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The circle action on topological Hochschild homology of complex cobordism and the Brown–Peterson spectrum

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Abstract

We specify exterior generators in $\pi_* THH(MU) = \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1)$ and $\pi_* THH(BP) = \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1)$, and calculate the action of the σ-operator on these graded rings. In particular, $\sigma(\lambda'_n) = 0$ and $\sigma(\lambda_n) = 0$, while the actions on $\pi_*(MU)$ and $\pi_*(BP)$ are expressed in terms of the right units η_R in the Hopf algebroids $(\pi_*(MU), \pi_*(MU \wedge MU))$ and $(\pi_*(BP), \pi_*(BP \wedge BP))$, respectively.

1. *Introduction*

Let S be the sphere spectrum. For any (associative) S-algebra R, the topological Hochschild homology spectrum $THH(R)$ is the geometric realization of a cyclic spectrum $[q] \mapsto THH(R)_q = R \wedge R^{\wedge q}$, see [**9**, **19**]. The skeleton filtration of $THH(R)$ leads to a spectral sequence

$$E^1_{q,*} = \pi_{q+*}(sk_q THH(R), sk_{q-1} THH(R)) \Longrightarrow \pi_{q+*} THH(R),$$

whose (E^1, d^1)-term is the normalized chain complex associated to the simplicial graded abelian group

$$[q] \mapsto \pi_* THH(R)_q = \pi_*(R \wedge R^{\wedge q}).$$

The cyclic structure specifies a natural circle action on $THH(R)$, which we shall treat as a right action. The cofiber sequence $1_+ \to S^1_+ \to S^1$ is split by a retraction $S^1_+ \to 1_+$ and a stable section $S^1 \to S^1_+$. We write σ for the composite map $THH(R) \wedge S^1 \to THH(R) \wedge S^1_+ \to THH(R)$ and call the induced homomorphism $\sigma\colon \pi_* THH(R) \to \pi_{*+1} THH(R)$ the (right) σ-operator. It satisfies $\sigma^2 = \eta\sigma$, where $\eta \in \pi_1(S)$ is the complex Hopf map, so if multiplication by η acts trivially on $\pi_* THH(R)$ then σ is a differential.

There is a spectral sequence

$$E^2_{*,*} = \hat{H}^{-*}(S^1; \pi_* THH(R)) = \mathbb{Z}[t, t^{-1}] \otimes \pi_* THH(R) \Longrightarrow \pi_* THH(R)^{tS^1} \tag{1.1}$$

converging to the homotopy of the circle Tate construction on $THH(R)$, see [**20**], more recently known [**23**] as the periodic topological cyclic homology $\pi_* TP(R)$. Its initial differential is given by

$$d^2(t^n \cdot x) = \begin{cases} t^{n+1} \cdot \sigma(x) & \text{for } n \text{ even} \\ t^{n+1} \cdot (\sigma(x) + \eta x) & \text{for } n \text{ odd.} \end{cases}$$

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Knowledge of the σ-operator therefore leads to knowledge of the $E^3 = E^4$-term of this spectral sequence. When η acts trivially on $\pi_* THH(R)$ we can write

$$E^4_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes H(\pi_* THH(R), \sigma).$$

In this paper, we determine the σ-operator on $\pi_* THH(MU)$ and $\pi_* THH(BP)$, where MU is the complex cobordism E_∞ ring spectrum [**33**, **37**, **40**] and BP is the Brown–Peterson E_4 ring spectrum [**5**, **12**]. In these cases $THH(R)$ is an E_∞, respectively, E_3, ring spectrum by [**13**], σ is a (right) derivation by [**2**], and the skeleton and Tate spectral sequences are algebra spectral sequences [**22**].

In Sections 2 and 3, we review the connection between complex cobordism and formal group laws, and their p-typical variants, including some explicit formulae in the Hopf algebroids

$$(\pi_*(MU), \pi_*(MU \wedge MU)) \cong (L, LB) \cong (L, LC)$$

and

$$(\pi_*(BP), \pi_*(BP \wedge BP)) \cong (V, VT).$$

We follow the expositions by Adams [**1**] and Landweber [**27**, **28**] of Quillen's theory [**42**], adding some less familiar details about the parametrization of strict isomorphisms of formal group laws by 'moving coordinates' using (L, LC), in place of 'absolute coordinates' using (L, LB).

In Section 4, we obtain isomorphisms of simplicial commutative rings

$$\pi_* THH(MU)_\bullet \cong \pi_*(MU) \otimes \beta(B)_\bullet \cong \pi_*(MU) \otimes \beta(C)_\bullet,$$

in the spirit of the equivalence $THH(MU) \simeq MU \wedge BBU_+$ of Blumberg, Cohen and Schlichtkrull [**6**]. Here, $\beta(B)_\bullet$ denotes the simplicial bar construction $[q] \mapsto \beta(B)_q = B^{\otimes q}$, and similarly for $\beta(C)_\bullet$. We also obtain analogous information for $\pi_* THH(BP)_\bullet$.

In Section 5, we recognize the circle action on the 0-simplices in $THH(MU)$ and $THH(BP)$ as being given by the right units $\eta_R \colon L \to LB \cong LC$ and $\eta_R \colon V \to VT$, respectively, and use this to determine the action of the σ-operator on $\pi_* THH(MU)$ and $\pi_* THH(BP)$. More precisely, in Proposition 5.1 we prove that for $x \in \pi_*(R)$ the homotopy class $\sigma(x) \in \pi_{*+1} THH(R)$ is detected in $E^\infty_{1,*}$ of the skeleton spectral sequence by the class of $(1 \wedge \pi)\eta_R(x) \in \pi_*(R \wedge R/S) = E^1_{1,*}$. Here, $\eta_R \colon R \cong S \wedge R \to R \wedge R$ and $\pi \colon R \to R/S$ are the evident maps.

According to McClure and Staffeld [**34**], who credit Andy Baker and Larry Smith, there are isomorphisms

$$\pi_* THH(MU) \cong \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1)$$

with λ'_n in degree $2n+1$, and

$$\pi_* THH(BP) \cong \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1)$$

with λ_n in degree $2p^n - 1$, at each prime p. We strengthen these results, in Theorems 5.3 and 5.6, to show that the exterior generators λ'_n and λ_n can be chosen so that $\sigma(\lambda'_n) = 0$ and $\sigma(\lambda_n) = 0$, for all $n \geqslant 1$. These choices are naturally connected to the moving coordinates on strict isomorphisms between formal group laws, or p-typical formal group laws, as made precise in Propositions 4.4 and 4.6. On the other hand, we show in Theorem 5.4 that $\sigma(e_3)$ and $\sigma(e_4)$ are nonzero for the alternative sequence of exterior generators e_n of $\pi_* THH(MU)$ associated, as in Proposition 4.5, to absolute coordinates.

We can summarize Proposition 4.6, Theorem 5.6 and equations (3.1) and (5.1) as follows.

Theorem 1.1. *Let $\pi_*(BP) = \mathbb{Z}_{(p)}[v_n \mid n \geqslant 1]$ where the v_n are the Hazewinkel generators. The σ-operator $\sigma \colon \pi_* THH(BP) \to \pi_{*+1} THH(BP)$ is a (right) derivation acting on*

$$\pi_* THH(BP) = \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1).$$

It satisfies $\sigma(\lambda_n) = 0$ for all $n \geqslant 1$, while $\sigma(v_n)$ is recursively determined by the equation

$$p\lambda_n = \sigma(v_n) + \sum_{i=1}^{n-1} \Big(v_{n-i}^{p^i} \lambda_i + (p^i \ell_i) v_{n-i}^{p^i - 1} \sigma(v_{n-i}) \Big).$$

Here, $p^i \ell_i \in \pi_*(BP)$ is recursively determined by

$$p\ell_n = \sum_{i=0}^{n-1} \ell_i v_{n-i}^{p^i} = v_n + \ell_1 v_{n-1}^p + \cdots + \ell_{n-1} v_1^{p^{n-1}}.$$

In Section 6, we evaluate the d^2-differential in the circle Tate spectral sequences for MU and BP, in a finite range of degrees, and use this to calculate the resulting $E^3 = E^4$-term. For example

$$E^4 = \mathbb{Z}[t, t^{-1}] \otimes H(\pi_* THH(BP), \sigma) \Longrightarrow \pi_* THH(BP)^{tS^1}$$

where

$$H(\pi_* THH(BP), \sigma) = \begin{cases} \mathbb{Z}_{(p)}\{1\} & \text{for } * = 0, \\ \mathbb{Z}/p\{v_1^{i-1}\lambda_1\} & \text{for } * = i(2p-2)+1,\ 1 \leqslant i \leqslant p-1, \\ \mathbb{Z}/p^2\{v_1^{p-1}\lambda_1\} & \text{for } * = 2p^2 - 2p + 1, \\ \mathbb{Z}/p^2\{\lambda_2\} & \text{for } * = 2p^2 - 1, \\ \mathbb{Z}_{(p)}/p^2(p+2)\{v_2\lambda_1 + v_1\lambda_2\} & \text{for } * = 2p^2 + 2p - 3, \\ \mathbb{Z}/p\{\lambda_1\lambda_2\} & \text{for } * = 2p^2 + 2p - 2, \\ 0 & \text{for the remaining } * \leqslant 2p^2 + 4p - 6. \end{cases}$$

This appears as Theorem 6.6 in the body of the paper. In particular, we see that while $H(\pi_* THH(BP), \sigma)$ is concentrated in odd degrees for $0 < * < 2p^2 + 2p - 2$, this ceases to be true in degree $|\lambda_1\lambda_2| = 2p^2 + 2p - 2$.

The cyclic structure on $THH(R)$ suffices to define the circle homotopy fixed points $THH(R)^{hS^1}$ and the circle Tate construction $THH(R)^{tS^1}$. When enriched to a cyclotomic structure [**9**, **39**], these data suffice to define the topological cyclic homology $TC(R)$, which is a powerful invariant [**17**] of the algebraic K-theory $K(R)$, especially for connective S-algebras R. The graded rings $\pi_* THH(\mathbb{F}_p)$ and $\pi_* THH(\mathbb{Z})$ were calculated by Bökstedt, and formed the basis for calculations of $TC(\mathbb{F}_p)$ and $TC(\mathbb{Z})$, see [**24**] for the case of the prime field $\mathbb{F}_p$, [**10**, **11**] for the integers localized at an odd prime p, and [**45**–**48**] for the integers localized at $p = 2$. The topological Hochschild homology of $R = \ell$ (the Adams summand in p-local connective complex K-theory) was worked out for $p \geqslant 5$ in [**34**] and promoted to a calculation of $TC(\ell)$ in [**4**]. In all of these cases, the σ-operator acts trivially on $\pi_* THH(R)$.

When $R = S$ the circle action on $THH(R) = S$ is trivial, so the d^2-differential in the circle Tate spectral sequence alternates between zero and multiplication by η in $\pi_* THH(R) = \pi_*(S)$. The resulting Tate spectral sequence agrees with the Atiyah–Hirzebruch spectral sequence for $S^{tS^1} \simeq \Sigma^2 \mathbb{C}P^{\infty}_{-\infty}$. Knowledge of the attaching maps in complex projective spaces translates to substantial knowledge [**38**] of the differential patterns in this spectral sequence. However, the limits on our knowledge of $\pi_*(S)$ put bounds on how well we can understand $TC(S)$ and $K(S)$ by this approach. Explicit calculations in low degrees were made in [**7**, **49**, **50**], but it would be desirable to place these in a context of systematic patterns, similar to the chromatic filtration in stable homotopy theory [**36**, **44**]. By the descent results of [**18**], a good understanding of $\pi_* TC(MU \wedge \cdots \wedge MU)$ (with one or more copies of MU) will also determine $\pi_* TC(S)$, through a homotopy limit or descent spectral sequence. The problem of determining

$TC(MU)$ and $K(MU)$ has therefore been frequently considered, for example, by Ausoni and the author at the time when [4] was completed. In this case the σ-operator acts nontrivially on $\pi_* THH(MU)$, but precise formulae seem not to have been worked out before this paper.

The author has also pursued a homological approach [14] to the calculations of $THH(R)^{hS^1}$ and $THH(R)^{tS^1}$ for S-algebras such as $R = MU$, working with continuous homology in the category of completed A_*-comodule algebras. This led, in [8, 31, 32], to a proof that there are p-adic equivalences

$$THH(MU)^{hS^1} \xleftarrow{\Gamma} TF(MU;p) \xrightarrow{\hat{\Gamma}} THH(MU)^{tS^1},$$

where $TF(MU;p) = \operatorname{holim}_n THH(MU)^{C_{p^n}}$. This provides the foundation for a calculation of $\pi_* THH(MU)^{C_{p^n}}$ by induction on n. We plan to discuss the homological approach to $THH(MU)^{tS^1}$ in a future paper.

2. *Formal group laws and moving coordinates*

2.1. *Formal group laws and complex cobordism*

The universal (commutative, 1-dimensional) formal group law

$$F(x,y) = x + y + \sum_{i,j \geqslant 1} a_{ij} x^i y^j$$

is defined over the Lazard ring $L = \mathbb{Z}[a_{ij} \mid i,j \geqslant 1]/I$ where

$$\begin{aligned} I = (&a_{12} - a_{21}, a_{13} - a_{31}, a_{14} - a_{41}, a_{23} - a_{32}, \dots, \\ &2a_{11}a_{12} + 3a_{13} - 2a_{22}, 2a_{12}^2 + 3a_{11}a_{13} + 4a_{14} - 2a_{23}, \\ &a_{11}^2 a_{12} + 3a_{12}^2 + 6a_{11}a_{13} - a_{11}a_{22} + 6a_{14} - 3a_{23}, \dots) \end{aligned}$$

is the ideal generated by the coefficients of $x^i y^j$ in $F(x,y) - F(y,x)$ and of $x^i y^j z^k$ in $F(F(x,y),z) - F(x,F(y,z))$. Each ring homomorphism $\theta\colon L \to R$ determines a formal group law

$$(\theta_* F)(x,y) = x + y + \sum_{i,j \geqslant 1} \theta(a_{ij}) x^i y^j$$

defined over R, and this specifies a natural bijection between ring homomorphisms $L \to R$ and formal group laws defined over R. We say that L classifies, or corepresents, formal group laws. We give L the grading where a_{ij} has degree $2(i+j-1)$. Lazard [29] proved that $L \cong \mathbb{Z}[x_n \mid n \geqslant 1]$ where x_n has degree $2n$. Following Adams [1, p. 57], augmented with a little computer algebra, we can take

$$\begin{aligned} x_1 &= a_{11} \\ x_2 &= a_{12} \\ x_3 &= a_{22} - a_{13} \\ x_4 &= a_{14} \end{aligned}$$

as the first four generators of the Lazard ring. Quillen [42, Theorem 2] showed that the tensor product formula for the first Chern class in complex cobordism theory specifies a formal group law over $\pi_*(MU)$, and that the homomorphism $L \to \pi_*(MU)$ that classifies this formal group law is an isomorphism.

