

STUFF ABOUT QUASICATEGORIES

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(Note: this is a draft, which can change daily. It includes some unorganized material which might later be incorporated into the narrative, or removed entirely. In particular, everything involving Cartesian and coCartesian fibrations needs to be completely reworked.)

1. INTRODUCTION TO ∞ -CATEGORIES

I'll give a brief discussion to motivate the notion of ∞ -categories.

1.1. Groupoids. Modern mathematics is based on sets. The most basic way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring R , we can consider the set $X(R)$ of tuples $(x, y, z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits; for instance, $X(R)$ is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y) \mapsto x^5 + y^5} R \xleftarrow{z^5 \leftarrow z} R.$$

Another way to construct new sets is by taking “quotients”; e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have “higher” structure standing behind them, which is often encoded in the form of a *groupoid*¹. Here are some examples.

- Given a topological space X , we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the *fundamental groupoid* $\Pi_1 X$, whose objects are points of X , and whose morphisms are path-homotopy classes of paths between two points.
- Given any category C , there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of C (also called the *maximal subgroupoid*), which is a groupoid having the same objects as C , but having as morphisms only the isomorphisms in C .
- As a special case of the above, let $C = \text{Vect}_F$ be the category of finite dimensional vector spaces and linear maps over some field F . Then isomorphism classes of objects correspond to non-negative integers, via the notion of dimension. The core $\text{Vect}_F^{\text{core}}$ is a groupoid whose objects are finite dimensional vector spaces, and whose morphisms are *invertible* linear maps.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal G -bundles on a space. In practice, one learns that when you try to classify some type of objects up to isomorphism, you will need to have a good handle on the isomorphisms between such objects, including the groups of automorphisms of such objects. So you will likely need to know about the groupoid, even if it is not the primary object of interest.

For instance, a problem such as: “describe the groupoid $\text{Bun}_G(M)$ of principal G -bundles on a space M ” is a more sophisticated analogue of: “find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring R ”. (In fact, the theory of “moduli stacks” exactly develops this analogy between the two problems.) To do this, you can imagine having a “groupoid-based mathematics”, generalizing the usual set-based one. Here are some observations about this.

- We regard two sets as “essentially the same” if they are *isomorphic*, i.e., if there is a bijection $f: X \rightarrow X'$ between them. Any such bijection has a unique inverse bijection $f^{-1}: X' \rightarrow X$.
On the other hand, we regard two categories as “essentially the same” if they are merely *equivalent*, i.e., if there is a functor $f: C \rightarrow C'$ which admits an inverse *up to natural*

¹I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

isomorphism. It is not the case that such an inverse up to natural isomorphism is itself unique. These same remarks apply in particular to equivalences of groupoids.

Although any equivalence of categories admits some kind of inverse, the failure to be unique leads to complications. For example, one goal of every course in abstract linear algebra is to demonstrate and exploit an equivalence of categories

$$f: \text{Mat}_F \rightarrow \text{Vect}_F.$$

Here Mat_F is the *matrix* category, whose objects are non-negative integers, and whose morphisms $n \rightarrow m$ are $m \times n$ -matrices with entries in F . The functor f is defined by an explicit construction; e.g., it sends the object n to the vector space F^n . However, there is no completely “natural” way to construct an inverse functor $f^{-1}: \text{Vect}_F \rightarrow \text{Mat}_F$: producing such an inverse functor requires making an arbitrary choice, for each abstract vector space V , of a basis for V .

- We can consider “solutions to equations” in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider “weak” or “homotopy” limits.

For example, suppose M is a space which is a union of two open sets U and V . The *weak pullback* of

$$\text{Bun}_G(U) \rightarrow \text{Bun}_G(U \cap V) \leftarrow \text{Bun}_G(V)$$

is a groupoid, whose objects are triples (P, Q, α) , where $P \rightarrow U$ and $Q \rightarrow U$ are G -bundles, and $\alpha: P|_{U \cap V} \xrightarrow{\sim} Q|_{U \cap V}$ is an isomorphism of G -bundles over $U \cap V$; the morphisms $(P, Q, \alpha) \rightarrow (P', Q', \alpha')$ are pairs $(f: P \rightarrow P', g: Q \rightarrow Q')$ are pairs of bundle maps which are compatible over $U \cap V$ with the isomorphisms α, α' . Compare this with the *strict pullback*, which consists of (P, Q) such that $P|_{U \cap V} = Q|_{U \cap V}$ as bundles; in particular, $P|_{U \cap V}$ and $Q|_{U \cap V}$ must be the *identical sets*.

A basic result about bundles is that $\text{Bun}_G(M)$ is equivalent to this weak pullback. The strict limit may fail to be equivalent to this; in fact, it is impossible to describe the strict pullback without knowing precisely what definition of G -bundle we are using, whereas the identification of weak pullback is insensitive to the precise definition of G -bundle. (The point being, there can exist many non-identical “precise definitions of G -bundle”, because what we really care about in the end is understanding $\text{Bun}_G(M)$ up to *equivalence*, rather than up to isomorphism.)

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. Higher groupoids. There is a category Gpd of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are *natural transformations* between functors $f, f': G \rightarrow G'$ of groupoids. That is, $\text{Fun}(G, G')$ forms not merely a set, but a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a *2-groupoid*². There is no reason to stop at 2-groupoids: there are n -groupoids, the totality of which are an example of an $(n + 1)$ -groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider ∞ -groupoids.

It turns out to be difficult (though not impossible) to construct an “algebraic” definition of n -groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid G has a *classifying space* BG . This is defined

²More precisely, a “quasistrict 2-groupoid”.

explicitly as a quotient space

$$G \mapsto BG := \left(\coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n} \Delta^n \right) / \sim,$$

where we glue in a topological n -simplex Δ^n for each n -fold sequence of composable arrows in G , modulo certain identifications. It turns out (i) the fundamental groupoid of BG is equivalent to G , and (ii) the higher homotopy groups π_k of BG are trivial, for $k \geq 2$. A space like this is called a *1-type*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I'll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define* n -groupoids as n -types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling ∞ -groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classifying space construction factors through a “combinatorial” construction, called the “nerve”. That is, we have

$$(G \in \mathbf{Gpd}) \mapsto (NG \in \mathbf{sSet}) \mapsto (|NG| = BG \in \mathbf{Top}),$$

where NG is the *nerve* of the groupoid, and is an example of a *simplicial set*; $|X|$ denotes the *geometric realization* of a *simplicial set* X . In fact, the nerve is a particular kind of simplicial set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that ∞ -groupoids are precisely the Kan complexes.

1.3. ∞ -categories. An ∞ -category is a generalization of ∞ -groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining ∞ -categories. Here are two which build on top of the identification of ∞ -groupoids with Kan complexes.

- A category C consists of a set $\text{ob } C$ of objects, and for each pair of objects a set $\text{hom}_C(x, y)$ of maps from x to y . If we replace the set $\text{hom}_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\text{map}_C(x, y)$, we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for ∞ -categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category C , we have a simplicial set NC . In general, NC is not a Kan complex; however, it does land in a special class of simplicial sets, which are called *quasicategories*. This leads to another model for ∞ -categories: *quasicategories*.

In this paper we focus on the second case: the quasicategory model for ∞ -categories.

1.4. Historical remarks. Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of ∞ -categories. This development began with the work of Joyal, first published in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls “ ∞ -categories”), recasting and extending Joyal’s work significantly.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a “categorical equivalence” between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I

follow here is essentially that of Joyal. However, I have tried to follow Lurie’s terminology and notation in most places.

1.5. Goal of this book. The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. No advanced training in homotopy theory is assumed: in particular, no knowledge of simplicial sets or model categories is assumed.
- The book is written in “lecture notes” style rather than “textbook” style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the “natural” place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)
- The idea is that, after you have read this book, you will be well-prepared to dip into the main references on quasicategories (e.g., Lurie’s books) without too much difficulty. Note that this book is not meant to (and does not) supplant any such reference.

1.6. Prerequisites. I assume only familiarity with basic concepts of category theory, such as those discussed in the first few chapters of [Rie16]. It is helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher’s textbook).

Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- general colimits and limits.
- adjoint functors.

1.7. References and other sources. As noted, the material depends mainly on the work of Joyal and Lurie.

- Joyal’s first paper [Joy02] on the subject explicitly introduces quasicategories as a model for ∞ -categories. It is worth looking at.
- There are several versions of unpublished lecture notes by Joyal [Joy08a], [Joy08b], which develop the theory of quasicategories from scratch. Also note the paper by Joyal and Tierney [JT08], which gives a summary of some of this unpublished work.
- Lurie’s “Higher topos theory” [Lur09] gives a complete development of ∞ -categories, including a number of topics not even touched in this book. The main general material on ∞ -categories is in Chapters 1–4, together with quite of bit of material from the appendices. It is also worth looking at Chapter 5, which develops the very important notions of accessible and presentable ∞ -categories. The final two chapters apply these ideas to the theory of ∞ -topoi.

- Lurie’s “Higher algebra” [Lur12] treats a number of “advanced topics”, including *stable ∞ -categories* (the ∞ -categorical foundations for derived categories in homological algebra and stable homotopy), various notions of monoidal structures on ∞ -categories (via the theory of *∞ -operads*), and other topics.
- After I came up with the first version of these notes, Cisinski published the book “Higher Categories and Homotopical Algebra”. It covers much of the material in these notes (and much more), on roughly similar lines: in his book model categories play a more prominent role from the start than they do here.
- Bergner’s “The homotopy theory of $(\infty, 1)$ -categories” is a survey of various approaches to higher categories and their interrelationships.
- Groth’s note “A short course on ∞ -categories” provides a brief survey to some of the basic ideas about quasicategories and their applications. It is not a complete treatment, but it does get very quickly to some of the more advanced topics.
- Riehl and Verity . . .

1.8. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of n -truncation and n -groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that $S^\triangleright \rightarrow \text{Fun}(D, C)$ is a colimit cone if each projection to $S^\triangleright \rightarrow \text{Fun}(\{d\}, C) \approx C$ is one.

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Part 1. Basic notions

2. SIMPLICIAL SETS

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*; relaxing this characterization will lead us to the definition of quasicategories.

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. We recommend Greg Friedman’s survey [Fri12] as a starting place for learning about this viewpoint, and we will discuss this point of view later on in §??. Here we will focus on what we need in order to develop quasicategories.

2.1. **The simplicial indexing category Δ .** We write Δ for the category whose

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- **objects** are the non-empty totally ordered sets $[n] := \{0 < 1 < \dots < n\}$ for $n \geq 0$, and
- **morphisms** $f: [n] \rightarrow [m]$ are weakly monotone functions, i.e., such that $x \leq y$ implies $f(x) \leq f(y)$.

Note that we exclude the empty set from Δ . Morphisms in Δ are often called **simplicial operators**.

Because $[n]$ is an ordered set, you can also think of it as a category: the objects are the elements of $[n]$, and there is a morphism (necessarily unique) $i \rightarrow j$ if and only if $i \leq j$. Thus, morphisms in the category Δ are precisely the functors between the categories $[n]$. We can, and will, also think of $[n]$ as the category “freely generated” by the picture

$$\boxed{0} \rightarrow \boxed{1} \rightarrow \dots \rightarrow \boxed{n-1} \rightarrow \boxed{n}.$$

Arbitrary non-identity morphisms in $[n]$ can be expressed uniquely as iterated composites of the arrows which are displayed in the picture.

We will often use the following notation for morphisms in Δ :

$$f = \langle f_0 \cdots f_n \rangle: [n] \rightarrow [m] \quad \text{with } f_0 \leq \cdots \leq f_n \quad \text{represents the function } k \mapsto f_k.$$

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators:

$$\begin{aligned} d^i &:= \langle 0, \dots, \widehat{i}, \dots, n \rangle: [n-1] \rightarrow [n], \quad 0 \leq i \leq n, \\ s^i &:= \langle 0 \dots i, i, \dots, n \rangle: [n+1] \rightarrow [n], \quad 0 \leq i \leq n. \end{aligned}$$

All maps in Δ can be obtained as a composition of face and degeneracy operators, and in fact Δ can be described as the category *generated* by the above symbols, subject to a set of *relations* called the “simplicial identities”, which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. **Simplicial sets.** A **simplicial set** is a functor $X: \Delta^{\text{op}} \rightarrow \text{Set}$, i.e., a contravariant functor (or “presheaf”) from Δ to sets.

It is typical to write X_n for $X([n])$, and call it the set of *n-simplices* in X . I generally prefer to call it the set of **n-dimensional elements** of X instead (because the word “simplices” also applies to the so called “standard *n*-simplices” defined below (2.7), and I would like to avoid to confusion between them). I will also speak of the set of all elements (or all simplices) of X , i.e., of the disjoint union $\coprod_{n \geq 0} X_n$ the sets X_n .

The 0-dimensional elements of a simplicial set are also called **vertices**, while the 1-dimensional elements are also called **edges**.

Given an element $a \in X_n$ and a simplicial operator $f: [m] \rightarrow [n]$, I will write $af \in X_m$ as shorthand for $X(f)(a)$. That is, I’ll think of simplicial operators as acting on elements from the right; this is a convenient choice given that X is a *contravariant* functor. In this language, a simplicial set consists of

- a sequence of sets X_0, X_1, X_2, \dots ,
- functions $a \mapsto af: X_n \rightarrow X_m$ for each simplicial operator $f: [m] \rightarrow [n]$, such that
- $a \text{ id} = a$, and $(af)g = a(fg)$ for any element a and simplicial operators f and g whenever this makes sense.

If I need to have the simplicial operator act from the left, I’ll write $f^*(a) = af$.

Sometimes I’ll use a subscript notation when speaking of the action of particular simplicial operators. So, given a simplicial operator of the form $f = \langle f_0 \cdots f_m \rangle: [m] \rightarrow [n]$, we can indicate the action of f on elements using subscripts:

$$a_{f_0 \cdots f_m} := af = a \langle f_0 \cdots f_m \rangle.$$

In particular, applying simplicial operators of the form $\langle i \rangle: [0] \rightarrow [n]$ gives vertices $a_0, \dots, a_n \in X_0$, which we call the “vertices of a ”, while applying simplicial operators of the form $\langle ij \rangle: [1] \rightarrow [n]$ for $0 \leq i < j \leq n$ gives edges $a_{ij} \in X_1$, which we call the “edges of a ”.

2.4. **The category of simplicial sets.** A simplicial set is a functor; therefore a **map of simplicial sets** is a natural transformation of functors. Explicitly, a map $\phi: X \rightarrow Y$ between simplicial sets is a collection of functions $\phi: X_n \rightarrow Y_n$, $n \geq 0$, which commute with simplicial operators:

$$(\phi a)f = \phi(af) \quad \text{for all simplicial operators } f \text{ and elements } a \text{ in } X, \text{ when this makes sense.}$$

I’ll write $s\text{Set}$ for the category of simplicial sets and maps between them³.

³Lurie [Lur09] uses Set_Δ to denote the category of simplicial sets. **Perhaps I should try to be consistent with this?**

2.5. Discrete simplicial sets. A simplicial set X is **discrete** if every simplicial operator f induces a bijection $f^*: X_n \rightarrow X_m$.

Every set S gives us a discrete simplicial set S^{disc} , defined so that $(S^{\text{disc}})_n = S$, and so that each simplicial operator acts according to the identity map of S . This construction defines a functor $S \mapsto S^{\text{disc}}: \text{Set} \rightarrow \text{sSet}$.

2.6. Exercise. Show that (i) every discrete simplicial set X is isomorphic to S^{disc} for some set S , (and that in fact you can take $S = X_0$), and (ii) for every pair of sets S and T , the evident function $\text{Hom}_{\text{Set}}(S, T) \rightarrow \text{Hom}_{\text{sSet}}(S^{\text{disc}}, T^{\text{disc}})$ is a bijection.

Let $\text{sSet}^{\text{disc}}$ denote the full subcategory of sSet spanned by discrete simplicial sets. That is, objects of $\text{sSet}^{\text{disc}}$ are discrete simplicial sets, and morphisms of $\text{sSet}^{\text{disc}}$ are all simplicial maps between them. Then the above exercise amounts to saying that *the full subcategory of discrete simplicial sets is equivalent to the category of sets*.

For this reason, it is often convenient to (at least informally) “identify” sets with their corresponding discrete simplicial sets (i.e., for a set S we also write S for the discrete simplicial set S^{disc} defined above).

2.7. Standard n -simplex. The **standard n -simplex** Δ^n is the simplicial set defined by

$$\Delta^n := \text{Hom}_{\Delta}(-, [n]).$$

That is, the standard n -simplex is exactly the functor *represented* by the object $[n]$. Explicitly, this means that

$$(\Delta^n)_m = \text{Hom}_{\Delta}([m], [n]) = \{\text{simplicial operators } a: [m] \rightarrow [n]\},$$

while the action of simplicial operators on elements of Δ^n is given by composition: $f: [m'] \rightarrow [m]$ sends $a \in (\Delta^n)_m$ to $(af): [m'] \rightarrow [n] \in (\Delta^n)_{m'}$.

The **generator** of Δ^n is the element

$$\iota_n := \langle 01 \dots n \rangle = \text{id}_{[n]} \in (\Delta^n)_n$$

corresponding to the identity map of $[n]$.

The **Yoneda lemma** (applied to the category Δ) asserts that the function

$$\begin{aligned} \text{Hom}_{\text{sSet}}(\Delta^n, X) &\rightarrow X_n, \\ g &\mapsto g(\iota_n), \end{aligned}$$

is a bijection for every simplicial set X . (*Exercise:* if this fact is not familiar to you, prove it.)

The Yoneda lemma can be stated this way: for each n -dimensional element $a \in X_n$ there exists a *unique* map $f_a: \Delta^n \rightarrow X$ of simplicial sets which sends the generator to it, i.e., such that $f_a(\iota_n) = a$. We call the map f_a the **representing map** of the element a .

We will often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as f_a , we will typically abuse notation and write $a: \Delta^n \rightarrow X$ for the representing map of the simplex $a \in X_n$. We reiterate that the map $a: \Delta^n \rightarrow X$ is characterized as the *unique* map sending the generator ι_n of Δ^n to a . Thus with our notation we have $a = a(\iota_n)$, where the two appearances of “ a ” denote respectively the element of X_n and the representing morphism $\Delta^n \rightarrow X$.

2.8. Exercise. Show that the representing map $f: \Delta^n \rightarrow X$ of $a \in X_n$ sends $\langle f_0 \dots f_k \rangle \in (\Delta^n)_k$ to $a \langle f_0 \dots f_k \rangle \in X_k$.

Note that if $X = \Delta^m$ is also a standard simplex, then the Yoneda lemma gives a bijection

$$\text{Hom}_{\text{sSet}}(\Delta^n, \Delta^m) \xrightarrow{\sim} (\Delta^m)_n = \text{Hom}_{\Delta}([n], [m]).$$

The inverse of this bijection sends a simplicial operator $f: [n] \rightarrow [m]$ to the map $\Delta^f: \Delta^n \rightarrow \Delta^m$ of simplicial sets defined on elements $g \in (\Delta^n)_k = \text{Hom}_{\Delta}([k], [n])$ by $g \mapsto fg$. (*Exercise:* prove this.)

I will commonly abuse notation, and write $f: \Delta^n \rightarrow \Delta^m$ instead of Δ^f for the map induced by the simplicial operator f , as it is also the representing map of the corresponding simplex $f \in (\Delta^m)_n$.

2.9. The standard 0-simplex and the empty simplicial set. The standard 0-simplex Δ^0 is the terminal object in $s\text{Set}$. Sometimes I write $*$ instead of Δ^0 for this object. Note that it is the only standard n -simplex which is discrete.

The **empty simplicial set** \emptyset is the functor $\Delta^{\text{op}} \rightarrow \text{Set}$ sending each $[n]$ to the empty set. It is the initial object in $s\text{Set}$.

2.10. Exercise. Show that a simplicial set X is isomorphic to the empty simplicial set if and only if X_0 is isomorphic to the empty set.

2.11. Standard simplices on totally ordered sets. The definition of the standard simplices Δ^n can be extended to simplicial sets “generated” by arbitrary totally ordered sets.

For instance, for *any* non-empty finite totally ordered set $S = \{s_0 < s_1 < \dots < s_n\}$, there is a *unique* order preserving bijection $S \approx [n]$ for a unique $n \geq 0$. We write Δ^S for the simplicial set with $(\Delta^S)_k = \{\text{order preserving } [k] \rightarrow S\}$. There is a unique isomorphism $\Delta^S \approx \Delta^n$ of simplicial sets (*Exercise*: prove this). We can also apply this idea to the empty ordered set $S = \emptyset$, in which case $(\Delta^\emptyset)_k = \emptyset$ for all k , i.e., Δ^\emptyset is the empty simplicial set.

This notation is especially convenient for subsets $S \subseteq [n]$ with induced ordering, as the simplicial set Δ^S is in a natural way a *subcomplex* of Δ^n (i.e., a collection of subsets of the $(\Delta^n)_k$ closed under action of simplicial operators; we will return to the notion of subcomplex below §4.9).

Furthermore, any simplicial operator $f: [m] \rightarrow [n]$ factors through its image $S = f([m]) \subseteq [n]$, giving a factorization

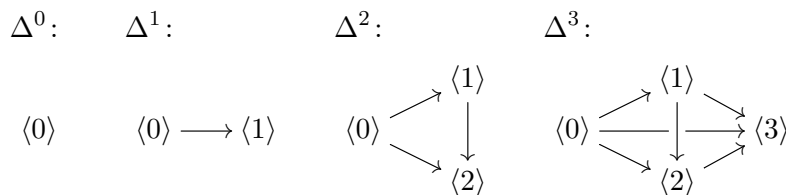
$$[m] \xrightarrow{f_{\text{surj}}} S \xrightarrow{f_{\text{inj}}} [n]$$

of maps between ordered sets, and thus a factorization $\Delta^m \xrightarrow{\Delta^{f_{\text{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\text{inj}}}} \Delta^n$ of the induced map Δ^f of simplicial sets.

2.12. Exercise. Show that $\Delta^{f_{\text{inj}}}$ and $\Delta^{f_{\text{surj}}}$ respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of k -dimensional elements for all k . (The case of $\Delta^{f_{\text{inj}}}$ is formal, but the case of $\Delta^{f_{\text{surj}}}$ is not completely formal.)

2.13. Pictures of standard simplices. When we draw a “picture” of Δ^n , we draw a geometric n -simplex: the convex hull of $n + 1$ points in general position, with vertices labelled by $0, \dots, n$. The faces of the geometric simplex correspond exactly to *injective* simplicial operators into $[n]$: these elements are called *non-degenerate*. For each non-degenerate simplex f in Δ^n , there is an infinite collection of *degenerate* elements with the same “image” as f (when viewed as a simplicial operator with target $[n]$).

Here are some “pictures” of standard simplices, which show their non-degenerate elements. Note that we draw the 1-dimensional elements of Δ^n as arrows; this lets us easily see the total ordering on the vertices of Δ^n .



We’ll extend the terminology of “degenerate” and “non-degenerate” elements to arbitrary simplicial sets in §15.5.

3. THE NERVE OF A CATEGORY

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat , the category of categories, into $s\text{Set}$, which means that we are able to think of categories as just a special kind of simplicial set.

3.1. Construction of the nerve. Given a category C , the **nerve** of C is the simplicial set NC defined so that

$$(NC)_n := \text{Hom}_{\text{Cat}}([n], C),$$

the set of functors from $[n]$ to C , and so that simplicial operators $f: [m] \rightarrow [n]$ act by precomposition: $a \mapsto af$ for an element $a: [n] \rightarrow C$ in $(NC)_n$.

3.2. Example. There is an evident isomorphism $N[n] \approx \Delta^n$.

Given a functor $F: C \rightarrow D$ between categories, we obtain a map $NF: NC \rightarrow ND$ of simplicial sets, sending $(a: [n] \rightarrow C) \in (NC)_n$ to $(Fa: [n] \rightarrow D) \in (ND)_n$. Thus the nerve construction defines a functor $N: \text{Cat} \rightarrow \text{sSet}$.

3.3. Structure of the nerve. We observe the following, whose verification we leave to the reader.

- $(NC)_0$ is canonically identified with the set of objects of C .
- $(NC)_1$ is canonically identified with the set of morphisms of C .
- The operators $\langle 0 \rangle^*, \langle 1 \rangle^*: (NC)_1 \rightarrow (NC)_0$ assign to a morphism its source and target respectively.
- The operator $\langle 00 \rangle^*: (NC)_0 \rightarrow (NC)_1$ assigns to an object its identity map.
- $(NC)_2$ is in bijective correspondence with the set of pairs (f, g) of morphisms such that gf is defined, i.e., such that the target of f is the source of g . This bijection is given by sending $a \in (NC)_2$ to $(a_{01}, a_{12}) \in (NC)_1 \times (NC)_1$.
- The operator $\langle 02 \rangle^*: (NC)_2 \rightarrow (NC)_1$ assigns, to an element corresponding to a pair (f, g) of morphisms, the composite morphism gf .

We have the following general description of n -dimensional elements in the nerve.

3.4. Proposition. *Let C be a category.*

(1) *There is a bijective correspondence*

$$(NC)_n \xrightarrow{\sim} \{ (g_1, \dots, g_n) \in (\text{mor } C)^{\times n} \mid \text{target}(g_{i-1}) = \text{source}(g_i) \},$$

which sends $(a: [n] \rightarrow C) \in (NC)_n$ to the sequence $(a\langle 0, 1 \rangle, \dots, a\langle n-1, n \rangle)$

(2) *With respect to the correspondence of (1), the map $f^*: (NC)_n \rightarrow (NC)_m$ induced by a simplicial operator $f: [m] \rightarrow [n]$ coincides with the function*

$$(g_1, \dots, g_n) \mapsto (h_1, \dots, h_m), \quad h_k = \begin{cases} \text{id} & \text{if } f(k-1) = f(k) \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

Proof. For (1), one verifies that an inverse is given by the function which sends a sequence (g_1, \dots, g_n) to $(a: [n] \rightarrow C) \in (NC)_n$ defined on objects by $a(k) = \text{target}(g_{k-1}) = \text{source}(g_k)$, and on morphisms by $a\langle i, j \rangle = g_j g_{j-1} \cdots g_{i+1}$ for $i < j$. For (2), note that for $a \in (NC)_n$ corresponding to the tuple (g_1, \dots, g_n) we can compute

$$(af)\langle k-1, k \rangle = a\langle f(k-1), f(k) \rangle = \begin{cases} \text{id} & \text{if } f(k-1) = f(k), \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

□

In particular, you can recover the category from its nerve, up to isomorphism, since the nerve contains all information about objects, morphisms, and composition of morphisms.

3.5. Remark. It is clear from the above remarks that most of the information in the nerve of C is redundant: we only needed $(NC)_k$ for $k = 0, 1, 2$ and certain simplicial operators between them to recover C .

3.6. Exercise. Show that for any discrete simplicial set X there exists a category C and an isomorphism $NC \approx X$.

3.7. Characterization of nerves. This leads to the question: given a simplicial set X , how can we detect that it is isomorphic to the nerve of some category?

3.8. Proposition. *A simplicial set X is isomorphic to the nerve of some category if and only if for all $n \geq 2$ the function*

$$\phi_n: X_n \rightarrow \{ (g_1, \dots, g_n) \in (X_1)^{\times n} \mid g_{i-1}\langle 1 \rangle = g_i\langle 0 \rangle, 1 \leq i \leq n \}$$

which sends $a \in X_n$ to $(a_{0,1}, \dots, a_{n-1,n})$ is a bijection.

Proof. First, suppose $X = NC$ for some category C . Then the function ϕ_n is precisely the bijection of (3.4)(1). Thus, if X is isomorphic to the nerve of some category then its ϕ_n are bijections.

Now suppose X is a simplicial set such that the ϕ_n are bijections. We *define* a category C , with

$$(\text{objects of } C) = X_0, \quad (\text{morphisms of } C) = X_1,$$

following the discussion in (3.3). Thus, the source and target of $g \in X_1$ are g_0 and g_1 in X_0 respectively, the identity map of $x \in X_0$ is $x_{00} \in X_1$, while the composite of (g, h) such that $g_1 = h_0$ is a_{02} , where $a \in X_2$ is the unique 2-dimensional element with $a_{01} = g$ and $a_{12} = h$. We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of elements of X_3 . (Or look ahead to (5.10), where we carry out the argument explicitly in a slightly different context.)

Next, we claim that for $a \in X_n$, and for $0 \leq i \leq j \leq k \leq n$, we have that

$$a_{i,k} = a_{j,k}a_{i,j},$$

where $a_{i,k}, a_{i,j}, a_{j,k} \in X_1$ are images of a under face operators $[1] \rightarrow [n]$, and right-hand side represents composition of two morphisms in C . To see this, note first that for $b \in X_2$, we have $b_{0,2} = b_{1,2}b_{0,1}$ by construction of C . The general case follows from this by setting $b = a_{i,j,k}$.

Now we can define maps $\psi_n: X_n \rightarrow (NC)_n$ by sending $a \in X_n$ to $\psi(a): [n] \rightarrow C$ defined by $\psi(a)(i \leq j) = a_{i,j}$, which is a functor by the above remarks. These maps ψ_n are seen to be bijections using the bijections ϕ_n and (3.4), since $\psi_n(a)(i-1 \leq i) = a_{i-1,i}$. If $f: [m] \rightarrow [n]$ is a simplicial operator, then we compute

$$\psi_m(af)(i \leq j) = (af)_{i,j} = a_{f(i),f(j)} = (\psi_n(a))(f(i) \leq f(j)) = (\psi_n(a)f)(i \leq j),$$

whence ψ is a map of simplicial sets. We have thus constructed an isomorphism $\phi: X \rightarrow NC$ of simplicial sets, as desired. □

3.9. A characterization of maps between nerves. Maps between nerves are the same as functors between categories.

3.10. Proposition. *The nerve functor $N: \text{Cat} \rightarrow \text{sSet}$ is fully faithful. That is, every simplicial set map $g: NC \rightarrow ND$ between nerves is of the form $g = N(f)$ for a unique functor $f: C \rightarrow D$.*

Proof sketch. We need to show that $\text{Hom}_{\text{Cat}}(C, D) \rightarrow \text{Hom}_{\text{sSet}}(NC, ND)$ is a bijection for all categories C and D . *Injectivity* is clear, as a functor f is determined by its effect on objects and morphisms, which is exactly the effect of $N(f)$ on 0- and 1-dimensional elements of the nerves.

For *surjectivity*, observe that for any map $g: NC \rightarrow ND$ of simplicial sets, we can define a candidate functor $f: C \rightarrow D$, defined on objects and morphisms by the action of g on 0-dimensional and 1-dimensional elements. That F has the correct action on identity maps follows from the fact that g commutes with the simplicial operator $\langle 00 \rangle: [1] \rightarrow [0]$. That f preserves composition uses (3.4) and the fact that g commutes with the simplicial operator $\langle 02 \rangle: [1] \rightarrow [2]$.

Note that given $g: NC \rightarrow ND$ and $f: C \rightarrow D$ as constructed above, the maps $g, N(f): NC \rightarrow ND$ coincide on 0-dimensional and 1-dimensional elements by construction. It follows that $g = N(f)$ by (3.11) below. Thus, we have shown that $N: \text{Hom}_{\text{Cat}}(C, D) \rightarrow \text{Hom}_{\text{sSet}}(NC, ND)$ is surjective as desired. □

3.11. *Exercise.* Show that if C is a category and X is any simplicial set (not necessarily a nerve), then two maps $g, g': X \rightarrow NC$ are equal if and only if $g_0 = g'_0$ and $g_1 = g'_1$, i.e., g and g' are equal if and only if they coincide on 0-dimensional and 1-dimensional elements. (Hint: use (3.4).)

4. SPINES

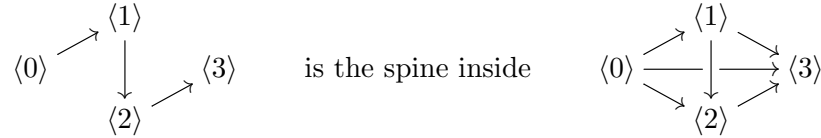
In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain “extension” condition. To state this condition we need the notion of a “spine” of a standard n -simplex.

4.1. **The spine of an n -simplex.** The **spine** of the n -simplex Δ^n is the simplicial set I^n defined by

$$(I^n)_k = \{ \langle a_0 \cdots a_k \rangle \in (\Delta^n)_k \mid a_k \leq a_0 + 1 \}.$$

That is, a k -dimensional element of I^n is a simplicial operator $a: [k] \rightarrow [n]$ whose image is of the form either $\{j\}$ or $\{j, j+1\}$. The action of simplicial operators on elements of I^n is induced by their action on Δ^n . (To see that this action is well defined, observe that for $a: [k] \rightarrow [n]$ in $(I^n)_k$ and $f: [p] \rightarrow [k]$, the image of the simplicial operator af is contained in the image of a .)

There is an evident injective map $I^n \rightarrow \Delta^n$ of simplicial sets. (In fact, I^n is another example of a *subcomplex* of Δ^n , see below §4.9.) Here is a picture of I^3 in Δ^3 :



Note that $I^0 = \Delta^0$ and $I^1 = \Delta^1$.

The key property of the spine is the following.

4.2. **Proposition.** *Given a simplicial set X , for every $n \geq 0$ there is a bijection*

$$\text{Hom}(I^n, X) \xrightarrow{\sim} \{ (a_1, \dots, a_n) \in (X_1)^{\times n} \mid a_i \langle 1 \rangle = a_{i+1} \langle 0 \rangle \},$$

defined by sending $f: I^n \rightarrow X$ to $(f(\langle 01 \rangle), f(\langle 12 \rangle), \dots, f(\langle n-1, n \rangle))$. (In the case $n = 0$, the target of the bijection is taken to be the set X_0 of vertices of X , and the bijection in this case sends $f \mapsto f(\langle 0 \rangle)$.)

We will give the proof at the end of this section, after we describe I^n as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices “glued” together at their ends.

4.3. **Nerves are characterized by unique spine extensions.** We can now state our new characterization of nerves: they are simplicial sets such that every map $I^n \rightarrow X$ from a spine extends *uniquely* along $I^n \subseteq \Delta^n$ to a map from the standard n -simplex. That is, nerves are precisely the simplicial sets with “unique spine extensions”.

4.4. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if the restriction map $\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(I^n, X)$ along $I^n \subseteq \Delta^n$ is a bijection for all $n \geq 2$.*

Proof. Immediate from (4.2) and (3.8). □

4.5. **Colimits of sets and simplicial sets.** Given any functor $F: C \rightarrow \text{Set}$ from a small category to sets, there is a “simple formula” for its colimit. First consider the coproduct (i.e., disjoint union) $\coprod_{c \in \text{ob } C} F(c)$ of the values of the functor; I’ll write (c, x) for a typical element of this coproduct, with $c \in \text{ob } C$ and $x \in F(c)$. Consider the relation \sim on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists \alpha: c \rightarrow c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Define

$$X := \left(\coprod_{c \in \text{ob } C} F(c) \right) / \approx,$$

the set obtained as the quotient by the equivalence relation “ \approx ” which is *generated* by the relation “ \sim ”. For each object c of C we have a function $i_c: F(c) \rightarrow X$ defined by $i_c(x) := [(c, x)]$, sending x to the equivalence class of (c, x) . Then the data $(X, \{i_c\})$ is a colimit of the functor F : i.e., for any set S and collection of functions

$$f_c: F(c) \rightarrow S \quad \text{such that} \quad f_{c'} \circ F(\alpha) = f_c \quad \text{for all } \alpha: c \rightarrow c'$$

there exists a *unique* function $f: X \rightarrow S$ such that $f \circ i_c = f_c$.

4.6. *Example.* Verify that $(X, \{i_c\})$ is in fact a colimit of F .

We write $\text{colim}_C F$ for a chosen colimit of F .

Note that the relation “ \sim ” is often not itself an equivalence relation, so it can be difficult to figure out what “ \approx ” actually is: the simple formula may not be so simple in practice.

4.7. *Exercise.* If C is a groupoid, then \sim is always an equivalence relation.

There are cases when things are more tractable.

4.8. **Proposition.** *Let \mathcal{A} be a collection of subsets of a set S , which is a partially ordered set under “ \subseteq ” and thus can be regarded as a category. Suppose \mathcal{A} has the following property: for all $s \in S$, and $T, U \in \mathcal{A}$ such that $s \in T \cap U$, there exists $V \in \mathcal{A}$ such that $s \in V \subseteq T \cap U$. Then the tautological map*

$$\text{colim}_{T \in \mathcal{A}} T \rightarrow \bigcup_{T \in \mathcal{A}} T$$

(sending $[(T, t)] \mapsto t$) is a bijection.

Proof sketch. Show that $(T, t) \approx (T', t')$ if and only if $t = t'$. □

Note: an easy way to satisfy the hypothesis of (4.8) is to show that \mathcal{A} is closed under finite intersection.

4.9. **Subcomplexes.** Given a simplicial set X , a **subcomplex** is just a subfunctor of X ; i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \rightarrow X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when A is a subcomplex of X .

4.10. *Example.* Examples we have already seen include the spines $I^n \subseteq \Delta^n$ and the $\Delta^S \subseteq \Delta^n$ associated to subsets $S \subseteq [n]$.

4.11. *Exercise.* For any map $f: X \rightarrow Y$ of simplicial sets, the image $f(X) \subseteq Y$ of f is a subcomplex of Y .

For every set S of elements of a simplicial set, there is a smallest subcomplex which contains the set, namely the intersection of all subcomplexes containing S .

4.12. *Example.* For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains x . This subcomplex has exactly one n -dimensional element for each $n \geq 0$, namely $x\langle 0 \cdots 0 \rangle$, and thus is isomorphic to Δ^0 .

More generally, for a collection of vertices $a, b, c, \dots \in X_0$, we write $\{a, b, c, \dots\} \subseteq X$ for the smallest subcomplex which contains a, b, c, \dots . This subcomplex is a discrete simplicial set. This choice of notation is supported by our informal identification of discrete sets with sets (2.5).

The result (4.8) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes.

4.13. Proposition. *Let \mathcal{A} be a collection of subcomplexes of a simplicial set X , which is a partially ordered set under “ \subseteq ” and thus can be regarded as a category. Suppose \mathcal{A} has the following property: for all $n \geq 0$, all $x \in X_n$, and all $K, L \in \mathcal{A}$ such that $x \in K_n \cap L_n$, there exists $M \in \mathcal{A}$ such that $x \in M_n$ and $M \subseteq K \cap L$. Then the tautological map*

$$\operatorname{colim}_{K \in \mathcal{A}} K \rightarrow \bigcup_{K \in \mathcal{A}} K$$

is a bijection.

Proof. Because simplicial sets are actually functors $\Delta^{\text{op}} \rightarrow \text{Set}$, colimits in simplicial sets are “computed degreewise”. That is, if $F: C \rightarrow \text{sSet}$ is a functor with colimit $Y = \operatorname{colim}_{c \in C} F(c) \in \text{sSet}$, then for each $n \geq 0$ there is a canonical bijection

$$Y_n \approx \operatorname{colim}_{c \in C} F(c)_n.$$

The proposition follows using this observation and (4.8). □

4.14. Remark (Pushouts of subcomplexes). A special case of (4.13) applied to simplicial sets which we will use constantly is the following. If K and L are subcomplexes of a simplicial set X , then so are both $K \cap L$ and $K \cup L$, and furthermore the evident commutative square

$$\begin{array}{ccc} K \cap L & \longrightarrow & L \\ \downarrow & & \downarrow \\ K & \longrightarrow & K \cup L \end{array}$$

is a pushout square in simplicial sets. (Proof: $\mathcal{A} = \{K, L, K \cap L\}$.)

4.15. Subcomplexes of Δ^n . For each $S \subseteq [n]$ we have a subcomplex $\Delta^S \subseteq \Delta^n$. The following says that every subcomplex of Δ^n is a union of Δ^S s.

4.16. Lemma. *Let $K \subseteq \Delta^n$ be a subcomplex. If $(f: [m] \rightarrow [n]) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.*

This the proof uses the following elementary fact.

4.17. Lemma. *Any order preserving surjection $f: S \rightarrow T$ between finite totally ordered sets admits an order preserving section, i.e., $s: T \rightarrow S$ such that $fs = \text{id}_T$.*

Proof. Let $s(t) = \min \{s \in S \mid f(s) = t\}$. □

Proof of (4.16). Given an order preserving map $g: [k] \rightarrow S$ a dotted arrow s exists making

$$\begin{array}{ccc} & [k] & \\ \swarrow s & \downarrow g & \searrow \bar{g} \\ [m] & \xrightarrow{f_{\text{surj}}} S & \xrightarrow{f_{\text{inj}}} [n] \end{array}$$

commute by (4.17). Therefore any element $\bar{g} \in (\Delta^S)_k$ is an element $\bar{g} = fs \in K_k$, and it is clear that $f \in (\Delta^S)_m$. □

4.18. Remark. Thus, a subcomplex $K \subseteq \Delta^n$ determines and is determined by a collection \mathcal{K} of subsets of $[n]$ with the property that $T \subseteq S$ and $S \in \mathcal{K}$ implies $T \in \mathcal{K}$: namely,

$$\mathcal{K} = \{S \subseteq [n] \mid \Delta^S \subseteq K\} \quad \text{and} \quad K = \bigcup_{S \in \mathcal{K}} \Delta^S.$$

In other words, a subcomplex of Δ^n is the “same thing” as an *abstract simplicial complex* whose vertex set is a subset of $[n]$.

We can sharpen the above: every subcomplex of Δ^n is a *colimit* of subcomplexes Δ^S .

4.19. Proposition. *Let $K \subseteq \Delta^n$ be a subcomplex. Let \mathcal{A} be the poset of all non-empty subsets $S \subseteq [n]$ such that the inclusion map $f: S \rightarrow [n]$ represents a $(|S| - 1)$ -dimensional element of K . Then the tautological map*

$$\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \rightarrow K$$

is an isomorphism.

