#} y_0 \in \bar{A} \cap \bar{B}$, $\bar{A} \cup \bar{B} = S^k(X \mathbin{\#} Y)$ and $\bar{f}_1 | \bar{A} = z_0$, $\bar{g}_1 | \bar{B} = z_0$. Since $\bar{f} \simeq \bar{f}_1$ and $\bar{g} \simeq \bar{g}_1$, we see that $\Lambda_k[f] = [\bar{f}_1]$, $\Lambda_k[g] = [\bar{g}_1]$. Also $\bar{h} | \bar{A} = \bar{g}_1 | \bar{A}$, $\bar{h} | \bar{B} = \bar{f}_1 | \bar{B}$ so $[\bar{h}] = [\bar{f}_1] + [\bar{g}_1]$. Therefore,

$$\Lambda_k([f] + [g]) = \Lambda_k[h] = [\bar{h}] = \Lambda_k[f] + \Lambda_k[g] \ ,$$

and Λ_k is homomorphic.

To complete the proof we must show that the isomorphisms Λ_k are consistent on passing to the limit. Consider the diagram

$$\begin{array}{ccc}
S^{k+1}(X \divideontimes Y) & = & S(S^k(X \divideontimes Y)) \\
\updownarrow & & \updownarrow \\
S^{k+1}X \divideontimes Y & \longleftrightarrow & S(S^kX \divideontimes Y) \\
{\scriptstyle Sf \divideontimes 1}\downarrow & & \downarrow{\scriptstyle S(f \divideontimes 1)} \\
SF(Y, S^kZ) \divideontimes Y & \longleftrightarrow & S(F(Y, S^kZ) \divideontimes Y) \\
{\scriptstyle \lambda \divideontimes 1}\downarrow & & \downarrow{\scriptstyle SE} \\
F(Y, S^{k+1}Z) \divideontimes Y & \xrightarrow{E'} & S^{k+1}Z.
\end{array}$$

The first two squares are commutative by the naturality of the associativity and commutativity of reduced products, and the last square commutes by the commutativity of (2.5). The composite down the right hand side is $S\bar{f}$ while the composite along the other edge of the diagram is $\overline{\lambda(Sf)}$. Therefore,

$$S\Lambda_k[f] = [S\bar{f}] = [\overline{\lambda(Sf)}] = \Lambda_{k+1}\lambda_\# S[f] \ ,$$

so the maps Λ_k are consistent and define the desired isomorphism

$$\Lambda\colon \{X, \mathbf{F}(Y, Z)\} \approx \{X \divideontimes Y, Z\}$$

by passage to the limit.

Let $\mathbf{X} = (X_k, \rho_k)$, $\mathbf{X}' = (X'_k, \rho'_k)$ be spectra. By a *map* $\mathbf{f}\colon \mathbf{X} \to \mathbf{X}'$ we mean a sequence of maps $f_k\colon X_k \to X'_k$ such that commutativity holds in each diagram

$$\begin{array}{ccc}
SX_k & \xrightarrow{\rho_k} & X_{k+1} \\
{\scriptstyle Sf_k}\downarrow & & \downarrow{\scriptstyle f_{k+1}} \\
SX'_k & \xrightarrow{\rho'_k} & X'_{k+1} \ .
\end{array}$$

It is clear such a map induces homomorphisms

$$\mathbf{f}_*\colon H_q(\mathbf{X}) \longrightarrow H_q(\mathbf{X}')$$

for every q and homomorphisms

$$\mathbf{f}_\#\colon \{Y, \mathbf{X}\}_p \longrightarrow \{Y, \mathbf{X}'\}_p$$

for every polyhedron Y and every p. $\mathbf{f}$ will be called a *weak equivalence* if $\mathbf{f}_\#$ is an isomorphism $\mathbf{f}_\#\colon \{Y, \mathbf{X}\}_p \approx \{Y, \mathbf{X}'\}_p$ for every polyhedron Y and every p.

THEOREM (3.5). *A map* $\mathbf{f}: \mathbf{X} \to \mathbf{X}'$ *is a weak equivalence if and only if it induces isomorphisms*

$$\mathbf{f}_*: H_q(\mathbf{X}) \approx H_q(\mathbf{X}') \qquad \textit{for every } q .$$

PROOF. Let m be a fixed integer. Choose N so that $N \geqq m - 2 \max(Q, Q')$ and such that for $k \geqq N$ we have isomorphisms

$$\left.\begin{array}{l} \rho_{k_*} S: H_{j+k}(X_k) \approx H_{j+k+1}(X_{k+1}) \\ \rho'_{k_*} S: H_{j+k}(X'_k) \approx H_{j+k+1}(X'_{k+1}) \end{array}\right\} \qquad \text{for } j \leqq m .$$

Such a choice of N is always possible in view of the comments following (3.2) and depends only on m (and $\mathbf{X}$, $\mathbf{X}'$). Then for $q \leqq m$ we have $H_q(\mathbf{X}) \approx H_{q+N}(X_N)$, $H_q(\mathbf{X}') \approx H_{q+N}(X'_N)$, and if Y is any polyhedron with dim $Y \leqq m + N$ then

$$\{Y, \mathbf{X}\}_{-N} \approx [Y, X_N], \quad \{Y, \mathbf{X}'\}_{-N} \approx [Y, X'_N] .$$

Therefore, $\mathbf{f}$ is a weak equivalence if and only if for every m, then with N as above, if Y is a polyhedron with dim $Y \leqq m + N$ then

(3.6) $$f_{N\#}: [Y, X_N] \approx [Y, X'_N] .$$

(3.6) implies

$$f_{N*}: H_{q+N}(X_N) \approx H_{q+N}(X'_N) \qquad \text{for } q \leqq m - 1$$

so $\mathbf{f}_*: H_q(\mathbf{X}) \approx H_q(\mathbf{X}')$ for $q \leqq m - 1$. Since m is arbitrary, the necessity is proved.

Conversely, if $\mathbf{f}_*: H_q(\mathbf{X}) \approx H_q(\mathbf{X}')$ for all q, then for fixed m if N is as before we have

$$f_{N*}: H_{q+N}(X_N) \approx H_{q+N}(X'_N) \qquad \text{for } q \leqq m .$$

This implies (3.6) if dim $Y < m + N$. Again since m is arbitrary, the sufficiency is proved.

If $\mathbf{X} = (X_k, \rho_k)$ is a spectrum, we define a spectrum $\mathbf{X}' = (X'_k, \rho'_k)$ by $X'_k = SX_k$ and $\rho'_k: S(SX_k) \to SX_{k+1}$ equals $S\rho_k$. There is a canonical map $\mathbf{f}: \mathbf{X}' \to S\mathbf{X}$ defined by $f_k = \rho_k$. Since $\mathbf{f}_*: H_q(\mathbf{X}') \approx H_q(S\mathbf{X})$ for all q, the map $\mathbf{f}$ is a weak equivalence. Hence, up to weak equivalence the suspension of a spectrum is just the spectrum of suspensions.

4. Functional duals

Given a polyedron X let $\mathbf{F}(X)$ denote the spectrum consisting of the sequence of function spaces $F(X, S^k)$ and maps $\rho_k: SF(X, S^k) \to F(X, S^{k+1})$ defined to be the map λ of §2. $\mathbf{F}(X)$ will be called the *functional dual* of X. Clearly $\mathbf{F}(X)$ is the same as $\mathbf{F}(X, S^0)$ defined in §3.

LEMMA (4.1). *If X is a connected polyhedron, there is an isomorphism*

$$\varphi\colon H_q(\mathbf{F}(X)) \approx H^{-q}(X)$$

such that commutativity holds in the diagram

$$\begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{\varphi_k} & H^{-q}(X) \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & H_q(\mathbf{F}(X)) & \end{array}$$

where ψ is the canonical map to the limit group.

PROOF. By (2.12) we have, for every n, a commutative diagram

$$\begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{\lambda_* S} & H_{k+q+1}(F(X, S^{k+1})) \\ & {\scriptstyle\varphi_k}\searrow \quad \swarrow{\scriptstyle\varphi_{k+1}} & \\ & H^{-q}(X)\ . & \end{array}$$

Therefore, the homomorphisms $\varphi_k\colon H_{k+q}(F(X, S^k)) \to H^{-q}(X)$ fit together to give a homomorphism of the limit group

$$\varphi\colon H_q(\mathbf{F}(X)) \longrightarrow H^{-q}(X)$$

commuting with the map $\psi\colon H_{k+q}(F(X, S^k)) \to H^{-q}(X)$.

By (2.8) φ_k is an isomorphism

$$\varphi_k\colon H_{k+q}(F(X, S^k)) \approx H^{-q}(X)$$

for $k + q < 2(k - \dim X)$ (or $k > q + 2\dim X$). Hence, for all q we have the isomorphism

$$\varphi\colon H_q(\mathbf{F}(X)) \approx H^{-q}(X)\ .$$

LEMMA (4.2). *There is an isomorphism*

$$\varphi\colon H_{q+1}(S\mathbf{F}(X)) \approx H^{-q}(X)$$

such that commutativity holds in the diagram

$$(4.3) \qquad \begin{array}{ccc} H_q(\mathbf{F}(X)) & \xrightarrow{S} & H_{q+1}(S\mathbf{F}(X)) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & H^{-q}(X)\ . & \end{array}$$

PROOF. By (2.12) each of the diagrams

$$\begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{\lambda_* S} & H_{k+q+1}(F(X, S^{k+1})) \\ & {\scriptstyle\varphi_k}\searrow \quad \swarrow{\scriptstyle\varphi_{k+1}} & \\ & H^{-q}(X) & \end{array}$$

is commutative. Hence, if we define φ: $H_{q+1}(S\mathbf{F}(X)) \to H^{-q}(X)$ by passing to the limit with the maps φ_{k+1}: $H_{k+q+1}(F(X, S^{k+1})) \to H^{-q}(X)$, we will obtain the commutativity of (4.3). Since S and φ: $H_q(\mathbf{F}(X)) \to H^{-q}(X)$ are isomorphisms in (4.3), so is φ: $H_{q+1}(S\mathbf{F}(X)) \to H^{-q}(X)$ and all is proved.

For a connected polyhedron X we define a map

$$\mathbf{f}\colon \mathbf{F}(X) \longrightarrow S\mathbf{F}(SX)$$

by the condition that f_k: $F(X, S^k) \to F(SX, S^{k+1})$ be the map defined by

$$(f_k\omega)(x \divideontimes t) = t \divideontimes \omega x \qquad \text{for } \omega \in F(X, S^k),\ x \in X,\ t \in S^1 .$$

It is easy to verify commutativity in the diagram

$$\begin{array}{ccc} SF(X, S^k) & \xrightarrow{\lambda} & F(X, S^{k+1}) \\ {\scriptstyle Sf_k}\downarrow & & \downarrow{\scriptstyle f_{k+1}} \\ SF(SX, S^{k+1}) & \xrightarrow{\lambda} & F(SX, S^{k+2}) \end{array}$$

so the maps f_k do define a map $\mathbf{f}$: $\mathbf{F}(X) \to S\mathbf{F}(SX)$.

