# INDUCTIVE PRESENTATIONS OF GENERALIZED REEDY CATEGORIES

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Abstract. This note explores the algebraic perspective on the notion of generalized Reedy category introduced by Berger and Moerdijk [BM08]. The aim is to unify inductive arguments by means of a canonical presentation of the hom bifunctor as a "generalized cell complex." This is analogous to the weighted (co)limits approach to strict Reedy category theory presented in [RV14], which inspired this work. These presentations are used to prove that various functors associated to categories of Reedy-indexed diagrams are "derived" preserving pointwise weak equivalences between appropriately fibrant or cofibrant diagrams.

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## 1. The algebraic perspective on Reedy categories

One aim of categorical homotopy theory is to separate categorical (up to isomorphism) statements from homotopical (merely up to weak equivalence) ones. The point is that there are more results of the former kind involving familiar machinery from abstract homotopy theory than one might expect and these are more easily understood as strict point-set-level phenomena.

The aim of this paper is to apply this approach to the theory of generalized Reedy categories (here simply "Reedy categories") introduced by Berger and Moerdijk [BM08]. Analogous work for ordinary (here "strict") Reedy categories was undertaken in [RV14]. Both varieties are small categories in which the objects are assigned a natural number degree equipped with specified "degree-decreasing" and "degree-increasing" morphisms. The category-theoretic results for strict Reedy categories extend fairly straightforwardly to the more general case. The main difference is that in the strict case the move from the algebraic results to the homotopical ones is considerably simplified by the fact that a strict Reedy category has no non-identity isomorphisms.

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**Algebraic perspectives on Reedy categories.** A Reedy category is a small category in which each object is assigned a natural number degree satisfying a few axioms that imply that diagrams and natural transformations may be defined inductively. The algebraic perspective on Reedy categories unifies these inductive arguments by means of a canonical (generalized) cell complex presentation of the hom bifunctor.

This presentation is most familiar for the strict Reedy category $\mathbb{\Delta}^{\mathrm{op}}$. A simplicial object $Y$ taking values in any cocomplete category admits a skeletal filtration

$$\emptyset \to \mathrm{sk}_0 Y \to \cdots \to \mathrm{sk}_{n-1} Y \to \mathrm{sk}_n Y \to \cdots \to Y$$

in which the step from stage $n-1$ to stage $n$ is given by a pushout

$$\begin{tikzcd} L_n Y \times \mathbb{\Delta}^n \cup Y_n \times \partial\mathbb{\Delta}^n \arrow[r] \arrow[d] & Y_n \times \mathbb{\Delta}^n \arrow[d] \\ \mathrm{sk}_{n-1} Y \arrow[r] & \mathrm{sk}_n Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $L_n Y \to Y_n$ is the object of "degenerate $n$-simplices." Considering the Yoneda embedding as a simplicial object $\mathbb{\Delta} \in (\mathsf{Set}^{\mathbb{\Delta}})^{\mathbb{\Delta}^{\mathrm{op}}}$, this specializes to the "canonical cell complex presentation" of the hom bifunctor:

$$\begin{tikzcd} & & \partial\mathbb{\Delta}_n \times \mathbb{\Delta}^n \cup \mathbb{\Delta}_n \times \partial\mathbb{\Delta}^n \arrow[r, hook] \arrow[d] & \mathbb{\Delta}_n \times \mathbb{\Delta}^n \arrow[d] & & \\ \emptyset \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{sk}_{n-1}\mathbb{\Delta} \arrow[r, hook] & \mathrm{sk}_n \mathbb{\Delta} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbb{\Delta} \end{tikzcd}$$

In the canonical cell complex presentation for $\mathbb{\Delta}$, there is a single "cell" attached at stage $n$, displayed in the pushout above. In an arbitrary strict Reedy category, there may be multiple cells attached at stage $n$, collected together via a coproduct indexed by the objects of degree $n$. In the non-strict case, this coproduct is replaced by a coend which quotients over the groupoid of isomorphisms between objects of degree $n$. We use the term *generalized cell complex* to refer to a sequential composite of pushouts of "generalized cells," which are constructed as groupoid-indexed coends, rather than mere coproducts, of basic cells. The dual notion is a *generalized Postnikov tower*, a sequential limit of pullbacks of "generalized layers," constructed as groupoid-indexed ends, rather than mere products, of basic maps. The canonical generalized cell complex presentation of the hom-bifunctor is constructed in section 4.

Interpreting the hom bifunctor as a "weight" for a limit or a colimit, this result immediately gives a pair of dual presentations for any diagram indexed by the Reedy category: the first as a generalized cell complex built using weighted colimits and the second as a generalized Postnikov tower built using weighted limits. The constructions can also be relativized to produce filtrations for natural transformations between Reedy category indexed diagrams.

The "cells" appearing in the generalized cell complexes or "layers" appearing in the generalized Postnikov presentations are built from the so-called relative latching and matching maps. It is not immediately obvious that these presentations assist with *homotopically invariant* inductive constructions because of the equivariance conditions imposed by the coherence isomorphisms between the maps defining these "cells" and "layers." Here we present three ways around this problem — two of which are well-known and a third that we find to be considerably more aesthetically satisfying, although it has no known applications just yet.

**Projectivity or injectivity.** One way to ensure good homotopy theoretic properties of automorphisms is to demand they act "freely" on the maps involved in the construction of the generalized cells or the generalized layers. This is the case if the relative latching or relative matching maps assemble, respectively, into projective cofibrant or injective fibrant diagrams in the category indexed by the groupoid of isomorphisms in the Reedy category. When an extra "dualizability" hypothesis is required of the Reedy category, both the projective and injective cases give rise to model structures on Reedy-indexed diagrams, first established by Berger and Moerdijk and reviewed in Corollary 8.6 below.

To state all of our model category theoretic results most concisely, we assume that we begin with a base model structure $(\mathcal{C},\mathcal{F},\mathcal{W})$ on a bicomplete category $\mathsf{M}$ that is *accessible* in the sense of [HKRS15]. This means that the category $\mathsf{M}$ is locally presentable and the weak factorization systems $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ and $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ are *accessible* (algebraic) weak factorization systems. We do not review this definition here. Instead, it suffices to note that every combinatorial model category is accessible; accessible model categories include additional model structures that are not cofibrantly generated, at least in the sense that this is traditionally understood.[1]

The point of this hypothesis is to assume that arbitrary categories of groupoid-indexed diagrams in $\mathsf{M}$ admit projective and injective model structures. If these model structures are known to exist for some other reason, the category $\mathsf{M}$ does not have to be locally presentable.

**Algebraic model categories.** A final tactic to detail with the automorphisms makes use of an "algebraic" framework for abstract homotopy theory, a more structured extension of Quillen's model categories. Briefly, a model structure on a category in which certain morphisms are deemed "weak equivalences" consists of a pair of interacting weak factorization systems. A *weak factorization system* consists of a "left" and a "right" class of maps, the former closed under certain colimits and the latter under the dual limits, that together satisfy a lifting property. This lifting property, together with a factorization axiom, enables many of the constructions of abstract homotopy theory.

In an *algebraic weak factorization system* [GT06, Gar09, Rie11, BG16i], we think of maps in the left class as "coalgebras" and maps in the right class as "algebras." The (co)algebra structures determine a canonical solution to any lifting problem that is preserved by maps of coalgebras or algebras. These (co)algebras structures acts as "witnesses" that the maps are members of the left or right classes. In an algebraic model category, once these witnesses are chosen, the other constructions of abstract homotopy theory are "algebraically" determined.

In particular, if the relative latching and relative matching maps are, respectively, coalgebras and algebras for some ambient algebraic weak factorization system on the target category, then these filtrations become sequences of coalgebras and towers of algebras, in which case they may be used to inductive solve lifting and extension problems, precisely as in the classical theory. This motivates the definition of the Reedy algebraic weak factorization system, formalized in an epilogue that will appear in the future. The projective and injective Reedy algebraic weak factorization systems of Berger and Moerdijk can be thought of as "special cases" of this intermediate, symmetrically defined, functorial factorization.

**Outline.** Before Reedy categories are introduced in §**??**, we review some of the necessary background material. Section 2 reviews weighted limits and colimits and the "pushout

[1]See however [BG16i].

product" and "pullback cotensor" constructions that are used to define the relative latching and matching maps. Here the fundamental example constructs the boundary of a product $\partial(A \times B)$ by gluing together $\partial A \times B$ and $A \times \partial B$. This boundary formula is often attributed to Leibniz; hence we refer to this gluing as the "Leibniz construction" in this case relative to the bifunctor $\times$. A second background section 3 reviews the weak factorization systems that comprise a model structure and develops the theory of Leibniz bifunctors between them.

In §**??**, we present the canonical inductive presentation of the hom bifunctor associated to any Reedy category and derive the generalized cell complex and generalized Postnikov presentations of any map between Reedy diagrams as an immediate corollary. Section 5 then specializes these results to the case of strict Reedy categories, reviewing the results of [RV14] and previewing the extension of this material to the generalized case that will follow.

The projective and injective Reedy weak factorization systems are introduced in §6 and their associated Leibniz bifunctors are established in §7. The dualizability hypothesis is introduced in §8 when homotopy theoretic considerations first appear. Finally, in §9, this theory is applied to the construction of derived functors, particularly involving homotopy limits and homotopy colimits.

**Acknowledgments.** The project was inspired by Peter May, who wrote to the author on the day that [RV14] hit the arXiv to complain that generalized Reedy categories were not included in its scope. The author gave a few talks along these lines during the fall of 2013, but abandoned the project after failing to generate much interest in these findings.

The author would like to credit Mona Merling for inspiring her to revisit this project and Yuri Sulyma for asking for a literature reference for the material that can now be found in §5.

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## 2. Weighted limits and colimits and the Leibniz construction

The notation introduced below is largely consistent with [RV14]. In particular, throughout we express covariant dependence on an object $a \in \mathcal{A}$ with a superscript $(-)^a$ and denote contravariant dependence with a subscript $(-)_a$. For instance, we let $\mathcal{A} \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$ stand as shorthand for the two sided representable (the hom bifunctor), specializing to the contravariant representables $\mathcal{A}^a \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ and the covariant representables $\mathcal{A}_a \in \mathsf{Set}^{\mathcal{A}}$. These conventions dictate that $\mathbb{\Delta}^n$ denotes the represented $n$-simplex in $\mathsf{Set}^{\mathbb{\Delta}^{\mathrm{op}}}$, as is standard (at least for this author).

Large categories will be assumed to be bicomplete so we need not worry whether the limits or colimits we define below exist. The category of $\mathcal{A}$-shaped diagrams in $\mathsf{M}$ is denoted $\mathsf{M}^{\mathcal{A}}$. In the special case of the category $\mathbb{2} = \bullet \to \bullet$, $\mathsf{M}^{\mathbb{2}}$ is the arrow category of maps in $\mathsf{M}$ and commutative squares.

**Leibniz products.** Given a bifunctor $- \otimes - \colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$, frequently arising as the left adjoint of a two-variable adjunction

$$\{-,-\} \colon \mathsf{K}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{L} \qquad\qquad \hom \colon \mathsf{L}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{K},$$

$$\mathsf{M}(K \otimes L, M) \cong \mathsf{L}(L, \{K, M\}) \cong \mathsf{K}(K, \hom(L, M)) \qquad \forall K \in \mathsf{K}, L \in \mathsf{L}, M \in \mathsf{M}$$

the "pushout-product" construction defines a bifunctor $-\mathbin{\hat{\otimes}}-\colon \mathsf{K}^{\mathbb{2}} \times \mathsf{L}^{\mathbb{2}} \to \mathsf{M}^{\mathbb{2}}$ that we refer to as the "Leibniz tensor" (when the bifunctor $\otimes$ is called a "tensor"). The "Leibniz cotensor" and "Leibniz hom"

$$\widehat{\{-,-\}}\colon (\mathsf{K}^{\mathbb{2}})^{\mathrm{op}} \times \mathsf{M}^{\mathbb{2}} \to \mathsf{L} \qquad \text{and} \qquad \widehat{\hom}(-,-)\colon (\mathsf{L}^{\mathbb{2}})^{\mathrm{op}} \times \mathsf{M}^{\mathbb{2}} \to \mathsf{K}^{\mathbb{2}}$$

are defined dually, using pullbacks in $\mathsf{L}$ and $\mathsf{K}$ respectively.

**Definition 2.1** (the Leibniz construction)**.** Given a bifunctor $-\otimes-\colon \mathsf{K}\times\mathsf{L} \to \mathsf{M}$, the **Leibniz tensor** of a map $k\colon I \to J$ in $\mathsf{K}$ and a map $\ell\colon A \to B$ in $\mathsf{L}$ is the map $k \mathbin{\hat{\otimes}} \ell$ in $\mathsf{M}$ induced by the pushout diagram

$$\begin{tikzcd}
I \otimes A \arrow[r, "I\otimes\ell"] \arrow[d, "k\otimes A"'] & I \otimes B \arrow[d] \arrow[ddr, bend left, "k\otimes B"] & \\
J \otimes A \arrow[r] \arrow[drr, bend right, "J\otimes\ell"'] & \bullet \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "k\hat{\otimes}\ell"] & \\
 & & J \otimes B
\end{tikzcd} \tag{2.2}$$

In the case of a bifunctor $\{-,-\}\colon \mathsf{K}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{L}$ contravariant in one of its variables, the **Leibniz cotensor** of a map $k\colon I \to J$ in $\mathsf{K}$ and a map $m\colon X \to Y$ in $\mathsf{M}$ is the map $\widehat{\{k,m\}}$ induced by the pullback diagram

$$\begin{tikzcd}
\{J,X\} \arrow[drr, bend left, "\{k,X\}"] \arrow[dr, dashed, "\widehat{\{k,m\}}"] \arrow[ddr, bend right, "\{J,m\}"'] & & \\
 & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \{I,X\} \arrow[d, "\{I,m\}"] \\
 & \{J,Y\} \arrow[r, "\{k,Y\}"'] & \{I,Y\}
\end{tikzcd} \tag{2.3}$$

**Proposition 2.4.** *The Leibniz construction preserves:*

*(i) structural isomorphisms: a natural isomorphism*

$$X * (Y \otimes Z) \cong (X \times Y) \square Z$$

*between suitable composable bifunctors extends to a natural isomorphism*

$$f \mathbin{\hat{*}} (g \mathbin{\hat{\otimes}} h) \cong (f \mathbin{\hat{\times}} g) \mathbin{\hat{\square}} h$$

*between the corresponding Leibniz products;*

*(ii) adjointness: if* $(\otimes, \{,\}, \hom)$ *define a two-variable adjunction, then the Leibniz functors* $(\hat{\otimes}, \widehat{\{,\}}, \widehat{\hom})$ *define a two-variable adjunction between the corresponding arrow categories;*

*(iii) colimits in the arrow category: if* $\otimes\colon \mathsf{K}\times\mathsf{L} \to \mathsf{M}$ *is cocontinuous in either variable, then so is* $\hat{\otimes}\colon \mathsf{K}^{\mathbb{2}} \times \mathsf{L}^{\mathbb{2}} \to \mathsf{M}^{\mathbb{2}}$*;*

*(iv) pushouts: if* $\otimes\colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ *is cocontinuous in its second variable, and if* $g'$ *is a pushout of* $g$*, then* $f \mathbin{\hat{\otimes}} g'$ *is a pushout of* $f \mathbin{\hat{\otimes}} g$*;*

(v) *composition, in a sense: the Leibniz tensor $f \hat{\otimes} (h \cdot g)$ factors as a composite of a pushout of $f \hat{\otimes} g$ followed by $f \hat{\otimes} h$*

$$\begin{tikzcd}
I \otimes A \arrow[r, "I\otimes g"] \arrow[d, "f\otimes A"'] & I \otimes B \arrow[r, "I\otimes h"] \arrow[d] & I \otimes C \arrow[d] \arrow[dddrr, bend left, "f\otimes C"] & & \\
J \otimes A \arrow[r] \arrow[drr, "J\otimes g"'] & \bullet \arrow[r] \arrow[dr, dashed, "f\hat{\otimes}g"] & \bullet \arrow[dr] \arrow[ddrr, bend left, "f\hat{\otimes}(h\cdot g)"] & & \\
& & J \otimes B \arrow[r] \arrow[drr, "J\otimes h"'] & \bullet \arrow[dr, dashed, "f\hat{\otimes}h"] & \\
& & & & J \otimes C
\end{tikzcd}$$

(vi) *(generalized) cell complex structures: if $f$ and $g$ may be presented as (generalized) cell complexes with cells $f_\alpha$ and $g_\beta$, respectively, and if $\otimes$ is cocontinuous in both variables, then $f \hat{\otimes} g$ may be presented as a (generalized) cell complex with cells $f_\alpha \hat{\otimes} g_\beta$.*

Proofs of these assertions and considerably more details are given in [RV14, §§4-5].

**Weighted limits and colimits.** Ordinary limits and colimits are objects representing the $\mathsf{Set}$-valued functor of cones with a given summit over or under a fixed diagram. Weighted limits and colimits are defined analogously, except that the cones over or under a diagram might come in exotic "shapes." These shapes are allowed to vary with the objects indexing the diagram. More formally, the **weight** (the "shape") of a cone over a diagram of shape $\mathcal{A}$ takes the form of a functor in $\mathsf{Set}^{\mathcal{A}}$. The weight for a cone under a diagram of shape $\mathcal{A}$ takes the form of a functor in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$.

*Example* 2.5 (tensors and cotensors). For example, in the case of a diagram of shape $\mathbb{1}$ in a category $\mathsf{M}$, the shape of a cone might be a set $S \in \mathsf{Set}$. Writing $X \in \mathsf{M}$ for the object in the image of the diagram, the $S$-weighted limit of $X$ is an object $\{S, X\} \in \mathsf{M}$ satisfying the universal property

$$\mathsf{M}(M, \{S, X\}) \cong \mathsf{Set}(S, \mathsf{M}(M, X))$$

while the $S$-weighted colimit of $X$ is an object $S * X \in \mathsf{M}$ satisfying the universal property

$$\mathsf{M}(S * X, M) \cong \mathsf{Set}(S, \mathsf{M}(X, M)).$$

For historical reasons, $\{S, X\}$ is called the **cotensor** and $S * X$ is called the **tensor** of $X \in \mathsf{M}$ by the set $S$.

If $\mathsf{M}$ has small products and coproducts, in this case guaranteed by our standing assumption that the large categories under consideration are bicomplete, then $\{S, X\}$ and $S * X$ are, respectively, the $S$-fold product and coproduct of the object $X$ with itself, and cotensors and tensors define bifunctors

$$\{-,-\} \colon \mathsf{Set}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{M} \qquad \text{and} \qquad - * - \colon \mathsf{Set} \times \mathsf{M} \to \mathsf{M}.$$

**Definition 2.6** (weighted limits and colimits, axiomatically)**.** For a general small category $\mathcal{A}$, the weighted limit and weighted colimit define bifunctors

$$\{-,-\}^{\mathcal{A}} \colon (\mathsf{Set}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M} \qquad \text{and} \qquad - *_{\mathcal{A}} - \colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$$

which are characterized by the following pair of axioms.