Proof. We must show that for each $m \geq 0$, the map $\operatorname{colim}_{S \in \mathcal{A}} (\Delta^S)_m \rightarrow K_m$ is a bijection. Each $(\Delta^S)_m = \{f: [m] \rightarrow [n] \mid f([m]) \subseteq S\}$ is a distinct subset of $K_m \subseteq (\Delta^n)_m$; i.e., $S \neq S'$ implies $(\Delta^S)_m \neq (\Delta^{S'})_m$. In view of (4.13), it suffices to show that for each $f \in K_m$ there is a minimal S in \mathcal{A} such that $f \in (\Delta^S)_m$. This is immediate from (4.16), which says that $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$ where $S = f([m])$, and it is obvious that this S is minimal with this property. \square

4.20. Proof of (4.2). Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

Proof of (4.2). Let \mathcal{A} be the poset of all non-empty $S \subseteq [n]$ which correspond to elements of I^n ; i.e., subsets of $[n]$ of the form $\{j\}$ or $\{j, j + 1\}$. Explicitly the poset \mathcal{A} has the form

$$\{0\} \rightarrow \{0, 1\} \leftarrow \{1\} \rightarrow \{1, 2\} \leftarrow \{2\} \rightarrow \cdots \leftarrow \{n - 1\} \rightarrow \{n - 1, n\} \leftarrow \{n\}.$$

By (4.19), $\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \rightarrow I^n$ is an isomorphism. Thus $\operatorname{Hom}(I^n, X) \approx \operatorname{Hom}(\operatorname{colim}_{S \in \mathcal{A}} \Delta^S, X) \approx \lim_{S \in \mathcal{A}} \operatorname{Hom}(\Delta^S, X)$, and an elementary argument gives the result. \square

5. HORNS AND INNER HORNS

We now are going to give another (less obvious!) characterization of nerves, in terms of “extending inner horns”, rather than “extending spines”. It will be this characterization that we “weaken” to obtain the definition of a quasicategory. W 16 Jan 2019

5.1. Definition of horns. We define a collection of subobjects of the standard simplices, called “horns”. For each $n \geq 1$, these are subsimplicial sets $\Lambda_j^n \subset \Delta^n$ for each $0 \leq j \leq n$. The **horn** Λ_j^n is the subcomplex of Δ^n defined by

$$(\Lambda_j^n)_k = \{f: [k] \rightarrow [n] \mid ([n] \setminus \{j\}) \not\subseteq f([k])\}.$$

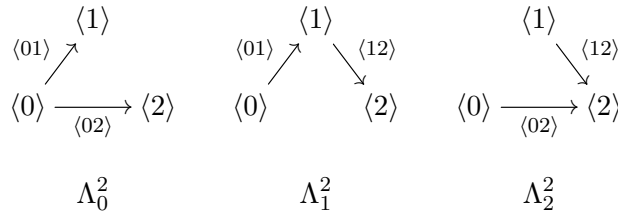
Using the fact (4.19) that subcomplexes of Δ^n are always unions of Δ^S s, we see that Λ_j^n is the union of “faces” $\Delta^{[n] \setminus i}$ of Δ^n other than the j th face:

$$\Lambda_j^n = \bigcup_{i \neq j} \Delta^{[n] \setminus i} \subset \Delta^n.$$

When $0 < j < n$ we say that $\Lambda_j^n \subset \Delta^n$ is an **inner horn**. We also say it is a **left horn** if $j < n$ and a **right horn** if $0 < j$.

5.2. Example. The horns inside Δ^1 are just the vertices viewed as subobjects: $\Lambda_0^1 = \Delta^{\{0\}} = \{0\} \subset \Delta^1$ and $\Lambda_1^1 = \Delta^{\{1\}} = \{1\} \subset \Delta^1$. Neither is an inner horn, the first is a left horn, and the second is a right horn.

5.3. Example. These are the three horns inside the 2-simplex.



Only Λ_1^2 is an inner horn, while Λ_0^2 and Λ_1^2 are left horns, and Λ_1^2 and Λ_2^2 are right horns. Note that Λ_1^2 is the same as the spine I^2 .

5.4. *Exercise.* Visualize the four horns inside the 3-simplex. The simplicial set Λ_j^3 actually kind of looks like a horn: you blow into the vertex $\langle j \rangle$, and sound comes out of the opposite missing face $\Delta^{[3] \setminus j}$.

5.5. *Exercise.* Show that Λ_j^n is the largest subobject of Δ^n which does not contain the element $\langle 0 \cdots \hat{j} \cdots n \rangle \in (\Delta^n)_{n-1}$, the “face opposite the vertex j ”.

We note that inner horns always contain spines: $I^n \subseteq \Lambda_j^n$ if $0 < j < n$. This is also true for non-inner horns if $n \geq 3$, but not for non-inner horns with $n = 1$ or $n = 2$.

5.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit “unique inner horn extensions”; this is different than, but analogous to, the characterization in terms of unique spine extensions (4.4).

5.7. **Proposition.** *A simplicial set X is isomorphic to the nerve of a category, if and only if $\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_j^n, X)$ is a bijection for all $n \geq 2$, $0 < j < n$.*

The proof will take up the rest of the section.

5.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

5.9. **Proposition.** *If C is a category, then for every inner horn $\Lambda_j^n \subset \Delta^n$ the evident restriction map*

$$\text{Hom}(\Delta^n, NC) \rightarrow \text{Hom}(\Lambda_j^n, NC)$$

is a bijection.

Proof. Since inner horns contain spines, we can consider restriction along $I^n \subseteq \Lambda_j^n \subseteq \Delta^n$. The composite

$$\text{Hom}(\Delta^n, NC) \rightarrow \text{Hom}(\Lambda_j^n, NC) \xrightarrow{r} \text{Hom}(I^n, NC)$$

of restriction maps is a bijection (4.4), so r is a surjection. Thus, it suffices to show that r is injective. This is immediate when $n = 2$, since $\Lambda_1^2 = I^2$, so we can assume $n \geq 3$.

We will show that for any inner horn Λ_j^n with $n \geq 3$ there exists a finite chain

$$I^n = F_0 \subset F_1 \subset \cdots \subset F_d = \Lambda_j^n$$

of subcomplexes, together with a list of subsets $S_1, \dots, S_d \subset [n]$, such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $I^{S_i} \subseteq F_{i-1} \cap \Delta^{S_i}$; here I^{S_i} denotes the spine of Δ^{S_i} . Given this, we see by (4.14) that F_i is isomorphic to a pushout:

$$F_i \approx \text{colim}(F_{i-1} \leftarrow F_{i-1} \cap \Delta^{S_i} \rightarrow \Delta^{S_i}).$$

We then obtain a commutative diagram of sets

$$\begin{array}{ccccc} \text{Hom}(F_i, NC) & \xrightarrow{b} & \text{Hom}(F_{i-1}, NC) & & \\ \downarrow & & \downarrow & & \\ \text{Hom}(\Delta^{S_i}, NC) & \xrightarrow{a} & \text{Hom}(F_{i-1} \cap \Delta^{S_i}, NC) & \longrightarrow & \text{Hom}(I^{S_i}, NC) \end{array}$$

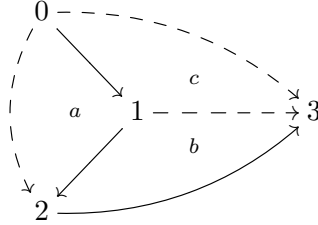
where all maps are induced by restriction, in which the square is a pullback (because F_i is a pushout), and such that the horizontal composition on the bottom is a bijection. It immediately follows that a , and hence b , are injective. We can thus conclude that $\text{Hom}(\Lambda_j^n, NC) \rightarrow \text{Hom}(I^n, NC)$ is injective as desired, since it is a composite of injective functions such as b .

Now we prove the claim about the filtration of Λ_j^n by suitable subcomplexes F_i .

When $n = 3$, we can “attach” simplices in order explicitly:

$$\Lambda_1^3 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,1,3\}}, \quad \Lambda_2^3 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,2,3\}}.$$

Note that, for instance, in building Λ_1^3 , we must add $\Delta^{\{0,1,3\}}$ after adding $\Delta^{\{1,2,3\}}$, so that the spine $I^{\{0,1,3\}}$ of $\Delta^{\{0,1,3\}}$ is already present.



When $n \geq 4$, we have that $(\Lambda_j^n)_1 = (\Delta^n)_1$ and $(\Lambda_j^n)_2 = (\Delta^n)_2$. The procedure to “build” Λ_j^n from I^n by adding subsimplices is: (1) first attach 2-simplices one at a time, in an allowable order; then (2) attach all needed higher dimensional subsimplices. In step (2) the order doesn’t matter since all 1-simplices (and hence all spines) are present in what has already been built. We leave the details of step (1) to the reader. \square

5.10. Nerves are characterized by unique inner horn extension. Let X be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each $\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_j^n, X)$ is a bijection for all $0 < j < n$ with $n \geq 2$.

Considering the unique extensions along $\Lambda_1^2 \subset \Delta^2$, we see that this defines a “composition law” on the set X_1 . That is, given $f, g \in X_1$ such that $f_1 = g_0$ in X_0 ,⁴ there is a unique map u

$$u: \Lambda_1^2 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X, \quad \langle 01 \rangle \mapsto f \in X_1, \langle 12 \rangle \mapsto g \in X_1.$$

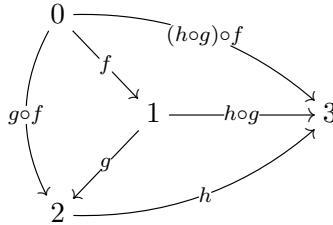
Let $\tilde{u}: \Delta^2 \rightarrow X$ be the *unique* extension of u along $\Lambda_1^2 \subset \Delta^2$, and define

$$g \circ f := \tilde{u}_{02}.$$

The 2-dimensional element \tilde{u} is uniquely characterized by: $\tilde{u}_{01} = f$, $\tilde{u}_{12} = g$, $\tilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x\langle 00 \rangle \in X_1$, so that $(1_x)_0 = x = (1_x)_1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider the 2-dimensional elements $f\langle 011 \rangle, g\langle 001 \rangle \in X_2$, and use the fact that their representing maps $\Delta^2 \rightarrow X$ are the unique extensions of their restrictions to $\Lambda_1^2 \subset \Delta^2$.)

Now consider $\Lambda_1^3 \subset \Delta^3$. Recall (4.19) that Λ_1^3 is a union (and colimit) of $\Delta^S \subseteq \Delta^3$ such that $S \not\supseteq \{0, 2, 3\}$. A map $\Lambda_1^3 \rightarrow X$ can be pictured as



so that the planar 2-cells in the picture correspond to non-degenerate 2-dimensional elements of Δ^3 which are contained in Λ_1^3 , while the edges are labelled according to their images in X , using the composition law defined above. Let $v: \Delta^3 \rightarrow X$ be any extension of the above picture along

⁴Recall that $f_1 = f(1)$ and $g_0 = g(0)$, regarded as maps $\Delta^0 \rightarrow X$ and thus as elements of X_0 , using the notation discussed in §2.3.

$\Lambda_1^3 \subset \Delta^3$, and consider the restriction $w := v\langle 023 \rangle: \Delta^2 \rightarrow X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of w demonstrates that

$$h(g \circ f) = (h \circ g)f.$$

In other words, the *existence* of extensions along $\Lambda_1^3 \subset \Delta^3$ implies that the composition law we defined above is associative. (We could carry out this argument using $\Lambda_2^3 \subset \Delta^3$ instead.)

Thus, given an X with unique inner horn extensions, we can construct a category C , so that objects of C are elements of X_0 , morphisms of C are elements of X_1 , and composition is given as above.

Next we construct a map $X \rightarrow NC$ of simplicial sets. There are obvious maps $\alpha_n: X_n \rightarrow (NC)_n$, corresponding to restriction along spines $I^n \subseteq \Delta^n$; i.e., $\alpha(x) = (x_{01}, \dots, x_{n-1,n})$. These maps are compatible with simplicial operators, so that they define a map $\alpha: X \rightarrow NC$ of simplicial sets. *Proof:* For any n -dimensional element $x \in X_n$, all of its edges are determined by edges on its spine via the composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \dots \circ x_{i,i+1}$, for all $0 \leq i \leq j \leq n$. Thus for $f: [m] \rightarrow [n]$ we have $\alpha(xf) = ((xf)_{01}, \dots, (xf)_{n-1,n}) = (x_{f_0 f_1}, \dots, x_{f_{n-1} f_n}) = (x_{01}, \dots, x_{n-1,n})_{f_0 \dots f_n} = (\alpha x)f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

Proof of (5.7). We have already shown (5.9) that nerves have unique extensions for inner horns. Consider a simplicial set X which has unique inner horn extension. By the discussion above, we obtain a category C and a map $\alpha: X \rightarrow NC$ of simplicial sets, which is clearly a bijection in degrees ≤ 2 . We will show $\alpha_n: X_n \rightarrow (NC)_n$ is bijective by induction on n .

Fix $n \geq 3$, and consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(\Delta^n, X) & \xrightarrow{\sim} & \mathrm{Hom}(\Lambda_1^n, X) \\ \alpha_{\Delta^n} \downarrow & & \downarrow \alpha_{\Lambda_1^n} \\ \mathrm{Hom}(\Delta^n, NC) & \xrightarrow{\sim} & \mathrm{Hom}(\Lambda_1^n, NC) \end{array}$$

The horizontal maps are induced by restriction, and are bijections (top by hypothesis, bottom by (5.9)). Because Λ_1^n is a colimit of standard simplices of dimension $< n$ (4.19), the map $\alpha_{\Lambda_1^n}$ is a bijection by the induction hypothesis. Therefore so is α_{Δ^n} . \square

6. QUASICATEGORIES

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

6.1. Identifying categories with their nerves. From this point on, I will (at least informally) often not distinguish a category C from its nerve. In particular, I may assert something like “let C be a simplicial set which is a category”, which should be read as “ C is a simplicial set which is isomorphic to the nerve of some category”. This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of Cat into sSet (3.10).

6.2. Definition of quasicategory. A **quasicategory** is a simplicial set C such that for every map $f: \Lambda_j^n \rightarrow C$ from an inner horn, there *exists* an extension of it to $g: \Delta^n \rightarrow C$. That is, C is a quasicategory if the function $\mathrm{Hom}(\Delta^n, C) \rightarrow \mathrm{Hom}(\Lambda_j^n, C)$ induced by restriction along $\Lambda_j^n \subset \Delta^n$ is surjective for all $0 < j < n$, $n \geq 2$, so there always exists a dotted arrow in any commutative diagram of the form

Any category (more precisely, the nerve of any category) is a quasicategory. In fact, by what we have shown (5.7) a category is precisely a quasicategory for which there exist *unique* extensions of inner horns.

Let C be a quasicategory. We refer to elements of C_0 as the **objects** of C , and elements of C_1 as the **morphisms** of C . Every morphism $f \in C_1$ has a **source** and **target**, namely its vertices

$f_0, f_1 \in C_0$. We write $f: f_0 \rightarrow f_1$, just as we would for morphisms in a category. Likewise, for every object $x \in C_0$, there is a distinguished morphism $1_x: x \rightarrow x$, called the **identity morphism**, defined by $1_x = x_{00}$. When C is (the nerve of) a category, all the above notions coincide with the usual ones. (Note, however, that we cannot generally define *composition* of morphisms in a quasicategory in the same way we do for a category.)

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

6.3. Products of quasicategories. Simplicial sets are functors, so the product of simplicial sets X and Y is just the product of the functors. Thus, $(X \times Y)_n = X_n \times Y_n$.

6.4. Proposition. *The product of two quasicategories (as simplicial sets) is a quasicategory.*

Proof. Exercise, using the bijective correspondence between the sets of (i) maps $K \rightarrow X \times Y$ and (ii) pairs of maps $(K \rightarrow X, K \rightarrow Y)$. \square

6.5. Exercise. If C and D are categories, then $N(C \times D) \approx NC \times ND$. Thus, the notion of product of quasicategories generalizes that of categories.

6.6. Coproducts of quasicategories. The coproduct of simplicial sets X and Y is just the coproduct of functors, whence $(X \amalg Y)_n = X_n \amalg Y_n$, i.e., the set n -dimensional elements of the coproduct is the disjoint union of the sets of n -dimensional elements of X and Y . More generally, $(\coprod_s X_s)_n = \coprod_s (X_s)_n$ for an indexed collection $\{X_s\}$ of simplicial sets.

6.7. Proposition. *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set X , define an equivalence relation \approx on the set $\coprod_{n \geq 0} X_n$ of elements of X , generated by the relation

$$a \approx af \quad \text{for all } n \geq 0, a \in X_n, f: [m] \rightarrow [n].$$

Thus two elements are related when you can get from one to another by a sequence of simplicial operators. An equivalence class for \approx is called a **connected component** of X , and we write $\pi_0 X$ for the set of connected components. This construction defines a functor $\pi_0: \mathit{sSet} \rightarrow \mathit{Set}$.

6.8. Exercise (Connected components are path components). Show that there is a canonical bijection

$$(X_0 / \approx_1) \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the set of equivalence classes in the vertex set X_0 with respect to the equivalence relation \approx_1 which is *generated* by the relation \sim_1 on X_0 , defined by

$$a \sim_1 b \quad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.$$

6.9. Exercise. Show that there is a bijection $\text{colim}_{\Delta^{\text{op}}} X \xrightarrow{\sim} \pi_0 X$, between the set of connected components of X and the colimit of the functor $X: \Delta^{\text{op}} \rightarrow \mathit{Set}$.

6.10. Exercise (Connected components respect colimits). Show that if X is the colimit of a functor $F: D \rightarrow \mathit{sSet}$ of some small category D , then $\pi_0 X \approx \text{colim}_D \pi_0 F$. In particular, $\pi_0(\coprod_s X_s) \approx \coprod_s \pi_0(X_s)$ for any collection $\{X_s\}$ of simplicial sets.

We say that a simplicial set X is **connected** if $\pi_0 X$ is a singleton.

6.11. Exercise. Show that every standard simplex Δ^n is connected, and that every horn Λ_j^n is connected.

6.12. Exercise (Every simplicial set is a coproduct of its connected components). Let X be a simplicial set. Given $a \in \pi_0 X$, let C_a denote its equivalence class (regarded as a subset of the set $\coprod_{n \geq 0} X_n$ of elements).

- (1) Show that C_a is closed under the action of simplicial operators, and thus describes a subcomplex of X .
- (2) Show that the evident map

$$\coprod_{a \in \pi_0 X} C_a \rightarrow X$$

is an isomorphism of simplicial sets.

Proof of (6.7). If $X = \coprod_s X_s$ is a coproduct of simplicial sets, then any connected component of X must be contained in exactly one of the X_s summands, by (6.10). The proof is now straightforward, using (6.12) and the fact that horns and standard simplices are connected (6.11). \square

6.13. *Exercise* (Important). Show that the evident map $\pi_0(X \times Y) \rightarrow \pi_0 X \times \pi_0 Y$ is a bijection.

6.14. **Full subquasicategories.** Given a category C and a set of objects $S \subseteq \text{ob } C$, the *full subcategory spanned by S* is the subcategory $C' \subseteq C$ with $\text{ob } C' = S$ and with $\text{mor } C' = \{f \in \text{mor } C \mid \text{source}(f), \text{target}(f) \in S\}$.

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This has a straightforward generalization to quasicategories. Given a simplicial set C and a set $S \subseteq C_0$ of vertices, let

$$C'_n = \{a \in C_n \mid a_j \in S \text{ for all } j = 0, \dots, n\},$$

the set of n -dimensional elements whose vertices are in S .

6.15. *Exercise.* Show that C' is a subcomplex of C , and that if C is a quasicategory then so is C' .

When C is a quasicategory, the subcomplex C' is called the **full subcategory spanned by S** . (Note: it would be more logical to say “subquasicategory”, but it is a mouthful. When constructing new terminology like this we often leave out the “quasi” if it won’t cause confusion.)

6.16. **Opposite of a quasicategory.** Given a category C , the *opposite* category C^{op} has $\text{ob } C^{\text{op}} = \text{ob } C$, and $\text{Hom}_{C^{\text{op}}}(x, y) = \text{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\text{op}}} f = f \circ_C g$.

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution $\text{op}: \Delta \rightarrow \Delta$ of the category Δ . This is the functor which on objects sends $[n] \mapsto [n]$, and on morphisms sends $\langle f_0, \dots, f_n \rangle: [n] \rightarrow [m]$ to $\langle m - f_n, \dots, m - f_0 \rangle$.

6.17. *Remark.* You can visualize this involution as the functor which “reverses the ordering” of the totally-ordered sets $[n]$. Note that the totally ordered set “[n] with the order of its elements reversed” isn’t actually an object of Δ , but rather is uniquely isomorphic to $[n]$, via the function $x \mapsto n - x$.

The **opposite** of a simplicial set $X: \Delta^{\text{op}} \rightarrow \text{Set}$ is the composite functor $X^{\text{op}} := X \circ \text{op}$. We have that $(\Delta^n)^{\text{op}} = \Delta^n$, while $(\Lambda_j^n)^{\text{op}} = \Lambda_{n-j}^n$, so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that $(NC)^{\text{op}} = N(C^{\text{op}})$, so the notion of opposite quasicategory generalizes the notion of opposite category. The functor $\text{op}: \Delta \rightarrow \Delta$ satisfies $\text{op} \circ \text{op} = \text{id}_\Delta$, so $(X^{\text{op}})^{\text{op}} = X$.

7. FUNCTORS AND NATURAL TRANSFORMATIONS

7.1. **Functors.** A **functor** between quasicategories is merely a map $f: C \rightarrow D$ between the simplicial sets.

We write qCat for the category of quasicategories and functors between them.⁵ Clearly $\text{qCat} \subset \text{sSet}$ is a full subcategory. Because the nerve functor is a full embedding of Cat into qCat , any functor between ordinary categories is also a functor between quasicategories.

7.2. *Exercise* (Mapping property of a full subcategory). Let C be a quasicategory, and $C' \subseteq C$ the full subcategory spanned by some subset $S \subseteq C_0$. Show that a functor $f: D \rightarrow C$ factors through a functor $f': D \rightarrow C' \subseteq C$ if and only if $f(D_0) \subseteq S$.

⁵Lurie [Lur09] denotes this category by Cat_Δ .

7.3. Natural transformations. Given functors $F, G: C \rightarrow D$ between categories, a *natural transformation* $\phi: F \Rightarrow G$ is a choice, for each object c of C , of a map $\phi(c): F(c) \rightarrow G(c)$ in D , such that for every morphism $\alpha: c \rightarrow c'$ in C the square

$$\begin{array}{ccc} F(c) & \xrightarrow{\phi(c)} & G(c) \\ f(\alpha) \downarrow & & \downarrow g(\alpha) \\ F(c') & \xrightarrow{\phi(c')} & G(c') \end{array}$$

commutes in D .

There is a standard convenient reformulation of this: a natural transformation $\phi: F \Rightarrow G$ is the same thing as a functor

$$H: C \times [1] \rightarrow D,$$

so that $H|C \times \{0\} = F$, $H|C \times \{1\} = G$, and $H|\{c\} \times [1] = \alpha(c)$ for each $c \in \text{ob } C$. (Here we make implicit use of the evident isomorphisms $C \times \{0\} \approx C \approx C \times \{1\}$.)

This reformulation admits a straightforward generalization to quasicategories. A **natural transformation** $f: f_0 \Rightarrow f_1$ of functors $f_0, f_1: C \rightarrow D$ between quasicategories is defined to be a map

$$f: C \times N[1] = C \times \Delta^1 \rightarrow D$$

of simplicial sets such that $f|C \times \{i\} = f_i$ for $i = 0, 1$. For ordinary categories this coincides with the classical notion.

8. EXAMPLES OF QUASICATEGORIES

There are many ways to produce quasicategories, as we will see. Unfortunately, “hands-on” constructions of quasicategories are relatively rare. Here I give a few reasonably explicit examples to play with.

8.1. Large vs. small. I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category C to be a simplicial set, we need $C_0 = \text{ob } C$ to be a set.

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a “proper class”, although for any pair of objects $\text{Hom}_C(X, Y)$ is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called **large** categories.

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) “universes”. A universe U is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe U , there is a larger universe U' such that $U \in U'$. Thus, if by “set” we mean “ U -set”, then the category Set is a “ U' -category”. This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals*.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the elements form a set), while the nerve of a large category is a large simplicial set.

I’m not going to be pedantic about this. I’ll usually assume categories like Set , Cat , sSet , etc., are categories whose objects are “small” sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe U of “small sets”. However, I sometimes need to consider examples of

sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions limits and colimits. Many typical examples of categories $C = \text{Set}, \text{Cat}, \text{sSet}, \text{etc.}$, in which objects are built out of small sets are **small complete** and **small cocomplete**: any functor $F: D \rightarrow C$ from a *small* category D has a limit and a colimit in C . This is *not* true if D is not assumed to be small. In this case care about the small/large distinction is necessary.

8.2. The Morita quasicategory. This is an example of a quasicategory in which *objects* are associative rings, *morphisms* between two rings are bimodules for the pair of rings, and *2-dimensional elements* are given by certain isomorphisms of bimodules.

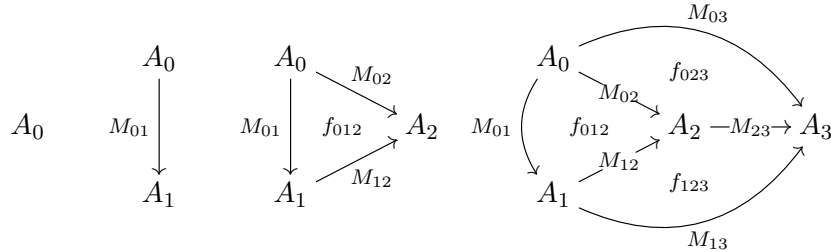
Define a simplicial set C , so that C_n is a set whose elements are data $x := (A_i, M_{ij}, f_{ijk})$, where

- for each $i \in [n]$, A_i is an associative ring,
- for each $i < j$ in $[n]$, M_{ij} is an (A_i, A_j) -bimodule,
- for each $i < j < k$ in $[n]$, $f_{ijk}: M_{ij} \otimes_{A_j} M_{jk} \rightarrow M_{ik}$ is an isomorphism of (A_i, A_k) -bimodules, such that
- for each $i < j < k < \ell$, the diagram

$$(8.3) \quad \begin{array}{ccc} M_{ij} \otimes M_{jk} \otimes M_{k\ell} & \xrightarrow{\text{id} \otimes f_{j k \ell}} & M_{ij} \otimes M_{j\ell} \\ f_{ijk} \otimes \text{id} \downarrow & & \downarrow f_{ij\ell} \\ M_{ik} \otimes M_{k\ell} & \xrightarrow{f_{ik\ell}} & M_{i\ell} \end{array}$$

commutes.

Here is a picture of the data of an n -simplex for $n \in \{0, 1, 2, 3\}$:



For an simplicial operator $\delta: [m] \rightarrow [n]$, we define $x\delta := (A_{\delta(i)}, M_{\delta(i)\delta(j)}, f_{\delta(i)\delta(j)\delta(k)})$. When δ is injective this stands as it is, but if δ is not injective, we must set $M_{ij} := A_{\delta(i)}$ when $\delta(i) = \delta(j)$, and set f_{ijk} to the canonical isomorphism $A_{\delta(i)} \otimes_{A_{\delta(j)}} M_{\delta(j)\delta(k)} \rightarrow M_{\delta(i)\delta(k)}$ if $\delta(i) = \delta(j)$ or $M_{\delta(i)\delta(j)} \otimes_{A_{\delta(j)}} A_{\delta(k)} \rightarrow M_{\delta(i)\delta(k)}$ if $\delta(j) = \delta(k)$.

I claim that C is a quasicategory. Fillers for $\Lambda_1^2 \subset \Delta^2$ always exist: a map $\Lambda_1^2 \rightarrow C$ is a choice of $(A_0, M_{01}, A_1, M_{12}, A_2)$, and an extension to Δ^2 can be given by setting M_{02} to be the tensor product, and f_{012} the identity map. Note that there can be more than one choice: even keeping M_{02} the same, there is a choice of isomorphism f_{012} .

Fillers for $\Lambda_1^3 \subset \Delta^3$ and $\Lambda_2^3 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $f_{023} = f_{ik\ell}$ (for Λ_1^3) or $f_{013} = f_{ij\ell}$ (for Λ_2^3) making (8.3) commute, and such choices are unique. Similarly, all fillers in higher dimensions $\Lambda_j^n \subset \Delta^n$ with $n \geq 4$ exist and are unique.

8.4. Quasicategory of categories. Define a simplicial set C so that C_n is a set whose elements are data $x := (C_i, F_{ij}, \phi_{ijk})$ where

- for each $i \in [n]$, C_i is a (small) category,
- for each $i < j$ in $[n]$, $F_{ij}: C_i \rightarrow C_j$ is a functor,

- for each $i < j < k$ in $[n]$, $\phi_{ijk}: F_{jk}F_{ij} \Rightarrow F_{ik}$ is a natural *isomorphism* of functors $C_i \rightarrow C_k$, such that
- for each $i < j < k < \ell$, the diagram

$$\begin{array}{ccc}
 F_{k\ell}F_{jk}F_{ij} & \xrightarrow{\phi_{jk\ell} \text{id}_{F_{ij}}} & F_{j\ell}F_{ij} \\
 \text{id}_{F_{k\ell}} \phi_{ijk} \downarrow & & \downarrow \phi_{ij\ell} \\
 F_{k\ell}F_{ik} & \xrightarrow{\phi_{ik\ell}} & F_{i\ell}
 \end{array}$$

commutes.

The action of simplicial operators is defined exactly as in the previous example, as is the proof that \mathcal{C} is a quasicategory.

8.5. Nerve of a crossed module. A **crossed module** is data (G, H, ϕ, ρ) , consisting of groups G and H , and homomorphisms $\phi: H \rightarrow G$ and $\rho: G \rightarrow \text{Aut}H$, such that

$$\phi(\rho(g)(h)) = g\phi(h)g^{-1}, \quad \rho(\phi(h))(h') = hh'h^{-1}, \quad \text{for all } g \in G, h, h' \in H.$$

(For instance: $G = H =$ the cyclic group of order 4, with $\phi(x) = x^2$ and ρ the non-trivial action.) From this we can construct a quasicategory (in fact, a “quasigroupoid”) much as in the last example: an n -simplex is data (g_{ij}, h_{ijk}) with $g_{ij} \in G$, $h_{ijk} \in H$, satisfying identities

$$g_{ij}g_{jk} = \phi(h_{ijk})g_{ik}, \quad h_{ijk}h_{ik\ell} = \rho(g_{ij})(h_{jkl})h_{ij\ell}.$$

8.6. Spans. (See [Bar14, §§2–3], where this is called the *effective Burnside ∞ -category*.) For each object $[n]$ of Δ , define $[n]^{\text{tw}}$ to be the category with

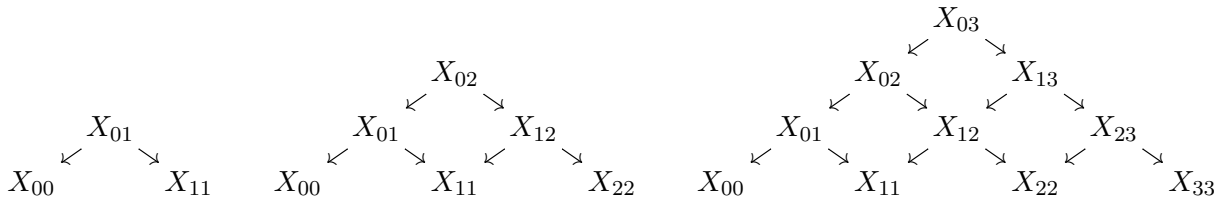
- **objects** pairs (i, j) with $0 \leq i \leq j \leq n$, and
- a unique **morphism** $(i, j) \rightarrow (i', j')$ whenever $i' \leq i \leq j \leq j'$.

The construction $[n] \mapsto [n]^{\text{tw}}$ defines a functor $\Delta \rightarrow \text{Cat}$. (The category $[n]^{\text{tw}}$ is called the *twisted arrow category* of $[n]$; in fact you can define a twisted arrow category C^{tw} for any category C .)

Let C be a category which has pullbacks; for an explicit example, think of the category of finite sets. Let $\mathcal{R}(C)$ be the simplicial set defined so that

$$\mathcal{R}(C)_n := \{\text{functors } ([n]^{\text{tw}})^{\text{op}} \rightarrow C\}.$$

Elements of $\mathcal{R}(C)_0$ are just objects of C . Elements of $\mathcal{R}(C)_1, \mathcal{R}(C)_2, \mathcal{R}(C)_3$ are respectively diagrams in C of shape



Let $\mathcal{A}(C)_n \subseteq \mathcal{R}(C)_n$ denote the subset whose n -dimensional elements are functors $X: ([n]^{\text{tw}})^{\text{op}} \rightarrow C$ such that for every $i' \leq i \leq j \leq j'$ the square

$$\begin{array}{ccc}
 X_{i'j'} & \longrightarrow & X_{ij'} \\
 \downarrow & & \downarrow \\
 X_{i'j} & \longrightarrow & X_{ij}
 \end{array}$$

is a pullback in C . Then $\mathcal{A}(C)$ is a subcomplex, and in fact is a quasicategory. This is another example in which extensions along inner horns $\Lambda_j^n \subset \Delta^n$ exist for $n \geq 2$, and are unique for $n \geq 3$.

8.7. **Singular complex of a space.** The **topological n -simplex** is

$$\Delta_{\text{top}}^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0 \right\},$$

the convex hull of the standard basis vectors. We get a functor $\Delta_{\text{top}}: \Delta \rightarrow \text{Top}$ by $\Delta_{\text{top}}([n]) = \Delta_{\text{top}}^n$. For a topological space T , we define $\text{Sing } T$ to be the simplicial set $[n] \mapsto \text{Hom}_{\text{Top}}(\Delta_{\text{top}}^n, T)$.

Define **topological horns**

$$(\Lambda_j^n)_{\text{top}} := \{ x \in \Delta_{\text{top}}^n \mid \exists i \in [n] \setminus \{j\} \text{ such that } x_i = 0 \} \subset \Delta_{\text{top}}^n,$$

and observe that continuous maps $(\Lambda_j^n)_{\text{top}} \rightarrow T$ correspond in a natural way with maps $\Lambda_j^n \rightarrow \text{Sing } T$. (*Exercise.* This is a consequence of the fact that Λ_j^n is a colimit of all the $\Delta^S \subseteq \Lambda_j^n$, and that $(\Lambda_j^n)_{\text{top}}$ is similarly a colimit of all the $\Delta_{\text{top}}^S \subseteq (\Lambda_j^n)_{\text{top}}$.) There exists a continuous retraction $\Delta_{\text{top}}^n \rightarrow (\Lambda_j^n)_{\text{top}}$ (*Exercise:* describe such a retraction), and thus we see that

$$\text{Hom}(\Delta^n, \text{Sing } T) \rightarrow \text{Hom}(\Lambda_j^n, \text{Sing } T)$$

is surjective for every horn (not just inner ones).

8.8. *Remark* (Kan complexes). A simplicial set X which has extensions for all horns is called a **Kan complex**. Thus, $\text{Sing } T$ is a Kan complex, and so in particular is a quasicategory (and as we will see below, a “quasigroupoid” (10.11)).

8.9. **Eilenberg-MacLane object.** Fix an abelian group A and an integer $d \geq 0$. We define a simplicial set $K = K(A, d)$, so that K_n is a set whose elements are data $a = (a_{i_0 \dots i_d})$ consisting of

- for each $0 \leq i_0 \leq \dots \leq i_d \leq n$, an element $a_{i_0 \dots i_d} \in A$, such that
- $a_{i_0 \dots i_d} = 0$ if $i_{u-1} = i_u$ for any u , and
- for each $0 \leq j_0 \leq \dots \leq j_{d+1} \leq n$ we have $\sum_u (-1)^u a_{j_0 \dots \widehat{j_u} \dots j_{d+1}} = 0$.

(Here “ $j_0 \dots \widehat{j_u} \dots j_{d+1}$ ” is shorthand for the subsequence $j_0, j_1, \dots, j_{u-1}, j_{u+1}, \dots, j_d, j_{d+1}$ with j_u omitted.)

For a map $\delta: [m] \rightarrow [n]$ we define

$$(a\delta)_{i_0 \dots i_d} = a_{\delta(i_0) \dots \delta(i_d)}.$$

The object $K(A, d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid). When $d = 0$, this is just a discrete simplicial set, equal to A in each dimension.

8.10. *Exercise.* Show that $K(A, 1)$ is isomorphic to the nerve of a category, namely the nerve of the group A regarded as a category with one object.

8.11. *Exercise.* Show that $K(A, d)$ is a Kan complex, i.e., that $\text{Hom}(\Delta^n, K(A, d)) \rightarrow \text{Hom}(\Lambda_j^n, K(A, d))$ is surjective for all horns $\Lambda_j^n \subset \Delta^n$. In fact, this map is *bijective* unless $n = d$. (Hint: there are four distinct cases to check, namely $n < d$, $n = d$, $n = d + 1$, and $n > d + 1$.)

8.12. *Exercise.* Given a simplicial set X , a **normalized d -cocycle** with values in an abelian group A is a function $f: X_d \rightarrow A$ such that

- (1) $f(x_{0, \dots, i, \dots, d-1}) = 0$ for all $x \in X_{d-1}$ and $0 \leq i \leq d-1$, and
- (2) $\sum (-1)^i f(x_{0, \dots, \widehat{i}, \dots, d+1}) = 0$ for all $x \in X_{d+1}$ and $0 \leq i \leq d+1$.

Show that the set $Z_{\text{norm}}^d(X; A)$ of normalized d -cocycles on X is in bijective correspondence with $\text{Hom}_{\text{sSet}}(X, K(A, d))$. (Hint: an element $a \in K_n$ is uniquely determined by the collection of elements $a\delta \in K_d = A$, as δ ranges over injective maps $[d] \rightarrow [n]$.)

8.13. *Remark.* Eilenberg-MacLane objects are an example of a *simplicial abelian group*: the map $+$: $K \times K \rightarrow K$ defined in each dimension by $(a+b)_{i_0 \dots i_d} = a_{i_0 \dots i_d} + b_{i_0 \dots i_d}$ is a map of simplicial sets which satisfies the axioms of an abelian group, reflecting the fact that $Z_{\text{norm}}^d(X; A)$ is an abelian group.

9. HOMOTOPY CATEGORY OF A QUASICATEGORY

Our next goal is to define the notion of an *isomorphism* in a quasicategory. This notion behaves much like that of *homotopy equivalence* in topology. We will define isomorphism by means of the *homotopy category* of a quasicategory. If we think of a quasicategory as “an ordinary category with higher structure”, then its homotopy category is the ordinary category obtained by “flattening out the higher structure”.

W 23 Jan 2019

9.1. The fundamental category of a simplicial set. The homotopy category of a quasicategory is itself a special case of the notion of the *fundamental category* of a simplicial set, which we turn to first.

A **fundamental category** for a simplicial set X consists of (i) a category hX , and (ii) a map $\alpha: X \rightarrow N(hX)$ of simplicial sets, such that for every category C , the map

$$\alpha^*: \text{Hom}(N(hX), NC) \rightarrow \text{Hom}(X, NC)$$

induced by restriction along α is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

9.2. Proposition. *Every simplicial set has a fundamental category.*

Proof sketch. Given X , we construct hX by generators and relations. First, consider the **free category** F , whose objects are the set X_0 , and whose morphisms are finite “composable” sequences $[a_n, \dots, a_1]$ of edges of X_1 . Thus, morphisms in F are “words”, whose “letters” are edges a_i with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n, \dots, a_1]$ is then a morphism $(a_1)_0 \rightarrow (a_n)_1$. (Note: we also suppose that there is an empty sequence $[\]_x$ in F for each vertex $x \in X_0$; these correspond to identity maps in F .)

Then hX is defined to be the largest quotient category of F subject to the following relations on morphisms:

- $[a] \sim [\]_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
- $[g, f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha: X \rightarrow N(hX)$ sends $x \in X_n$ to the equivalence class of $[x_{n-1,n}, \dots, x_{0,1}]$. Given this, verifying the desired universal property of α is formal.

(We will give another construction of the fundamental category in (13.18).) □

9.3. Exercise. Complete the proof of (9.2) by showing that $\alpha^*: \text{Hom}(N(hX), NC) \rightarrow \text{Hom}(X, NC)$ is a bijection for any category C .

As a consequence: the fundamental category construction describes a functor $h: s\text{Set} \rightarrow \text{Cat}$, which is left adjoint to the nerve functor $N: \text{Cat} \rightarrow s\text{Set}$.

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a “quotient category” induced by a relation on its morphisms. We will not be making much use of it. When C is a quasicategory, there is a more concrete construction of hC , which in this context is called the *homotopy category* of C . *Warning:* Sometimes people will not distinguish “fundamental category” from “homotopy category” as I have here, and just call either the homotopy category.

9.4. The homotopy relation on morphisms. Fix a quasicategory C . For $x, y \in C_0$, let $\text{hom}_C(x, y) := \{f \in C_1 \mid f_0 = x, f_1 = y\}$ denote the set of “morphisms” in C from x to y . We write 1_x for the degenerate element $x_{00} \in \text{hom}_C(x, x)$.

Define relations \sim_ℓ, \sim_r on $\text{hom}_C(x, y)$ (called **left homotopy** and **right homotopy**) by

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

Pictorially:

$$f \sim_\ell g: \begin{array}{ccc} x & \xrightarrow{f} & y \\ 1_x \downarrow & \searrow a & \nearrow \\ x & \xrightarrow{g} & y \end{array} \qquad f \sim_r g: \begin{array}{ccc} & & y \\ & \xrightarrow{f} & \downarrow 1_y \\ x & \xrightarrow{b} & y \\ & \searrow g & \nearrow \end{array}$$

Note that $f \sim_\ell g$ in $\text{hom}_C(x, y)$ coincides with $g \sim_r f$ in $\text{hom}_{C^{\text{op}}}(y, x)$.

9.5. *Remark.* If C is an ordinary category, then the left homotopy and right homotopy relations reduce to the equality relation on morphisms $x \rightarrow y$.

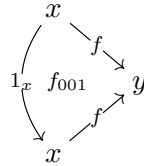
9.6. **Proposition.** *The relations \sim_ℓ and \sim_r are identical, and are an equivalence relation on $\text{hom}_C(x, y)$.*

Proof. Given $f, g, h: x \rightarrow y$ in a quasicategory C , we will prove

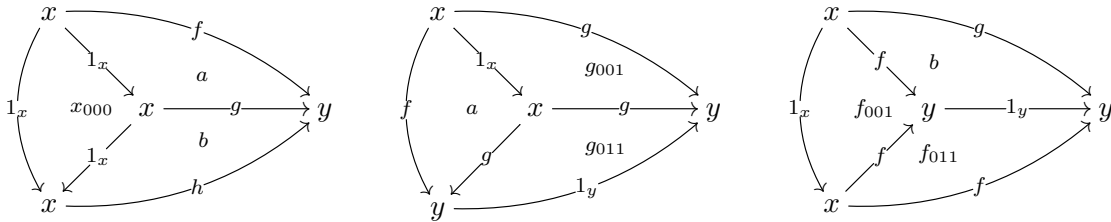
- (1) $f \sim_\ell f$,
- (2) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$,
- (3) $f \sim_\ell g$ implies $f \sim_r g$,
- (4) $f \sim_r g$ implies $g \sim_\ell f$.

These show that \sim_ℓ is an equivalence relation, and also that \sim_r and \sim_ℓ coincide. The idea is to use the inner-horn extension condition for C to produce the appropriate relations.

- (1) $f \sim_\ell f$ is exhibited by $f_{001} \in C_2$.