LEMMA (4.4). *For a connected polyhedron X the map* $\mathbf{f}$: $\mathbf{F}(X) \to S\mathbf{F}(SX)$ *is a weak equivalence. Furthermore, commutativity holds in the diagram*

$$(4.5) \qquad \begin{array}{ccc} H_q(\mathbf{F}(X)) & \xrightarrow{\mathbf{f}_*} & H_q(S\mathbf{F}(SX)) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle (-1)^q\varphi} \\ H^{-q}(X) & \xrightarrow{S} & H^{-q+1}(SX) . \end{array}$$

PROOF. In order to prove the lemma it suffices to prove the commutativity of (4.5) because we know that all the homomorphisms in (4.5), except possibly for $\mathbf{f}_*$, are isomorphisms. Hence, commutativity of (4.5) would imply $\mathbf{f}_*$ is an isomorphism so it would follow from (3.5) that $\mathbf{f}$ is a weak equivalence.

To prove the commutativity of (4.5) it suffices to prove commutativity of

$$(4.6) \qquad \begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{f_{k*}} & H_{k+q}(F(SX, S^{k+1})) \\ {\scriptstyle \varphi_k}\downarrow & & \downarrow{\scriptstyle (-1)^q\varphi_k} \\ H^{-q}(X) & \xrightarrow{S} & H^{-q+1}(SX) \end{array}$$

for every k because (4.5) is the limit of the above diagrams. Let ρ: $S^{k+1} \to S^{k+1}$ be defined by

$$\rho(t_1 \divideontimes t_2 \divideontimes \cdots \divideontimes t_{k+1}) = t_2 \divideontimes \cdots \divideontimes t_{k+1} \divideontimes t_1 .$$

From the definition of f_k and μ of §2 we obtain a commutative diagram

$$\begin{array}{ccc} F(X, S^k) \wedge SX & \xrightarrow{\mu \wedge 1} & F(SX, S^{k+1}) \wedge SX \\ {\scriptstyle f_k \wedge 1}\downarrow & & \downarrow {\scriptstyle E} \\ F(SX, S^{k+1}) \wedge SX & \xrightarrow{E} S^{k+1} \xrightarrow{\rho} & S^{k+1} . \end{array}$$

It then follows from the naturality properties of the slant product and the fact that $\rho^* s_{k+1}^* = (-1)^k s_{k+1}^*$ (because ρ has degree $(-1)^k$) that we have a commutative diagram

$$\begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{\mu_*} & H_{k+q}(F(SX, S^{k+1})) \\ {\scriptstyle f_{k*}}\downarrow & & \downarrow {\scriptstyle \varphi_k} \\ H_{k+q}(F(SX, S^{k+1})) & \xrightarrow{(-1)^k \varphi_k} & H^{-q+1}(SX) . \end{array}$$

Therefore, the commutativity of (4.6) is equivalent to that of

$$\begin{array}{ccc} H_{k+q}(F(X, S^k)) & \xrightarrow{\mu_*} & H_{k+q}(F(SX, S^{k+1})) \\ {\scriptstyle \varphi_k}\downarrow & & \downarrow {\scriptstyle (-1)^{k+q}\varphi_k} \\ H^{-q}(X) & \xrightarrow{S} & H^{-q+1}(SX) , \end{array}$$

which is identical with the commutative diagram (2.13).

LEMMA (4.7). *Let X and Y be polyhedra. There is an isomorphism*

$$\Lambda\colon \{X, S^n\mathbf{F}(Y)\} \approx \{X \wedge Y, S^n\}$$

such that if $f\colon S^kX \to F(Y, S^{n+k})$ represents an element $\{f\} \in \{X, S^n\mathbf{F}(Y)\}$ then $\Lambda\{f\}$ is represented by the composite

$$S^k(X \wedge Y) \longleftrightarrow S^kX \wedge Y \xrightarrow{f \wedge 1} F(Y, S^{n+k}) \wedge Y \xrightarrow{E} S^{n+k} .$$

PROOF. Since $S^n\mathbf{F}(Y) = \mathbf{F}(Y, S^n)$, this is just a restatement of (3.4) for the special case $Z = S^n$.

Consider the map $\gamma_k\colon [S^k(Y \wedge X), S^{n+k}] \to [S^k(Y \wedge SX), S^{n+k+1}]$ which assigns to the homotopy class $[f]$, where $f\colon S^k(Y \wedge X) \to S^{n+k}$, the homotopy class of the composite

$$S^k(Y \wedge SX) \xrightarrow{g_k} S^{k+1}(Y \wedge X) \xrightarrow{Sf} S^{n+k+1}$$

where $g_k(y \wedge (x \wedge t) \wedge t_1 \wedge \cdots \wedge t_k) = (y \wedge x) \wedge t \wedge t_1 \cdots \wedge t_k$ (so g_k equals $(-1)^k$ times the canonical homeomorphism

$$S^k(Y \wedge SX) \to S(S^k(Y \wedge X)) = S^{k+1}(Y \wedge X)) .$$

γ_k is an isomorphism for k large enough because it equals the composite

$$[S^k(Y \wedge X), S^{n+k}] \xrightarrow{S} [S^{k+1}(Y \wedge X), S^{n+k+1}] \xrightarrow{g_k^\#} [S^k(Y \wedge SX), S^{n+k+1}] ,$$

and the first map is an isomorphism for $k \geqq \dim Y + \dim X - 2(n-1)$, while the second map, being induced by a homeomorphism, is always an isomorphism. Since $Sg_k = g_{k+1}$: $S^{k+1}(Y \divideontimes SX) \to S^{k+2}(Y \divideontimes X)$, it follows that $S\gamma_k = \gamma_{k+1}S$ so the maps γ_k define, in the limit, an isomorphism

$$\bar{S}\colon \{Y \divideontimes X, S^n\} \approx \{Y \divideontimes SX, S^{n+1}\} .$$

LEMMA (4.8). *Let X and Y be connected polyhedra and let* $\mathbf{f}\colon \mathbf{F}(X) \to S\mathbf{F}(SX)$ *be the weak equivalence of* (4.4). *Then we have a commutative diagram*

$$\begin{array}{ccc} \{Y, S^n\mathbf{F}(X)\} & \xrightarrow{\mathbf{f}_\#} & \{Y, S^{n+1}\mathbf{F}(SX)\} \\ \Lambda \downarrow & & \downarrow \Lambda \\ \{Y \divideontimes X, S^n\} & \xrightarrow{\bar{S}} & \{Y \divideontimes SX, S^{n+1}\} . \end{array}$$

PROOF. Let $f\colon S^kY \to F(X, S^{n+k})$ represent the element $\{f\} \in \{Y, S^n\mathbf{F}(x)\}$. Then $\Lambda\{f\} = \{\bar{f}\}$ where $\bar{f}$ is the composite

$$S^k(Y \divideontimes X) \longleftrightarrow S^kY \divideontimes X \xrightarrow{f \divideontimes 1} F(X, S^{n+k}) \divideontimes X \xrightarrow{E} S^{n+k}$$

and $\mathbf{f}_\#\{\bar{f}\} = \{f_kf\}$ where $f_kf\colon S^kY \to F(SX, S^{n+k+1})$. Consider the diagram

$$\begin{array}{ccc} S^k(Y \divideontimes SX) & \longleftrightarrow & S(S^k(Y \divideontimes X)) \\ \updownarrow & & \updownarrow \\ S^kY \divideontimes SX & \longleftrightarrow & S(S^kY \divideontimes X) \\ f \divideontimes 1 \downarrow & & \downarrow S(f \divideontimes 1) \\ F(X, S^{n+k}) \divideontimes SX & \longleftrightarrow & S(F(X, S^{n+k}) \divideontimes X) \\ \mu \divideontimes 1 \downarrow & & \downarrow SE \\ F(SX, S^{n+k+1}) \divideontimes SX & \xrightarrow{E''} & S^{n+k+1} \end{array}$$

which is commutative in view of the naturality of the commutativity and associativity of the reduced product and (2.6). Going from $S^k(Y \divideontimes SX)$ to S^{n+k+1} by going across and down is, by definition, $(-1)^k\bar{S}\{\bar{f}\}$ and going down and across is, by definition, $(-1)^k\Lambda\{f_kf\}$ (because $f_k \simeq (-1)^k\mu$). Therefore,

$$\bar{S}\Lambda\{f\} = \bar{S}\{\bar{f}\} = \Lambda\{f_kf\} = \Lambda\mathbf{f}_\#\{f\} .$$

THEOREM (4.9). *Let X and Y be connected polyhedra. There is an isomorphism*

$$D\colon \{Y, S^n\mathbf{F}(X)\} \approx \{X, S^n\mathbf{F}(Y)\}$$

characterized by the property that for k large enough $f\colon S^kY \to$

$F(X, S^{n+k})$ and g: $S^kX \to F(Y, S^{n+k})$ represent elements corresponding under D if and only if the following diagram is homotopy commutative

$$\text{(4.10)}\qquad \begin{array}{ccc} S^kY \wedge X & \longleftrightarrow & S^kX \wedge Y \\ {\scriptstyle f\wedge 1}\downarrow & & \downarrow{\scriptstyle g\wedge 1} \\ F(X, S^{n+k}) \wedge X \xrightarrow{E} & S^{n+k} & \xleftarrow{E} F(Y, S^{n+k}) \wedge Y . \end{array}$$

The isomorphism D: $\{X, S^n\mathbf{F}(Y)\} \approx \{Y, S^n\mathbf{F}(X)\}$ is the inverse of the one above.

PROOF. Define D to be the composite

$$\{Y, S^n\mathbf{F}(X)\} \xrightarrow{\Lambda} \{Y \wedge X, S^n\} \longrightarrow \{X \wedge Y, S^n\} \xrightarrow{\Lambda^{-1}} \{X, S^n\mathbf{F}(Y)\}$$

where the middle map is induced by the canonical homeomorphism $X \wedge Y \to Y \wedge X$. Then commutativity of (4.10) characterizes D in view of the definition of Λ. The last statement is an immediate consequence of the symmetry of (4.10).

We define an isomorphism

$$\bar{S}\colon \{X, S^n\mathbf{F}(Y)\} \approx \{SX, S^{n+1}\mathbf{F}(Y)\}$$

so that commutativity holds in the diagram

$$\begin{array}{ccc} \{X, S^n\mathbf{F}(Y)\} & \xrightarrow{\bar{S}} & \{SX, S^{n+1}\mathbf{F}(Y)\} \\ \Lambda\downarrow & & \downarrow\Lambda \\ \{X \wedge Y, S^n\} & & \{SX \wedge Y, S^{n+1}\} \\ \downarrow & & \downarrow \\ \{Y \wedge X, S^n\} & \xrightarrow{\bar{S}} & \{Y \wedge SX, S^{n+1}\} . \end{array}$$

If f: $S^kX \to F(Y, S^{n+k})$ represents an element $\{f\} \in \{X, S^n\mathbf{F}(Y)\}$, then it is easily verified that $\bar{S}\{f\}$ is represented by the composite

$$S^k(SX) \xleftrightarrow{(-1)^k} S(S^kX) \xrightarrow{Sf} SF(Y, S^{n+k}) \xrightarrow{\lambda} F(Y, S^{n+k+1}) .$$

THEOREM (4.11). *For connected polyhedra X, Y there is a commutative diagram*

$$\begin{array}{ccc} \{SY, S^n\mathbf{F}(X)\} & \xrightarrow{\mathbf{f}_\#} & \{SY, S^{n+1}\mathbf{F}(SX)\} \\ D\downarrow & & \downarrow D \\ \{X, S^n\mathbf{F}(SY)\} & \xrightarrow{\bar{S}} & \{SX, S^{n+1}\mathbf{F}(SY)\} . \end{array}$$

PROOF. This follows from (4.8) and the definitions of D, $\bar{S}$.