(i) Weighted (co)limits with representable weights evaluate at the representing object: $\{\mathcal{A}_a, X\}^{\mathcal{A}} \cong X^a$ and $\mathcal{A}^a *_{\mathcal{A}} X \cong X^a$.

(ii) The weighted (co)limit bifunctors are cocontinuous in the weight: for any diagram $X \in \mathsf{M}^{\mathcal{A}}$, the functor $- *_{\mathcal{A}} X$ preserves colimits, while the functor $\{-, X\}^{\mathcal{A}}$ carries colimits to limits.

We interpret axiom (ii) to mean that weights can be "made-to-order": a weight constructed as a colimit of representables — as all $\mathsf{Set}$-valued functors are — will stipulate the expected universal property.

**Definition 2.7** (weighted limits and colimits, constructively)**.** Equivalently, the weighted colimit is a functor tensor product and the weighted colimit is a functor cotensor product[2]:

$$\{W, X\}^{\mathcal{A}} \cong \int_{a\in\mathcal{A}} \{W^a, X^a\} \qquad\qquad W *_{\mathcal{A}} X \cong \int^{a\in\mathcal{A}} W_a * X^a. \tag{2.8}$$

The limit $\{W, X\}^{\mathcal{A}}$ of the diagram $X$ weighted by $W$ and the colimit $W *_{\mathcal{A}} X$ of $X$ weighted by $W$ are characterized by the universal properties:

$$\mathsf{M}(M, \{W, X\}^{\mathcal{A}}) \cong \mathsf{Set}^{\mathcal{A}}(W, \mathsf{M}(M, X)) \qquad \mathsf{M}(W *_{\mathcal{A}} X, M) \cong \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}(W, \mathsf{M}(X, M)).$$

*Example* 2.9. When $W$ is, respectively, the constant $\mathcal{A}$-diagram or $\mathcal{A}^{\mathrm{op}}$-diagram at the terminal object $\mathbb{1} \in \mathsf{Set}$, we see from the defining formulae (2.8) that

$$\{\mathbb{1}, X\}^{\mathcal{A}} \cong \lim X \qquad \text{and} \qquad \mathbb{1} *_{\mathcal{A}} X \cong \operatorname{colim} X.$$

Here the weight $\mathbb{1}$ stipulates that the cones should have their ordinary "shapes" with one leg pointing to or from each object in the diagram $X$.

*Example* 2.10. Suppose that the indexing category $\mathcal{G}$ is a one-object groupoid (i.e., a group) and, for simplicity, that the target category is $\mathsf{Set}$. The weighted colimit $W *_{\mathcal{G}} X$, constructed by a coequalizer

$$W \times \mathcal{G} \times X \rightrightarrows W \times X \twoheadrightarrow W *_{\mathcal{G}} X$$

takes "orbits," identifying the actions of a left $\mathcal{G}$-set $X$ and a right $\mathcal{G}$-set $W$. The weighted limit $\{W, X\}^{\mathcal{G}}$ of two left $\mathcal{G}$-sets, constructed by an equalizer

$$\{W, X\}^{\mathcal{G}} \rightarrowtail \{W, X\} \rightrightarrows \{\mathcal{G} \times W, X\}$$

is the set of $\mathcal{G}$-equivariant maps from $W$ to $X$. These are the $\mathcal{G}$-fixed points in the set $\{W, X\}$ of all maps from $W$ to $X$, with the conjugation action.

*Example* 2.11. Let $K\colon \mathcal{A} \to \mathcal{D}$ denote a functor between small categories, suppose $\mathsf{M}$ is bicomplete, and consider a diagram $X \in \mathsf{M}^{\mathcal{A}}$. The right and left Kan extensions of $X$ along $K$ are the limit and colimit of $X$ weighted by restricted hom bifunctors:

$$\mathrm{Ran}_K X \cong \{\mathcal{D}(-, K-), X\}^{\mathcal{A}} \qquad \mathrm{Lan}_K X \cong \mathcal{D}(K-, -) *_{\mathcal{A}} X.$$

In the special case where $K = \mathrm{id}_{\mathcal{A}}$, we have

$$X \cong \mathrm{Ran}_{\mathrm{id}} X \cong \{\mathcal{A}, X\}^{\mathcal{A}} \qquad X \cong \mathrm{Lan}_{\mathrm{id}} X \cong \mathcal{A} *_{\mathcal{A}} X,$$

the former result generalizing the Yoneda lemma and dual one accordingly called the coYoneda lemma.

[2]Here the left-hand and right-hand "$W$"s must denote different functors as the weights for limits and colimits have contrasting variance.

*Remark* 2.12 (profunctorial weights). A **profunctor** from $\mathcal{B}$ to $\mathcal{A}$, meaning simply a functor $W \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}$, can serve as a weight for $\mathcal{A}$-indexed colimits and $\mathcal{B}$-indexed limits. In this case, the weighted colimit and weighted limit define adjoint functors

$$\begin{tikzcd} \mathsf{M}^{\mathcal{A}} \arrow[r, shift left, "W *_{\mathcal{A}} -"] \arrow[r, phantom, "\perp"] & \mathsf{M}^{\mathcal{B}} \arrow[l, shift left, "\{W,-\}^{\mathcal{B}}"] \end{tikzcd}$$

In the case where $W = \mathcal{A} \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$ is the hom bifunctor (the identity profunctor), this is the identity adjunction, by the coYoneda lemma and the Yoneda lemma.

*Remark* 2.13. In the present unenriched context, weighted limits and colimits reduce to ordinary limits and colimits via the Grothendieck construction: the limit of $X \in \mathsf{M}^{\mathcal{A}}$ weighted by $W \in \mathsf{Set}^{\mathcal{A}}$ is the limit of the composite diagram $\mathsf{el}W \to \mathcal{A} \xrightarrow{X} \mathsf{M}$ which restricts the domain of $X$ along the functor defined by the pullback of categories

$$\begin{tikzcd} \mathsf{el}W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Set}_* \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "W"'] & \mathsf{Set} \end{tikzcd}$$

A similar construction computes weighted colimits as ordinary colimits. Nevertheless, the weighted framework can provide a useful conceptual simplification in Reedy category theory, as we shall now discover.

## 3. Weak factorization systems and Leibniz bifunctors

In this section, we review the theory of weak factorization systems, which are pervasive in abstract homotopy theory. The familiar factorization and lifting properties axiomatized, for instance, in a Quillen model structure can be summarized by saying that a model category is equipped with an interacting pair of weak factorization systems. We then develop the basic theory of morphisms of weak factorization systems, focusing in particular on bifunctors that satisfy a particular "Leibniz property" to be detailed below.

**Weak factorization systems.**

**Definition 3.1.** A **weak factorization system** on a category $\mathsf{M}$ is comprised of a pair of classes of maps $(\mathcal{L}, \mathcal{R})$, referred to as **left class** and the **right class**, so that:

(i) Any morphism in $\mathcal{M}$ can be factored as a map in the left class followed by a map in the right class.

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L} \ni \ell"'] & & \bullet \\ & \bullet \arrow[ur, "r \in \mathcal{R}"'] & \end{tikzcd}$$

(ii) The morphisms in $\mathcal{L}$ have the **left lifting property** with respect to the morphisms in $\mathcal{R}$, and equivalently, the morphisms in $\mathcal{R}$ have the **right lifting property** with respect to the morphisms in $\mathcal{L}$, which is to say that any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L} \ni \ell"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \tag{3.2}$$

admits a dashed diagonal **lift**, making both triangles commute.

The notation $\mathcal{L} \boxslash \mathcal{R}$ asserts that the classes $\mathcal{L}$ and $\mathcal{R}$ have the lifting property just described.

(iii) Moreover, every morphism $f$ with the left lifting property $f \boxslash \mathcal{R}$ is in the class $\mathcal{L}$, and every morphism $g$ with the right lifting property $\mathcal{L} \boxslash g$ is in the class $\mathcal{R}$.[3]

*Remark* 3.3. Because the left class of a weak factorization system is characterized by a right lifting property, namely $\mathcal{L} = \mathcal{R}^{\boxslash}$, $\mathcal{L}$ is closed under coproducts, pushouts, transfinite composition, retracts, and contains the isomorphisms. Dually, the right class $\mathcal{R} = {}^{\boxslash}\mathcal{L}$ is closed under products, pullbacks, transfinite inverse limits, retracts, and contains the isomorphisms; see [Rie14, §11.1].

**Definition 3.4.** A weak factorization system $(\mathcal{L}, \mathcal{R})$ is **cofibrantly generated** by a set of morphisms $\mathcal{J}$ if

(i) its right class is comprised precisely of those maps with the right lifting property with respect to $\mathcal{J}$, i.e., if $\mathcal{R} = \mathcal{J}^{\boxslash}$, and if
(ii) each morphism in its left class is a retract of a cell complex whose cells are in $\mathcal{J}$.

Dually, $(\mathcal{L}, \mathcal{R})$ is **fibrantly generated** by a set of morphisms $\mathcal{X}$ if $\mathcal{L} = {}^{\boxslash}\mathcal{X}$ and if each morphism in $\mathcal{R}$ is a retract of a Postnikov tower whose cells are in $\mathcal{X}$.

Axioms (ii) and (iii) implies that every weak factorization system is "class-cofibrantly generated" by its left class and "class-fibrantly generated" by its right class.

*Example* 3.5. There is a weak factorization system $(\mathcal{M}, \mathcal{E})$ on $\mathsf{Set}$ whose left class is the class of monomorphisms and whose right class is the class of epimorphisms. This weak factorization system is cofibrantly generated by the single morphism $\{\emptyset \hookrightarrow 1\}$ from the empty set to the singleton. Among the many potential functorial factorizations is the "cograph factorization" of a morphism through the coproduct of its domain and codomain.

*Example* 3.6. Any Quillen model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on a category $\mathsf{M}$ provides two weak factorization systems:

- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ whose left class is the class of **trivial cofibrations** and whose right class is the class of **fibrations**, and
- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ whose left class is the class of **cofibrations** and whose right class is the class of **trivial fibrations**.

Conversely, given any category $\mathsf{M}$ with a class $\mathcal{W}$ of maps satisfying the 2-of-3 property, to define a Quillen model structure with weak equivalences $\mathcal{W}$ is to specify a class of cofibrations $\mathcal{C}$ and fibrations $\mathcal{F}$ defining weak factorization systems, as above [Rie14, §11.3].

*Example* 3.7 (projective and injective weak factorization systems). Now suppose $\mathsf{M}$ is locally presentable and the weak factorization system $(\mathcal{L}, \mathcal{R})$ admits an accessible functorial factorization[4], as is the case when $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated.

Then for any small category $\mathcal{A}$ the following pullbacks define the **projective** and **injective** weak factorization systems, respectively:[5]

$$\begin{tikzcd} \mathcal{R}^{\mathcal{A}}_{\mathrm{proj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathcal{R} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathsf{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}^{\mathcal{A}}_{\mathrm{inj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathcal{L} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathsf{M} \end{tikzcd}$$

That is:

[3] In the presence of (i) and (ii), (iii) is equivalent to the assertion that $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

[4] A functorial factorization is *accessible* if it preserves $\kappa$-filtered colimits for some regular cardinal $\kappa$.

[5] In the projective case, it suffices to assume that $\mathsf{M}$ is cocomplete but not necessarily locally presentable.

- In the projective weak factorization system $(\mathcal{L}^{\mathcal{A}}_{\mathrm{proj}}, \mathcal{R}^{\mathcal{A}}_{\mathrm{proj}})$, the right class is comprised of those natural transformations whose components are in $\mathcal{R}$.
- In the injective weak factorization system $(\mathcal{L}^{\mathcal{A}}_{\mathrm{inj}}, \mathcal{R}^{\mathcal{A}}_{\mathrm{inj}})$, the left class is comprised of those natural transformations whose components are in $\mathcal{L}$.

When $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by $\mathcal{J}$, the projective weak factorization system is cofibrantly generated by the set

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{J} = \{\mathcal{A}_a * j \mid a \in \mathcal{A}, j \in \mathcal{J}\}$$

of tensors of maps in $\mathcal{J}$ with covariant representables.

**Left and right Leibniz bifunctors.**

**Definition 3.8.** Let $\mathsf{K}$ and $\mathsf{M}$ be categories equipped with weak factorization systems $(\mathcal{M}, \mathcal{E})$ and $(\mathcal{L}, \mathcal{R})$ respectively. An **adjunction of weak factorization systems** is an adjunction

$$\begin{tikzcd} \mathsf{K} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{M} \arrow[l, shift left, "U"] \end{tikzcd}$$

so that the following equivalent conditions are satisfied

(i) The left adjoint preserves the left classes, i.e., the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathrm{mor}\,\mathsf{K} \arrow[r, "F"] & \mathrm{mor}\,\mathsf{M} \end{tikzcd}$$

commutes.

(ii) The right adjoint preserves the left classes, i.e., the diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "U"] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{mor}\,\mathsf{M} \arrow[r, "U"] & \mathrm{mor}\,\mathsf{K} \end{tikzcd}$$

commutes.

The equivalence of (i) and (ii) is a consequence of the equivalence between lifting problems $Ff \boxslash g$ in $\mathsf{M}$ and $f \boxslash Ug$ in $\mathsf{K}$ and the fact that the left and right classes are characterized by such lifting properties.

**Definition 3.9.** Let $\mathsf{K}$, $\mathsf{L}$, and $\mathsf{M}$ be cocomplete categories each equipped with weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathcal{R})$, respectively. A **left Leibniz bifunctor** is a bifunctor

$$\otimes \colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$$

that is

(i) cocontinuous in each variable separately, and
(ii) has the **Leibniz property**: $\otimes$-pushout products (2.2) of a map in $\mathcal{M}$ with a map in $\mathcal{C}$ are in $\mathcal{L}$, i.e., the diagram

$$\begin{tikzcd} \mathcal{M} \times \mathcal{C} \arrow[r, "\hat{\otimes}"] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathrm{mor}\,\mathsf{K} \times \mathrm{mor}\,\mathsf{L} \arrow[r, "\hat{\otimes}"] & \mathrm{mor}\,\mathsf{M} \end{tikzcd}$$

commutes.

Dually, a bifunctor between complete categories equipped with weak factorization systems is a **right Leibniz bifunctor** if it is continuous in each variable separately and if pullback cotensors (2.3) of maps in the right classes land in the right class. We most frequently apply this definition in the case of a bifunctor

$$\{-,-\}\colon \mathsf{K}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{L}$$

that is contravariant in one of its variables, in which we case the relevant hypothesis is that $\mathsf{K}$ is cocomplete and colimits in the first variable are carried to limits in $\mathsf{L}$. The nature of the duality between left and right Leibniz bifunctors is somewhat subtle to articulate. We leave this as a puzzle for the reader, with the hint to see [CGR14].

**Lemma 3.10.** *If the bifunctors*

$$\mathsf{K} \times \mathsf{L} \xrightarrow{\otimes} \mathsf{M}, \quad \mathsf{K}^{\mathrm{op}} \times \mathsf{M} \xrightarrow{\{-,-\}} \mathsf{L}, \quad \text{and} \quad \mathsf{L}^{\mathrm{op}} \times \mathsf{M} \xrightarrow{\hom} \mathsf{K}$$

*define a two-variable adjunction, and* $(\mathcal{M}, \mathcal{E})$*,* $(\mathcal{C}, \mathcal{F})$*, and* $(\mathcal{L}, \mathcal{R})$ *are three weak factorization systems on* $\mathsf{K}$*,* $\mathsf{L}$*, and* $\mathsf{M}$ *respectively, then the following are equivalent*

*(i)* $\otimes\colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ *defines a left Leibniz bifunctor.*
*(ii)* $\{-,-\}\colon \mathsf{K}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{L}$ *defines a right Leibniz bifunctor.*
*(iii)* $\hom\colon \mathsf{L}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{K}$ *defines a right Leibniz bifunctor.*

*When these conditions are satisfied, we say that* $(\otimes, \{,\}, \hom)$ *defines a* ***Leibniz two-variable adjunction****.*

*Proof.* The presence of the adjoints ensures that each of the bifunctors satisfy the required (co)continuity hypotheses. Note that, for instance, $\mathcal{M} \hat{\otimes} \mathcal{C} \subset \mathcal{L}$ if and only if $\mathcal{M} \hat{\otimes} \mathcal{C} \boxslash \mathcal{R}$. Now the equivalence of (i), (ii), and (iii) follows from Proposition 2.4(ii), which implies that the following lifting problems are adjunct:

$$\mathcal{M} \hat{\otimes} \mathcal{C} \boxslash \mathcal{R} \quad \Leftrightarrow \quad \mathcal{C} \boxslash \widehat{\{\mathcal{M}, \mathcal{R}\}} \quad \Leftrightarrow \quad \mathcal{M} \boxslash \widehat{\hom}(\mathcal{C}, \mathcal{R}). \qquad \square$$

*Remark* 3.11. By Proposition 2.4(vi) and Remark 3.3, to show that a cocontinuous bifunctor $\otimes$ satisfies the Leibniz property, it suffices to show that $\otimes$-Leibniz products of generating morphisms are in the left class of the codomain weak factorization system.

*Example* 3.12. If $\mathsf{K}$, $\mathsf{L}$, and $\mathsf{M}$ are model categories, a **left Quillen bifunctor** is no more and no less than a bifunctor $\otimes\colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ that is left Leibniz with respect to all possible choices of constituent weak factorization systems, with the exception of choosing the trivial cofibrations only for $\mathsf{M}$.