(2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of Δ^3 (respectively Λ_1^3 , Λ_1^3 , and Λ_2^3) to C constructed from the given data. The restriction of any extension to Δ^3 along the remaining face gives the conclusion.



□

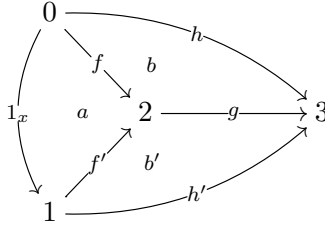
9.7. **Composition of homotopy classes of morphisms.** We now define $f \approx g$ to mean $f \sim_\ell g$ (equivalently $f \sim_r g$). We speak of **homotopy classes** $[f]$ of morphisms $f \in \text{hom}_C(x, y)$, meaning equivalence classes under \approx . Next we observe that we can compose homotopy classes.

Given $f \in \text{hom}_C(x, y)$, $g \in \text{hom}_C(y, z)$, $h \in \text{hom}_C(x, z)$, we say that h is a **composite** of (g, f) if there exists a 2-dimensional element $a \in C_2$ with $a\langle 01 \rangle = f$, $a\langle 12 \rangle = g$, $a\langle 02 \rangle = h$; thus composition is a three-fold relation on $\text{hom}(x, y) \times \text{hom}(y, z) \times \text{hom}(x, z)$. The composite relation is compatible with the homotopy relation, as shown by the following.

9.8. **Lemma.** *If $f \approx f'$, $g \approx g'$, h a composite of (g, f) , and h' a composite of (g', f') , then $h \approx h'$.*

Proof. Since \approx is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b).

Let $a \in C_2$ exhibit $f \sim_\ell f'$, and let $b, b' \in C_2$ exhibit h as a composite of (g, f) and h' as a composite of (g, f') respectively. The inner horn $\Lambda_2^3 \rightarrow C$ defined by



extends to $u: \Delta^3 \rightarrow C$, and $u|_{\Delta^{\{0,1,3\}}}$ exhibits $h \sim_\ell h'$. □

Thus, composites of (g, f) live in a unique homotopy class of morphisms in C , which only depends on the homotopy classes of g and f . I will write $[g] \circ [f]$ for this class.

I'll leave the following as exercises; the proofs are much like what we have already seen.

9.9. **Lemma.** *Given $f: x \rightarrow y$, we have $[f] \circ [1_x] = [f] = [1_y] \circ [f]$.*

9.10. **Lemma.** *If $[g] \circ [f] = [u]$, $[h] \circ [g] = [v]$, then $[h] \circ [u] = [v] \circ [f]$.*

9.11. **The homotopy category of a quasicategory.** For any quasicategory, we define its **homotopy category** hC , so that $\text{ob}(hC) := C_0$, while $\text{hom}_{hC}(x, y) := \text{hom}_C(x, y) / \approx$, with composition defined by $[g] \circ [f]$. The above lemmas (9.9) and (9.10) exactly imply that hC is a category.

We define a map $\pi: C \rightarrow N(hC)$ of simplicial sets as follows. On vertices, π is the identity map $C_0 = N(hC)_0 = \text{ob } hC$. On edges, the map is defined by the tautological quotient maps $\text{hom}_C(x, y) \rightarrow \text{hom}_C(x, y) / \approx$ sending $f \mapsto [f]$. The map π sends an n -dimensional element $a \in C_n$ to the unique element $\pi(a) \in N(hC)_n$ such that $\pi(a)_{i-1, i} = \pi(a_{i-1, i})$. These functions are seen to be compatible with simplicial operators using the following exercise.

9.12. *Exercise.* Let C be a quasicategory and $a \in C_n$ an n -dimensional element, and define $f_i := a_{i-1, i} \in C_1$ for $i = 1, \dots, n$ and $g := a_{0, n} \in C_1$. Show that $[f_n] \circ \dots \circ [f_1] = [g]$ in the homotopy category hC .

Note that if C is an ordinary category, then $f \approx g$ if and only if $f = g$. Thus, $\pi: C \rightarrow N(hC)$ is an isomorphism of simplicial sets if and only if C is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation “ hC ”.

9.13. **Proposition.** *Let C be a quasicategory and D a small category, and let $\phi: C \rightarrow N(D)$ be a map of simplicial sets. Then there exists a unique map $\psi: N(hC) \rightarrow N(D)$ such that $\psi\pi = \phi$.*

Proof. We first show existence, by constructing a suitable map ψ , which being a map between nerves can be described as a functor $hC \rightarrow D$. On objects, let ψ send $x \in \text{ob}(hC) = C_0$ to $\phi(x) \in \text{ob}(D) = (ND)_0$. On morphisms, let ψ send $[f] \in \text{hom}_{hC}(x, y)$ to $\phi(f) \in \text{hom}_D(\phi(x), \phi(y)) \subseteq (ND)_1$. Observe that the function on morphisms is well-defined since if $f \sim_\ell f'$, exhibited by some $a \in C_2$, then $\phi(a) \in (ND)_2$ exhibits the identity $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$ in D . It is straightforward to show that ψ so defined is actually a functor, and that $\psi\pi = \phi$ as maps $C \rightarrow N(D)$.

The functor ψ defined above is the unique solution: the value of ψ on objects and morphisms is uniquely determined, and $\pi: C_k \rightarrow (hC)_k$ is bijective for $k = 0$ and surjective for $k = 1$. □

In particular, the homotopy category construction gives a pair of adjoint functors

$$h: \text{qCat} \rightleftarrows \text{Cat} : N.$$

9.14. *Exercise.* Understand the homotopy categories of the various examples described in §8.

9.15. *Exercise* (Easy but important). Show that for quasicategories C and D there is an isomorphism $hC \times hD \approx h(C \times D)$.

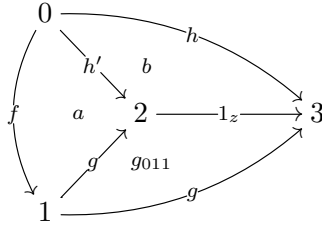
9.16. A criterion for composition. We have observed that for morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$ in a quasicategory that we can define a composite “ $g \circ f$ ” using extension along $\Lambda_1^2 \subset \Delta^2$, and that though such compositions are not unique, they are unique up to homotopy, so we get a well-defined homotopy class $[g] \circ [f]$. The following proposition says that every element in this homotopy class is obtained from this construction.

9.17. Proposition. *If $f: x \rightarrow y$, $g: y \rightarrow z$, and $h: x \rightarrow z$ are morphisms in a quasicategory C , then $[h] = [g] \circ [f]$ if and only if there exists $u: \Delta^2 \rightarrow C$ such that*

$$u|\Delta^{\{0,1\}} = f, \quad u|\Delta^{\{1,2\}} = g, \quad u|\Delta^{\{0,2\}} = h.$$

Thus, every morphism in the homotopy class of h can be interpreted as a composite of g with f .

Proof. Clearly if u exists then $[h] = [g] \circ [f]$. Conversely, suppose given f, g, h with $h \in [g] \circ [f]$, and choose a $a: \Delta^2 \rightarrow C$ with $a_{01} = f$ and $a_{12} = g$, whence $[g] \circ [f] = [h']$ for $h' = a_{02}$. Since $h \in [h']$ there is a $b \in C_2$ witnessing the relation $h' \sim_r h$, and using this we can construct a map $\Lambda_2^3 \rightarrow C$ according to the diagram



Extend to a map $v: \Delta^3 \rightarrow C$; then $u = v|\Delta^{\{0,1,3\}}$ exhibits h as a composite of (g, f) as desired. \square

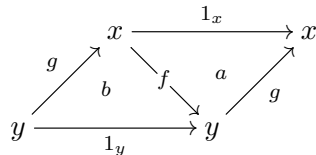
10. ISOMORPHISMS IN A QUASICATEGORY

Let C be a quasicategory. We say that an edge $f \in C_1$ is an **isomorphism**⁶ if its image in the homotopy category hC is an isomorphism in the usual sense of category theory.

Explicitly, $f: x \rightarrow y$ is an isomorphism if and only if there exists an edge $g: y \rightarrow x$ such that $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, where equality is in the homotopy category hC .

10.1. Example. Consider $f \in C_1$. If we can produce $g \in C_1$ and $a, b \in C_2$ such that

$$a_{01} = f = b_{12}, \quad a_{12} = g = b_{01}, \quad a_{02} = x_{00}, \quad b_{02} = y_{00} :$$



then $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, so f isomorphism. The converse also holds: if f is an isomorphism, then there exist $g \in C_1$ and $a, b \in C_2$ as above, which can be proved using (9.17).

10.2. Example (Identity maps are isomorphisms). For every $x \in C_0$ the identity map $1_x: x \rightarrow x$ is an isomorphism: for instance, use $a = b = x_{000}$ in the above diagram.

⁶Lurie [Lur09, §1.2.4] uses the term “equivalence” for this. I prefer to go with “isomorphism” here, because it is in fact a generalization of the classical notion of isomorphism, and because so many other things also get to be called some kind of equivalence. Other authors also use “isomorphism” in this context.

10.3. Preinverses and postinverses. Let C be a quasicategory. Given $f: x \rightarrow y \in C_1$, a **postinverse**⁷ of f is a $g: y \rightarrow x \in C_1$ such that $[g] \circ [f] = [1_x]$, and a **preinverse**⁸ of f is an $e: y \rightarrow x \in C_1$ such that $[f] \circ [e] = [1_y]$. An **inverse** is an $f' \in C_1$ which is both a postinverse and a preinverse. The following is trivial, but very handy.

10.4. Proposition. *In a quasicategory C consider $f \in C_1$. The following are equivalent.*

- f is an isomorphism.
- f admits an inverse f' .
- f admits a postinverse g and a preinverse e .
- f admits a postinverse g and g admits a postinverse h .
- f admits a preinverse e and e admits a preinverse d .

If these equivalent conditions hold, then $f \approx d \approx h$ and $f' \approx e \approx g$, and all of them are isomorphisms.

Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category hC , where they are seen to be equivalent by elementary arguments. \square

Note that inverses to a morphism in a quasicategory are generally *not unique*, though necessarily they are *unique up to homotopy*.

10.5. Quasigroupoids. A **quasigroupoid** is a quasicategory C such that hC is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism.

10.6. Exercise. If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

10.7. The core of a quasicategory. For an ordinary category A , the **core** (or **maximal subgroupoid**) of A is the subcategory $A^{\text{core}} \subseteq A$ consisting of all the objects, and all the *isomorphisms* between the objects.

For a quasicategory C , we define the core⁹ $C^{\text{core}} \subseteq C$ to be the subsimplicial set consisting of elements all of whose edges are all isomorphisms. That is, C^{core} is defined so that the diagram

$$\begin{array}{ccc} C^{\text{core}} & \xrightarrow{\quad} & C \\ \downarrow & & \downarrow \pi \\ (hC)^{\text{core}} & \xrightarrow{\quad} & hC \end{array}$$

is a pullback of simplicial sets. Observe that $N(A^{\text{core}}) = (NA)^{\text{core}}$ for a category A .

10.8. Proposition. *For a quasicategory C , its core C^{core} is a quasigroupoid, and every subcomplex of C which is a quasigroupoid is contained in C^{core} .*

Proof. First, note that C^{core} is a subcomplex by construction: if $a \in C_n$ is such that all edges are isomorphisms, then the same is true for $af \in C_m$ for any $f: [m] \rightarrow [n]$, since $(af)_{i,j} = a_{f(i),f(j)}$ for any $0 \leq i \leq j \leq m$.

Next, we show that C^{core} is a quasicategory. In fact, we show that given $f: \Delta^n \rightarrow C$ such that $f(\Lambda_j^n) \subseteq C^{\text{core}}$ for some $0 < j < n$, then $f(\Delta^n) \subseteq C^{\text{core}}$, so that inner-horn-filling for C implies inner-horn-filling for C^{core} . For $n = 2$ this is the fact that composites of isomorphisms are isomorphisms (true in any quasicategory because it is true in its homotopy category), while for $n \geq 3$ it is just the fact that an inner horn of Δ^n contains all the edges of Δ^n .

Thus, C^{core} is a quasicategory, and is easily seen to be a quasigroupoid, since an inverse of an isomorphism in C is also an isomorphism.

⁷or *left inverse*, or *retraction*,

⁸or *right inverse*, or *section*,

⁹Lurie uses the notation C^{\simeq} for what we are calling C^{core} .

The final statement is clear: if $G \subseteq C$ is a subcomplex which is a quasigroupoid, then every edge in G has an inverse in G , and hence an inverse in C . \square

10.9. Kan complexes. Recall that a Kan complex (8.8) is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, K is a Kan complex iff

$$\mathrm{Hom}(\Delta^n, K) \rightarrow \mathrm{Hom}(\Lambda_j^n, K)$$

is surjective for all $0 \leq j \leq n$, $n \geq 1$.

10.10. Exercise. Show that every simplicial set X has extensions for 1-dimensional horns; i.e., every $\Lambda_j^1 \rightarrow X$ extends over Δ^1 , where $j \in \{0, 1\}$. Thus, X is a Kan complex if and only if it has extensions just for the horns inside simplices of dimension ≥ 2 .

10.11. Proposition. *Every Kan complex is a quasigroupoid.*

Proof. It is immediate that a Kan complex K is a quasicategory. To show K is a quasigroupoid, note that the extension condition for $\Lambda_0^2 \subset \Delta^2$ implies that every morphism in hK admits a postinverse. Explicitly, if $f: x \rightarrow y$ is an edge in K , let $u: \Lambda_0^2 \rightarrow K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v: \Delta^2 \rightarrow K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (10.6). \square

This proposition has a converse.

A. Deferred Proposition. *Quasigroupoids are precisely the Kan complexes.*

This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will give the proof in (29.2).

Recall (§8.7) that we observed that the singular complex $\mathrm{Sing} T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\mathrm{Sing} T$ as the **fundamental quasigroupoid** of the space T .

10.12. Exercise (for topologists). Show that if T is a space, then $h \mathrm{Sing} T$, the homotopy category of the singular complex of T , is precisely the usual fundamental groupoid of T .

10.13. Quasigroupoids, components, and isomorphism classes. We say that two objects in a quasicategory are **isomorphic** if there exists an isomorphism between one. This is clearly an equivalence relation on C_0 , and thus we speak of **isomorphism classes** of objects.

Recall (6.8) that the set of connected components of a simplicial set is given by

$$\pi_0 X \approx \left(\left(\prod_{n \geq 0} X_n \right) / \sim \right) \approx (X_0 / \sim_1),$$

the equivalence classes of elements of X under the equivalence relation generated by “related by a simplicial operator”, or equivalently the equivalence classes of vertices of X under the equivalence relation generated by “connected by an edge”. Note that if T is a topological space, then elements of $\pi_0 \mathrm{Sing} T$ correspond exactly to path components of T .

For *quasigroupoids*, π_0 recovers the set of isomorphism classes of objects.

10.14. Proposition. *If C is a quasicategory, then*

$$\pi_0(C^{\mathrm{core}}) \approx \text{isomorphism classes of objects of } C.$$

Proof. Straightforward: edges in C^{core} are precisely the isomorphisms in C . \square

10.15. Exercise. Show that for a quasicategory C , $\pi_0(C^{\mathrm{core}}) \approx \pi_0(h(C^{\mathrm{core}})) \approx \pi_0((hC)^{\mathrm{core}})$.

11. FUNCTION COMPLEXES AND THE FUNCTOR QUASICATEGORY

Given ordinary categories C and D , the *functor category* $\text{Fun}(C, D)$ has

- as objects, the functors $C \rightarrow D$, and
- as morphisms $f \rightarrow f'$, natural transformations of functors.

Furthermore, for any category A there is a bijective correspondence between sets of functors

$$\{A \times C \rightarrow D\} \longleftrightarrow \{A \rightarrow \text{Fun}(C, D)\}.$$

Explicitly, a functor $\phi: A \rightarrow \text{Fun}(C, D)$ corresponds to $\tilde{\phi}: A \times C \rightarrow D$, given on objects by $\tilde{\phi}(a, c) = \phi(a)(c)$ for $a \in \text{ob } A$ and $c \in \text{ob } C$, and on morphisms by $\tilde{\phi}(\alpha, \gamma) = \phi(a')(\gamma) \circ \phi(a)(c) = \phi(\alpha)(c') \circ \phi(a)(\gamma): \phi(a)(c) \rightarrow \phi(a')(\gamma)$ for $\alpha: a \rightarrow a' \in \text{mor } A$ and $\gamma: c \rightarrow c' \in \text{mor } C$.

The generalization of the functor category to quasicategories admits a similar adjunction, and in fact can be defined for arbitrary simplicial sets.

11.1. Function complexes. Given simplicial sets X and Y , we may form the **function complex** (or **mapping space**) $\text{Map}(X, Y)$. This is a simplicial set with

$$\text{Map}(X, Y)_n = \text{Hom}(\Delta^n \times X, Y),$$

so that the action of a simplicial operator $\delta: [m] \rightarrow [n]$ on $\text{Map}(X, Y)$ is induced by $\text{Hom}(\delta \times \text{id}_X, Y): \text{Hom}(\Delta^n \times X, Y) \rightarrow \text{Hom}(\Delta^m \times X, Y)$. In particular, the set $\text{Map}(X, Y)_0$ of vertices of the function complex is precisely the set of maps $X \rightarrow Y$ of simplicial sets.

11.2. Proposition. *The function complex construction defines a functor*

$$\text{Map}: s\text{Set}^{\text{op}} \times s\text{Set} \rightarrow s\text{Set}.$$

Proof. Left as an exercise. □

By construction, for each n , there is a bijective correspondence

$$\{\Delta^n \times X \rightarrow Y\} \longleftrightarrow \{\Delta^n \rightarrow \text{Map}(X, Y)\}.$$

In fact, we can replace Δ^n with an arbitrary simplicial set.

11.3. Proposition. *For simplicial sets X, Y, Z , there is a bijection*

$$\text{Hom}(X \times Y, Z) \xrightarrow{\sim} \text{Hom}(X, \text{Map}(Y, Z))$$

natural in all three variables.

Proof. The bijection sends $f: X \times Y \rightarrow Z$ to $\tilde{f}: X \rightarrow \text{Map}(Y, Z)$ defined so that for $x \in X_n$, the element $\tilde{f}(x) \in \text{Map}(Y, Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \text{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g: X \rightarrow \text{Map}(Y, Z)$ to $\tilde{g}: X \times Y \rightarrow Z$, defined so that for $(x, y) \in X_n \times Y_n$, the element $\tilde{g}(x, y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\text{id}, y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both \tilde{f} and \tilde{g} are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality. □

11.4. Exercise. Show, using the previous proposition, that there are natural isomorphisms

$$\text{Map}(X \times Y, Z) \approx \text{Map}(X, \text{Map}(Y, Z)).$$

of simplicial sets. This implies that the function complex construction makes $s\text{Set}$ into a *cartesian closed category*. (Hint: show that both objects represent *isomorphic* functors $s\text{Set}^{\text{op}} \rightarrow \text{Set}$, and apply the Yoneda lemma.)

11.5. *Remark.* The construction of the function complex is not special to simplicial sets. The construction of $\text{Map}(X, Y)$ (and its properties as described above) works the same way in any category of functors $C^{\text{op}} \rightarrow \text{Set}$, where C is a small category (e.g., $C = \Delta$). In this general setting, the role of the standard n -simplices is played by the representable functors $\text{Hom}_C(-, c): C^{\text{op}} \rightarrow \text{Set}$.

11.6. **Functor quasicategories.** Thus, we may expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if C and D are quasicategories, then the vertices of $\text{Map}(C, D)$ are precisely the functors $C \rightarrow D$, and the edges of $\text{Map}(C, D)$ are precisely the natural transformations. Furthermore, for ordinary categories, the function complex recovers the functor category.

11.7. *Exercise.* Show that for ordinary categories C and D that $N\text{Fun}(C, D) \approx \text{Map}(NC, ND)$. (Hint: use that $N([n]) = \Delta^n$, and the fact that the nerve preserves finite products (6.5).)

It turns out that a function complex between quasicategories is again a quasicategory. In fact, we have the following.

B. Deferred Proposition. *Let K be any simplicial set and C a quasicategory. Then $\text{Map}(K, C)$ is a quasicategory.*

For this reason, we will sometimes write $\text{Fun}(K, C)$ for $\text{Map}(K, C)$ when C is a quasicategory.

To prove (B), we need to take a detour to develop some technology about “weakly saturated” classes of maps and “lifting properties”. After this, we will complete the proof in §16.

Part 2. Lifting properties

12. WEAKLY SATURATED CLASSES AND INNER-ANODYNE MAPS

Quasicategories are defined by an “extension property”: they are the simplicial sets C such that any map $K \rightarrow C$ extends over L , whenever $K \subset L$ is an inner horn inclusion $\Lambda_j^n \subset \Delta^n$. The set of inner horns “generates” a larger class of maps (which will be called the class of *inner anodyne* maps), which “automatically” shares the extension property of the inner horns. This class of inner anodyne maps is called the *weak saturation* of the set of inner horns.

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For instance, we will observe that the spine inclusions $I^n \subset \Delta^n$ are inner anodyne, so that quasicategories admit “spine extensions”, i.e., any $I^n \rightarrow C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \rightarrow C$.

12.1. **Weakly saturated classes.** Consider a category (such as $s\text{Set}$) which has all small colimits. A **weakly saturated class** is a class \mathcal{A} of morphisms in the category, which

- (1) contains all isomorphisms,
- (2) is closed under cobase change,
- (3) is closed under composition,
- (4) is closed under transfinite composition,
- (5) is closed under coproducts, and
- (6) is closed under retracts.

Given a class of maps S , its **weak saturation** \overline{S} is the smallest weakly saturated class containing S .

We need to explain some of the elements of this definition.

- **Closed under cobase change** is also called **closed under pushout**: it means that if f' is the pushout of $f: X \rightarrow Y$ along some map $g: X \rightarrow Z$, then $f \in \mathcal{A}$ implies $f' \in \mathcal{A}$.
- **Closed under composition** means that if $g, f \in \mathcal{A}$ and gf is defined, then $gf \in \mathcal{A}$.
- We say that \mathcal{A} is **closed under countable composition** if given a countable sequence of composable morphisms, i.e., maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$$

such that each $f_k \in \mathcal{A}$ for all $k \in \mathbb{Z}_{>0}$, the induced map $X_0 \rightarrow \operatorname{colim}_k X_k$ to the colimit is in \mathcal{A} .

The notion **closed under transfinite composition** is the generalization of this, in which \mathbb{N} is replaced by an arbitrary ordinal λ (i.e., a well-ordered set). This means that for any ordinal λ and any functor $X: \lambda \rightarrow \mathbf{sSet}$, if for every $i \in \lambda$ with $i \neq 0$ the evident map

$$(\operatorname{colim}_{j < i} X(j)) \rightarrow X(i)$$

is in \mathcal{A} , then the induced map $X(0) \rightarrow \operatorname{colim}_{j \in \lambda} X(j)$ is in \mathcal{A} .

- **Closed under coproducts** means that if $\{f_i: X_i \rightarrow Y_i\}$ is a set of maps in \mathcal{A} , then $\coprod_i f_i: \coprod_i X_i \rightarrow \coprod_i Y_i$ is in \mathcal{A} .
- We say that f is a **retract** of g if there exists a commutative diagram in C of the form

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \\
 & \downarrow f & & \downarrow g & & \downarrow f \\
 & \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \\
 & & \text{id} & &
 \end{array}$$

This is really a special case of the notion of a retract of an object in the functor category $\operatorname{Fun}([1], \mathbf{sSet})$. We say that \mathcal{A} is **closed under retracts** if for every diagram as above, $g \in \mathcal{A}$ implies $f \in \mathcal{A}$.

12.2. *Remark.* This list of properties is not minimal: (3) is actually the special case of (4) when $\lambda = [2]$, and (5) can be deduced from (2) and (4).

12.3. *Example.* Consider the category of sets. The class of all surjective maps is weakly saturated, and in fact is the weak saturation of $\{\{0, 1\} \rightarrow \{1\}\}$. Likewise, the class of injective maps is weakly saturated, and in fact is the weak saturation of $\{\emptyset \rightarrow \{1\}\}$.

12.4. *Example.* The classes of monomorphisms and epimorphisms of simplicial sets are weakly saturated classes. Later we will identify the class of monomorphisms of simplicial sets as the weak saturation of the set of “cell inclusions” (15.25).

There is a dual notion of a **weakly cosaturated class**: a weakly cosaturated class is the same thing as a weakly saturated class in the opposite category, and is characterized by being closed under properties formally dual to (1)–(6).

12.5. **Classes of “anodyne” morphisms.** We use the following notation for sets of types of horns:

$$\begin{aligned}
 \operatorname{InnHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 < k < n, n \geq 2 \}, & (\text{inner horns}), \\
 \operatorname{LHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \}, & (\text{left horns}), \\
 \operatorname{RHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \}, & (\text{right horns}), \\
 \operatorname{Horn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k \leq n, n \geq 1 \}, & (\text{horns}).
 \end{aligned}$$

The weak saturation of each of these sets will play an important role in what follows. Right now, we focus on the weak saturation $\overline{\operatorname{InnHorn}}$ of the set of inner horns, which is called the class of **inner anodyne**¹⁰ morphisms. (There are also classes of “left anodyne”, “right anodyne”, and plain old “anodyne” morphisms, about which we have more to say later.) Note that inner anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets themselves form a weakly saturated class.

¹⁰The “anodyne” terminology for the weak saturation of a set of horns was introduced by Gabriel and Zisman [GZ67]. “Anodyne” derives from ancient Greek, meaning “without pain”; we leave it to the reader to decide whether this choice of terminology is appropriate.

12.6. Proposition. *If C is a quasicategory and $A \subseteq B$ is an inner anodyne inclusion, then any $f: A \rightarrow C$ admits an extension to $g: B \rightarrow C$ so that $g|_A = f$.*

Proof. It suffices to show that the collection \mathcal{A} of monomorphisms $i: A \rightarrow B$ such that every map from A to a quasicategory extends along i is weakly saturated. Since $\text{InnHorn} \subseteq \mathcal{A}$ it then follows that $\text{InnHorn} \subseteq \mathcal{A}$. To prove this claim is a relatively straightforward exercise, which we leave for the reader: check that the class \mathcal{A} satisfies each of the conditions (1)–(6) of a weakly saturated class. It is highly recommended that you work through this argument this if you haven't seen it before. \square

12.7. Exercise (Easy but important). Show that every inner anodyne map induces a bijection on vertices. (Hint: show that the class of maps of simplicial sets which are a bijection on vertices is weakly saturated.)

12.8. Examples of inner anodyne morphisms. It is crucial to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. A **generalized horn** the subcomplex $\Lambda_S^n \subset \Delta^n$ defined by

$$\Lambda_S^n := \bigcup_{i \in S} \Delta^{[n] \setminus i},$$

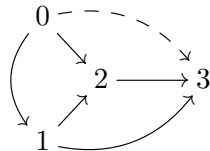
i.e., the union of codimension one faces of the n -simplex indexed by elements of S . In particular, $\Lambda_{[n] \setminus \{j\}}^n$ is the usual horn Λ_j^n . I'll generalize this notation to arbitrary totally ordered sets, so $\Lambda_S^T = \bigcup_{i \in S} \Delta^{T \setminus i}$ when $S \subseteq T$.

We call $\Lambda_S^n \subset \Delta^n$ a **generalized inner horn** if S is not an “interval” in $[n]$, i.e., if there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

12.9. Lemma. *All generalized inner horn inclusions $\Lambda_S^n \subset \Delta^n$ are inner anodyne.*

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (57.1).

12.10. Example. Consider $\Lambda_{\{0,3\}}^3$, which can be pictured as the solid diagram in



We can get from this to Δ^3 in two steps:

$$\begin{array}{ccc} \Lambda_{\{0,3\}}^{\{0,2,3\}} & \xrightarrow{\cong} & \Delta^{\{0,2,3\}} \\ \downarrow & & \downarrow \\ \Lambda_{\{0,3\}}^3 & \xrightarrow{\cong} & \Lambda_2^3 \xrightarrow{\cong} \Delta^3 \end{array}$$

The square is a pushout of subcomplexes since $\Lambda_{\{0,3\}}^3 \cap \Delta^{\{0,2,3\}} = \Lambda_{\{0,3\}}^{\{0,2,3\}}$, and the map along the top is isomorphic to $\Lambda_1^2 \subset \Delta^2$, an inner horn inclusion. This proves that $\Lambda_{\{0,3\}}^3 \subset \Delta^3$ is inner anodyne.

Recall that every standard n -simplex contains a spine $I^n \subseteq \Delta^n$.

12.11. Lemma. *The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all n . Thus, for a quasicategory C , any $I^n \rightarrow C$ extends to $\Delta^n \rightarrow C$.*

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix (57.2).

12.12. *Example.* To show that $I^3 \subset \Delta^3$ is inner anodyne, observe that we can get from I^3 to a generalized inner horn two steps by gluing 2-simplices along inner horns inclusions:

$$\begin{array}{ccc} \Lambda_{\{0,2\}}^{\{0,1,2\}} & \xrightarrow{\quad} & \Delta^{\{0,1,2\}} \\ \downarrow & & \downarrow \\ I^3 & \xrightarrow{\quad} & I^3 \cup \Delta^{\{0,1,2\}} \end{array} \quad \begin{array}{ccc} \Lambda_{\{1,3\}}^{\{1,2,3\}} & \xrightarrow{\quad} & \Delta^{\{1,2,3\}} \\ \downarrow & & \downarrow \\ I^3 \cup \Delta^{\{0,1,2\}} & \xrightarrow{\quad} & \Lambda_{\{0,3\}}^3 \end{array}$$

since $I^3 \cap \Delta^{\{0,1,2\}} = \Lambda_{\{0,2\}}^{\{0,1,2\}}$ and $(I^3 \cup \Delta^{\{0,1,2\}}) \cap \Delta^{\{1,2,3\}} = \Lambda_{\{1,3\}}^{\{1,2,3\}}$.

12.13. *Exercise.* Use (12.11) to show that the tautological map $\pi: C \rightarrow N(hC)$ from a quasicategory to (the nerve of) its homotopy category is surjective in every degree.

13. LIFTING CALCULUS AND INNER FIBRATIONS

We have defined quasicategories by an “extension property”: in general, we say that X has satisfies the extension property for $A \rightarrow B$ if in any diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow s & \\ B & & \end{array}$$

there exists a morphism s making the diagram commute. In this section, we discuss a “relative” version of this, called a “lifting property”.

13.1. **The lifting relation.** Given morphisms $f: A \rightarrow B$ and $g: X \rightarrow Y$ in a category, a **lifting problem** for (f, g) is a pair of morphisms (u, v) such that $vf = gu$. That is, a lifting problem is any commutative square of solid arrows of the form

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow s & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

A **lift** for the lifting problem is a morphism s such that $sf = u$ and $gs = v$, i.e., a dotted arrow making the diagram commute.

We may thus define the **lifting relation** on morphisms in our category: we write “ $f \square g$ ” if every lifting problem for (f, g) admits a lift¹¹. Equivalently, $f \square g$ exactly if

$$\text{Hom}(B, X) \xrightarrow{s \mapsto (sf, gs)} \text{Hom}(A, X) \times_{\text{Hom}(A, Y)} \text{Hom}(B, Y)$$

is a surjection, where the target is the set of pairs $(u: A \rightarrow X, v: B \rightarrow Y)$ such that $gu = vf$ (i.e., the target is exactly the set of lifting problems for (f, g)).

When $f \square g$ holds, one sometimes says f has the **left lifting property** relative to g , or that g has the **right lifting property** relative to f . Or we just say that f **lifts against** g .

We extend the notation to classes of maps, so “ $\mathcal{A} \square \mathcal{B}$ ” means: $a \square b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Given a class of morphisms \mathcal{A} , define the **right complement** \mathcal{A}^\square and **left complement** ${}^\square\mathcal{A}$ by

$$\mathcal{A}^\square = \{ g \mid a \square g \text{ for all } a \in \mathcal{A} \}, \quad {}^\square\mathcal{A} = \{ f \mid f \square a \text{ for all } a \in \mathcal{A} \}.$$

13.2. **Proposition.** *For any class \mathcal{B} , the left complement ${}^\square\mathcal{B}$ is a weakly saturated class.*

13.3. *Exercise (Important).* Prove (13.2). (This is a “relative” version of the proof of (12.6).)

¹¹Sometimes one sees the notation “ $f \perp g$ ” or “ $f \pitchfork g$ ” used instead. Our notation is taken from [Rie14, §11].

The above proposition (13.2) has a dual statement: any right complement \mathcal{B}^\square is weakly cosaturated, i.e., satisfies dual versions of the closure properties of a weakly saturated class, or equivalently, corresponds to a weakly saturated class in the opposite category.

13.4. *Exercise* (Easy). Prove that if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^\square \supseteq \mathcal{B}^\square$ and ${}^\square\mathcal{A} \supseteq {}^\square\mathcal{B}$. Use this to show $\mathcal{A}^\square = ({}^\square(\mathcal{A}^\square))^\square$ and ${}^\square\mathcal{A} = {}^\square({}^\square\mathcal{A})^\square$.

13.5. *Exercise* (for those who know a little homological algebra). Fix an abelian category \mathcal{C} (e.g., the category of modules over some ring R). Let \mathcal{P} be the class of morphisms in \mathcal{C} of the form $0 \rightarrow P$ where P is projective, and let \mathcal{B} be the class of epimorphisms in \mathcal{C} . Show that $\mathcal{P} \square \mathcal{B}$; also, show that $\mathcal{B} = \mathcal{P}^\square$ if \mathcal{C} has enough projectives.

13.6. *Exercise*. In the setting of the previous exercise, identify the class ${}^\square\mathcal{B}$.

13.7. **Inner fibrations.** A map p of simplicial sets is an **inner fibration** if $\text{InnHorn} \square p$. The class of inner fibrations $\text{InnFib} = \text{InnHorn}^\square$ is thus the right complement of the set of inner horns. Note that C is a quasicategory if and only if $C \rightarrow *$ is an inner fibration.

Because InnFib is a right complement, it is weakly cosaturated. In particular, it is closed under composition. This implies that if $p: C \rightarrow D$ is an inner fibration and D is a quasicategory, then C is also a quasicategory.

13.8. *Exercise*. Show that if $f: C \rightarrow D$ is any functor from a quasicategory C to a category D , then f is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

13.9. **Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

13.10. **Proposition** (“Small object argument”). *Let S be a set of morphisms in $s\text{Set}$. Every map f between simplicial sets admits a factorization $f = pj$ with $j \in \overline{S}$ and $p \in S^\square$.*

The proof of this proposition is by means of what is known as the “small object argument”. I’ll give the proof in the next section. For now we record a consequence.

13.11. **Corollary.** *For any set S of morphisms in $s\text{Set}$, we have that $\overline{S} = {}^\square(S^\square)$.*

Proof. That $\overline{S} \subseteq {}^\square(S^\square)$ is immediate from (13.2). Given f such that $f \square S^\square$, use the small object argument (13.10) to choose $f = pj$ with $j \in \overline{S}$ and $p \in S^\square$. We have a commutative diagram of solid arrows

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \bullet & \xrightarrow{\text{id}} & \bullet & \xrightarrow{\text{id}} & \bullet \\
 \downarrow f & & \downarrow j & & \downarrow f \\
 \bullet & \xrightarrow{s} & \bullet & \xrightarrow{p} & \bullet \\
 & \curvearrowleft & & \curvearrowright & \\
 & & \text{id} & &
 \end{array}$$

A map s exists making the diagram commute, because $f \square p$, so there is a lift in

$$\begin{array}{ccc}
 \bullet & \xrightarrow{j} & \bullet \\
 \downarrow f & \nearrow s & \downarrow p \\
 \bullet & \xrightarrow{\text{id}} & \bullet
 \end{array}$$

The diagram exhibits f as a retract of j , whence $f \in \overline{S}$ since weak saturations are closed under retracts. \square

The proof of the corollary is called the “retract trick”: given $f = pj$, $f \sqsupseteq p$ implies that f is a retract of j , while $j \sqsupseteq f$ implies that f is a retract of p .

In the case we are currently interested in, we have that $\overline{\text{InnHorn}} = \square \text{InnFib}$ and $\overline{\text{InnHorn}}^{\square} = \text{InnFib}$, and thus any map can be factored into an inner anodyne map followed by an inner fibration.

13.12. Weak factorization systems. A **weak factorization system** in a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that F 1 Feb 2019

- every map f admits a factorization $f = r\ell$ with $r \in \mathcal{R}$ and $\ell \in \mathcal{L}$, and
- $\mathcal{L} = \square \mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\square}$.

Thus, in any weak factorization the “left” class \mathcal{L} is weakly saturated and the “right” class \mathcal{R} is weakly cosaturated. The small object argument implies that $(\overline{S}, S^{\square})$ is a weak factorization in $s\text{Set}$ for every set of maps S . In particular, $(\overline{\text{InnHorn}}, \text{InnFib})$ is a weak factorization system.

13.13. Exercise (for those who know some homological algebra). In an abelian category, let \mathcal{A} be the class of injective maps with projective cokernel, and let \mathcal{B} be the class of surjections. Show that the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorization system if and only if the category has enough projectives. (This exercise is related to (13.5).)

13.14. Uniqueness of liftings. The relation $f \sqsupseteq g$ says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map $f: A \rightarrow B$, let $f^{\vee} := (f, f): B \amalg_A B \rightarrow B$ be the “fold” map, i.e., the unique map such that the composition with either of the canonical maps $B \rightarrow B \amalg_A B$ is f . It is straightforward to show that for a map $g: X \rightarrow Y$ we have that $\{f, f^{\vee}\} \sqsupseteq g$ if and only if in every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & \nearrow s & \downarrow g \\ B & \longrightarrow & Y \end{array}$$

there exists a *unique* lift s .

13.15. Example. Consider the category of topological spaces. Let \mathcal{A} be the class of morphisms of the form $A \times \{0\} \rightarrow A \times [0, 1]$, where A is an arbitrary space. Then $(\mathcal{A} \cup \mathcal{A}^{\vee})^{\square}$ contains all covering maps (by the “Covering Homotopy Theorem”).

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in which liftings of type $\mathcal{L} \sqsupseteq \mathcal{R}$ are always unique is called an **orthogonal factorization system**.

13.16. Exercise. Show that in an orthogonal factorization system, the factorizations $f = r\ell$ are unique up to *unique isomorphism*.

13.17. Exercise. Show that $(\{\text{surjections}\}, \{\text{injections}\})$ is an orthogonal factorization system for Set .

The small object argument implies that $(\overline{S \cup S^{\vee}}, (S \cup S^{\vee})^{\square})$ is an orthogonal factorization system for every set S of morphisms.

13.18. Example (The fundamental category via a weak factorization system). In simplicial sets, the projection map $C \rightarrow *$ is in the right complement to $S := \text{InnHorn} \cup \text{InnHorn}^{\vee}$ if and only if C is *isomorphic* to a nerve of a category (5.7). The small object argument using S , applied to a projection $X \rightarrow *$, thus produces a morphism $\pi: X \rightarrow Y$ in \overline{S} with Y the nerve of a category.

Uniqueness of liftings in this case implies that $\pi: X \rightarrow Y$ has precisely the universal property of the fundamental category of X defined in §9.1: given $f: X \rightarrow C$ with C a category, a unique extension of f over $X \rightarrow Y$ exists. Thus, the small object argument applied to S gives another construction of the fundamental category (9.1) of an arbitrary simplicial set S .

13.19. *Exercise.* Prove that if $f: X \rightarrow Y$ is any inner anodyne map, then the induced functor $h(f): hX \rightarrow hY$ between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to $h(f)$.)

14. THE SMALL OBJECT ARGUMENT

In this section we give the proof of (13.10), i.e., that given a fixed set $S = \{s_i: A_i \rightarrow B_i\}$ of maps of simplicial sets, we can factor any map $f: X \rightarrow Y$ as $f = pj$ with $j \in \bar{S}$ and $p \in S^\square$. For the reader: it may be helpful to first work through the special case where $Y = \Delta^0$ (the terminal object in simplicial sets).

14.1. **A factorization construction.** Given any map $f: X \rightarrow Y$, we first produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \quad (Rf)(Lf) = f$$

as follows. Consider the set

$$[S, f] := \{ (s_i, u, v) \mid s_i \in S, fu = vs_i \} = \left\{ \begin{array}{ccc} A_i & \xrightarrow{u} & X \\ s_i \downarrow & & \downarrow f \\ B_i & \xrightarrow{v} & Y \end{array} \right\}$$

of all commutative squares which have an arrow from S on the left-hand side, and f on the right-hand side. We define Ef , Lf , and Rf using the diagram

$$\begin{array}{ccccc} \coprod_{(s_i, u, v)} A_i & \xrightarrow{(u)} & X & & \\ \downarrow \coprod s_i & & \downarrow Lf & \searrow f & \\ \coprod_{(s_i, u, v)} B_i & \xrightarrow{\quad} & Ef & \xrightarrow{Rf} & Y \end{array}$$

(A curved arrow labeled (v) goes from $\coprod B_i$ to Y .)

where the the coproducts are indexed by the set $[S, f]$, and the square is a pushout. Note that $Lf \in \bar{S}$ by construction; however, we do not expect that Rf in S^\square .

We can iterate the construction:

$$\begin{array}{ccccccc} X & & & & & & \\ \parallel & \searrow Lf & \searrow L^2 f & \searrow L^3 f & \searrow L^\omega f & & \\ X & \xrightarrow{Lf} & Ef & \xrightarrow{LRf} & E^2 f & \xrightarrow{LR^2 f} & E^3 f & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & E^\omega f \\ \downarrow f & \nearrow Rf & \nearrow R^2 f & \nearrow R^3 f & \nearrow R^\omega f & & & & & & \\ Y & & & & & & & & & & \end{array}$$

Here each triple $(E^\alpha f, L^\alpha f, R^\alpha f)$ is obtained by factoring the “ R ” map of the previous one, so that

$$(14.2) \quad E^{\alpha+1} f := E(R^\alpha f), \quad L^{\alpha+1} f := L(R^\alpha f) \circ (L^\alpha f), \quad R^{\alpha+1} f := R(R^\alpha f).$$

Taking direct limits gives a factorization $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$ of f , with $E^\omega f = \text{colim}_{n \rightarrow \infty} E^n f$.

We can go even further, using the magic of transfinite induction, and define compatible factorizations $(E^\lambda f, L^\lambda f, R^\lambda f)$ for each ordinal¹² λ . For successor ordinals $\alpha + 1$ use the prescription of (14.2), while for limit ordinals β take a direct limit $E^\beta f := \text{colim}_{\alpha < \beta} E^\alpha f$ as in the construction of $E^\omega f$ above.

¹²For a treatment of ordinals, see for instance the chapter on sets in [TS14].