COROLLARY (4.12). *Commutativity holds in the diagram*

$$\begin{array}{ccc} \{Y, S^n\mathbf{F}(SX)\} & \xrightarrow{\bar{S}} & \{SY, S^{n+1}\mathbf{F}(SX)\} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \{SX, S^n\mathbf{F}(Y)\} & \xrightarrow{\mathbf{f}_\#} & \{SX, S^{n+1}\mathbf{F}(SY)\} \end{array} .$$

PROOF. Since $D = D^{-1}$, this follows from (4.11) on interchanging X and Y.

5. Duality

Let X and X' be connected polyhedra. Let u: $X' ⨳ X \to S^n$ be a continuous map. If u has the property that the map

$$\varphi_u\colon H_q(X') \longrightarrow H^{n-q}(X)$$

defined by $\varphi_u(z) = u^* s_n^* / z$ is an isomorphism, then u will be called *a duality map* and X' will be called an *n-dual* of X by means of u.

LEMMA (5.1). *Let X be a connected subpolyhedron of S^{n+1} and X' a connected subpolyhedron of $S^{n+1} - X$ such that the inclusion map $X' \subset S^{n+1} - X$ induces isomorphisms of all the singular homology groups $H_q(X') \approx H_q(S^{n+1} - X)$. Then there is a duality map*

$$u\colon X' ⨳ X \longrightarrow S^n$$

showing that X' is an n-dual[5] *of X.*

PROOF. This follows from [11; (12.1) and the remarks following (12.2)].

The lemma above shows that the present concept of dual is more general than the one in [12]. It also shows that for sufficiently large n a connected polyhedron has an n-dual.

LEMMA (5.2). *Let u: $X' ⨳ X \to S^n$ be a duality map. For $p \geqq 0, q \geqq 0$ we define*

$$u_{p,q}\colon S^p X' ⨳ S^q X \longrightarrow S^{n+p+q}$$

by

$$u_{p,q}((x' ⨳ t_1 ⨳ \cdots ⨳ t_p) ⨳ (x ⨳ \tau_1 ⨳ \cdots ⨳ \tau_q)) = u(x' ⨳ x) ⨳ t_1 ⨳ \cdots ⨳ t_p ⨳ \tau_1 ⨳ \cdots ⨳ \tau_q .$$

Then $u_{p,q}$ is also a duality map and for $z' \in H_r(X')$ we have the relation

(5.3) $$\varphi_{u_{p,q}}(S^p z') = (-1)^{q(p+r)} S^q(\varphi_u(z')) .$$

[5] This lemma says that every $(n+1)$-dual of X in the sense of [12] is an n-dual of X in the present sense. In view of the definition of dual used in the present paper it seems more natural to relabel duals so that the old $(n+1)$-dual now becomes an n-dual.

PROOF. By (1.8) we know that under the homeomorphism $S^{n+p+q} \longleftrightarrow S^n \divideontimes S^p \divideontimes S^q$ the cohomology class s^*_{n+p+q} corresponds to

$$(-1)^{np+nq+pq} s^*_n \divideontimes s^*_p \divideontimes s^*_q$$

and under the homeomorphism $S^pX' \longleftrightarrow X' \divideontimes S^p$ the homology class S^pz' corresponds to $(-1)^{pr} z' \divideontimes s_p$. Then under the homeomorphism $S^qX \longleftrightarrow X \divideontimes S^q$, $\varphi_{u_{p,q}}(S^pz')$ corresponds to a cohomology class $a \in H^{n+q-r}(X \divideontimes S^q)$ such that

$$\langle a, z \divideontimes s_q \rangle = (-1)^{np+nq+pq+pr} \langle \bar{u}^*_{p,q}(s^*_n \divideontimes s^*_p \divideontimes s^*_q), z' \divideontimes s_p \divideontimes z \divideontimes s_q \rangle$$

where $\bar{u}_{p,q}$: $X' \divideontimes S^p \divideontimes X \divideontimes S^q \rightarrow S^n \divideontimes S^p \divideontimes S^q$ corresponds to $u_{p,q}$. So

$$\begin{aligned} \langle a, z \divideontimes s_q \rangle &= (-1)^{np+nq+pq+pr} \langle s^*_n \divideontimes s^*_p \divideontimes s^*_q, \bar{u}_{p,q*}(z' \divideontimes s_p \divideontimes z \divideontimes s_q) \rangle \\ &= (-1)^{nq+pq} \langle s^*_n \divideontimes s^*_p \divideontimes s^*_q, u_*(z' \divideontimes z) \divideontimes s_p \divideontimes s_q \rangle \\ &= (-1)^{nq+pq} \langle s^*_n, u_*(z' \divideontimes z) \rangle \\ &= (-1)^{nq+pq} \langle \varphi_u(z'), z \rangle = (-1)^{nq+pq} \langle \varphi_u(z') \divideontimes s^*_q, z \divideontimes s_q \rangle . \end{aligned}$$

Therefore, $a = (-1)^{nq+pq} \varphi_u(z') \divideontimes s^*_q$, and since $\varphi_u(z') \divideontimes s^*_q$ corresponds to $(-1)^{q(n-r)} S^q(\varphi_u(z'))$, (5.3) is proved. This, in turn, shows that $u_{p,q}$ is a duality map.

LEMMA (5.4). *Let X, X' be connected polyhedra and let $\bar{u}$: $X' \divideontimes X \rightarrow S^n$ be a duality map. Then the map $\bar{u}$: $X \divideontimes X' \rightarrow S^n$ defined by $\bar{u}(x \divideontimes x') = u(u' \divideontimes x)$ is also a duality map.*

PROOF. Let $c \in C^n(X' \divideontimes X)$ be an n-cochain representing $u^* s^*_n$. Let $\overline{\varphi}$: $C_q(X') \rightarrow C^{n-q}(X)$ be defined by $\overline{\varphi}(a) = c/a$ for $a \in C_q(X')$. Then the isomorphism φ_u: $H_q(X') \approx H^{n-q}(X)$ is identical with the homomorphism $\overline{\varphi}^*$ induced by $\overline{\varphi}$. Since X, X' are polyhedra, we can identify $C_{n-q}(X) = \text{Hom}\,(C^{n-q}(X), Z)$ and $C^q(X') = \text{Hom}\,(C_q(X'), Z)$. Then, letting $\overline{\varphi}^*$ also denote the homomorphism induced by $\overline{\varphi}$ on the groups of homomorphisms, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}\,(B^{n-q+1}(X), Z) & \longrightarrow & C_{n-q}(X) & \longrightarrow & \text{Hom}\,(Z^{n-q}(X), Z) & \longrightarrow & 0 \\ & & \downarrow \overline{\varphi}^* & & \downarrow \overline{\varphi}^* & & \downarrow \overline{\varphi}^* & & \\ 0 & \longrightarrow & \text{Hom}\,(B_{q-1}(X'), Z) & \longrightarrow & C^q(X') & \longrightarrow & \text{Hom}\,(Z_q(X'), Z) & \longrightarrow & 0 \end{array}$$

where each row is exact. Passing to homology we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ext}\,(H^{n-q+1}(X), Z) & \longrightarrow & H_{n-q}(X) & \longrightarrow & \text{Hom}\,(H^{n-q}(X), Z) & \longrightarrow & 0 \\ & & \downarrow \overline{\varphi}^* & & \downarrow \overline{\varphi}^* & & \downarrow \overline{\varphi}^* & & \\ 0 & \longrightarrow & \text{Ext}\,(H_{q-1}(X'), Z) & \longrightarrow & H^q(X') & \longrightarrow & \text{Hom}\,(H_q(X'), Z) & \longrightarrow & 0 \end{array}$$

with the following properties:

(a) Each row is exact (by the universal coefficient theorem).

(b) Each of the outside vertical maps is induced by φ_u so is an isomorphism by assumption on u.

(c) The middle vertical map is, up to sign, the map $\varphi_{\bar{u}}$ because if $b \in C_{n-q}(X)$, $a \in C_q(X')$, then

$$\begin{aligned} \langle \overline{\varphi}^* b, a \rangle = \langle \overline{\varphi} a, b \rangle = \langle c/a, b \rangle &= \langle c, a \divideontimes b \rangle \\ = \pm \langle c', b \divideontimes a \rangle &= \pm \langle c'/b, a \rangle \end{aligned}$$

where $c' \in C^n(X \divideontimes X')$ represents $\bar{u}^* s_n^*$.

From (a), (b) and the 5-lemma [4] it follows that the middle vertical map is an isomorphism. By (c) this implies that $\bar{u}$ is a duality map.

The last two results show that associated to every duality map

$$u: X' \divideontimes X \longrightarrow S^n$$

there are other duality maps $u_{p,q}$, $\bar{u}$, $\bar{u}_{p,q}$. When we start with a duality map u by means of which X' is an n-dual of X, then we shall implicitly understand that S^pX' is $(n + p + q)$-dual to S^qX by $u_{p,q}$.

If X', X are connected subpolyhedra of S^{n+1} with $X' \subset S^{n+1} - X$ so that $H_q(X') \approx H_q(S^{n+1} - X)$ then, as in (5.1), we have a duality map

$$u: X' \divideontimes X \longrightarrow S^n .$$

Then S^pX', S^qX are similarly related in $S^{n+p+q+1}$ [12; (3.2)] so we have a map

$$u': S^pX' \divideontimes S^qX \longrightarrow S^{n+p+q}$$

and also X has the same relation to X' that X' has to X [12; (3.2)] so we have a map

$$u'': X \divideontimes X' \longrightarrow S^n .$$

It is easy to verify that $u' \simeq (-1)^p u_{p,q}$ and $u'' \simeq (-1)^n \bar{u}$. The fact that these signs are present accounts for the differences in sign between some of the theorems of [12] and the corresponding theorems of the present paper.

