On the Non-Existence of Kervaire Invariant One Manifolds

M. Hill[1], M. Hopkins[2], D. Ravenel[3]

[1]University of Virginia

[2]Harvard University

[3]University of Rochester

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- 2: $S^1 \times S^1$
- 6: $SU(2) \times SU(2)$
- 14: $S(\mathbb{O}) \times S(\mathbb{O})$
- 30: (Bökstedt) Related to $E_6/\big(U(1) \times Spin(10)\big)$
- 62: Possibly a similar construction.

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Except possibly in dimension $4k + 2$, where there is an obstruction: Kervaire Invariant.

Adams Spectral Sequence

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- (Adem) $\mathrm{Ext}^1(\mathbb{F}_2, \mathbb{F}_2)$ is generated by classes h_i, $i \geq 0$.
- h_j survives the Adams SS if $\mathbb{R}^{2^j}$ admits a division algebra structure.

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$$d_2(h_{j+1}) = h_0 h_j^2.$$

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Theorem (H.-Hopkins-Ravenel)

For $j \geq 7$, h_j^2 does not survive the Adams SS.

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Reduction to Simpler Cases

Adams SS

Contains our classes $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

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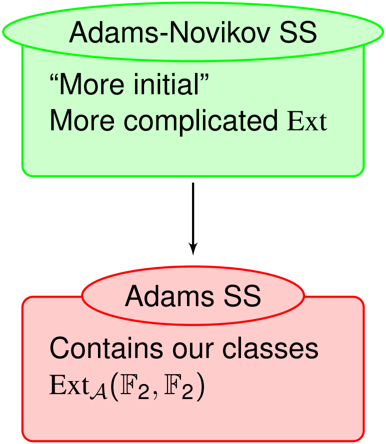

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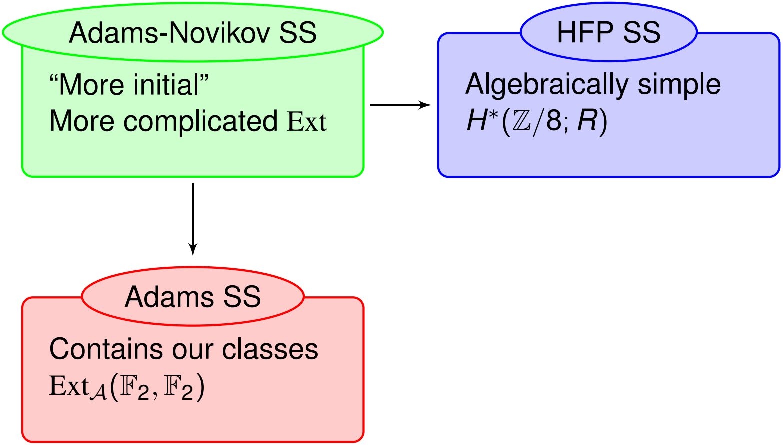

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So good choice of R gives us something that is

- easily computable
- strong enough to detect the classes.

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Example

If $G = \mathbb{Z}/2$, then have $S^{\rho_2} = \mathbb{C}^+$ and S^2.

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We care only about $\rho_8 = 1 \oplus \sigma \oplus L \oplus L^2 \oplus L^3$. Plus the regular reps for subgroups.

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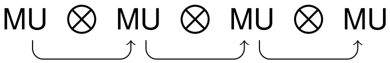

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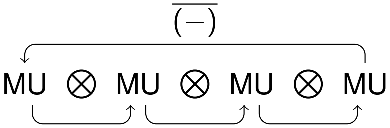

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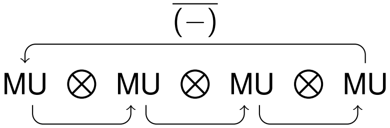

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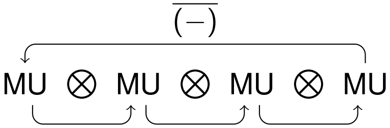

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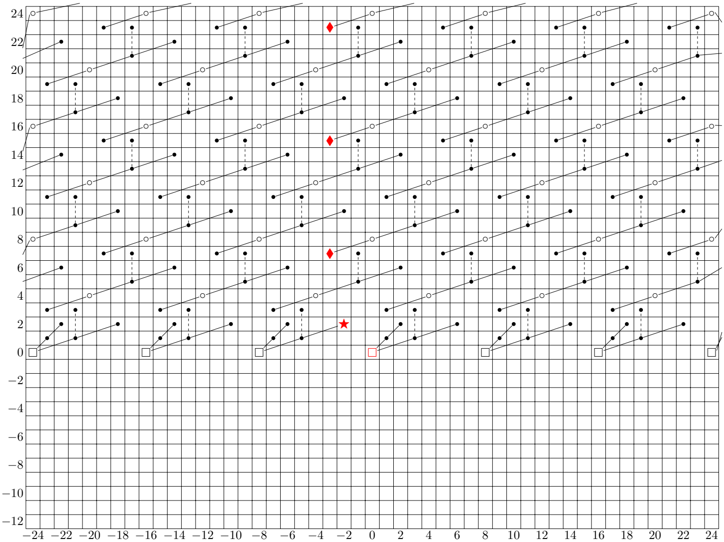
4. Inverting an equivariant class Δ makes the fixed points and homotopy fixed points agree.

Advantages of the Slice SS



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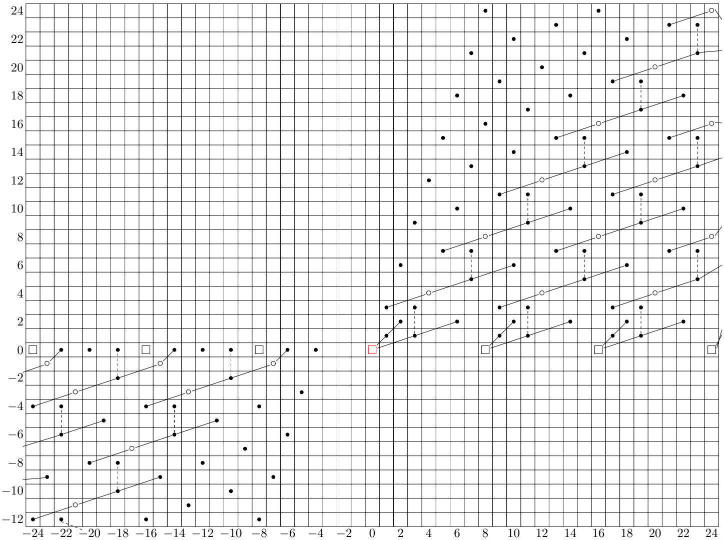

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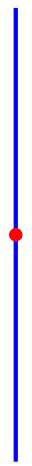

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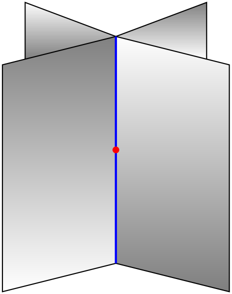

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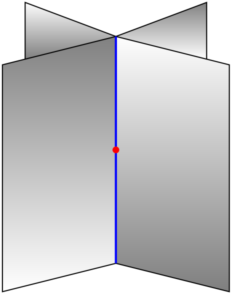

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Maps determined by $H_*(C_\bullet) = H_*(S^3)$.

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2. If $k \leq 0$, then we look at the associated *co*chain algebra.
3. In the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1, 1)$.

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- Inversion is a colimit and first steps show $\pi_{-2} = 0$. □

Take Home Message

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Happy A_5 Birthday,
Bob and Ron!