It is immediate that every $L^\alpha f \in \overline{S}$, because weak saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in S^\square , though they do satisfy a “partial lifting property”: whenever $\alpha < \beta$ there exists *by construction* a dashed arrow making

$$\begin{array}{ccccc}
 A_i & \xrightarrow{u'} & E^\alpha f & \longrightarrow & E^{\alpha+1} f & \longrightarrow & E^\beta f \\
 s_i \downarrow & & \nearrow \text{---} & & & & \downarrow R^\beta f \\
 B_i & & & \xrightarrow{v} & & & Y
 \end{array}$$

commute, for any u' and v making the square commute. Thus, we get a solution to a lifting problem (u, v) of s_i against $R^\beta f$ whenever the map $u: A_i \rightarrow E^\beta f$ on the top of a commutative square that we want a lift for can be factored through one of the maps $E^\alpha f \rightarrow E^\beta f$ with $\alpha < \beta$. This is so exactly because $E^{\alpha+1} f$ was obtained from $E^\alpha f$ by “formally adjoining” a solution to *every* such lifting problem.

The “small object argument” amounts to the following.

Claim. There exists an ordinal κ such that for every domain A_i of a map in S , every map $A_i \rightarrow E^\kappa f$ factors through some $E^\alpha f \rightarrow E^\kappa f$ with $\alpha < \kappa$.

Given this, it follows from the “partial lifting property” that $S \square R^\kappa f$, and so we obtain the desired factorization: $f = (R^\kappa f) \circ (L^\kappa f)$ with $L^\kappa f \in \overline{S}$ and $R^\kappa f \in S^\square$.

It remains to prove the claim, which we will do by choosing κ to be a *regular cardinal* which is “bigger” than all the simplicial sets A_i .

14.3. Regular cardinals. The **cardinality** of a set X is the smallest ordinal λ such that there exists a bijection between X and λ ; we write $|X|$ for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal ω is the countable cardinal.

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol \aleph_α denotes the α th infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that λ is a **regular cardinal**¹³ if it is an infinite cardinal, and if for every set A of ordinals such that (i) $\alpha < \lambda$ for all $\alpha \in A$, and (ii) $|A| < \lambda$, then $\sup A < \lambda$. For instance, ω is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular¹⁴; however, there exist arbitrarily large regular cardinals¹⁵.

Every ordinal α defines a category, which is the poset of ordinals strictly less than α . Colimits of functors $Y: \kappa \rightarrow \text{Set}$ with κ a regular cardinal have the following property: the map

$$(14.4) \quad \text{colim}_{\alpha < \kappa} \text{Hom}(X, Y_\alpha) \rightarrow \text{Hom}(X, \text{colim}_{\alpha < \kappa} Y_\alpha)$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

14.5. Exercise. Prove that (14.4) is a bijection when $|X| < \kappa$.

14.6. Small simplicial sets. Given a regular cardinal κ , we say that a simplicial set is κ -**small** if it is isomorphic to the colimit of some functor $F: C \rightarrow s\text{Set}$, such that (i) $|\text{ob } C|, |\text{mor } C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex Δ^n . Morally, we are saying that a simplicial set is κ -small if it can be “presented” with fewer than κ generators and fewer than κ relations.

Given a functor $Y: \kappa \rightarrow s\text{Set}$ and a κ -small simplicial set X , we have a bijection as in (14.4). (This is sometimes phrased as: *κ -small simplicial sets are κ -compact.*) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal κ greater than $\sup\{|A_i|\}$.

¹³In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

¹⁴For instance, $\aleph_\omega = \sup\{\aleph_k \mid k < \omega\}$ is not regular.

¹⁵For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.

14.7. *Example.* The standard simplices Δ^n , as well as any subcomplex such as the horns Λ_j^n , are ω -small: this is a consequence of (4.19). Thus, when we carry out the small object argument for $S = \text{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

14.8. **Functoriality.** The construction $f \mapsto (X \xrightarrow{L^f} Ef \xrightarrow{R^f} Y)$ is a functor $\text{Fun}([1], s\text{Set}) \rightarrow \text{Fun}([2], s\text{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any α . Because the choice of regular cardinal κ depends only on S , not on the map f , we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of \overline{S} with an element S^\square . We will have use of this later.

15. NON-DEGENERATE ELEMENTS AND THE SKELETAL FILTRATION

We have noted that monomorphisms of simplicial sets form a weakly saturated class. Here we identify an important set of maps called Cell , so that the weak saturation of Cell is precisely the class of monomorphisms. We do so by getting a very explicit handle on monomorphisms of simplicial sets. This will involve the notion of *degenerate* and *non-degenerate* elements of a simplicial set.

15.1. **Boundary of a standard simplex.** For each $n \geq 0$, we define

$$\partial\Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \setminus \{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the n -simplex. Equivalently,

$$(\partial\Delta^n)_k = \{f: [k] \rightarrow [n] \mid f([k]) \neq [n]\}.$$

We call $\partial\Delta^n$ the **boundary** of Δ^n . Note that $\partial\Delta^0 = \emptyset$ and $\partial\Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}}$.

15.2. *Exercise.* Show that $\partial\Delta^n$ is the largest subcomplex of Δ^n which does not contain the “generator” $(0 \dots n) \in (\Delta^n)_n$. In other words, $\partial\Delta^n$ is the maximal *proper* subcomplex of Δ^n .

15.3. *Exercise.* Show that if C is a category, then the evident maps $\text{Hom}(\Delta^n, C) \rightarrow \text{Hom}(\partial\Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

15.4. **Trivial fibrations and monomorphisms.** Let Cell be the set consisting of the inclusions $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is $\text{TrivFib} := \text{Cell}^\square$, the class of **trivial fibrations** (also sometimes called **acyclic fibrations**). By the small object argument, we obtain a weak factorization system $(\overline{\text{Cell}}, \text{TrivFib})$.

Since the elements of Cell are monomorphisms, and the class of all monomorphisms is weakly saturated, we see that all elements of $\overline{\text{Cell}}$ are monomorphisms. We are going to prove the converse, i.e., we will show that $\overline{\text{Cell}}$ is precisely equal to the class of monomorphisms.

15.5. **Degenerate and non-degenerate elements.** Recall $\Delta^{\text{surj}}, \Delta^{\text{inj}} \subset \Delta$, the subcategories of the category Δ of simplicial operators, consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively, and that every operator factors uniquely as $f = f^{\text{inj}} f^{\text{surj}}$, a surjection followed by an injection.

An element $a \in X_n$ is said to be **degenerate** if there exists a *non-injective* simplicial operator $f \in \Delta$ and an element b in X such that $a = bf$. In view of the factorization $f = f^{\text{inj}} f^{\text{surj}}$, we see that a is degenerate if and only if there exists a *non-identity surjective* simplicial operator $f \in \Delta^{\text{surj}}$ and an element b in X such that $a = b\sigma$.

Likewise, an element $a \in X_n$ is said to be **non-degenerate** if it is not degenerate, i.e., if $a = bf$ for some f in Δ and b in X we must have $f \in \Delta^{\text{inj}}$. Equivalently, a is non-degenerate if $a = bf$ for some f in Δ^{surj} and b in X implies $f = \text{id}$.

We write $X_n = X_n^{\text{deg}} \amalg X_n^{\text{nd}}$ for the decomposition of X_n into complementary subsets of degenerate and non-degenerate elements. Note that if $f: A \rightarrow X$ is a map of simplicial sets, then $f(A_n^{\text{deg}}) \subseteq X_n^{\text{deg}}$,

while $f^{-1}(X_n^{\text{nd}}) \subseteq A_n^{\text{nd}}$. Note that neither X_n^{deg} nor X_n^{nd} assemble to give a subcomplex of X (unless X is empty).

15.6. Proposition. *If X is a simplicial set and $A \subseteq X$ is a subcomplex, then $A_n^{\text{nd}} = X_n^{\text{nd}} \cap A_n$ and $A_n^{\text{deg}} = X_n^{\text{deg}} \cap A_n$.*

Proof. The first statement is a consequence of the second, since subsets of degenerate and non-degenerate element are complementary. It is clear that $A_n^{\text{deg}} \subseteq X_n^{\text{deg}} \cap A_n$. Conversely, suppose $a \in X_n^{\text{deg}} \cap A_n$, so $a \in A_n$ and $a = bg$ for some non-identity $g: [n] \rightarrow [k] \in \Delta^{\text{surj}}$ and $b \in X_k$. Any surjection in Δ has a section (4.17), so there exists $s: [k] \rightarrow [n]$ such that $gs = 1_{[k]}$. Then $b = bgs = as \in A_k$, whence $a \in A_n^{\text{deg}}$ as desired. \square

15.7. Exercise (easy). For any simplicial set X , we have $X_0^{\text{deg}} = \emptyset$ and $X_0^{\text{nd}} = X_0$, while X_1^{deg} is the image of $\langle 00 \rangle^*: X_0 \rightarrow X_1$ and X_1^{nd} is its complement.

15.8. Example. Here are all elements in the standard 2-simplex up to dimension 3, with the non-degenerate ones indicated by a box.

$(\Delta^2)_0$	$(\Delta^2)_1$	$(\Delta^2)_2$	$(\Delta^2)_3$
$\langle 0 \rangle$	$\langle 00 \rangle$	$\langle 000 \rangle$	$\langle 0000 \rangle$
$\langle 1 \rangle$	$\langle 11 \rangle$	$\langle 111 \rangle$	$\langle 1111 \rangle$
$\langle 2 \rangle$	$\langle 22 \rangle$	$\langle 222 \rangle$	$\langle 2222 \rangle$
	$\langle 01 \rangle$	$\langle 001 \rangle \langle 011 \rangle$	$\langle 0001 \rangle \langle 0011 \rangle \langle 0111 \rangle$
	$\langle 02 \rangle$	$\langle 002 \rangle \langle 022 \rangle$	$\langle 0002 \rangle \langle 0022 \rangle \langle 0222 \rangle$
	$\langle 12 \rangle$	$\langle 112 \rangle \langle 122 \rangle$	$\langle 1112 \rangle \langle 1122 \rangle \langle 1222 \rangle$
		$\langle 012 \rangle$	$\langle 0012 \rangle \langle 0112 \rangle \langle 0122 \rangle$

15.9. Exercise. Describe the degenerate and non-degenerate elements of all the standard n -simplices Δ^n .

15.10. Exercise. For every $n \geq 0$, let S^n be the pushout of the diagram $\Delta^n \leftarrow \partial\Delta^n \rightarrow \Delta^0$, where $\partial\Delta^n \hookrightarrow \Delta^n$ is the usual inclusion and $\partial\Delta^n \rightarrow \Delta^0$ is the unique map to the terminal object. Describe all degenerate and non-degenerate elements of S^n .

15.11. Exercise. Show that if C is an ordinary category, then an element $a \in N(C)_k$ of the nerve is non-degenerate if and only if it is represented by a composable sequence of *non-identity* maps $c_0 \rightarrow \dots \rightarrow c_k$ in the category C .

The following exercises show that the subcomplexes of a simplicial set X can be completely characterized by the sets of non-degenerate elements of X that they contain.

15.12. Exercise. Let $X^{\text{nd}} = \coprod_{n \geq 0} X_n^{\text{nd}}$ be the set of non-degenerate elements of X . For $x, y \in X^{\text{nd}}$ write $y \leq x$ if there exists $f \in \Delta^{\text{inj}}$ such that $y = xf$. Show that “ \leq ” is a partial order on the set X^{nd} ; it is called the *face relation*.

15.13. Exercise. Show that if $xf = yg$ for some $x, y \in X^{\text{nd}}$, $f \in \Delta$ and $g \in \Delta^{\text{surj}}$, then $y \leq x$.

15.14. Exercise. Let $S \subseteq X^{\text{nd}}$ be a subset of non-degenerate elements which is closed downward under “ \leq ”, i.e., $y \leq x$ and $x \in S$ implies $y \in S$. Show that there exists a unique subcomplex $A \subseteq X$ such that $A^{\text{nd}} = S$. (Hint: the elements of A are of the form xg where $x \in S$ and $g \in \Delta^{\text{surj}}$.)

15.15. Simplicial sets are canonically free with respect to surjective operators. The key observation is that degenerate elements in a simplicial set are precisely determined by knowledge of the non-degenerate elements.

15.16. Proposition (Eilenberg-Zilber lemma). *Let a be an element of X . Then there exists a unique pair (b, σ) consisting of a non-degenerate element b and a map σ in Δ^{surj} such that $a = b\sigma$.*

Proof. [GZ67, §II.3]. Given $\sigma: [n] \rightarrow [m]$, let $\Gamma(\sigma) = \{ \delta: [m] \rightarrow [n] \mid \sigma\delta = \text{id}_{[m]} \}$ denote the set of sections of σ . The sets $\Gamma(\sigma)$ is non-empty when $\sigma \in \Delta^{\text{surj}}$ (4.17). We note the following elementary observation, whose proof is left for the reader:

$$\text{If } \sigma, \sigma' \in \Delta^{\text{surj}} \text{ are such that } \Gamma(\sigma) = \Gamma(\sigma'), \text{ then } \sigma = \sigma'.$$

Let $a \in X_n$ be such that $a = b_i\sigma_i$ for $b_i \in X_{m_i}^{\text{nd}}$, $\sigma_i \in \Delta^{\text{surj}}([n], [m_i])$, for $i = 1, 2$. We want to show that $m_1 = m_2$, $b_1 = b_2$, and $\sigma_1 = \sigma_2$.

Pick any $\delta_1 \in \Gamma(\sigma_1)$ and $\delta_2 \in \Gamma(\sigma_2)$. Then we have

$$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1, \quad b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2,$$

so b_1 and b_2 are related by the simplicial operators $\sigma_2\delta_1$ and $\sigma_1\delta_2$. Since b_1 and b_2 are both non-degenerate, $\sigma_2\delta_1: [m_1] \rightarrow [m_2]$ and $\sigma_1\delta_2: [m_2] \rightarrow [m_1]$ must be injective. This implies $m_1 = m_2$, and since the only order-preserving injective map $[m] \rightarrow [m]$ is the identity map, we must have $\sigma_2\delta_1 = \text{id} = \sigma_1\delta_2$, from which it follows that $b_1 = b_2$. This also shows that $\delta_1 \in \Gamma(\sigma_2)$ and $\delta_2 \in \Gamma(\sigma_1)$. Since δ_1 and δ_2 were arbitrarily chosen sections, we have shown $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and therefore $\sigma_1 = \sigma_2$. \square

We can reinterpret the Eilenberg-Zilber lemma as follows.

15.17. Corollary. *For any simplicial set X , the evident maps*

$$\prod_{j \geq 0} X_j^{\text{nd}} \times \text{Hom}_{\Delta^{\text{surj}}}([n], [j]) \rightarrow X_n$$

defined by $(j, x, \sigma) \mapsto x\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators $[n'] \rightarrow [n]$.

Proof. The bijection is a restatement of (15.16). For the second statement, note that if $\tau: [n'] \rightarrow [n]$ is a surjective simplicial operator, then $(k, x, \sigma\tau) \mapsto (x\sigma)\tau$. \square

Another way to say this: the restricted functor $X|_{(\Delta^{\text{surj}})^{\text{op}}}: (\Delta^{\text{surj}})^{\text{op}} \rightarrow \text{Set}$ is *canonically isomorphic* to a coproduct of representable functors $\text{Hom}_{\Delta^{\text{surj}}}(-, [k])$ indexed by the nondegenerate simplices of X . Or more simply: simplicial sets are *canonically free* with respect to surjective simplicial operators.

15.18. Remark. A simplicial set can be recovered up to isomorphism if you only know (i) its sets of non-degenerate elements, and (ii) the faces of the non-degenerate elements. The proposition we proved above tells how to reconstruct the degenerate elements; simplicial operators on degenerate elements are computed using the fact that any simplicial operator factors into a surjection followed by an injection. M 4 Feb 2019

Warning. The faces of a non-degenerate element can be degenerate; this happens for instance in (15.10) when $n \geq 2$. If X is such that all faces of non-degenerate elements are also non-degenerate, then we get a functor $X^{\text{nd}}: (\Delta^{\text{inj}})^{\text{op}} \rightarrow \text{Set}$, and the full simplicial set X can be recovered from X^{nd} . For instance, this is so for the standard simplices Δ^n , as well as any subcomplexes of such. Functors $(\Delta^{\text{inj}})^{\text{op}} \rightarrow \text{Set}$ are the combinatorial data behind the notion of a Δ -*complex*, as seen in Hatcher's textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

15.19. Exercise. Fix an object $[n]$ in Δ , and consider the category $\Delta_{[n]/}^{\text{surj}}$, which has

- **objects** the surjective morphisms $\sigma: [n] \rightarrow [k]$ in Δ , and

- **morphisms** commutative triangles in Δ of the form

$$\begin{array}{ccc} & & [k] \\ & \sigma \nearrow & \downarrow \tau \\ [n] & & \\ & \sigma' \searrow & [k'] \end{array}$$

Show that the category $\Delta_{[n]}^{\text{surj}}$ is *isomorphic* to the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1, \dots, n\}$. In particular, $\Delta_{[n]}^{\text{surj}}$ is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).

15.20. *Exercise.* Let X be a simplicial set. Given $n \geq 0$ and $\sigma: [n] \rightarrow [k]$ in Δ^{surj} , let $X_n^\sigma := \sigma^*(X_k)$, the image of the operator σ^* in X_n . Show that $X_n^{\sigma \vee \sigma'} = X_n^\sigma \cap X_n^{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta_{[n]}^{\text{surj}}$. Conclude that for each $x \in X_n$ there exists a maximal σ such that $x \in X_n^\sigma$.

15.21. **Skeleta.** Given a simplicial set X , the k -**skeleton** $\text{Sk}_k X \subseteq X$ is the subcomplex with n -dimensional elements.

$$(\text{Sk}_k X)_n = \bigcup_{0 \leq j \leq k} \{yf \mid y \in X_j, f: [n] \rightarrow [j] \in \Delta\}.$$

It is immediate that this defines a subcomplex of X , which is in fact the smallest subcomplex containing all elements of dimensions $\leq k$. Note that $X = \bigcup_k \text{Sk}_k X$, and that a map $X \rightarrow Y$ of simplicial sets restricts to a map $\text{Sk}_k X \rightarrow \text{Sk}_k Y$.

In view of (15.16) and (15.17), we see that

$$(\text{Sk}_k X)_n \approx \prod_{0 \leq j \leq k} X_j^{\text{nd}} \times \text{Hom}_{\Delta^{\text{surj}}}([n], [j]).$$

Note that $X = \text{colim}_{k \rightarrow \infty} \text{Sk}_k X$. The complement of $\text{Sk}_{k-1} X$ in $\text{Sk}_k X$ consists precisely of the *nondegenerate* k -dimensional elements of X together with their degeneracies (in dimensions $> k$).

The skeleta constructions define functors $\text{Sk}_k: s\text{Set} \rightarrow s\text{Set}$.

15.22. *Example.* The $(n-1)$ -skeleton of the standard n -simplex is precisely what we have called its boundary: $\text{Sk}_{n-1} \Delta^n = \partial \Delta^n$. The only simplices of Δ^n not contained in its boundary are the generator $\iota = \langle 0 \dots n \rangle \in (\Delta_n)_n$ together with the degenerate elements associated to it.

15.23. **Proposition.** *The evident square*

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}}} \partial \Delta^k & \longrightarrow & \text{Sk}_{k-1} X \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}}} \Delta^n & \longrightarrow & \text{Sk}_k X \end{array}$$

is a pushout of simplicial sets. More generally, for any subcomplex $A \subseteq X$, the evident square

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \partial \Delta^k & \longrightarrow & A \cup \text{Sk}_{k-1} X \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \Delta^k & \longrightarrow & A \cup \text{Sk}_k X \end{array}$$

is a pushout.

Proof. In each of the above squares, the complements of the vertical inclusions coincide precisely. In particular, the complement of the inclusion $(A \cup \text{Sk}_{k-1} X)_n \subseteq (A \cup \text{Sk}_k X)_n$ is in bijective correspondence with $(X_k^{\text{nd}} \setminus A_k^{\text{nd}}) \times \text{Hom}_{\Delta^{\text{surj}}}([n], [k])$, and thus the square is a pushout (15.24). \square

In the proof, we used the following fact, which is worth recording.

15.24. **Lemma.** *If*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow i & & \downarrow j \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a pullback of simplicial sets such that (i) j is a monomorphism, and (ii) f induces in each degree n a bijection $Y'_n \setminus i(X'_n) \xrightarrow{\sim} Y_n \setminus j(X_n)$, then the square is a pushout square.

Proof. Verify the analogous statement for a pullback square of sets. □

15.25. **Corollary.** $\overline{\text{Cell}}$ *is precisely the class of monomorphisms.*

Proof. We know all elements of $\overline{\text{Cell}}$ are monomorphisms. Any monomorphism is isomorphic to an inclusion $A \subseteq X$ of a subcomplex, so we only need show that such inclusions are contained in $\overline{\text{Cell}}$. Since $X \approx \text{colim}_k A \cup \text{Sk}_k X$, (15.23) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell . □

15.26. **Geometric realization.** Recall the singular complex functor $\text{Sing}: \text{Top} \rightarrow \text{sSet}$ (8.7). This functor has a left adjoint $|-|: \text{sSet} \rightarrow \text{Top}$, called **geometric realization**, constructed explicitly by

$$(15.27) \quad |X| := \text{Cok} \left[\coprod_{f: [m] \rightarrow [n]} X_n \times \Delta_{\text{top}}^m \rightrightarrows \coprod_{[p]} X_p \times \Delta_{\text{top}}^p \right];$$

that is, take a collection of topological simplices indexed by elements of X , and make identifications according to the simplicial operators in X . (Here the symbol “Cok” represents taking a “coequalizer”, i.e., the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$.)

15.28. *Exercise.* Describe the two unlabelled maps in (15.27). Then show that $|-|$ is in fact left adjoint to Sing .

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that $|\Delta^n| \approx \Delta_{\text{top}}^n$, and that $|\partial\Delta^n| \approx \partial\Delta_{\text{top}}^n$. Applying this to the skeletal filtration, we discover that there are pushouts

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}}} \partial\Delta_{\text{top}}^k & \longrightarrow & |\text{Sk}_{k-1} X| \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}}} \Delta_{\text{top}}^k & \longrightarrow & |\text{Sk}_k X| \end{array}$$

of spaces, and that $|X| = \bigcup |\text{Sk}_k X|$ with the direct limit topology. Thus, $|X|$ is presented to us as a *CW-complex*, whose cells are in an evident bijective correspondence with the set of non-degenerate elements of X .

16. PUSHOUT-PRODUCT AND PULLBACK-POWER

We are going to prove several “enriched” versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we’ll be able to prove that function complexes of quasicategories are themselves quasicategories.

16.1. Definition of pushout-product and pullback-hom. Given maps $f: A \rightarrow B$, $g: K \rightarrow L$ and $h: X \rightarrow Y$ of simplicial sets, we define new maps $f \square g$ and $g \square h$ called the **pushout-product**¹⁶ and the **pullback-hom**^{17,18}. The pushout-product $f \square g: (A \times L) \amalg_{A \times K} (B \times K) \rightarrow B \times L$ is the unique map fitting in the diagram

$$\begin{array}{ccccc}
 A \times K & \xrightarrow{\text{id} \times g} & A \times L & & \\
 \downarrow f \times \text{id} & & \downarrow & \searrow f \times \text{id} & \\
 B \times K & \longrightarrow & (A \times L) \amalg_{A \times K} (B \times K) & \xrightarrow{f \square g} & B \times L \\
 & \searrow \text{id} \times g & & &
 \end{array}$$

while the pullback-hom $h \square g: \text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$ is the unique map fitting in the diagram

$$\begin{array}{ccccc}
 \text{Map}(L, X) & & & & \\
 \downarrow h \square g & \searrow & & \searrow & \\
 \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y) & \longrightarrow & \text{Map}(K, X) & & \\
 \downarrow & & \downarrow \text{Map}(\text{id}, h) & & \\
 \text{Map}(L, Y) & \xrightarrow{\text{Map}(f, \text{id})} & \text{Map}(K, Y) & &
 \end{array}$$

16.2. Remark. Typically we form $f \square g$ when f and g are monomorphisms, in which case $f \square g$ is also a monomorphism. In this case, the elements $(b, \ell) \in B_n \times L_n$ which are *not* in the image of $f \square g$ are exactly those such that $b \in B_n \setminus A_n$ and $\ell \in L_n \setminus K_n$.

16.3. Remark (Important!). On vertices, the pullback-hom $h \square g$ is just the “usual” map $\text{Hom}(L, X) \rightarrow \text{Hom}(K, X) \times_{\text{Hom}(K, Y)} \text{Hom}(L, Y)$ sending $s \mapsto (sg, hs)$. Thus, $h \square g$ is surjective on vertices if and only if $g \square h$.

We think of the pullback-hom as encoding an “enriched” version of the lifting problem for (g, h) . Thus, the target of $h \square g$ is an object which “parameterizes families” of commutative squares involving g and h . Similarly, the source of $h \square g$ “parameterizes families” of commutative squares together with lifts.

The product/mapping object adjunction gives rise to the following relationship between lifting problems.

16.4. Proposition. *We have that $(f \square g) \square h$ if and only if $f \square (h \square g)$.*

¹⁶This is sometimes called the **box-product**. Some also call it the **Leibniz-product**, as its form is that of the Leibniz rule for boundary of a product space: $\partial(X \times Y) = (\partial X \times Y) \cup_{\partial X \times \partial Y} (X \times \partial Y)$ (which is itself reminiscent of the original Leibniz rule $D(fg) = (Df)g + f(Dg)$ of calculus).

¹⁷Sometimes called the **box-power** or **pullback-power**. A common alternate notation is $g \pitchfork h$. This may also be called the **Leibniz-hom**, though I don’t know what rule of calculus it is related to.

¹⁸This notation for pullback-hom is kinda awkward, and I’d like to change it. However, a new notation ought to admit compatible variants to describe the “pullback-slice” and “alternate pullback-slice” constructions which appear later on. I don’t see a good way to do this.

Proof. Compare the two lifting problems using the product/map adjunction.

$$\begin{array}{ccc}
 (A \times L) \amalg_{A \times K} (B \times K) & \xrightarrow{(u,v)} & X \\
 f \square g \downarrow & \dashrightarrow s & \downarrow h \\
 B \times L & \xrightarrow{w} & Y
 \end{array}
 \iff
 \begin{array}{ccc}
 A & \xrightarrow{\tilde{u}} & \text{Map}(L, X) \\
 f \downarrow & \dashrightarrow \tilde{s} & \downarrow h \square g \\
 B & \xrightarrow[(\tilde{v}, \tilde{w})]{} & \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)
 \end{array}$$

On the left-hand side are maps

$$u: A \times L \rightarrow X, \quad v: B \times K \rightarrow X, \quad w: B \times L \rightarrow Y, \quad s: B \times L \rightarrow X,$$

while on the right-hand side are maps

$$\tilde{u}: A \rightarrow \text{Map}(L, X), \quad \tilde{v}: B \rightarrow \text{Map}(K, X), \quad \tilde{w}: B \rightarrow \text{Map}(L, Y), \quad \tilde{s}: B \rightarrow \text{Map}(L, X).$$

The data of (u, v, w) giving a commutative square as on the left corresponds bijectively to data $(\tilde{u}, \tilde{v}, \tilde{w})$ giving a commutative square as on the right. Similarly, lifts s correspond bijectively to lifts \tilde{s} . \square

It is important to note the special cases where one or more of $A = \emptyset$, $K = \emptyset$, or $Y = *$ hold. For instance, if $K = \emptyset$ and $Y = *$, the proposition implies

$$(A \times L \xrightarrow{f \times L} B \times L) \square (X \rightarrow *) \quad \text{iff} \quad (A \xrightarrow{f} B) \square (\text{Map}(L, X) \rightarrow *).$$

This is the kind of case we are interested in for proving that $\text{Map}(K, C)$ is a quasicategory whenever C is. The more general statement of the proposition is a kind of “relative” version of the thing we want; it is especially handy for carrying out inductive arguments.

16.5. *Exercise* (if you like monoidal categories). Let $\mathcal{C} := \text{Fun}([1], \text{sSet})$, the “arrow category” of simplicial sets. Show that $\square: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ defines a symmetric monoidal structure on \mathcal{C} , with unit object $(\emptyset \subset \Delta^0)$. Furthermore, show that this is a *closed* monoidal structure, with $-\square g$ left adjoint to $(-)^{\square g}: \mathcal{C} \rightarrow \mathcal{C}$.

16.6. **Inner anodyne maps and pushout-products.** The key fact we want to prove is the following.

16.7. **Proposition.** *We have that $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$, i.e., that $i \square j$ is inner anodyne whenever i is inner anodyne and j is a monomorphism.*

To set up the proof we need the following.

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16.8. **Proposition.** *For any sets of maps S and T , we have $\overline{S \square T} \subseteq \overline{S} \square \overline{T}$.*

Proof. Let $\mathcal{F} = (S \square T)^{\square}$. From the small object argument we have that $\overline{S \square T} = {}^{\square} \mathcal{F}$ (13.11), so we will show $(\overline{S \square T}) \square \mathcal{F}$. First we show that $(\overline{S} \square T) \square \mathcal{F}$. Consider

$$\begin{aligned}
 \mathcal{A} &:= \{ a \mid (a \square T) \square \mathcal{F} \} \\
 &\approx \{ a \mid a \square (\mathcal{F}^{\square T}) \}
 \end{aligned}$$

by correspondence between lifting problems for pushout-products and pullback-homs (16.4). Thus \mathcal{A} is a left complement, and so is weakly saturated. Since $S \subseteq \mathcal{A}$ then $\overline{S} \subseteq \mathcal{A}$, i.e., $(\overline{S} \square T) \square \mathcal{F}$. The same idea applied to

$$\mathcal{B} := \{ b \mid (\overline{S} \square b) \square \mathcal{F} \} \approx \left\{ b \mid b \square (\mathcal{F}^{\square \overline{S}}) \right\},$$

gives $\overline{T} \subseteq \mathcal{B}$, whence $(\overline{S} \square \overline{T}) \square \mathcal{F}$. \square

16.9. **Lemma.** *We have $\text{InnHorn} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$.*

Proof. This is a calculation, given in [Joy08a, App. H], and presented in the appendix (58.3). \square

Proof of (16.7). We have that

$$\overline{\text{InnHorn}\square\text{Cell}} \subseteq \overline{\text{InnHorn}\square\text{Cell}} \subseteq \overline{\text{InnHorn}}.$$

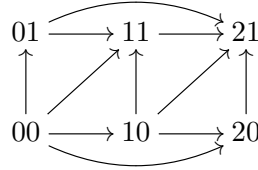
The first inclusion is (16.8), while the second is an immediate consequence of $\text{InnHorn}\square\text{Cell} \subseteq \overline{\text{InnHorn}}$ (16.9). \square

Let's carry out a proof of (16.9) explicitly in one case, by showing that $(\Lambda_1^2 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ is inner anodyne. This map is the inclusion

$$(\Lambda_1^2 \times \Delta^1) \cup_{\Lambda_1^2 \times \partial\Delta^1} (\Delta^2 \times \partial\Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a ‘‘prism’’, and whose source is a ‘‘trough’’. To show this is in $\overline{\text{InnHorn}}$, we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that $\Delta^2 \times \Delta^1 = N([2] \times [1])$, so we are working inside the nerve of a poset, whose elements (objects) are ‘‘ ij ’’ with $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. Here is a picture of the trough, showing all the non-degenerate simplices.



The complement of this in the prism consists of three non-degenerate 3-dimensional elements, five non-degenerate 2-dimensional elements (two of which form the ‘‘lid’’ of the trough, while the other three are in the interior of the prism), and one non-degenerate edge element (separating the two 2-dimensional elements which form the lid).

The following chart lists all non-degenerate elements in the complement of the trough, along with their codimension one faces (in order). The ‘‘ \surd ’’ marks elements which are contained in the trough.

$\langle 00, 21 \rangle$	$\langle 00, 20, 21 \rangle$	$\langle 00, 01, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 10, 20, 21 \rangle$	$\langle 00, 10, 11, 21 \rangle$	$\langle 00, 01, 11, 21 \rangle$
$\surd\langle 21 \rangle$	$\surd\langle 20, 21 \rangle$	$\surd\langle 01, 21 \rangle$	$\surd\langle 10, 21 \rangle$	$\surd\langle 11, 21 \rangle$	$\surd\langle 10, 20, 21 \rangle$	$\surd\langle 10, 11, 21 \rangle$	$\surd\langle 01, 11, 21 \rangle$
$\surd\langle 10 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 20, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 11, 21 \rangle$
	$\surd\langle 00, 20 \rangle$	$\surd\langle 00, 01 \rangle$	$\surd\langle 00, 10 \rangle$	$\surd\langle 00, 11 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 01, 21 \rangle$
					$\surd\langle 00, 10, 20 \rangle$	$\surd\langle 00, 10, 11 \rangle$	$\surd\langle 00, 01, 11 \rangle$

Note that the elements $\langle 00, 21 \rangle$, $\langle 00, 10, 21 \rangle$, and $\langle 00, 11, 21 \rangle$ of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

- ① $\langle 00, 10, 21 \rangle$,
- ② $\langle 00, 10, 20, 21 \rangle$,
- ③ $\langle 00, 10, 11, 21 \rangle$,
- ④ $\langle 00, 01, 11, 21 \rangle$.

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits $(\Lambda_1^2 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ as an inner anodyne map.

17. FUNCTION COMPLEXES OF QUASICATEGORIES ARE QUASICATEGORIES

17.1. Enriched lifting properties. We record the immediate consequences of $\overline{\text{InnHorn}\square\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.7).

17.2. Proposition.

- (1) *If $i: A \rightarrow B$ is inner anodyne and $j: K \rightarrow L$ a monomorphism, then*

$$i\square j: (A \times L) \cup_{A \times K} (B \times K) \rightarrow B \times L$$

is inner anodyne.

- (2) *If $j: K \rightarrow L$ is a monomorphism and $p: X \rightarrow Y$ is an inner fibration, then*

$$p\square j: \text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is an inner fibration.

(3) If $i: A \rightarrow B$ is inner anodyne and $p: X \rightarrow Y$ is an inner fibration, then

$$p^{\square i}: \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a trivial fibration.

These can be summarized as

$$\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}, \quad \text{InnFib}^{\square \overline{\text{Cell}}} \subseteq \text{InnFib}, \quad \text{InnFib}^{\square \overline{\text{InnHorn}}} \subseteq \text{TrivFib}.$$

Statement (1) is just restating (16.7). The other two statements follow from (1) using the correspondence between lifting problems for pushout-products and pullback-homs (16.4), together with the facts that $\text{InnFib} = \text{InnHorn}^{\square}$ and $\text{TrivFib} = \text{Cell}^{\square}$. For instance, (2) follows from the observation that $i \not\square p^{\square j}$ iff $(i \square j) \not\square p$, and that $i \in \overline{\text{InnHorn}}$ and $j \in \overline{\text{Cell}}$ imply $i \square j \in \overline{\text{InnHorn}}$. Likewise (3) follows a similar argument using that $j \not\square p^{\square i}$ iff $(i \square j) \not\square p$.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert “ $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” without other explanation; alternately, to indicate an application of statements (2) and (3), I will call it “enriched lifting”. The following gives the most general statement, of which (16.7) amounts to the special case of $S = U = \text{InnHorn}$ and $T = \text{Cell}$.

17.3. Proposition. *Let $S, T,$ and U be sets of morphisms in $s\text{Set}$. Write $\overline{S}, \overline{T},$ and \overline{U} for the weak saturations of these sets, and let $S\text{Fib} := S^{\square}, T\text{Fib} := T^{\square},$ and $U\text{Fib} := U^{\square}$ denote the respective right complements. If $S \square T \subseteq \overline{U}$, then*

$$\overline{S} \square \overline{T} \subseteq \overline{U}, \quad U\text{Fib}^{\square \overline{T}} \subseteq S\text{Fib}, \quad U\text{Fib}^{\square \overline{S}} \subseteq T\text{Fib}.$$

Proof. Exercise using (16.4). □

There are many useful special cases of (17.2), obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i: A \rightarrow B$ is inner anodyne, so is $i \times \text{id}_L: A \times L \rightarrow B \times L$.
- If $p: X \rightarrow Y$ is an inner fibration, then so is $\text{Map}(L, p): \text{Map}(L, X) \rightarrow \text{Map}(L, Y)$.
- If $j: K \rightarrow L$ is a monomorphism and C a quasicategory, then $\text{Map}(j, C): \text{Map}(L, C) \rightarrow \text{Map}(K, C)$ is an inner fibration.
- If $i: A \rightarrow B$ is inner anodyne and C a quasicategory, then $\text{Map}(i, C): \text{Map}(B, C) \rightarrow \text{Map}(A, C)$ is a trivial fibration.
- If C is a quasicategory, so is $\text{Map}(L, C)$. Thus we have proved (B).

Let’s spell out the proof of (B) in a little more detail. Because $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$, we have (16.7) that

$$(\Lambda_j^n \subset \Delta^n) \square (\emptyset \subseteq K) = (\Lambda_j^n \times K \rightarrow \Delta^n \times K)$$

is inner anodyne for any K and $0 < j < n$. Thus, for any diagram

$$\begin{array}{ccc} \Lambda_j^n \times K & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n \times K & & \end{array}$$

with C a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend $\Lambda_j^n \rightarrow \text{Map}(K, C)$ along $\Lambda_j^n \subset \Delta^n$. That is, we have proved that $\text{Map}(K, C)$ is a quasicategory.

17.4. Remark. Most weakly saturated classes \overline{S} that we will explicitly discuss in these notes will have the property that $S \square \text{Cell} \subseteq \overline{S}$, and thus analogues of the above remarks will hold for such classes.

17.5. Exercise (Important). Show that $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. (Hint: (15.25).) State the analogue of (17.2) associated to this inclusion.

17.6. Composition functors. *Rewrite this to use $n = 2$ as the basic example.*

We can use the above theory to construct “composition functors”. If C is an ordinary category, the operation of composing a sequence of n maps can be upgraded to a functor:

$$\mathrm{Fun}([1], C) \times_C \cdots \times_C \mathrm{Fun}([1], C) \rightarrow \mathrm{Fun}([1], C)$$

which on *objects* describes composition of a sequence of maps. The source of this functor is the evident inverse limit in Cat ; it can be identified (using simplicial set language) with $\mathrm{Fun}(I^n, C)$.

We can generalize such composition functors to quasicategories. We use the following observation: any trivial fibration admits a section, since $(\emptyset \rightarrow Y) \sqsupset (p: X \rightarrow Y)$ if p is a trivial fibration (17.2).

Let C be a quasicategory. Then map $r: \mathrm{Fun}(\Delta^n, C) \rightarrow \mathrm{Fun}(I^n, C)$ induced by restriction along $I^n \subseteq \Delta^n$ is a trivial fibration by (17.2), since the spine inclusion is inner anodyne (12.11). Therefore r admits a section s , so we get a diagram

$$\mathrm{Fun}(I^n, C) \xleftarrow[r]{} \mathrm{Fun}(\Delta^n, C) \xrightarrow[r']{} \mathrm{Fun}(\Delta^{\{0,n\}}, C)$$

where r' is restriction along $\Delta^{\{0,n\}} \subset \Delta^n$. The composite $r's$ can be thought of as a kind of “ n -fold composition” functor. It is not unique, since s isn't, but we'll see (??) that this is ok: all functors constructed this way are “naturally isomorphic” to each other.

17.7. A useful variant. The proof of (16.7) actually proves something a little stronger.

17.8. Proposition ([Joy08a, §2.3.1], [Lur09, §2.3.2]). *We have that $\overline{\{\Lambda_1^2 \subset \Delta^2\} \square \mathrm{Cell}} = \overline{\mathrm{InnHorn}}$.*

Proof. We give a proof in the appendix (58.3). \square

A consequence of this is another characterization of quasicategories.

17.9. Corollary. *A simplicial set C is a quasicategory if and only if $f: \mathrm{Map}(\Delta^2, C) \rightarrow \mathrm{Map}(\Lambda_1^2, C)$ is a trivial fibration.*

Proof. First notice that $(\partial\Delta^k \subset \Delta^k) \sqsupset f$ for all $k \geq 0$ iff $(\partial\Delta^k \subset \Delta^k) \square (\Lambda_1^2 \subset \Delta^2) \sqsupset (C \rightarrow *)$ for all $k \geq 0$, since $f = (C \rightarrow *) \square \{\Lambda_1^2 \subset \Delta^2\}$. Therefore $f \in \mathrm{TrivFib} = \mathrm{InnHorn}^{\square}$ if and only if $(C \rightarrow *) \in (\mathrm{Cell} \square \{\Lambda_1^2 \subset \Delta^2\})^{\square}$. The conclusion immediately follows using (17.8). \square

18. NATURAL ISOMORPHISMS

18.1. Natural isomorphisms of functors. Let C and D be quasicategories. Recall that a natural transformation between functors $f_0, f_1: C \rightarrow D$ is defined to be a morphism $\alpha: f_0 \rightarrow f_1$ in the functor quasicategory $\mathrm{Fun}(C, D)$, or equivalently a map $\tilde{\alpha}: C \times \Delta^1 \rightarrow D$ such that $\tilde{\alpha}|_{C \times \Delta^{\{i\}}} = f_i$, $i = 0, 1$.

Say that $\alpha: f_0 \rightarrow f_1$ is a **natural isomorphism** if α is an isomorphism in the quasicategory of functors $\mathrm{Fun}(C, D)$. Thus, α is a natural isomorphism iff there exists a natural transformation $\beta: f_1 \rightarrow f_0$ such that $\beta\alpha \approx 1_{f_0}$ and $\alpha\beta \approx 1_{f_1}$, where “ \approx ” is homotopy between morphisms in the quasicategory $\mathrm{Fun}(C, D)$.

This notion of natural isomorphism corresponds with the usual one for ordinary categories, since in that case homotopy of morphisms is the same as equality of morphisms.

Observe that “there exists a natural isomorphism $f_0 \rightarrow f_1$ ” is an equivalence relation on the set of all functors $C \rightarrow D$, as this relation precisely coincides with “there exists an isomorphism $f_0 \rightarrow f_1$ ” in the category $h\mathrm{Fun}(C, D)$. We say that f_0 and f_1 are **naturally isomorphic** functors.

Furthermore, the “naturally isomorphic” relation is compatible with composition: if f, f' are naturally isomorphic and g, g' are naturally isomorphic, then so are gf and $g'f'$. You can read this off from the fact the operation of composition of functors extends to a functor $\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D) \rightarrow \mathrm{Fun}(C, E)$ between quasicategories, and so induces a functor

$$h\mathrm{Fun}(D, E) \times h\mathrm{Fun}(C, D) \approx h(\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D)) \rightarrow h\mathrm{Fun}(C, E).$$

(This uses (9.15) to identify the homotopy category of the product with the product of homotopy categories.)