THEOREM (5.5). *Let u: $X' \divideontimes X \to S^n$ be a duality map. There is a map* $\mathbf{g}$: $\mathrm{S}(X') \to S^k\mathrm{F}(X)$ *which is a weak equivalence and such that for every $k \geqq 0$ we have a commutative diagram*

$$\begin{array}{ccc} S^kX' \divideontimes X & & \\ & \searrow{\scriptstyle u_{k,0}} & \\ {\scriptstyle g_k \divideontimes 1}\Big\downarrow & & S^{n+k} \\ & \nearrow{\scriptstyle E_{n+k}} & \\ F(X, S^{n+k}) \divideontimes X & & \end{array} \tag{5.6}$$

PROOF. By (2.2) there is a unique map $g_k: S^kX' \to F(X, S^{n+k})$ such that (5.6) is commutative. We shall show that the collection $(g_k)_{k=0,1,\dots}$ defines a map $\mathbf{g}: \mathbf{S}(X') \to S^n\mathbf{F}(X)$ which is a weak equivalence. First we verify that for every k we have a commutative diagram

$$\begin{array}{ccc} & S^{k+1}X' & \\ {\scriptstyle Sg_k}\swarrow & & \searrow{\scriptstyle g_{k+1}} \\ SF(X, S^{n+k}) & \xrightarrow{\lambda} & F(X, S^{n+k+1}) . \end{array}$$

This follows, using (2.3), from the commutativity of the diagram

$$\begin{array}{ccc} S^{k+1}X' \circledast X & \xrightarrow{u_{k+1,0}} & S^{n+k+1} \\ {\scriptstyle Sg_k \circledast 1}\downarrow & & \uparrow{\scriptstyle E_{n+k+1}} \\ SF(X, S^{n+k}) \circledast X & \xrightarrow{\lambda \circledast 1} & F(X, S^{n+k+1}) \circledast X , \end{array}$$

which is commutative because it combines the suspension of the commutative diagram (5.6) with the commutative diagram (2.5) and the commutativity of

$$\begin{array}{ccc} S(S^kX' \circledast X) & \xrightarrow{Su_{k,0}} & S^{n+k+1} \\ \updownarrow & \nearrow{\scriptstyle u_{k+1,0}} & \\ S^{k+1}X' \circledast Y . & & \end{array}$$

This shows that $\mathbf{g}$ is a map. To show it is a weak equivalence we use (3.5). We have the homomorphism

$$\varphi_{u_{k,0}}: H_{q+k}(S^kX') \longrightarrow H^{n-q}(X)$$

and, by 5.3, we have a commutative diagram

$$\begin{array}{ccc} H_{q+k}(S^kX') & \xrightarrow{S} & H_{q+k+1}(S^{k+1}X') \\ {\scriptstyle \varphi u_{k,0}}\searrow & & \swarrow{\scriptstyle \varphi u_{k+1,0}} \\ & H^{n-q}(X) & \end{array}$$

in which each map is an isomorphism. Passing to the limit we obtain an isomorphism

$$\varphi': H_q(\mathbf{S}(X')) \approx H^{n-q}(X) .$$

From (5.6) and the definition of $\varphi_{u_{k,0}}$, φ_{n+k} we have a commutative diagram

$$\begin{array}{ccc} H_{q+k}(S^kX') & \xrightarrow{g_{k*}} & H_{q+k}(F(X, S^{n+k})) \\ {\scriptstyle \varphi u_{k,0}}\searrow & & \swarrow{\scriptstyle \varphi_{n+k}} \\ & H^{n-q}(X) . & \end{array}$$

Passing to the limit gives the commutative diagram

$$\begin{array}{ccc} H_q(\mathrm{S}(X')) & \xrightarrow{\mathbf{g}_*} & H_q(S^n\mathbf{F}(X)) \\ & \searrow^{\varphi'} \quad \swarrow^{\varphi} & \\ & H^{n-q}(X)\,. & \end{array}$$

Since φ, φ' are isomorphisms, so is $\mathbf{g}_*$, which completes the proof.

The converse of (5.5) is also valid as the next result shows.

THEOREM (5.7). *Let X' be a connected polyhedron and suppose* $\mathbf{g}$: $\mathrm{S}(X')\to S^n\mathbf{F}(X)$ *is a weak equivalence. If* u: $X' \divideontimes X \to S^n$ *is defined so that commutativity holds in the diagram*

$$\begin{array}{ccc} X' \divideontimes X & & \\ {\scriptstyle g_0 \divideontimes 1}\downarrow & \searrow^{u} & \\ & & S^n \\ & \nearrow_{E_n} & \\ F(X, S^n) \divideontimes X & & \end{array}$$

then u is a duality map.

PROOF. Such a map u exists (merely let $u = E \circ (g_0 \divideontimes 1)$, and it is easy to prove inductively that for $k \geqq 0$ we have a commutative diagram

$$\begin{array}{ccc} S^kX' \divideontimes X & & \\ {\scriptstyle g_k \divideontimes 1}\downarrow & \searrow^{u_{k,0}} & \\ & & S^{n+k} \\ & \nearrow_{E_{n+k}} & \\ F(X, S^{n+k}) \divideontimes X\,. & & \end{array}$$

It follows that we have a commutative diagram

$$\begin{array}{ccc} H_q(X') & = & H_q(\mathrm{S}(X')) \\ {\scriptstyle \varphi_u}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_*} \\ H^{n-q}(X) & \xleftarrow{\varphi} & H_q(S^n\mathbf{F}(X)) \end{array}$$

so φ_u is an isomorphism and u is a duality map.

This last result gives a procedure for obtaining a dual of the connected polyhedron X. Namely, first construct the spectrum $\mathbf{F}(X)$; then find an n and a connected polyhedron X' such that there is a weak equivalence $\mathbf{g}$: $\mathrm{S}(X') \to S^n\mathbf{F}(X)$. This can be done inductively, an intermediate step being to find an integer n', a polyhedron X'_m, and a map $\mathbf{g}_m$: $\mathrm{S}(X'_m) \to S^{n'}\mathbf{F}(X)$ such that $\mathbf{g}_{m*}$: $H_q(\mathrm{S}(X'_m)) \approx H_q(S^{n'}\mathbf{F}(X))$ for $q \leqq m$ and $H_q(\mathrm{S}(X'_m)) = 0$ for $q > m$.

LEMMA (5.8). *Let $u\colon X' \divideontimes X \to S^n$ be a duality map and let Y be a polyhedron. There is an isomorphism*

$$\Gamma_u\colon \{Y, X'\} \approx \{Y \divideontimes X, S^n\}$$

such that if $f\colon S^kY \to S^kX'$ represents an element $\{f\} \in \{Y, X'\}$, then $\Gamma_u\{f\}$ is represented by the composite

$$S^k(Y \divideontimes X) \longleftrightarrow S^kY \divideontimes X \xrightarrow{f \divideontimes 1} S^kX' \divideontimes X \xrightarrow{u_{k,0}} S^{n+k} .$$

PROOF. Using (5.5) we have an isomorphism

$$\mathbf{g}_\sharp\colon \{Y, X'\} \approx \{Y, S^n\mathbf{F}(X)\}$$

and, using (4.7), we have an isomorphism

$$\Lambda\colon \{Y, S^n\mathbf{F}(X)\} \approx \{Y \divideontimes X, S^n\} .$$

We define $\Gamma_u = \Lambda\mathbf{g}_\sharp$. If $f\colon S^kY \to S^kX'$, then $g_\sharp\{f\} = \{g_kf\}$. Consider the diagram

$$\begin{array}{ccccccc} S^k(Y \divideontimes X) & \overset{h}{\longleftrightarrow} & S^kY \divideontimes X & \xrightarrow{f \divideontimes 1} & S^kX' \divideontimes X & \xrightarrow{u_{k,0}} & S^{n+k} \\ & & & & {\scriptstyle g_k \divideontimes 1}\big\downarrow & \nearrow{\scriptstyle E} & \\ & & & & F(X, S^{n+k}) \divideontimes X . & & \end{array}$$

By the definition of Λ we have $\Lambda\mathbf{g}_\sharp\{f\}$ is represented by the composite $E(g_k \divideontimes 1)(f \divideontimes 1)h$, and by the commutativity of (5.6), this equals the composite $u_{k,0}(f \divideontimes 1)h$, completing the proof.

THEOREM (5.9). *Let X, X', Y, Y' be connected polyhedra and let $u\colon X' \divideontimes X \to S^n$, $v\colon Y' \divideontimes Y \to S^n$ be duality maps. There is an isomorphism*

$$D_n(u, v)\colon \{X, Y\} \approx \{Y', X'\}$$

such that for sufficiently large k if $f\colon S^kX \to S^kY$ and $f'\colon S^kY' \to S^kX'$ then $D_n\{f\} = \{f'\}$ if and only if the following diagram is homotopy commutative

$$\begin{array}{ccccc} S^kY' \divideontimes X & \longleftrightarrow & & & Y' \divideontimes S^kY \\ {\scriptstyle f' \divideontimes 1}\big\downarrow & & & & \big\downarrow{\scriptstyle 1 \divideontimes f} \\ S^kX' \divideontimes X & \xrightarrow{u_{k,0}} & S^{n+k} & \xleftarrow{v_{k,0}} & Y' \divideontimes S^kY . \end{array} \tag{5.10}$$

The isomorphism $D_n(v, u)\colon \{Y', X'\} \approx \{X, Y\}$ is the inverse of the one above.

PROOF. Let $\bar{v}\colon Y \divideontimes Y' \to S^n$ be the duality map defined by v as in (5.4). There is then an isomorphism $\Gamma_{\bar{v}}\colon \{X, Y\} \approx \{X \divideontimes Y', S^n\}$. We also have an isomorphism $\Gamma_u\colon \{Y', X'\} \approx \{Y' \divideontimes X, S^n\}$. We define $D_n(u, v)$ to be the composite

$$\{X, Y\} \xrightarrow{\Gamma_{\bar v}} \{X \divideontimes Y', S^n\} \longrightarrow \{Y' \divideontimes X, S^n\} \xrightarrow{\Gamma_u^{-1}} \{Y', X'\} ,$$

where the middle map is induced by the canonical homeomorphism $Y' \divideontimes X \to X \divideontimes Y'$. Choose k large enough so that $\{X, Y\} \approx [S^kX, S^kY]$, $\{X \divideontimes Y', S^n\} \approx [S^k(X \divideontimes Y'), S^{n+k}]$, $\{Y', X'\} \approx [S^kY', S^kX']$. Let $f\colon S^kX \to S^kY$ and $f'\colon S^kY' \to S^kX'$. Consider the diagram

$$\begin{array}{ccccccc}
S^k(Y' \divideontimes X) & \xrightarrow{h_1} & S^kY' \divideontimes X & \xrightarrow{f' \divideontimes 1} & S^kX' \divideontimes X & & \\
{\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow & & & \searrow{\scriptstyle u_{k,0}} & \\
S^k(X \divideontimes Y') & \xrightarrow{h_2} & S^kX \divideontimes Y' & \xrightarrow{f \divideontimes 1} & S^kY \divideontimes Y' & \xrightarrow{\bar v_{k,0}} & S^{n+k} .
\end{array}$$

Clearly $hh_1 = h_2h'$. By definition of Γ we have $\Gamma_u\{f'\} = \{u_{k,0}(f' \divideontimes 1)h_1\}$ and $\Gamma_{\bar v}\{f\} = \{\bar v_{k,0}(f \divideontimes 1)h_2\}$. Therefore, $D_n\{f\} = \{f'\}$ if and only if the above diagram is homotopy commutative (i.e., $\bar v_{k,0}(f \divideontimes 1)h \simeq u_{k,0}(f' \divideontimes 1)$). On the other hand, by the definition of $\bar v$ we have a commutative diagram

$$\begin{array}{ccccc}
S^kX \divideontimes Y' & \xrightarrow{f \divideontimes 1} & S^kY \divideontimes Y' & & \\
\updownarrow & & \updownarrow & \searrow{\scriptstyle \bar v_{k,0}} & \\
 & & & & S^{n+k} \\
 & & & \nearrow{\scriptstyle v_{0,k}} & \\
Y' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & Y' \divideontimes S^kY . & &
\end{array}$$

Combining these diagrams, we see that $D_n\{f\} = \{f'\}$ if and only if (5.10) is homotopy commutative.