**Lemma 3.13.** *For any category* $\mathsf{M}$ *with a wfs* $(\mathcal{L}, \mathcal{R})$ *the set-tensor, set-cotensor, and hom*

$$*\colon \mathsf{Set} \times \mathsf{M} \to \mathsf{M}, \quad \{-,-\}\colon \mathsf{Set}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{M}, \quad \text{and} \quad \hom\colon \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{Set}$$

*respectively define a Leibniz two-variable adjunction relative to the mono-epi weak factorization system* $(\mathcal{M}, \mathcal{E})$ *on* $\mathsf{Set}$*.*

*Proof.* By Lemma 3.10, it suffices to prove any one of these bifunctors is Leibniz. We focus on the left case because this sort of analysis will reappear later. When $A \hookrightarrow B$ is a monomorphism in $\mathsf{Set}$, the Leibniz tensor with $f\colon X \to Y$ decomposes as a coproduct of maps that are manifestly in $\mathcal{L}$.

$$\begin{tikzcd}
A*X \arrow[r, hook] \arrow[d, "A*f"'] & B*X \cong A*X \coprod B\backslash A*X \arrow[d, "A*f \coprod \mathrm{id}"] \arrow[dr, bend left, "B*f"] & \\
A*Y \arrow[r] \arrow[dr, bend right] & A*Y \coprod B\backslash A*X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\mathrm{id} \coprod B\backslash A*f"] & \\
& & B*Y \cong A*Y \coprod B\backslash A*Y
\end{tikzcd} \tag{3.14}$$

A slicker proof is also possible. By Remark 3.11 and Example 3.5, it suffices to consider Leibniz tensor with the generating monomorphism $\emptyset \hookrightarrow 1$. But note that the functor

$$\mathsf{M}^{\mathbb{2}} \xrightarrow{(\emptyset \hookrightarrow 1)\hat{*}-} \mathsf{M}^{\mathbb{2}}$$

is naturally isomorphic to the identity, which certainly preserves the left class $\mathcal{L}$. $\square$

*Remark* 3.15. To prove that $\hom\colon \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{Set}$ is right Leibniz is to show that for any $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$, the morphism

$$\mathsf{M}(\mathrm{cod}\,\ell, \mathrm{dom}\,r) \xrightarrow{r \circ - \circ \ell} \mathsf{M}(\mathrm{dom}\,\ell, \mathrm{dom}\,r) \underset{\mathsf{M}(\mathrm{dom}\,\ell, \mathrm{cod}\,r)}{\times} \mathsf{M}(\mathrm{cod}\,\ell, \mathrm{cod}\,r)$$

is an epimorphism. The target of this map is the set of commutative squares in $\mathsf{M}$ of the form (3.2), while the fiber over any element is the set of solutions to the lifting problem so-presented. The fact that this is an epimorphism follows from the lifting property $\mathcal{L} \boxslash \mathcal{R}$.

**Proposition 3.16.** *When* $(\otimes, \{,\}, \hom)\colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ *defines a Leibniz two-variable adjunction* $\mathcal{M} \hat{\otimes} \mathcal{C} \subset \mathcal{L}$ *then the functor tensor product, functor cotensor product, and hom*

$$\mathsf{K}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}} \times \mathsf{L}^{\mathcal{A}} \xrightarrow{\otimes_{\mathcal{A}}} \mathsf{M}^{\mathcal{B}}, \quad (\mathsf{K}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{B}} \xrightarrow{\{-,-\}^{\mathcal{B}}} \mathcal{L}^{\mathcal{A}}, \quad \text{and} \quad (\mathsf{L}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{B}} \xrightarrow{\hom} \mathsf{K}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}$$

*define a Leibniz two-variable adjunction with respect to either:*

*(i) the weak factorization systems*

$$(\mathcal{M}_{\mathrm{proj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}, \mathcal{E}_{\mathrm{proj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}), \quad (\mathcal{C}_{\mathrm{inj}}^{\mathcal{A}}, \mathcal{F}_{\mathrm{inj}}^{\mathcal{A}}), \quad \text{and} \quad (\mathcal{L}_{\mathrm{proj}}^{\mathcal{B}}, \mathcal{R}_{\mathrm{proj}}^{\mathcal{B}}),$$

*or*

*(ii) the weak factorization systems*

$$(\mathcal{M}_{\mathrm{inj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}, \mathcal{E}_{\mathrm{inj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}), \quad (\mathcal{C}_{\mathrm{proj}}^{\mathcal{A}}, \mathcal{F}_{\mathrm{proj}}^{\mathcal{A}}), \quad \text{and} \quad (\mathcal{L}_{\mathrm{inj}}^{\mathcal{B}}, \mathcal{R}_{\mathrm{inj}}^{\mathcal{B}}).$$

*Proof.* For (i), note that for $j^\bullet \in \mathsf{L}^{\mathcal{A}}$ and $p^\bullet \in \mathsf{M}^{\mathcal{B}}$, $\widehat{\hom}(j^\bullet, p^\bullet) \in \mathsf{K}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}$ is the bifunctor with components $\widehat{\hom}(j^a, p^b)$. If each $j^a \in \mathcal{C}$ and each $p^b \in \mathcal{R}$, then since hom is right Leibniz it follows that $\widehat{\hom}(j^a, p^b) \in \mathcal{E}$, which tells us that if $j^\bullet \in \mathcal{C}_{\mathrm{inj}}^{\mathcal{A}}$ and $p^\bullet \in \mathcal{R}_{\mathrm{proj}}^{\mathcal{B}}$ then $\widehat{\hom}(j^\bullet, p^\bullet) \in \mathcal{E}_{\mathrm{proj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}$, as claimed.

For (ii), we wish to show that if $i^\bullet_\bullet \in \mathsf{M}_{\mathrm{inj}}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{B}}$ and $p^\bullet \in \mathcal{R}_{\mathrm{inj}}^{\mathcal{B}}$, then $\widehat{\{i^\bullet_\bullet, p^\bullet\}}^{\mathcal{B}} \in \mathcal{F}_{\mathrm{proj}}^{\mathcal{A}}$. That is, for each $a \in \mathcal{A}$, we must show that if $i^\bullet_a \in \mathsf{M}_{\mathrm{inj}}^{\mathcal{B}}$ and $p^\bullet \in \mathcal{R}_{\mathrm{inj}}^{\mathcal{B}}$, then $\widehat{\{i^\bullet_a, p^\bullet\}}^{\mathcal{B}} \in \mathcal{F}$. By adjunction, it suffices to show that $\mathcal{M}_{\mathrm{inj}}^{\mathcal{B}} \hat{\otimes} \mathcal{C} \subset \mathcal{L}_{\mathrm{inj}}^{\mathcal{B}}$, which follows directly from the inclusion $\mathcal{M} \hat{\otimes} \mathcal{C} \subset \mathcal{L}$, because both injective left classes are defined pointwise. $\square$

Taking $\mathcal{A}$ or $\mathcal{B}$ to be the terminal category, Proposition 3.16 specializes to the following results:

**Corollary 3.17.** *When* $(\otimes, \{,\}, \hom)\colon \mathsf{K}\times\mathsf{L} \to \mathsf{M}$ *defines a Leibniz two-variable adjunction* $\mathcal{M}\,\hat{\otimes}\,\mathcal{C} \subset \mathcal{L}$ *then*

*(i) The functor tensor product* $(\otimes_{\mathcal{A}}, \{,\}, \hom)\colon \mathsf{K}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{L}^{\mathcal{A}} \to \mathsf{M}$ *defines a Leibniz two-variable adjunction relative to*
  - *(a) the injective and projective weak factorization systems on* $\mathsf{K}^{\mathcal{A}^{\mathrm{op}}}$ *and* $\mathsf{L}^{\mathcal{A}}$*, and*
  - *(b) the projective and injective weak factorization systems on* $\mathsf{K}^{\mathcal{A}^{\mathrm{op}}}$ *and* $\mathsf{L}^{\mathcal{A}}$*.*

*(ii) The functor cotensor product* $\{-,-\}^{\mathcal{A}}\colon (\mathsf{K}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{L}$ *defines a Leibniz two-variable adjunction relative to*
  - *(a) the projective weak factorization systems on* $\mathsf{K}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*, and*
  - *(b) the injective weak factorization systems on* $\mathsf{K}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*.*

*(iii) The functor cotensor product* $\hom^{\mathcal{A}}\colon (\mathsf{L}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{K}$ *defines a Leibniz two-variable adjunction relative to*
  - *(a) the projective weak factorization systems on* $\mathsf{L}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*, and*
  - *(b) the injective weak factorization systems on* $\mathsf{L}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*.*

*Proof.* The six statements are dual; we prove (i)(a). By adjunction

$$\mathcal{M}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{inj}}\,\hat{\otimes}\,\mathcal{C}^{\mathcal{A}}_{\mathrm{proj}} \boxslash \mathcal{R} \qquad \Leftrightarrow \qquad \mathcal{C}^{\mathcal{A}}_{\mathrm{proj}} \boxslash \widehat{\{\mathcal{M}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{inj}}, \mathcal{R}\}}.$$

The right adjoint here is the Leibniz cotensor

$$\begin{aligned} (\mathsf{K}^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} \times \mathsf{M} &\xrightarrow{\widehat{\{-,-\}}} \mathsf{L}^{\mathcal{A}} \\ (i_\bullet, p) &\longmapsto \widehat{\{i_\bullet, p\}} \end{aligned}$$

To say that $i_\bullet \in \mathcal{M}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{inj}}$ is to say that $i_a \in \mathcal{M}$ for each $a \in \mathcal{A}$. Since $\{-,-\}$ is right Leibniz, this says that the components $\widehat{\{i_a, p\}}$ are in $\mathcal{F}$, i.e., that $\widehat{\{i_\bullet, p\}} \in \mathcal{F}^{\mathcal{A}}_{\mathrm{proj}}$. Thus, the adjunct lifting problems have a solution. □

*Remark* 3.18. Applying Proposition 3.16 or Corollary 3.17 to Leibniz two-variable adjunction of Lemma 3.13, it follows that the weighted colimit defines a left Leibniz bifunctor and the weighted limit defines a right Leibniz bifunctor relative to appropriately chosen pairs of projective and injective weak factorization systems defined relatively to the monomorphism–epimorphism weak factorization system on $\mathsf{Set}$ and an arbitrary weak factorization system $(\mathcal{L},\mathcal{R})$ on a bicomplete category $\mathsf{M}$. Analogous results hold if $\mathsf{M}$ is tensored, cotensored, and enriched over a monoidal category $\mathsf{V}$. For instance, the implications of this result for the theory of homotopy limits and homotopy colimits in a simplicial model category is discussed in [Rie14, §11.5].

## 4. Reedy categories and cell complexes

The following definition relaxes the Berger-Moerdijk notion of Reedy structure, omitting their axioms (iv) and (iv)'. This last pair of axioms, which we refer to somewhat unfaithfully as "dualizability," will reappear in section 8, where we first turn to homotopy theoretic, as opposed to algebraic, considerations.

**Definition 4.1.** A **Reedy structure** on a small category $\mathcal{A}$ consists of a **degree function**[6] $\deg\colon \mathrm{ob}\,\mathcal{A} \to \omega$ together with a pair of wide subcategories $\overrightarrow{\mathcal{A}}$ and $\overleftarrow{\mathcal{A}}$ of **degree-increasing** and **degree-decreasing** arrows respectively so that

[6] The degree function can take values in a different ordinal with no substantial effect on the mathematics.

(i) Isomorphisms preserve the degree, whereas non-invertible morphisms in $\overrightarrow{\mathcal{A}}$ or $\overleftarrow{\mathcal{A}}$ strictly raise and lower the degree, respectively.
(ii) $\overrightarrow{\mathcal{A}} \cap \overleftarrow{\mathcal{A}} = \mathsf{iso}\mathcal{A}$.
(iii) Every $f \in \operatorname{mor}\mathcal{A}$ may be factored as

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\overleftarrow{\mathcal{A}}\ni\overleftarrow{f}"'] & & \bullet \\ & \bullet \arrow[ur, "\overrightarrow{f}\in\overrightarrow{\mathcal{A}}"'] & \end{tikzcd} \tag{4.2}$$

and this factorization is unique up to isomorphism.

A **strict** (or classical) Reedy category is a Reedy category in which the subgroupoid $\mathcal{G} \subset \mathcal{A}$ of isomorphisms consists only of identities.

*Remark* 4.3. If $\mathcal{A}$ is a Reedy category, then so is $\mathcal{A}^{\mathrm{op}}$: its Reedy structure has the same degree function but has the degree-increasing and degree-decreasing arrows interchanged.

**Reedy factorizations.** Call any factorization of the form (4.2) a **Reedy factorization** of the map $f$. The degree of the object $\operatorname{cod}\overleftarrow{f} = \operatorname{dom}\overrightarrow{f}$ will be called the **degree** of $f$. From axioms (i) and (iii), it is clear that the degree is well-defined. Moreover:

(i) It is the minimal degree of an object through which $f$ factors.
(ii) Any factorization of $f$ with this degree is a Reedy factorization.

To prove these assertions, consider the category $\mathsf{fact}\, f$ whose objects are factorizations $a \xrightarrow{g} c \xrightarrow{h} b$ of $f$ and whose morphisms $h \cdot g \to h' \cdot g'$ are maps $k\colon c \to c'$ so that the triangles

$$\begin{tikzcd} & c \arrow[dd, "k"] \arrow[dr, "h"] & \\ a \arrow[ur, "g"] \arrow[dr, "g'"'] & & b \\ & c' \arrow[ur, "h'"'] & \end{tikzcd}$$

commute. Write $\mathsf{fact}_n f \subset \mathsf{fact}\, f$ for the subcategory of factorizations through an object of degree at most $n$.

**Lemma 4.4.** *The category* $\mathsf{fact}\, f$ *is connected, and each subcategory* $\mathsf{fact}_n f$ *is either empty or connected. The minimal $n$ with* $\mathsf{fact}_n f$ *non-empty is the degree of $f$, and each object in* $\mathsf{fact}_n f$ *is a Reedy factorization.*

*Proof.* Consider $h \cdot g \in \mathsf{fact}\, f$ and choose Reedy factorizations:

$$\begin{tikzcd} \bullet \arrow[rr, "g"] \arrow[dr, "\overleftarrow{g}"'] & & \bullet \arrow[rr, "h"] \arrow[dr, "\overleftarrow{h}"'] & & \bullet \\ & \bullet \arrow[ur, "\overrightarrow{g}"'] \arrow[rr, "k=\overleftarrow{h}\cdot\overrightarrow{g}"'] \arrow[dr, "\overleftarrow{k}"'] & & \bullet \arrow[ur, "\overrightarrow{h}"'] & \\ & & \bullet \arrow[ur, "\overrightarrow{k}"'] & & \end{tikzcd} \tag{4.5}$$

In this way, we define a zig-zag of morphisms in $\mathsf{fact}\, f$ connecting $h \cdot g$ to a Reedy factorization $\overrightarrow{h}\overrightarrow{k} \cdot \overleftarrow{k}\overleftarrow{g}$. By (iii), this shows that $\mathsf{fact}\, f$ is connected. Moreover, (4.5) and axiom (i) imply that the degree of $\operatorname{cod}(g) = \operatorname{dom}(h)$ is at least the degree of $f$. If these degrees coincide, then $\overrightarrow{g}$ and $\overleftarrow{h}$ must be isomorphisms, from which we deduce that $g \in \overleftarrow{\mathcal{A}}$ and $h \in \overrightarrow{\mathcal{A}}$: i.e., that $h \cdot g$ is a Reedy factorization.

Now if $h \cdot g \in \mathsf{fact}_n f$, each of the factorizations in (4.5) is as well, proving that $\mathsf{fact}_n f$ is connected if it is non-empty. This diagram also shows that each non-empty category $\mathsf{fact}_n f$ contains a Reedy factorization. Hence, the minimal such $n$ is the degree of $f$. □

**A cellular decomposition for two-sided representables.** Lemma 4.4 will be used to establish a "cellular decomposition" for the hom bifunctor $\mathcal{A} \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. That is, we shall use the Reedy structure to present the bifunctor $\mathcal{A}$ as a generalized cell complex: a sequential composite of pushouts of groupoid-indexed *coends*[7] of basic "cells" that have a particular form.

Lemma 4.4 implies that the subset of arrows of degree at most $n$ assembles into a sub*functor* of the hom-bifunctor.

**Definition 4.6** ($n$-skeleton of the hom bifunctor)**.** For any Reedy category $\mathcal{A}$, the $n$**-skeleton** is the subfunctor

$$\mathrm{sk}_n\mathcal{A} \hookrightarrow \mathcal{A} \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$$

of arrows of degree at most $n$. Equivalently, $\mathrm{sk}_n\mathcal{A}$ is the left Kan extension of the restriction of the hom bifunctor from $\mathcal{A}$ to the full subcategory $\mathcal{A}^{\leq n}$ spanned by objects of degree at most $n$.

$$\begin{tikzcd} & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[dr, dashed, "\mathrm{sk}_n\mathcal{A}"] \arrow[d, phantom, "\Uparrow\cong"] & \\ (\mathcal{A}^{\leq n})^{\mathrm{op}} \times \mathcal{A}^{\leq n} \arrow[ur, hook] \arrow[r, hook] & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[r, "\mathcal{A}"'] & \mathsf{Set} \end{tikzcd}$$

There are obvious inclusions $\mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A}$. The colimit of the sequence

$$\emptyset \hookrightarrow \mathrm{sk}_0\mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A} \hookrightarrow \cdots \hookrightarrow \mathrm{colim} \cong \mathcal{A}$$

is $\mathcal{A}$. The morphisms of degree $n$ first appear in $\mathrm{sk}_n\mathcal{A}$. It remains to express each inclusion $\mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A}$ as a pushout of a coend of basic "cells."

The external (pointwise) product defines a bifunctor $\mathsf{Set}^{\mathcal{A}} \times \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \xrightarrow{-\underline{\times}-} \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$. For any $a \in \mathcal{A}$, there is a natural "composition" map whose domain is the external product of the contravariant and covariant representables

$$\mathcal{A}_a \underline{\times} \mathcal{A}^a \xrightarrow{\circ} \mathcal{A}. \tag{4.7}$$

Its image is the subfunctor of arrows in $\mathcal{A}$ that factor through $a$, but (4.7) is not in general a monomorphism: e.g., this fails to be the case whenever $a$ has non-identity automorphisms.

*Notation* 4.8. For any $n \in \omega$ and Reedy category $\mathcal{A}$, let $\mathcal{G}(n)$ denote the subgroupoid of isomorphisms between objects of degree $n$. Write

$$\mathcal{A}_n \colon \mathcal{G}(n)^{\mathrm{op}} \to \mathsf{Set}^{\mathcal{A}} \qquad \text{and} \qquad \mathcal{A}^n \colon \mathcal{G}(n) \to \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

for the restricted Yoneda embeddings, i.e., for the $\mathcal{G}(n)$-indexed diagrams of covariant and contravariant representable functors $\mathcal{A}_a$ and $\mathcal{A}^a$ spanned by the objects $a$ of degree $n$. Finally, write

$$\mathcal{A}_n \underline{\times}_{\mathcal{G}(n)} \mathcal{A}^n := \int^{a \in \mathcal{G}(n)} \mathcal{A}_a \underline{\times} \mathcal{A}^a \qquad \in \qquad \mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$$

for the functor tensor product of $\mathcal{A}_n$ and $\mathcal{A}^n$, a coend indexed by the groupoid $\mathcal{G}(n)$.

[7]The coends in a generalized cell complex take the place of coproducts in an ordinary cell complex, as defined in [RV14, 5.3].