18.2. Objectwise criterion for natural isomorphisms. Recall that if C and D are ordinary categories, a natural transformation $\alpha: f_0 \rightarrow f_1$ between functors $f_0, f_1: C \rightarrow D$ is a natural isomorphism iff and only if α is “an isomorphism objectwise”; i.e., if for each object c of C the evident map $\alpha(c): f_0(c) \rightarrow f_1(c)$ is an isomorphism in D . That natural isomorphisms are “objectwise isomorphisms” is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its “values on objects”. Thus, given $\alpha: f_0 \rightarrow f_1$ such that each $\alpha(c): f_0(c) \rightarrow f_1(c)$ is an isomorphism, we may explicitly construct an inverse transformation $\beta: f_1 \rightarrow f_0$ by setting $\beta(c) := \alpha(c)^{-1}: f_1(c) \rightarrow f_0(c)$. Note that this β is in fact the *unique* inverse to α (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

18.3. Proposition. *Let C and D be quasicategories. If $\alpha: C \times \Delta^1 \rightarrow D$ is a natural isomorphism between functors $f_0, f_1: C \rightarrow D$, then for each object c of C the induced map $\alpha(c): f_0(c) \rightarrow f_1(c)$ is an isomorphism in D .*

Proof. The map $\text{Fun}(C, D) \rightarrow \text{Fun}(\{c\}, D) = D$ induced by restriction along $\{c\} \subseteq C$ is a functor between quasicategories, so it takes isomorphisms to isomorphisms. It sends α to $\alpha(c)$. \square

The converse to this proposition is also true.

C. Deferred Proposition. *A natural transformation $\alpha: C \times \Delta^1 \rightarrow D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in D .*

Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$ s to produce an inverse to α , which though it exists is not at all unique. We will prove this converse later (30).

18.4. Remark. An immediate consequence of (C) is that if D is a *quasigroupoid*, then so is $\text{Fun}(C, D)$.

18.5. Remark. The objectwise criterion (C) can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation $\alpha: f_0 \rightarrow f_1$ of functors $f_0, f_1: C \rightarrow D$ between quasicategories (i.e., a functor $\alpha: C \times \Delta^1 \rightarrow D$ such that $(\alpha|_{C \times \{j\}}) = f_j$), we obtain an induced transformation $h\alpha: hf_0 \rightarrow hf_1$ of functors $hf_0, hf_1: hC \rightarrow hD$ between their homotopy categories (so that the value of $h\alpha$ at an object $c \in \text{ob } hC = C_0$ is the homotopy class of the edge $\alpha(\{c\} \times \Delta^1) \subseteq D$). Then (C) asserts that α is a natural isomorphism of functors between quasicategories if and only if $h\alpha$ is a natural isomorphism of functors between ordinary categories.

19. CATEGORICAL EQUIVALENCE

We are now in position to define the correct generalization of the notion of “equivalence” of categories. This will be called *categorical equivalence* of quasicategories, and will be a direct generalization of the classical notion.

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Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets. Finally, we will show that the two definitions agree for maps between quasicategories.

19.1. Categorical equivalences between quasicategories. A **categorical inverse** to a functor $f: C \rightarrow D$ between quasicategories is a functor $g: D \rightarrow C$ such that gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D . We provisionally say that a functor f between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

19.2. Remark. Categorical equivalence between quasicategories is a kind of “homotopy equivalence”, where homotopies are natural isomorphisms between functors.

If C and D are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

If quasicategories are equivalent, then their homotopy categories are equivalent.

19.3. Proposition. *If $f: C \rightarrow D$ is a categorical equivalence between quasicategories, then $h(f): hC \rightarrow hD$ is an equivalence of categories.*

Proof. Immediate, given that natural transformations $f \Rightarrow g: C \rightarrow D$ induce natural transformations $h(f) \Rightarrow h(g): hC \rightarrow hD$. \square

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to Δ^0 , but whose homotopy categories are: e.g., $\text{Sing } T$ for any non-contractible simply connected space T , or $K(A, d)$ for any non-trivial abelian group A and $d \geq 2$.

19.4. Exercise (Categorical inverses are unique up to natural isomorphism). Let $f: C \rightarrow D$ be a functor between quasicategories, and suppose $g, g': D \rightarrow C$ are both categorical inverses to f . Show that g and g' are naturally isomorphic.

19.5. General categorical equivalence. We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map $f: X \rightarrow Y$ between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory C , the induced functor $\text{Fun}(f, C): \text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$ of quasicategories admits a categorical inverse.

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

19.6. Lemma. *For a map $f: C \rightarrow D$ between quasicategories, the two notions of categorical equivalence described above coincide. That is, the following are equivalent:*

- (1) f admits a categorical inverse.
- (2) For every quasicategory E , the functor $\text{Fun}(f, E): \text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$ admits a categorical inverse.

To prove this, we will need the following observation. The construction $X \mapsto \text{Map}(X, E)$ is a functor $s\text{Set}^{\text{op}} \rightarrow s\text{Set}$, and so in particular induces a natural map

$$\gamma_0: \text{Hom}(X, Y) \rightarrow \text{Hom}(\text{Map}(Y, E), \text{Map}(X, E))$$

of sets, which sends $f: X \rightarrow Y$ to $\text{Map}(f, E): \text{Map}(Y, E) \rightarrow \text{Map}(X, E)$. The observation we need is that this construction admits an “enrichment”, to a map

$$\gamma: \text{Map}(X, Y) \rightarrow \text{Map}(\text{Map}(Y, E), \text{Map}(X, E)),$$

which coincides with γ_0 on vertices. The map γ is defined to be adjoint to the “composition” map $\text{Map}(X, Y) \times \text{Map}(Y, E) \rightarrow \text{Map}(X, E)$. (*Exercise:* Describe explicitly what γ does to n -dimensional elements.) We say that the functor $\text{Map}(-, E)$ is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on vertices in function complexes), but in fact gives a map between function complexes.

Proof. (1) \implies (2). When C , D , and E are quasicategories so are the function complexes between them. In this case, the above map γ takes functors $C \rightarrow D$ to functors $\text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$ between quasicategories, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Using this observation, it is straightforward to show that a categorical inverse $g: D \rightarrow C$ to $f: C \rightarrow D$ gives rise to a categorical inverse $\text{Map}(g, E): \text{Map}(C, E) \rightarrow \text{Map}(C, D)$ to the induced functor $\text{Map}(f, E): \text{Map}(D, E) \rightarrow \text{Map}(C, E)$.

(2) \implies (1). Conversely, suppose $f: C \rightarrow D$ is a categorical equivalence in the general sense, so that $f^* = \text{Map}(f, E)$ admits a categorical inverse for every quasicategory E , which implies that each functor

$$h(f^*): h\text{Fun}(D, E) \rightarrow h\text{Fun}(C, E)$$

is an equivalence of ordinary categories (19.3). In particular, it follows that f^* induces a bijection of sets

$$f^*: \pi_0(\text{Fun}(D, E)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(C, E)^{\text{core}});$$

recall that $\pi_0(\text{Fun}(D, E)^{\text{core}}) \approx \pi_0((h\text{Fun}(D, E))^{\text{core}})$ is precisely the set of *natural isomorphism classes* of functors $D \rightarrow E$.

Taking $E = C$, this implies that there must exist $g \in \text{Fun}(D, C)_0$ such that there exists a natural isomorphism $gf \rightarrow \text{id}_C$ in $\text{Fun}(C, C)_1$. Taking $E = D$, we note that since

$$f^*(\text{id}_D) = \text{id}_D f = f \text{id}_C \approx f g f = f^*(f g),$$

we must have that $\text{id}_D \approx f g$, i.e., there exists a natural isomorphism $\text{id}_D \rightarrow f g$ in $\text{Fun}(D, D)_1$. Thus, we have shown that g is a categorical inverse of f , as desired. \square

19.7. Remark. The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie [Lur09, §2.2.5]. It is also slightly different from the notion of “weak categorical equivalence” used by Joyal [Joy08a, 1.20]. As we will show soon (22.12), Joyal’s weak categorical equivalence is equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show’s that Lurie’s and Joyal’s definitions are equivalent, and so they are both equivalent to the one we have used.

19.8. Exercise. Let $f: C \rightarrow D$ be a functor between quasicategories. Show that f is a categorical equivalence if and only if for all simplicial sets X , the induced functor $f_*: \text{Map}(X, C) \rightarrow \text{Map}(X, D)$ is a categorical equivalence.

20. TRIVIAL FIBRATIONS AND INNER ANODYNE MAPS

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

20.1. Trivial fibrations to the terminal object. Recall that a trivial fibration $p: X \rightarrow Y$ of simplicial sets is a map such that $(\partial\Delta^k \subset \Delta^k) \sqsupseteq p$ for all $k \geq 0$. That is, $\text{TrivFib} = \text{Cell}^{\square}$, so p is a trivial fibration if and only if $\text{Cell} \sqsupseteq p$.

20.2. Exercise. Consider an indexed collection of trivial fibrations $p_i: X_i \rightarrow Y_i$. Show that $p := \coprod p_i: \coprod X_i \rightarrow \coprod Y_i$ is a trivial fibration. (Hint: similar to proof of (6.7).)

20.3. Proposition. *Let X be a simplicial set and $p: X \rightarrow *$ be a trivial fibration whose target is the terminal simplicial set. Then X is a Kan complex (and thus a quasigroupoid) and p is a categorical equivalence.*

Proof. Enriched lifting (17.3) applied to $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ (17.5) means that for any monomorphism $i: A \rightarrow B$ of subcomplexes the pullback-hom map

$$p^{\square i} = \text{Map}(i, X): \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, *)} \text{Map}(B, *) = \text{Map}(A, X)$$

is a trivial fibration. In particular, it implies that $\text{Map}(i, X)$ is surjective on vertices, so $\text{Hom}(B, X) \rightarrow \text{Hom}(A, X)$ is surjective.

It follows immediately that X is a Kan complex, by taking i to be any horn inclusion.

To show that p is a categorical equivalence, first note that X is non-empty, since $\text{Hom}(\Delta^0, X) \rightarrow \text{Hom}(\emptyset, X) = *$ is surjective. Choose any $s \in \text{Hom}(\Delta^0, X)$. Clearly $ps = \text{id}_{\Delta^0}$. We will show that $sp: X \rightarrow X$ is naturally isomorphic to id_X . Consider the commutative diagram

$$\begin{array}{ccccc}
 & & (\text{id}_X, sp) & & \\
 & & \curvearrowright & & \\
 X \amalg X & \xrightarrow{\quad} & X \times \partial\Delta^1 & \xrightarrow{\quad} & X \\
 & & \downarrow & \nearrow h & \downarrow p \\
 & & X \times \Delta^1 & \xrightarrow{\quad} & *
 \end{array}$$

Since p is a trivial fibration, a lift h exists, which exhibits a natural transformation $\text{id}_X \rightarrow sp$; note that h represents a morphism in $\text{Fun}(X, X)$. To show that h represents an isomorphism, it's enough to know that $\text{Fun}(X, X)$ is actually a quasigroupoid. In fact, restriction along $\emptyset \rightarrow X$ a trivial fibration

$$\text{Fun}(X, X) \rightarrow \text{Fun}(\emptyset, X) = *,$$

whence $\text{Fun}(X, X)$ is a Kan complex by the argument given above. □

We will prove a partial converse to this later (34.10): if C is a quasicategory which is categorically equivalent to $*$, then $C \rightarrow *$ is a trivial fibration.

20.4. Preisomorphisms. We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let X be a simplicial set. Say that an edge $a \in X_1$ is a **preisomorphism** if it projects to an isomorphism under $\alpha: X \rightarrow hX$, the tautological map to the (nerve of the) fundamental category (9.1). If X is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

20.5. Proposition. *An edge $a \in X_1$ is a preisomorphism if and only if for every map $g: X \rightarrow C$ to a quasicategory C , the image $g(a)$ is an isomorphism in C .*

Proof. Isomorphisms in C are exactly the edges which are sent to isomorphisms under $\gamma: C \rightarrow hC$. Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that hX is itself a category and hence a quasicategory. □

As a consequence, any map $X \rightarrow Y$ of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

20.6. Example. Consider the subcomplex $\Lambda_{\{0,3\}}^3 = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3 . Define Z to be the pushout of the diagram

$$\Lambda_{\{0,3\}}^3 \xleftarrow{j} \Delta^{\{0,2\}} \amalg \Delta^{\{1,3\}} \xrightarrow{p} \Delta^{\{x\}} \amalg \Delta^{\{y\}}$$

where j is the evident inclusion, $\Delta^{\{x\}}$ and $\Delta^{\{y\}}$ are simplicial sets isomorphic to Δ^0 , but with vertices labelled “ x ” and “ y ” respectively, and p is induced by the evident projections $\Delta^{\{0,2\}} \rightarrow \Delta^{\{x\}}$ and $\Delta^{\{1,3\}} \rightarrow \Delta^{\{y\}}$. The resulting complex Z looks like

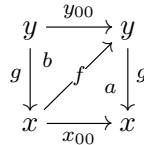
$$\begin{array}{ccc}
 y & \xrightarrow{y_{00}} & y \\
 g \downarrow & \nearrow f & \downarrow h \\
 x & \xrightarrow{x_{00}} & x
 \end{array}$$

with seven non-degenerate elements: $x, y \in Z_0, f, g, h \in Z_1, a, b \in Z_2$. The simplicial set Z is not a quasicategory (*why not?*). However, any map $\phi: Z \rightarrow C$ to a quasicategory sends f, g, h to morphisms $\phi(f), \phi(g), \phi(h)$ of C so that $\phi(g)$ is a preinverse of $\phi(f)$ and $\phi(h)$ is a postinverse of $\phi(f)$. Therefore these (and thus all) edges of Z are preisomorphisms.

20.7. *Example.* Here is a variant of the previous example. Consider the subcomplex $\Lambda_{\{0,3\}}^3 = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3 . Define Z' to be the pushout of the diagram

$$\Lambda_{\{0,3\}}^3 \xleftarrow{j} \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,3\}} \cup \Delta^{\{2,3\}} \xrightarrow{p} \Delta^{\{y<x\}},$$

where j is the evident inclusion, $\Delta^{\{y<x\}}$ is a simplicial set isomorphic to Δ^1 but with vertices labelled “ y ” and “ x ” instead of “0” and “1”, and p is the unique map which on vertices sends $0, 2 \mapsto y, 1, 3 \mapsto x$. The resulting complex Z' looks like



with six non-degenerate elements: $x, y \in Z'_0, f, g \in Z'_1, a, b \in Z'_2$. Again, Z' is not a quasicategory, but all edges of Z' are preisomorphisms.

Say that vertices in a simplicial set X are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g: X \rightarrow C$ to a quasicategory takes preisomorphic vertices of X to isomorphic objects of C .

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We can apply this to function complexes. If two maps $f_0, f_1: X \rightarrow Y$ are preisomorphic (viewed as vertices in $\text{Map}(X, Y)$), then for any quasicategory C , the induced functors $\text{Map}(f_0, C), \text{Map}(f_1, C): \text{Map}(Y, C) \rightarrow \text{Map}(X, C)$ are naturally isomorphic. To see this, consider

$$\Delta^1 \xrightarrow{a} \text{Map}(X, Y) \xrightarrow{b} \text{Map}(\text{Map}(Y, C), \text{Map}(X, C))$$

where b is adjoint to the composition map $\text{Map}(Y, C) \times \text{Map}(X, Y) \rightarrow \text{Map}(X, C)$. If a represents a preisomorphism $f_0 \rightarrow f_1$ in $\text{Map}(X, Y)$, then ba represents an isomorphism $\text{Map}(f_0, C) \rightarrow \text{Map}(f_1, C)$, since the target of b is a quasicategory. As a consequence we get the following.

20.8. **Lemma.** *If $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are maps of simplicial sets such that gf is preisomorphic to id_X in $\text{Map}(X, X)$ and fg is preisomorphic to id_Y in $\text{Map}(Y, Y)$, then f and g are categorical equivalences.*

It is important to note that this is a sufficient condition for a map to be a categorical equivalence, but not a necessary one: there are many categorical equivalences of simplicial sets to which the lemma cannot be applied (see (21.3) below).

20.9. **Trivial fibrations are always categorical equivalences.**

20.10. **Proposition.** *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps $f: A \rightarrow Y$ and $g: B \rightarrow Y$, we write $\text{Map}_Y(f, g)$ or $\text{Map}_Y(A, B)$ for the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{Map}_Y(A, B) & \longrightarrow & \text{Map}(A, B) \\ \downarrow & & \downarrow g_* = \text{Map}(A, g) \\ \{f\} & \longrightarrow & \text{Map}(A, Y) \end{array}$$

Note that vertices of $\text{Map}_{/Y}(A, B)$ correspond exactly to “sections of g over f ”, i.e., to $s: A \rightarrow B$ such that $gs = f$. You can think of $\text{Map}_{/Y}(A, B)$ as a simplicial set which “parameterizes” sections of g over f . I’ll call this the **relative function complex over Y** .

20.11. *Exercise.* Show that n -dimensional elements of $\text{map}_{/Y}(A, B)$ correspond to maps $a: \Delta^n \times A \rightarrow B$ such that $ga = \pi(\text{id} \times f)$, where $\pi: \Delta^n \times Y \rightarrow Y$ is the projection.

Proof of (20.10). Fix a trivial fibration $p: X \rightarrow S$. We regard both X and S as objects *over* S , via p and id_S , and consider various relative function complexes over S .

Note that since p is a trivial fibration, so are $\text{Map}(X, p) = p^{\square(\emptyset \subset X)}$ and $\text{Map}(S, p) = p^{\square(\emptyset \subset X)}$ by enriched lifting $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. The maps

$$\text{Map}_{/S}(S, X) \rightarrow \text{Map}_{/S}(S, S) = * \quad \text{and} \quad \text{Map}_{/S}(X, X) \rightarrow \text{Map}_{/S}(X, S) = *$$

are (by construction) base changes of $\text{Map}(S, p)$ and $\text{Map}(X, p)$ respectively, and so are also trivial fibrations since TrivFib is closed under base change. It follows from (20.3) that both $\text{Map}_{/S}(S, X)$ and $\text{Map}_{/S}(X, X)$ are quasigroupoids which are categorically equivalent to the terminal object (and so are non-empty and such that all objects are isomorphic). Note that these are subcomplexes of simplicial sets $\text{Map}(S, X)$ and $\text{Map}(X, X)$ respectively, which however need not be quasicategories. The edges of $\text{Map}_{/S}(S, X)$ and $\text{Map}_{/S}(X, X)$ are preisomorphisms in $\text{Map}(S, X)$ and $\text{Map}(X, X)$.

Pick any vertex s of $\text{Map}_{/S}(S, X)$, so that s can be regarded as a map $s: S \rightarrow X$ such that $ps = \text{id}_S$. Pick any isomorphism $a: \text{id}_X \rightarrow sp$ in $\text{Map}_{/S}(X, X)$, which is hence a preisomorphism in $\text{Map}(X, X)$.

Thus, we have exhibited maps p and s whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (20.8). \square

20.12. *Remark* (“Uniqueness” of sections of trivial fibrations). Suppose that $p: C \rightarrow D$ is a trivial fibration between quasicategories. As we have noted, the relative function complex $\text{Map}_{/D}(D, C)$ “parameterizes sections of p ”. Since this is a quasigroupoid equivalent to the terminal quasicategory (20.10), not only is p a categorical equivalence, but also

- p admits a section, which is a categorical inverse to p , and
- any two sections of p are naturally isomorphic.

We will often make use of this observation.

20.13. **Inner anodyne maps are always categorical equivalences.**

20.14. **Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

Proof. Let $j: X \rightarrow Y$ be a map in $\overline{\text{InnHorn}}$, and let C be any quasicategory. The induced map $\text{Map}(j, C): \text{Map}(Y, C) \rightarrow \text{Map}(X, C)$ is a trivial fibration by enriched lifting and $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (17.2), and therefore is a categorical equivalence. \square

20.15. **Every simplicial set is categorically equivalent to a quasicategory.**

20.16. **Proposition.** *Fix a simplicial set X .*

- (1) *There exists a quasicategory C and an inner anodyne map $f: X \rightarrow C$, which is therefore a categorical equivalence.*
- (2) *For any two $f_i: X \rightarrow C_i$ as in (1), there exists a categorical equivalence $g: C_1 \rightarrow C_2$ such that $gf_1 = f_2$.*

Here is some more notation. Given maps $f: X \rightarrow A$ and $g: X \rightarrow B$, we write $\text{Map}_{X/}(f, g)$ or $\text{Map}_{X/}(A, B)$ for the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{Map}_{X/}(A, B) & \longrightarrow & \text{Map}(A, B) \\ \downarrow & & \downarrow f^* = \text{Map}(f, B) \\ \{g\} & \longrightarrow & \text{Map}(X, B) \end{array}$$

This is the **relative function complex under X** .

20.17. *Exercise.* Show that n -dimensional elements of $\text{map}_{X/}(A, B)$ correspond to maps $a: \Delta^n \times A \rightarrow B$ such that $a(\text{id} \times f) = g\pi$, where $\pi: \Delta^n \times X \rightarrow X$ is the projection

Proof of (20.16). (1) By the small object argument (13.10), we can factor $X \rightarrow *$ into $X \xrightarrow{j} C \xrightarrow{p} *$ where $j \in \text{InnHorn}$ and $p \in \text{InnFib}$. The inner anodyne map j is the desired categorical equivalence to a quasicategory.

(2) For $i, j \in \{1, 2\}$, we have a restriction map $f_{i,j}^*: \text{Map}(C_i, C_j) \rightarrow \text{Map}(X, C_j)$, which is necessarily a trivial fibration by enriched lifting since $\overline{\text{Cell} \square \text{Cell}} \subseteq \overline{\text{Cell}}$. Therefore the maps $\text{Map}_{X/}(C_i, C_j) \rightarrow *$ (obtained by base-change from the $f_{i,j}^*$) are all trivial fibrations, i.e., each $\text{Map}_{X/}(C_i, C_j)$ is a quasigroupoid with only one isomorphism class of objects (20.3). As in the proof of (20.10) we construct $g: C_1 \rightarrow C_2$ and $g': C_2 \rightarrow C_1$ which are categorically inverse to each other; details are left to the reader. \square

Thus, we can always “replace” a simplicial set X by a categorically equivalent quasicategory C . Although such C is not unique, it is unique up to categorical equivalence.

You can think of such a replacement $X \rightarrow C$ of X as a quasicategory “freely generated” by the simplicial set X , an idea which is validated by the fact that $\text{Fun}(j, D): \text{Fun}(C, D) \rightarrow \text{Map}(X, D)$ is a categorical equivalence for every quasicategory D .

21. SOME EXAMPLES OF CATEGORICAL EQUIVALENCES

21.1. **Free monoid on one generator.** Let F denote the free monoid on one generator g . This is a category with one object x , and morphism set $\{g^n \mid n \geq 0\}$.

Associated to the generator g is a map

$$\gamma: S := \Delta^1 / \partial \Delta^1 \rightarrow N(F)$$

sending the image of the generator $\iota \in (\Delta^1)_1$ in S to g . (We use “ L/K ” as a shorthand for “ $L \amalg_K *$ ” whenever $K \subseteq L$. The object S is called the “simplicial circle”, which has exactly two nondegenerate simplices, one in dimension 0 and one in dimension 1.)

It is not hard to see that F is “freely generated” as a category by S , in the sense that $hS = F$ (the fundamental category of S is F). It turns out that $N(F)$ is actually freely generated as a *quasicategory* by S .

21.2. **Proposition.** *The map $\gamma: S \rightarrow N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

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Proof. This is an explicit calculation. Note that a general element in $N(F)_d$ corresponds to a sequence $(g^{m_1}, \dots, g^{m_d})$ of elements of the monoid F , where $m_1, \dots, m_d \geq 0$. Let $a_k \in N(F)_k$ denote the k -dimensional element corresponding to the sequence (g, g, \dots, g) , and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of the representing map $a_k: \Delta^k \rightarrow N(F)$. For $f: [d] \rightarrow [k]$ we compute that $a_k f = (g^{m_1}, \dots, g^{m_d})$ where $m_i = f(i) - f(i-1)$, so that

$$(Y_k)_d = \{a_k f \mid f: [d] \rightarrow [k]\} = \{(g^{m_1}, \dots, g^{m_d}) \mid m_1 + \dots + m_d \leq k\},$$

Clearly $N(F) = \bigcup_{k \geq 1} Y_k$, with $Y_1 \approx S$ and $Y_2 \approx Y_1 \cup_{\Lambda_1^2} \Delta^2$. Furthermore we have the following:

- An element f of $(\Delta^k)_d$ is such that $a_k f$ is in the subcomplex Y_{k-1} of Y_k if and only if $f(d) - f(0) < k$, i.e., if and only if f is in the subcomplex $\Lambda_{\{0,k\}}^k = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}}$.
- Every element y of Y_k not in Y_{k-1} is the image under a_k of a *unique* element in Δ^k . (I.e., if $f: [d] \rightarrow [k]$, then $m_1 + \dots + m_d = f(d) - f(0)$, which is equal to k if and only if $f(0) = 0$ and $f(d) = k$, and if this is the case then $f(i) = m_1 + \dots + m_i$.)

In other words, the square

$$\begin{array}{ccc} \Lambda_{\{0,k\}}^k & \longrightarrow & Y_{k-1} \\ \downarrow & & \downarrow \\ \Delta^k & \xrightarrow{a_k} & Y_k \end{array}$$

is a pullback, and a_k induces in each dimension d a bijection $(\Delta^k)_d \setminus (\Lambda_{\{0,k\}}^k)_d \xrightarrow{\sim} (Y_k)_d \setminus (Y_{k-1})_d$. It follows (15.24) that the square is a pushout.

The inclusion $\Lambda_{\{0,k\}}^k \subset \Delta^k$ is a *generalized inner horn*, and we have noted this is inner anodyne when $k \geq 2$ (12.9). It follows that each $Y_{k-1} \rightarrow Y_k$ is inner anodyne for $k \geq 2$, whence $S \rightarrow N(F)$ is inner anodyne. \square

21.3. Remark. This gives an explicit example of a categorical equivalence to which (20.8) does not apply: γ does not admit an “inverse up to preisomorphisms”. There is only one map $\delta: N(F) \rightarrow S$, namely the composite $N(F) \rightarrow * \rightarrow S$, and it is clear that neither $\delta\gamma: N(F) \rightarrow N(F)$ nor $\gamma\delta: S \rightarrow S$ are preisomorphic to identity functors.

21.4. Free categories. We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let S be a **1-dimensional** simplicial set, i.e., one such that $S = \text{Sk}_1 S$. These are effectively the same thing as *directed graphs* (allowed to have multiple parallel edges and loops): S_0 corresponds to the set of vertices of the directed graph, and S_1^{nd} corresponds to the set of edges of the directed graph.

Let $F := hS$. We call F the **free category** on the 1-dimensional simplicial set S . In this case, the morphisms of the fundamental category are precisely the words in the edges S_1^{nd} of the directed graph (including empty words for each vertex, corresponding to identity maps).

21.5. Proposition. *The evident map $\gamma: S \rightarrow N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is virtually the same as the proof of (21.2). In this case, $Y_k \subseteq N(F)$ is the subcomplex generated by all $a: \Delta^k \rightarrow N(F)$ such that each spine-edge $a_{i-1,i}$ is in S_1^{nd} , and Y_k is obtained by attaching a generalized horn to Y_{k-1} for each such a . \square

As a consequence, it is “easy” to construct functors $F \rightarrow C$ from a free category to a quasicategory: start with a map $S \rightarrow C$, which amounts to specifying vertices and edges in C corresponding to elements S_0 and S_1^{nd} , and extend over $S \subseteq F$. The evident restriction map $\text{Fun}(F, C) \rightarrow \text{Map}(S, C)$ is a categorical equivalence, and in fact a trivial fibration.

21.6. Exercise. Describe the ordinary category $A := h\Lambda_0^3$ “freely generated” by Λ_0^3 . Show that the tautological map $\Lambda_0^3 \rightarrow N(A)$ is inner anodyne.

21.7. Free commutative monoids. Let F be the free monoid on one generator again, with generator corresponding to simplicial circle $S = \Delta^1/\partial\Delta^1 \subset F$. Recall that $F^{\times n}$ is the free commutative monoid on n generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map

$$\delta = \gamma^{\times n}: S^{\times n} \rightarrow N(F^{\times n})$$

from the “simplicial n -torus”.

21.8. Proposition. *The map $\delta: S^{\times n} \rightarrow N(F^{\times n})$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is a consequence of the fact that if $j: A \rightarrow B$ is inner anodyne and K an arbitrary simplicial set, then $j \times K \times: A \times K \rightarrow B \times K$ is inner anodyne (because $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$). It follows that $A^{\times n} \rightarrow B^{\times n}$ is a composite of inner anodyne maps, and so is inner anodyne and thus a categorical equivalence (20.14). Also use the fact that the nerve construction preserves products (6.5), so $N(F^{\times n}) = N(F)^{\times n}$. \square

21.9. Exercise. Let $S \vee S \subset S^{\times 2}$ be the subcomplex obtained as the evident “one-point union” of the two “coordinate circles”; i.e., $S \vee S = (S \times \{*\}) \cup (\{*\} \times S)$. Suppose given a map $\phi: S \vee S \rightarrow C$ to a quasicategory C , corresponding to a choice of object $x \in C_0$ together with two morphisms $f, g: x \rightarrow x$ in C_1 . Show that there exists an extension of ϕ along $S \vee S \subset N(F^{\times 2})$ if and only if $[f][g] = [g][f]$ in hC .

21.10. Remark. The analogue of the above exercise for $n = 3$ isn’t true. That is, consider the subcomplex $S \vee S \vee S \subset S^{\times 3}$ which is a one-point union of three circles, suppose we have $S \vee S \vee S \rightarrow C$ corresponding to three morphisms $f, g, h: x \rightarrow x$ in C , and suppose we also know that $[f][g] = [g][f]$, $[g][h] = [h][g]$, and $[f][h] = [h][f]$ in hC . Then you can show that there exists an extension to a map $\text{Sk}_2(S^{\times 3}) \rightarrow C$ as in (21.9). However, there need not exist an extension to a map $S^{\times 3} \rightarrow C$, and thus there may not exist an extension to a map $N(F^{\times 3}) \rightarrow C$. (For an explicit example where this fails, take $C = \text{Sing} T$, where $T \subseteq (S^1)^{\times 3}$ is the subspace of the 3-torus consisting of tuples (x_1, x_2, x_3) such that at least one x_i is the basepoint of S^1 .)

Thus, this is a situation where the “higher structure” of a quasicategory plays a role. When C is an ordinary category, it is easy to show that the desired extension does always exist. However, for a general quasicategory C , three pairwise-commuting endomorphisms of an object do not generally give rise to a functor $N(F^{\times 3}) \rightarrow C$ from the free commutative monoid on 3 generators.

21.11. Finite groups are not finite. If A is any ordinary category, then $\text{Sk}_2 N(A)$ “freely generates $N(A)$ as a category”, in the sense that $h(\text{Sk}_2 N(A)) \approx A$, or equivalently that $\text{Map}(N(A), N(B)) \rightarrow \text{Map}(\text{Sk}_2 N(A), N(B))$ is an isomorphism for any category B . However, it is often the case that *no finite dimensional subcomplex* “freely generates $N(A)$ as a quasicategory”. In fact, this is the case for every non-trivial finite group.

21.12. Example. Let G be a finite group, and let $C = N(G)$. The geometric realization $BG := |N(G)|$ is the *classifying space* of G . I want to show that if G is not the trivial group, then NG is not categorically equivalent to any *finite dimensional simplicial set* K (i.e., one with no non-degenerate elements above a certain dimension). We need to use some homotopy theory, along with a fact to be proved later¹⁹: if $f: X \rightarrow Y$ is any categorical equivalence of simplicial sets, then the induced map $|f|: |X| \rightarrow |Y|$ on geometric realizations is a homotopy equivalence of spaces.

First consider $G = \mathbb{Z}/n$ with $n > 1$. A standard calculation in topology says that $H^{2k}(|N(G)|, \mathbb{Z}) \approx \mathbb{Z}/n \not\approx 0$ for all $k > 0$. This implies that $|N(G)|$ cannot be homotopy equivalent to any finite dimensional complex.

Now consider a general non-trivial finite group G , and choose a non-trivial cyclic subgroup $H \leq G$. We know the fundamental group: $\pi_1 |K| \approx \pi_1 |N(G)| = G$. Covering space theory tells us we can construct a covering map $p: E \rightarrow |N(G)|$ so that $\pi_1 E \rightarrow \pi_1 |N(G)|$ is the inclusion $H \rightarrow G$. In fact, E is homotopy equivalent to the classifying space BH (because $\pi_k E \approx 0$ for $k \geq 2$). But if $|N(G)|$ is finite dimensional then so is E , but this would then contradicting the observation that $H^*(BH, \mathbb{Z}) \approx H^*(E, \mathbb{Z}) \approx 0$ for infinitely many $*$.

Thus, non-trivial *finite* groups are never “freely generated as a quasicategory” by finite dimensional complexes.

¹⁹I don’t know if this will actually get proved later. It is proved in [GJ09]

21.13. *Remark.* Let T be a *finite* CW-complex, and G a finite group. A theorem of Haynes Miller (the “Sullivan conjecture”) implies that every functor $N(G) \rightarrow \text{Sing } T$ is naturally isomorphic to a constant functor (i.e., one which factors through Δ^0). Thus, the singular complex of a finite CW-complex cannot “contain” any non-trivial finite group, even if its fundamental group contains a non-trivial finite subgroup.

22. THE HOMOTOPY CATEGORY OF QUASICATEGORIES

22.1. **The homotopy category of qCat .** The homotopy category $h\text{qCat}$ of quasicategories is defined as follows. The objects of $h\text{qCat}$ are the quasicategories. Morphisms $C \rightarrow D$ in $h\text{qCat}$ are natural isomorphism classes of functors. That is,

$$\text{Hom}_{h\text{qCat}}(C, D) := \text{isomorphism classes of objects in } h\text{Fun}(C, D) = \pi_0(\text{Fun}(C, D)^{\text{core}}).$$

That this defines a category results from the fact that composition of functors passes to a functor $h\text{Fun}(D, E) \times h\text{Fun}(C, D) \rightarrow h\text{Fun}(C, E)$, and thus is compatible with natural isomorphism.

It comes with an obvious functor $\text{qCat} \rightarrow h\text{qCat}$. Note that a map $f: C \rightarrow D$ of quasicategories is a categorical equivalence if and only if its image in $h\text{qCat}$ is an isomorphism.

22.2. *Remark.* We can similarly define a category $h\text{Cat}$, whose objects are categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding $h\text{Cat} \rightarrow h\text{qCat}$.

22.3. *Warning.* Although we use the word “homotopy”, the definition of $h\text{qCat}$ given above is not an example of the notion of the homotopy category of a quasicategory defined in §9: qCat is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

For future reference, we note that $h\text{qCat}$ has finite products, which just amount to the usual products of simplicial sets.

22.4. **Proposition.** *The terminal simplicial set Δ^0 is a terminal object in $h\text{qCat}$. If C_1, C_2 are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in $h\text{qCat}$.*

Proof. This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (6.13), and the fact that $\text{Map}(X, C_1 \times C_2) \xrightarrow{\sim} \text{Map}(X, C_1) \times \text{Map}(X, C_2)$. \square

22.5. **The 2-out-of-6 and 2-out-of-3 properties.** A class of morphisms \mathcal{W} in a category is said to satisfy the **2-out-of-6 property** if (i) \mathcal{W} contains all identity maps, and (ii) given sequence (h, g, f) of maps such that the composites gf and hg are defined, if $gf, hg \in \mathcal{W}$ then also $f, g, h, hgf \in \mathcal{W}$.

A class of morphisms \mathcal{W} in a category is said to satisfy the **2-out-of-3 property** if (i) \mathcal{W} contains all identity maps, and (ii) given a sequence (g, f) of maps such that the composite gf is defined, if any two of (f, g, gf) are in \mathcal{W} , so is the third.

22.6. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-6 property and the 2-out-of-3 property. The class of identity maps satisfies 2-out-of-3, but does not generally satisfy 2-out-of-6.

22.7. **Proposition.** *If \mathcal{W} satisfies 2-out-of-6, then it satisfies 2-out-of-3.*

Proof. Given f, g such that gf is defined, apply 2-out-of-6 to the composable sequences (id, g, f) , (g, id, f) , (g, f, id) . \square

22.8. *Exercise.* Given a functor $f: C \rightarrow D$ between categories, let \mathcal{W} be the class of maps in C that f takes to isomorphisms in D . Show that \mathcal{W} satisfies 2-out-of-6, and thus 2-out-of-3.

22.9. *Example* (2-out-of-6 for equivalences of categories). In Cat , the category of small categories and functors, the class of equivalences satisfies 2-out-of-6, and thus 2-out-of-3.

To see this, first suppose (h, g, f) is a triple of functors such that there are natural isomorphisms $gf \approx \text{id}$ and $hg \approx \text{id}$. Then, since (i) natural isomorphism is an equivalence relation on functors and (ii) is compatible with composition, we see that

$$h = h \text{id} \approx h(gf) = (hg)f \approx \text{id} f = f,$$

and thus that g is an equivalence since $hg \approx \text{id}$ and $gh \approx gf \approx \text{id}$.

Next, note that composites of equivalences are equivalences, by a straightforward argument: if g and f are equivalences and composable, and g' and f' are categorical inverses to them, then $f'g'$ is easily seen to be a categorical inverse to gf .

Now suppose that (h, g, f) are such that gf and hg are categorical equivalences. Choose categorical inverses u and v for these, so that

$$gfu \approx \text{id}, \quad ugf \approx \text{id}, \quad hgv \approx \text{id}, \quad vhg \approx \text{id}.$$

Apply the above remarks to the triples (ug, f, ug) , (vh, g, fu) , (gv, h, gv) , and (ugv, hgf, vgu) to show that f, g, h are equivalences, where we use that

$$fug \approx (vhg)fug = vh(gfu)g \approx vhg \approx \text{id}, \quad gvh \approx gvh(gfu) = g(vhg)fu \approx gfu \approx \text{id}.$$

It follows that the composite hgf is also an equivalence.

Alternately, we can apply (22.8) to the tautological functor $\text{Cat} \rightarrow h\text{Cat}$, which sends a functor to an isomorphism in $h\text{Cat}$ if and only if it is an equivalence.

22.10. Proposition. *The class CatEq of categorical equivalences in $s\text{Set}$ satisfies 2-out-of-6, and thus 2-out-of-3.*

Proof. It is immediate that the identity map of a simplicial set is a categorical equivalence.

Next consider functors f, g, h between *quasicategories* such that gf and hg are defined and are categorical equivalences. Then f, g, h and hgf are categorical equivalences by an argument which is word-for-word the same as in (22.9).

For the general case, we reduce to the quasicategory case by applying $\text{Fun}(-, C)$, where C is an arbitrary quasicategory. \square

22.11. Weak categorical equivalence. Joyal [Joy08a, 1.20] uses a variant of the notion of categorical equivalence, which turns out to be equivalent to what we are using. A map $f: X \rightarrow Y$ of simplicial sets is a **weak categorical equivalence**²⁰ if for every quasicategory C , the induced map $h\text{Fun}(Y, C) \rightarrow h\text{Fun}(X, C)$ is an equivalence of ordinary categories. Note that, like categorical equivalences, the class of weak categorical equivalences also satisfies 2-out-of-3.

22.12. Proposition. *A map is a categorical equivalence if and only if it is a weak categorical equivalence.*

Proof. (\implies) Straightforward. (\impliedby) In the case that f is a weak categorical equivalence between *quasicategories*, this is exactly what the second half of the proof of (19.6) really shows. For a general map f , use factorization to construct a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ u \downarrow & & \downarrow v \\ X' & \xrightarrow{f'} & Y' \end{array}$$

²⁰This is not to be confused with “weak equivalence”, which we will talk about later (33.1).

so that u and v are inner anodyne (and so categorical equivalences), and X' and Y' are quasicategories. Applying $h\text{Fun}(-, C)$ to the square with C a quasicategory, we see that the vertical maps become equivalences of categories, so if f is weak categorical equivalence so is f' , which is then a categorical equivalence by what we have already proved, whence f is a categorical equivalence by 2-out-of-3. \square

22.13. The homotopy 2-category of qCat . A **2-category** E is a category which is itself “enriched” over Cat . That is,

- for each pair of objects $x, y \in \text{ob } E$, there is a category $\underline{\text{Hom}}_E(x, y)$, so that
- the objects of $\underline{\text{Hom}}_E(x, y)$ are precisely the set $\text{Hom}_E(x, y)$ of morphisms of E , and
- there are “composition functors” $\text{Hom}_E(y, z) \times \text{Hom}_E(x, y) \rightarrow \text{Hom}_E(x, z)$ for all $x, y, z \in \text{ob } E$ which on objects is just ordinary composition of morphisms in E , which
- is unital and associative in the evident sense.

One refers to the objects of $\text{Hom}_E(x, y)$ as **1-morphisms** $f: x \rightarrow y$ of E , and the morphisms of $\text{Hom}_E(x, y)$ as **2-morphisms** $\alpha: f \Rightarrow g$ of E . The **underlying category** of E consists of the objects and 1-morphisms only.

The standard example of a 2-category is Cat , the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category qCat of quasicategories to a **homotopy 2-category** $h_2\text{qCat}$, so that

$$\text{Hom}_{h_2\text{qCat}}(C, D) := h\text{Fun}(C, D).$$

That is,

- **objects** of $h_2\text{qCat}$ are quasicategories,
- **1-morphisms** of $h_2\text{qCat}$ are functors between quasicategories,
- **2-morphisms** of $h_2\text{qCat}$ are *isomorphism classes* of natural transformations of functors.

Note that qCat sits inside $h_2\text{qCat}$ as its underlying category; thus, $h_2\text{qCat}$ contains all the information of qCat . On the other hand $h\text{qCat}$ is obtained from $h_2\text{qCat}$ by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms.

Part 3. Joins, slices, and Joyal’s extension and lifting theorems

In this part we describe and apply two new methods to construct new quasicategories from old, called “joins” and “slices”. They are both generalizations of constructions which can be carried out on categories: the most familiar of these classical constructions is *slice category* C/x associated to an object x of a category C , in which objects of slice C/x are morphisms $c \rightarrow x$ in C .

With these constructions in hand, we will be able to define notions of *limits* and *colimits* in quasicategories. We will also be able to prove some of the results we have deferred up until now, including the equivalence of quasigroupoids and Kan complexes (A) and the objectwise criterion for natural isomorphisms (C). Much of the material in this part comes from Joyal’s seminal paper [Joy02].