2.2. *Strict isomorphisms*

Let F and F' be formal group laws defined over R. A strict isomorphism $f: F \to F'$ over R is a formal power series

$$f(x) = x + \sum_{n \geqslant 1} b_n x^{n+1}$$

with $b_n \in R$, such that $f(F(x,y)) = F'(f(x), f(y))$. If R is torsion-free then there is at most one strict isomorphism from F to F'. Let $B = \mathbb{Z}[b_n \mid n \geqslant 1]$, graded so that b_n has degree $2n$. The tensor product $LB = L \otimes B$ then classifies diagrams

$$F \xrightarrow{f} F'$$

where F and F' are formal group laws and f is a strict isomorphism. Restriction along the inclusions $\eta_L: L \to LB$ and $\iota: B \to LB$ lets us recover F and f, respectively, while restriction along the right unit $\eta_R: L \to LB$ classifies F'. Continuing Adams' calculations [**1**, p. 63], we have

$$\begin{aligned}
\eta_R(a_{11}) &= a_{11} + 2b_1 \\
\eta_R(a_{12}) &= a_{12} + a_{11}b_1 + (3b_2 - 2b_1^2) \\
\eta_R(a_{13}) &= a_{13} + a_{11}(2b_2 - 2b_1^2) + (4b_3 - 8b_1b_2 + 4b_1^3) \\
\eta_R(a_{22}) &= a_{22} + (2a_{12} + a_{11}^2)b_1 + a_{11}(6b_2 - 3b_1^2) + (6b_3 - 6b_1b_2 + 2b_1^3) \\
\eta_R(a_{14}) &= a_{14} - a_{13}b_1 + a_{12}(b_2 - b_1^2) + a_{11}(3b_3 - 8b_1b_2 + 5b_1^3) \\
&\quad + (5b_4 - 14b_1b_3 - 6b_2^2 + 25b_1^2b_2 - 10b_1^4) \\
\eta_R(a_{23}) &= a_{23} + a_{13}b_1 + 2a_{11}a_{12}b_1 + a_{12}(8b_2 - 6b_1^2) + a_{11}^2(3b_2 - 2b_1^2) \\
&\quad + a_{11}(12b_3 - 16b_1b_2 + 6b_1^3) + (10b_4 - 16b_1b_3 - 3b_2^2 + 14b_1^2b_2 - 4b_1^4),
\end{aligned}$$

where we have corrected a (rare) typographical error in Adams' formula for $\eta_R(a_{22})$. It follows that

$$\begin{aligned}
\eta_R(x_1) &= x_1 + 2b_1 \\
\eta_R(x_2) &= x_2 + x_1b_1 + (3b_2 - 2b_1^2) \\
\eta_R(x_3) &= x_3 + (2x_2 + x_1^2)b_1 + x_1(4b_2 - b_1^2) + (2b_3 + 2b_1b_2 - 2b_1^3) \\
\eta_R(x_4) &= x_4 + (2x_1x_2 - 2x_3)b_1 + x_2(b_2 - b_1^2) + x_1(3b_3 - 8b_1b_2 + 5b_1^3) \\
&\quad + (5b_4 - 14b_1b_3 - 6b_2^2 + 25b_1^2b_2 - 10b_1^4).
\end{aligned} \tag{2.1}$$

2.3. *Hopf algebroids*

The formal group laws and strict isomorphisms defined over R form the objects and morphisms of a small groupoid $\mathcal{G}(R)$, depending functorially on the commutative ring R. The identity morphism $id_F: F \to F$ is the formal power series $id_F(x) = x$, which is classified by an augmentation $\epsilon: LB \to L$. It has the form $id \otimes \epsilon$, where $\epsilon: B \to \mathbb{Z}$ maps each b_n to zero. The composite of two strict isomorphisms $f: F \to F'$ and $f': F' \to F''$ is a strict isomorphism $f'f: F \to F''$. The composition pairing $\circ$ is a natural function

$$\begin{aligned}
\mathcal{G}(R)(F', F'') \times \mathcal{G}(R)(F, F') &\longrightarrow \mathcal{G}(R)(F, F'') \\
\circ: (f', f) &\longrightarrow f'f.
\end{aligned}$$

The opposite pairing $\bullet\colon (f, f') \mapsto f'f$ is classified by a coproduct $\psi\colon LB \to LB \otimes_L LB$, where L acts through η_R on the left-hand tensor factor and through η_L on the right-hand tensor factor. When composed with the obvious isomorphism $LB \otimes_L LB \cong L \otimes B \otimes B$ it takes the form $id \otimes \psi$, where $\psi\colon B \to B \otimes B$ defines the coproduct in a Hopf algebra structure on B. In low degrees,

$$\begin{aligned}
\psi(b_1) &= b_1 \otimes 1 + 1 \otimes b_1 \\
\psi(b_2) &= b_2 \otimes 1 + 2b_1 \otimes b_1 + 1 \otimes b_2 \\
\psi(b_3) &= b_3 \otimes 1 + (b_1^2 + 2b_2) \otimes b_1 + 3b_1 \otimes b_2 + 1 \otimes b_3 \\
\psi(b_4) &= b_4 \otimes 1 + (2b_1b_2 + 2b_3) \otimes b_1 + (3b_1^2 + 3b_2) \otimes b_2 + 4b_1 \otimes b_3 + 1 \otimes b_4,
\end{aligned}$$

see [1, p. 91]. The inverse $f^{-1}\colon F' \to F$ of a strict isomorphism $f\colon F \to F'$ is classified by a homomorphism $\chi\colon LB \to LB$. Its restriction along $\eta_L\colon L \to LB$ is η_R, while its restriction along $\iota\colon B \to LB$ is $\iota\chi$, where $\chi\colon B \to B$ is the conjugation in the Hopf algebra structure on B. Following [1, p. 65], for $f(x) = x + \sum_{n\geqslant 1} b_n x^{n+1}$ we have

$$f^{-1}(x) = x + \sum_{n\geqslant 1} \bar{b}_n x^{n+1},$$

where $\bar{b}_n = \chi(b_n)$ is given in low degrees by

$$\begin{aligned}
\bar{b}_1 &= -b_1 \\
\bar{b}_2 &= 2b_1^2 - b_2 \\
\bar{b}_3 &= -5b_1^3 + 5b_1b_2 - b_3 \\
\bar{b}_4 &= 14b_1^4 - 21b_1^2b_2 + 3b_2^2 + 6b_1b_3 - b_4.
\end{aligned}$$

One might now like to say that L, LB, η_L, η_R and ψ classify the objects, morphisms, sources, targets and composition in the groupoid $\mathcal{G}(R)$, but due to the reversal of ordering in the pairing $\bullet$, this is not quite correct. Instead, L and LB classify the objects and morphisms in the opposite groupoid, $\mathcal{G}^{op}(R)$. A homomorphism $LB \to R$ corresponds to a diagram $f\colon F \to F'$, as above, which we can view as a morphism in $\mathcal{G}^{op}(R)$ with source F' and target F. Then η_L and η_R classify the target and source, respectively, and ψ classifies the composition $(f, f') \mapsto f \bullet f'$ in $\mathcal{G}^{op}(R)$, where $f'\colon F' \to F''$ is as before.

Alternatively, we can focus on the inverse strict isomorphism $\phi = f^{-1}\colon F' \to F$, in place of $f\colon F \to F'$. We think of L as classifying formal group laws in the same way as before, but now we think of LB as classifying diagrams

$$F \xleftarrow{\phi} F',$$

where F and F' are formal group laws and ϕ is the strict isomorphism

$$\phi(x) = x + \sum_{n\geqslant 1} \bar{b}_n x^{n+1}$$

with $\bar{b}_n \in B$ included by $\iota\colon B \to LB$. The $\bar{b}_n$ provide a second polynomial basis for LB over L, so that $LB \cong L\bar{B} = L[\bar{b}_n \mid n \geqslant 1]$. Then L and LB corepresent the objects and morphisms in $\mathcal{G}(R)$, η_L and η_R corepresent the target and source of a morphism, in that order, ϵ corepresents the identity morphism, ψ corepresents the composition pairing

$$\begin{aligned}
\mathcal{G}(R)(F', F) \times \mathcal{G}(R)(F'', F') &\longrightarrow \mathcal{G}(R)(F'', F) \\
\circ\colon (\phi, \phi') &\longrightarrow \phi\phi',
\end{aligned}$$

and χ corepresents passage to the inverse of a morphism.

Novikov [41] and Landweber [26] studied the cohomology operations in complex cobordism, which are represented by classes in $MU^*(MU)$. Turning instead to homology, Adams [1, Lemma 4.5(ii)] showed that

$$\pi_*(MU \wedge MU) \cong \pi_*(MU)[b_n \mid n \geqslant 1]$$

for specific classes $b_n \in \pi_*(MU \wedge MU)$, so that Quillen's isomorphism $L \cong \pi_*(MU)$ extends to an isomorphism $LB \cong \pi_*(MU \wedge MU)$. By the results of [1, § 11], the left and right units $\eta_L, \eta_R \colon L \to LB$ correspond to the homomorphisms induced by the maps $MU \cong MU \wedge S \to MU \wedge MU$ and $MU \cong S \wedge MU \to MU \wedge MU$, respectively. Likewise, the augmentation $\epsilon \colon LB \to L$ is induced by the multiplication $MU \wedge MU \to MU$. The coproduct $\psi \colon LB \to LB \otimes_L LB$ is induced by the map $MU \wedge MU \cong MU \wedge S \wedge MU \to MU \wedge MU \wedge MU$, via the isomorphism

$$\pi_*(MU \wedge MU) \otimes_{\pi_*(MU)} \pi_*(MU \wedge MU) \xrightarrow{\cong} \pi_*(MU \wedge MU \wedge MU).$$

Finally, the conjugation $\chi \colon LB \to LB$ is induced by the twist map $\tau \colon MU \wedge MU \cong MU \wedge MU$. In all cases, the unlabeled map $S \to MU$ is the unit map in the ring spectrum structure. Using the terminology introduced by Haynes Miller [35], (L, LB) and $(\pi_*(MU), \pi_*(MU \wedge MU))$ are isomorphic as Hopf algebroids.

2.4. *Moving coordinates*

So far we have classified strict isomorphisms $f(x) = x + \sum_{n\geqslant 1} b_n x^{n+1}$ or $\phi(x) = x + \sum_{n\geqslant 1} \bar{b}_n x^{n+1}$ in a way that is independent of the source and target of f and ϕ. Such 'absolute coordinates' exist, because the Hopf algebroid (L, LB) is split. Following Araki [3, Proposition 2.10] and Landweber [28], we can instead classify strict isomorphisms in terms of 'moving coordinates'. This will lead to nicer formulae for the σ-operator in $\pi_* THH(MU)$. The strict isomorphism

$$F \xleftarrow{\phi} F'$$

can be uniquely written as a formal sum

$$\phi(x) = x +_F \sum_{n\geqslant 1}^{F} c_n x^{n+1},$$

with respect to the target formal group law, for a sequence of elements $c_n \in LB$ with c_n in degree $2n$. (In spite of the notation, these are essentially unrelated to the Chern classes.) In low degrees,

$$\begin{aligned}
c_1 &= \bar{b}_1 \\
c_2 &= -a_{11}\bar{b}_1 + \bar{b}_2 \\
c_3 &= -a_{12}\bar{b}_1 + a_{11}^2\bar{b}_1 - a_{11}\bar{b}_2 + \bar{b}_3 \\
c_4 &= (a_{11}^2 - a_{12})\bar{b}_1^2 - (a_{11}^3 - 2a_{11}a_{12} + a_{13})\bar{b}_1 + (a_{11}^2 - a_{11}\bar{b}_1 - a_{12})\bar{b}_2 - a_{11}\bar{b}_3 + \bar{b}_4
\end{aligned}$$

so that

$$\begin{aligned}
c_1 &= -b_1 \\
c_2 &= a_{11}b_1 + (2b_1^2 - b_2) \\
c_3 &= a_{12}b_1 - a_{11}^2 b_1 + a_{11}(b_2 - 2b_1^2) + (-5b_1^3 + 5b_1b_2 - b_3) \\
c_4 &= 14b_1^4 + (a_{11}^2 - a_{12})b_1^2 - 21b_1^2b_2 + (a_{11}^2 + a_{11}b_1 - a_{12})(2b_1^2 - b_2) \\
&\quad + a_{11}(5b_1^3 - 5b_1b_2 + b_3) + (a_{11}^3 - 2a_{11}a_{12} + a_{13})b_1 + 3b_2^2 + 6b_1b_3 - b_4.
\end{aligned}$$

Hence,

$$
\begin{aligned}
c_1 &= -b_1 \\
c_2 &= x_1 b_1 + (2b_1^2 - b_2) \\
c_3 &= x_2 b_1 - x_1^2 b_1 + x_1(b_2 - 2b_1^2) + (-5b_1^3 + 5b_1 b_2 - b_3) \\
c_4 &= 14b_1^4 + (x_1^2 - x_2)b_1^2 - 21b_1^2 b_2 + (x_1^2 + x_1 b_1 - x_2)(2b_1^2 - b_2) \\
&\quad + x_1(5b_1^3 - 5b_1 b_2 + b_3) + (x_1^3 - 4x_1 x_2 + 2x_3)b_1 + 3b_2^2 + 6b_1 b_3 - b_4.
\end{aligned} \tag{2.2}
$$

The moving coordinates c_n form yet another polynomial basis for LB over L, so that $LB \cong LC = L[c_n \mid n \geqslant 1]$. This specifies an isomorphism of Hopf algebroids $(L, LB) \cong (L, LC)$. The left unit $\eta_L \colon L \to LC$ is given by the evident inclusion, and the augmentation $\epsilon \colon LC \to L$ sends each c_n to zero, for $n \geqslant 1$. The right unit $\eta_R \colon L \to LC$ corepresents the source F' of the strict isomorphism $\phi \colon F' \to F$ defined as above. In the next subsection, we shall obtain a useful formula for this right unit homomorphism.