The symmetry of (5.10) establishes the last statement of the theorem.

It is convenient to express the last result in the following form.

THEOREM (5.11). *Let $u\colon X' \divideontimes X \to S^n$, $v\colon Y' \divideontimes Y \to S^n$ be duality maps. Given maps $f\colon S^kX \to S^kY$ and $f'\colon S^{k'}Y' \to S^{k'}X'$ with k, k' large enough then $D_n\{f\} = \{f'\}$ if and only if the following diagram is homotopy commutative*

(5.12)
$$\begin{array}{ccc}
S^{k'}Y' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & S^{k'}Y' \divideontimes S^kY \\
{\scriptstyle f' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle v_{k,k}} \\
S^{k'}X' \divideontimes S^kX & \xrightarrow{u_{k',k}} & S^{n+k+k'} .
\end{array}$$

PROOF. We choose k, k' so large that $\{X, Y\} \approx [S^kX, S^kY]$, $\{Y', X'\} \approx [S^{k'}Y', S^{k'}X']$ and consider the case where $k \leqq k'$ (a similar argument applies if $k' < k$). Let $f\colon S^kX \to S^kY$, $f'\colon S^{k'}Y' \to S^{k'}X'$. The commutativity of (5.10) gives us the homotopy commutativity of

$$\begin{array}{ccccc}
S^{k'}Y' \divideontimes X & \longleftrightarrow & & & Y' \divideontimes S^{k'}X \\
{\scriptstyle f' \divideontimes 1}\swarrow & & & & \searrow{\scriptstyle 1 \divideontimes S^{k'-k}f} \\
S^{k'}X' \divideontimes X & \xrightarrow{u_{k',0}} & S^{n+k'} & \xrightarrow{v_{0,k'}} & Y' \divideontimes S^{k'}Y .
\end{array}$$

By suspending this diagram k times and combining it with other diagrams which are easily seen to be homotopy commutative we find that the homotopy commutativity of the above diagram is equivalent to the homotopy commutativity of the following diagram.

$$\begin{array}{ccccc}
S^{k'}Y' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & S^{k'}Y' \divideontimes S^kY & \xrightarrow{v_{k',k}} & S^{n+k+k'} \\
\updownarrow & & \updownarrow & & \|\,{\scriptstyle (-1)^{k(k'-k)}} \\
S^{k'-k}(S^kY' \divideontimes S^kX) & \xrightarrow{S^{k'-k}(1 \divideontimes f)} & S^{k'-k}(S^kY' \divideontimes S^kY) & \xrightarrow{v_{k,k}} & S^{n+k+k'} \\
\updownarrow & & \updownarrow & & \| \\
S^kY' \divideontimes S^{k'}X & \xrightarrow{1 \divideontimes S^{k'-k}f} & S^kY' \divideontimes S^{k'}Y & \xrightarrow{v_{k,k'}} & S^{n+k+k'} \\
\updownarrow & & \updownarrow & & \|\,{\scriptstyle (-1)^{kk'}} \\
S^k(Y' \divideontimes S^{k'}X) & \xrightarrow{S^k(1 \divideontimes S^{k'-k}f)} & S^k(Y' \divideontimes S^{k'}X) & \xrightarrow{S^kv_{0,k'}} & S^{n+k+k'} \\
\updownarrow & & & & \| \\
S^k(S^{k'}Y' \divideontimes X) & \xrightarrow{S^k(f \divideontimes 1)} & S^k(S^{k'}X' \divideontimes X) & \xrightarrow{S^ku_{k',0}} & S^{n+k+k'} \\
\updownarrow & & \updownarrow & & \| \\
S^{k'}Y \divideontimes S^kX & \xrightarrow{f' \divideontimes 1} & S^{k'}X' \divideontimes S^kX & \xrightarrow{u_{k',k}} & S^{n+k+k'} .
\end{array}$$

The composite down the first column corresponds to the homeomorphism of $S^{k'-k}(S^kY') \divideontimes S^kX$ into itself which switches the two sets of S^k-coordinates so equals $(-1)^k$. The composite down the last column also equals $(-1)^k$. Therefore, we can drop both signs and have the result that the homotopy commutativity of the above diagram is equivalent to that of

$$\begin{array}{ccc}
S^{k'}Y' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & S^{k'}Y' \divideontimes S^kY \\
{\scriptstyle f' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle v_{k',k}} \\
S^{k'}X' \divideontimes S^kX & \xrightarrow{u_{k',k}} & S^{n+k+k'} ,
\end{array}$$

and this completes the proof.

We want to see how the duality $D_u(u, v)$ is altered by changing one of the duality maps. The next result answers this in the most important case, namely when u is altered by following it by a homeomorphism of S^n.

THEOREM (5.13). *Let* $u\colon X' \ast X \to S^n$, $v\colon Y' \ast Y \to S^n$ *be dualities. Let* $\eta\colon S^n \to S^n$ *have degree* ± 1. *Then* $\eta u = u'\colon X' \ast X \to S^n$ *is also a duality map and* $D_n(u', v) = (\text{degree } \eta) D_n(u, v)$.

PROOF. u' is also a duality map because $\varphi_{u'} = (\text{degree } \eta)\varphi_u$. Since $u' = \eta u$, it follows that $u'_{k,k} = (S^{2k}\eta)u_{k,k}$. Let $f\colon S^kX \to S^kY$, $f'\colon S^kY' \to S^kX'$ be such that

$$u_{k,k}(1 \ast f) \simeq v_{k,k}(f' \ast 1) .$$

Then $D_n(u, v)\{f\} = \{f'\}$. Also we see that

$$u'_{k,k}(1 \ast f) \simeq (\text{degree } \eta)u_{k,k}(1 \ast f) \simeq v_{k,k}((\text{degree } \eta)f' \ast 1)$$

showing that $D_n(u', v)\{f\} = (\text{degree } \eta)\{f'\}$, and completing the proof.

6. Properties of the duality

In this section we summarize some of the properties of the duality D_n of the last section. We shall assume that all the spaces X, Y, X', Y', etc., of this section are connected polyhedra and shall use $u\colon X' \ast X \to S^n$, $v\colon Y' \ast Y \to S^n$ for duality maps.

THEOREM (6.1). *Let* $\alpha \in \{X, Y\}$. *Then* $D_n(u, v)\alpha \in \{Y', X'\}$ *and we have a commutative diagram*

$$\begin{array}{ccc} H_q(Y') & \xrightarrow{\varphi_v} & H^{n-q}(Y) \\ \downarrow{\scriptstyle (D_n(u,v)\alpha)_*} & & \downarrow{\scriptstyle \alpha^*} \\ H_q(X') & \xrightarrow{\varphi_u} & H^{n-q}(X) . \end{array}$$

PROOF. Let $f\colon S^kX \to S^kY$, $f'\colon S^kY' \to S^kX'$ be such that (5.12) is homotopy commutative. Then $D_n\{f\} = \{f'\}$, and it follows from the commutativity of (5.12) and naturality properties of the slant product that

$$f^*\varphi_{v_{k,k}} = \varphi_{u_{k,k}} f'_* .$$

Since $\{f\}^*$ is defined to be the composite

$$H^{n-q}(Y) \xrightarrow{S^k} H^{p-q+k}(S^kY) \xrightarrow{f^*} H^{n-q+k}(S^kX) \xrightarrow{(S^k)^{-1}} H^{n-q}(X)$$

and $\{f'\}_*$ is defined to be the commposite

$$H_q(Y') \xrightarrow{S^k} H_{q+k}(S^kY') \xrightarrow{f'_*} H_{q+k}(S^kX') \xrightarrow{(S^k)^{-1}} H_q(X') ,$$

it follows from (5.3) and the equality above that

$$\{f\}^*\varphi_v = \varphi_u\{f'\}_* ,$$

which is the desired result.

THEOREM (6.2). *Commutativity holds in each diagram*

$$\begin{array}{ccc} \{X, Y\} & \xrightarrow{D_n(u,v)} & \{Y', X'\}, \\ & \searrow^{D_{n+1}(u_{1,0},\, v_{1,0})} & \downarrow S \\ & & \{SY', SX'\}, \end{array} \qquad \begin{array}{ccc} \{X, Y\} & \xrightarrow{D_n(u,v)} & \{Y', X'\} \\ S \downarrow & \nearrow_{D_{n+1}(u_{0,1},\, v_{0,1})} & \\ \{SX, SY\} & & \end{array} .$$

PROOF. We prove only the first commutativity as the second follows from the first by interchanging X with X', Y with Y', u with $\bar{u}$, and v with $\bar{v}$. Let $f\colon S^kX \to S^kY$, $f'\colon S^kY' \to S^kX'$ be such that (5.12) is homotopy commutative. Then $D_n(u, v)\{f\} = \{f'\}$. Suspending (5.12) and using obvious commutativity properties we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} S^{k+1}Y' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & S^{k+1}Y' \divideontimes S^kY & \xrightarrow{v_{k+1,k}} & S^{n+2k+1} \\ \updownarrow & & \updownarrow & & \| \, (-1)^k \\ S(S^kY' \divideontimes S^kX) & \xrightarrow{S(1 \divideontimes f)} & S(S^kY' \divideontimes S^kY) & \xrightarrow{Sv_{k,k}} & S^{n+2k+1} \\ \| & & & & \| \\ S(S^kY' \divideontimes S^kX) & \xrightarrow{S(f' \divideontimes 1)} & S(S^kX' \divideontimes S^kX) & \xrightarrow{Su_{k,k}} & S^{n+2k+1} \\ \updownarrow & & \updownarrow & & \| \, (-1)^k \\ S^{k+1}Y' \divideontimes S^kX & \xrightarrow{Sf' \divideontimes 1} & S^{k+1}X' \divideontimes S^kX & \xrightarrow{u_{k+1,k}} & S^{n+2k+1} \end{array} .$$

Under the homeomorphism $S^k(SY') \longleftrightarrow S^{k+1}Y'$ defined by

$$(y' \divideontimes t) \divideontimes t_1 \divideontimes \cdots \divideontimes t_k \longleftrightarrow y' \divideontimes t \divideontimes t_1 \divideontimes \cdots \divideontimes t_k$$

we see that $v_{k+1,k}$ corresponds to $(v_{1,0})_{k,k}$. Under a similar homeomorphism $u_{k+1,k}$ corresponds to $(u_{1,0})_{k,k}$. Then homotopy commutativity of the above diagram implies that $D_{n+1}(u_{1,0}, v_{1,0})\{f\} = \{Sf'\} = SD_n(u, v)\{f\}$.