The morphisms (4.7) assemble to define a map

$$\begin{tikzcd} \coprod_{a\in\mathcal{G}(n)} \mathcal{A}_a \underline{\times} \mathcal{A}^a \arrow[rr, "\circ"] \arrow[dr, two heads] & & \mathrm{sk}_n\mathcal{A} \\ & \mathcal{A}_n \underline{\times}_{\mathcal{G}(n)} \mathcal{A}^n \arrow[ur, "\circ"'] & \end{tikzcd}$$

which factors through the regular epimorphism defining the coend as a quotient of the coproduct. The quotient map $\circ$ again fails to be a monomorphism though Lemma 4.4 implies that it is one-to-one on the subset of arrows with degree $n$ (and not less).

**Definition 4.9** (boundaries of representable functors)**.** If $a \in \mathcal{A}$ has degree $n$, write

$$\partial\mathcal{A}_a := \mathrm{sk}_{n-1}\mathcal{A}_a \qquad \in \mathsf{Set}^{\mathcal{A}} \qquad \text{and}$$
$$\partial\mathcal{A}^a := \mathrm{sk}_{n-1}\mathcal{A}^a \qquad \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}.$$

*Remark* 4.10. Any isomorphism in $\mathcal{A}$ restricts in the obvious way to a natural isomorphism between the boundaries of the corresponding representable functors, which thus assemble into functors

$$\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n \colon \mathcal{G}(n)^{\mathrm{op}} \to \mathsf{Set}^{\mathcal{A}} \qquad \text{and} \qquad \partial\mathcal{A}^n \hookrightarrow \mathcal{A}^n \colon \mathcal{G}(n) \to \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

indexed by the groupoid $\mathcal{G}(n)$.

**Lemma 4.11.** *The complements of the inclusions*

$$\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a \qquad \text{and} \qquad \partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$$

*are the functors*

$$\mathcal{A}_a \backslash \partial\mathcal{A}_a = \overrightarrow{\mathcal{A}}_a \qquad \text{and} \qquad \mathcal{A}^a \backslash \partial\mathcal{A}^a = \overleftarrow{\mathcal{A}}^a$$

*of degree-increasing morphisms with domain a and degree-decreasing morphisms with codomain a, respectively.*

*Proof.* We prove the second of these dual statements. If $f \colon \overline{a} \to a$ is not in $\partial\mathcal{A}^a$ its Reedy factorization must have degree $\deg(a)$. The morphism $\overrightarrow{f}$ then preserves degrees and so must be an isomorphism, which implies that $f \in \overleftarrow{\mathcal{A}}^a$. Conversely, if $f \in \overleftarrow{\mathcal{A}}^a$, then $\mathrm{id}_a \cdot f$ is a Reedy factorization, so the degree of $f$ equals the degree of $a$, and $f$ is not in $\partial\mathcal{A}^a$. □

In particular, the exterior Leibniz product

$$\mathcal{A}_a \underline{\times} \partial\mathcal{A}^a \cup \partial\mathcal{A}_a \underline{\times} \mathcal{A}^a \xhookrightarrow{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\hat{\underline{\times}}(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)} \mathcal{A}_a \underline{\times} \mathcal{A}^a \tag{4.12}$$

defines the subfunctor of pairs of morphisms $h \cdot g$ with $\mathrm{dom}\, h = \mathrm{cod}\, g = a$ in which at least one of the morphisms $g$ and $h$ has degree less than the degree of $a$.

**Proposition 4.13.** *The square*

$$\begin{tikzcd} \partial\mathcal{A}_n \underline{\times}_{\mathcal{G}(n)} \mathcal{A}^n \cup \mathcal{A}_n \underline{\times}_{\mathcal{G}(n)} \partial\mathcal{A}^n \arrow[r, hook] \arrow[d, "\circ"'] & \mathcal{A}_n \underline{\times}_{\mathcal{G}(n)} \mathcal{A}^n \arrow[d, "\circ"] \\ \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n\mathcal{A} \end{tikzcd} \tag{4.14}$$

*is both a pullback and a pushout in* $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}$.

The fact that (4.14) is a pullback is used to facilitate the proof that it is also a pushout.

*Proof.* We first argue that the top horizontal map, a colimit of a diagram in the category of arrows in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}$, is a pointwise monomorphism. We find this easiest to prove using the theory of algebraic weak factorization systems introduced in §**??**; the reader is invited to supply their own argument or take this result on faith. By Remark **??**, the objects in this diagram, the exterior Leibniz products (4.12), are pointwise monomorphisms and hence coalgebras for the comonad $\mathbb{M}$ of the monomorphism–epimorphism algebraic weak factorization system on $\mathsf{Set}$. The maps in this diagram are isomorphisms and hence $\mathbb{M}$-coalgebra maps, because any commutative square of monomorphisms in which two opposite sides are isomorphisms is a pullback; see Example 3.5. Because the forgetful functor $\mathbb{M}\text{-}\mathsf{coalg} \to \mathsf{Set}^{\mathbb{2}}$ creates colimits, it follows that the top horizontal map of (4.14) is a pointwise $\mathbb{M}$-coalgebra and in particular a monomorphism.

An element of the pullback consists of $f \in \mathrm{sk}_{n-1}\mathcal{A}$ together with a factorization $f = h\cdot g$ through an object $a$ of degree $n$. If both $h$ and $g$ have degree $n$, then Lemma 4.4 tells us that $h \cdot g$ is a Reedy factorization, contradicting the fact that $f$ has degree at most $n-1$. So we must have either $h \in \partial\mathcal{A}_a$ or $g \in \partial\mathcal{A}^a$, which tells us that the map from the upper left corner of (4.14) surjects onto the pullback. Because the top-horizontal map is monic, the comparison is therefore an isomorphism; i.e., (4.14) is a pullback square.

To see that it is a pushout, it suffices now to show that the right-hand vertical is one-to-one on the complement of $\mathrm{sk}_{n-1}\mathcal{A} \hookrightarrow \mathrm{sk}_n\mathcal{A}$. This follows from the connectedness of $\mathsf{fact}_n f$ established in Lemma 4.4. □

As a corollary of Proposition 4.13, the two-sided representable $\mathcal{A}$ has a canonical presentation as a generalized cell complex.

**Theorem 4.15.** *The inclusion $\emptyset \hookrightarrow \mathcal{A}$ has a canonical presentation as a generalized cell complex:*

$$\begin{tikzcd}
 & & \partial\mathcal{A}_n\underline{\times}_{\mathcal{G}(n)}\mathcal{A}^n \cup \mathcal{A}_n\underline{\times}_{\mathcal{G}(n)}\partial\mathcal{A}^n \arrow[r, hook] \arrow[d, "\circ"'] & \mathcal{A}_n\underline{\times}_{\mathcal{G}(n)}\mathcal{A}^n \arrow[d, "\circ"] \arrow[dl, phantom, very near end, "\ulcorner"] & \\
\emptyset \arrow[r, hook] & \mathrm{sk}_0\mathcal{A} \arrow[r, dotted] & \mathrm{sk}_{n-1}\mathcal{A} \arrow[r, hook] & \mathrm{sk}_n\mathcal{A} \arrow[r, dotted] & \mathrm{colim}_n\mathrm{sk}_n\mathcal{A} \cong \mathcal{A}
\end{tikzcd}$$

*i.e., a composite of pushouts of cells constructed as coends of exterior Leibniz products*

$$(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n)\ \hat{\underline{\times}}_{\mathcal{G}(n)}\ (\partial\mathcal{A}^n \hookrightarrow \mathcal{A}^n) := \int^{a\in\mathcal{G}(n)} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)\ \hat{\underline{\times}}\ (\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a),$$

*attached at stage n.*

*Remark* 4.16. One meaning of "canonical" should be "functorial." Indeed, a morphism of Reedy categories—a functor preserving degree and the subcategories of degree-increasing and degree-decreasing maps—induces a morphism of generalized cell complexes: given a morphism $\mathcal{A} \to \mathcal{A}'$ of Reedy categories, there is a natural transformation in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}$ between the generalized cell complex presentation for $\mathcal{A}$ and the restriction the generalized cell complex presentation for $\mathcal{A}'$.

As a corollary of Theorem 4.15, any morphism $f \in \mathsf{M}^{\mathcal{A}}$ is itself a generalized cell complex: the cellular decomposition of $\mathcal{A}$ is translated into a cellular decomposition for $f$ by taking weighted colimits. Taking weighted limits instead transforms the cellular decomposition of $\mathcal{A}$ into a "generalized Postnikov presentation" for $f$ as the limit of a countable tower of pullbacks of ends of a particular form. This sort of result is exemplary of the slogan of [RV14] that "it's all in the weights." Before proving this corollary, let us introduce notation for the maps appearing as the generalized cells.

**Definition 4.17** (latching and matching objects)**.** Let $a \in \mathcal{A}$. The **latching** and **matching objects** of diagram $X \in \mathsf{M}^{\mathcal{A}}$ are defined to be the colimits and limits, respectively, weighted by the boundary representables of appropriate variance:

$$L^a X := \partial\mathcal{A}^a *_{\mathcal{A}} X \qquad M^a X := \{\partial\mathcal{A}_a, X\}^{\mathcal{A}}.$$

The boundary inclusions $\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a$ and $\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a$ induce the **latching** and **matching maps** $L^aX \to X^a$ and $X^a \to M^aX$, on account of the isomorphisms $\mathcal{A}^a *_{\mathcal{A}} X \cong X^a \cong \{\mathcal{A}_a, X\}^{\mathcal{A}}$.

**Definition 4.18** (relative latching and matching maps)**.** The **relative latching** and **relative matching maps** of a natural transformation $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ are defined to be the Leibniz weighted colimits and limits

$$\widehat{\ell}^a f := (\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)\ \widehat{*}_{\mathcal{A}}\ f \qquad \widehat{m}^a f := \{\partial\mathcal{A}_a \widehat{\hookrightarrow \mathcal{A}_a}, f\}^{\mathcal{A}},$$

i.e., by the pullbacks and pushouts:

$$\begin{tikzcd}
L^aX \arrow[r] \arrow[d, "L^a f"'] & X^a \arrow[d] \arrow[ddr, bend left, "f^a"] & \\
L^aY \arrow[r] \arrow[drr, bend right] & \ell^a f \arrow[dr, dashed, "\widehat{\ell}^a f"'] & \\
 & & Y^a
\end{tikzcd}
\qquad
\begin{tikzcd}
X^a \arrow[dr, dashed, "\widehat{m}^a f"] \arrow[drr, bend left] \arrow[ddr, bend right, "f^a"'] & & \\
 & m^a f \arrow[r] \arrow[d] & M^aX \arrow[d, "M^a f"] \\
 & Y^a \arrow[r] & M^aY
\end{tikzcd}$$

of the maps $L^a f := \partial\mathcal{A}^a *_{\mathcal{A}} f$ and $M^a f := \{\partial\mathcal{A}_a, f\}^{\mathcal{A}}$.

*Remark* 4.19. The functoriality observed in Remark 4.10 extends to the relative latching and matching maps, which thus assemble into functors

$$\widehat{\ell}^n f\colon \mathcal{G}(n) \to \mathsf{M} \qquad \text{and} \qquad \widehat{m}^n f\colon \mathcal{G}(n) \to \mathsf{M}.$$

*Notation* 4.20. For any diagram $X \in \mathsf{M}^{\mathcal{A}}$ let

$$\mathrm{sk}_n X := \mathrm{sk}_n\mathcal{A} *_{\mathcal{A}} X \qquad \text{and} \qquad \mathrm{cosk}_n X := \{\mathrm{sk}_n\mathcal{A}, X\}^{\mathcal{A}}$$

denote the results of applying the weighted colimit and weighted limit bifunctors

$$- *_{\mathcal{A}} -\colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}^{\mathcal{A}} \qquad \text{and} \qquad (\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}^{\mathcal{A}}$$

to the diagram $X$ with weight $\mathrm{sk}_n\mathcal{A}$.

**Corollary 4.21.** *Let $\mathcal{A}$ be a Reedy category and let $\mathsf{M}$ be bicomplete. Any morphism $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ is a generalized cell complex*

$$X \to X \bigcup_{\mathrm{sk}_0 X} \mathrm{sk}_0 Y \to \cdots \to X \bigcup_{\mathrm{sk}_{n-1} X} \mathrm{sk}_{n-1} Y \to X \bigcup_{\mathrm{sk}_n X} \mathrm{sk}_n Y \to \cdots \to \mathrm{colim} \cong Y$$

*with the generalized cell*

$$(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n)\ \widehat{*}_{\mathcal{G}(n)}\ \widehat{\ell}^n f \tag{4.22}$$

*attached at stage n and a generalized Postnikov tower*

$$X \cong \lim \to \cdots \to \mathrm{cosk}_n X \underset{\mathrm{cosk}_n Y}{\times} Y \to \mathrm{cosk}_{n-1} X \underset{\mathrm{cosk}_{n-1} Y}{\times} Y \to \cdots \to \mathrm{cosk}_0 X \underset{\mathrm{cosk}_0 Y}{\times} Y \to Y$$

*whose n-th layer is*

$$\{\partial\mathcal{A}^n \widehat{\hookrightarrow \mathcal{A}^n}, \widehat{m}^n f\}^{\mathcal{G}(n)}. \tag{4.23}$$

*Proof.* These dual results follow immediately by applying the weighted colimit and weighted limit bifunctors

$$- *_{\mathcal{A}} -\colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}^{\mathcal{A}} \qquad \text{and} \qquad (\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}^{\mathcal{A}}$$

to the generalized cell complex presentations of Theorem 4.15; recall both bifunctors are cocontinuous in the weight.

To see that the generalized cell complex presentation for $f$ has the asserted form, note that for any diagram $X \in \mathsf{M}^{\mathcal{A}}$ and weight defined by an exterior product of $U \in \mathsf{Set}^{\mathcal{A}}$ and $V \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, there is a natural isomorphism

$$(U \underline{\times} V) *_{\mathcal{A}} X \cong U * (V *_{\mathcal{A}} X),$$

which extends to a natural isomorphism between Leibniz products (Proposition 2.4(i)).

By the coYoneda lemma, $f \cong \mathcal{A} *_{\mathcal{A}} f \cong (\emptyset \hookrightarrow \mathcal{A}) \hat{*}_{\mathcal{A}} f$. By cocontinuity, $- \hat{*}_{\mathcal{A}} f$ preserves generalized cell structures (Proposition 2.4(vi)). It follows that $f$ admits a canonical presentation as a generalized cell complex with cells

$$\begin{aligned}[(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n) \hat{\underline{\times}}_{\mathcal{G}(n)} (\partial\mathcal{A}^n \hookrightarrow \mathcal{A}^n)] \hat{*}_{\mathcal{A}} f &\cong (\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n) \hat{*}_{\mathcal{G}(n)} [(\partial\mathcal{A}^n \hookrightarrow \mathcal{A}^n) \hat{*}_{\mathcal{A}} f] \\ &\cong (\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n) \hat{*}_{\mathcal{G}(n)} \widehat{\ell^n} f.\end{aligned}$$

□

In summary, Corollary 4.21 tells us that we may express a generic natural transformation between diagrams of shape $\mathcal{A}$ as

(i) a generalized cell complex whose cells are Leibniz tensors built from boundary inclusions of covariant representables and relative latching maps,
(ii) and dually as a generalized Postnikov tower whose layers are Leibniz cotensors built from boundary inclusions of contravariant representables and relative matching maps.

This explains the importance of these maps to Reedy category theory, as we shall soon discover.

*Remark* 4.24 (on the cells). By Corollary 4.21, $f$ is the composite of maps

$$\begin{tikzcd}
\partial\mathcal{A}_n *_{\mathcal{G}(n)} Y^n \cup \mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[r, "(\partial\mathcal{A}_n \hookrightarrow \partial\mathcal{A}_n)\hat{*}_{\mathcal{G}(n)}\widehat{\ell^n} f"] \arrow[d] & \mathcal{A}_n *_{\mathcal{G}(n)} Y^n \arrow[d] \\
X \underset{\mathrm{sk}_{n-1}X}{\cup} \mathrm{sk}_{n-1} Y \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & X \underset{\mathrm{sk}_n X}{\cup} \mathrm{sk}_n Y
\end{tikzcd}$$

The lower right corner is the domain of the Leibniz weighted colimit $(\mathrm{sk}_n\mathcal{A} \hookrightarrow \mathcal{A}) \hat{*}_{\mathcal{A}} f$. The top horizontal map—the cell attached at step $n$—is the map defined by the pushout:

$$\begin{tikzcd}
\partial\mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[r] \arrow[d] & \mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[d] \arrow[ddr, bend left] & \\
\partial\mathcal{A}_n *_{\mathcal{G}(n)} Y^n \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[dr] & \\
& & \mathcal{A}_n *_{\mathcal{G}(n)} Y^n
\end{tikzcd}$$

## 5. Cell complex presentations of strict Reedy categories

In this section, we explore the homotopical implications of Theorem 4.15 and its corollary in the case where $\mathcal{A}$ is a strict Reedy category. In this case, the coend (4.22) and end (4.23) reduce to ordinary coproducts and products

$$\coprod_{a\in\mathcal{G}(n)} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*} \widehat{\ell^a} f \qquad \text{and} \qquad \prod_{a\in\mathcal{G}(n)} \{\partial\mathcal{A}^a \widehat{\hookrightarrow \mathcal{A}^a}, \widehat{m^a} f\}$$

indexed by the objects in $\mathcal{A}$ of degree $n$, and Corollary 4.21 filters any $f\colon X \to Y$ in $\mathsf{M}^{\mathcal{A}}$ as an ordinary cell complex and Postnikov tower. Our presentation follows [RV14], though Theorem 5.9 was unfortunately omitted from that work, and serves as an outline for the extension of these results to the Reedy categories of Berger–Moerdijk in the sections that follow.

We first explain how any weak factorization system on $\mathsf{M}$ gives rise to a **Reedy weak factorization system** on $\mathsf{M}^{\mathcal{A}}$. We then prove an inductive result that allows us to prove that the Reedy weak factorization systems associated to a model structure on $\mathsf{M}$ define the **Reedy model structure** on $\mathsf{M}^{\mathcal{A}}$. Finally, we prove that the weighted limit and weighted colimit bifunctors define Quillen bifunctors, as a consequence of a more general algebraic result, and discuss the consequences of this result for the theory of homotopy limits and homotopy colimits indexed by strict Reedy categories.

**Reedy weak factorization systems.** Let $\mathsf{M}$ be a category with a weak factorization system $(\mathcal{L},\mathcal{R})$, comprised of a *left* and *right* class of maps, both closed under retracts, that satisfy the factorization and lifting properties recalled in Definition 3.1. Let $\mathcal{A}$ be a strict Reedy category.