23. JOINS

23.1. Join of categories. If A and B are ordinary categories, we can define a category $A \star B$ called the join. This has

$$\text{ob}(A \star B) = \text{ob } A \amalg \text{ob } B, \quad \text{mor}(A \star B) = \text{mor } A \amalg (\text{ob } A \times \text{ob } B) \amalg \text{mor } B,$$

so that we put in a *unique* map from each object of A to each object of B . Explicitly,

$$\mathrm{Hom}_{A \star B}(x, y) := \begin{cases} \mathrm{Hom}_A(x, y) & \text{if } x, y \in \mathrm{ob} A, \\ \mathrm{Hom}_B(x, y) & \text{if } x, y \in \mathrm{ob} B, \\ \{*\} & \text{if } x \in \mathrm{ob} A, y \in \mathrm{ob} B, \\ \emptyset & \text{if } x \in \mathrm{ob} B, y \in \mathrm{ob} A, \end{cases}$$

with composition defined so that the evident inclusions $A \rightarrow A \star B \leftarrow B$ are functors. (Check that this really defines a category, and that A and B are identified with full subcategories of $A \star B$.)

23.2. *Example.* We have that $[p] \star [q] \approx [p + 1 + q]$.

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23.3. *Exercise* (Functors from a join of categories). Show that functors $f: A \star B \rightarrow C$ are in bijective correspondence with triples $(f_A: A \rightarrow C, f_B: B \rightarrow C, \gamma: f_A \circ \pi_A \Rightarrow f_B \circ \pi_B)$, where f_A and f_B are functors, and γ is a natural transformation of functors $A \times B \rightarrow C$, where $\pi_A: A \times B \rightarrow A$ and $\pi_B: A \times B \rightarrow B$ denote the evident projection functors.

23.4. *Exercise* (Functors to a join of categories). Show that functors $f: C \rightarrow A \star B$ are in bijective correspondence with triples of functors $(\pi: C \rightarrow [1], f_{\{0\}}: C^{\{0\}} \rightarrow A, f_{\{1\}}: C^{\{1\}} \rightarrow B)$, where $C^{\{j\}} := \pi(\{j\}) \subseteq C$ is the fiber of π over $j \in \mathrm{ob}[1]$, i.e., the subcategory of C consisting of objects which π sends to j and morphisms which π sends to id_j .

23.5. *Exercise.* Describe an isomorphism $(A \star B)^{\mathrm{op}} \approx B^{\mathrm{op}} \star A^{\mathrm{op}}$.

An important special case are the **left cone** and **right cone** of a category, defined by $A^\triangleleft := [0] \star A$ and $A^\triangleright := A \star [0]$. For instance, the right cone A^\triangleright is the category obtained by adjoining one additional object v to A , as well as a unique map $x \rightarrow v$ for each object x of A^\triangleright . In this case, v becomes a terminal object for A^\triangleright , and we can say that $A \mapsto A^\triangleright$ *freely adjoins a terminal object* to A . (Note that a terminal object of A will not be terminal in A^\triangleright anymore.) Likewise, $A \mapsto A^\triangleleft$ *freely adjoins an initial object* to A .

Limits and colimits of functors can be characterized using cones: if $p: A \rightarrow C$ is a functor, a *colimit* of p is a functor $\widehat{p}: A^\triangleright \rightarrow C$ which is initial among functors which extend p , and likewise, a *limit* of p is a functor $\widehat{p}: A^\triangleleft \rightarrow C$ which is terminal among functors which extend p .

23.6. *Remark.* It is worthwhile to spell this out in detail. Given a functor $p: A \rightarrow C$, to describe a functor $q: A^\triangleright \rightarrow C$ which *extends* p , it suffices to give

- (1) an object $q(v)$ in C ,
- (2) for each object $a \in \mathrm{ob} A$ a morphism $q(a \rightarrow v): p(a) = q(a) \rightarrow q(v)$ in C , such that
- (3) for each morphism $\alpha: a \rightarrow a'$ in A we have an equality $q(a' \rightarrow v) \circ p(\alpha) = q(a \rightarrow v)$ of morphisms $p(a) \rightarrow q(v)$ in C .

$$\begin{array}{ccc} a & & p(a) \\ \alpha \downarrow & \Longrightarrow & \downarrow p(\alpha) \\ a' & & p(a') \end{array} \quad \begin{array}{ccc} & & q(a \rightarrow v) \\ & \searrow & \\ & & q(v) \\ & \nearrow & \\ & & q(a' \rightarrow v) \end{array}$$

Given functors $q, q': A^\triangleright \rightarrow C$, we may consider natural transformations $\phi: q \rightarrow q'$ which *extend the identity transformation* of p . Explicitly, such a transformation ϕ is exactly determined by

- (1) a morphism $\phi(v): q(v) \rightarrow q'(v)$ in C such that

- (2) for each object $a \in \text{ob } A$ we have an equality $q'(a \rightarrow v) = \phi(v) \circ q(a \rightarrow v)$ of morphisms $p(a) \rightarrow q'(v)$ in C .

$$a \quad \Longrightarrow \quad \begin{array}{ccc} & & q(v) \\ & q(a \rightarrow v) \nearrow & \downarrow \phi(v) \\ p(a) & & q'(v) \\ & q'(a \rightarrow v) \searrow & \\ & & \end{array}$$

An extension $\widehat{p}: A^\triangleright \rightarrow C$ of p is a *colimit* of p if for every q extending p there exists a unique map $\phi(v): \widehat{p}(v) \rightarrow q(v)$ in C such that $q(a \rightarrow v) = \phi(v) \circ \widehat{p}(a \rightarrow v)$ for all $a \in \text{ob } A$. The object $\widehat{p}(v)$ is what is colloquially known as “the colimit of p ”, although the full data of a colimit of p is actually the functor \widehat{p} . We will call the functor \widehat{p} a *colimit cone* in what follows.

23.7. Ordered disjoint union. As noted above (23.2), the join operation on categories effectively descends to Δ . We will call this the **ordered disjoint union**. It is a functor $\sqcup: \Delta \times \Delta \rightarrow \Delta$, defined so that $[p] \sqcup [q] := [p + 1 + q]$, to be thought of as the disjoint union of underlying sets, ordered so that the subsets $[p]$ and $[q]$ retain their ordering, and elements of $[p]$ come before elements of $[q]$.

It is handy to extend this to the category Δ_+ , the full subcategory of ordered sets obtained by adding the empty set $[-1] := \emptyset$ to Δ . The functor \sqcup extends in an evident way to $\sqcup: \Delta_+ \times \Delta_+ \rightarrow \Delta_+$. This extended functor makes Δ_+ into a (nonsymmetric but strict) monoidal category, with unit object $[-1]$.

Note that for any map $f: [p] \rightarrow [q_1] \sqcup [q_2]$ in Δ_+ , there is a *unique* decomposition $[p] = [p_1] \sqcup [p_2]$ such that $f = f_1 \sqcup f_2$ for some (necessarily unique) $f_i: [p_i] \rightarrow [q_i]$ in Δ_+ . (We need an object $[-1]$ to be able to say this, even if $p, q_1, q_2 \geq 0$; if $f([p]) \subseteq [q_1]$ then $p_2 = -1$.)

23.8. Join of simplicial sets. Let X and Y be simplicial sets. The **join** of X and Y is a simplicial set $X \star Y$ defined as follows.

The join of simplicial sets X and Y is a simplicial set $X \star Y$ with n -dimensional elements

$$(X \star Y)_n := \coprod_{[n]=[n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2},$$

where $[n_1], [n_2] \in \text{ob } \Delta_+$, and we declare $X_{-1} = * = Y_{-1}$ to be a one-point set. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: for $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \star Y)_n$ and $f: [m] \rightarrow [n]$, we have $(x, y)f = (xf_1, yf_2) \in X_{m_1} \times Y_{m_2} \subseteq (X \star Y)_m$, where $f = f_1 \sqcup f_2$, $f_j: [m_j] \rightarrow [n_j]$ is the unique decomposition of f over $[n] = [n_1] \sqcup [n_2]$.

23.9. Exercise. Check that the above defines a simplicial set.

In particular,

$$\begin{aligned} (X \star Y)_0 &= X_0 \amalg Y_0, \\ (X \star Y)_1 &= X_1 \amalg X_0 \times Y_0 \amalg Y_1, \\ (X \star Y)_2 &= X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2, \end{aligned}$$

and so on.

Note that there are evident maps $X \rightarrow X \star Y \leftarrow Y$, which give isomorphisms from X and Y to subcomplexes of $X \star Y$, and these subcomplexes are disjoint from each other.

There are isomorphisms

$$(X \star Y) \star Z \xrightarrow{\sim} X \star (Y \star Z),$$

natural in X, Y, Z : on either side, the set of n -dimensional simplices can be described as $\coprod_{[n]=[n_1] \sqcup [n_2] \sqcup [n_3]} X_{n_1} \times Y_{n_2} \times Z_{n_3}$. Together with the evident isomorphisms $\emptyset \star X \approx X \star X \star \emptyset$, the

join gives a monoidal structure on $s\text{Set}$ with unit object $\Delta^{-1} := \emptyset$. Note that \star is *not* symmetric monoidal, though it is true that $(Y \star X)^{\text{op}} \approx X^{\text{op}} \star Y^{\text{op}}$. We have that

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

An important example are the cones. The **left cone** and **right cone** of a simplicial set X are

$$X^{\triangleleft} := \Delta^0 \star X, \quad X^{\triangleangleright} := X \star \Delta^0.$$

Note that outer horns are examples of cones:

$$(\partial\Delta^n)^{\triangleleft} = \Delta^0 \star \partial\Delta^n \approx \Lambda_0^{n+1}, \quad (\partial\Delta^n)^{\triangleangleright} = \partial\Delta^n \star \Delta^0 \approx \Lambda_{n+1}^{n+1}.$$

23.10. Exercise. Let $f: [m] \rightarrow [n]$ be any simplicial operator. Show that the induced map $f: \Delta^m \rightarrow \Delta^n$ on standard simplices is uniquely isomorphic to a join of maps $f_0 \star f_1 \star \cdots \star f_n$, with $f_j: \Delta^{m_j} \rightarrow \Delta^0$, where each $m_j \geq -1$.

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \star B) \approx N(A) \star N(B)$, and thus $N(A^{\triangleleft}) \approx (NA)^{\triangleleft}$ and $N(A^{\triangleangleright}) \approx (NA)^{\triangleangleright}$. (*Exercise:* prove this.)

23.11. The join of quasicategories is a quasicategory. Here is a handy rule for constructing maps into a join (compare (23.4)). Note that every join admits a canonical map $\pi: X \star Y \rightarrow \Delta^0 \star \Delta^0 \approx \Delta^1$, namely the join applied to the projections $X \rightarrow \Delta^0$ and $Y \rightarrow \Delta^0$.

23.12. Lemma ([Joy08a, Prop. 3.5], compare (23.4)). *Maps $f: K \rightarrow X \star Y$ are in bijective correspondence with the set of triples*

$$(\pi: K \rightarrow \Delta^1, \quad f_{\{0\}}: K^{\{0\}} \rightarrow X, \quad f_{\{1\}}: K^{\{1\}} \rightarrow Y),$$

where $K^{\{j\}} := \pi^{-1}(\{j\}) \subseteq K$, the pullback of $\{j\} \rightarrow \Delta^1$ along π .

Proof. This is a straightforward exercise. In one direction, the correspondence sends f to $(\bar{\pi}f, f|_{K^{\{0\}}}, f|_{K^{\{1\}}})$, where $\bar{\pi}: X \star Y \rightarrow \Delta^0 \star \Delta^0 = \Delta^1$. \square

23.13. Proposition. *If C and D are quasicategories, so is $C \star D$.*

Proof. Use the previous lemma (23.12), together with the observations (which we leave as an exercise) that for any map $\pi: \Lambda_j^n \rightarrow \Delta^1$ from an *inner* horn, the preimages $\pi^{-1}(\{0\})$ and $\pi^{-1}(\{1\})$ are either inner horns, standard simplices, or are empty, and for any map $\pi: \Delta^n \rightarrow \Delta^1$ from a standard simplex, the preimages are either a standard simplex or empty. \square

24. SLICES

24.1. Slices of categories. Given an ordinary category C , and an object $x \in \text{ob } C$, we may form the **slice categories** $C_{x/}$ and $C_{/x}$, also called **undercategory** and **overcategory**, or **slice-over category** and **slice-under category**.

For instance, the slice-over category $C_{/x}$ is the category whose *objects* are maps $f: c \rightarrow x$ with target x , and whose *morphisms* $(f: c \rightarrow x) \rightarrow (f': c' \rightarrow x)$ are maps $g: c \rightarrow c'$ such that $f'g = f$.

This can be reformulated in terms of joins. Let “ T ” denote the terminal category (isomorphic to $[0]$). Note that $\text{ob } C_{/x}$ corresponds to the set of functors $f: [0] \star T \rightarrow C$ such that $f|T = x$, and $\text{mor } C_{/x}$ corresponds to the set of functors $g: [1] \star T \rightarrow C$ such that $g|T = x$.

More generally, given a functor $p: A \rightarrow C$ of categories, we obtain slice categories $C_{p/}$ and $C_{/p}$ defined as follows. The category $C_{/p}$ has

- **objects:** functors $f: [0] \star A \rightarrow C$ such that $f|A = p$,
- **morphisms** $f \rightarrow f'$: functors $g: [1] \star A \rightarrow C$ such that $g|A = p$.

Likewise, the category $C_{p/}$ has

- **objects:** functors $f: A \star [0] \rightarrow C$ such that $f|A = p$,
- **morphisms** $f \rightarrow f'$: functors $g: A \star [1] \rightarrow C$ such that $g|A = p$.

24.2. *Exercise.* Describe composition of morphisms in $C_{/p}$ and $C_{p/}$.

24.3. *Exercise.* Show that $(C_{p/})^{\text{op}} \approx (C^{\text{op}})_{p^{\text{op}}/}$ (isomorphism of categories).

24.4. *Exercise.* Fix a functor $p: A \rightarrow C$, and let B be a category. Construct bijections

$$\{\text{functors } f: B \rightarrow C_{/p}\} \leftrightarrow \{\text{functors } g: B \star A \rightarrow C \text{ s.t. } g|_A = p\}$$

and

$$\{\text{functors } f: B \rightarrow C_{p/}\} \leftrightarrow \{\text{functors } g: A \star B \rightarrow C \text{ s.t. } g|_A = p\}.$$

24.5. *Remark.* The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices. Thus, given a functor $p: A \rightarrow C$, a *colimit* of p is the same data as an initial object of $C_{p/}$, while a *limit* of p is the same data as a terminal object of $C_{/p}$. (*Exercise:* prove this; this will be the starting case for formulating notions of limits and colimits for quasicategories. Compare (23.6).)

24.6. **Joins and colimits of simplicial sets.** The join functor $\star: s\text{Set} \times s\text{Set} \rightarrow s\text{Set}$ is in some ways analogous to the product functor \times , e.g., it is a monoidal functor.

The product operation $(-) \times (-)$ on simplicial sets commutes with colimits in each input, and the functors $X \times -$ and $- \times X$ admit right adjoints (in both cases, the right adjoint is $\text{Map}(X, -)$). The join functor does not commute with colimits in each variable, but *almost* does so; the only obstruction is the value on the initial object

More precisely, the functors $X \star -$ and $- \star X: s\text{Set} \rightarrow s\text{Set}$ do not preserve the initial object, since $X \star \emptyset \approx X \approx \emptyset \star X$. However, (the identity map of) X is tautologically the initial object of $s\text{Set}_{X/}$, the slice category of simplicial sets under X .

24.7. **Proposition.** *For every simplicial set X , the induced functors*

$$X \star -, \quad - \star X: s\text{Set} \rightarrow s\text{Set}_{X/}$$

preserve colimits.

Proof. This follows from the degreewise formula for the join, which has the form:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg (X_0 \times Y_{n-1}) \amalg Y_n = X_n \amalg (\text{terms which are "linear" in } Y).$$

That is, for each $n \geq 0$ the functor $Y \mapsto (X \star Y)_n: s\text{Set} \rightarrow \text{Set}_{X_n/}$ is seen to be colimit preserving, since each functor $X_k \times (-): \text{Set} \rightarrow \text{Set}$ is colimit preserving. \square

24.8. *Exercise* (Trivial, but important). Show that the functors $X \star -$ and $- \star X: s\text{Set} \rightarrow s\text{Set}$ preserve *pushouts*.

24.9. **Slices of simplicial sets.** We have seen that the functors

$$S \star -: s\text{Set} \rightarrow s\text{Set}_{S/} \quad \text{and} \quad - \star T: s\text{Set} \rightarrow s\text{Set}_{T/}$$

preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted

$$(p: S \rightarrow X) \mapsto X_{p/}: s\text{Set}_{S/} \rightarrow s\text{Set}$$

and

$$(q: T \rightarrow X) \mapsto X_{/q}: s\text{Set}_{T/} \rightarrow s\text{Set}.$$

I will sometimes distinguish these as **slice-under** and **slice-over**, respectively. Explicitly, there are bijective correspondences

$$(24.10) \quad \left\{ \begin{array}{ccc} S & \xrightarrow{p} & X \\ \downarrow & \dashrightarrow & \nearrow \\ S \star K & & \end{array} \right\} \iff \{K \dashrightarrow X_{p/}\}, \quad \left\{ \begin{array}{ccc} T & \xrightarrow{q} & X \\ \downarrow & \dashrightarrow & \nearrow \\ K \star T & & \end{array} \right\} \iff \{K \dashrightarrow X_{/q}\}.$$

Here we write “ $S \rightarrow S \star K$ ” and “ $T \rightarrow K \star T$ ” for the inclusions $S \star \emptyset \subseteq S \star K$ and $\emptyset \star T \subseteq K \star T$, using the canonical isomorphisms $S \star \emptyset = S$ and $\emptyset \star T = T$.

Taking $K = \Delta^n$ we obtain the formulas

$$(X_{p/})_n = \text{Hom}_{s\text{Set}_{S/}}(S \star \Delta^n, X), \quad (X_{/q})_n = \text{Hom}_{s\text{Set}_{T/}}(\Delta^n \star T, X),$$

which we regard as the definition of slices. (I.e., these formulas specify the n -dimensional elements of the slices, and naturality in “ Δ^n ” specifies the action of simplicial operators.)

24.11. *Exercise.* Given this explicit definition of slices in terms of their elements and the action of simplicial operators, verify the bijective correspondences (24.10).

In particular, we note the special cases associated to $x: \Delta^0 \rightarrow X$:

$$\text{Hom}_{s\text{Set}}(K, (X_{x/})) = \text{Hom}_{s\text{Set}_{\Delta^0/}}(\Delta^0 \star K, X) \approx \text{Hom}_{s\text{Set}_*}((K^\triangleleft, v), (X, x)),$$

$$\text{Hom}_{s\text{Set}}(K, X_{/x}) = \text{Hom}_{s\text{Set}_{\Delta^0/}}(K \star \Delta^0, X) \approx \text{Hom}_{s\text{Set}_*}((K^\triangleright, v), (X, x)).$$

The notation (X, x) with $x \in X_0$ represents a **pointed simplicial set**, the category of which is $s\text{Set}_* := s\text{Set}_{\Delta^0/}$. We write v for the cone point of K^\triangleleft and K^\triangleright .

The slice construction for simplicial sets agrees with that for categories.

24.12. **Proposition.** *The nerve preserves slices; i.e., if $p: A \rightarrow C$ is a functor, then $N(C_{p/}) \approx (NC)_{Np/}$ and $N(C_{/p}) \approx (NC)_{/Np}$.*

Proof. Left as an exercise. □

24.13. **Slice as a functor.** The function complex construction $\text{Map}(-, -)$ is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated, because the slice constructions also depend on a map between the two objects. A precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$\begin{array}{ccc} \begin{array}{ccc} S & \xrightarrow{p} & X \\ \uparrow j & & \downarrow f \\ T & \xrightarrow{fpj} & Y \end{array} & \implies & \begin{array}{ccc} X_{/p} & \longrightarrow & Y_{/fp} \\ \downarrow & & \downarrow \\ X_{/pj} & \longrightarrow & Y_{/fpj} \end{array} \quad \begin{array}{ccc} X_{p/} & \longrightarrow & Y_{fp/} \\ \downarrow & & \downarrow \\ X_{pj/} & \longrightarrow & Y_{fpj/} \end{array} \end{array}$$

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this.

24.14. *Remark.* A very precise formulation is that each kind of slice defines a functor $s\text{Set}^{\text{tw}} \rightarrow s\text{Set}$ from the *twisted arrow category* of simplicial sets, whose objects are maps p of simplicial sets, and whose morphisms are pairs $(j, f): p \rightarrow fpj$, where j and f are themselves maps of simplicial sets.

Let’s spell this out in terms of the correspondence between “maps into slices” and “maps from joins”. Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider “restriction map” $X_{p/} \rightarrow Y_{fpj/}$. The composite of a map $u: K \rightarrow X_{p/}$ with this restriction map is described in terms of the bijection of (24.10) as follows. The map u corresponds to a dotted arrow in

$$\begin{array}{ccccc} T & \xrightarrow{j} & S & \xrightarrow{p} & X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow & & \downarrow & \nearrow & \\ T \star K & \xrightarrow{j \star K} & S \star K & & & & \end{array}$$

\tilde{u}

The composite $K \xrightarrow{u} X_{p/} \rightarrow Y_{fpj/}$ corresponds to $f\tilde{u}(j \star K)$.

A particular special case which we will see a lot of are the “restriction” functors

$$X_{/p} \rightarrow X \quad \text{and} \quad X_{p/} \rightarrow X$$

induced by sequence $\emptyset \xrightarrow{j} S \xrightarrow{p} X$, using that $X_{/\emptyset} = X = X_{\emptyset/}$. For instance, $X_{/p} \rightarrow X$ sends an n -dimensional element $x \in (X_{/p})_n$ corresponding to $\tilde{x}: \Delta^n \star S \rightarrow X$ extending p to the n -dimensional element of X represented by the map $\tilde{x}|(\Delta^n \star \emptyset)$ defined as the composite

$$\Delta^n = \Delta^n \star \emptyset \rightarrow \Delta^n \star S \xrightarrow{x} X.$$

24.15. *Exercise.* Let $p: S \rightarrow X$ and $q: T \rightarrow X$ be maps of simplicial sets. Describe and prove bijections between the following sets of solutions to lifting problems:

$$\left\{ \begin{array}{ccc} & X_{p/} & \\ \nearrow & \downarrow & \\ T & \xrightarrow{q} & X \end{array} \right\} \iff \left\{ \begin{array}{ccc} S \amalg T & \xrightarrow{(p,q)} & X \\ \downarrow & \nearrow & \\ S \star T & & \end{array} \right\} \iff \left\{ \begin{array}{ccc} & X_{/q} & \\ \nearrow & \downarrow & \\ S & \xrightarrow{p} & X \end{array} \right\}$$

Here $X_{p/} \rightarrow X$ and $X_{/q} \rightarrow X$ are the evident restriction functors, and $S \amalg T \rightarrow S \star T$ the tautological inclusion.

25. INITIAL AND TERMINAL OBJECTS

We now show that for a vertex x in a quasicategory C , the slice objects $C_{/x}$ and $C_{x/}$ are also quasicategories. Using this, we can make a definition of initial and terminal object in a quasicategory.

25.1. **Initial and terminal objects.** An **initial object**²¹ of a quasicategory C is an $x \in C_0$ such that every $f: \partial\Delta^n \rightarrow C$ (for all $n \geq 1$) such that $f|\{0\} = x$, there exists an extension $f': \Delta^n \rightarrow C$.

A **terminal object** of C is an initial object of C^{op} . That is, a $y \in C_0$ such that every $f: \partial\Delta^n \rightarrow C$ with $f|\{n\} = y$ extends to Δ^n .

$$\begin{array}{ccc} & x & \\ \{0\} \xrightarrow{\quad} & \partial\Delta^n & \xrightarrow{\quad} C \\ & \downarrow & \nearrow \\ & \Delta^n & \end{array} \quad \exists$$

$$\begin{array}{ccc} & y & \\ \{n\} \xrightarrow{\quad} & \partial\Delta^n & \xrightarrow{\quad} C \\ & \downarrow & \nearrow \\ & \Delta^n & \end{array} \quad \exists$$

Let’s spell out the first parts of the definition of initial object applied to $x \in C_0$:

- The condition for $n = 1$ says that for every object c in C there exists $f: x \rightarrow c$,
- The condition for $n = 2$ says that for every triple of maps $f: x \rightarrow c$, $g: c \rightarrow c'$, and $h: x \rightarrow c'$, we must have $[h] = [g][f]$. In particular (taking $f = 1_x$), we see there is at most one homotopy class of maps from x to any object.

If C is the nerve of an ordinary category, then $\text{Hom}(\Delta^n, C) \xrightarrow{\sim} \text{Hom}(\partial\Delta^n, C)$ for all $n \geq 3$. Thus, for ordinary categories, this definition coincides with the usual notion of initial object.

For general quasicategories, we see that an initial object $x \in C_0$ necessarily satisfies $\text{Hom}_{hC}(x, y) \approx *$ for all $y \in C_0$, so that x represents an initial object in the homotopy category hC , but this is not sufficient to be initial in C : there are also an infinite sequence of “higher” conditions that an initial object of a quasicategory must satisfy.

We will now reformulate these notions using slice categories.

²¹We use Joyal’s definition of initial and terminal object [Joy02, §4] here. Lurie’s definition [Lur09, 1.2.12.1] is different, but is equivalent to what we use, by [Lur09, 1.2.12.5] and (25.5).

25.2. **Left and right horns, fibrations, and anodyne maps.** We recall the sets **left horns**

$$\text{LHorn} := \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda_0^n \subset \Delta^n \mid n \geq 1 \}$$

and the **right horns**

$$\text{RHorn} := \{ \Lambda_k^n \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda_n^n \subset \Delta^n \mid n \geq 1 \}.$$

The associated weak saturations $\overline{\text{LHorn}}$ and $\overline{\text{RHorn}}$ are the **left anodyne** and **right anodyne** maps. The associated right complements

$$\text{LFib} := \text{LHorn}^{\square}, \quad \text{RFib} := \text{RHorn}^{\square}$$

are the **left fibrations** and **right fibrations**. Note that

$$\overline{\text{InnHorn}} \subseteq \overline{\text{LHorn}} \cap \overline{\text{RHorn}} \quad \text{and} \quad \text{LFib} \cup \text{RFib} \subseteq \text{InnFib}.$$

These classes correspond to each other under the opposite involution $(-)^{\text{op}}: \text{sSet} \rightarrow \text{sSet}$; i.e., $\text{LHorn}^{\text{op}} = \text{RHorn}$, $\text{LFib}^{\text{op}} = \text{RFib}$.

25.3. **Proposition.** *Let C be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \rightarrow C$ and $C_{/x} \rightarrow C$ which “forget x ” (i.e., induced by the sequence $\emptyset \rightarrow \{x\} \rightarrow C$) are left fibration and right fibration respectively. In particular, $C_{x/}$ and $C_{/x}$ are also quasicategories.*

Proof. I claim that $\pi: C_{/x} \rightarrow C$ is a right fibration. Explicitly, this map sends the n -dimensional element $a: \Delta^n \rightarrow C_{/x}$, which corresponds to $\tilde{a}: \Delta^n \star \Delta^0 \rightarrow C$ such that $\tilde{a}|(\emptyset \star \Delta^0) = x$, to the n -dimensional element $\tilde{a}|(\Delta^n \star \emptyset) \rightarrow C$. Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$\begin{array}{ccc} \Lambda_j^n & \xrightarrow{f} & C_{/x} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \xrightarrow{g} & C \end{array} \iff \begin{array}{ccc} & & x \\ & \searrow & \nearrow \\ \emptyset \star \Delta^0 & \xrightarrow{\quad} & (\Lambda_j^n \star \Delta^0) \cup_{\Lambda_j^n \star \emptyset} (\Delta^n \star \emptyset) \xrightarrow{(\tilde{f}, g)} C \\ & \downarrow & \nearrow \\ & \Delta^n \star \Delta^0 & \end{array}$$

Note that there is a unique isomorphism $\Delta^n \star \Delta^0 \approx \Delta^{n+1}$. For any subset $S \subset [n]$, the above isomorphism identifies the subcomplex $\Delta^S \star \Delta^0 \subset \Delta^n \star \Delta^0$ with $\Delta^{S \cup \{n+1\}} \subset \Delta^{n+1}$, while $\Delta^S \star \emptyset \subset \Delta^n \star \Delta^0$ is identified with $\Delta^S \subseteq \Delta^{n+1}$. Since $\Lambda_j^n = \bigcup_{k \in [n] \setminus j} \Delta^{[n] \setminus k}$, we see that

- (1) the subcomplex $(\Lambda_j^n \star \Delta^0) \cup_{\Lambda_j^n \star \emptyset} (\Delta^n \star \emptyset)$ of $\Delta^n \star \Delta^0$ is the horn $\Lambda_j^{n+1} \subset \Delta^{n+1}$, and
- (2) the subcomplex $\emptyset \star \Delta^0$ of $\Delta^n \star \Delta^0$ is the vertex $\{n+1\}$.

Thus, the right hand diagram above is isomorphic to

$$\begin{array}{ccc} & & x \\ & \searrow & \nearrow \\ \{n+1\} & \xrightarrow{\quad} & \Lambda_j^{n+1} \xrightarrow{\quad} C \\ & \downarrow & \nearrow \\ & \Delta^{n+1} & \end{array}$$

If C is a quasicategory, then an extension exists for $0 < j \leq n$.

Since right fibrations are inner fibrations, the composite $C_{/x} \rightarrow C \rightarrow *$ is an inner fibration, and thus $C_{/x}$ is a quasicategory.

The case of $C_{x/} \rightarrow C$ is similar, using the correspondence

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & C_{x/} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \longrightarrow & C \end{array} \iff \begin{array}{ccc} & \xrightarrow{x} & \\ \{0\} & \longrightarrow & \Lambda_{j+1}^{n+1} \longrightarrow C \\ & \searrow & \downarrow \\ & & \Delta^{n+1} \end{array}$$

□

25.4. Reformulation of initial/terminal via slices. We can restate the definition of initial/terminal object using the “forgetful” functor of the relevant slice.

25.5. Proposition. *If C is a quasicategory, then $x \in C_0$ is initial if and only if $C_{x/} \rightarrow C$ is a trivial fibration, and terminal if and only if $C_{/x} \rightarrow C$ is a trivial fibration.*

Proof. This is an application of the join/slice adjunction. Applied to $\partial\Delta^n \subset \Delta^n$ with $n \geq 0$ and $C_{x/} \rightarrow C$, this has the form

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & C_{x/} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \xrightarrow{g} & C \end{array} \iff \begin{array}{ccc} & \xrightarrow{x} & \\ \Delta^0 \star \emptyset & \longrightarrow & (\Delta^0 \star \partial\Delta^n) \cup_{\emptyset \star \partial\Delta^n} (\emptyset \star \Delta^n) \longrightarrow C \\ & \searrow & \downarrow (\tilde{f}, g) \\ & & \Delta^0 \star \Delta^n \end{array}$$

The right-hand diagram is isomorphic to

$$\begin{array}{ccc} & \xrightarrow{x} & \\ \{0\} & \longrightarrow & \partial\Delta^{n+1} \longrightarrow C \\ & \searrow & \downarrow \\ & & \Delta^{n+1} \end{array}$$

Thus $C_{x/}$ is in $\text{TrivFib} = \text{Cell}^\square$ if and only if x is an initial object of C , as desired.

□

25.6. Remark. This implies that if x is initial, then $C_{x/} \rightarrow C$ is a categorical equivalence. Later (37.10) we’ll be able to show the converse: if $C_{x/} \rightarrow C$ is a categorical equivalence, then x is initial.

25.7. Uniqueness of initial and terminal objects. A crucial fact about initial and terminal objects in an ordinary category is that they are *unique up to unique isomorphism*. One way to formulate this is as follows: given a category C , let $C^{\text{init}} \subseteq C$ be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so C^{init} is empty, or there is at least one initial object, and C^{init} is equivalent to the terminal category $[0]$.

This leads to an analogous formulation for quasicategories.

25.8. Proposition. *Let C be a quasicategory. Let C^{init} and C^{term} denote respectively the full subcategories spanned by initial objects and terminal objects. Then (i) either C^{init} is empty or is categorically equivalent to the terminal quasicategory Δ^0 , and (ii) either C^{term} is empty or is categorically equivalent to the terminal quasicategory Δ^0 .*

Proof. Since $C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}}$, we just need to consider the case of initial objects. By definition of initial object, any $f: \partial\Delta^n \rightarrow C^{\text{init}}$ with $n \geq 1$ can be extended to $g: \Delta^n \rightarrow C$, and the image of g must lie in the full subcategory C^{init} since all of its vertices do. If $C^{\text{init}} \neq \emptyset$, then this extension condition also holds for $n = 0$, whence $C \rightarrow \Delta^0$ is a trivial fibration, and thus C is categorically equivalent to Δ^0 by (20.1). □

There are some seemingly obvious facts about initial and terminal objects that we can't prove just yet.

D. Deferred Proposition.

- (1a) Let $f: x \rightarrow y$ be a morphism in a quasicategory C , and let $\tilde{f} \in (C_{x/})_0$ be the object of the slice which corresponds to $f \in C_1$. Then \tilde{f} is initial in $C_{x/}$ if and only if f is an isomorphism.
- (1b) Let $f: x \rightarrow y$ be a morphism in a quasicategory C , and let $\tilde{f} \in (C_{/y})_0$ be the object of the slice which corresponds to $f \in C_1$. Then \tilde{f} is terminal in $C_{/y}$ if and only if f is an isomorphism.
- (2) In a quasicategory, every object which is isomorphic to an initial object is initial, and any object isomorphic to a terminal object is terminal.

Proofs will be given in (29.6).

25.9. Initial and terminal objects in functor categories. Here is a sample of a property of initial/terminal objects that we can now prove. A functor between ordinary categories whose values are all initial (or terminal) objects is itself initial (or terminal) as an object of the functor category. The same holds with categories replaced by quasicategories.

25.10. Proposition. Consider a map $f: X \rightarrow C$ from a simplicial set to a quasicategory. Suppose that for every vertex $x \in X_0$ the object $f(x) \in C_0$ is initial (resp. terminal) in C . Then the functor f is initial (resp. terminal) viewed as an object of $\text{Fun}(X, C)$.

As a consequence, if C has an initial (or terminal) object c_0 , then $\text{Fun}(X, C)$ also has an initial (or terminal) object, e.g., the functor which is the composite of $X \rightarrow \{c_0\} \rightarrow C$.

Proof. Assume $f(x) \in C_0$ is initial in C for all $x \in X_0$. Suppose given $g: \partial\Delta^n \rightarrow \text{Fun}(X, C)$ with $n \geq 1$ and $g|\{0\} = f$. We want to show that there exists an extension $g': \Delta^n \rightarrow \text{Fun}(X, C)$ of g . We convert this to the adjoint lifting problem:

$$\begin{array}{ccc}
 & & f \\
 & \curvearrowright & \\
 \{0\} \times X & \longrightarrow & \partial\Delta^n \times X \xrightarrow{\tilde{g}} C \\
 & & \downarrow \tilde{g}' \\
 & & \Delta^n \times X
 \end{array}$$

The strategy is to construct the extension by inductively constructing extensions $\tilde{g}_k: F_k \rightarrow C$ where $F_k = (\partial\Delta^n \times X) \cup \text{Sk}_k(\Delta^n \times X)$, $k \geq 0$ is the skeletal filtration (15.23) of the inclusion $\partial\Delta^n \times X \rightarrow \Delta^n \times X$. Note that any k -dimensional element $h = (a, b): \Delta^k \rightarrow \Delta^n \times X$ which is not contained in the subcomplex $\partial\Delta^n \times X$ must necessarily be such that $a: \Delta^k \rightarrow \Delta^n$ is surjective, whence a takes the vertex $0 \in (\Delta^k)_0$ to $0 \in (\Delta^n)_0$, and therefore $\tilde{g}h(0) = \tilde{g}(0, b(0)) = f(b(0))$ is an initial object in C by hypothesis. Thus we can lift in

$$\begin{array}{ccc}
 \coprod \partial\Delta^k \xrightarrow{(h|\partial\Delta^k)} F_{k-1} & \xrightarrow{\tilde{g}_{k-1}} & C \\
 \downarrow & & \downarrow \\
 \coprod \Delta^k \xrightarrow{(h)} F_k & \xrightarrow{\tilde{g}_k} & C
 \end{array}$$

since the square is a pushout and for each $h: \Delta^k \rightarrow F_k$ we have that $(\tilde{g}_{k-1}h)|\partial\Delta^k$ sends the vertex 0 to an initial object of C , so an extension of $(\tilde{g}_{k-1}h)|\partial\Delta^k$ to Δ^k always exists. \square

26. JOINS AND SLICES IN LIFTING PROBLEMS

Recall that for an object x in a quasicategory C , the slice objects $C_{x/}$ and $C_{/x}$ are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

26.1. Proposition. *Let $p: S \rightarrow C$ be a map of simplicial sets, and suppose C is a quasicategory. Then both $C_{p/}$ and $C_{/p}$ are quasicategories.*

The proof is just like that of (25.3): we will show below (26.14) that $C_{p/} \rightarrow C$ is a left fibration and $C_{/p} \rightarrow C$ is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

26.2. Pushout-joins. We define an analogue of the pushout-product for the the join. Given maps $i: A \rightarrow B$ and $j: K \rightarrow L$ of simplicial sets, the **pushout-join** (or **box-join**) $i \boxtimes j$ is the map

$$i \boxtimes j: (A \star L) \amalg_{A \star K} (B \star K) \xrightarrow{(i \star L, B \star j)} B \star L.$$

Warning: unlike the pushout-product, the pushout-join is not symmetric, since the join is not symmetric: $i \boxtimes j \neq j \boxtimes i$.

26.3. Example. We have already observed examples of pushout-joins in the proof of (25.3), namely

$$(\Lambda_j^n \subset \Delta^n) \boxtimes (\emptyset \subset \Delta^0) \approx (\Lambda_j^{n+1} \subset \Delta^{n+1}), \quad (\emptyset \subset \Delta^0) \boxtimes (\Lambda_j^n \subset \Delta^n) \approx (\Lambda_{1+j}^{1+n} \subset \Delta^{1+n}),$$

and also

$$(\emptyset \subset \Delta^0) \boxtimes (\partial \Delta^n \subset \Delta^n) \approx (\partial \Delta^{1+n} \subset \Delta^{1+n}), \quad (\partial \Delta^n \subset \Delta^n) \boxtimes (\emptyset \subset \Delta^0) \approx (\partial \Delta^{n+1} \subset \Delta^{n+1})$$

in the proof of (25.5). These generalize to arbitrary horns and cells. The pushout-join of a horn with a cell is always a horn:

$$\begin{aligned} (\Lambda_j^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) &\approx (\Lambda_j^{n+1+k} \subset \Delta^{n+1+k}), \\ (\partial \Delta^k \subset \Delta^k) \boxtimes (\Lambda_j^n \subset \Delta^n) &\approx (\Lambda_{k+1+j}^{k+1+n} \subset \Delta^{k+1+n}). \end{aligned}$$

Also, the pushout-join of a cell with a cell is always a cell:

$$(\partial \Delta^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) \approx (\partial \Delta^{n+1+k} \subset \Delta^{n+1+k}).$$

We leave proofs as an exercise for the reader.

26.4. Exercise. Prove the isomorphisms asserted in (26.3).

26.5. Remark. Both pushout-product and pushout-join are special cases of a general construction: given any functor $F: s\text{Set} \times s\text{Set} \rightarrow s\text{Set}$ of two variables, you get a corresponding “pushout-F” functor: $F_{\square}: \text{Fun}([1], s\text{Set}) \times \text{Fun}([1], s\text{Set}) \rightarrow \text{Fun}([1], s\text{Set})$. We will meet more examples later.

26.6. Pullback-slices. Just as the pushout-product is associated to the pullback-hom, so the pushout-join is associated to two kinds of **pullback-slices** (or **box-slices**). Given maps $K \xrightarrow{j} L \xrightarrow{q} X \xrightarrow{h} Y$, we define the map

$$h^{\boxtimes qj}: X_{/q} \rightarrow X_{/qj} \times_{Y_{/hqj}} Y_{/hq},$$

where the components $X_{/q} \rightarrow X_{/qj}$ and $X_{/q} \rightarrow Y_{/hq}$ are the evident maps constructed by “restricting” along $p: K \rightarrow L$ in the first case, and “composing” with $h: X \rightarrow Y$ in the second case.

Similarly, given maps $A \xrightarrow{i} B \xrightarrow{p} X \xrightarrow{h} Y$ we have

$$h^{i \boxtimes p}: X_{p/} \rightarrow X_{pi/} \times_{Y_{hpi/}} Y_{hp/}.$$

26.7. Remark. When $Y = *$, these pullback-slice maps are just the restriction maps $X_{/q} \rightarrow X_{/qj}$ and $X_{p/} \rightarrow X_{pi/}$.

26.8. *Remark.* Both pullback-hom and pullback-slices are special cases of a general construction: given any functor $F: s\text{Set}^{\text{tw}} \rightarrow s\text{Set}$ from the twisted arrow category (24.14), you get a corresponding “pullback- F ” functor $F^\square: s\text{Set}^{\text{tw}} \rightarrow s\text{Set}$. In the case of pullback-hom, the F in question is a composite functor $s\text{Set}^{\text{tw}} \rightarrow s\text{Set}^{\text{op}} \times s\text{Set} \xrightarrow{\text{Map}} s\text{Set}$.

26.9. **Joins, slices, and lifting problems.** The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-hom do.

26.10. **Proposition.** *Given $i: A \rightarrow B$, $j: K \rightarrow L$, and $h: X \rightarrow Y$, the following are equivalent.*

- (1) $(i \boxtimes j) \boxtimes h$.
- (2) $i \boxtimes (h^{\boxtimes q} j)$ for all $q: L \rightarrow X$.
- (3) $j \boxtimes (h^{i \boxtimes p})$ for all $p: B \rightarrow X$.

Proof. A straightforward exercise. The equivalence of (1) and (2) is

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \longrightarrow & X/q \\
 \downarrow i & \nearrow & \downarrow h^{\boxtimes q} \\
 B & \longrightarrow & X/qj \times_{Y/hqj} Y/hq
 \end{array} & \iff & \begin{array}{ccc}
 \emptyset \star L & \xrightarrow{q} & (A \star L) \cup_{A \star K} (B \star K) \longrightarrow X \\
 & & \downarrow i \boxtimes j \\
 & & B \star L \longrightarrow Y \\
 & & \downarrow h
 \end{array}
 \end{array}$$

□

26.11. **Proposition.** *Let S and T be sets of maps in $s\text{Set}$. Then $\overline{S \boxtimes T} \subseteq \overline{S \boxtimes T}$.*

Proof. This is formal and nearly identical to the proof of the weak saturation result for box-products (16.8). □

26.12. **Proposition.** *We have*

$$\overline{\text{Cell} \boxtimes \text{Cell}} \subseteq \overline{\text{Cell}}, \quad \overline{\text{RHorn} \boxtimes \text{Cell}} \subseteq \overline{\text{InnHorn}}, \quad \text{and} \quad \overline{\text{Cell} \boxtimes \text{LHorn}} \subseteq \overline{\text{InnHorn}}.$$

Proof. Immediate from (26.3) and (26.11). □

26.13. **Corollary.** *Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider the pullback-slice maps*

$$\ell: X_{p/} \rightarrow X_{pj/} \times_{Y_{fpj/}} Y_{fp/}, \quad r: X_{/p} \rightarrow X_{/pj} \times_{Y_{/fpj}} Y_{/fp}.$$

We have the following.