THEOREM (6.3). *Let* $u\colon X' \divideontimes X \to S^n$, $v\colon Y' \divideontimes Y \to S^n$, $w\colon Z' \divideontimes Z \to S^n$ *be duality maps. Let* $\alpha \in \{X, Y\}$, $\beta \in \{Y, Z\}$. *Then* $\beta\alpha \in \{X, Z\}$ *and*

$$D_n(u, w)\beta\alpha = (D_n(u, v)\alpha)(D_n(v, w)\beta) .$$

PROOF. Choose k large enough so that α is represented by $f\colon S^kX \to S^kY$, β is represented by $g\colon S^kY \to S^kZ$, $D_n(u, v)\alpha$ is represented by $f'\colon S^kY' \to S'X'$, $D_n(v, w)\beta$ is represented by $g'\colon S^kZ' \to S^kY'$ and $v_{k,k}(1 \divideontimes f) \simeq u_{k,k}(f' \divideontimes 1)$, $v_{k,k}(g' \divideontimes 1) \simeq w_{k,k}(1 \divideontimes g)$. Then

$$\begin{aligned} u_{k,k}(f'g' \divideontimes 1) &= u_{k,k}(f' \divideontimes 1)(g' \divideontimes 1) \simeq v_{k,k}(1 \divideontimes f)(g' \divideontimes 1) \\ &= v_{k,k}(g' \divideontimes 1)(1 \divideontimes f) \simeq w_{k,k}(1 \divideontimes g)(1 \divideontimes f) = w_{k,k}(1 \divideontimes gf) \end{aligned}$$

showing that

$$D_n(u, w)\{gf\} = \{f'g'\} .$$

Since $\{gf\} = \beta\alpha$ and $\{f'g'\} = (D_n(u, v)\alpha)(D_n(v, w)\beta)$, this completes the proof.

If $p \geqq 0$ then $u_{0,p}: X' ⨳ S^pX \to S^{n+p}$, $v_{p,0}: S^pY' ⨳ Y \to S^{n+p}$ are dualities so we have the isomorphism

$$D_{n+p}(u_{0,p}, v_{p,0}): \{X, Y\}_p \approx \{Y', X'\}_p .$$

Similarly if $p \leqq 0$ then $u_{-p,0}: S^{-p}X' ⨳ X \to S^{n-p}$, $v_{0,-p}: Y' ⨳ S^{-p}Y \to S^{n-p}$ are dualities and we have the isomorphism

$$D_{n-p}(u_{-p,0}, v_{0,-p}): \{X, Y\}_p \approx \{Y', X'\}_p .$$

We define

$$D_{n+|p|}(u, v) = \begin{cases} D_{n+p}(u_{0,p}, v_{p,0}) & \text{if } p \geqq 0 \\ D_{n-p}(u_{-p,0}, v_{0,-p}) & \text{if } p \leqq 0 . \end{cases}$$

Then for any p we have the isomorphism

$$D_{n+|p|}(u, v): \{X, Y\}_p \approx \{Y', X'\}_p .$$

We then have the following extension of (6.3).

THEOREM (6.4). *Let $u: X' ⨳ X \to S^n$, $v: Y' ⨳ Y \to S^n$, $w: Z' ⨳ Z \to S^u$ be duality maps. Let $\alpha \in \{X, Y\}_p$, $\beta \in \{Y, Z\}_p$. Then $\beta\alpha \in \{X, Z\}_{p+q}$ and*

$$D_{n+|p+q|}(u, w)\beta\alpha = (D_{n+|p|}(u, v)\alpha)(D_{n+|q|}(v, w)\beta) .$$

PROOF. The proof involves consideration of several cases depending on the signs of p, q, $p + q$ and follows from (6.2) and (6.3) and the definition of the composition operation. We omit the details.

The next results show how to construct new dualities from old ones. Roughly, the sum of dualities is a duality, and the product of dualities is a duality.

THEOREM (6.5) *Let $u: X' ⨳ X \to S^n$, $v: Y' ⨳ Y \to S^n$ be dualities. Define $u \oplus v: (X' \vee Y') ⨳ (X \vee Y) \to S^n$ by $u \oplus v | X' ⨳ X = u$, $u \oplus v | Y' ⨳ Y = v$, $u \oplus v(X' ⨳ Y \cup Y' ⨳ X) = 0$. Then $u \oplus v$ is a duality map such that the dual of the inclusion map of one of the summands (such as $i: X \subset X \vee Y$ or $j': Y' \subset X' \vee Y'$) is the retraction of the dual onto the corresponding summand ($r': X' \vee Y' \to X'$ or $s: X \vee Y \to Y$).*

PROOF. The map $u \oplus v$ was defined so that we have a commutative diagram

$$(6.6)\qquad \begin{array}{ccccc} (X' \vee Y') \divideontimes X & \xrightarrow{1 \divideontimes i} & (X' \vee Y') \divideontimes (X \vee Y) & \xleftarrow{j' \divideontimes 1} & Y' \divideontimes (X \vee Y) \\ \downarrow{\scriptstyle r' \divideontimes 1} & & \downarrow{\scriptstyle u \oplus v} & & \downarrow{\scriptstyle 1 \divideontimes s} \\ X' \divideontimes X & \xrightarrow{u} & S^n & \xleftarrow{v} & Y' \divideontimes Y . \end{array}$$

From this and the naturality of the slant product we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_q(Y') & \xrightarrow{j'_*} & H_q(X' \vee Y') & \xrightarrow{r'_*} & H_q(X') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_v} & & \downarrow{\scriptstyle \varphi_{u\oplus v}} & & \downarrow{\scriptstyle \varphi_u} & & \\ 0 & \longrightarrow & H^{n-q}(Y) & \xrightarrow{s^*} & H^{n-q}(X \vee Y) & \xrightarrow{i^*} & H^{n-q}(X) & \longrightarrow & 0 . \end{array}$$

It follows from the direct sum theorem that each row is exact. Since φ_u, φ_v are isomorphisms, the 5-lemma implies that $\varphi_{u\oplus v}$ is an isomorphism showing that $u \oplus v$ is a duality map. The rest of the theorem follows from the commutativity of (6.6) (and of a similar diagram with X, Y and X', Y' interchanged) and (5.11).

COROLLARY (6.7). *Given dualities $u: X' \divideontimes X \to S^n$, $v: Y' \divideontimes Y \to S^n$, $u_1: X'_1 \divideontimes X_1 \to S^n$, $v_1: Y'_1 \divideontimes Y_1 \to S^n$ then if $\alpha \in \{X, X_1\}$, $\beta \in \{Y, Y_1\}$, we have $\alpha \vee \beta \in \{X \vee Y, X_1 \vee Y_1\}$ and*

$$D_n(u \oplus v, u_1 \oplus v_1)(\alpha \vee \beta) = (D_n(u, u_1)\alpha) \vee (D_n(v, v_1)\beta) .$$

PROOF. This is an immediate consequence of (6.5) and the fact that $\{X \vee Y, X_1 \vee Y_1\}$ is isomorphic to the direct sum

$$\{X, X_1\} + \{X, Y_1\} + \{Y, X_1\} + \{Y, Y_1\} .$$

THEOREM (6.8). *Let $u: X' \divideontimes X \to S^n$, $v: Y' \divideontimes Y \to S^m$ be dualities. Define $u \otimes v: (X' \divideontimes Y') \divideontimes (X \divideontimes Y) \to S^n \divideontimes S^m \longleftrightarrow S^{n+m}$ by*

$$(u \otimes v)((x' \divideontimes y') \divideontimes (x \divideontimes y)) = u(x' \divideontimes x) \divideontimes v(y' \divideontimes y) .$$

Then $u \otimes v$ is a duality map. Given also duality maps $u_1: X'_1 \divideontimes X_1 \to S^n$, $v_1: Y'_1 \divideontimes Y_1 \to S^m$ then if $\alpha \in \{X, X_1\}$, $\beta \in \{Y, Y_1\}$ we have $\alpha \divideontimes \beta \in \{X \divideontimes Y, X_1 \divideontimes Y_1\}$ and

$$D_{n+m}(u \otimes v, u_1 \otimes v_1)(\alpha \divideontimes \beta) = (D_n(u, u_1)\alpha) \divideontimes (D_m(v, v_1)\beta) .$$

PROOF. A direct calculation shows that $\varphi_{u\otimes v}$ is induced by a chain transformation $C_{p+q}(X' \divideontimes Y') \to C^{n+m-p-q}(X \divideontimes Y)$ which is, up to sign, the same as the tensor product of the chain transformations

$$C_p(X') \longrightarrow C^{n-p}(X),\quad C_q(Y') \longrightarrow C^{n-q}(Y)$$

which induce φ_u, φ_v. Since φ_u, φ_v are isomorphisms, so is their tensor product. This show that $\varphi_{u\otimes v}$ is an isomorphism, so $u \otimes v$ is a duality map.

To prove the second part of the theorem let $f\colon S^kX \to S^kX_1$, $g\colon S^kY \to S^kY_1$, $f'\colon S^kX_1' \to S^kX'$, $g'\colon S^kY_1' \to S^kY'$ be maps such that the following are homotopy commutative

$$\begin{array}{ccc} S^kX_1' \divideontimes S^kX & \xrightarrow{1 \divideontimes f} & S^kX_1' \divideontimes S^kX_1, \\ {\scriptstyle f' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle (u_1)_{k,k}} \\ S^kX' \divideontimes S^kX & \xrightarrow{u_{k,k}} & S^{n+2k}, \end{array} \qquad \begin{array}{ccc} S^kY_1' \divideontimes S^kY & \xrightarrow{1 \divideontimes g} & S^kY_1' \divideontimes S^kY_1 \\ {\scriptstyle g' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle (v_1)_{k,k}} \\ S^kY' \divideontimes S^kY & \xrightarrow{v_{k,k}} & S^{m+2k} \end{array}$$

From these we get a homotopy commutative diagram

$$\begin{array}{ccc} S^kX_1' \divideontimes S^kX \divideontimes S^kY_1' \divideontimes S^kY & \xrightarrow{1 \divideontimes f \divideontimes 1 \divideontimes g} & S^kX_1' \divideontimes S^kX_1 \divideontimes S^kY_1' \divideontimes S^kY_1 \\ {\scriptstyle f' \divideontimes 1 \divideontimes g' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle (u_1)_{k,k} \divideontimes (v_1)_{k,k}} \\ S^kX' \divideontimes S^kX \divideontimes S^kY' \divideontimes S^kY & \xrightarrow{u_{k,k} \divideontimes v_{k,k}} & S^{n+m+4k}\,. \end{array}$$

From the definition of $u \otimes v$ we get a commutative diagram

$$\begin{array}{ccc} S^kX' \divideontimes S^kX \divideontimes S^kY' \divideontimes S^kY & \xrightarrow{u_{k,k} \divideontimes v_{k7k}} & S^{n+m+4k} \\ \updownarrow & & \\ S^kX' \divideontimes S^kY' \divideontimes S^kX \divideontimes S^kY & & \Vert\,{\scriptstyle (-1)^k} \\ \updownarrow & & \\ S^{2k}(X' \divideontimes Y') \divideontimes S^{2k}(X \divideontimes Y) & \xrightarrow{(u \otimes v)_{2k,2k}} & S^{n+m+4k} \end{array}$$

with a similar diagram for u_1, v_1. Combining these with the preceding diagram with the middle terms commuted we get a homotopy commutative diagram

$$\begin{array}{ccc} S^{2k}(X_1' \divideontimes Y_1') \divideontimes S^{2k}(X \divideontimes Y) & \xrightarrow{1 \divideontimes h} & S^{2k}(X_1' \divideontimes Y_1') \divideontimes S^{2k}(X_1 \divideontimes Y_1) \\ {\scriptstyle h' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle (u_1 \otimes v_1)_{2k,2k}} \\ S^{2k}(X' \divideontimes Y') \divideontimes S^{2k}(X \divideontimes Y) & \xrightarrow{(u \otimes v)_{2k,2k}} & S^{n+m+4k} \end{array}$$

where $h\colon S^{2k}(X \divideontimes Y) \to S^{2k}(X_1 \divideontimes Y_1)$ is the composite

$$S^{2k}(X \divideontimes Y) \longleftrightarrow S^kX \divideontimes S^kY \xrightarrow{f \divideontimes g} S^kX_1 \divideontimes S^kY_1 \longleftrightarrow S^{2k}(X_1 \divideontimes Y_1)$$

and h' is a similar composite. Then, by (5.11),

$$D_{n+m}(u \otimes v,\ u' \otimes v')(\{h\}) = \{h'\}\ .$$

Since $\{h\} = \{f\} \divideontimes \{g\}$, $\{h'\} = \{f'\} \divideontimes \{g'\}$ and $\{f'\} = D_n(u, v)\{f\}$, $\{g'\} = D_n(u_1, v_1)\{g\}$, the proof is complete.