**Definition 5.1.** The **Reedy weak factorization system** $(\mathcal{L}[\mathcal{A}],\mathcal{R}[\mathcal{A}])$ on $\mathsf{M}^{\mathcal{A}}$ defined relative to the weak factorization system $(\mathcal{L},\mathcal{R})$ on $\mathsf{M}$ has:

- as left class $\mathcal{L}[\mathcal{A}]$ those maps $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ whose relative latching maps $\widehat{\ell^a} f\colon \ell^a f \to Y^a \in \mathsf{M}$ are in $\mathcal{L}$, and
- as right class $\mathcal{R}[\mathcal{A}]$ those maps $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ whose relative matching maps $\widehat{m^a} f\colon X^a \to m^a f \in \mathsf{M}$ are in $\mathcal{R}$.

We say a map $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ is **Reedy in $\mathcal{L}$** or **Reedy in $\mathcal{R}$** if its relative latching or relative matching maps are in $\mathcal{L}$ or $\mathcal{R}$, respectively. The following pair of lemmas, which are proven in the general case as Lemmas 6.8 and 6.9, imply that these two classes indeed define a weak factorization system on the category of Reedy diagrams in $\mathsf{M}$.

**Lemma 5.2** ([RV14, 7.3])**.** *The maps $i \in \mathcal{L}[\mathcal{A}]$ have the left lifting property with respect to the maps $p \in \mathcal{R}[\mathcal{A}]$.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & L \end{tikzcd}$$

**Lemma 5.3** ([RV14, 7.4])**.** *Every map $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ can be factored as a map in $\mathcal{L}[\mathcal{A}]$ followed by a map in $\mathcal{R}[\mathcal{A}]$.*

**Proposition 5.4.**

*(i) If $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ is Reedy in $\mathcal{L}$, that is, if the relative latching maps $\widehat{\ell^a} f$ are in $\mathcal{L}$, then each of the components $f^a\colon X^a \to Y^a$ and each of the latching maps $L^a f\colon L^a X \to L^a Y$ are also in $\mathcal{L}$.*

*(ii) If $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ is Reedy in $\mathcal{R}$, that is, if the relative matching maps $\widehat{m}^a f$ are in $\mathcal{L}$, then each of the components $f^a\colon X^a \to Y^a$ and each of the matching maps $M^a f\colon M^a X \to M^a Y$ are also in $\mathcal{R}$.*

*Proof.* We prove the first of these dual statements. The maps $f^a$ and $L^a f$ are the Leibniz weighted colimits of $f$ with the maps $\emptyset \hookrightarrow \mathcal{A}^a$ and $\emptyset \hookrightarrow \partial\mathcal{A}^a$ respectively. Evaluating the covariant variable of the cell complex presentation of Theorem 4.15 at $a \in \mathcal{A}$, we see that $\emptyset \hookrightarrow \mathcal{A}^a$ is a cell complex whose cells have the form

$$((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)\ \hat{\underline{\times}}\ (\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x), \tag{5.5}$$

indexed by the objects $x \in \mathcal{A}$. In fact, it suffices to consider those objects with $\deg(x) \leq \deg(a)$; when $\deg(x) > \deg(a)$ the inclusion $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x$, and hence the cell (5.5), is an isomorphism. Similarly, since $\partial\mathcal{A}^a = \mathrm{sk}_{\deg(a)-1}\mathcal{A}^a$, Theorem 4.15 implies that $\emptyset \hookrightarrow \partial\mathcal{A}^a$ is a cell complex whose cells have the form (5.5) with $\deg(x) < \deg(a)$.

By Proposition 2.4(vi), the maps $f^a$ and $L^a f$ are then cell complexes whose cells, indexed by the objects $x \in \mathcal{A}$ with the degree bounds just discussed, have the form

$$\Big(((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)\ \hat{\underline{\times}}\ (\partial\mathcal{A}^x \hookrightarrow \mathcal{A}^x)\Big)\ \hat{\circledast}_{\mathcal{A}}\ f \cong ((\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)\ \hat{\circledast}\ \widehat{\ell}^x f, \tag{5.6}$$

the isomorphism arising from Proposition 2.4(i). By Lemma 3.13, the Leibniz tensor of a monomorphism with a map in the left class of a weak factorization system is again in the left class. Thus, since $(\partial\mathcal{A}_x)^a \hookrightarrow \mathcal{A}^a_x)$ is a monomorphism and $\widehat{\ell}^x f$ is in $\mathcal{L}$, these cells, and thus the maps $f^a$ and $L^a f$ are in $\mathcal{L}$ as well. ◻

**The Reedy model structure.** Recall that a **model structure** on a category $\mathsf{M}$ with a class of weak equivalences $\mathcal{W}$ satisfying the 2-of-3 property is given by two classes of maps $\mathcal{C}$ and $\mathcal{F}$ so that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ define weak factorization systems. To show that the Reedy weak factorization systems on $\mathsf{M}^{\mathcal{A}}$ relative to a model structure on $\mathsf{M}$ define a model structure on $\mathsf{M}^{\mathcal{A}}$ with the weak equivalences defined pointwise, one lemma is needed.

**Lemma 5.7.** *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathsf{M}$. Then a map $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$*

*(i) is Reedy in $\mathcal{C} \cap \mathcal{W}$ if and only if $f$ is Reedy in $\mathcal{C}$ and a pointwise weak equivalence, and*

*(ii) is Reedy in $\mathcal{F} \cap \mathcal{W}$ if and only if $f$ is Reedy in $\mathcal{F}$ and a pointwise weak equivalence.*

*Proof.* We prove the first of these dual statements. If $f$ is Reedy in $\mathcal{C} \cap \mathcal{W}$, then it is obviously Reedy in $\mathcal{C}$, and Proposition 5.4 implies that its components $f^a$ are also in $\mathcal{C} \cap \mathcal{W}$. Thus $f$ is a pointwise weak equivalence.

For the converse, we make use of the diagram

$$\begin{tikzcd}
L^a X \arrow[r, "L^a f"] \arrow[d] & L^a Y \arrow[d] \arrow[ddr, bend left] & \\
X^a \arrow[r] \arrow[drr, bend right, "f^a"'] & \bullet \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\widehat{\ell}^a f"] & \\
 & & Y^a
\end{tikzcd}$$

which relates the maps $L^a f$, $\widehat{\ell}^a f$, and $f^a$ for any $a \in \mathcal{A}$; this is an instance of Proposition 2.4(v) applied to $(\emptyset \hookrightarrow \partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)\ \hat{\circledast}_{\mathcal{A}}\ f$. Suppose that $f$ is Reedy in $\mathcal{C}$ and a pointwise weak equivalence. By Proposition 5.4, it follows that $L^a f$ is in $\mathcal{C}$. We will show that $L^a f$

is in fact in $\mathcal{C} \cap \mathcal{W}$ and then apply pushout stability of the left class of a weak factorization system and the 2-of-3 property, to conclude that $\widehat{\ell^a} f \in \mathcal{W}$ and hence that $f$ is Reedy in $\mathcal{C} \cap \mathcal{W}$. We argue by induction. If $a$ has degree zero, then $L^a f$ is the identity at the initial object, which is certainly a weak equivalence, and $\widehat{\ell^a} f = f^a$ is in $\mathcal{C} \cap \mathcal{W}$. If $a$ has degree $n$, we may now assume that $\widehat{\ell^x} f \in \mathcal{C} \cap \mathcal{W}$ for any $x$ with degree less than the degree of $a$. By the proof of Proposition 5.4, $L^a f$ may be presented as a cell complex whose cells (5.6) are Leibniz tensors of monomorphisms with maps in $\mathcal{C} \cap \mathcal{W}$, and thus lie in $\mathcal{C} \cap \mathcal{W}$. Thus, we conclude that $L^a f \in \mathcal{C} \cap \mathcal{W}$, completing the proof. □

Lemmas 5.2, 5.3, and 5.7 assemble to prove:

**Theorem 5.8.** *If $\mathcal{A}$ is a strict Reedy category and $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathsf{M}$, then the Reedy weak factorization systems $(\mathcal{C} \cap \mathcal{W}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F} \cap \mathcal{W}[\mathcal{A}])$ define a model structure on $\mathsf{M}^{\mathcal{A}}$ with pointwise weak equivalences.*

**Reedy diagrams and Leibniz bifunctors.**

**Theorem 5.9.** *Let $\mathcal{A}$ be a strict Reedy category and let $\otimes \colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ be a left Leibniz bifunctor with respect to weak factorization systems $(\mathcal{M}, \mathcal{E})$, $(\mathcal{C}, \mathcal{F})$, and $(\mathcal{L}, \mathcal{R})$. Then the functor tensor product*

$$\otimes_{\mathcal{A}} \colon \mathsf{K}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{L}^{\mathcal{A}} \to \mathsf{M}$$

*is left Leibniz with respect to the Reedy weak factorization systems $(\mathcal{M}[\mathcal{A}^{\mathrm{op}}], \mathcal{E}[\mathcal{A}^{\mathrm{op}}])$ and $(\mathcal{C}[\mathcal{A}], \mathcal{F}[\mathcal{A}])$ and $(\mathcal{L}, \mathcal{R})$.*

*Proof.* The reasons for the cocontinuity of the functor tensor product are well-understood. We argue that $\otimes_{\mathcal{A}}$ has the Leibniz property. Corollary 4.21 asserts that the maps $f \in \mathsf{K}^{\mathcal{A}^{\mathrm{op}}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\hat{*}} \widehat{\ell_a} f,$$

and the maps $g \in \mathsf{L}^{\mathcal{A}}$ can be built as cell complexes whose cells are Leibniz products

$$(\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \mathbin{\hat{*}} \widehat{\ell^b} g.$$

By Proposition 2.4(vi), $f \mathbin{\hat{\otimes}_{\mathcal{A}}} g$ is then a cell complex whose cells have the form

$$\begin{aligned} &\Big((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\hat{*}} \widehat{\ell_a} f\Big) \mathbin{\hat{\otimes}_{\mathcal{A}}} \Big((\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b) \mathbin{\hat{*}} \widehat{\ell^b} g\Big) \\ &\qquad\qquad \cong ((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \mathbin{\hat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_b \hookrightarrow \mathcal{A}_b)) \mathbin{\hat{*}} (\widehat{\ell_a} f \mathbin{\hat{\otimes}} \widehat{\ell^b} g) \end{aligned}$$

To say that $f$ is Reedy in $\mathcal{M}$ and $g$ is Reedy in $\mathcal{C}$ means that $\widehat{\ell_a} f \in \mathcal{M}$ and $\widehat{\ell^b} g \in \mathcal{C}$. Since $\otimes$ is left Leibniz, it follows that $\widehat{\ell_a} f \mathbin{\hat{\otimes}} \widehat{\ell^b} g \in \mathcal{L}$. The Leibniz functor tensor product

$$(\partial\mathcal{A}^b \hookrightarrow \mathcal{A}^b) \mathbin{\hat{\times}_{\mathcal{A}}} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a)$$

of the maps in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ and in $\mathsf{Set}^{\mathcal{A}}$ amounts to the inclusion into the hom-set $\mathcal{A}_b^a = \mathcal{A}(b, a)$ of the subset of morphisms from $b$ to $a$ that factor through an object of degree strictly less than $a$ or strictly less than $b$; in particular, this map is a monomorphism. Now Lemma 3.13 applies to the weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathsf{M}$ to prove that the Leibniz tensor of this monomorphism with $\widehat{\ell_a} f \mathbin{\hat{\otimes}} \widehat{\ell^b} g$ remains in $\mathcal{L}$, completing the proof. □

**Homotopy limits and colimits of Reedy shape.** Applying Theorem 5.9 to Lemma 3.13, with (monomorphism, epimorphism) taken as the default weak factorization system on $\mathsf{Set}$, we conclude:

**Corollary 5.10.** *For any bicomplete category* $\mathsf{M}$ *with a weak factorization system* $(\mathcal{L}, \mathcal{R})$ *and any strict Reedy category, the weighted colimit and weighted limit*

$$*_{\mathcal{A}} \colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M} \qquad \text{and} \qquad \{-,-\}^{\mathcal{A}} \colon (\mathsf{Set}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$$

*define left and right Leibniz bifunctors relative to the Reedy weak factorization systems.*

In the setting of a model category, a monoidal model category, or a $\mathsf{V}$-model category (which subsumes the previous two cases by taking $\mathsf{V}$ to be $\mathsf{Set}$ or the model category itself), Corollary 5.10 specializes to the following result.

**Corollary 5.11.** *Let* $\mathsf{M}$ *be a* $\mathsf{V}$*-model category and let* $\mathcal{A}$ *be a strict Reedy category. Then for any weight* $W$ *in* $\mathsf{M}^{\mathcal{A}^{\mathrm{op}}}$ *or* $\mathsf{M}^{\mathcal{A}}$ *as appropriate that is Reedy cofibrant*[8]*, the weighted colimit and weighted limit functors*

$$W *_{\mathcal{A}} - \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M} \qquad \text{and} \qquad \{W, -\}^{\mathcal{A}} \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$$

*are respectively left and right Quillen with respect to the Reedy model structure on* $\mathsf{M}^{\mathcal{A}}$.

*Example* 5.12 (geometric realization and totalization). The Yoneda embedding defines a Reedy cofibrant weight $\mathbb{\Delta}^{\bullet} \in \mathsf{sSet}^{\mathbb{\Delta}}$. The weighted colimit and weighted limit functors

$$\mathbb{\Delta}^{\bullet} *_{\mathbb{\Delta}^{\mathrm{op}}} - \colon \mathsf{M}^{\mathbb{\Delta}^{\mathrm{op}}} \to \mathsf{M} \qquad \text{and} \qquad \{\mathbb{\Delta}^{\bullet}, -\}^{\mathbb{\Delta}} \colon \mathsf{M}^{\mathbb{\Delta}} \to \mathsf{M}$$

typically go by the names of **geometric realization** and **totalization**. Corollary 5.11 proves that if $\mathsf{M}$ is a simplicial model category, then these functors are left and right Quillen.

*Example* 5.13 (homotopy limits and colimits). Taking the terminal weight 1 in $\mathsf{Set}^{\mathcal{A}}$, the weighted limit reduces to the ordinary limit functor. The functor $1 \in \mathsf{Set}^{\mathcal{A}}$ is Reedy monomorphic just when, for each $a \in \mathcal{A}$, the category of elements for the weight $\partial\mathcal{A}^{a}$ is either empty or connected. This is the case if and only if $\mathcal{A}$ has **cofibrant constants**, meaning that the constant $\mathcal{A}$-indexed diagram at any cofibrant object in any model category is Reedy cofibrant. Thus, we conclude that if $\mathcal{A}$ has cofibrant constants, then the limit functor $\lim \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ is right Quillen.

Dually, the colimit functor is a special case of the weighted colimit functor with the terminal weight $1 \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$. This is Reedy monomorphic just when each category of elements for the weights $\partial\mathcal{A}_{a}$ is either empty or connected, which is the case if and only if $\mathcal{A}$ has **fibrant objects**, meaning that the constant $\mathcal{A}$-indexed diagram at any fibrant object in any model category is Reedy fibrant. Thus, we conclude that if $\mathcal{A}$ has fibrant constants, then the colimit functor $\operatorname{colim} \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ is left Quillen. See [RV14, §9] for more discussion.

## 6. Projective and injective Reedy weak factorization systems

Corollary 4.21 states that $f$ is canonically:

- the colimit of a countable sequence of pushouts of functor tensor products

  $$(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n) \mathbin{\hat{*}_{\mathcal{G}(n)}} \widehat{\ell^n} f,$$

  and also

[8] In the case of $\mathsf{V} = \mathsf{Set}$, "Reedy cofibrant" should be read as "Reedy monomorphic."

- the limit of a countable tower of pullbacks of functor cotensor products

$$\{\partial\mathcal{A}^n \widehat{\hookrightarrow \mathcal{A}^n}, \widehat{m}^n f\}^{\mathcal{G}(n)},$$

built from the relative latching and relative matching maps and the covariant and contravariant boundary inclusions,

$$\mathcal{A}_a \hookrightarrow \mathcal{A}_a \quad \in \mathsf{Set}^{\mathcal{A}} \qquad \text{and} \qquad \mathcal{A}^a \hookrightarrow \mathcal{A}^a \quad \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

defined for each object $a \in \mathcal{A}$.

In this section, we pay close attention to the bifunctoriality of these boundary functors, in both the domain and codomain variables, as first discussed in Remark 4.19. For this reason we add an extra decoration to our notation so we may write simply

$$\overleftarrow{\partial\mathcal{A}}{}^x_a := (\partial\mathcal{A}_a)^x \qquad \text{and} \qquad \overrightarrow{\partial\mathcal{A}}{}^a_x := (\partial\mathcal{A}^a)_x$$

for the functors of morphisms from $a \to x$ and $x \to a$, respectively, that have degree less than the degree of $a$.

The natural pre- and post-composition actions of the hom bifunctor $\mathcal{A} \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}$ restrict to the boundary functors $\overleftarrow{\partial\mathcal{A}}$ and $\overrightarrow{\partial\mathcal{A}}$ in the case where the morphism being composed into the contravariant and covariant variables, respectively, is an isomorphism. This observation motivates consideration of the following weights.

*Notation* 6.1 (groupoid of isomorphisms). For any Reedy category $\mathcal{A}$, let $\mathcal{G} \subset \mathcal{A}$ denote the groupoid of isomorphisms. Since $\mathcal{A}$ is a Reedy category, its groupoid of isomorphisms decomposes as a direct sum:

$$\mathcal{G} \cong \coprod_{n\geq 0} \mathcal{G}(n),$$

that is to say, there are no isomorphisms between objects of differing degrees.

**Definition 6.2.**

$$\overleftarrow{W} \colon \overleftarrow{\partial\mathcal{A}} \hookrightarrow \mathcal{A} \quad \in \quad \mathsf{Set}^{\mathcal{G}^{\mathrm{op}}\times\mathcal{A}} \qquad \text{and} \qquad \overrightarrow{W} \colon \overrightarrow{\partial\mathcal{A}} \hookrightarrow \mathcal{A} \quad \in \quad \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{G}}$$

for the covariant and contravariant boundary inclusions, respectively.