2.5. *Logarithms*

The additive formal group law F_a is defined by $F_a(x, y) = x + y$. Working over $L \otimes \mathbb{Q} \cong \pi_*(MU) \otimes \mathbb{Q}$ there is a unique strict isomorphism

$$
F_a \xleftarrow{\log} F
$$

from the universal formal group law to the additive one, which we can write as

$$
\log(x) = x + \sum_{n \geqslant 1} m_n x^{n+1}
$$

for unique elements $m_n \in L \otimes \mathbb{Q}$, with m_n in degree $2n$. See [**1**, Corollary 7.15] or [**43**, Theorem A2.1.6]. Let $\exp(x) = x + \sum_{n \geqslant 1} \bar{m}_n x^{n+1}$ be the inverse strict isomorphism, from F_a to F. Then $\log F(x, y) = \log(x) + \log(y)$ and

$$
F(x, y) = \exp(\log(x) + \log(y))
$$

over $L \otimes \mathbb{Q}$. We can express the $\bar{m}_n$, the a_{ij} and the x_n as integer polynomials in the logarithmic coefficients m_n. In low degrees,

$$
\begin{aligned}
\bar{m}_1 &= -m_1 \\
\bar{m}_2 &= 2m_1^2 - m_2 \\
\bar{m}_3 &= -5m_1^3 + 5m_1 m_2 - m_3 \\
\bar{m}_4 &= 14m_1^4 - 21m_1^2 m_2 + 3m_2^2 + 6m_1 m_3 - m_4
\end{aligned}
$$

and

$$
\begin{aligned}
a_{11} &= -2m_1 \\
a_{12} &= 4m_1^2 - 3m_2 \\
a_{13} &= -8m_1^3 + 12m_1 m_2 - 4m_3 \\
a_{22} &= -20m_1^3 + 24m_1 m_2 - 6m_3 \\
a_{14} &= 16m_1^4 - 36m_1^2 m_2 + 9m_2^2 + 16m_1 m_3 - 5m_4 \\
a_{23} &= 72m_1^4 - 132m_1^2 m_2 + 27m_2^2 + 44m_1 m_3 - 10m_4.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
x_1 &= -2m_1 \\
x_2 &= 4m_1^2 - 3m_2 \\
x_3 &= -12m_1^3 + 12m_1m_2 - 2m_3 \\
x_4 &= 16m_1^4 - 36m_1^2m_2 + 9m_2^2 + 16m_1m_3 - 5m_4.
\end{aligned}
\tag{2.3}
$$

The resulting homomorphism $L \to \mathbb{Z}[m_n \mid n \geqslant 1]$ then induces an isomorphism $L \otimes \mathbb{Q} \cong \mathbb{Q}[m_n \mid n \geqslant 1]$, so after rationalization the classes m_n serve as another set of polynomial generators for $L \cong \mathbb{Z}[x_n \mid n \geqslant 1]$. The rational classes m_n are canonically defined, as opposed to the integral classes x_n.

The right unit $\eta_R \colon L \to LB$ can be calculated using the identity

$$
\sum_{n\geqslant 0} \eta_R(m_n) = \sum_{i\geqslant 0} m_i \left(\sum_{j \geqslant 0} \bar{b}_j \right)^{i+1} \tag{2.4}
$$

in $LB \otimes \mathbb{Q}$, where $m_0 = 1$, $\bar{b}_0 = 1$ and $\eta_R(m_n)$ is equal to the degree $2n$ part of either side of the formula. See [**1**, Proposition 9.4] or [**43**, Theorem A2.1.16]. Working instead with moving coordinates we obtain the following formula, which does not seem to appear in the standard references.

Proposition 2.1. *The right unit $\eta_R \colon L \to LC$ is determined by the formula*

$$
\eta_R(m_n) = \sum_{(i+1)(j+1)=n+1} m_i c_j^{i+1}
$$

in $LC \otimes \mathbb{Q}$, where $m_0 = 1$ and $c_0 = 1$. The sum runs over the indices $i, j \geqslant 0$ with $(i+1)(j+1) = n+1$.

Proof. The proofs of [**1**, Theorem 16.1(i); **28**, Theorem 3(i)] readily carry over from the p-typical situation to the general one. The formal sum

$$
\phi(x) = \sum_{n\geqslant 0}^{F} c_n x^{n+1}
$$

defines a strict isomorphism

$$
(\eta_L)_*(F) \xleftarrow{\phi} (\eta_R)_*(F)
$$

of formal group laws over LC. The strict isomorphism $\log \colon F \to F_a$ over $L \otimes \mathbb{Q}$ induces strict isomorphisms

$$
\begin{aligned}
(\eta_L)_*(\log) \colon (\eta_L)_*(F) &\longrightarrow (\eta_L)_*(F_a) = F_a \\
(\eta_R)_*(\log) \colon (\eta_R)_*(F) &\longrightarrow (\eta_R)_*(F_a) = F_a
\end{aligned}
$$

over $LC \otimes \mathbb{Q}$. By their uniqueness we must have

$$
(\eta_R)_*(\log) = (\eta_L)_*(\log) \circ \phi.
$$

Hence,

$$\sum_{n\geqslant 0}\eta_R(m_n)x^{n+1} = \log\left(\sum_{j\geqslant 0}^{F} c_j x^{j+1}\right) = \sum_{j\geqslant 0}\log(c_j x^{j+1}) = \sum_{i,j\geqslant 0} m_i (c_j x^{j+1})^{i+1}.$$

Concentrating on the coefficients of x^{n+1} yields the formula. □

3. *The p-typical case*

Let p be any prime. A (1-dimensional, commutative) formal group law F over a torsion-free $\mathbb{Z}_{(p)}$-algebra R is p-typical [**15**, **16**; **43**, Definition A2.1.17] if its logarithmic coefficients satisfy $m_n = 0$ unless $n+1$ is a power of p. In other words,

$$\log(x) = x + \sum_{n\geqslant 1}\ell_n x^{p^n}$$

for a sequence of coefficients $\ell_n \in R\otimes\mathbb{Q}$, with ℓ_n in degree $2(p^n-1)$. There is a universal p-typical formal group law F, defined over the $\mathbb{Z}_{(p)}$-subalgebra $V\subset\mathbb{Q}[\ell_n \mid n\geqslant 1]$ generated by the coefficients of the formal power series

$$F(x,y) = \log^{-1}(\log(x)+\log(y)).$$

By the universal property of the Lazard ring, there is a ring homomorphism $\alpha\colon L\otimes\mathbb{Z}_{(p)}\to V$ classifying the underlying formal group law of F. Conversely, each formal group law over a $\mathbb{Z}_{(p)}$-algebra is strictly isomorphic to a unique p-typical one. Hence, there is a ring homomorphism $\beta\colon V\to L\otimes\mathbb{Z}_{(p)}$ classifying the p-typification of the universal formal group law. The composite $\alpha\beta\colon V\to L\otimes\mathbb{Z}_{(p)}\to V$ is the identity, and the composite $e=\beta\alpha\colon L\otimes\mathbb{Z}_{(p)}\to V\to L\otimes\mathbb{Z}_{(p)}$ is (the Quillen) idempotent. After rationalization, $e(m_n)=m_n$ if $n+1$ is a power of p, and $e(m_n)=0$ otherwise. It follows that

$$V = \mathbb{Z}_{(p)}[v_n \mid n\geqslant 1],$$

with v_n in degree $2(p^n-1)$, and $V\otimes\mathbb{Q}\to\mathbb{Q}[\ell_n\mid n\geqslant 1]$ is an isomorphism. One choice of generators v_n, due to Hazewinkel [**21**, (4.3.1)], is recursively defined by

$$p\ell_n = \sum_{i=0}^{n-1}\ell_i v_{n-i}^{p^i} = v_n + \ell_1 v_{n-1}^p + \cdots + \ell_{n-1}v_1^{p^{n-1}}. \tag{3.1}$$

Here, $\ell_0 = 1$. In low degrees,

$$\begin{aligned} p\ell_1 &= v_1 \\ p^2\ell_2 &= pv_2 + v_1^{p+1} \\ p^3\ell_3 &= p^2v_3 + p(v_1v_2^p + v_1^{p^2}v_2) + v_1^{p^2+p+1}. \end{aligned} \tag{3.2}$$

These formulae exhibit one advantage of the p-typical context over the general one. We do not have similar formulae characterizing polynomial generators $x_n\in L$ in terms of the logarithmic coefficients $m_n\in L\otimes\mathbb{Q}$.

Lemma 3.1. *$p^n\ell_n\in V$ for each $n\geqslant 1$.*

Proof. This follows from (3.1) by induction on n. □

Quillen [**42**, Theorem 4] constructed the Brown–Peterson spectrum BP with maps $\alpha\colon MU_{(p)}\to BP$ and $\beta\colon BP\to MU_{(p)}$, such that $\alpha\beta$ is homotopic to the identity and $\beta\alpha$

induces the idempotent e on homotopy. The resulting formal group law over $\pi_*(BP)$ is then p-typical, and the homomorphism $V \to \pi_*(BP)$ that classifies this p-typical formal group law is an isomorphism.

Consider a strict isomorphism

$$F \xleftarrow{\phi} F'$$

given in moving coordinates by

$$\phi(x) = \sum_{n \geqslant 0}^{F} c_n x^{n+1}.$$

By Araki [3, Theorem 3.6] and Landweber [28, Lemma 1], F' is p-typical if and only if $c_n = 0$ unless $n+1$ is a power of p. In other words,

$$\phi(x) = \sum_{n \geqslant 0}^{F} t_n x^{p^n}$$

for a sequence of coefficients t_n, with $t_0 = 1$. Let

$$T = \mathbb{Z}_{(p)}[t_n \mid n \geqslant 1]$$

and $VT = V \otimes T$, with t_n in degree $2(p^n - 1)$. Then V and VT corepresent the objects and morphisms in the full subgroupoid $\mathcal{T}(R) \subset \mathcal{G}(R)$ of p-typical formal group laws and their strict isomorphisms. The inclusion $\eta_L \colon V \to VT$ and a right unit homomorphism $\eta_R \colon V \to VT$ classify the target F and the source F'. The augmentation $\epsilon \colon VT \to V$ mapping each t_n to zero classifies the identity morphism. A coproduct $\psi \colon VT \to VT \otimes_V VT$ classifies the composition

$$\circ \colon \mathcal{T}(R)(F', F) \times \mathcal{T}(R)(F'', F') \longrightarrow \mathcal{T}(R)(F'', F),$$

and a conjugation $\chi \colon VT \to VT$ classifies the function sending ϕ to $\phi^{-1} \colon F \to F'$. The pair (V, VT), equipped with these structure maps, is then a Hopf algebroid, corepresenting $\mathcal{T}(R)$ as a functor of commutative $\mathbb{Z}_{(p)}$-algebras. The full inclusion $\mathcal{T}(R) \subset \mathcal{G}(R)$ and the p-typification functor $\mathcal{G}(R) \to \mathcal{T}(R)$ are classified by morphisms $\alpha \colon (L, LC) \otimes \mathbb{Z}_{(p)} \to (V, VT)$ and $\beta \colon (V, VT) \to (L, LC) \otimes \mathbb{Z}_{(p)}$ of Hopf algebroids. Here, $\alpha \colon L \otimes \mathbb{Z}_{(p)} \to V$ is given rationally by $\alpha(m_n) = 0$ if $n+1$ is not a power of p and $\alpha(m_n) = \ell_i$ if $n+1 = p^i$. Similarly, $\alpha(c_n) = 0$ if $n+1$ is not a power of p and $\alpha(c_n) = t_i$ if $n+1 = p^i$. Conversely, $\beta(\ell_i) = m_{p^i-1}$ and $\beta(t_i) = c_{p^i-1}$.

Adams [1, Theorem 16.1(ii)] showed that

$$\pi_*(BP \wedge BP) \cong \pi_*(BP)[t_n \mid n \geqslant 1]$$

for specific classes $t_n \in \pi_*(BP \wedge BP)$, so that Quillen's isomorphism $V \cong \pi_*(BP)$ extends to an isomorphism $VT \cong \pi_*(BP \wedge BP)$. Adams showed that the formulae for the flat Hopf algebroid $(\pi_*(BP), \pi_*(BP \wedge BP))$, associated to the (homotopy commutative or better) ring spectrum BP, agree with those specified by Quillen. Landweber [28, § 3] then verified that these agree with the structure maps of (V, VT), corepresenting $\mathcal{T}(R)$.

In particular, [1, Theorem 16.1(i); 42, Theorem 5(iii)] gave a formula for the right unit $\eta_R \colon V \to VT$ after rationalization, which in our notation reads

$$\eta_R(\ell_n) = \sum_{i+j=n} \ell_i t_j^{p^i} = \ell_n + \ell_{n-1} t_1^{p^{n-1}} + \cdots + \ell_1 t_{n-1}^{p} + t_n \tag{3.3}$$

for $n \geqslant 1$. When combined with (3.1), this will give us good formulae for the σ-operator in $\pi_* THH(BP)$.