- (1) $j \in \overline{\text{Cell}}$, $f \in \text{TrivFib}$ implies $\ell, r \in \text{TrivFib}$.
- (2) $j \in \overline{\text{Cell}}$, $f \in \text{InnFib}$ implies $\ell \in \text{LFib}$, $r \in \text{RFib}$.
- (3) $j \in \overline{\text{RHorn}}$, $f \in \text{InnFib}$ implies $\ell \in \text{TrivFib}$.
- (4) $j \in \overline{\text{LHorn}}$, $f \in \text{InnFib}$ implies $r \in \text{TrivFib}$.

Proof. Exercise, using (26.12). □

We are mostly interested in special cases when $X = C$ is a quasicategory, and $Y = *$.

26.14. **Corollary.** *Given $T \xrightarrow{j} S \xrightarrow{p} C$ with C a quasicategory and j a monomorphism, the induced map $C_{/p} \rightarrow C_{/pj}$ is a right fibration, and $C_{p/} \rightarrow C_{pj/}$ is a left fibration. In particular, $C_{/p} \rightarrow C$ is a right fibration and $C_{p/} \rightarrow C$ is a left fibration (case $T = \emptyset$).*

26.15. **Corollary.** *Given $T \xrightarrow{j} S \xrightarrow{p} C$ with C a quasicategory, if j is right anodyne then $C_{p/} \rightarrow C_{pj/}$ is a trivial fibration, while if j is left anodyne then $C_{/p} \rightarrow C_{/pj}$ is a trivial fibration.*

Another case we will need is when $T = \emptyset$.

26.16. **Corollary.** Given $S \xrightarrow{p} X \xrightarrow{f} Y$ where f is a trivial fibration, all of the maps in

$$X_{p/} \rightarrow X \times_Y Y_{fp/} \rightarrow Y_{fp/} \quad \text{and} \quad X_{/p} \rightarrow X \times_Y Y_{/fp} \rightarrow Y_{/fp}$$

are trivial fibrations.

Proof. The two pullback-slice maps are trivial fibrations by (26.13). The projections are each base changes of the trivial fibration f , and so are trivial fibrations. \square

26.17. **Composition functors for slices.** Here is a nice consequence of the above. Let C be a quasicategory and $f: x \rightarrow y$ a morphism in it; we represent f by a map $\Delta^1 \rightarrow C$ of simplicial sets, which we also call f . We obtain two functors

$$C_{/x} \xleftarrow{p} C_{/f} \xrightarrow{q} C_{/y}$$

associated to the inclusions $\{0\} \subset \Delta^1 \supset \{1\}$. The first inclusion $\{0\} \subset \Delta^1$ is a left-horn inclusion, and thus by (26.15) the restriction map p is a trivial fibration, and hence we can choose a section $s: C_{/x} \rightarrow C_{/f}$ of p .

The resulting composite $qs: C_{/x} \rightarrow C_{/y}$ can be thought of as a functor realizing the operation which sends an object $(c \xrightarrow{g} x)$ of $C_{/x}$ to “the object” $(c \xrightarrow{fg} y)$ of $C_{/y}$ defined by “composing f and g ” (but remember that such composition is not uniquely defined in a quasicategory C ; the choice of section s gives a collection of such choices for all g .)

26.18. *Exercise.* Show that if C is a category, then p is an isomorphism, and that qs is precisely the functor $C_{/x} \rightarrow C_{/y}$ described above.

27. LIMITS AND COLIMITS IN QUASICATEGORIES

27.1. **Definition of limits and colimits.** Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map $p: K \rightarrow C$ where C is a quasicategory, a **colimit** of p is defined to be an initial object of the slice quasicategory $C_{p/}$. Explicitly, a colimit of $p: K \rightarrow C$ is a map $\tilde{p}: K \star \Delta^0 = K^\triangleright \rightarrow C$ extending p , such that for $n \geq 1$ a lift exists in every diagram of the form

$$\begin{array}{ccc} & \tilde{p} & \\ & \curvearrowright & \\ K \star \{0\} & \longrightarrow & K \star \partial\Delta^n \longrightarrow C \\ & \searrow & \nearrow \\ & & K \star \Delta^n \end{array}$$

Sometimes it is better to call \tilde{p} a **colimit cone**, in which case the restriction $\tilde{p}|_{\emptyset \star \Delta^0}$ to the cone point is an object in C which can be called a “colimit of p ”.

Similarly, a **limit** of p is a terminal object of $C_{/p}$; explicitly, this is a map $\tilde{p}: \Delta^0 \star K = K^\triangleleft \rightarrow C$ extending p such that for $n \geq 1$ a lift exists in every

$$\begin{array}{ccc} & \tilde{p} & \\ & \curvearrowright & \\ \{n\} \star K & \longrightarrow & \partial\Delta^n \star K \longrightarrow C \\ & \searrow & \nearrow \\ & & \Delta^n \star K \end{array}$$

27.2. *Example.* Consider the empty simplicial set $K = \emptyset$. Then $C_{\emptyset/} = C$, so a colimit of $p: \emptyset \rightarrow C$ is precisely the same as an initial object of C . Likewise, a limit of p is precisely the same as a terminal object of C .

27.3. *Example.* Consider $K = \Lambda_0^2$, which is the nerve of a category which we can draw as the picture $(1 \leftarrow 0 \rightarrow 2)$. Then $(\Lambda_0^2)^\triangleright \approx \Delta^1 \times \Delta^1$ is also an ordinary category; explicitly it has the form of a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \searrow & \downarrow \\ 2 & \longrightarrow & v \end{array}$$

where v is the ‘‘cone vertex’’. A colimit cone $(\Lambda_0^2)^\triangleright \rightarrow C$ is called a **pushout diagram** in C .

Similar considerations give $(\Lambda_2^2)^\triangleleft \approx \Delta^1 \times \Delta^1$; a limit cone $(\Lambda_2^2)^\triangleleft \rightarrow C$ is called a **pullback diagram** in C .

27.4. *Exercise.* Let $C' \subseteq C$ be an inclusion of a full subcategory. Show that if $p: K \rightarrow C'$ has a colimit \tilde{p} in C , and if the image of \tilde{p} is contained in C' , then \tilde{p} is in fact a colimit of p in C' .

27.5. **Uniqueness of limits and colimits.** Limits and colimits are unique if they exist.

27.6. **Proposition.** *Let $p: K \rightarrow C$ be a map to a quasicategory, and let $(C_{p/})^{\text{colim}} \subseteq C_{p/}$ and $(C_{/p})^{\text{lim}} \subseteq C_{/p}$ denote the full subcategories spanned by colimit cones and limit cones respectively. Then (i) either $(C_{p/})^{\text{colim}}$ is empty or is categorically equivalent to Δ^0 , and (ii) either $(C_{/p})^{\text{lim}}$ is empty or is categorically equivalent to Δ^0 .*

Proof. This is just the uniqueness of initial and terminal objects (25.8), since $(C_{p/})^{\text{colim}} = (C_{p/})^{\text{init}}$ and $(C_{/p})^{\text{lim}} = (C_{/p})^{\text{term}}$. \square

We have noted above (25.5) that an object x in a quasicategory C is initial iff $C_{x/} \rightarrow C$ is a trivial fibration, and terminal iff $C_{/x} \rightarrow C$ is a trivial fibration. There is a similar characterization of limit and colimit cones.

27.7. **Proposition.** *Let C be a quasicategory. Let $\tilde{p}: K^\triangleright \rightarrow C$ be a map, and write $p := \tilde{p}|K$. Then \tilde{p} is a colimit diagram if and only if $C_{\tilde{p}/} \rightarrow C_{p/}$ is a trivial fibration.*

Likewise, let $\tilde{q}: K^\triangleleft \rightarrow C$ be a map, and write $q := \tilde{q}|K$. Then \tilde{q} is a limit diagram if and only if $C_{/\tilde{q}} \rightarrow C_{/q}$ is a trivial fibration.

Proof. I’ll just do the case of colimits.

We make an elementary observation about iterated slices (see (27.8) below). There is an isomorphism $(C_{p/})_{\tilde{p}/} \approx C_{\tilde{p}/}$, where the symbol ‘‘ \tilde{p} ’’ refers to both a morphism $\tilde{p}: K^\triangleright \rightarrow C_{p/}$ (on the right-hand side of the isomorphism) and the corresponding object $\tilde{p} \in (C_{p/})_0$ (on the left-hand side of the isomorphism). The point is that in either simplicial set, a k -dimensional element corresponds to a map $K \star \Delta^0 \star \Delta^k \rightarrow C$ which restricts to \tilde{p} on $K \star \Delta^0 \star \emptyset$.

Using this, the statement amounts to the special case for initial and terminal objects (25.5). \square

27.8. *Exercise (Iterated slices).* Let $f: A \star B \rightarrow C$ be a map of simplicial sets. Describe isomorphisms

$$C_{f/} \approx (C_{f_A/})_{\tilde{f}_B/}, \quad C_{/f} \approx (C_{/f_B})_{/\tilde{f}_A},$$

where $f_A: A \rightarrow C$ and $f_B: B \rightarrow C$ are the evident restrictions of f to subcomplexes, and $\tilde{f}_A: A \rightarrow C_{/f_B}$ and $\tilde{f}_B: B \rightarrow C_{f_A/}$ are the adjoints to f .

27.9. **Limits and colimits in slices.** Given a map $p: S \rightarrow C$ to a quasicategory, we have ‘‘forgetful functors’’ $\pi: C_{/p} \rightarrow C$ and $\pi: C_{p/} \rightarrow C$ from the slices to C .

The following proposition says that an initial object of C implies a compatible initial object of $C_{/p}$, and a terminal object of C implies a compatible terminal object of $C_{p/}$.

27.10. **Proposition.** *Let $p: S \rightarrow C$ be a map from a simplicial set to a quasicategory.*

- (1) *Suppose C has an initial object. Then*

- (a) $C_{/p}$ has an initial object, and
 - (b) if $x \in (C_{/p})_0$ is an object such that $\pi(x) \in C_0$ is initial in C , then x is initial in $C_{/p}$.
- (2) Suppose C has a terminal object. Then
- (a) $C_{p/}$ has a terminal object, and
 - (b) if $x \in (C_{p/})_0$ is an object such that $\pi(x) \in C_0$ is terminal in C , then x is terminal in $C_{p/}$.

Proof. (See [Lur09, 1.2.13.8].) I'll only prove (1), as the proof of (2) is analogous.

I prove (1b) first. Let $x \in (C_{/p})_0$ and $y = \pi(x) \in C_0$; we need to show that if y is initial then so is x . To show that x is initial we must show that a lift in any diagram of the form

$$\begin{array}{ccc}
 & x & \\
 & \curvearrowright & \\
 \Delta^0 \star \emptyset & \longrightarrow & (\Delta^0 \star \partial\Delta^n) \cup (\emptyset \star \Delta^n) \longrightarrow C_{/p} \\
 & & \downarrow \\
 & & \Delta^0 \star \Delta^n
 \end{array}$$

for $n \geq 0$, using the identification $(\Delta^0 \star \partial\Delta^n) \cup (\emptyset \star \Delta^n) \approx \partial\Delta^{n+1}$. This lifting problem is equivalent to one of the form

$$\begin{array}{ccc}
 & x' & \\
 & \curvearrowright & \\
 \Delta^0 \star \emptyset \star S & \longrightarrow & (\Delta^0 \star \partial\Delta^n \star S) \cup (\emptyset \star \Delta^n \star S) \longrightarrow C \\
 & & \downarrow \\
 & & \Delta^0 \star \Delta^n \star S
 \end{array}$$

which in turn is equivalent to one of the form

$$\begin{array}{ccc}
 & x'' & \\
 & \curvearrowright & \\
 S & \longrightarrow & \partial\Delta^n \star S \longrightarrow C_{y/} \\
 & & \downarrow \quad \searrow \\
 & & \Delta^n \star S \longrightarrow C \quad \downarrow q
 \end{array}$$

(Here the maps marked x, x', x'' are all adjoints of each other.) Since y is initial, q is a trivial fibration (25.5), and therefore a lift exists since $\partial\Delta^n \star S \rightarrow \Delta^n \star S$ is a monomorphism. We conclude that x is initial if y is.

Next we prove (1a). Suppose $y \in C_0$ is an initial object. This implies $q: C_{y/} \rightarrow C$ is a trivial fibration (25.5). In particular, a lift exists in

$$\begin{array}{ccc}
 & C_{y/} & \\
 & \nearrow x'' & \downarrow q \\
 S & \xrightarrow{p} & C
 \end{array}$$

By an adjunction argument (24.15), x'' corresponds to a map $x: \Delta^0 \rightarrow C_{/p}$ such that $\pi(x) = y$. By what we have already proved, x must be initial since $\pi(x) = y$ is initial. \square

27.11. *Remark.* In fact, the converses of (1b) and (2b) in (27.10) are also true. The proof of these converses requires (D), which we have not established yet.

We can now generalize the above to arbitrary limits in colimits.

The following proposition says that colimits in $C_{/p}$ or limits in $C_{p/}$ can be “computed in the underlying quasicategory” C (if the corresponding colimit or limit in C exists).

27.12. Proposition. *Let $p: S \rightarrow C$ be a map from a simplicial set to a quasicategory.*

- (1) *Let $f: K \rightarrow C_{/p}$ be a map such that the composite map $f_0 = \pi f: K \xrightarrow{f} C_{/p} \xrightarrow{\pi} C$ has a colimit cone in C . Then*
 - (a) *f admits a colimit cone, and*
 - (b) *if $\tilde{f}: K^\triangleright \rightarrow C_{/p}$ is such that the composite map $K^\triangleright \xrightarrow{\tilde{f}} C_{/p} \rightarrow C$ is a colimit cone, then \tilde{f} is a colimit cone.*
- (2) *Let $f: K \rightarrow C_{p/}$ be a map such that the composite map $f_0 = \pi f: K \xrightarrow{f} C_{p/} \xrightarrow{\pi} C$ has a limit cone in C . Then*
 - (a) *f admits a limit cone, and*
 - (b) *if $\tilde{f}: K^\triangleright \rightarrow C_{p/}$ is such that the composite map $K^\triangleright \xrightarrow{\tilde{f}} C_{p/} \rightarrow C$ is a limit cone, then \tilde{f} is a limit cone.*

The proof will make use an observation sketched in the following exercise: any composite of a slice-over followed by a slice-under can be reinterpreted as a slice-under followed by a slice-over.

27.13. Exercise (Two-sided slice). Fix a map $p: A \star B \rightarrow X$ of simplicial sets. Describe a simplicial set $X_{/p/}$ which admits bijective correspondences

$$\left\{ \begin{array}{ccc} A \star B & \xrightarrow{p} & X \\ \downarrow & \dashrightarrow & \uparrow \\ A \star K \star B & & \end{array} \right\} \iff \{K \dashrightarrow X_{/p/}\},$$

natural in K . Then construct natural isomorphisms

$$(X_{p_A/})_{/\tilde{p}_B} \approx X_{/p/} \approx (X_{/p_B})_{\tilde{p}_A/},$$

where $p_A: A \rightarrow X$ and $p_B: B \rightarrow X$ are the evident restrictions of p to subcomplexes, and $\tilde{p}_A: A \rightarrow X_{/p_B}$ and $\tilde{p}_B: B \rightarrow X_{p_A/}$ are adjoints to p .

Proof of (27.12). I prove (1), as (2) is analogous. Note that $f: K \rightarrow C_{/p}$ is adjoint to a map $g: K \star S \rightarrow C$, which in turn is adjoint to a map $q: S \rightarrow C_{f_0/}$. Colimit cones of f_0 correspond precisely to initial objects of $C_{f_0/}$; in particular, the hypothesis of (1) asserts that $C_{f_0/}$ has an initial object. Likewise, colimit cones of f correspond exactly to initial objects of $(C_{/p})_{f/}$. As in (27.13) we have isomorphisms

$$(C_{/p})_{f/} \approx C_{/g/} \approx (C_{f_0/})_{/q}.$$

To prove (1a) here it suffices to show that $(C_{f_0/})_{/q}$ has an initial object, which since $C_{f_0/}$ does using (27.10)(1a). To prove (1b) here it suffices to show that the projection $(C_{f_0/})_{/q} \rightarrow C_{f_0/}$ has the property that objects sent to initial objects of $C_{f_0/}$ are initial in $(C_{f_0/})_{/q}$, which is immediate from (27.10)(1b). \square

Limits/colimits are invariant under categorical equivalence. Where does this get proved?

28. THE JOYAL EXTENSION AND LIFTING THEOREMS

We are now at the point where we can state and prove Joyal’s theorems about extending or lifting maps along outer horns. This will allow us to prove many of the results whose proofs we have deferred up to now.

28.1. Joyal extension theorem. The following gives a condition for extending maps from *outer horns* into a quasicategory.

28.2. Theorem (Joyal extension). [Joy02, Thm. 1.3] *Let C be a quasicategory, and fix a map $f: \Delta^1 \rightarrow C$. The following are equivalent.*

- (1) *The edge represented by f is an isomorphism in C .*
- (2) *Every $a: \Lambda_0^n \rightarrow C$ with $n \geq 2$ such that $f = a|_{\Delta^{\{0,1\}}}: \Delta^1 \rightarrow C$ admits an extension to a map $\Delta^n \rightarrow C$.*
- (3) *Every $b: \Lambda_n^n \rightarrow C$ with $n \geq 2$ such that $f = b|_{\Delta^{\{n-1,n\}}}: \Delta^1 \rightarrow C$ admits an extension to a map $\Delta^n \rightarrow C$.*

I'll call $\langle 01 \rangle \in \Delta^n$ the **leading edge**, and $\langle n-1, n \rangle \in \Delta^n$ the **trailing edge**. Thus, the implications (1) \Rightarrow (2) and (1) \Rightarrow (3) say that we can always extend $\Lambda_0^n \rightarrow C$ to an n -simplex if the *leading edge* goes to an isomorphism in C , and extend $\Lambda_n^n \rightarrow C$ to an n -simplex if the *trailing edge* goes to an isomorphism in C .

The implications (2) \Rightarrow (1) and (3) \Rightarrow (1) are easy, and are left as an exercise.

28.3. Exercise (Easy part of Joyal extension). Suppose C is a quasicategory with edge $f \in C_1$, and suppose that every map $a: \Lambda_0^n \rightarrow C$ with $n \in \{2, 3\}$ and $f = a|_{\Delta^{\{0,1\}}}$ admits an extension along $\Lambda_0^n \subset \Delta^n$. Prove that f is an isomorphism.

The non-trivial implications of Joyal extension will lead to proofs of the deferred propositions (A), (C), and (D).

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations*.

28.4. Conservative functors. A functor $p: C \rightarrow D$ between categories is **conservative** if whenever f is a morphism in C such that $p(f)$ is an isomorphism in D , then f is an isomorphism in C . The definition of a conservative functor between *quasicategories* is precisely the same.

28.5. Proposition. *All left fibrations and right fibrations between quasicategories are conservative.*

Proof. Consider a right fibration $p: C \rightarrow D$, and a morphism $f: x \rightarrow y$ in C such that $p(f)$ is an isomorphism. We first show that f admits a preinverse.

Let $a: \Lambda_2^2 \rightarrow C$ such that $a_{12} = f$ and $a_{02} = 1_y$. Let $b: \Delta^2 \rightarrow C$ be any 2-dimensional element exhibiting a preinverse of $p(f)$, i.e., such that $b_{12} = p(f)$ and $b_{02} = 1_{p(y)}$, so that b_{01} is a preinverse. Now have a diagram with a lift

$$\begin{array}{ccc} \Lambda_2^2 & \xrightarrow{a} & C \\ \downarrow & \nearrow & \downarrow p \\ \Delta^2 & \xrightarrow{b} & D \end{array}$$

which exhibits a preinverse of f , which we will call g .

Because $p(f)$ was assumed to be an isomorphism in D , its preinverse $p(g)$ is also an isomorphism, and therefore by the above argument g admits a preinverse as well. We conclude that f is invertible by (10.4). \square

28.6. Isofibrations. We say that a functor $p: C \rightarrow D$ of quasicategories is an **isofibration**²² if

- (1) p is an inner fibration, and
- (2) we have “isomorphism lifting” along p . That is, for any $c \in C_0$ and *isomorphism* $g: p(c) \rightarrow d'$, there exists a $c' \in C_0$ and *isomorphism* $f: c \rightarrow c'$ such that $p(f) = g$.

²²Joyal uses the term “quasifibration” in [Joy02]. Later in [Joy08a] this is called a “pseudofibration”. Lurie uses this notion, but never names it. The term “isofibration” is used by Riehl and Verity [RV15].

Recall that if C and D are nerves of ordinary categories, then any functor $C \rightarrow D$ is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories we can replace (2) with the dual condition

(2') for any $c \in C_0$ and isomorphism $g': d' \rightarrow p(c)$, there exists a $c' \in C_0$ and isomorphism $f': c' \rightarrow c$ such that $p(f) = g'$.

To prove (2) from (2'), just apply condition (2') to the (unique) inverse of g .

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

28.7. Proposition. *An inner fibration $p: C \rightarrow D$ between quasicategories is an isofibration if and only if $h(p): h(C) \rightarrow h(D)$ is an isofibration of ordinary categories.*

Proof. (\implies) Straightforward. (\impliedby) Suppose given an isomorphism $g: p(c) \rightarrow d'$ in D . If $h(p)$ is an isofibration, there exists an isomorphism $f': c \rightarrow c'$ in C such that $p(f') \sim_r g$. Now choose a lift in

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{a} & C \\ \downarrow & \nearrow f & \downarrow p \\ \Delta^2 & \xrightarrow{b} & D \end{array}$$

where b exhibits $p(f') \sim_r g$ and $a(\langle 01 \rangle) = f'$ and $a(\langle 12 \rangle) = 1_{c'}$. The edge $f = s_{02}$ is a lift of g , and is an isomorphism since $f' \sim_r f$. \square

28.8. Exercise. (i) Let Group denote the category of groups, whose objects are pairs $G = (S, \mu)$ consisting of a set S and a function $\mu: S \times S \rightarrow S$ satisfying a well-known list of axioms. Show that the functor $U: \text{Group} \rightarrow \text{Set}$ which on objects sends $(S, \mu) \mapsto S$ is an isofibration between ordinary categories.

(ii) Consider the functor $U': \text{Group} \rightarrow \text{Set}$ defined on objects by $G \mapsto \text{Hom}(\mathbb{Z}, G)$. Explain why, although U' is naturally isomorphic to U , you don't know how to show whether U' is an isofibration without explicit reference to the axioms of your set theory. The moral is that the property of being an isofibration is not "natural isomorphism invariant".

28.9. Left and right fibrations are isofibrations.

28.10. Proposition. *All left fibrations and right fibrations between quasicategories are isofibrations.*

Proof. Suppose $p: C \rightarrow D$ is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$\begin{array}{ccc} \{1\} & \longrightarrow & C \\ \downarrow & \nearrow f & \downarrow p \\ \Delta^1 & \xrightarrow{g} & D \end{array}$$

where g represents an isomorphism. Because p is a right fibration, there exists a lift f . Because right fibrations are conservative, f represents an isomorphism. \square

Note that the above proof checked explicitly isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

28.11. Proof of the Joyal extension theorem.

Proof of (28.2). We prove (1) \implies (2). Suppose given $a: \Lambda_0^n \rightarrow C$ such that $f = a|_{\Delta^{\{0,1\}}}$ represents an isomorphism. Observe (26.3) that $(\Lambda_0^n \subset \Delta^n)$ is the pushout-join of a 1-horn with an $(n-2)$ -cell:

$$(\Lambda_0^n \subset \Delta^n) \approx (\Delta^{\{0\}} \subset \Delta^{\{0,1\}}) \boxtimes (\partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}),$$

since $\Lambda_0^n \approx (\Delta^{\{0\}} \star \Delta^{\{2, \dots, n\}}) \cup (\Delta^{\{0,1\}} \star \partial \Delta^{\{2, \dots, n\}})$ inside $\Delta^n \approx \Delta^{\{0,1\}} \star \Delta^{\{2, \dots, n\}}$. Using this, we get a correspondence of lifting problems

$$\begin{array}{ccc}
 \Delta^{\{0,1\}} \xrightarrow{\quad} \Lambda_0^n \xrightarrow{a} C & \iff & \begin{array}{ccc}
 \Delta^{\{0\}} & \xrightarrow{h} & C/(a|\Delta^{\{2, \dots, n\}}) \\
 \downarrow & \nearrow & \downarrow q \\
 \Delta^{\{0,1\}} & \xrightarrow{g} & C/(a|\partial \Delta^{\{2, \dots, n\}}) \\
 \searrow f & & \downarrow p \\
 & & C
 \end{array}
 \end{array}$$

where g is adjoint to $a|(\Delta^{\{0,1\}} \star \partial \Delta^{\{2, \dots, n\}})$, and h is adjoint to $a|(\Delta^{\{0\}} \star \Delta^{\{2, \dots, n\}})$. Because C is a quasicategory, both p and q are right fibrations (26.14), and therefore are conservative isofibrations (28.5), (28.10). Thus since f represents an isomorphism, so does g since p is conservative, and therefore a lift exists since q is an isofibration.

The proof of (2) \implies (1) is left as an exercise (28.3). The proof of (1) \iff (3) is similar. \square

28.12. The Joyal lifting theorem. There is a relative generalization.

28.13. Theorem (Joyal lifting). *Let $p: C \rightarrow D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in D . The following are equivalent.*

- (1) *The edge f is an isomorphism in C .*
- (2) *For all $n \geq 2$, every diagram of the form*

$$\begin{array}{ccc}
 \Delta^{\{0,1\}} \xrightarrow{\quad} \Lambda_0^n \xrightarrow{\quad} C & & \\
 \downarrow & \nearrow & \downarrow p \\
 \Delta^n \xrightarrow{\quad} D & &
 \end{array}$$

admits a lift.

- (3) *For all $n \geq 2$, every diagram of the form*

$$\begin{array}{ccc}
 \Delta^{\{n-1, n\}} \xrightarrow{\quad} \Lambda_n^n \xrightarrow{\quad} C & & \\
 \downarrow & \nearrow & \downarrow p \\
 \Delta^n \xrightarrow{\quad} D & &
 \end{array}$$

admits a lift.

Proof. The implications (2) \implies (1) and (3) \implies (1) are elementary, as in (28.3).

For (1) \implies (2), the first step is to prove that

$$C/(a|\Delta^{\{2, \dots, n\}}) \xrightarrow{q} C/(a|\partial \Delta^{\{2, \dots, n\}}) \times_{D/(pa|\partial \Delta^{\{2, \dots, n\}})} D/(pa|\Delta^{\{2, \dots, n\}}) \xrightarrow{p} C$$

are both right fibrations. For instance, the map q is the slice-power of the inner fibration p by a monomorphism, so is a right fibration by (26.13). The map p is the composite

$$C/(a|\partial \Delta^{\{2, \dots, n\}}) \times_{D/(pa|\partial \Delta^{\{2, \dots, n\}})} D/(pa|\Delta^{\{2, \dots, n\}}) \xrightarrow{p'} C/(a|\Delta^{\{2, \dots, n\}}) \xrightarrow{p''} C,$$

where p' is the base change of the right fibration $D/(pa|\Delta^{\{2, \dots, n\}}) \rightarrow D/(pa|\partial \Delta^{\{2, \dots, n\}})$, and p'' is a right fibration (in both cases by (26.14)) Then the proof of (1) \implies (2) proceeds exactly as in (28.2). \square

We note here that as a consequence of Joyal lifting we get a characterization of conservative inner fibrations between quasicategories in terms of a lifting property.

28.14. Corollary. *Let $p: C \rightarrow D$ be an inner fibration between quasicategories. The following are equivalent.*

- (1) *The functor p is conservative.*
- (2) *For all $n \geq 2$, every diagram of the form*

$$\begin{array}{ccccc}
 & & f & & \\
 & & \curvearrowright & & \\
 \Delta^{\{0,1\}} & \longrightarrow & \Lambda_0^n & \longrightarrow & C \\
 & & \downarrow & \nearrow & \downarrow p \\
 & & \Delta^n & \longrightarrow & D
 \end{array}$$

such that $p(f)$ represents an isomorphism in D admits a lift.

- (3) *For all $n \geq 2$, every diagram of the form*

$$\begin{array}{ccccc}
 & & f & & \\
 & & \curvearrowright & & \\
 \Delta^{\{n-1,n\}} & \longrightarrow & \Lambda_n^n & \longrightarrow & C \\
 & & \downarrow & \nearrow & \downarrow p \\
 & & \Delta^n & \longrightarrow & D
 \end{array}$$

such that $p(f)$ represents an isomorphism in D admits a lift.

29. APPLICATIONS OF THE JOYAL EXTENSION THEOREM

We can now prove all the statements whose proofs we have deferred until now, as well as some others. I'll prove (A) and (D) in this section, and (C) in the next section.

29.1. Quasigroupoids are Kan complexes. First we prove (A), the identification of quasigroupoids with Kan complexes.

29.2. Proposition. *Every quasigroupoid is a Kan complex.*

Proof. In a quasigroupoid, the Joyal extension property (28.2) applies to all maps from Λ_0^n and Λ_n^n , since every edge is an isomorphism. \square

From now on we will use terms “quasigroupoid” and “Kan complex” interchangeably.

29.3. Invariance of slice categories. Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

29.4. Exercise (Reformulation of Joyal extension). If $f: x \rightarrow y$ is an edge in a quasicategory C , then the following are equivalent: (1) f is an isomorphism; (2) $C_{f/} \rightarrow C_{x/}$ is a trivial fibration; (3) $C_{/f} \rightarrow C_{/y}$ is a trivial fibration.

29.5. Corollary. *If $f: x \rightarrow y$ is an isomorphism in a quasicategory C , then $C_{x/}$ and $C_{y/}$ are categorically equivalent, and $C_{/x}$ and $C_{/y}$ are categorically equivalent.*

Proof. Consider $C_{x/} \xleftarrow{\pi} C_{f/} \xrightarrow{\rho} C_{y/}$. We have already observed (26.15) that $\rho \in \text{TrivFib}$, since $\{1\} \subset \Delta^1$ is right anodyne. The reformulation of Joyal extension (29.4) implies that $\pi \in \text{TrivFib}$ when f is an isomorphism. \square

29.6. Initial objects. Now we prove (D) about initial and terminal objects. We will explicitly prove the statements about initial objects, as the case of terminal objects is similar.

29.7. Proposition. *Let $f: x \rightarrow y$ be a morphism in a quasicategory C , and let $\tilde{f} \in (C_{x/})_0$ be the object of the slice which corresponds to $f \in C_1$. Then \tilde{f} is initial in $C_{x/}$ if and only if f is an isomorphism.*

Proof. For all $n \geq 1$ we have a correspondence of lifting problems

$$\begin{array}{ccc} \{0\} \xrightarrow{\quad} \partial\Delta^n \xrightarrow{\quad} C_{x/} & \iff & \Delta^0 \star \{0\} \xrightarrow{\quad} \Delta^0 \star \partial\Delta^n \xrightarrow{\quad} C \\ \downarrow \quad \nearrow & & \downarrow \quad \nearrow \\ \Delta^n & & \Delta^0 \star \Delta^n \end{array}$$

\tilde{f} f

and $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda_0^{1+n} \subseteq \Delta^{1+n})$, so a lift exists by the Joyal extension theorem. \square

In fact, (29.7), together with dual statement for terminal objects, is merely a reformulation of the Joyal extension theorem (28.2).

Note that (29.7) implies that the slice $C_{x/}$ necessarily has an initial object, namely the vertex corresponding to the edge $1_x \in C_1$.

29.8. Proposition. *Any object in a quasicategory isomorphic to an initial object is also initial.*

Proof. Let x be an initial object in C , and let c be an object isomorphic to x . It is easy to see that x is initial in the homotopy category hC , and therefore c is initial in hC also. This has a useful consequence: *any* map between x and c (in either direction) must be an isomorphism in C .

We next note another fact: if x is initial, any map $f: S \rightarrow C$ extends along $S \subset \Delta^0 \star S$ to a map $f': \Delta^0 \star S \rightarrow C$ such that $f'|_{\Delta^0}$ represents x . This is a consequence of the fact (25.5) that $p: C_{/x} \rightarrow C$ is a trivial fibration whence, there exists a map $s: C \rightarrow C_{/x}$ such that $ps = \text{id}_C$; set f' be the adjoint to $sf: S \rightarrow C_{/x}$.

To show c is initial in C , we need to extend any $a: \partial\Delta^n \rightarrow C$ with $a_0 = c$ to a map $\tilde{a}: \Delta^n: C$. This follows from a succession of two extension problems:

$$\begin{array}{ccc} (\Delta^0 \star \emptyset) \amalg (\emptyset \star \partial\Delta^n) & \xrightarrow{(x,a)} & C \\ \downarrow & \nearrow \text{---} g \text{---} & \\ \Delta^0 \star \partial\Delta^n & & \\ \downarrow & \nearrow \text{---} h \text{---} & \\ \emptyset \star \Delta^n & \xrightarrow{\quad} & \Delta^0 \star \Delta^n \end{array}$$

The extension g exists by the remarks of the previous paragraph since x is initial. The extension h exists because the leading edge of g is a map $x \rightarrow c$ in C , which is an isomorphism by the remarks of the first paragraph. The desired extension \tilde{a} is $h|_{(\emptyset \star \Delta^n)}$. \square

30. PROOF OF OBJECTWISE CRITERION FOR NATURAL ISOMORPHISMS

Next we prove (C). We say that a natural transformation $\alpha: \Delta^1 \rightarrow \text{Fun}(C, D)$ functors between quasicategories is an **objectwise isomorphism** for each object $c \in C_0$, the composite $\Delta^1 \xrightarrow{\alpha} \text{Fun}(C, D) \xrightarrow{\text{res}} \text{Fun}(\{c\}, D) \approx D$ represents an isomorphism in D . Then (C) asserts that every objectwise natural isomorphism is a natural isomorphism (i.e., that the adjoint map $\tilde{\alpha}: \Delta^1 \rightarrow \text{Fun}(C, D)$ represents an isomorphism in the functor category).

30.1. Some properties of objectwise natural isomorphisms. We first establish a “lifting property” for objectwise natural isomorphisms.

30.2. Lemma. *Let C be a quasicategory, and let $i: S \rightarrow T$ be a monomorphism such that $i: S_0 \rightarrow T_0$ is a bijection. Then for every diagram*

$$\begin{array}{ccc} \{0\} & \longrightarrow & \text{Fun}(T, C) \\ \downarrow & \nearrow t & \downarrow \text{Fun}(i, C) \\ \Delta^1 & \xrightarrow{v} & \text{Fun}(S, C) \end{array}$$

such that v is an objectwise natural isomorphism of functors $S \rightarrow C$, a lift t exists, and furthermore any such lift t is an objectwise natural isomorphism of functors $T \rightarrow C$.

Proof. First note that the last statement is automatic: any lift t is an objectwise natural isomorphism if v is, because $i: S_0 \rightarrow T_0$ is a bijection.

Let \mathcal{C} denote the class of monomorphisms $i: S \rightarrow T$ such that (i) $i: S_0 \rightarrow T_0$ is a bijection, and (ii) such that the conclusion of the lemma holds. A straightforward argument shows that \mathcal{C} is a weakly saturated class. (We need condition (i) to ensure that any lift of an objectwise natural isomorphism is also an objectwise isomorphisms, needed for showing that \mathcal{C} is closed under composition and transfinite composition.)

Let $\text{Cell}_{\geq 1} := \{\partial\Delta^n \subset \Delta^n\}_{n \geq 1}$. From the skeletal filtration (15.23) we see that $\overline{\text{Cell}_{\geq 1}}$ is precisely the class of monomorphisms which are bijections on vertices. Thus the lemma amounts to showing that $\text{Cell}_{\geq 1} \subseteq \mathcal{C}$, which in turn follows from the following proposition (30.3) applied to the case of $D = *$ and $(i, j) = (0, 0)$. \square

The following lemma is a kind of “pushout-product” version of Joyal lifting, where we replace a horn such as $\Lambda_0^n \subset \Delta^n$ with the inclusion $(\{0\} \subset \Delta^1) \square (\partial\Delta^n \subset \Delta^n)$, with the role of the “leading edge” played by $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^n$.

30.3. Proposition (Pushout-product Joyal lifting). *Suppose $p: C \rightarrow D$ is an inner fibration of quasicategories, and suppose $n \geq 1$, and either $(i, j) = (0, 0)$ or $(i, j) = (1, n)$. For any diagram*

$$\begin{array}{ccc} \Delta^1 \times \{j\} & \xrightarrow{f} & C \\ \downarrow & \searrow & \downarrow p \\ \Delta^1 \times \Delta^n & \xrightarrow{\quad} & D \end{array}$$

$(\{i\} \times \Delta^n) \cup_{\{i\} \times \partial\Delta^n} (\Delta^1 \times \partial\Delta^n)$

such that f represents an isomorphism in C , a lift exists.

Proof. This is a calculation, given in the appendix (58.5), which itself relies on Joyal lifting. \square

30.4. Example. To give an idea of the proof (30.3), consider the case of $n = 1$ and $(i, j) = 0$, in which case $K = (\{0\} \times \Delta^1) \cup_{\{0\} \times \partial\Delta^1} (\Delta^1 \times \partial\Delta^1)$ can be pictured the solid-arrow part of the diagram

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow a & \nearrow b & \uparrow \\ (0, 0) & \xrightarrow{e} & (1, 0) \end{array}$$

To lift to a map $\Delta^1 \times \Delta^1 \rightarrow C$, we first choose a lift on the 2-simplex a , which is attached along an inner horn $\Lambda_1^2 \subset \Delta^2$; then we choose a lift on the 2-simplex b , which is a non-inner horn $\Lambda_0^2 \subset \Delta^2$ such that $K \rightarrow C$ sends the edge marked e to an isomorphism in C , so Joyal-lifting applies.

30.5. Proof of the objectwise criterion. We can reformulate (C) as follows: for a quasicategory C and simplicial set K , the evident map

$$\mathrm{Fun}(j, C): \mathrm{Fun}(K, C) \rightarrow \mathrm{Fun}(\mathrm{Sk}_0 K, C) \approx \prod_{k \in K_0} C$$

induced by restriction along $j: \mathrm{Sk}_0 K \rightarrow K$ is conservative. This is justified by **insert justification here**.

What we need to prove is a special case of the following.

30.6. Proposition. *Let $j: K \rightarrow L$ be a monomorphism of simplicial sets such that $j: K_0 \xrightarrow{\sim} L_0$ is a bijection. Then for every quasicategory C the restriction map $\mathrm{Fun}(j, C): \mathrm{Fun}(L, C) \rightarrow \mathrm{Fun}(K, C)$ is conservative.*

The proof we give uses ideas from [Lur09, §3.1.1].

Possible alternate proof: Inner fibration p between quasicategories is conservative iff $(\Delta^1 \subset N\mathrm{Iso}) \boxtimes p$; this uses Joyal lifting applied to cell structure of $N\mathrm{Iso}$. Let \mathcal{C} be the class of $i: K \rightarrow L$ such that $p^{\square i}$ is conservative, so \mathcal{C} is left complement of $p^{\square(\Delta^1 \subset N\mathrm{Iso})}$ and thus weakly saturated. Then $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}$ for $n \geq 1$ by pushout-product Joyal lifting.

Proof. Note that $p = \mathrm{Fun}(j, C)$ is always an inner fibration between quasicategories (since j is a monomorphism and $\overline{\mathrm{InnHorn}} \square \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$). Therefore we can apply the criterion of (26.15) to show p is conservative: we show that for every $n \geq 2$ and every diagram of the form

$$\begin{array}{ccc} & & f \\ \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^n \xrightarrow{\quad} \mathrm{Fun}(L, C) \\ & \searrow & \downarrow \scriptstyle p = \mathrm{Fun}(j, C) \\ & & \Delta^n \xrightarrow{\quad} \mathrm{Fun}(K, C) \end{array}$$

such that $p(f)$ represents an isomorphism in $\mathrm{Fun}(K, C)$, a lift a exists.

We reduce this lifting problem to a different one using a retraction. For $n \geq 2$ we can define maps

$$\Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n$$

uniquely characterized by their effect on vertices: $s(x) = (1, x)$, and $r(0, 1) = 0$ and $r(x, y) = y$ if $(x, y) \neq (0, 1)$. We check that

- $rs = \mathrm{id}$,
- $s(\Lambda_0^n) = \{1\} \times \Lambda_0^n \subseteq (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n)$,
- $r(\{0\} \times \Delta^n) = \Delta^{[n] \setminus 1} \subseteq \Lambda_0^n$, and
- $r(\Delta^1 \times \Delta^{[n] \setminus j}) = \Delta^{[n] \setminus j}$ if $j \neq 0$, whence $r(\Delta^1 \times \Lambda_0^n) = \Lambda_0^n$.

Therefore we can form the solid arrow diagram

$$\begin{array}{ccccccc} \Lambda_0^n & \xrightarrow{s} & (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n) & \xrightarrow{r} & \Lambda_0^n & \xrightarrow{u} & \mathrm{Fun}(L, C) \\ \downarrow & & \downarrow & & \downarrow & \nearrow \scriptstyle a & \downarrow \scriptstyle p = \mathrm{Fun}(j, C) \\ \Delta^n & \xrightarrow{s} & \Delta^1 \times \Delta^n & \xrightarrow{r} & \Delta^n & \xrightarrow{\quad} & \mathrm{Fun}(K, C) \end{array}$$

and observe that to produce a lift a , it suffices to produce a map b which is a lift in its rectangle: given b , take $a = bs$.

Note that r sends each edge $\Delta^1 \times \{k\}$ in $(\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n)$ to either: the degenerate edge $\langle kk \rangle$ in Λ_0^n (if $k \neq 1$), or the leading edge $\langle 01 \rangle$ in Λ_0^n (if $k = 1$). Thus for all $k \in [n]$ the restriction of pur to $\Delta^1 \times \{k\}$ represents an isomorphism in $\mathrm{Fun}(K, C)$.

By adjunction producing a lift b amounts to showing that a lift exists in

$$\begin{array}{ccc}
 \{0\} & \longrightarrow & \text{Fun}(T, C) \\
 \downarrow & \nearrow & \downarrow \text{Fun}(i, C) \\
 \Delta^1 & \xrightarrow{v} & \text{Fun}(S, C)
 \end{array}$$

where $i: S \rightarrow T$ is the monomorphism $(\Lambda_0^n \times L) \cup (\Delta^n \times K) \rightarrow \Delta^n \times L$, and the edge in $\text{Fun}(S, C)$ represented by v is such that for every $k \in [n]$ the restriction $v|_{\{k\} \times K}$ represents an isomorphism in $\text{Fun}(\{k\} \times K)$.