*Remark* 6.3 (latching and matching functoriality, revisited). The weighted limit and weighted colimit weighted by $\overleftarrow{\partial\mathcal{A}}$ and $\overrightarrow{\partial\mathcal{A}}$ respectively define functors

$$\begin{array}{ccc} \mathcal{M}^{\mathcal{A}} \xrightarrow{\overrightarrow{\partial\mathcal{A}} *_{\mathcal{A}} -} \mathcal{M}^{\mathcal{G}} & \qquad & \mathcal{M}^{\mathcal{A}} \xrightarrow{\{\overleftarrow{\partial\mathcal{A}}, -\}^{\mathcal{A}}} \mathcal{M}^{\mathcal{G}} \\ X \longmapsto L^\bullet X & & X \longmapsto M^\bullet X \end{array}$$

Hence, the Leibniz weighted colimit weighted by $\overrightarrow{W}$ and the Leibniz weighted limit weighted by $\overleftarrow{W}$ respectively define functors

$$\begin{array}{ccc} (\mathcal{M}^{\mathcal{A}})^2 \xrightarrow{\overrightarrow{W} \widehat{*}_{\mathcal{A}} -} (\mathcal{M}^{\mathcal{G}})^2 & \qquad & (\mathcal{M}^{\mathcal{A}})^2 \xrightarrow{\widehat{\{\overleftarrow{W}, -\}}^{\mathcal{A}}} (\mathcal{M}^{\mathcal{G}})^2 \\ f \longmapsto \widehat{\ell}^\bullet f & & f \longmapsto \widehat{m}^\bullet f \end{array}$$

Composition defines the components of a natural transformation

$$\overleftarrow{\partial\mathcal{A}} *_{\mathcal{G}} \overrightarrow{\partial\mathcal{A}} \xrightarrow{\circ} \mathcal{A} \quad \in \quad \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{A}}. \tag{6.4}$$

**Lemma 6.5.** *The composition map* $\circ\colon \overleftarrow{\partial\mathcal{A}} *_{\mathcal{G}} \overrightarrow{\partial\mathcal{A}} \to \mathcal{A}$ *induces canonical natural transformations*

$$L^\bullet X \to R^\bullet X \quad \in \quad \mathcal{M}^{\mathcal{G}} \qquad \text{and} \qquad \widehat{\ell}^\bullet f \to \widehat{m}^\bullet f \quad \in \quad (\mathcal{M}^{\mathcal{G}})^2$$

*between the latching and matching functors and relative latching and relative matching functors of Remark 6.3.*

*Proof.* The second statement follows from the first by the functoriality of the Leibniz construction. To prove the first, observe that the $\overleftarrow{\partial A}$-weighted limit functor admits a left adjoint

$$\begin{tikzcd} \mathsf{M}^{\mathcal{A}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\{\overleftarrow{\partial\mathcal{A}},-\}^{\mathcal{A}}"'] & \mathsf{M}^{\mathcal{G}} \arrow[l, shift right=1.5ex, "\overleftarrow{\partial\mathcal{A}}*_{\mathcal{G}}-"'] \end{tikzcd}$$

Hence, to define a natural transformation

$$L^\bullet X := \overrightarrow{\partial\mathcal{A}} *_{\mathcal{A}} X \to \{\overleftarrow{\partial\mathcal{A}}, X\}^{\mathcal{A}} =: R^\bullet X \quad \in \quad \mathsf{M}^{\mathcal{G}}$$

is to define a natural transformation

$$(\overleftarrow{\partial\mathcal{A}} *_{\mathcal{G}} \overrightarrow{\partial\mathcal{A}}) *_{\mathcal{A}} X \cong \overleftarrow{\partial\mathcal{A}} *_{\mathcal{G}} (\overrightarrow{\partial\mathcal{A}} *_{\mathcal{A}} X) \to X \quad \in \quad \mathsf{M}^{\mathcal{A}}.$$

By the Yoneda lemma, $X \cong \mathcal{A} *_{\mathcal{A}} X$, so the map of weights $\circ\colon \overleftarrow{\partial\mathcal{A}} *_{\mathcal{G}} \overrightarrow{\partial\mathcal{A}} \to \mathcal{A}$ induces the desired natural transformation. ◻

*Remark* 6.6. The map (6.4) is an instance of what Shulman [Shu06] calls **abstract bigluing data** from $\mathcal{A}$ to $\mathcal{G}$: a triple comprised of a profunctor $\overrightarrow{\partial\mathcal{A}}\colon \mathcal{G} \nrightarrow \mathcal{A}$, a profunctor $\overleftarrow{\partial\mathcal{A}}\colon \mathcal{A} \nrightarrow \mathcal{G}$, and a natural transformation $\circ\colon \overleftarrow{\partial\mathcal{A}} *_{\mathcal{A}} \overrightarrow{\partial\mathcal{A}} \to \mathcal{A}$ from their functor tensor product to the hom-bifunctor. Such data equivalently defines a functor from $\mathcal{G}$ into the **Isbell envelope** of $\mathcal{A}$, a category that combines both the covariant and contravariant Yoneda embeddings [Is66].

**Definition 6.7** (projective and injective Reedy weak factorization systems)**.** For any accessible weak factorization system $(\mathcal{L}, \mathcal{R})$ on a locally presentable category $\mathsf{M}$ and any Reedy category $\mathcal{A}$, the **projective Reedy weak factorization system** is defined by the pullbacks:

$$\begin{tikzcd} \mathcal{R}[\mathcal{A}]_{\mathrm{proj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{R}^{\mathcal{G}}_{\mathrm{proj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathcal{R} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{\{W,-\}}^{\mathcal{A}}"] & \mathrm{mor}\,(\mathcal{M}^{\mathcal{G}}) \arrow[r] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathcal{M} \end{tikzcd} = \begin{tikzcd} \mathcal{R}[\mathcal{A}]_{\mathrm{proj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathcal{R} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{m}^\bullet"] & \prod\limits_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathcal{M} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{L}[\mathcal{A}]_{\mathrm{proj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}^{\mathcal{G}}_{\mathrm{proj}} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{\ell}^\bullet"] & \mathrm{mor}\,(\mathcal{M}^{\mathcal{G}}) \end{tikzcd}$$

Dually, the **injective Reedy weak factorization system** is defined by the pullback:

$$\begin{tikzcd} \mathcal{L}[\mathcal{A}]_{\mathrm{inj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}^{\mathcal{G}}_{\mathrm{inj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{\mathrm{ob}\,\mathcal{A}} \mathcal{L} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\vec{W}\hat{\circledast}_{\mathcal{A}}-"] & \mathrm{mor}\,(\mathsf{M}^{\mathcal{G}}) \arrow[r] & \prod_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathsf{M} \end{tikzcd} = \begin{tikzcd} \mathcal{L}[\mathcal{A}]_{\mathrm{inj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{\mathrm{ob}\,\mathcal{A}} \mathcal{L} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{\ell^{\bullet}}"] & \prod_{\mathrm{ob}\,\mathcal{A}} \mathrm{mor}\,\mathsf{M} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{R}[\mathcal{A}]_{\mathrm{inj}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{R}^{\mathcal{G}}_{\mathrm{inj}} \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{m^{\bullet}}"] & \mathrm{mor}\,(\mathcal{M}^{\mathcal{G}}) \end{tikzcd}$$

The first of each pair of pullbacks characterize the projective Reedy right maps and the injective Reedy left maps: these are exactly the pointwise right and left maps, respectively. The second of each of pullbacks then characterize the projective Reedy left maps and the injective Reedy right maps: these are the diagrams whose relative latching and relative matching maps assemble into projective and injective $\mathcal{G}$-indexed diagrams of left and right maps, respectively.

It remains to argue that the four classes of maps introduced in Definition 6.7 assemble into a pair of weak factorization systems

$$(\mathcal{L}[\mathcal{A}]_{\mathrm{proj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{proj}}) \qquad \text{and} \qquad (\mathcal{L}[\mathcal{A}]_{\mathrm{inj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{inj}}).$$

This is accomplished by the following pair of lemmas.

**Lemma 6.8** (lifting)**.** *Let $\mathcal{A}$ be a Reedy category with groupoid of isomorphisms $\mathcal{G} \cong \coprod_n \mathcal{G}(n)$. Let $(\mathcal{L}(n), \mathcal{R}(n))$ denote any weak factorization system on $\mathsf{M}^{\mathcal{G}(n)}$, assembling into a weak factorization system $(\mathcal{L}(\bullet), \mathcal{R}(\bullet))$ on $\mathsf{M}^{\mathcal{G}} \cong \prod_n \mathsf{M}^{\mathcal{G}(n)}$. Define a pair of class of maps by the pullbacks*

$$\begin{tikzcd} \mathcal{L}^{\dagger} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{L}(\bullet) \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{\ell^{\bullet}}"] & \mathrm{mor}\,(\mathcal{M}^{\mathcal{G}}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}^{\dagger} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{R}(\bullet) \arrow[d] \\ \mathrm{mor}\,(\mathsf{M}^{\mathcal{A}}) \arrow[r, "\widehat{m^{\bullet}}"] & \mathrm{mor}\,(\mathcal{M}^{\mathcal{G}}) \end{tikzcd}$$

*Then $\mathcal{L}^{\dagger} \boxslash \mathcal{R}^{\dagger}$.*

*Proof.* By Corollary 4.21, to show that $f \boxslash g$ for any pair of morphisms $f, g \in \mathsf{M}^{\mathcal{A}}$, it suffices to solve the lifting problems

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n)\hat{\circledast}_{\mathcal{G}(n)}\widehat{\ell^n} f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

in $\mathsf{M}^{\mathcal{A}}$ for each $n$. By adjunction, it suffices to solve the transposed lifting problem

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\widehat{\ell^n} f"'] & \bullet \arrow[d, "\widehat{\{\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n, g\}}^{\mathcal{A}} \cong \widehat{m^n} g"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

in $\mathsf{M}^{\mathcal{G}(n)}$ for each $n$. If $f \in \mathcal{L}^{\dagger}$ and $g \in \mathcal{R}^{\dagger}$, then by definition $\widehat{\ell^n} f \in \mathcal{L}(n)$ and $\widehat{m^n} g \in \mathcal{R}(n)$, so a solution exists. $\square$

**Lemma 6.9** (factorization)**.** *Any morphism $f \in \mathsf{M}^{\mathcal{A}}$ can be factored as a map in $\mathcal{L}[\mathcal{A}]_{\mathrm{proj}}$ followed by a map in $\mathcal{R}[\mathcal{A}]_{\mathrm{proj}}$, or as a map in $\mathcal{L}[\mathcal{A}]_{\mathrm{inj}}$ followed by a map in $\mathcal{R}[\mathcal{A}]_{\mathrm{inj}}$.*

*Proof.* The constructions are the same mutatis mutandis. We discuss the first. The map $f \in \mathsf{M}^{\mathcal{A}}$ restricts to $f^0 \in \mathsf{M}^{\mathcal{G}(0)}$, which we factor using the projective weak factorization system $(\mathcal{L}^{\mathcal{G}(0)}_{\mathrm{proj}}, \mathcal{R}^{\mathcal{G}(0)}_{\mathrm{proj}})$. We denote this factorization by

$$\begin{tikzcd} X^0 \arrow[rr, "f^0"] \arrow[dr, "\ell^0"'] & & Y^0 \\ & Z_0 \arrow[ur, "r^0"'] & \end{tikzcd} \qquad \in \mathsf{M}^{\mathcal{G}(0)}$$

As for classical Reedy categories, $f^0 = \widehat{\ell^0} f = \widehat{m}^0 f$. In particular, the 0-th relative latching maps of the left factor and relative matching maps of the right factor are in the classes $\mathcal{L}^{\mathcal{G}(0)}_{\mathrm{proj}}$ and $\mathcal{R}^{\mathcal{G}(0)}_{\mathrm{proj}}$ respectively.

Continuing inductively, suppose we have factored the restriction $f^{<n} \in \mathsf{M}^{\mathcal{A}_{<n}}$ as

$$\begin{tikzcd} X^{<n} \arrow[rr, "f^{<n}"] \arrow[dr, "\ell^{<n}"'] & & Y^{<n} \\ & Z_{<n} \arrow[ur, "r^{<n}"'] & \end{tikzcd} \qquad \in \mathsf{M}^{\mathcal{A}_{<n}}$$

with

$$\widehat{\ell}^k \ell^{<n} \in \mathcal{L}^{\mathcal{G}(k)}_{\mathrm{proj}} \qquad \text{and} \qquad \widehat{m}^k r^{<n} \in \mathcal{R}^{\mathcal{G}(k)}_{\mathrm{proj}}.$$

By Lemma 6.5, this data assembles into the following solid-arrow diagram in $\mathsf{M}^{\mathcal{G}(n)}$:

$$\begin{tikzcd} L^n X \arrow[rr] \arrow[dd] & & L^n Z \arrow[rr] \arrow[dl] & & L^n Y \arrow[dd] \\ & X_n \underset{L^n X}{\cup} L^n Z \arrow[dr, dashed] & & & \\ X^n \arrow[ur] \arrow[rr, dashed, "\ell^n"'] \arrow[dd] & & Z^n \arrow[rr, dashed, "r^n"] \arrow[dr, dashed] & & Y^n \arrow[dd] \\ & & & M^n Z \underset{M^n Y}{\times} Y^n \arrow[ur] \arrow[dl] & \\ M^n X \arrow[rr] & & M^n Z \arrow[rr] & & M^n Y \end{tikzcd}$$

We factor the diagonal map from the pushout to the pullback using $(\mathcal{L}^{\mathcal{G}(n)}_{\mathrm{proj}}, \mathcal{R}^{\mathcal{G}(n)}_{\mathrm{proj}})$. The diagonal factors become the $n$-th relative latching map and matching map of the composite morphisms $\ell^n$ and $r^n$ so-defined, and in particular lie in the classes $\mathcal{L}^{\mathcal{G}(n)}_{\mathrm{proj}}$ and $\mathcal{R}^{\mathcal{G}(n)}_{\mathrm{proj}}$, respectively. It follows from the universal properties of the pushout and the pullback that $f^n = r^n \cdot \ell^n$. As in [BM08], these definitions extend the natural transformations $\ell$ and $r$ to degree $n$. $\square$

**Lemma 6.10.** *If the weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathsf{M}$ is cofibrantly generated by a set of arrows $\mathcal{J}$, then the projective Reedy weak factorization system $(\mathcal{L}[\mathcal{A}]_{\mathrm{proj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{proj}})$ on $\mathsf{M}^{\mathcal{A}}$ is cofibrantly generated by the set of arrows*

$$\mathcal{B} \hat{*} \mathcal{J} := \{(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*} j \mid a \in \mathrm{ob}\,\mathcal{A}, j \in \mathcal{J}\}.$$

*Proof.* Recall from Example 3.7 that the projective weak factorization system $(\mathcal{L}^{\mathcal{G}}_{\text{proj}}, \mathcal{R}^{\mathcal{G}}_{\text{proj}})$ is cofibrantly generated by the set of arrows

$$\{(\partial\mathcal{G}_a \hookrightarrow \mathcal{G}_a) \hat{*}\, j \mid a \in \operatorname{ob}\mathcal{G} = \operatorname{ob}\mathcal{A},\, j \in \mathcal{J}\}.$$

Since the right class of the projective Reedy weak factorization system is created from $\mathcal{R}^{\mathcal{G}}_{\text{proj}}$ by applying a right adjoint, it follows that the Reedy weak factorization system is also cofibrantly generated, by the set of arrows obtained by applying the left adjoint to the generators for $(\mathcal{L}^{\mathcal{G}}_{\text{proj}}, \mathcal{R}^{\mathcal{G}}_{\text{proj}})$. By Remark 2.12, the adjunction in question is

$$\begin{tikzcd} \mathsf{M}^{\mathcal{A}} \arrow[rr, bend right=0, shift right=1.5ex, "\widehat{m}^\bullet \cong \widehat{\{\overleftarrow{\partial\mathcal{A}} \hookrightarrow \mathcal{A}, -\}}^{\mathcal{A}}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{M}^{\mathcal{G}} \arrow[ll, shift right=1.5ex, "(\overleftarrow{\partial\mathcal{A}} \hookrightarrow \mathcal{A}) \hat{*}_{\mathcal{G}} -"'] \end{tikzcd}$$

By the coYoneda lemma

$$(\overleftarrow{\partial\mathcal{A}} \hookrightarrow \mathcal{A}) \hat{*}_{\mathcal{G}} ((\partial\mathcal{G}_a \hookrightarrow \mathcal{G}_a) \hat{*}\, j) \cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*}\, j. \qquad \square$$

This result appears as [BM08, 7.5] under the hypotheses that $\mathcal{A}$ is a dualizable "EZ" Reedy category. In the cofibrantly generated case, the generalized cell complex presentations of Corollary 4.21 reduce to ordinary cell complexes on account of the following computation.

**Lemma 6.11.** *If $\widehat{\ell^n} f \in (\mathsf{M}^{\mathcal{G}(n)})^{\mathbb{2}}$ is a cell complex built from tensors of representables with maps in $\mathcal{L}$, then the generalized cell*

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*}_{\mathcal{G}(n)} \widehat{\ell^a} f \quad \in (\mathsf{M}^{\mathcal{A}})^{\mathbb{2}} \tag{6.12}$$

*is a cell complex whose cells are Leibniz tensors of boundary inclusions of $\mathcal{A}$-representables with maps in $\mathcal{L}$.*

*Proof.* By hypothesis, $\widehat{\ell^n} f$ is a composite of pushouts of coproducts of cells of the form $\mathcal{G}(n)_g * j$, with $j \colon A \to B$ in $\mathcal{L}$ and $g \in \mathcal{G}(n)$. Suppose, as a base case, that $\widehat{\ell^n} f$ is just comprised of a single one these cells, i.e., that $\widehat{\ell^a} f = \mathcal{G}(n)^a_g * j$. By the coYoneda lemma,

$$(\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*}_{\mathcal{G}(n)} (\mathcal{G}(n)^a_g * j) \cong (\partial\mathcal{A}_a \underline{\times}_{\mathcal{G}(n)} \mathcal{G}(n)^a_g \hookrightarrow \mathcal{A}_a \underline{\times}_{\mathcal{G}(n)} \mathcal{G}(n)^a_g) \hat{*}\, j \cong (\partial\mathcal{A}_g \hookrightarrow \mathcal{A}_g) \hat{*}\, j.$$

Now given a composite of pushouts of coproducts of cells like $\mathcal{G}(n)_g \hat{*}\, j$, [RV14, 5.6] implies that the cell (6.12) obtained by applying $\int^{a \in \mathcal{G}(n)} (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \hat{*} -$ is a cell complex whose cells are the maps $(\partial\mathcal{A}_g \hookrightarrow \mathcal{A}_g) \hat{*}\, j$. $\square$

*Example* 6.13. Specializing to the strict Reedy category $\mathbb{\Delta}^{\text{op}}$ and the weak factorization system $(\mathcal{M}, \mathcal{E})$ of Example 3.5 on $\mathsf{Set}$, observe that

$$\{\partial\Delta^n \hookrightarrow \Delta^n\} \hat{*} \{\emptyset \to *\} = \{\partial\Delta^n \hookrightarrow \Delta^n\}.$$

So the Reedy weak factorization system $(\mathcal{M}[\mathbb{\Delta}^{\text{op}}], \mathcal{E}[\mathbb{\Delta}^{\text{op}}])$ on simplicial sets is the familiar weak factorization system, generated by the inclusions of boundaries of simplices, whose left class is the monomorphisms. The latching map $L_n X \hookrightarrow X_n$ for a simplicial set is the inclusion of the set of degenerate $n$-simplices, while the matching map $X_n \to M_n X = \{\partial\mathbb{\Delta}^n \to X\}$ is the function that sends an $n$-simplex to its boundary sphere.