COROLLARY (6.9). *Let $v\colon Y' \wedge Y \to S^n$ be a duality map. For polyhedra X, Z there is an isomorphism*

$$\Gamma\colon \{X, Z \wedge Y'\} \approx \{X \wedge Y, S^n Z\}$$

such that if $f\colon S^k X \to S^k(Z \wedge Y')$ represents an element of $\{X, Z \wedge Y'\}$ then $\Gamma\{f\}$ is represented by the composite

$$\begin{array}{ccccccc}
S^k(X \wedge Y) & & & & & & S^{n+k}Z \\
\updownarrow & & & & & & \updownarrow \\
S^k X \wedge Y & \xrightarrow{f \wedge 1} & S^k(Z \wedge Y') \wedge Y & \longleftrightarrow & Z \wedge (S^k Y' \wedge Y) & \xrightarrow{1 \wedge v_{k,0}} & Z \wedge S^{n+k}
\end{array}$$

PROOF. For sufficiently large m there exists Z' and a duality $w\colon Z \wedge Z' \to S^m$. Then, by (6.8),

$$w \otimes v\colon (Z \wedge Y') \wedge (Z' \wedge Y) \longrightarrow S^{n+m}$$

is a duality. By (5.8) we have isomorphisms

$$\Gamma_{w \otimes v}\colon \{X, Z \wedge Y'\} \approx \{X \wedge (Z' \wedge Y), S^{n+m}\}$$
$$\Gamma_{w_{n,0}}\colon \{X \wedge Y, S^n Z\} \approx \{(X \wedge Y) \wedge Z', S^{n+m}\} .$$

The associativity and commutativity of the reduced product gives a canonical isomorphism $\{X \wedge (Z' \wedge Y), S^{n+m}\} \approx \{(X \wedge Y) \wedge Z', S^{n+m}\}$. We define $\Gamma\colon \{X, Z \wedge Y'\} \to \{X \wedge Y, S^n Z\}$ to be the composite

$$\{X, Z \wedge Y'\} \xrightarrow{\Gamma_{w \otimes v}} \{X \wedge (Z' \wedge Y), S^{n+m}\} \approx \{(X \wedge Y) \wedge Z', S^{n+m}\} \xrightarrow{\Gamma^{-1}_{w_{n,0}}} \{X \wedge Y, S^n Z\} .$$

Let $f\colon S^k X \to S^k(Z \wedge Y')$ and let g be the composite

$$S^k(X \wedge Y) \longleftrightarrow S^k X \wedge Y \xrightarrow{f \wedge 1} S^k(Z \wedge Y') \wedge Y \longleftrightarrow Z \wedge (S^k Y' \wedge Y) \xrightarrow{1 \wedge v_{k,0}} Z \wedge S^{n+k} \longleftrightarrow S^{n+k} Z .$$

Then $\Gamma_{w_{n,0}}\{g\}$ is represented by the composite

$$S^k((X \wedge Y) \wedge Z') \longleftrightarrow S^k(X \wedge Y) \wedge Z' \xrightarrow{g \wedge 1} S^{n+k} Z \wedge Z' \xrightarrow{w_{n+k,0}} S^{n+m+k}$$

and $\Gamma_{w \otimes v}\{f\}$ is represented by the composite

$$S^k(X \wedge (Z' \wedge Y)) \longleftrightarrow S^k X \wedge (Z' \wedge Y) \xrightarrow{f \wedge 1} S^k(Z \wedge Y') \wedge (Z' \wedge Y) \xrightarrow{(w \otimes v)_{k,0}} S^{n+m+k} .$$

Under the canonical homeomorphism $S^k((X \wedge Y) \wedge Z') \longleftrightarrow S^k(X \wedge (Z' \wedge Y))$ these correspond, showing that $\Gamma\{f\} = \{g\}$ and completing the proof.

Let $f\colon X \to Y$. We form a space Z_f equal to the quotient space of the disjoint union of TX and Y by the identification $x \divideontimes 1 = fx$. There is then a canonical injection i: $Y \to Z_f$ and a canonical projection p: $Z_f \to SX$ defined by

$$p(x \divideontimes t) = x \divideontimes t, \;\; py = x_0 \,.$$

It is easy to verify that we have an exact sequence

$$\cdots \longrightarrow H_q(X) \xrightarrow{f_*} H_q(Y) \xrightarrow{i_*} H_q(Z_f) \xrightarrow{p_*} H_q(SX) \xrightarrow{(Sf)_*} H_q(SX) \longrightarrow \cdots$$

and a similar exact sequence for cohomology.

It is also easy to verify that there is a canonical homeomorphism $Z_{Sf} \longleftrightarrow SZ_f$ which combines with the suspension map S to map the above diagram isomorphically onto the diagram

$$\cdots \longrightarrow H_{q+1}(SX) \xrightarrow{(Sf)_*} H_{q+1}(SY) \longrightarrow H_{q+1}(Z_{Sf}) \longrightarrow H_{q+1}(S^2X) \longrightarrow \cdots .$$

Let u: $X' \divideontimes X \to S^n$, v: $Y' \divideontimes Y \to S^n$ be duality maps and assume f: $X \to Y$, f': $Y' \to X'$ are such that the diagram

$$\begin{array}{ccc} Y' \divideontimes X & \xrightarrow{1 \divideontimes f} & Y' \divideontimes Y \\ {\scriptstyle f' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle v} \\ X' \divideontimes X & \xrightarrow{u} & S^n \end{array}$$

is homotopy commutative. Let $Z = Z_f$, $Z' = Z_{f'}$ and let H: $(Y' \divideontimes X) \times I \to S^n$ be a homotopy of $u(f' \divideontimes 1)$ to $v(1 \divideontimes f)$. Therefore,

$$H(y' \divideontimes x, 0) = u(f'g' \divideontimes x), \quad H(y' \divideontimes x, 1) = v(y' \divideontimes fx)$$
$$H(y'_0 \divideontimes x, t) = H(y' \divideontimes x'_0, t) = 0 \,.$$

We define a continuous map

$$w\colon Z' \divideontimes Z \longrightarrow S^{n+1}$$

by

$$w((x' \divideontimes t') \divideontimes (x \divideontimes t)) = \begin{cases} H\Big(y' \divideontimes x, \dfrac{1}{2}\dfrac{1-t'}{1-t}\Big) \divideontimes t & \text{if } t \leqq t',\ t \neq 1 \\ H\Big(y' \divideontimes x, 1 - \dfrac{1}{2}\dfrac{1-t}{1-t'}\Big) \divideontimes t' & \text{if } t \geqq t',\ t' \neq 1 \\ 0 & \text{if } t = t' = 1 \,. \end{cases}$$

$$\begin{aligned} w(x' \divideontimes (x \divideontimes t)) &= u(x' \divideontimes x) \divideontimes t \\ w((y' \divideontimes t') \divideontimes y) &= v(y' \divideontimes y) \divideontimes t' \\ w(x' \divideontimes y) &= 0 \,. \end{aligned}$$

Then w is well defined (i.e., it is consistent with the identifications in Z', Z) and is continuous.

THEOREM (6.10). *w is a duality map and relative to the dualities $u_{1,0}\colon SX' \divideontimes X \to S^{n+1}$, $u_{0,1}\colon X' \divideontimes SX \to S^{n+1}$, $v_{1,0}\colon SY' \divideontimes Y \to S^{n+1}$, $v_{0,1}\colon Y' \divideontimes SY \to S^{n+1}$ we have the $(n+1)$-dual sequences*

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & Z & \xrightarrow{p} & SX & \xrightarrow{Sf} & SY \\ u_{1,0} & & v_{1,0} & & w & & u_{0,1} & & v_{0,1} \\ SX' & \xleftarrow{Sf'} & SY' & \xleftarrow{p'} & Z' & \xleftarrow{i'} & X' & \xleftarrow{f'} & Y' \end{array}$$

(*where the duality map has been inserted in the middle row*).

PROOF. By (5.11) we know $D_n(u, v)\{f\} = \{f'\}$ so, by (6.2),

$$D_{n+1}(u_{1,}, v_{1,0})\{f\} = \{Sf'\}, \quad D_{n+1}(u_{0,1}, v_{0,1})\{Sf\} = \{f'\}$$

and the end maps are dual. From the definition of w we have commutative diagrams

$$\begin{array}{ccc} X' \divideontimes Z & \xrightarrow{1 \divideontimes p} & X' \divideontimes SX \\ {\scriptstyle i' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle u_{0,1}} \\ Z' \divideontimes Z & \xrightarrow{w} & S^{n+1} \end{array}, \qquad \begin{array}{ccc} Z' \divideontimes Y & \xrightarrow{1 \divideontimes i} & Z' \divideontimes Z \\ {\scriptstyle p' \divideontimes 1}\downarrow & & \downarrow{\scriptstyle w} \\ SY' \divideontimes Y & \xrightarrow{v_{1,0}} & S^{n+1} \end{array}$$

which would show, by (5.11), that $D_{n+1}(w, u_{0,1})\{p\} = \{i'\}$, $D_{n+1}(v_{1,0}, w)\{i\} = \{p'\}$ once we know w is a duality map. Hence, all that remains is to prove that w is a duality map.

From the above properties and the naturality of the slant product we obtain a commutative diagram

$$\begin{array}{ccc} H_q(Y') & \xrightarrow{\varphi_{v_{0,1}}} & H^{n-q+1}(SY) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle (Sf)^*} \\ H_q(X') & \xrightarrow{\varphi_{u_{0,1}}} & H^{n-q+1}(SX) \\ {\scriptstyle i'_*}\downarrow & & \downarrow{\scriptstyle p^*} \\ H_q(Z') & \xrightarrow{\varphi_w} & H^{n-q+1}(Z) \\ {\scriptstyle p'_*}\downarrow & & \downarrow{\scriptstyle i^*} \\ H_q(SY') & \xrightarrow{\varphi_{v_{1,0}}} & H^{n-q+1}(Y) \\ {\scriptstyle (Sf')^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H_q(SX') & \xrightarrow{\varphi_{u_{1,0}}} & H^{n-q+1}(X)\,. \end{array}$$

All horizontal maps except possibly φ_w are isomorphisms because u, v are duality maps. Since each row is exact, it follows from the 5-lemma that φ_w is an isomorphism, so w is a duality map.