4. *Topological Hochschild homology and the bar construction*

4.1. *Chains of composable strict isomorphisms*

Recall that $THH(MU)_\bullet$ is a simplicial (E_∞ ring) spectrum $[q] \mapsto THH(MU)_q$. We shall analyze the simplicial graded commutative ring $[q] \mapsto \pi_* THH(MU)_q$ in terms of the Hopf algebroids (L, LB) and (L, LC). We emphasize the latter case, since it is closer to the Hopf algebroid (V, VT) that we need to consider in the p-typical case. However, we shall also state some of the results for (L, LB), in part to illustrate the advantage of using moving coordinates for these calculations.

The product map

$$L \otimes L^{\otimes q} = \pi_*(MU) \otimes \pi_*(MU)^{\otimes q} \longrightarrow \pi_*(MU \wedge MU^{\wedge q}) = \pi_* THH(MU)_q$$

is not an isomorphism (for $q \geqslant 1$), but becomes one after rationalization. Since $LB \cong LC$ is flat over L, we can rewrite its target as

$$\pi_*((MU \wedge MU) \wedge_{MU} \cdots \wedge_{MU} (MU \wedge MU)) \cong LC \otimes_L \cdots \otimes_L LC,$$

with q copies of $MU \wedge MU$ and LC. Here, $LC \otimes_L \cdots \otimes_L LC \cong L \otimes C^{\otimes q}$ by iterated use of standard isomorphisms of the form $X \otimes_L L \otimes Y \cong X \otimes Y$, for suitable X and Y. The tensor product $\pi_* THH(MU)_q \cong L \otimes C^{\otimes q}$ classifies chains

$$(F_0; \phi_1, \ldots, \phi_q) = (F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \ldots \xleftarrow{\phi_q} F_q)$$

of q composable strict isomorphisms between formal group laws, with L classifying the formal group law F_0 and the ith copy of C classifying the strict isomorphism $\phi_i \colon F_i \to F_{i-1}$, presented in moving coordinates with respect to its target. The composite homomorphism $L \otimes L^{\otimes q} \to L \otimes C^{\otimes q}$ classifies the function taking $(F_0; \phi_1, \ldots, \phi_q)$ to the $(1+q)$-tuple of formal group laws $(F_0, F_1, \ldots, F_q)$.

The ith face map $d_i \colon \pi_* THH(MU)_{q+1} \to THH(MU)_q$ is a homomorphism $L \otimes C^{\otimes q+1} \to L \otimes C^{\otimes q}$. When $0 \leqslant i \leqslant q$ it is induced by the multiplication $MU \wedge MU \to MU$ of the ith and $(i+1)$th copies of MU (counting from zero), and is compatible with the multiplication $L \otimes L \to L$ of the ith and $(i+1)$th copies of L in $L \otimes L^{\otimes q+1}$. Hence, it corepresents the function taking $(F_0; \phi_1, \ldots, \phi_q)$ to a chain where F_i has been duplicated in the ith and $(i+1)$th positions. Since the only strict automorphism of a formal group law over a torsion-free ring is the identity, the chain of composable strict isomorphisms must be of the form

$$(F_0; \phi_1, \ldots, \phi_i, id, \phi_{i+1}, \ldots, \phi_q),$$

where $id \colon F_i \to F_i$ denotes the identity isomorphism of the repeated formal group law. The last face map, with $i = q+1$, is induced by multiplying the final and initial copies of MU. It is compatible with the multiplication of the final and initial copies of L, and therefore corepresents the function taking $(F_0; \phi_1, \ldots, \phi_q)$ to a chain where a copy of F_0 has been added at the end. This chain must have the form

$$(F_0; \phi_1, \ldots, \phi_q, (\phi_1 \cdots \phi_q)^{-1}),$$

where $(\phi_1 \cdots \phi_q)^{-1} \colon F_0 \to F_q$.

The jth degeneracy map $s_j \colon \pi_* THH(MU)_{q-1} \to THH(MU)_q$ is a homomorphism $L \otimes C^{\otimes q-1} \to L \otimes C^{\otimes q}$. It is induced by inserting the unit map $S \to MU$ between the jth and $(j+1)$th copies of MU (counting from zero), and is compatible with the unit map $\mathbb{Z} \to L$ to the $(j+1)$th copy of L in $L \otimes L^{\otimes q}$. It therefore corepresents the function taking $(F_0; \phi_1, \ldots, \phi_q)$ to a chain where F_{j+1} has been omitted. For $0 \leqslant j \leqslant q-2$ this is the chain $(F_0; \phi_1, \ldots, \phi_{j+1}\phi_{j+2}, \ldots, \phi_q)$, while for $j = q-1$ we obtain $(F_0; \phi_1, \ldots, \phi_{q-1})$.

4.2. *Strict isomorphisms with common target*

Let $\beta(C)_\bullet$ be the simplicial bar construction

$$[q] \mapsto \beta(C)_q = C^{\otimes q}$$

on the augmented algebra C. The homology of its normalized chain complex $N\beta(C)_*$ calculates

$$\mathrm{Tor}_*^C(\mathbb{Z},\mathbb{Z}) = E([c_n] \mid n \geqslant 1),$$

since C is flat over $\mathbb{Z}$. Here, $[c_n]$ is the class in $\mathrm{Tor}_1^C(\mathbb{Z},\mathbb{Z}) = I(C)/I(C)^2$ of the bar 1-cycle $c_n \in I(C)$, where $I(C) = \ker(C \to \mathbb{Z}) \cong \mathrm{cok}(\mathbb{Z} \to C)$ is the positive-degree part of C, and $E(-)$ denotes the exterior algebra on the indicated generators.

Then $L \otimes \beta(C)_\bullet$ is a simplicial graded commutative ring with

$$[q] \mapsto L \otimes \beta(C)_q = L \otimes C^{\otimes q},$$

and the homology of $L \otimes N\beta(C)_*$ calculates $L \otimes \mathrm{Tor}_*^C(\mathbb{Z},\mathbb{Z})$. We think of $L \otimes \beta(C)_q = L \otimes C^{\otimes q}$ as classifying a q-tuple

$$(F_0; \gamma_1, \ldots, \gamma_q) = (F_0 \xleftarrow{\gamma_i} F_i)_{i=1}^q$$

of strict isomorphisms $\gamma_i \colon F_i \to F_0$ in moving coordinates, all with the same target.

The ith face map $d_i \colon L \otimes C^{\otimes q+1} \to L \otimes C^{\otimes q}$ is given for $i = 0$ by the augmentation $C \to \mathbb{Z}$ of the first copy of C, for $1 \leqslant i \leqslant q$ by the multiplication $C \otimes C \to C$ of the ith and $(i+1)$th copy of C (counting from one), and for $i = q+1$ by the augmentation of the last copy of C. This corepresents the function that takes $(F_0; \gamma_1, \ldots, \gamma_q)$ to $(F_0; id, \gamma_1, \ldots, \gamma_q)$ for $i = 0$, to $(F_0; \gamma_1, \ldots, \gamma_i, \gamma_i, \ldots, \gamma_q)$ for $1 \leqslant i \leqslant q$, and to $(F_0; \gamma_1, \ldots, \gamma_q, id)$ for $i = q+1$. In the first and last cases, $id \colon F_0 \to F_0$ refers to the identity isomorphism for F_0.

The jth degeneracy map $s_j \colon L \otimes C^{\otimes q-1} \to L \otimes C^{\otimes q}$, for $0 \leqslant j \leqslant q-1$, is induced by the unit $\mathbb{Z} \to C$ to the $(j+1)$th copy of C (counting from one). It corepresents the function that omits $\gamma_{j+1} \colon F_{j+1} \to F_0$ from the q-tuple of strict isomorphisms with target F_0, leaving $(F_0; \gamma_1, \ldots, \gamma_j, \gamma_{j+2}, \ldots, \gamma_q)$.

4.3. *A shearing isomorphism*

To each chain of q composable strict isomorphisms

$$F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \ldots \xleftarrow{\phi_q} F_q$$

we can associate a q-tuple of strict isomorphisms

$$(F_0 \xleftarrow{\gamma_i} F_i)_{i=1}^q,$$

having the same underlying sequence $F_0, F_1, \ldots, F_q$ of formal group laws. This one-to-one correspondence is classified by the following isomorphism.

Proposition 4.1. *There is an isomorphism of simplicial graded commutative rings*

$$L \otimes \beta(C)_\bullet \xrightarrow{\cong} \pi_* THH(MU)_\bullet$$

that, in degree q, classifies the bijection

$$(F_0; \phi_1, \ldots, \phi_q) \overset{\cong}{\longmapsto} (F_0; \gamma_1, \ldots, \gamma_q),$$

where $\gamma_i = \phi_1 \cdots \phi_i$ for $1 \leqslant i \leqslant q$. Its inverse is given by $\phi_1 = \gamma_1$ and $\phi_i = \gamma_{i-1}^{-1}\gamma_i$ for $2 \leqslant i \leqslant q$.

Proof. The stated bijection is natural (in the ring over which the formal group laws and the strict isomorphisms are defined), hence is corepresented by an isomorphism

$$L \otimes \beta(C)_q = L \otimes C^{\otimes q} \xrightarrow{\cong} L \otimes C^{\otimes q} \cong \pi_* THH(MU)_q.$$

This is the identity for $q = 0$ and $q = 1$, but has a more complex expression for $q \geqslant 2$, which we do not need to make explicit.

It remains to verify that these isomorphisms $L \otimes \beta(C)_q \cong \pi_* THH(MU)_q$ are compatible with the simplicial structure maps. This follows from the explicit descriptions given in the previous two subsections: On both sides of the isomorphism the ith face map, except for the last one, classifies the function that repeats F_i in the underlying sequence of formal group laws. Similarly, on both sides the last face map classifies the function that appends F_0 to the underlying sequence. This ensures that all face maps are compatible under these isomorphisms. Finally, on both sides the jth degeneracy map classifies the function that omits F_{j+1} from the underlying sequence. This ensures that all degeneracy maps are compatible. □

Turning to the split case, we have an isomorphism

$$\pi_* THH(MU)_q \cong LB \otimes_L \cdots \otimes_L LB \cong L \otimes B^{\otimes q}$$

and $L \otimes B^{\otimes q}$ classifies chains

$$(F_0; f_1, \dots, f_q) = (F_0 \xrightarrow{f_1} F_1 \longrightarrow \dots \xrightarrow{f_q} F_q)$$

of q composable strict isomorphisms. On the other hand, $L \otimes \beta(B)_q = L \otimes B^{\otimes q}$ classifies q-tuples

$$(F_0, g_1, \dots, g_q) = (F_0 \xrightarrow{g_i} F_i)_{i=1}^q$$

of strict isomorphisms, all with the same source.

Proposition 4.2. *There is an isomorphism of simplicial graded commutative rings*

$$L \otimes \beta(B)_\bullet \xrightarrow{\cong} \pi_* THH(MU)_\bullet$$

that, in degree q, classifies the bijection

$$(F_0; f_1, \dots, f_q) \overset{\cong}{\longmapsto} (F_0; g_1, \dots, g_q)$$

where $g_i = f_i \cdots f_1$ for $1 \leqslant i \leqslant q$. Its inverse is given by $f_1 = g_1$ and $f_i = g_i g_{i-1}^{-1}$ for $2 \leqslant i \leqslant q$.

Proof. The proof follows the same lines in the split case as for moving coordinates. One difference is that in the split case the isomorphism

$$L \otimes \beta(B)_q = L \otimes B^{\otimes q} \cong L \otimes B^{\otimes q} \cong \pi_* THH(MU)_q$$

can easily be made explicit as the tensor product of the identity on L and an isomorphism $B^{\otimes q} \cong B^{\otimes q}$, given by the $(q + 1 - i)$-fold coproduct

$$B \longrightarrow B^{\otimes q+1-i}$$

from the ith copy of B, followed by a permutation and the i-fold product

$$B^{\otimes i} \longrightarrow B$$

to the ith copy of B. □

In the p-typical case, the product map

$$V \otimes V^{\otimes q} = \pi_*(BP) \otimes \pi_*(BP)^{\otimes q} \longrightarrow \pi_*(BP \wedge BP^{\wedge q}) = \pi_* THH(BP)_q$$

becomes an isomorphism after rationalization. Since VT is flat over V we can rewrite the target as

$$\pi_*((BP \wedge BP) \wedge_{BP} \cdots \wedge_{BP} (BP \wedge BP)) \cong VT \otimes_V \cdots \otimes_V VT.$$

There is an evident isomorphism

$$VT \otimes_V \cdots \otimes_V VT \cong V \otimes T^{\otimes q},$$

and $V \otimes T^{\otimes q}$ classifies chains

$$(F_0; \phi_1, \ldots, \phi_q) = (F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \ldots \xleftarrow{\phi_q} F_q)$$

of q composable strict isomorphisms between p-typical formal group laws. On the other hand, $V \otimes \beta(T)_q = V \otimes T^{\otimes q}$ classifies q-tuples

$$(F_0, \gamma_1, \ldots, \gamma_q) = (F_0 \xleftarrow{\gamma_i} F_i)_{i=1}^q$$

of strict isomorphisms between p-typical formal group laws, all with the same target.

Proposition 4.3. *There is an isomorphism of simplicial graded commutative rings*

$$V \otimes \beta(T)_\bullet \xrightarrow{\cong} \pi_* THH(BP)_\bullet$$

that, in degree q, classifies the bijection

$$(F_0; \phi_1, \ldots, \phi_q) \xmapsto{\cong} (F_0; \gamma_1, \ldots, \gamma_q)$$

where $\gamma_i = \phi_1 \cdots \phi_i$ for $1 \leqslant i \leqslant q$. Its inverse is given by $\phi_1 = \gamma_1$ and $\phi_i = \gamma_{i-1}^{-1}\gamma_i$ for $2 \leqslant i \leqslant q$.

Proof. The proof is the same as for $\pi_* THH(MU)_\bullet$ with moving coordinates, except that all formal group laws in sight are p-typical. □

4.4. *The skeleton spectral sequence*

We return to MU with moving coordinates.