The following result proves that projective Reedy left maps are pointwise left maps. Dual remarks apply to injective Reedy right maps.

**Proposition 6.14.** *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system on $\mathsf{M}$ and let $\mathcal{A}$ be a Reedy category.*

*(i) Let $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ be a projective Reedy left map. Then for each $a \in \mathcal{A}$, the component $f^a$, the latching map $L^a f$, and the relative latching map $\widehat{\ell^a} f$ are in the left class $\mathcal{L}$.*

*(ii) Dually, let $f\colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ be an injective Reedy right map. Then for each $a \in \mathcal{A}$, the component $f^a$, the matching map $M^a f$, and the relative matching map $\widehat{m^a} f$ are in the right class $\mathcal{R}$.*

*Proof.* We prove the first of these dual statements. Note that the hypothesis that $f$ is projective Reedy in $\mathcal{L}$ implies in particular that its relative latching maps $\widehat{\ell^a} f$ are in $\mathcal{L}$, this being a strictly weaker hypothesis than the hypothesis that the diagram $\widehat{\ell^n} f$ is a $\mathcal{G}(n)$-projective $\mathcal{L}$-map for each $n$.

The maps $f^a$ and $L^a f$ are the Leibniz weighted colimits of $f$ with the maps $\emptyset \hookrightarrow \mathcal{A}^a$ and $\emptyset \hookrightarrow \partial\mathcal{A}^a$ in $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ respectively. Fixing the covariant variable of the cell complex presentation of Theorem 4.15 at $a$, we see that $\emptyset \hookrightarrow \mathcal{A}^a$ is a cell complex whose cells have the form

$$((\partial\mathcal{A}_k)^a \hookrightarrow \mathcal{A}^a_k) \mathbin{\hat{\underline{\times}}_{\mathcal{G}(k)}} (\partial\mathcal{A}^k \hookrightarrow \mathcal{A}^k), \tag{6.15}$$

indexed by the degrees $k \in \omega$. In fact, it suffices to consider those degrees $k \leq \deg(a)$; when $k > \deg(a)$ the inclusion $(\partial\mathcal{A}_k)^a \hookrightarrow \mathcal{A}^a_k$, and hence the cell (6.15), is an isomorphism. Similarly, since $\partial\mathcal{A}^a = \mathrm{sk}_{\deg(a)-1}\mathcal{A}^a$, Theorem 4.15 implies that $\emptyset \hookrightarrow \partial\mathcal{A}^a$ is a cell complex whose cells have the form (5.5) with $k < \deg(a)$.

By Proposition 2.4(vi), the maps $f^a$ and $L^a f$ are then generalized cell complexes whose cells, indexed by the degrees with the bounds just discussed, have the form

$$\Big(((\partial\mathcal{A}_k)^a \hookrightarrow \mathcal{A}^a_k) \mathbin{\hat{\underline{\times}}_{\mathcal{G}(k)}} (\partial\mathcal{A}^k \hookrightarrow \mathcal{A}^k)\Big) \mathbin{\hat{\circledast}_{\mathcal{A}}} f \cong ((\partial\mathcal{A}_k)^a \hookrightarrow \mathcal{A}^a_k) \mathbin{\hat{\circledast}_{\mathcal{G}(k)}} \widehat{\ell^k} f, \tag{6.16}$$

the isomorphism arising from Proposition 2.4(i).

The inclusions $(\partial\mathcal{A}_k)^a \hookrightarrow \mathcal{A}^a_k$ define a pointwise monomorphism, i.e., an element in $\mathcal{M}^{\mathcal{G}(k)^{\mathrm{op}}}_{\mathrm{inj}}$. By hypothesis, $\widehat{\ell^k} f \in \mathsf{M}^{\mathcal{G}(k)}$ is in $\mathcal{L}^{\mathcal{G}(k)}_{\mathrm{proj}}$, so Proposition 3.16(ii) implies that the Leibniz weighted colimit (6.16) is an injective (i.e., pointwise) $\mathcal{L}$-map, as claimed. Pushouts and transfinite composites of maps in $\mathcal{L}$ are again in $\mathcal{L}$, so since $f^a$ and $L^a f$ are cell complexes build from the cells (6.16), the maps $f^a$ and $L^a f$ are in $\mathcal{L}$ as well. □

By contrast, additional hypotheses will be needed on the Reedy category to ensure that the generalized Postnikov tower of a projective Reedy right map has good properties, e.g., that its layers are pointwise in $\mathcal{R}$; dual remarks apply to the generalized cell complex presentation of an injective Reedy left map. This result appears as Proposition 8.3, making use of hypotheses that will be introduced in section 8.

## 7. Leibniz functors of Reedy weak factorization systems

**Theorem 7.1.** *When $(\otimes, \{,\}, \hom)\colon \mathsf{K} \times \mathsf{L} \to \mathsf{M}$ defines a Leibniz two-variable adjunction $\mathcal{M} \mathbin{\hat{\otimes}} \mathcal{C} \subset \mathcal{L}$, and $\mathcal{A}$ and $\mathcal{B}$ are Reedy categories, then the functor tensor product, functor cotensor product, and hom*

$$\mathsf{K}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}} \times \mathsf{L}^{\mathcal{A}} \xrightarrow{\otimes_{\mathcal{A}}} \mathsf{M}^{\mathcal{B}}, \quad (\mathsf{K}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{B}} \xrightarrow{\{-,-\}^{\mathcal{B}}} \mathcal{L}^{\mathcal{A}}, \quad \text{and} \quad (\mathsf{L}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{B}} \xrightarrow{\hom} \mathsf{K}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}}$$

*define a Leibniz two-variable adjunction with respect to either:*

*(i) the Reedy weak factorization systems*

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}} \times \mathcal{B}]_{\mathrm{proj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}} \times \mathcal{B}]_{\mathrm{proj}}), \quad (\mathcal{C}[\mathcal{A}]_{\mathrm{inj}}, \mathcal{F}[\mathcal{A}]_{\mathrm{inj}}), \quad \text{and} \quad (\mathcal{L}[\mathcal{B}]_{\mathrm{proj}}, \mathcal{R}[\mathcal{B}]_{\mathrm{proj}}),$$

*or*

*(ii) the Reedy weak factorization systems*

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}}\times\mathcal{B}]_{\mathrm{inj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}}\times\mathcal{B}]_{\mathrm{inj}}), \quad (\mathcal{C}[\mathcal{A}]_{\mathrm{proj}}, \mathcal{F}[\mathcal{A}]_{\mathrm{proj}}), \quad \text{and} \quad (\mathcal{L}[\mathcal{B}]_{\mathrm{inj}}, \mathcal{R}[\mathcal{B}]_{\mathrm{inj}}).$$

*Proof.* For (i), we must show that if the relative latching maps of $j^\bullet \in \mathsf{L}^{\mathcal{A}}$ are in $\mathcal{C}$ and the relative matching maps of $p^\bullet \in \mathsf{M}^{\mathcal{B}}$ are in $\mathcal{R}$, then the relative matching maps of $\widehat{\hom}(j^\bullet, p^\bullet) \in \mathsf{K}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}}$ are in $\mathcal{E}$. The relative matching map at an object $(a,b) \in \mathcal{A}^{\mathrm{op}}\times\mathcal{B}$ is the Leibniz weighted colimit with the inclusion of the covariant boundary of the representable $(\mathcal{A}^{\mathrm{op}}\times\mathcal{B})_{(a,b)} \cong \mathcal{A}^a \underline{\times} \mathcal{B}_b$ associated to this object. A quick computation reveals that this boundary inclusion is isomorphic to the Leibniz external product

$$\partial(\mathcal{A}^{\mathrm{op}}\times\mathcal{B})_{(a,b)} \hookrightarrow (\mathcal{A}^{\mathrm{op}}\times\mathcal{B})_{(a,b)} \cong (\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)\, \hat{\underline{\times}}\, (\partial\mathcal{B}_b \hookrightarrow \mathcal{B}_b). \tag{7.2}$$

Applying Proposition 2.4(i),(iii), the relative matching map is then

$$\begin{aligned} &\widehat{\{(\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a)\, \hat{\underline{\times}}\, (\partial\mathcal{B}_b \hookrightarrow \mathcal{B}_b), \widehat{\hom}(j^\bullet, p^\bullet)\}}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}} \\ &\quad \cong \widehat{\hom}\left((\partial\mathcal{A}^a \hookrightarrow \mathcal{A}^a) \,\hat{\circledast}_{\mathcal{A}}\, j^\bullet, \widehat{\{\partial\mathcal{B}_b \hookrightarrow \mathcal{B}_b, p^\bullet\}}^{\mathcal{B}}\right) \\ &\quad \cong \widehat{\hom}(\widehat{\ell}^a j, \widehat{m}^b p). \end{aligned}$$

Since hom is right Leibniz, the fact that $\widehat{\ell}^a j \in \mathcal{C}$ and $\widehat{m}^b p \in \mathcal{R}$ implies that $\widehat{\hom}(\widehat{\ell}^a j, \widehat{m}^b p) \in \mathcal{E}$ as claimed.

For (ii), we first observe that the dual of the isomorphism (7.2) implies that the diagram commutes:

$$\begin{tikzcd} \mathsf{K}^{\mathcal{A}^{\mathrm{op}}\times\mathcal{B}} \arrow[rr, "\widehat{\ell}_a"] \arrow[dr, "\widehat{\ell}^{(a,b)}"'] & & \mathsf{K}^{\mathcal{B}} \arrow[dl, "\widehat{\ell}^b"] \\ & \mathsf{K} & \end{tikzcd}$$

It follows that the Leibniz weighted colimit $\widehat{\ell}_a - \cong (\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a) \,\hat{\circledast}_{\mathcal{A}^{\mathrm{op}}}\, -$ carries the class $\mathcal{M}[\mathcal{A}^{\mathrm{op}}\times\mathcal{B}]_{\mathrm{inj}}$ into the the class $\mathcal{M}[\mathcal{B}]_{\mathrm{inj}}$, which is created from $\mathcal{M} \subset \mathrm{mor}\,\mathsf{K}$ by the relative latching maps for each $b \in \mathcal{B}$.

Now we are asked to show that if $i^\bullet_\bullet \in \mathsf{M}[\mathcal{A}^{\mathrm{op}}\times\mathcal{B}]_{\mathrm{inj}}$ and $p^\bullet \in \mathcal{R}[\mathcal{B}]_{\mathrm{inj}}$, then $\widehat{\{i^\bullet_\bullet, p^\bullet\}}^{\mathcal{B}} \in \mathcal{F}[\mathcal{A}]_{\mathrm{proj}}$, i.e., that the relative matching map

$$\widehat{\{\partial\mathcal{A}_a \hookrightarrow \mathcal{A}_a, \widehat{\{i^\bullet_\bullet, p^\bullet\}}^{\mathcal{B}}\}}^{\mathcal{A}} \cong \widehat{\{\widehat{\ell}_a i^\bullet, p^\bullet\}}^{\mathcal{B}} \in \mathcal{F}.$$

By hypothesis, $\widehat{\ell}_a i^\bullet \in \mathcal{M}[\mathcal{B}]_{\mathrm{inj}}$ and $p^\bullet \in \mathcal{R}[\mathcal{B}]_{\mathrm{inj}}$. By adjunction, it suffices to show that $\mathcal{M}[\mathcal{B}]_{\mathrm{inj}} \hat{\otimes} \mathcal{C} \subset \mathcal{L}[\mathcal{B}]_{\mathrm{inj}}$, which follows from the inclusion $\mathcal{M}\hat{\otimes}\mathcal{C} \subset \mathcal{L}$ and the commutative diagram (a consequence of the fact that $\otimes$ preserves colimits in each variable):

$$\begin{tikzcd} (\mathsf{K}^{\mathcal{B}})^{\mathbf{2}} \times \mathsf{L}^{\mathbf{2}} \arrow[r, "\hat{\otimes}"] \arrow[d, "\widehat{\ell}^b\times 1"'] & (\mathsf{M}^{\mathcal{B}})^{\mathbf{2}} \arrow[d, "\widehat{\ell}^b"] \\ \mathsf{K}^{\mathbf{2}} \times \mathsf{L}^{\mathbf{2}} \arrow[r, "\hat{\otimes}"] & \mathsf{M}^{\mathbf{2}} \end{tikzcd}$$

because both injective left classes are defined pointwise. □

As before, taking $\mathcal{A}$ or $\mathcal{B}$ to be the terminal category, Theorem 7.1 specializes to the Reedy analogous of the weak factorization systems enumerated in Corollary 3.17. In particular:

**Theorem 7.3.** *For any Reedy category $\mathcal{A}$ and any weak factorization system $(\mathcal{L},\mathcal{R})$ on a bicomplete category $\mathsf{M}$, the weighted colimit bifunctor $- *_{\mathcal{A}} - \colon \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ defines Leibniz two-variable adjunctions*

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{proj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{proj}}) \times (\mathcal{L}[\mathcal{A}]_{\mathrm{inj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{inj}}) \xrightarrow{-\hat{*}_{\mathcal{A}}-} (\mathcal{L},\mathcal{R}), \quad \text{and}$$

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{inj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{inj}}) \times (\mathcal{L}[\mathcal{A}]_{\mathrm{proj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{proj}}) \xrightarrow{-\hat{*}_{\mathcal{A}}-} (\mathcal{L},\mathcal{R}).$$

*Dually, the weighted limit bifunctor $\{-,-\}^{\mathcal{A}} \colon (\mathsf{Set}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ defines Leibniz two-variable adjunctions*

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{proj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{proj}}) \times (\mathcal{L}[\mathcal{A}]_{\mathrm{proj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{proj}}) \xrightarrow{\widehat{\{-,-\}}^{\mathcal{A}}} (\mathcal{L},\mathcal{R}), \quad \text{and}$$

$$(\mathcal{M}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{inj}}, \mathcal{E}[\mathcal{A}^{\mathrm{op}}]_{\mathrm{inj}}) \times (\mathcal{L}[\mathcal{A}]_{\mathrm{inj}}, \mathcal{R}[\mathcal{A}]_{\mathrm{inj}}) \xrightarrow{\widehat{\{-,-\}}^{\mathcal{A}}} (\mathcal{L},\mathcal{R}).$$

## 8. Homotopy invariance and the Reedy model structure

In this section we expose the Berger–Moerdijk proof of the existence of the Reedy model structure on $\mathsf{M}^{\mathcal{A}}$ associated to an accessible model structure on $\mathsf{M}$. Definition 6.7 supplies the candidate weak factorization systems: the *projective Reedy model structure* will be comprised of the projective Reedy weak factorization systems associated to the weak factorization systems defining the model structure on $\mathsf{M}$; an injective Reedy model structure is defined similarly. One detail remains: to prove the analogue of Lemma 5.7, which shows that a map that is a projective Reedy cofibration and a pointwise weak equivalence is a projective Reedy trivial cofibration: i.e., its relative latching maps are define groupoid-indexed diagrams of projective trivial cofibrations. This argument, and its dual in the injective Reedy case, will require a more detailed analysis of the cellular and Postnikov presentations of Corollary 4.21 and an additional hypothesis which we somewhat unfaithfully call "dualizablity" — subjects to which we now turn.

*Remark* 8.1 (back to the cells). Recall any morphism $f \colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ is a generalized cell complex

$$\begin{tikzcd}
 & & \partial\mathcal{A}_n *_{\mathcal{G}(n)} Y^n \cup \mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[r, "(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n)\hat{*}_{\mathcal{G}(n)}\widehat{\ell^n f}"] \arrow[d] & \mathcal{A}_n *_{\mathcal{G}(n)} Y^n \arrow[d] & \\
X \arrow[r] & X \underset{\mathrm{sk}_0 X}{\cup} \mathrm{sk}_0 Y \arrow[r, dotted] & X \underset{\mathrm{sk}_{n-1} X}{\cup} \mathrm{sk}_{n-1} Y \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X \underset{\mathrm{sk}_n X}{\cup} \mathrm{sk}_n Y \arrow[r, dotted] & Y
\end{tikzcd}$$

where the generalized cell attached at stage $n$ is the map defined by the pushout

$$\begin{tikzcd}
\partial\mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[r] \arrow[d] & \mathcal{A}_n *_{\mathcal{G}(n)} \ell^n f \arrow[d] \arrow[ddr, bend left] & \\
\partial\mathcal{A}_n *_{\mathcal{G}(n)} Y^n \arrow[r] \arrow[drr] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet \arrow[dr, dashed] & \\
 & & \mathcal{A}_n *_{\mathcal{G}(n)} Y^n
\end{tikzcd}$$

Pointwise at $x \in \mathcal{A}$, we have a coproduct decomposition $\mathcal{A}_n^x \cong \overleftarrow{\partial\mathcal{A}_n^x} \coprod \overrightarrow{\mathcal{A}_n^x}$ which is respected by pre-composition with the groupoid $\mathcal{G}(n)$ of automorphisms between objects of degree $n$. Note, however, that this splitting is not respected by composition in the covariant variable.[9] Nevertheless, as in (3.14), the component at $x$ of the attached cell decomposes as

$$(\overleftarrow{\partial\mathcal{A}}^x_n \hookrightarrow \mathcal{A}^x_n) \mathbin{\hat{*}}_{\mathcal{G}(n)} \widehat{\ell^n} f \cong \mathrm{id}_{\overleftarrow{\partial\mathcal{A}}^x_n *_{\mathcal{G}(n)} Y^n} \coprod \overrightarrow{\mathcal{A}}^x_n *_{\mathcal{G}(n)} \widehat{\ell^n} f.$$

The identity term contributes nothing to the pushout. The way in which the remaining term $\overrightarrow{\mathcal{A}}^x_n *_{\mathcal{G}(n)} \widehat{\ell^n} f$ attached cell contributes to the filtration of $f^x \colon X^x \to Y^x$ depends on the degree of $x$ relative to $n$:

- When $\deg(x) < n$, $\overrightarrow{\mathcal{A}}^x_n = \emptyset$ and the component of the attached cell is an identity, which contributes nothing to the pushout.
- When $\deg(x) = n$, $\overrightarrow{\mathcal{A}}^x_n = \mathcal{G}(n)^x_n$ and so $\overrightarrow{\mathcal{A}}^x_n *_{\mathcal{G}(n)} \widehat{\ell^n} f \cong \widehat{\ell^x} f$, by the coYoneda lemma.
- When $\deg(x) > n$, an additional hypothesis introduced in Definition 8.2 will enable a simplified description of the attached cell.