The fact that (6.10) is true without a change of sign of $\{p'\}$, as distinguished from [12; (6.2)], is due to the fact that in [12; (6.2)] the dualities used are $u_{0,1}$, $v_{0,1}$, $-u_{1,0}$, $-v_{1,0}$ (see that comment following (5.4)) and (5.13).

7. Relative theory

Let X be a polyhedron. By a *polyhedral lattice* $\mathfrak{A}$ on X we mean a lattice of subcomplexes A of X each containing the base point and such that $\mathfrak{A}$ contains $\{x_0\}$ and X. Let $\mathfrak{A}'$ be a polyhedral lattice on X' and assume there is an anti-isomorphism $\mathfrak{a}\colon \mathfrak{A} \to \mathfrak{A}'$. Let $G(\mathfrak{a})$ denote the subset of $X' \divideontimes X$ composed of the union of all $\mathfrak{a}A \divideontimes A$ for $A \in \mathfrak{A}$. Let

$$u\colon X' \divideontimes X/G(\mathfrak{a}) \longrightarrow S^n$$

be a continuous map. If $A_1, A_2 \in \mathfrak{A}$ with $A_1 \subset A_2$, then $\mathfrak{a}A_2 \subset \mathfrak{a}A_1$ and u defines a continuous map

$$u_{A_1,A_2}\colon \mathfrak{a}A_1 \divideontimes A_2/(\mathfrak{a}A_1 \divideontimes A_1 \cup \mathfrak{a}A_2 \divideontimes A_2) \longrightarrow S^n .$$

Since there is a canonical homeomorphism

$$\mathfrak{a}A_1 \divideontimes A_2/(\mathfrak{a}A_1 \divideontimes A_1 \cup \mathfrak{a}A_2 \divideontimes A_2) \longleftrightarrow \mathfrak{a}A_1/\mathfrak{a}A_2 \divideontimes A_2/A_1 ,$$

the map u_{A_1,A_2} can be regarded as a map

$$u_{A_1,A_2}\colon \mathfrak{a}A_1/\mathfrak{a}A_2 \divideontimes A_2/A_1 \longrightarrow S^n .$$

The pair $(u, \mathfrak{a})$ is called a *duality map* if $\mathfrak{a}$ is an anti-isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$ and if for every $A_1 \subset A_2$ in $\mathfrak{A}$ the map u_{A_1,A_2} is a duality map in the sense of §5. This is a generalization of the concept of external duality of [14].

Let $\mathfrak{a}_{q,p}\colon S^p\mathfrak{A} \to S^q\mathfrak{A}'$ (where $S^p\mathfrak{A}$, $S^q\mathfrak{A}'$ are polyhedral lattices on S^pX, S^qX', respectively) be defined by $\mathfrak{a}_{q,p}(S^pA) = S^q(aA)$. Then

$$u_{q,p}\colon S^qX' \divideontimes S^pX/G(\mathfrak{a}_{q,p}) \longrightarrow S^{n+p+q}$$

is also a duality map. Similarly $(\bar{u}, \mathfrak{a}^{-1})$ is a duality map (where

$$\bar{u}\colon X \divideontimes X'/G(\mathfrak{a}^{-1}) \longrightarrow S^n$$

is defined by $\bar{u}(x \divideontimes x') = u(x' \divideontimes x)$).

Let $\mathfrak{B}$ be a polyhedral lattice on Y, $\mathfrak{B}'$ be a polyhedral lattice on Y', $\mathfrak{b}$ be an anti-isomorphism $\mathfrak{b}\colon \mathfrak{B} \to \mathfrak{B}'$, and let

$$v\colon Y' \divideontimes Y/G(\mathfrak{b}) \longrightarrow S^n$$

be such that $(v, \mathfrak{b})$ is a duality map. Let $\mathfrak{f}\colon \mathfrak{A} \to \mathfrak{B}$ be a join-homomorphism (by [14; (3.3)] every carrier $\mathfrak{A} \to \mathfrak{B}$ such that $\{x_0\} \to \{y_0\}$ is equivalent to a join-homomorphism). Let $\mathfrak{f}'\colon \mathfrak{B}' \to \mathfrak{A}'$ be dual to $\mathfrak{f}$ under $\mathfrak{a}$, $\mathfrak{b}$ (as in §4 of [14]). This means that $\mathfrak{f}'\mathfrak{b}B = \mathfrak{a}\mathfrak{f}^{\#}B$ where $\mathfrak{f}^{\#}B$ is the largest element of $\mathfrak{A}$

mapped into B by $\mathfrak{f}$. We define $\Gamma(\mathfrak{f}) \subset Y' \wedge X$ by

$$\Gamma(\mathfrak{f}) = \bigcup \{B' \wedge A \mid B' \in \mathfrak{B}', \ A \in \mathfrak{A} \text{ and } B' \subset \mathfrak{b}\mathfrak{f}A\} \ .$$

Since $B' \subset \mathfrak{b}\mathfrak{f}A \Longleftrightarrow \mathfrak{b}^{-1}B' \supset \mathfrak{f}A \Longleftrightarrow A \subset \mathfrak{f}^{\#}\mathfrak{b}^{-1}B' \Longleftrightarrow \mathfrak{a}A \supset \mathfrak{f}'B' \Longleftrightarrow A \subset \mathfrak{a}^{-1}\mathfrak{f}'B$, it follows that under the canonical homeomorphism $Y' \wedge X \longleftrightarrow X \wedge Y'$ the subset $\Gamma(\mathfrak{f})$ corresponds to $\Gamma(\mathfrak{f}')$. We let $\Gamma_{k',k}(\mathfrak{f}) \subset S^{k'}Y' \wedge S^k X$ denote the subset $\bigcup\{S^{k'}B' \wedge S^k A \mid B' \subset \mathfrak{b}\mathfrak{f}A\}$. Then the main result on relative duality is the following analogue of (5.11).

THEOREM (7.1). *There is an isomorphism*

$$D_n(u, \mathfrak{a}, v, \mathfrak{b})\colon \{\mathfrak{f}\} \approx \{\mathfrak{f}'\}$$

such that for k, k' large enough the $\mathfrak{f}$-map $f\colon S^k X \to S^k Y$ and the $\mathfrak{f}'$-map $f'\colon S^{k'} Y' \to S^{k'} X'$ are such that $D_n\{f\} = \{f'\}$ if and only if the following diagram is homotopy commutative

$$\text{(7.2)} \qquad \begin{array}{ccc} S^{k'}Y' \wedge S^k X/\Gamma_{k',k}(\mathfrak{f}) & \xrightarrow{1 \wedge f} & S^{k'}Y' \wedge S^k Y/G(\mathfrak{b}_{k',k}) \\ {\scriptstyle f' \wedge 1}\big\downarrow & & \big\downarrow{\scriptstyle v_{k',k}} \\ S^{k'}X' \wedge S^k X/G(\mathfrak{a}_{k',k}) & \xrightarrow{u_{k',k}} & S^{n+k+k'} \end{array}$$

(*Note that* $(1 \wedge f)(\Gamma_{k',k}(\mathfrak{f})) \subset G(\mathfrak{b}_{k',k})$ *because* $fA \subset \mathfrak{f}A$ *by the assumption that* f *is an* $\mathfrak{f}$*-map. Similarly* $(f' \wedge 1)(\Gamma_{k',k}(\mathfrak{f})) \subset G(\mathfrak{a}_{k',k})$ *because* f' *is an* $\mathfrak{f}'$*-map.*)

The proof of (7.1) involves a repetition for relative theory of the steps leading to (5.11). This entails a development of the theory of spectra with partially ordered collections of subspectra, an analogue of the equivalence theorem (3.5) for such collections of spectra, and the theory of the functional dual $\mathbf{F}(\mathfrak{A})$ of a polyhedral lattice. We define the latter. For each $A \in \mathfrak{A}$ let $\mathbf{A}'$ denote the subspectrum of $\mathbf{F}(X)$ whose k^{th} term equals the space $\{\omega \in F(X, S^k) \mid \omega A = 0\}$. The map $A \to \mathbf{A}'$ is order-reversing and the collection of all $\mathbf{A}'$ forms the functional dual of $\mathfrak{A}$. Though $\mathfrak{A}$ is a lattice, the functional dual need not be (because the set of functions vanishing on $A_1 \cap A_2$ need not equal the union of the set of functions vanishing on A_1 with the set of those vanishing on A_2). This does not cause any trouble as we only need the equivalence theorem analogous to (3.5), and this is true for partially ordered collections of subspectra (which need not be lattices).

Having (7.1), the results of §6 are valid with minor modifications. In particular, the adjunction theorem (6.10) holds for the relative theory, and then the results of [14] can be derived.

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REFERENCES

1. M. G. BARRATT, *Track groups* I, Proc. London Math. Soc. (3), 5 (1955), 71–106.
2. N. BOURBAKI, Topologie Générale, Livre III, Chap. X, Hermann, Paris, 1949.
3. D. E. COHEN, *Products and carrier theory*, Proc. London Math. Soc. (3), 7 (1957), 219-248.
4. S. EILENBERG AND N. E. STEENROD, Foundations of Algebraic Topology, Princeton University Press, 1952.
5. R. H. FOX, *On topologies for function spaces*, Bull. Amer. Math. Soc., 51 (1945), 429–432.
6. E. L. LIMA, Duality and Postinikov invariants, (doctorial dissertation) University of Chicago, 1958.
7. J. MOORE, *On a theorem of Borsuk*, Fund. Math., 43 (1955), 195-201.
8. ———, Homotopy theory (mimeographed notes), Princeton, 1953.
9. D. PUPPE, *Homotopiemengen und ihre induzierten Abbildungen* I, Math. Zeit., 69 (1958), 299–344.
10. E. SPANIER, *Duality and S-theory*, Bull. Amer. Math. Soc., 62, (1956), 194-203.
11. ———, *Infinite symmetric products, function spaces, and duality*, Ann. of Math., 69 (1959), 142-198.
12. ——— and J. H. C. WHITEHEAD, *Duality in homotopy theory*, Mathematika, 2 (1955), 56-80.
13. ——— and ———, *Carriers and S-theory*, in Algebraic Geometry and Topology (A symposium in honor of S. Lefschetz), Princeton University Press, 1957, 330-360.
14. ——— and ———, *Duality in relative homotopy theory*, Ann. of Math., 67 (1958), 203-238.
15. G. W. WHITEHEAD, Homotopy theory (mimeographed notes), Massachusetts Institute of Technology, 1953.
16. J. H. C. WHITEHEAD, *On the homotopy type of ANR*'s, Bull. Amer. Math. Soc., 54 (1948), 1133-1145.
17. ———, *Combinatorial homotopy* I, Bull. Amer. Math. Soc., 55 (1949), 213-245.