Proposition 4.4. *The skeleton spectral sequence for $\pi_* THH(MU)$ collapses at the E^2-term*

$$E^2 = E^\infty \cong L \otimes \mathrm{Tor}_*^C(\mathbb{Z}, \mathbb{Z}).$$

For each $n \geqslant 1$ there is a unique class $\lambda'_n \in \pi_{2n+1} THH(MU)$ detected by $[c_n]$ in $E^\infty_{1,2n}$, and

$$\pi_* THH(MU) \cong \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1).$$

Proof. The isomorphism of simplicial commutative rings

$$\pi_* THH(MU)_\bullet \cong L \otimes \beta(C)_\bullet$$

induces an isomorphism

$$E^1 = N\pi_* THH(MU)_* \cong L \otimes N\beta(C)_*$$

of normalized differential graded algebras, hence also of homology algebras

$$E^2 \cong L \otimes \mathrm{Tor}_*^C(\mathbb{Z}, \mathbb{Z}) = L \otimes E([c_n] \mid n \geqslant 1).$$

The skeleton spectral sequence is a multiplicative first quadrant spectral sequence. It follows that it collapses at the E^2-term, since the algebra generators are concentrated in filtrations 0 and 1.

The class $[c_n] \in E^2_{1,2n} = E^\infty_{1,2n}$ detects a class λ'_n in the image of

$$\pi_{2n+1}(sk_1 THH(MU)) \longrightarrow \pi_{2n+1} THH(MU)$$

and is well-defined modulo the image of

$$\pi_{2n+1}(sk_0 THH(MU)) \longrightarrow \pi_{2n+1} THH(MU).$$

Since $\pi_{2n+1}(sk_0 THH(MU)) = \pi_{2n+1}(MU) = 0$, the class $\lambda'_n \in \pi_{2n+1} THH(MU)$ is in fact well defined by this condition. (We could also have used the fact that MU splits off from $THH(MU)$, using the augmentation $THH(MU) \to MU$, to arrange that λ'_n maps to zero under the augmentation, but this method of normalization is irrelevant for the current investigation.)

Since each λ'_n is in an odd degree, and $\pi_* THH(MU)$ is graded-commutative, it follows that the λ'_n for $n \geqslant 1$ freely generate $\pi_* THH(MU)$ over $L \cong \pi_*(MU)$, concluding the proof. □

Here is the split analogue.

Proposition 4.5. *The skeleton spectral sequence for $\pi_* THH(MU)$ collapses at the E^2-term*

$$E^2 = E^\infty \cong L \otimes \mathrm{Tor}^B_*(\mathbb{Z}, \mathbb{Z}).$$

For each $n \geqslant 1$ there is a unique class $e_n \in \pi_{2n+1} THH(MU)$ detected by $[b_n]$ in $E^\infty_{1,2n}$, and

$$\pi_* THH(MU) = L \otimes E(e_n \mid n \geqslant 1).$$

The expressions (2.2) for the c_n in terms of the absolute coordinates in LB lead to relations in $L \otimes \mathrm{Tor}^B_*(\mathbb{Z}, \mathbb{Z})$ which detect the following identities in $\pi_* THH(MU)$:

$$\begin{aligned}
\lambda'_1 &= -e_1 \\
\lambda'_2 &= x_1 e_1 - e_2 \\
\lambda'_3 &= (x_2 - x_1^2) e_1 + x_1 e_2 - e_3 \\
\lambda'_4 &= (2x_3 - 4x_1 x_2 + x_1^3) e_1 + (x_2 - x_1^2) e_2 + x_1 e_3 - e_4.
\end{aligned}$$

Here is the p-typical statement.

Proposition 4.6. *The skeleton spectral sequence for $\pi_* THH(BP)$ collapses at the E^2-term*

$$E^2 = E^\infty \cong V \otimes \mathrm{Tor}^T_*(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}).$$

For each $n \geqslant 1$ there is a unique class $\lambda_n \in \pi_{2p^n-1} THH(BP)$ detected by $[t_n]$ in $E^\infty_{1,2p^n-2}$, and

$$\pi_* THH(BP) \cong \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1).$$

The proof is the same as for MU.

Remark 4.7. Following Andy Baker and Larry Smith, Jim McClure and Ross Staffeldt [**34**, Remark 4.3] calculated $\pi_* THH(MU) \cong \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1)$ and $\pi_* THH(BP) \cong \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1)$ as graded rings, where the classes λ'_n in degree $2n+1$ and λ_n in degree $2p^n - 1$ were only specified in terms of their mod p Hurewicz images. Our choices of generators λ'_n and λ_n are uniquely defined, and have the feature that $\sigma(\lambda'_n) = 0$ and $\sigma(\lambda_n) = 0$.

5. *The circle action and the right unit*

For any S-algebra R, the cyclic structure on $THH(R)_\bullet$ induces a circle action on its realization. Its restriction

$$R \wedge S^1_+ \longrightarrow THH(R) \wedge S^1_+ \longrightarrow THH(R)$$

to the 0-skeleton $R = sk_0 THH(R) \subset THH(R)$ factors through the 1-skeleton $sk_1 THH(R) \subset THH(R)$ as the map induced by the right unit $\eta_R \colon R \cong S \wedge R \to R \wedge R$, from the pushout $R \wedge S^1_+ \cong R \wedge (\Delta^1/\partial\Delta^1)_+$ of the maps

$$R \longleftarrow R \wedge \partial\Delta^1_+ \longrightarrow R \wedge \Delta^1_+$$

to the pushout $sk_1 THH(R)$ of the maps

$$R \longleftarrow (R \wedge R \wedge \partial\Delta^1_+) \cup (R \wedge S \wedge \Delta^1_+) \longrightarrow R \wedge R \wedge \Delta^1_+.$$

This follows from the definition of the circle action, which for a 0-simplex x traces out the loop given by the 1-simplex $t_1 s_0(x)$. Hence, we have a map of horizontal cofiber sequences

$$\begin{tikzcd} R \arrow[r] \arrow[d, "="'] & R \wedge S^1_+ \arrow[r] \arrow[d] & \Sigma R \arrow[d] \\ R \arrow[r] & sk_1 THH(R) \arrow[r] & \Sigma(R \wedge R/S) \end{tikzcd}$$

where the right-hand vertical map is the suspension of the composite

$$R \cong S \wedge R \xrightarrow{\eta_R} R \wedge R \xrightarrow{1\wedge\pi} R \wedge R/S.$$

Using the splitting of the upper row, we see that the right-hand vertical map is also the composite

$$\Sigma R \xrightarrow{\sigma} sk_1 THH(R) \to \Sigma(R \wedge R/S).$$

This proves the following result.

PROPOSITION 5.1. *For $x \in \pi_*(R)$ the homotopy class $\sigma(x) \in \pi_{*+1} THH(R)$ is detected in $E^\infty_{1,*}$ of the skeleton spectral sequence by the class of the infinite cycle*

$$(1 \wedge \pi)\eta_R(x) \in \pi_*(R \wedge R/S) = E^1_{1,*}.$$

We now specialize to $R = MU$. In terms of moving coordinates, the maps η_R and π induce the homomorphisms

$$L \xrightarrow{\eta_R} LC = L \otimes C \xrightarrow{1\otimes\pi} L \otimes I(C),$$

where η_R is the right unit and $\pi \colon C \to I(C)$ is the projection away from $\mathbb{Z} \to C$. We can also view $LC \to L \otimes I(C)$ as the cokernel of the left unit $\eta_L \colon L \to LC$. The split case is practically the same.

PROPOSITION 5.2. *The rationalized σ-operator*

$$\pi_*(MU) \otimes \mathbb{Q} \longrightarrow \pi_{*+1} THH(MU) \otimes \mathbb{Q}$$

is the (right) derivation given by

$$\sigma(m_n) = \lambda'_n.$$

Proof. By Propositions 2.1 and 5.1, $\sigma(m_n)$ is detected in $E^\infty_{1,*} \otimes \mathbb{Q}$ by the image of

$$\eta_R(m_n) = \sum_{(i+1)(j+1)=n+1} m_i c_j^{i+1}$$

under the projections

$$L \otimes C \otimes \mathbb{Q} \longrightarrow L \otimes I(C) \otimes \mathbb{Q} \longrightarrow L \otimes I(C)/I(C)^2 \otimes \mathbb{Q} = L \otimes \mathrm{Tor}_1^C(\mathbb{Z}, \mathbb{Z}) \otimes \mathbb{Q}.$$

The term with $j = 0$ maps to zero in $L \otimes I(C) \otimes \mathbb{Q}$, and the terms with $i \geqslant 1$ map to zero in $L \otimes I(C)/I(C)^2 \otimes \mathbb{Q}$, so only the term with $i = 0$ and $j = n$ remains. Hence, $\sigma(m_n)$ is detected by $[c_n]$ in $E^\infty_{1,*} \otimes \mathbb{Q} = L \otimes \mathrm{Tor}_1^C(\mathbb{Z}, \mathbb{Z}) \otimes \mathbb{Q}$, and this characterizes the homotopy class $\lambda'_n \in \pi_{2n+1} THH(MU) \subset \pi_{2n+1} THH(MU) \otimes \mathbb{Q}$. □

Theorem 5.3. *The σ-operator*

$$\sigma \colon \pi_* THH(MU) \longrightarrow \pi_{*+1} THH(MU)$$

is the (right) $\mathbb{Z}$-linear derivation acting on

$$\pi_* THH(MU) \cong \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1)$$

by taking $x \in L \cong \pi_*(MU) \subset \pi_* THH(MU)$ to the homotopy class $\sigma(x) \in \pi_{*+1} THH(MU)$ detected by the image of $\eta_R(x) \in LC$ in $L \otimes \mathrm{Tor}_1^C(\mathbb{Z}, \mathbb{Z}) = E^\infty_{1,*}$, while

$$\sigma(\lambda'_n) = 0$$

for all $n \geqslant 1$. In low degrees,

$$\begin{aligned}
\sigma(x_1) &= -2\lambda'_1 \\
\sigma(x_2) &= -4x_1\lambda'_1 - 3\lambda'_2 \\
\sigma(x_3) &= -(4x_2 + 5x_1^2)\lambda'_1 - 6x_1\lambda'_2 - 2\lambda'_3 \\
\sigma(x_4) &= -4(2x_3 - x_1x_2)\lambda'_1 - 3(2x_2 + x_1^2)\lambda'_2 - 8x_1\lambda'_3 - 5\lambda'_4.
\end{aligned}$$

Proof. The general statements summarize Propositions 4.4, 5.1 and 5.2. We know that $\sigma(\lambda'_n) = \sigma^2(m_n) = 0$ in $\pi_* THH(MU) \otimes \mathbb{Q}$, since σ acts as a differential. Hence, $\sigma(\lambda'_n) = 0$ in $\pi_* THH(MU)$, since these groups are torsion-free.

For the explicit formulae, we first calculate in $\pi_* THH(MU) \otimes \mathbb{Q} \cong L \otimes E(\lambda'_n \mid n \geqslant 1) \otimes \mathbb{Q}$, using the expressions (2.3) for the x_n in terms of the m_n, and applying the derivation σ:

$$\begin{aligned}
\sigma(x_1) &= -2\lambda'_1 \\
\sigma(x_2) &= 8m_1\lambda'_1 - 3\lambda'_2 \\
\sigma(x_3) &= 12(m_2 - 3m_1^2)\lambda'_1 + 12m_1\lambda'_2 - 2\lambda'_3 \\
\sigma(x_4) &= 8(8m_1^3 - 9m_1m_2 + 2m_3)\lambda'_1 + 18(m_2 - 2m_1^2)\lambda'_2 + 16m_1\lambda'_3 - 5\lambda'_4.
\end{aligned}$$

The asserted formulae in $\pi_* THH(MU) \cong L \otimes E(\lambda'_n \mid n \geqslant 1)$ then follow, by rewriting the polynomials in the m_n as elements of L. □

Here is the analogous result in the split case. We do not have a closed formula for $\sigma(e_n)$.

Theorem 5.4. *The σ-operator $\sigma \colon \pi_* THH(MU) \to \pi_{*+1} THH(MU)$ is the (right) $\mathbb{Z}$-linear derivation acting on*

$$\pi_* THH(MU) \cong \pi_*(MU) \otimes E(e_n \mid n \geqslant 1)$$

by taking $x \in L \cong \pi_*(MU) \subset \pi_* THH(MU)$ to the homotopy class $\sigma(x) \in \pi_{*+1} THH(MU)$ detected by the image of $\eta_R(x) \in LB$ in $L \otimes \mathrm{Tor}_1^B(\mathbb{Z},\mathbb{Z}) = E^\infty_{1,*}$. The classes $\sigma(e_n)$ are inductively determined by the relation $\sigma^2(x_n) = 0$. In low degrees,

$$\sigma(x_1) = 2e_1$$

$$\sigma(x_2) = x_1 e_1 + 3e_2$$

$$\sigma(x_3) = (2x_2 + x_1^2)e_1 + 4x_1 e_2 + 2e_3$$

$$\sigma(x_4) = (2x_1 x_2 - 2x_3)e_1 + x_2 e_2 + 3x_1 e_3 + 5e_4$$

and

$$\sigma(e_1) = 0$$

$$\sigma(e_2) = 0$$

$$\sigma(e_3) = e_1 e_2$$

$$\sigma(e_4) = 2e_1 e_3.$$

Proof. For x_n we use (2.1) to calculate

$$(1 \otimes \pi)\eta_R(x_1) = 2b_1$$

$$(1 \otimes \pi)\eta_R(x_2) = x_1 b_1 + (3b_2 - 2b_1^2)$$

$$(1 \otimes \pi)\eta_R(x_3) = (2x_2 + x_1^2)b_1 + x_1(4b_2 - b_1^2) + (2b_3 + 2b_1 b_2 - 2b_1^3)$$

$$(1 \otimes \pi)\eta_R(x_4) = (2x_1 x_2 - 2x_3)b_1 + x_2(b_2 - b_1^2) + x_1(3b_3 - 8b_1 b_2 + 5b_1^3) + (5b_4 - 14b_1 b_3 - 6b_2^2 + 25b_1^2 b_2 - 10b_1^4)$$

in $E^1_{1,*} = L \otimes I(B)$. Hence, $\sigma(x_n)$ is detected by

$$[(1 \otimes \pi)\eta_R(x_1)] = 2[b_1]$$

$$[(1 \otimes \pi)\eta_R(x_2)] = x_1[b_1] + 3[b_2]$$

$$[(1 \otimes \pi)\eta_R(x_3)] = (2x_2 + x_1^2)[b_1] + 4x_1[b_2] + 2[b_3]$$

$$[(1 \otimes \pi)\eta_R(x_4)] = (2x_1 x_2 - 2x_3)[b_1] + x_2[b_2] + 3x_1[b_3] + 5[b_4]$$

in $E^\infty_{1,*} = L \otimes \mathrm{Tor}_1^B(\mathbb{Z},\mathbb{Z})$. Since $e_n \in \pi_{2n+1} THH(MU)$ is characterized by being detected by $[b_n] \in E^1_{1,2n}$, the stated formulae for $\sigma(x_n)$ hold. Furthermore, $\sigma^2 = 0$ when acting on $\pi_* THH(MU)$, and the e_n generate an exterior algebra, so it follows that

$$0 = \sigma(2e_1) = 2\sigma(e_1)$$

$$0 = \sigma(x_1 e_1 + 3e_2) = 3\sigma(e_2)$$

$$0 = \sigma((2x_2 + x_1^2)e_1 + 4x_1 e_2 + 2e_3) = -2e_1 e_2 + 2\sigma(e_3)$$

$$0 = \sigma((2x_1 x_2 - 2x_3)e_1 + x_2 e_2 + 3x_1 e_3 + 5e_4) = -10e_1 e_3 + 5\sigma(e_4).$$

Here, we have used the form of the Leibniz rule that is appropriate for right actions, that is, $\sigma(xy) = x\sigma(y) + (-1)^{|y|}\sigma(x)y$. Since $\pi_* THH(MU)$ is torsion-free, this implies the stated formulae for $\sigma(e_n)$. □

We now turn to the p-typical case.