The notion of Reedy category defined by Berger and Moerdijk in [BM08, 1.2] is stronger than our Definition 4.1, including condition (i) of Definition 8.2 below. If both conditions are included, then they refer to the categories of Definition 8.2 as "dualizable" Reedy categories, which inspires the name we adopt here.

**Definition 8.2.** A Reedy category is **dualizable** if

(i) If $\theta f = f$ for $f \in \overleftarrow{\mathcal{A}}$ and $\theta$ an isomorphism, then $\theta = \mathrm{id}_{\mathrm{cod}\, f}$.
(ii) If $f\theta = f$ for $f \in \overrightarrow{\mathcal{A}}$ and $\theta$ an isomorphism, then $\theta = \mathrm{id}_{\mathrm{dom}\, f}$.

We know of no Reedy categories that fail to be dualizable. A prototypical example is $\mathsf{Fin}$ or $\mathsf{Fin}_*$: the symmetric group $\Sigma_n$ acts freely on the set of monomorphisms with domain $n$ and on the set of epimorphisms with codomain $n$. Note, however, that symmetric groups do not act freely on general domains or codomains of morphisms between finite sets.

Our first use of dualizability is to extend Proposition 6.14 to injective Reedy left maps and projective Reedy right maps.

**Proposition 8.3.** *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system on $\mathsf{M}$ and let $\mathcal{A}$ be a dualizable Reedy category.*

*(i) Let $f \colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ be an injective Reedy left map. Then for each $a \in \mathcal{A}$, the component $f^a$, the latching map $L^a f$, and the relative latching map $\widehat{\ell^a} f$ are in the left class $\mathcal{L}$.*
*(ii) Dually, let $f \colon X \to Y \in \mathsf{M}^{\mathcal{A}}$ be a projective Reedy right map. Then for each $a \in \mathcal{A}$, the component $f^a$, the matching map $M^a f$, and the relative matching map $\widehat{m^a} f$ are in the right class $\mathcal{R}$.*

*Proof.* We prove the first of these dual statements using the condition 8.2(ii). As argued in the proof of Proposition 6.14, the maps $f^a$ and $L^a f$ are generalized cell complexes whose cells, indexed by the degrees with the bounds just discussed, have the form

$$\Big( (\overleftarrow{\partial\mathcal{A}}^a_k \hookrightarrow \mathcal{A}^a_k) \mathbin{\hat{\underline{\times}}}_{\mathcal{G}(k)} (\partial\mathcal{A}^k \hookrightarrow \mathcal{A}^k) \Big) \mathbin{\hat{*}}_{\mathcal{A}} f \cong (\overleftarrow{\partial\mathcal{A}}^a_k \hookrightarrow \mathcal{A}^a_k) \mathbin{\hat{*}}_{\mathcal{G}(k)} \widehat{\ell^k} f,$$

with $k \leq \deg(a)$ in the case of $f^a$ and $k < \deg(a)$ in the case of $L^a f$.

[9] In the special case of a generalized Reedy category such as $\mathsf{FI}$, of finite sets and injections, which has $\mathcal{A} = \overrightarrow{\mathcal{A}}$, this coproduct decomposition is preserved. If the maps in $\mathcal{A}$ are monomorphisms, then "orbits" with respect to $\mathcal{G}(n)$ are also respected.

Remark 8.1 implies that this cell splits as a coproduct of an identity term, which contributes nothing to the pushout and the weighted colimit

$$\overrightarrow{\mathcal{A}}^a_k *_{\mathcal{G}(k)} \widehat{\ell^k} f.$$

When $\mathcal{A}$ is dualizable, the groupoid $\mathcal{G}(k)$ acts freely on $\overrightarrow{\mathcal{A}}^a_k$ by precomposition into the domain variable. So we may decompose $\overrightarrow{\mathcal{A}}^a_k$ into $\mathcal{G}(k)$-orbits, each represented by some strictly degree-increasing map $m_i \colon a_i \to a$. The contributing cell is now

$$\overrightarrow{\mathcal{A}}^a_k *_{\mathcal{G}(k)} \widehat{\ell^k} f \cong \coprod_{m_i \colon a_i \to a} \widehat{\ell^{a_i}} f. \tag{8.4}$$

By hypothesis, each $\widehat{\ell^{a_i}} f \in \mathcal{L}$, so this coproduct, and its pushout, and thus the cell complex defining $f^a$ and $L^a f$ are as well. ◻

A similar argument establishes the missing ingredient in the proof of the Reedy model structure.

**Proposition 8.5.** *Let $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ define a model structure on $\mathsf{M}$ and suppose $\mathcal{A}$ is a dualizable Reedy category. Then a map $f \colon X \to Y \in \mathsf{M}^{\mathcal{A}}$*

*(i) is projective or injective Reedy in $\mathcal{C} \cap \mathcal{W}$ if and only if $f$ is projective or injective Reedy in $\mathcal{C}$ and a pointwise weak equivalence, and*

*(ii) is projective or injective Reedy in $\mathcal{F} \cap \mathcal{W}$ if and only if $f$ is projective or injective Reedy in $\mathcal{F}$ and a pointwise weak equivalence.*

*Proof.* We prove both parts of the first of these dual statements. If $f$ is projective or injective Reedy in $\mathcal{C} \cap \mathcal{W}$, this means that the diagrams $\widehat{\ell^n} f \in \mathsf{M}^{\mathcal{G}(n)}$ are projective or injective trivial cofibrations. Using the projective or injective model structure on $\mathsf{M}^{\mathcal{G}(n)}$, it is clear that this implies that $\widehat{\ell^n} f \in \mathsf{M}^{\mathcal{G}(n)}$ is then a projective or injective cofibration, and thus that $f$ is projective or injective Reedy in $\mathcal{C}$. Propositions 6.14 or8.3, applied in the projective or injective cases respectively to the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ implies further that its components $f^a$ are also in $\mathcal{C} \cap \mathcal{W}$. Thus $f$ is a pointwise weak equivalence.

For the converse, we assume that $f$ is a projective or injective Reedy cofibration and a pointwise weak equivalence. In the first case, our hypothesis is that $\widehat{\ell^n} f \in \mathsf{M}^{\mathcal{G}(n)}$ is a projective cofibration. If we can show that each component $\widehat{\ell^a} f$ is a weak equivalence, then the projective model structure on $\mathsf{M}^{\mathcal{G}(n)}$ implies that $\widehat{\ell^n} f$ is a projective trivial cofibration, and thus $f$ is a projective Reedy trivial cofibration, as desired. In the injective case, our hypothesis is that $\widehat{\ell^a} f$ is a cofibration. If we can show that $\widehat{\ell^a} f$ is a weak equivalence, then $f$ is an injective Reedy trivial cofibration, as desired. So, in summary, we wish in both cases to prove that $\widehat{\ell^a} f$ is a weak equivalence, and we may as well assume only the weaker hypothesis: that $f$ is an injective Reedy cofibration and pointwise weak equivalence.

We make use of the diagram

$$\begin{tikzcd}
L^a X \arrow[r, "L^a f"] \arrow[d] & L^a Y \arrow[d] \arrow[ddr, bend left] & \\
X^a \arrow[r] \arrow[drr, bend right, "f^a"'] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\widehat{\ell^a} f"] & \\
& & Y^a
\end{tikzcd}$$

which relates the maps $L^a f$, $\widehat{\ell^a} f$, and $f^a$ for any $a \in \mathcal{A}$. Since we have assumed that $f$ is an injective Reedy cofibration, Proposition 8.3 implies that $L^a f$ is in $\mathcal{C}$. We will show that $L^a f$ is in fact in $\mathcal{C} \cap \mathcal{W}$ and then apply pushout stability of the left class of a weak factorization system and the 2-of-3 property, to conclude that $\widehat{\ell^a} f \in \mathcal{W}$ and hence that $f$ is Reedy in $\mathcal{C} \cap \mathcal{W}$. We argue by induction. If $a$ has degree zero, then $L^a f$ is the identity at the initial object, which is certainly a weak equivalence, and $\widehat{\ell^a} f = f^a$ is in $\mathcal{C} \cap \mathcal{W}$, as $f$ was assumed to be a pointwise weak equivalence. If $a$ has degree $n$, we may now assume that $\widehat{\ell^x} f \in \mathcal{C} \cap \mathcal{W}$ for any $x$ with degree less than the degree of $a$. By the proof of Proposition 8.3, $L^a f$ may be presented as a generalized cell complex whose contributing cells, by the dualizability hypothesis (ii), reduce to coproducts (8.4) of relative latching maps indexed by objects of strictly smaller degree than the degree of $a$. By induction, these maps lie in $\mathcal{C} \cap \mathcal{W}$. Thus, we conclude that $L^a f \in \mathcal{C} \cap \mathcal{W}$, completing the proof. □

As an immediate corollary.

**Corollary 8.6.** *Suppose $\mathcal{A}$ is a dualizable Reedy category and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is an accessible model structure on $\mathsf{M}$. Then the category $\mathsf{M}^{\mathcal{A}}$ admits model structures where the weak equivalences are defined pointwise and:*

- *(i) the cofibrations, trivial cofibrations, fibrations, and trivial fibrations are defined by the projective Reedy weak factorization systems, or*
- *(ii) the cofibrations, trivial cofibrations, fibrations, and trivial fibrations are defined by the injective Reedy weak factorization systems.*

## 9. Derived limit and colimit functors

In this final section, we combine the algebraic results of §7 with the homotopical results of §8 to conclude that the weighted limit and colimit define Quillen bifunctors relative to various Reedy model structures associated with a dualizable Reedy category. Note that the underlying "algebraic" statements, that these constructions define Leibniz bifunctors relative to the Reedy weak factorization systems, do not require the dualizability hypothesis. However, it is necessary to construct derived functors from these Leibniz bifunctors. By Ken Brown's lemma, a functor that preserves cofibrations and trivial cofibrations will preserve all weak equivalences between cofibrant objects, but this argument makes use of the fact that a cofibration that is also a weak equivalence is a trivial cofibration—hence in the Reedy context the model structures of Theorem 8.6 will be needed.

**Homotopy limits and colimits of Reedy shape.** All of the results in this section are corollaries of Theorem 7.1, establishing the basic Reedy Leibniz two-variable adjunctions, and Corollary 8.6, establishing the projective and injective Reedy model structures. We begin by stating the model category version of Theorem 7.3, which is one of

**Theorem 9.1.** *For any dualizable Reedy category $\mathcal{A}$ and accessible model category $\mathsf{M}$, the weighted colimit*

$$\mathsf{M}^{\mathcal{A}} \xrightarrow{W *_{\mathcal{A}} -} \mathsf{M}$$

*defines a left Quillen adjoint relative to*

- *(i) the injective Reedy model structure on $\mathsf{M}^{\mathcal{A}}$, assuming $W \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is projective Reedy monomorphic, and*
- *(ii) the projective Reedy model structure on $\mathsf{M}^{\mathcal{A}}$, assuming $W \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is injective Reedy monomorphic.*

*Dually, the weighted limit*

$$\mathsf{M}^{\mathcal{A}} \xrightarrow{\{W,-\}^{\mathcal{A}}} \mathsf{M}$$

*defines a right Quillen adjoint relative to*

*(i) the projective Reedy model structure on* $\mathsf{M}^{\mathcal{A}}$*, assuming* $W \in \mathsf{Set}^{\mathcal{A}}$ *is projective Reedy monomorphic, and*

*(ii) the injective Reedy model structure on* $\mathsf{M}^{\mathcal{A}}$*, assuming* $W \in \mathsf{Set}^{\mathcal{A}}$ *is injective Reedy monomorphic.*

*Proof.* Specialize Theorem 7.1 to the weak factorization systems of Corollary 8.6 and the Leibniz two-variable adjunction of Lemma 3.13. □

In the setting of a model category, a monoidal model category, or a $\mathsf{V}$-model category (which subsumes the previous two cases by taking $\mathsf{V}$ to be $\mathsf{Set}$ or the model category itself), Theorem 7.1 specializes to the following result.

**Theorem 9.2.** *Let* $\mathsf{M}$ *be an accessible* $\mathsf{V}$*-model category and let* $\mathcal{A}$ *be a dualizable Reedy category. Then the weighted colimit*

$$\mathsf{V}^{\mathcal{A}^{\mathrm{op}}} \times \mathsf{M}^{\mathcal{A}} \xrightarrow{-\otimes_{\mathcal{A}}-} \mathsf{M}$$

*defines a Quillen two-variable adjunction relative to*

*(i) the projective Reedy model structure on* $\mathsf{V}^{\mathcal{A}^{\mathrm{op}}}$ *and the injective Reedy model structure on* $\mathsf{M}^{\mathcal{A}}$*, and*

*(ii) the injective Reedy model structure on* $\mathsf{V}^{\mathcal{A}^{\mathrm{op}}}$ *and the projective Reedy model structure on* $\mathsf{M}^{\mathcal{A}}$*.*

*Dually, the weighted limit*

$$(\mathsf{V}^{\mathcal{A}})^{\mathrm{op}} \times \mathsf{M}^{\mathcal{A}} \xrightarrow{\{-,-\}^{\mathcal{A}}} \mathsf{M}$$

*defines a Quillen two-variable adjunction relative to*

*(i) the projective Reedy model structures on* $\mathsf{V}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*, and*

*(ii) the injective Reedy model structures on* $\mathsf{V}^{\mathcal{A}}$ *and* $\mathsf{M}^{\mathcal{A}}$*.*

*Proof.* Specialize Theorem 7.1 to the weak factorization systems of Corollary 8.6 and the Leibniz two-variable adjunction

$$\mathsf{V} \times \mathsf{M} \xrightarrow{\otimes} \mathsf{M}, \quad \mathsf{V}^{\mathrm{op}} \times \mathsf{M} \xrightarrow{\{-,-\}} \mathsf{M}, \quad \text{and} \quad \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \xrightarrow{\mathrm{hom}} \mathsf{V}$$

defining the tensor, cotensor, and hom of a $\mathsf{V}$-model category. □

**Derived functors.** Quillen bifunctors are a fertile source of derived functors. For instance:

**Corollary 9.3.** *Let* $\mathsf{M}$ *be an accessible* $\mathsf{V}$*-model category and let* $\mathcal{A}$ *be a dualizable Reedy category.*

*(i) When* $W \in \mathsf{V}^{\mathcal{A}^{\mathrm{op}}}$ *is projective Reedy cofibrant, the weighted colimit* $W \otimes_{\mathcal{A}} - \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ *preserves pointwise weak equivalences between injective Reedy cofibrant diagrams.*

*(ii) When* $W \in \mathsf{V}^{\mathcal{A}^{\mathrm{op}}}$ *is injective Reedy cofibrant, the weighted colimit* $W \otimes_{\mathcal{A}} - \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ *preserves pointwise weak equivalences between projective Reedy cofibrant diagrams.*

*(iii) When* $W \in \mathsf{V}^{\mathcal{A}}$ *is projective Reedy cofibrant, the weighted limit* $\{W,-\}^{\mathcal{A}} \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ *preserves pointwise weak equivalences between projective Reedy cofibrant diagrams.*

*(iv) When $W \in \mathsf{V}^{\mathcal{A}}$ is injective Reedy cofibrant, the weighted limit $\{W, -\}^{\mathcal{A}} \colon \mathsf{M}^{\mathcal{A}} \to \mathsf{M}$ preserves pointwise weak equivalences between injective Reedy cofibrant diagrams.*

In the case where $\mathsf{V} = \mathsf{Set}$, the condition on the weights is that $W$ is projective or injective Reedy monomorphic, meaning the relative latching maps are projective or injective monomorphisms.

*Example* 9.4. Corollary 9.3 suggests a natural question: when is $1 \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, the weight for the ordinary colimit functor, projective or injective Reedy monomorphic? The argument in the injective case proceeds as in Example 5.13. In the projective case, this is so just when the latching maps $(\partial\mathcal{A}_n \hookrightarrow \mathcal{A}_n) *_{\mathcal{A}} 1$ define a projective monomorphism in $\mathsf{Set}^{\mathcal{G}(n)}$ for each $n$. As in [RV14, §9], the codomains of the latching maps are the constant diagrams at 1, so these maps are pointwise monomorphisms if and only if each $\mathsf{el}\partial\mathcal{A}_n$ is connected (see Example 2.9 and Remark 2.13). The monomorphism is $1 \hookrightarrow 1$ if $\mathsf{el}\partial\mathcal{A}_n$ is connected and non-empty and $\emptyset \hookrightarrow 1$ if $\mathsf{el}\partial\mathcal{A}_n = \emptyset$. Unless the objects of $\mathcal{G}(n)$ have no non-identity automorphisms, only the former is a projective monomorphism. So, in summary, $1 \in \mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is projective Reedy monomorphic if and only if for each $a \in \mathcal{A}$, the category $\mathsf{el}\partial\mathcal{A}_a$ is connected and moreover either

- it is non-empty, or
- $a$ admits no non-identity automorphisms.

This is the case, for instance, when $\mathcal{A}$ has a terminal object with degree zero, e.g., for $\mathcal{A} = \mathsf{Fin}_*$ or $\mathsf{Fin}_*^{\mathrm{op}}$. A non-example is $\mathsf{Fin}^{\mathrm{op}}$: the boundary of the contravariant functor represented by the two-element set is not connected.

*Example* 9.5. Let $A \in \mathsf{V}$ be a cofibrant object and consider the functor

$$A^\bullet \colon \mathsf{Fin}_*^{\mathrm{op}} \to \mathsf{V}$$

defined by sending the based finite set $n_+ = \{*, 1, \ldots, n\}$ to the $n$-fold power $A^n$.[10] The latching object $L_nA \to A^n$ is the "fat diagonal," informally the subspace in which some coordinate equals the basepoint or some pair of coordinates are duplicated. To say that $A^\bullet$ is injective Reedy cofibrant is to say that this map $L_nA \to A^n$ is a cofibrant in $\mathsf{V}$, in which case the weighted colimit

$$\mathsf{M}^{\mathsf{Fin}_*} \xrightarrow{A^\bullet \otimes_{\mathsf{Fin}_*} -} \mathsf{M}$$

would define a left Quillen functor for any $\mathsf{V}$-model category $\mathsf{M}$. Note that $A^\bullet$ is never *projective* Reedy cofibrant because $\Sigma_n$ does not act freely on the fat diagonal.

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[10] The use of the superscript here contradicts our usual notational convention of reserving superscripts for covariant functoriality, but here this notation $A^{n_+} := A^n$ is really unavoidable.

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