Proposition 5.5. *The rationalized σ-operator $\pi_*(BP) \otimes \mathbb{Q} \to \pi_{*+1} THH(BP) \otimes \mathbb{Q}$ is the (right) derivation given by*

$$\sigma(\ell_n) = \lambda_n.$$

Proof. By (3.3) and Proposition 5.1, $\sigma(\ell_n)$ is detected in $E^\infty_{1,*} \otimes \mathbb{Q}$ by the image of

$$\eta_R(\ell_n) = \sum_{i+j=n} \ell_i t_j^{p^i}$$

under the projections

$$V \otimes T \otimes \mathbb{Q} \longrightarrow V \otimes I(T) \otimes \mathbb{Q} \longrightarrow V \otimes I(T)/I(T)^2 \otimes \mathbb{Q} = V \otimes \mathrm{Tor}_1^T(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \otimes \mathbb{Q}.$$

The term with $j = 0$ maps to zero in $V \otimes I(T) \otimes \mathbb{Q}$, and the terms with $i \geqslant 1$ map to zero in $V \otimes I(T)/I(T)^2 \otimes \mathbb{Q}$, so only the term with $i = 0$ and $j = n$ remains. Hence, $\sigma(\ell_n)$ is detected by $[t_n]$ in $E^\infty_{1,*} \otimes \mathbb{Q} = V \otimes \mathrm{Tor}_1^T(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \otimes \mathbb{Q}$, and this characterizes the homotopy class $\lambda_n \in \pi_{2p^n-1} THH(BP) \subset \pi_{2p^n-1} THH(BP) \otimes \mathbb{Q}$. □

Theorem 5.6. *The σ-operator*

$$\sigma \colon \pi_* THH(BP) \longrightarrow \pi_{*+1} THH(BP)$$

is the (right) $\mathbb{Z}_{(p)}$-linear derivation acting on

$$\pi_* THH(BP) \cong \pi_*(BP) \otimes E(\lambda_n \mid n \geqslant 1)$$

by taking $x \in V \cong \pi_*(BP) \subset \pi_* THH(BP)$ to the class $\sigma(x) \in \pi_{*+1} THH(BP)$ detected by the image of $\eta_R(x) \in VT$ in $V \otimes \mathrm{Tor}_1^T(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) = E^\infty_{1,*}$, while

$$\sigma(\lambda_n) = 0$$

for all $n \geqslant 1$. In low degrees,

$$\sigma(v_1) = p\lambda_1$$
$$\sigma(v_2) = p\lambda_2 - (p+1)v_1^p\lambda_1$$
$$\sigma(v_3) = p\lambda_3 - (pv_1v_2^{p-1} + v_1^{p^2})\lambda_2 - (v_2^p - (p+1)v_1^{p+1}v_2^{p-1} + p^2v_1^{p^2-1}v_2 + pv_1^{p^2+p})\lambda_1.$$

Proof. The general results are proved as for MU with moving coordinates. For the explicit formulae, we apply the derivation σ to (3.1), to obtain

$$p\lambda_n = \sigma(v_n) + \sum_{i=1}^{n-1} \Big(v_{n-i}^{p^i}\lambda_i + (p^i\ell_i)v_{n-i}^{p^i-1}\sigma(v_{n-i}) \Big). \tag{5.1}$$

Here, $p^i\ell_i$ lies in V by Lemma 3.1, and is listed in low degrees in (3.2). This leads to

$$p\lambda_1 = \sigma(v_1)$$
$$p\lambda_2 = \sigma(v_2) + (v_1^p\lambda_1 + v_1v_1^{p-1}\sigma(v_1))$$
$$p\lambda_3 = \sigma(v_3) + (v_2^p\lambda_1 + v_1v_2^{p-1}\sigma(v_2)) + (v_1^{p^2}\lambda_2 + (pv_2 + v_1^{p+1})v_1^{p^2-1}\sigma(v_1))$$

which we can rewrite as stated. □

6. *The circle Tate construction*

We can now calculate the d^2-differential and $E^3 = E^4$-term of the circle Tate spectral sequence

$$E^2_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes \pi_* THH(MU) \Longrightarrow \pi_* THH(MU)^{tS^1},$$

in the first few degrees. Since η acts trivially on $\pi_* THH(MU)$, the d^2-differential is given by the σ-operator, and

$$E^3_{*,*} = E^4_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes H(\pi_* THH(MU), \sigma).$$

Let us first note that after rationalization the spectral sequence collapses after the d^2-differential.

Proposition 6.1. *Rationally,*

$$\begin{aligned} \pi_*(MU) \otimes \mathbb{Q} &\cong \mathbb{Q}[m_n \mid n \geqslant 1] \\ \pi_* THH(MU) \otimes \mathbb{Q} &\cong \mathbb{Q}[m_n \mid n \geqslant 1] \otimes E(\lambda'_n \mid n \geqslant 1) \\ H(\pi_* THH(MU), \sigma) \otimes \mathbb{Q} &\cong \mathbb{Q}. \end{aligned}$$

Proof. We know that $\pi_*(MU) \otimes \mathbb{Q} \cong L \otimes \mathbb{Q} \cong \mathbb{Q}[m_n \mid n \geqslant 1]$ and

$$\pi_* THH(MU) \otimes \mathbb{Q} \cong \pi_*(MU) \otimes E(\lambda'_n \mid n \geqslant 1) \otimes \mathbb{Q} \cong \mathbb{Q}[m_n \mid n \geqslant 1] \otimes E(\lambda'_n \mid n \geqslant 1),$$

with $\sigma(m_n) = \lambda'_n$. Here, $H(\mathbb{Q}[m_n] \otimes E(\lambda'_n), \sigma) = \mathbb{Q}$, for each $n \geqslant 1$, so the near-vanishing of $H(\pi_* THH(MU) \otimes \mathbb{Q}, \sigma)$ follows by the Künneth theorem. $\square$

Integrally, the situation is more complicated.

Theorem 6.2.

$$H(\pi_* THH(MU), \sigma) = \begin{cases} \mathbb{Z}\{1\} & \text{for } * = 0, \\ 0 & \text{for } * = 1, 2, 4, 6, 8, \\ \mathbb{Z}/2\{\lambda'_1\} & \text{for } * = 3, \\ \mathbb{Z}/4\{x_1\lambda'_1\} \oplus \mathbb{Z}/3\{\lambda'_2\} & \text{for } * = 5, \\ \mathbb{Z}/4\{\lambda'_3\} \oplus \mathbb{Z}/3\{2x_1^2\lambda'_1\} & \text{for } * = 7, \\ \mathbb{Z}/16\{(x_3 - 2x_1x_2)\lambda'_1 + x_1\lambda'_3\} & \\ \quad \oplus \mathbb{Z}/6\{(x_1^2 - x_2)\lambda'_2\} \oplus \mathbb{Z}/5\{\lambda'_4\} & \text{for } * = 9, \\ \mathbb{Z}/2\{\lambda'_1\lambda'_3\} & \text{for } * = 10. \end{cases}$$

Proof. Additively,

$$\pi_* THH(MU) = (\mathbb{Z}\{1\}, 0, \mathbb{Z}\{x_1\}, \mathbb{Z}\{\lambda'_1\}, \mathbb{Z}\{x_1^2, x_2\}, \mathbb{Z}\{x_1\lambda'_1, \lambda'_2\},$$
$$\mathbb{Z}\{x_1^3, x_1x_2, x_3\}, \mathbb{Z}\{x_1^2\lambda'_1, x_2\lambda'_1, x_1\lambda'_2, \lambda'_3\}, \mathbb{Z}\{\lambda'_1\lambda'_2, \dots\}, \dots).$$

As a cochain complex with differential given by the σ-operator, this breaks up as a direct sum of the shorter complexes

$$\mathbb{Z}\{1\}$$

$$\mathbb{Z}\{x_1\} \xrightarrow{(-2)} \mathbb{Z}\{\lambda'_1\}$$

$$\mathbb{Z}\{x_1^2, x_2\} \xrightarrow{\begin{pmatrix} -4 & -4 \\ 0 & -3 \end{pmatrix}} \mathbb{Z}\{x_1\lambda_1', \lambda_2'\}$$

$$\mathbb{Z}\{x_1^3, x_1x_2, x_3\} \xrightarrow{\begin{pmatrix} -6 & -4 & -5 \\ 0 & -2 & -4 \\ 0 & -3 & -6 \\ 0 & 0 & -2 \end{pmatrix}} \mathbb{Z}\{x_1^2\lambda_1', x_2\lambda_1', x_1\lambda_2', \lambda_3'\} \xrightarrow{\begin{pmatrix} 0 & -3 & 2 & 0 \end{pmatrix}} \mathbb{Z}\{\lambda_1'\lambda_2'\}$$

and

$$\mathbb{Z}\{x_1^4, x_1^2x_2, x_1x_3, x_2^2, x_4\} \xrightarrow{\begin{pmatrix} -8 & -4 & -5 & 0 & 0 \\ 0 & -4 & -4 & -8 & 4 \\ 0 & 0 & -2 & 0 & -8 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 0 & -2 & 0 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}}$$

$$\mathbb{Z}\{x_1^3\lambda_1', x_1x_2\lambda_1', x_3\lambda_1', x_1^2\lambda_2', x_2\lambda_2', x_1\lambda_3', \lambda_4'\} \xrightarrow{\begin{pmatrix} 0 & -3 & -6 & 4 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 2 & 0 \end{pmatrix}}$$

$$\mathbb{Z}\{x_1\lambda_1'\lambda_2', \lambda_1'\lambda_3'\}.$$

By rational considerations, σ acts injectively on the remaining summand

$$\mathbb{Z}\{x_1^5, x_1^3x_2, \dots, x_2x_3, x_5\}$$

of $\pi_{10}THH(MU)$. The result then follows by comparing images and kernels in these complexes. $\square$

Here is the same calculation in absolute coordinates.

THEOREM 6.3.

$$H(\pi_*THH(MU), \sigma) = \begin{cases} \mathbb{Z}\{1\} & \textit{for } * = 0, \\ 0 & \textit{for } * = 1, 2, 4, 6, 8, \\ \mathbb{Z}/2\{e_1\} & \textit{for } * = 3, \\ \mathbb{Z}/12\{e_2\} & \textit{for } * = 5, \\ \mathbb{Z}/12\{e_3'\} & \textit{for } * = 7, \\ \mathbb{Z}/240\{e_4'\} \oplus \mathbb{Z}/2\{e_4''\} & \textit{for } * = 9, \\ \mathbb{Z}/2\{e_1e_3\} & \textit{for } * = 10, \end{cases}$$

where

$$e_3' = e_3 + 2x_1e_2 + x_2e_1$$

$$e_4' = e_4 - x_1^2e_2 - x_3e_1$$

$$e_4'' = x_1x_2e_1 + 3x_2e_2.$$

Proof. Additively,

$$\pi_*THH(MU) = (\mathbb{Z}\{1\}, 0, \mathbb{Z}\{x_1\}, \mathbb{Z}\{e_1\}, \mathbb{Z}\{x_1^2, x_2\}, \mathbb{Z}\{x_1e_1, e_2\},$$
$$\mathbb{Z}\{x_1^3, x_1x_2, x_3\}, \mathbb{Z}\{x_1^2e_1, x_2e_1, x_1e_2, e_3\}, \mathbb{Z}\{e_1e_2, \dots\}, \dots).$$

The cochain complex $(\pi_*THH(MU), \sigma)$ breaks up as the direct sum of the complexes

$$\mathbb{Z}\{1\}$$

$$\mathbb{Z}\{x_1\} \xrightarrow{(2)} \mathbb{Z}\{e_1\}$$

$$\mathbb{Z}\{x_1^2, x_2\} \xrightarrow{\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}} \mathbb{Z}\{x_1e_1, e_2\}$$

$$\mathbb{Z}\{x_1^3, x_1x_2, x_3\} \xrightarrow{\begin{pmatrix} 6 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}} \mathbb{Z}\{x_1^2e_1, x_2e_1, x_1e_2, e_3\} \xrightarrow{\begin{pmatrix} 0 & 3 & -2 & 1 \end{pmatrix}} \mathbb{Z}\{e_1e_2\}$$

and

$$\mathbb{Z}\{x_1^4, x_1^2x_2, x_1x_3, x_2^2, x_4\} \xrightarrow{\begin{pmatrix} 8 & 1 & 1 & 0 & 0 \\ 0 & 4 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}}$$

$$\mathbb{Z}\{x_1^3e_1, x_1x_2e_1, x_3e_1, x_1^2e_2, x_2e_2, x_1e_3, e_4\} \xrightarrow{\begin{pmatrix} 0 & 3 & 4 & -4 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -2 & 2 \end{pmatrix}}$$

$$\mathbb{Z}\{x_1e_1e_2, e_1e_3\}.$$

The result then follows by comparing images and kernels. □

REMARK 6.4. Ignoring decomposables, one might have expected that the σ-operator acts on $\pi_*THH(MU)$ by $\sigma(x_n) = d_ne_n$, where $d_n = p$ if $n+1$ is a power of a prime p and $d_n = 1$ otherwise, and that $\sigma(e_n) = 0$. This would alter the group structure of $H(\pi_*THH(MU), \sigma)$ in degree 7 to $\mathbb{Z}/2\{e_3\} \oplus \mathbb{Z}/6\{x_1^2e_1\}$, and in degree 9 to $\mathbb{Z}/120 \oplus (\mathbb{Z}/2)^2$, and is therefore not a permissible simplification. Any expectation that $H(\pi_*THH(MU), \sigma)$ might be trivial in all positive even degrees, or cyclic in all positive odd degrees, is also dispelled by these calculations.

We can also calculate the d^2-differential and $E^3 = E^4$-term of the circle Tate spectral sequence

$$E^2_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes \pi_*THH(BP) \Longrightarrow \pi_*THH(BP)^{tS^1},$$

in the first few degrees. Since η acts trivially on $\pi_*THH(BP)$, the d^2-differential is given by the σ-operator, and

$$E^3_{*,*} = E^4_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes H(\pi_*THH(BP), \sigma).$$

Rationally, this spectral sequence collapses after the d^2-differential.

PROPOSITION 6.5.

$$\begin{aligned}\pi_*(BP)\otimes\mathbb{Q} &\cong \mathbb{Q}[\ell_n \mid n\geqslant 1]\\ \pi_*THH(BP)\otimes\mathbb{Q} &\cong \mathbb{Q}[\ell_n \mid n\geqslant 1]\otimes E(\lambda_n \mid n\geqslant 1)\\ H(\pi_*THH(BP),\sigma)\otimes\mathbb{Q} &\cong \mathbb{Q}.\end{aligned}$$

The proof is the same as for Proposition 6.1.

THEOREM 6.6.

$$H(\pi_*THH(BP),\sigma) = \begin{cases} \mathbb{Z}_{(p)}\{1\} & \text{for } * = 0,\\ \mathbb{Z}/p\{v_1^{i-1}\lambda_1\} & \text{for } * = i(2p-2)+1,\ 1\leqslant i\leqslant p-1,\\ \mathbb{Z}/p^2\{v_1^{p-1}\lambda_1\} & \text{for } * = 2p^2-2p+1,\\ \mathbb{Z}/p^2\{\lambda_2\} & \text{for } * = 2p^2-1,\\ \mathbb{Z}_{(p)}/p^2(p+2)\{v_2\lambda_1+v_1\lambda_2\} & \text{for } * = 2p^2+2p-3,\\ \mathbb{Z}/p\{\lambda_1\lambda_2\} & \text{for } * = 2p^2+2p-2,\\ 0 & \text{for the remaining } * \leqslant 2p^2+4p-6.\end{cases}$$

The group in degree $* = 2p^2+2p-3$ is $\mathbb{Z}/p^2$ for p odd, and $\mathbb{Z}/16$ for $p=2$.

Proof. The cochain complex $(\pi_*THH(BP),\sigma)$ is the direct sum of a sequence of smaller complexes, which begin with

$$\mathbb{Z}_{(p)}\{1\}$$

$$\mathbb{Z}_{(p)}\{v_1\} \xrightarrow{(p)} \mathbb{Z}_{(p)}\{\lambda_1\}$$

$$\mathbb{Z}_{(p)}\{v_1^2\} \xrightarrow{(2p)} \mathbb{Z}_{(p)}\{v_1\lambda_1\}$$

$$\vdots$$

$$\mathbb{Z}_{(p)}\{v_1^p\} \xrightarrow{(p^2)} \mathbb{Z}_{(p)}\{v_1^{p-1}\lambda_1\}$$

$$\mathbb{Z}_{(p)}\{v_1^{p+1}, v_2\} \xrightarrow{\begin{pmatrix} p(p+1) & -(p+1)\\ 0 & p\end{pmatrix}} \mathbb{Z}_{(p)}\{v_1^p\lambda_1, \lambda_2\}$$

and

$$\mathbb{Z}_{(p)}\{v_1^{p+2}, v_1v_2\} \xrightarrow{\begin{pmatrix} p(p+2) & -(p+1)\\ 0 & p\\ 0 & p\end{pmatrix}} \mathbb{Z}_{(p)}\{v_1^{p+1}\lambda_1, v_2\lambda_1, v_1\lambda_2\} \xrightarrow{\begin{pmatrix}0 & p & -p\end{pmatrix}} \mathbb{Z}_{(p)}\{\lambda_1\lambda_2\}.$$

It is elementary to calculate the cohomology of these complexes. ∎

7. *Algebraic de Rham cohomology*

For any ring R there is a linearization map $\pi_*THH(R) \to HH_*(R)$ to Hochschild homology, which is a rational isomorphism. If R is commutative, then there is also a multiplicative homomorphism $\Omega^*_R \to HH_*(R)$ from the algebra of de Rham forms to Hochschild homology, which by the Hochschild–Kostant–Rosenberg theorem [**25**] is an isomorphism when R is smooth. The σ-operator on $\pi_*THH(R)$ is compatible with the Connes B-operator acting on $HH_*(R)$ and the exterior differential d acting on Ω^*_R, as proved by Loday–Quillen [**30**, Proposition 2.2]. Hence, the linearization map from the Tate spectral sequence (1.1) for $THH(R)$ to the corresponding spectral sequence

$$E^2_{*,*} = \mathbb{Z}[t, t^{-1}] \otimes HH_*(R) \Longrightarrow HP_*(R), \tag{7.1}$$

converging to the periodic cyclic homology $HP_*(R)$, becomes an isomorphism after rationalization. In particular, the map of E^3-terms

$$\mathbb{Z}[t, t^{-1}] \otimes H(\pi_*THH(R), \sigma) \longrightarrow \mathbb{Z}[t, t^{-1}] \otimes H(HH_*(R), B)$$

is a rational isomorphism. Furthermore, the induced homomorphism

$$H^*_{dR}(R) = H(\Omega^*_R, d) \longrightarrow H(HH_*(R), B)$$

from the algebraic de Rham cohomology of R is an isomorphism for R smooth. It is known [**30**, Theorem 2.9] that after rationalization the spectral sequence (7.1) collapses after the d^2-differentials, so that $E^3 \otimes \mathbb{Q} = E^\infty \otimes \mathbb{Q}$.

In view of these classical results, it would be interesting to obtain a more intrinsic algebraic description of the E^3-terms $\mathbb{Z}[t, t^{-1}] \otimes H(\pi_*THH(MU), \sigma)$ and $\mathbb{Z}[t, t^{-1}] \otimes H(\pi_*THH(BP), \sigma)$ of the Tate spectral sequences (1.1) converging to $\pi_*THH(MU)^{tS^1}$ and $\pi_*THH(BP)^{tS^1}$, respectively, than those offered in Theorems 6.2 and 6.6. As first approximations to such descriptions we observe below that there are natural homomorphisms

$$H^*_{dR}(\pi_*(MU)) \longrightarrow H(\pi_*THH(MU), \sigma) \longrightarrow H^*_{dR}(H_*(MU))$$

and

$$H^*_{dR}(\pi_*(BP)) \longrightarrow H(\pi_*THH(BP), \sigma) \longrightarrow H^*_{dR}(H_*(BP)),$$

relating the de Rham cohomology of the graded commutative rings $\pi_*(MU) \cong L$, $H_*(MU) \cong C$, $\pi_*(BP) \cong V$ and $H_*(BP) \cong T$ to the E^3-terms of interest. These are rational isomorphisms, in a trivial way, but fail to be integral isomorphisms. Finally, we observe that the Tate spectral sequences (1.1) for MU and BP do not collapse after the d^2-differential, due to the presence of nonzero d^4-differentials for $THH(MU)^{tS^1}$ and nonzero d^{2p}-differentials for $THH(BP)^{tS^1}$.

7.1. *The Hurewicz homomorphism*

Let HR denote the Eilenberg–Mac Lane ring spectrum of a ring R. There is a unique map $MU \to H\mathbb{Z}$ of E_∞ ring spectra, and a unique map $BP \to H\mathbb{Z}_{(p)}$ of E_4 ring spectra. The following two lemmas are well known.

Lemma 7.1. *There is a commutative diagram of graded commutative rings*

$$\begin{tikzcd}
 & B \arrow[d, "\iota"'] \arrow[dr, "\cong"] & \\
L \arrow[r, "\eta_R"] \arrow[d] \arrow[rr, bend right, "h"] & LB \arrow[r] & H_*(MU) \arrow[d] \\
L \otimes \mathbb{Q} \arrow[rr, "h\otimes\mathbb{Q}", "\cong"'] & & H_*(MU) \otimes \mathbb{Q},
\end{tikzcd}$$

where $LB \to H_*(MU)$ is the surjective homomorphism

$$\pi_*(MU \wedge MU) \longrightarrow \pi_*(H\mathbb{Z} \wedge MU) = H_*(MU)$$

induced by $MU \to H\mathbb{Z}$. The composition $h\colon \pi_*(MU) = L \to H_*(MU)$ is the Hurewicz homomorphism, and $h \otimes \mathbb{Q}$ sends $m_n \in L \otimes \mathbb{Q}$ to the image of $\bar{b}_n \in B$ in $H_*(MU) \otimes \mathbb{Q}$. There is a similar commutative diagram with C and LC in place of B and LB, where $h \otimes \mathbb{Q}$ sends m_n to the image of $c_n \in C$.

Proof. The Hurewicz homomorphism $h\colon \pi_*(MU) \to H_*(MU)$ is induced by the composition

$$MU \cong S \wedge MU \longrightarrow MU \wedge MU \longrightarrow H\mathbb{Z} \wedge MU.$$

The first map induces the right unit $\eta_R\colon L \to LB$. The second map induces the surjective homomorphism

$$LB \longrightarrow \mathbb{Z} \otimes_L LB \cong \pi_*(H\mathbb{Z} \wedge_{MU} (MU \wedge MU)) \cong \pi_*(H\mathbb{Z} \wedge MU),$$

where we use that $LB \cong \pi_*(MU \wedge MU)$ is flat as a (left) module over $L \cong \pi_*(MU)$. The composition $B \to LB \to \mathbb{Z} \otimes_L LB$ is evidently an isomorphism, and similarly for $C \to LC \to \mathbb{Z} \otimes_L LC$.

Using (2.4) and Proposition 2.1, we see that the image of $\eta_R(m_n)$ in $H_*(MU) \otimes \mathbb{Q}$ is equal to the images of $\bar{b}_n \in B$ and $c_n \in C$, since the remaining terms in each sum are sent to zero under $\pi_*(MU \wedge MU) \to \pi_*(H\mathbb{Z} \wedge MU)$. □

Lemma 7.2. *There is a commutative diagram of graded commutative $\mathbb{Z}_{(p)}$-algebras*

$$\begin{tikzcd}
 & T \arrow[d] \arrow[dr, "\cong"] & \\
V \arrow[r, "\eta_R"] \arrow[d] \arrow[rr, bend right, "h"] & VT \arrow[r] & H_*(BP) \arrow[d] \\
V \otimes \mathbb{Q} \arrow[rr, "h\otimes\mathbb{Q}", "\cong"'] & & H_*(BP) \otimes \mathbb{Q},
\end{tikzcd}$$

where $VT \to H_*(BP)$ is the surjective homomorphism

$$\pi_*(BP \wedge BP) \longrightarrow \pi_*(H\mathbb{Z}_{(p)} \wedge BP) = H_*(BP)$$

induced by $BP \to H\mathbb{Z}_{(p)}$. The composition $h\colon \pi_*(BP) = V \to H_*(BP)$ is the Hurewicz homomorphism, and $h \otimes \mathbb{Q}$ sends $\ell_n \in V \otimes \mathbb{Q}$ to the image of $t_n \in T$ in $H_*(BP) \otimes \mathbb{Q}$.

Proof. The proof is similar to that of Lemma 7.1, using (3.3) to calculate the image of $h(\ell_n)$ in $\mathbb{Z}_{(p)} \otimes_V VT \otimes \mathbb{Q} \cong H_*(BP) \otimes \mathbb{Q}$. □

7.2. *Algebraic de Rham complexes*

The Hurewicz homomorphism $h\colon \pi_*(MU) \to H_*(MU)$ maps $\pi_*(MU) \cong L = \mathbb{Z}[x_n \mid n \geqslant 1]$ injectively to $H_*(MU) \cong C = \mathbb{Z}[c_n \mid n \geqslant 1]$. Let

$$\Omega^1_L \cong L\{dx_n \mid n \geqslant 1\} \cong L \otimes \mathrm{Tor}^L_1(\mathbb{Z}, \mathbb{Z})$$

be the module of Kähler differentials of L over $\mathbb{Z}$, and let Ω^*_L be the algebraic de Rham complex, with $\Omega^q_L \cong L \otimes \mathrm{Tor}^L_q(\mathbb{Z}, \mathbb{Z})$ in codegree q. The exterior differential $d\colon \Omega^q_L \to \Omega^{q+1}_L$ is given by $d(x_{n_0}\, dx_{n_1} \cdots dx_{n_q}) = dx_{n_0} dx_{n_1} \cdots dx_{n_q}$. Let us view

$$\pi_* THH(MU) \cong L \otimes \mathrm{Tor}^C_*(\mathbb{Z}, \mathbb{Z})$$

as a cohomologically graded object (in addition to the internal, homotopical grading), with $L \otimes \mathrm{Tor}^C_q(\mathbb{Z}, \mathbb{Z})$ in codegree q. Let $\sigma'(x) = (-1)^{|x|}\sigma(x)$ denote the left derivation associated to σ. We then have inclusions

$$(\Omega^*_L, d) \longrightarrow (\pi_* THH(MU), \sigma') \longrightarrow (\Omega^*_C, d)$$

of cocomplexes, given in codegree q by

$$L \otimes \mathrm{Tor}^L_q(\mathbb{Z}, \mathbb{Z}) \subset L \otimes \mathrm{Tor}^C_q(\mathbb{Z}, \mathbb{Z}) \subset C \otimes \mathrm{Tor}^C_q(\mathbb{Z}, \mathbb{Z}).$$

The first inclusion maps $dx_n \in \Omega^1_L$ to $\sigma'(x_n) \in \pi_* THH(MU)$, while the second inclusion maps $\lambda'_n \in \pi_* THH(MU)$ to $dc_n \in \Omega^1_C$, which corresponds to $[c_n] \in \mathrm{Tor}^C_1(\mathbb{Z}, \mathbb{Z})$.

Similarly, $\pi_*(BP) \cong V = \mathbb{Z}_{(p)}[v_n \mid n \geqslant 1]$ maps injectively by the Hurewicz homomorphism to $H_*(BP) \cong T = \mathbb{Z}_{(p)}[t_n \mid n \geqslant 1]$. We view

$$\pi_* THH(BP) \cong V \otimes \mathrm{Tor}^T_*(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$$

as a cohomologically graded object, with $V \otimes \mathrm{Tor}^T_q(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$ in codegree q. We then have inclusions

$$(\Omega^*_V, d) \longrightarrow (\pi_* THH(BP), \sigma') \longrightarrow (\Omega^*_T, d)$$

of cocomplexes, given in codegree q by

$$V \otimes \mathrm{Tor}^V_q(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \subset V \otimes \mathrm{Tor}^T_q(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \subset T \otimes \mathrm{Tor}^T_q(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}).$$

The first inclusion sends $dv_n \in \Omega^1_V$ to $\sigma'(v_n) \in \pi_* THH(BP)$, while the second inclusion sends $\lambda_n \in \pi_* THH(BP)$ to $dt_n \in \Omega^1_T$, which corresponds to $[t_n] \in \mathrm{Tor}^T_1(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)})$.

Hence, $(\pi_* THH(MU), \sigma')$ is bracketed between the de Rham complexes (Ω^*_L, d) and (Ω^*_C, d), while $(\pi_* THH(BP), \sigma')$ is bracketed between (Ω^*_V, d) and (Ω^*_T, d). The induced homomorphisms in cohomology

$$H^*_{dR}(L) = \bigoplus_q H^q(\Omega^*_L, d) \longrightarrow H(\pi_* THH(MU), \sigma) \longrightarrow \bigoplus_q H^q(\Omega^*_C, d) = H^*_{dR}(C)$$

and

$$H^*_{dR}(V) = \bigoplus_q H^q(\Omega^*_V, d) \longrightarrow H(\pi_* THH(BP), \sigma) \longrightarrow \bigoplus_q H^q(\Omega^*_T, d) = H^*_{dR}(T)$$

are, however, far from isomorphisms.

7.3. *Further differentials*

The E_∞ ring spectrum map $MU \to H\mathbb{Z}$ induces a homomorphism $(\pi_* THH(MU), \sigma) \to (\pi_* THH(\mathbb{Z}), \sigma)$ of differential graded algebras, sending $\lambda'_1 \in \pi_3 THH(MU) \cong \mathbb{Z}$ to a generator

$g_3 \in \pi_3 THH(\mathbb{Z}) \cong \mathbb{Z}/2$. In the circle Tate spectral sequence for $THH(\mathbb{Z})$ there is a nonzero differential

$$d^4(t^{-1}) = tg_3,$$

see [**45**, Theorem 1.3; **47**, Theorem 1.9(2)]. By naturality, it follows that there is a nonzero differential

$$d^4(t^{-1}) = t\lambda_1'$$

in the circle Tate spectral sequence for $THH(MU)$. It also follows that there are nonzero differentials

$$d^4(t^i\lambda_3') = t^{i+2}\lambda_1'\lambda_3',$$

for all i of one parity.

Similarly, the E_4 ring spectrum map $BP \to H\mathbb{Z}_{(p)}$ induces a differential graded algebra homomorphism $(\pi_* THH(BP), \sigma) \to (\pi_* THH(\mathbb{Z}_{(p)}), \sigma)$ sending $\lambda_1 \in \pi_{2p-1} THH(BP) \cong \mathbb{Z}_{(p)}$ to a generator $g_{2p-1} \in \pi_{2p-1} THH(\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$. In the circle Tate spectral sequence for $THH(\mathbb{Z}_{(p)})$ there is a nonzero differential

$$d^{2p}(t^{1-p}) \doteq tg_{2p-1}$$

(see [**10**, p. 100] in the odd case), hence there is a nonzero differential

$$d^{2p}(t^{1-p}) \doteq t\lambda_1$$

in the circle Tate spectral sequence for $THH(BP)$. It follows that there are also nonzero differentials

$$d^{2p}(t^i\lambda_2) \doteq t^{i+p}\lambda_1\lambda_2,$$

for i in all but one congruence class of integers modulo p.

These observations show that after the d^2-differentials given by the σ-operator there will also be later differentials in these Tate spectral sequences, originating not only on the horizontal axis. To determine the precise differential structure will require other methods than those of the present paper. A good beginning would be given by determining the differentials in the C_p-Tate spectral sequence

$$E^2_{*,*} = \hat{H}^{-*}(C_p, \pi_* THH(MU)) \Longrightarrow \pi_* THH(MU)^{tC_p},$$

where we know by [**31**] that the target is p-adically equivalent to $THH(MU)$.

References

1. J. F. Adams, *Stable homotopy and generalized homology*, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1974) x+373.

2. V. Angeltveit and J. Rognes, 'Hopf algebra structure on topological Hochschild homology', *Algebr. Geom. Topol.* 5 (2005) 1223–1290.

3. S. Araki, *Typical formal groups in complex cobordism and K-theory*, Lectures in Mathematics (Kinokuniya Book-Store Co. Ltd., Tokyo, 1973) v+101 pp. (loose errata).

4. C. Ausoni and J. Rognes, 'Algebraic K-theory of topological K-theory', *Acta Math.* 188 (2002) 1–39.

5. M. Basterra and M. A. Mandell, 'The multiplication on BP', *J. Topol.* 6 (2013) 285–310.

6. A. J. Blumberg, R. L. Cohen and C. Schlichtkrull, 'Topological Hochschild homology of Thom spectra and the free loop space', *Geom. Topol.* 14 (2010) 1165–1242.

7. A. J. Blumberg and M. A. Mandell, 'The homotopy groups of the algebraic K-theory of the sphere spectrum', *Geom. Topol.* 23 (2019) 101–134.

8. M. Bökstedt, R. R. Bruner, S. Lunøe-Nielsen and J. Rognes, 'On cyclic fixed points of spectra', *Math. Z.* 276 (2014) 81–91.

9. M. Bökstedt, W. C. Hsiang and I. Madsen, 'The cyclotomic trace and algebraic K-theory of spaces', *Invent. Math.* 111 (1993) 465–539.

10. M. Bökstedt and I. Madsen, 'Topological cyclic homology of the integers, K-theory (Strasbourg, 1992)', *Astérisque* 7–8 (1994) 57–143.

11. M. Bökstedt and I. Madsen, 'Algebraic K-theory of local number fields: the unramified case', *Prospects in topology*, Annals of Mathematics Studies 138 (Princeton University Press, Princeton, NJ, 1995) 28–57.
12. E. H. Brown, Jr. and F. P. Peterson, 'A spectrum whose Z_p cohomology is the algebra of reduced pth powers', *Topology* 5 (1966) 149–154.
13. M. Brun, Z. Fiedorowicz and R. M. Vogt, 'On the multiplicative structure of topological Hochschild homology', *Algebr. Geom. Topol.* 7 (2007) 1633–1650.
14. R. R. Bruner and J. Rognes, 'Differentials in the homological homotopy fixed point spectral sequence', *Algebr. Geom. Topol.* 5 (2005) 653–690.
15. P. Cartier, 'Groupes formels associés aux anneaux de Witt généralisés', *C. R. Acad. Sci. Paris Sér. A-B* 265 (1967) A49–A52 (French).
16. P. Cartier, 'Modules associés à un groupe formel commutatif. Courbes typiques', *C. R. Acad. Sci. Paris Sér. A-B* 265 (1967) A129–A132 (French).
17. B. I. Dundas, T. G. Goodwillie and R. McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications 18 (Springer London, Ltd., London, 2013) xvi+435.
18. B. I. Dundas and J. Rognes, 'Cubical and cosimplicial descent', *J. Lond. Math. Soc.* (2) 98 (2018) 439–460.
19. A. D. Elmendorf, I. Kriz, M. A. Mandell and J. P. May, '*Rings, modules, and algebras in stable homotopy theory*', Mathematical Surveys and Monographs 47, With an appendix by M. Cole (American Mathematical Society, Providence, RI, 1997) xii+249.
20. J. P. C. Greenlees and J. P. May, 'Generalized Tate cohomology', *Mem. Amer. Math. Soc.* 113 (1995) viii+178.
21. M. Hazewinkel, 'Constructing formal groups. I. The local one dimensional case', *J. Pure Appl. Algebra* 9 (1976/77) 131–149.
22. A. Hedenlund and J. Rognes, Multiplicative Tate spectral sequences for compact Lie group actions, in preparation.
23. L. Hesselholt, 'Topological Hochschild homology and the Hasse-Weil zeta function', *An alpine bouquet of algebraic topology*, Contemporary Mathematics 708 (American Mathematical Society, Providence, RI, 2018) 157–180.
24. L. Hesselholt and I. Madsen, 'On the K-theory of finite algebras over Witt vectors of perfect fields', *Topology* 36 (1997) 29–101.
25. G. Hochschild, B. Kostant and A. Rosenberg, 'Differential forms on regular affine algebras', *Trans. Amer. Math. Soc.* 102 (1962) 383–408. MR142598.
26. P. S. Landweber, 'Cobordism operations and Hopf algebras', *Trans. Amer. Math. Soc.* 129 (1967) 94–110.
27. P. S. Landweber, 'Associated prime ideals and Hopf algebras', *J. Pure Appl. Algebra* 3 (1973) 43–58.
28. P. S. Landweber, '$\mathrm{BP}_*(\mathrm{BP})$ and typical formal groups', *Osaka J. Math.* 12 (1975) 357–363.
29. M. Lazard, 'Sur les groupes de Lie formels à un paramètre', *Bull. Soc. Math. France* 83 (1955) 251–274 (French).
30. J.-L. Loday and D. Quillen, 'Cyclic homology and the Lie algebra homology of matrices', *Comment. Math. Helv.* 59 (1984) 569–591. MR 780077.
31. S. Lunøe-Nielsen and J. Rognes, 'The Segal conjecture for topological Hochschild homology of complex cobordism', *J. Topol.* 4 (2011) 591–622.
32. S. Lunøe-Nielsen and J. Rognes, 'The topological Singer construction', *Doc. Math.* 17 (2012) 861–909.
33. J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics 577, With contributions by Frank Quinn, Nigel Ray and Jørgen Tornehave (Springer, Berlin, 1977).
34. J. E. McClure and R. E. Staffeldt, 'On the topological Hochschild homology of bu. I', *Amer. J. Math.* 115 (1993) 1–45.
35. H. R. Miller, 'Some algebraic aspects of the Adams–Novikov spectral sequence', PhD Thesis, Princeton University, ProQuest LLC, Ann Arbor, MI, 1975.
36. H. R. Miller, D. C. Ravenel and W. S. Wilson, 'Periodic phenomena in the Adams-Novikov spectral sequence', *Ann. of Math.* (2) 106 (1977) 469–516.
37. J. Milnor, 'On the cobordism ring Ω^* and a complex analogue. I', *Amer. J. Math.* 82 (1960) 505–521.
38. R. E. Mosher, 'Some stable homotopy of complex projective space', *Topology* 7 (1968) 179–193.
39. T. Nikolaus and P. Scholze, 'On topological cyclic homology', *Acta Math.* 221 (2018) 203–409.
40. S. P. Novikov, 'Some problems in the topology of manifolds connected with the theory of Thom spaces', *Soviet Math. Dokl.* 1 (1960) 717–720.
41. S. P. Novikov, 'Rings of operations and spectral sequences of Adams type in extraordinary cohomology theories. U-cobordism and K-theory', *Dokl. Akad. Nauk SSSR* 172 (1967) 33–36 (Russian).
42. D. Quillen, 'On the formal group laws of unoriented and complex cobordism theory', *Bull. Amer. Math. Soc.* 75 (1969) 1293–1298.
43. D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121 (Academic Press Inc., Orlando, FL, 1986) xx+413.
44. D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies 128, Appendix C by Jeff Smith (Princeton University Press, Princeton, NJ, 1992) xiv+209.
45. J. Rognes, 'Trace maps from the algebraic K-theory of the integers (after Marcel Bökstedt)', *J. Pure Appl. Algebra* 125 (1998) 277–286.
46. J. Rognes, 'The product on topological Hochschild homology of the integers with mod 4 coefficients', *J. Pure Appl. Algebra* 134 (1999) 211–218.

47. J. Rognes, 'Topological cyclic homology of the integers at two', *J. Pure Appl. Algebra* 134 (1999) 219–286.
48. J. Rognes, 'Algebraic K-theory of the two-adic integers', *J. Pure Appl. Algebra* 134 (1999) 287–326.
49. J. Rognes, 'Two-primary algebraic K-theory of pointed spaces', *Topology* 41 (2002) 873–926.
50. J. Rognes, 'The smooth Whitehead spectrum of a point at odd regular primes', *Geom. Topol.* 7 (2003) 155–184.

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