Note that $i: S \rightarrow T$ induces a bijection on vertices, and that the hypothesis on v implies that the restriction $v|_{\{x\}}$ of v to every vertex $x \in S_0$ represents an isomorphism in $\text{Fun}(\{x\}, C) \approx C$. Thus the proposition follows from the analogous claim for lifting objectwise natural isomorphisms, which we proved as (30.2). □

Now that we have proved the objectwise criterion for natural isomorphisms (C), we can reinterpret the conclusion of (30.2): the functor $\text{Fun}(T, C) \rightarrow \text{Fun}(S, C)$ is an isofibration when $S_0 \rightarrow T_0$ is a bijection. **Perhaps show here that the bijection condition can be dropped, and do a relative version here, rather than in a later section.**

30.7. Older proof. (This material can probably be removed now.)

Proof. Let \mathcal{C} be the class of monomorphisms $j: K \rightarrow L$ for which the conclusion holds, i.e., such that $\text{Fun}(j, C)$ is conservative for all quasicategories C . To prove the claim, it suffices to show (a) that \mathcal{C} is a weakly saturated class, and (b) that \mathcal{C} contains the inclusions $\partial\Delta^n \subset \Delta^n$ for all $n \geq 1$. Given this, the result follows using the skeletal filtration (15.23).

Note that $\text{Fun}(j, C)$ is always an inner fibration between quasicategories, and recall the “lifting characterization” (28.14) of conservative inner fibrations between quasicategories which follows from the Joyal lifting theorem. Using this, the proof of (a) is a straightforward exercise.

To show (b) that $\text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(\partial\Delta^n, C)$ is conservative for $n \geq 1$, we again use (28.14), which we convert into its adjoint form. In fact, it suffices to prove that in any diagram of the form

$$\begin{array}{ccc}
 & \xrightarrow{f_0} & \\
 \Delta^{\{0,1\}} \times \{0\} & \longrightarrow & (\Lambda_0^m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \longrightarrow C \\
 & \searrow j & \nearrow \\
 & \Delta^2 \times \Delta^n &
 \end{array}$$

with $m \geq 2$ and $n \geq 1$ such that f_0 represents an isomorphism in C , a lift exists. This reduction uses that f_0 is the composite

$$\Delta^{\{0,1\}} \xrightarrow{f} \text{Fun}(\Delta^n, C) \xrightarrow{p} \text{Fun}(\partial\Delta^n, C) \rightarrow \text{Fun}(\{0\}, C)$$

induced by restriction along $\{0\} \subset \partial\Delta^n \subset \Delta^n$, so that if $p(f)$ represents an isomorphism in $\text{Fun}(\partial\Delta^n, C)$, then f_0 represents an isomorphism in C .

This statement is a consequence of the following lemma (30.8). □

30.8. Proposition (Pushout-product version of Joyal lifting). *Suppose $p: C \rightarrow D$ is an inner fibration between quasicategories. Suppose $m, n \geq 1$. For any diagram*

$$\begin{array}{ccc}
 \Delta^{\{0,1\}} \times \{0\} & \xrightarrow{\quad f \quad} & C \\
 \downarrow & \searrow & \downarrow p \\
 (\Lambda_0^m \times \Delta^n) \cup_{\Lambda_0^m \times \partial \Delta^n} (\Delta^m \times \partial \Delta^n) & \xrightarrow{\quad} & C \\
 \downarrow j & \searrow & \downarrow p \\
 \Delta^m \times \Delta^n & \xrightarrow{\quad} & D
 \end{array}$$

such that f represents an isomorphism in C , a lift exists.

Note: if we take instead $m \geq 2$ and $n = 0$, the above statement becomes the Joyal lifting theorem.

Proof. Proved as [Joy08a, 5.8] (though there it is the $\Lambda_m^m \subset \Delta^m$ case that is proved). \square

The idea is to produce the map j by iteratively attaching simplices along horns, which in each case are either:

- (1) an inner horn $\Lambda_i^k \subset \Delta^k$ for $0 < i < k$, or
- (2) a horn $\Lambda_0^k \subset \Delta^k$ with $k \geq 2$ along a map from Λ_0^k such that the restriction to its leading edge $\Delta^{\{0,1\}} \subset \Lambda_0^k$ of the map to C is identical to f .

The existence of the lifting follows, by applying the definition of inner fibration in cases of type (1), or the Joyal extension theorem in cases of type (2).

Here is a picture for the case $m = 1, n = 1$. The source of $(\Lambda_0^1 \subset \Delta^1) \square (\partial \Delta^1 \subset \Delta^1)$ looks like

$$\begin{array}{ccc}
 01 & \longrightarrow & 11 \\
 \uparrow & & \\
 00 & \xrightarrow{\sim} & 10
 \end{array}$$

First attach the 2-simplex $\langle 00, 01, 11 \rangle$, which intersects the source along the inner horn Λ_1^2 . Then attach the 2-simplex $\langle 00, 10, 11 \rangle$, which intersects what we have already built along the horn Λ_0^2 , whose leading edge $\langle 00, 11 \rangle$ is sent to an isomorphism in C .

Here are pictures for the case $m = 2, n = 1$. Here is the source of $(\Lambda_0^2 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$.

$$\begin{array}{ccccc}
 & & \curvearrowright & & \\
 11 & \longleftarrow & 01 & \longrightarrow & 21 \\
 \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
 10 & \xrightarrow{\sim} & 00 & \longrightarrow & 20 \\
 & & \curvearrowleft & &
 \end{array}$$

The following chart lists all non-degenerate elements in the complement, with “ \surd ” marking those in the source.

$\langle 10, 21 \rangle$	$\langle 10, 20, 21 \rangle$	$\langle 10, 11, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 10, 20, 21 \rangle$	$\langle 00, 10, 11, 21 \rangle$	$\langle 00, 01, 11, 21 \rangle$
$\surd \langle 21 \rangle$	$\surd \langle 20, 21 \rangle$	$\surd \langle 11, 21 \rangle$	$\langle 10, 21 \rangle$	$\surd \langle 11, 21 \rangle$	$\langle 10, 20, 21 \rangle$	$\langle 10, 11, 21 \rangle$	$\surd \langle 01, 11, 21 \rangle$
$\surd \langle 10 \rangle$	$\langle 10, 21 \rangle$	$\langle 10, 21 \rangle$	$\surd \langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\surd \langle 00, 20, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 11, 21 \rangle$
	$\surd \langle 10, 20 \rangle$	$\surd \langle 10, 11 \rangle$	$\surd \langle 00, 10 \rangle$	$\surd \langle 00, 11 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\surd \langle 00, 01, 21 \rangle$
					$\surd \langle 00, 10, 20 \rangle$	$\surd \langle 00, 10, 11 \rangle$	$\surd \langle 00, 01, 11 \rangle$

Note that the elements $\langle 10, 21 \rangle$, $\langle 00, 10, 21 \rangle$, and $\langle 00, 11, 21 \rangle$ of the complement appear multiple times as faces. We attach elements to the domain in the following order:

- ① $\langle 10, 11, 21 \rangle$,
- ② $\langle 00, 01, 11, 21 \rangle$,
- ③ $\langle 00, 10, 11, 21 \rangle$,
- ④ $\langle 00, 10, 20, 21 \rangle$.

Only the final step involves attaching along a non-inner horn; in that case, the attaching map sends the leading edge of Λ_0^3 to $\langle 00, 10 \rangle$.

Part 4. The fundamental theorem

Recall that a functor $f: C \rightarrow D$ between quasicategories is said to be an *equivalence* there exists a $g: D \rightarrow C$ such that gf and fg are naturally isomorphic to the respective identity functors. When C and D are ordinary categories, there is a well-known criterion for the existence of such a g , namely: *f is an equivalence if and only if f is fully faithful and essentially surjective.* Here

- *fully faithful* means that $\text{Hom}_C(x, y) \rightarrow \text{Hom}_D(f(x), f(y))$ is a bijection of sets for every pair of objects $x, y \in \text{ob } C$, and
- *essentially surjective* means that for every object $d \in \text{ob } D$ there exists an object $c \in \text{ob } C$ such that $f(c)$ is isomorphic to d .

I like to call this fact the *Fundamental Theorem of Category Theory*. This is non-standard and frankly pretentious terminology (I am unaware of any standard descriptive name for this result²³). I want to give this fact a fancy name in order to signpost it, as it is quite *nonconstructive*: to prove it requires making a *choice* for each object d in D of an object c of C such that $f(c) \approx d$ (so it in fact relies on an appropriate form of the axiom of choice). (*Exercise*: prove the “Fundamental theorem” by positing a choice of object $g(d) \in \text{ob } C$ and isomorphism $\alpha(d): f(g(d)) \rightarrow d$ for each object of d , and extending this to the data of a categorical inverse of f .)

Example. Fix a field k . Let Mat be the category whose objects are non-negative integers $n \geq 0$, and whose morphisms $A: n \rightarrow m$ are $(m \times n)$ -matrices with entries in k , so that composition is matrix multiplication. Let Vect be the category of finite dimensional k -vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor $F: \text{Mat} \rightarrow \text{Vect}$ is fully faithful and essentially surjective. Therefore F is an equivalence of categories. However, there is no *canonical* choice of an inverse functor, whose construction relies on making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the *quasigroupoid* of maps between two objects.

31. MAPPING SPACES OF A QUASICATEGORY

Given a quasicategory C and objects $x, y \in C_0$, the **mapping space** (or **mapping quasigroupoid**) from x to y is the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{map}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & C \times C \end{array}$$

That is, $\text{map}_C(x, y)$ is the fiber of the restriction map $\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\partial\Delta^1, C)$ over the point $(x, y) \in (C \times C)_0$, where we use the isomorphism $\text{Fun}(\partial\Delta^1, C) \approx C \times C$ induced by the isomorphism $\partial\Delta^1 \approx \Delta^0 \amalg \Delta^0$.

If $C = N(A)$ is the nerve of a category, then $\text{map}_C(x, y)$ is a discrete simplicial set (2.5) corresponding to the set $\text{Hom}_C(x, y)$.

31.1. Mapping spaces are Kan complexes. The terminology “space” is justified by the following

31.2. Proposition. *The simplicial sets $\text{map}_C(x, y)$ are quasigroupoids (and hence Kan complexes by (A)).*

This is a special case of the following, applied to $\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\partial\Delta^1, C)$, the restriction along $j = (\partial\Delta^1 \subset \Delta^1)$.

²³I also don’t know when it was first formulated, or who first stated it.

31.3. Proposition. *Let C be a quasicategory, and let $j: K \rightarrow L$ be a monomorphism which is a bijection on vertices. Then the fibers of the restriction map $p = \text{Fun}(j, C): \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$ are quasigroupoids.*

Proof. Consider a vertex $g \in \text{Fun}(K, C)_0$, and let F denote the fiber of p over g , i.e., the pullback of p along the inclusion $\{g\} \rightarrow \text{Fun}(K, C)$.

Note that $p: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$ is an inner fibration, a case of enriched lifting using $\text{InnHorn} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.7). Therefore $F \rightarrow \{g\}$, which is a pullback of p , is also an inner fibration, so F is a quasicategory.

To show that the quasicategory F is a quasigroupoid, it suffices to show that every $\Lambda_0^2 \rightarrow F$ extends to a 2-simplex. Thus we must consider the lifting problem

$$\begin{array}{ccccc}
 & & f & & \\
 \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^2 & \xrightarrow{\quad} & F & \xrightarrow{\quad} & \text{Fun}(L, C) \\
 & & \downarrow & \nearrow & \downarrow & & \downarrow p \\
 & & \Delta^2 & \xrightarrow{\quad} & \{g\} & \xrightarrow{\quad} & \text{Fun}(K, C)
 \end{array}$$

Since the right hand square is a pullback, it suffices to produce a lift $\Delta^2 \rightarrow \text{Fun}(L, C)$. Since $p(f)$ is an identity map and hence an isomorphism in $\text{Fun}(K, C)$, by the lifting criterion for conservative inner fibrations (28.14) it suffices to show that p is conservative, which is the case by (30.6) since j is a bijection on vertices. \square

31.4. Mapping spaces and homotopy classes. The set of morphisms $x \rightarrow y$ in a quasicategory is precisely the set of vertices of $\text{map}_C(x, y)$.

31.5. Proposition. *Let C be a quasicategory. For any two maps $f, g: x \rightarrow y$ in C , we have that $f \approx g$ (equivalence under the relation used to define the homotopy category hC) if and only if f and g are isomorphic as objects of the quasigroupoid $\text{map}_C(x, y)$. That is,*

$$\text{Hom}_{hC}(x, y) \approx \pi_0 \text{map}_C(x, y)$$

for every pair x, y of objects of C .

Proof. Suppose $f, g \in \text{map}_C(x, y)_0$ are isomorphic, so that in particular there is a morphism $f \rightarrow g$ in the quasigroupoid $\text{map}_C(x, y)$. This amounts to a map $\Delta^1 \times \Delta^1 \rightarrow C$ which can be represented by a diagram of elements of C of the form:

$$\begin{array}{ccc}
 x & \xrightarrow{g} & y \\
 \downarrow 1_x & \nearrow h & \downarrow 1_y \\
 x & \xrightarrow{f} & y
 \end{array}$$

This explicitly exhibits a chain $f \sim_r h \sim_\ell g$ of homotopies, so $f \approx g$ as desired.

Conversely, if $f \approx g$, we can explicitly construct a map $H: f \rightarrow g$ in $\text{map}_C(x, y)$: in terms of the above picture, let $h = g$, let b be an explicit choice of right-homotopy $f \sim_r g$, and let $a = g_{001}$. \square

31.6. Extended mapping spaces and composition. Given a finite list $x_0, \dots, x_n \in C_0$ of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

$$\begin{array}{ccc}
 \text{map}_C(x_0, \dots, x_n) & \longrightarrow & \text{Fun}(\Delta^n, C) \\
 \downarrow & & \downarrow \\
 \{(x_0, \dots, x_n)\} & \longrightarrow & C^{\times(n+1)}
 \end{array}$$

where the right-hand vertical arrow is induced by restriction along $\mathrm{Sk}_0 \Delta^n \rightarrow \Delta^n$, using the isomorphism $\mathrm{Sk}_0 \Delta^n \approx \coprod_{n+1} \Delta^0$, whence $\mathrm{Fun}(\mathrm{Sk}_0 \Delta^n, C) \approx C^{\times(n+1)}$. By (31.3) the extended mapping spaces are quasigroupoids.

We can compare the extended mapping spaces to the fibers of $\mathrm{Fun}(I^n, C) \rightarrow C^{\times(n+1)}$, which is seen to be an n -fold product of mapping spaces.

31.7. Lemma. *The map*

$$g_n: \mathrm{map}_C(x_0, \dots, x_n) \rightarrow \mathrm{map}_C(x_{n-1}, x_n) \times \cdots \times \mathrm{map}_C(x_0, x_1)$$

induced by restriction along the spine inclusion $I^n \subseteq \Delta^n$ is a trivial fibration. In particular, this map is a categorical equivalence between Kan complexes.

Proof. The map g_n is a base change of $p: \mathrm{Fun}(\Delta^n, C) \rightarrow \mathrm{Fun}(I^n, C)$. Since $I^n \subset \Delta^n$ is inner anodyne (12.11), and C is a quasicategory, the map p is a trivial fibration by enriched lifting using $\overline{\mathrm{InnHorn}} \square \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (16.7). \square

The inclusions $I^2 \subset \Delta^2 \supset \Delta^{\{0,2\}}$ induce restriction maps

$$\mathrm{Fun}(I^2, C) \leftarrow \mathrm{Fun}(\Delta^2, C) \rightarrow \mathrm{Fun}(\Delta^{\{0,2\}}, C).$$

Restricting to the fibers over a triple (x_0, x_1, x_2) of objects, we obtain a zig-zag of maps of Kan complexes

$$\mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \xleftarrow{g_2} \mathrm{map}_C(x_0, x_1, x_2) \rightarrow \mathrm{map}_C(x_0, x_2),$$

where the second map is induced by restriction along $\Delta^{\{0,2\}} \subset \Delta^2$, and the first map g_2 is a categorical equivalence, and in fact a trivial fibration. After choosing a categorical inverse to g_2 (e.g., a section of g_2 using (20.12)), we obtain a “composition” map

$$(31.8) \quad \mathrm{comp}: \mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \rightarrow \mathrm{map}_C(x_0, x_2).$$

This map is *not* uniquely determined, since it depends on a choice of categorical inverse to g_2 . However, any two categorical inverses to g_2 are naturally isomorphic (19.4), and therefore comp is defined up to natural isomorphism. That is, it is a well-defined map in $h\mathrm{Kan}$, the homotopy category of Kan complexes (defined to be the full subcategory of $hq\mathrm{Cat}$ spanned by Kan complexes).

31.9. Proposition. *The two maps obtained by composing the sides of the square*

$$\begin{array}{ccc} \mathrm{map}_C(x_2, x_3) \times \mathrm{map}_C(x_2, x_1) \times \mathrm{map}_C(x_0, x_1) & \xrightarrow{\mathrm{id} \times \mathrm{comp}} & \mathrm{map}_C(x_2, x_3) \times \mathrm{map}_C(x_0, x_2) \\ \mathrm{comp} \times \mathrm{id} \downarrow & & \downarrow \mathrm{comp} \\ \mathrm{map}_C(x_1, x_3) \times \mathrm{map}_C(x_0, x_1) & \xrightarrow{\mathrm{comp}} & \mathrm{map}_C(x_0, x_3) \end{array}$$

are naturally isomorphic. That is, the diagram commutes in $h\mathrm{Kan} \subset hq\mathrm{Cat}$.

Proof. Here is a diagram of simplicial sets which actually commutes on the nose, i.e., not merely in the homotopy category, but actually commutes in $q\mathrm{Cat}$. I use “ $\langle x, y, z \rangle$ ” as shorthand for “ $\mathrm{map}_C(x, y, z)$ ”, etc.

$$\begin{array}{ccccc} \langle x_2, x_3 \rangle \times \langle x_1, x_2 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_2, x_3 \rangle \times \langle x_0, x_1, x_2 \rangle & \rightarrow & \langle x_2, x_3 \rangle \times \langle x_0, x_2 \rangle \\ \uparrow \sim & & \uparrow \sim & & \uparrow \sim \\ \langle x_1, x_2, x_3 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_0, x_1, x_2, x_3 \rangle & \longrightarrow & \langle x_0, x_2, x_3 \rangle \\ \downarrow & & \downarrow & & \downarrow \\ \langle x_1, x_3 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_0, x_1, x_3 \rangle & \longrightarrow & \langle x_0, x_3 \rangle \end{array}$$

The maps labelled “ $\xrightarrow{\sim}$ ” are categorical equivalences, and in fact are trivial fibrations. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{array}{ccccc}
 \Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} & \longleftarrow & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 \Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^3 & \longleftarrow & \Delta^{\{0,2,3\}} \\
 \uparrow & & \uparrow & & \uparrow \\
 \Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^{\{0,1,3\}} & \longleftarrow & \Delta^{\{0,3\}}
 \end{array}$$

where the maps labelled “ $\xrightarrow{\sim}$ ” are inner anodyne, and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define comp . After passing to $h\text{Kan}$ the categorical equivalences become isomorphisms, and the result follows. \square

31.10. Segal categories. Thus, a quasicategory does not quite give rise to a category “enriched over quasigroupoids”. Although we can define a composition law, it is not uniquely determined, and is only associative “up to homotopy”.

What we do get is a Segal category. A **Segal category** is a functor

$$M: \Delta^{\text{op}} \rightarrow s\text{Set}$$

such that

- (1) the simplicial set $M([0])$ is discrete, i.e., $M([0]) = \text{Sk}_0 M([0])$, and
- (2) for each $n \geq 1$ the “Segal map”

$$M([n]) \xrightarrow{\langle (n-1, n)^*, \dots, (0, 1)^* \rangle} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a “weak equivalence” of simplicial sets.

We will define “weak equivalence” of simplicial sets below. For now, we note that a map between *Kan complexes* is a weak equivalence if and only if it is a categorical equivalence, and that if each $M([n])$ is a Kan complex, then so are the fiber products which appear in the above definition.

Given a quasicategory C , we obtain a functor $M_C: \Delta^{\text{op}} \rightarrow s\text{Set}$ by

$$\begin{aligned}
 M_C([0]) &:= \text{Sk}_0 C, \\
 M_C([n]) &:= \text{Fun}(\Delta^n, C) \times_{\text{Fun}(\text{Sk}_0 \Delta^n, C)} \text{Fun}(\text{Sk}_0 \Delta^n, \text{Sk}_0 C) \\
 &\approx \coprod_{x_0, \dots, x_n \in C_0} \text{map}_C(x_0, \dots, x_n).
 \end{aligned}$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0])} M_C([1]) \xleftarrow{\langle (12)^*, (01)^* \rangle} M_C([2]) \xrightarrow{(02)^*} M_C([1])$$

is a coproduct over all triples $x_0, x_1, x_2 \in C_0$ of the zig-zag (31.8) used to define “composition”.

You also get a Segal category from any “simplicially enriched” category. Suppose \mathcal{C} is a (small) category which is enriched over the category of simplicial sets, with object set $\text{ob } \mathcal{C}$, and function objects $\mathcal{C}(x, x') \in s\text{Set}$ for each x, x' . Then we can define $M_{\mathcal{C}}: \Delta^{\text{op}} \rightarrow s\text{Set}$ by

$$\begin{aligned}
 M_{\mathcal{C}}([0]) &:= \text{ob } \mathcal{C}, \\
 M_{\mathcal{C}}([n]) &:= \coprod_{x_0, \dots, x_n \in \text{ob } \mathcal{C}} \mathcal{C}(x_{n-1}, x_n) \times \cdots \times \mathcal{C}(x_0, x_1).
 \end{aligned}$$

We thus obtain functors

$$\text{qCat} \rightarrow \text{SeCat} \leftarrow s\text{Cat}$$

relating quasicategories, Segal categories, and simplicially enriched categories. Simplicially enriched categories were proposed as a model for ∞ -categories by Dwyer and Kan²⁴, while Segal categories were proposed as a model for ∞ -categories by Hirschowitz and Simpson [HS01]²⁵. All of these models are known to be equivalent in a suitable sense; see [Ber10] for more about these models and their comparison.

31.11. The enriched homotopy category of a quasicategory. Given a quasicategory C we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of C and denoted $\mathcal{H}C$.²⁶ This object will be a category *enriched over* $h\mathbf{Kan}$, the homotopy category of Kan complexes, whose underlying category is hC .

We now define $\mathcal{H}C$. The objects of $\mathcal{H}C$ are just the objects of C . For any two objects $x, y \in C_0$, we have the quasigroupoid

$$\mathcal{H}C(x, y) := \text{map}_C(x, y)$$

which we will regard as an object of the homotopy category $h\mathbf{Kan}$ of Kan complexes. Composition $\mathcal{H}C(x_1, x_2) \times \mathcal{H}C(x_0, x_1) \rightarrow \mathcal{H}C(x_0, x_2)$ is the composition map defined above, which is well-defined as a morphism in $h\mathbf{Kan}$. Composition is associative as shown above (31.9).

The *underlying* ordinary category of $\mathcal{H}C$ is just the ordinary homotopy category hC , since

$$\text{Hom}_{h\mathbf{Kan}}(\Delta^0, \text{map}_C(x, y)) \approx \pi_0 \text{map}_C(x, y) \approx \text{Hom}_{hC}(x, y).$$

31.12. Warning. A quasicategory C *cannot* be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. In fact, there exist $h\mathbf{Kan}$ -enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory C . A proof is outside the scope of these notes; however, we note that counterexamples may be produced from associative H -spaces which are not loop spaces.

31.13. Exercise. Let C and D be quasicategories. Show that there is an isomorphism $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$ of $h\mathbf{Kan}$ -enriched categories.

32. THE FUNDAMENTAL THEOREM OF QUASICATEGORY THEORY

32.1. Fully faithful and essentially surjective functors between quasicategories. Note that any functor $f: C \rightarrow D$ of quasicategories induces functors $\text{map}_C(x, y) \rightarrow \text{map}_D(f(x), f(y))$ for every pair of objects x, y in C . We say that a functor $f: C \rightarrow D$ between quasicategories is

- **fully faithful** if for every pair $c, c' \in C_0$, the resulting map $\text{map}_C(c, c') \rightarrow \text{map}_D(fc, fc')$ is a categorical equivalence, and
- **essentially surjective** if the induced functor $hf: hC \rightarrow hD$ is essentially surjective; i.e., if for every $d \in D_0$ there exists $c \in C_0$ and an isomorphism $fc \rightarrow d$ in D_1 .

Another way to say this: $f: C \rightarrow D$ is fully faithful and essentially surjective iff the induced $h\mathbf{Kan}$ -enriched functor $\mathcal{H}f: \mathcal{H}C \rightarrow \mathcal{H}D$ is an equivalence of *enriched* categories.

32.2. Proposition. *If $f: C \rightarrow D$ is a categorical equivalence between quasicategories, then f is fully faithful and essentially surjective.*

Proof. We already know that $hf: hC \rightarrow hD$ is an equivalence of ordinary categories, which implies essential surjectivity.

²⁴They called them “homotopy theories” instead of “ ∞ -categories; see [DS95, §11.6].

²⁵In fact, they generalize this to “Segal n -categories”, which were the first effective model for (∞, n) -categories.

²⁶Lurie usually calls this “ hC ”, though he also uses that notation for the ordinary homotopy category of C that we have already discussed. I prefer to have two separate notations.

To show that f is fully faithful, choose a categorical inverse g of f . Given $x, y \in C_0$, consider the induced diagram of quasigroupoids

$$\begin{array}{ccccccc}
 & & gf & & & & \\
 & \curvearrowright & & \curvearrowleft & & & \\
 \text{map}_C(x, y) & \xrightarrow{f} & \text{map}_D(fx, fy) & \xrightarrow{g} & \text{map}_C(gfx, gfy) & \xrightarrow{f} & \text{map}_D(fgfx, fgfy) \\
 & & & & fg & & \\
 & \curvearrowleft & & \curvearrowright & & &
 \end{array}$$

By the 2-out-of-6 property for categorical equivalences (22.10), it will suffice to show that the maps marked gf and fg are categorical equivalences between the respective mapping spaces. Since gf and fg are naturally isomorphic to the identity maps of $\text{map}_C(x, y)$ and $\text{map}_D(fx, fy)$ respectively, the claim follows from (32.3) which we prove below. \square

32.3. Proposition. *If $f_0, f_1: C \rightarrow D$ are functors which are naturally isomorphic, then f_0 is fully faithful if and only if f_1 is.*

32.4. Path category. For the proof of the fact that natural isomorphisms preserve the fully-faithful property, we will need to consider the **path category** of a quasicategory D . This is the full sub(quasi)category

$$\widehat{D} \subseteq \text{Fun}(\Delta^1, D)$$

spanned by the objects which are $\Delta^1 \rightarrow D$ which represent isomorphisms in D . (A generalization of this construction will be used later (??).) The restriction maps along $\{0\} \subset \Delta^1 \supset \{1\}$ induce functors $D \xleftarrow{\pi_0} \widehat{D} \xrightarrow{\pi_1} D$. Note that a functor $\tilde{H}: C \rightarrow \widehat{D}$ corresponds exactly to giving a natural isomorphism $H: C \times \Delta^1 \rightarrow D$ of functors $f_0, f_1: C \rightarrow D$, where $f_i = \pi_i \tilde{H}$ (because natural isomorphisms are the same as objectwise natural isomorphisms (C)).

32.5. Lemma. *Let D be a quasicategory. Then both restriction functors*

$$D \xleftarrow{\pi_0} \widehat{D} \xrightarrow{\pi_1} D$$

are trivial fibrations.

Proof. We need to solve the lifting problem

$$\begin{array}{ccc}
 \partial\Delta^n & \xrightarrow{\quad} & \widehat{D} \xrightarrow{\quad} \text{Fun}(\Delta^1, D) \\
 \downarrow & \nearrow \text{dashed} & \downarrow \pi_i \\
 \Delta^n & \xrightarrow{\quad} & \text{Fun}(\{i\}, D)
 \end{array}$$

for all $n \geq 0$ and $i = 0, 1$. When $n = 0$ this is easy: any object of D is the source and target of an isomorphism in D , namely its identity map. For $n \geq 1$ it suffices to find a lifting in the adjoint lifting problem o

$$\begin{array}{ccc}
 \Delta^1 \times \{j\} & \xrightarrow{\quad} & (\{i\} \times \Delta^n) \cup_{\{i\} \times \partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \xrightarrow{\quad} D \\
 & & \downarrow \\
 & & \Delta^1 \times \Delta^n
 \end{array}$$

where $j = 0$ if $i = 0$ and $j = n$ if $i = 1$. In either case we know by hypothesis that f represents an isomorphism in D , so a lift exists by the “pushout-product version” of Joyal lifting (30.3). \square

Proof of (32.3). Consider a natural isomorphism $H: C \times \Delta^1 \rightarrow D$ between f_0 and f_1 , and write $\tilde{H}: C \rightarrow \hat{D} \subseteq \text{Fun}(\Delta^1, D)$ for its adjoint. The lemma (32.5) implies that in the commutative diagram

$$\begin{array}{ccc}
 & & \text{Fun}(\{0\}, D) = D \\
 & \nearrow^{f_0} & \\
 C & \xrightarrow{\tilde{H}} \hat{D} & \\
 & \searrow_{f_1} & \\
 & & \text{Fun}(\{1\}, D) = D
 \end{array}$$

both π_0 and π_1 are trivial fibrations, whence for any $x, y \in C_0$ we get a commutative diagram

$$\begin{array}{ccc}
 & & \text{map}_D(f_0(x), f_0(y)) \\
 & \nearrow^{f_0} & \\
 \text{map}_C(x, y) & \longrightarrow \text{map}_{\hat{D}}(\tilde{H}(x), \tilde{H}(y)) & \\
 & \searrow_{f_1} & \\
 & & \text{map}_D(f_1(x), f_1(y))
 \end{array}$$

in which the indicated maps are categorical equivalences. Using the 2-out-of-3 property of categorical equivalences (22.10), we see that the mapped marked f_0 is a categorical equivalence if and only if the map marked f_1 is. Thus we have shown that $f_0: C \rightarrow D$ is fully faithful if and only if $f_1: C \rightarrow D$ is fully faithful. \square

To check that a functor is fully faithful, it suffices to check the defining property on representatives of isomorphism classes of objects.

32.6. Proposition. *Let $f: C \rightarrow D$ be a functor between quasicategories, and let $S \subset C_0$ be a subset of objects which includes a representative of every isomorphism class in C . Then f is fully faithful if and only if $\text{map}_C(c, c') \rightarrow \text{map}_D(fc, fc')$ is a categorical equivalence for all $c \in S$.*

Proof. The only-if direction is immediate from the definition of fully faithful. To prove the if direction, let $x, x' \in C_0$ and choose isomorphisms $\alpha: x \rightarrow c$ and $\alpha': x' \rightarrow c'$ where $c, c' \in S$. We may interpret α and α' as objects of $\hat{C} \subseteq \text{Fun}(\Delta^1, C)$. We obtain a commutative diagram

$$\begin{array}{ccccc}
 \text{map}_C(x, x') & \xleftarrow{\pi_0} & \text{map}_{\hat{C}}(\alpha, \alpha') & \xrightarrow{\pi_1} & \text{map}_C(c, c') \\
 f \downarrow & & \hat{f} \downarrow & & f \downarrow \\
 \text{map}_D(fx, fx') & \xleftarrow{\pi_0} & \text{map}_{\hat{D}}(\hat{f}\alpha, \hat{f}\alpha') & \xrightarrow{\pi_1} & \text{map}_D(fc, fc')
 \end{array}$$

where the vertical arrows are induced by $f: C \rightarrow D$ and $\hat{f}: \hat{C} \rightarrow \hat{D}$, where \hat{f} is the restriction of $\text{Fun}(\Delta^1, f): \text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\Delta^1, D)$ to full subcategories. The maps marked π_0 and π_1 are categorical equivalences using (32.5). Therefore the left-hand vertical arrow is a categorical equivalence using the hypothesis on f and 2-out-of-3 for categorical equivalences (22.10). \square

32.7. The fundamental theorem for quasicategories. The converse to (32.2) is true, but nowhere near as straightforward.

E. Deferred Proposition (Fundamental Theorem of Quasicategory Theory). *A map $f: C \rightarrow D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for $f: C \rightarrow D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse. After many preliminaries, we will give the proof in §40.

32.8. 2-out-of-6 for fully faithful essentially surjective functors. The following result will be useful in the proof of the fundamental theorem. Recall the *2-out-of-6* and *2-out-of-3* properties of a class of morphisms (22.5), and that the class of categorical equivalences has these properties (22.10).

32.9. Proposition. *The class \mathcal{C} of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-6 property, and thus the 2-out-of-3 property.*

Proof. Any identity functor $\text{id}: C \rightarrow C$ is manifestly fully faithful and essentially surjective.

Next note that if a functor $f: C \rightarrow D$ between quasicategories is fully faithful and essentially surjective, then the induced $hf: hC \rightarrow hD$ is an equivalence of ordinary categories. Conversely, if hf is an equivalence, then f is essentially surjective.

Suppose $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} F$ is a sequence of functors between quasicategories such that gf and hg are fully faithful and essentially surjective. The induced sequence $hC \rightarrow hD \rightarrow hE \rightarrow hF$ of functors on homotopy categories has the same property, and thus all the functors between homotopy categories are equivalences. From this we conclude immediately that f, g, h, hgf are essentially surjective.

Given objects $x, x' \in C_0$, we have induced maps

$$\begin{array}{ccccccc}
 & & gf & & & & \\
 & \curvearrowright & & \curvearrowleft & & & \\
 \text{map}_C(x, y) & \xrightarrow{f} & \text{map}_D(fx, fy) & \xrightarrow{g} & \text{map}_E(gfx, gfy) & \xrightarrow{h} & \text{map}_F(hgfx, hgy) \\
 & & & & hg & & \\
 & \curvearrowleft & & \curvearrowright & & &
 \end{array}$$

The hypothesis that gf and hg are fully faithful implies that the indicated arrows are categorical equivalences, and hence all arrows are by (22.10). Because f and gf are essentially surjective, we may use (32.6) to conclude that f, g, h and thus hgf are fully faithful. \square

Introduce path fibration here, and reduce to isofibration case.

32.10. Fully faithful and essentially surjective functors between quasigroupoids. The special case for quasigroupoids is already interesting. Here, (E) specializes to the “Fundamental Theorem of Quasigroupoid Theory”, which says that a map $f: X \rightarrow Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\text{map}_X(x_0, x_1) \rightarrow \text{map}_Y(fx_0, fx_1)$ on “path spaces”, and
- induces a surjection $\pi_0 X \rightarrow \pi_0 Y$.

32.11. Remark. A mild variant of this says that a map $f: X \rightarrow Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\Omega_x X \rightarrow \Omega_{fx} Y$ on “loop spaces”, and
- induces an isomorphism $\pi_0 X \rightarrow \pi_0 Y$.

Here $\Omega_x X := \text{map}_X(x, x)$. Under the correspondence between quasigroupoids and classical homotopy theory, this turns out to be an analogue of the Whitehead theorem²⁷, which says that a map between CW-complexes is a homotopy equivalence iff it induces an isomorphism on all homotopy groups.

We will prove (E) after first considering the special case of quasigroupoids=Kan complexes.

33. ANODYNE MAPS AND KAN FIBRATIONS

In the next few sections, we will develop some properties related to Kan complexes. As a byproduct, we’ll obtain the proof of the specialization of (E) to Kan complexes.

²⁷The “Fundamental Theorem of Classical Homotopy Theory”?

33.1. Weak equivalence. Say that a map $f: X \rightarrow Y$ is a **weak equivalence** of simplicial sets if and only if $\text{Map}(f, G): \text{Map}(Y, G) \rightarrow \text{Map}(X, G)$ is a categorical equivalence for every *quasigroupoid* (i.e., every *Kan complex*) G .

Every categorical equivalence is a weak equivalence, but not conversely. For maps between Kan complexes, weak equivalences and categorical equivalences are the same thing.

33.2. Proposition. *If $f: X \rightarrow Y$ is a map between Kan complexes, then f is a weak equivalence if and only if it is a categorical equivalence.*

Proof. The proof is straightforward, using the same ideas as the proof of (19.6). \square

33.3. Proposition. *Weak equivalences of simplicial sets satisfy the 2-out-of-6 property, and thus the 2-out-of-3 property.*

Proof. Proved just as for categorical equivalences (22.10). \square

33.4. Remark. Given the analogy to categorical equivalence, a more sensible name for weak equivalence is “groupoidal equivalence”. However, the term “weak equivalence” here is historically well-established.

33.5. Simplicial homotopy equivalence. A **simplicial homotopy inverse** to a map $f: X \rightarrow Y$ of simplicial sets is a map $g: Y \rightarrow X$ such that there exists a chain of edges in $\text{Map}(X, X)$ connecting id_X with gf , and a chain of edges in $\text{Map}(Y, Y)$ connecting id_Y with fg . Such an f is called a **simplicial homotopy equivalence**, and of course any simplicial homotopy inverse to f is also a simplicial homotopy equivalence.

33.6. Proposition. *Any simplicial homotopy equivalence is a weak equivalence.*

Proof. First, if $f: X \rightarrow Y$ is a simplicial homotopy equivalence between Kan complexes, then it is clearly a categorical equivalence, because $\text{Map}(X, X)$ and $\text{Map}(Y, Y)$ are quasigroupoids, and so any simplicial homotopy inverse $g: Y \rightarrow X$ for f must satisfy $gf \approx \text{id}_X$ and $fg \approx \text{id}_Y$ and so is a categorical inverse for f .

In general, suppose K is a Kan complex and consider $f^*: \text{Map}(Y, K) \rightarrow \text{Map}(X, K)$. By the same reasoning as used in the proof of (19.6), we see that f^* is a simplicial homotopy equivalence between Kan complexes, so a categorical equivalence. \square

33.7. Anodyne maps and Kan fibrations. Let

$$\text{Horn} = \{ \Lambda_j^n \subset \Delta^n \mid n \geq 1, 0 \leq j \leq n \} = \text{RHorn} \cup \text{LHorn}$$

denote the set of all horn inclusions. A map is **anodyne** if it is in $\overline{\text{Horn}}$, and is a **Kan fibration** if it is in $\text{KanFib} := \text{Horn}^{\square}$.

Since Horn is a set, the small object argument (13.10) applies to it: any map can be factored $f = pj$ with $j \in \overline{\text{Horn}}$ and $p \in \text{KanFib}$.

33.8. Proposition. *We have that $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$.*

Proof. This amounts to showing $\text{Horn} \square \text{Cell} \subseteq \overline{\text{Horn}}$, which is proved in [JT08, Theorem 3.2.2], or [GZ67]. We give a proof in the appendix (58). \square

Thus, we have that

$$\text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is a Kan fibration whenever $K \subseteq L$ and $X \rightarrow Y$ is a Kan fibration, and is a trivial fibration if $K \subseteq L$ is also anodyne.

As a special case, we learn that if X is a Kan complex and $K \subseteq L$, then $\text{Fun}(L, X) \rightarrow \text{Fun}(K, X)$ is a Kan fibration. Here is another consequence.

33.9. Proposition. *Every anodyne map is a weak equivalence.*

Proof. If $f: A \rightarrow B$ is anodyne, then $\text{Map}(f, G)$ is a trivial fibration for every Kan complex G , and hence a categorical equivalence (20.10). \square

33.10. Exercise. Show that the inclusion $\{j\} \subseteq \Delta^n$ of any vertex into any standard n -simplex is anodyne. (Hint: for $0 \leq k \leq n$ let $F_k \subseteq \Delta^n$ be the subcomplex which is the union of all $\Delta^S \subseteq \Delta^n$ with $j \in S$ and $|S| \leq k + 1$, and show that each inclusion $F_{k-1} \rightarrow F_k$ is anodyne.)

33.11. Example. From (33.10) it follows that any map $f: \Delta^m \rightarrow \Delta^n$ between standard simplices is a weak equivalence, using the 2-out-of-3 property (33.3). Any such f which is not an identity map gives an example of a weak equivalence between quasicategories which is not a categorical equivalence.

33.12. Exercise. Let $f: X \rightarrow Y$ be any map between Kan complexes. Show that f is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting.)

33.13. Exercise. Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

33.14. The universal isomorphism. Let Iso be the “walking isomorphism”, i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Let $u: \Delta^1 \rightarrow N\text{Iso}$ be the inclusion representing the unique map $0 \rightarrow 1$ in Iso .

33.15. Proposition. *The map $u: \Delta^1 \rightarrow N\text{Iso}$ is anodyne, and hence a weak equivalence.*

Proof. The k -dimensional elements of $N(\text{Iso})$ are in one-to-one correspondence with sequences $(x_0 x_1 \cdots x_k)$ with $x_i \in \{0, 1\}$. For each $k \geq 0$ there are exactly two *non-degenerate* k -dimensional elements, corresponding to the alternating sequences $(0101 \dots)$ and $(1010 \dots)$ of length $k + 1$.

Let $u_k, v_k: \Delta^k \rightarrow N\text{Iso}$ be the non-degenerate elements $u_k = (0101 \dots)$ and $v_k = (1010 \dots)$ in $(N\text{Iso})_k$. Let $F_k \subseteq N\text{Iso}$ be the smallest subcomplex containing u_k . Observe that for a simplicial operator $f: [d] \rightarrow [k]$ we have $u_k f = (x_0 x_1 \cdots x_d)$ with $x_i \equiv f(i) \pmod 2$. In particular,

- $u_k \langle 1 \cdots k \rangle = v_{k-1}$,
- $u_k \langle 0 \cdots k - 1 \rangle = u_{k-1}$,
- $u_k \langle 01 \cdots \widehat{i} \cdots k - 1, k \rangle$ is a degenerate element associated to u_{k-2} if $i = 1, \dots, k - 1$.

From this we can see that the only non-degenerate elements of $F_k \setminus F_{k-1}$ are u_k and $v_{k-1} = u_k \langle 1 \cdots k \rangle$. Therefore $N\text{Iso} = \bigcup F_k$, $F_1 = u(\Delta^1)$, and the commutative square

$$\begin{array}{ccc} \Lambda_0^k & \longrightarrow & F_{k-1} \\ \downarrow & & \downarrow \\ \Delta^k & \xrightarrow{u_k} & F_k \end{array}$$

is a pushout square for all $k \geq 1$ by (15.24), since (1) it is a pullback, and (2) any element in the complement of $F_{k-1} \subset F_k$ is the image of a unique element under the map u_k .

It follows that u is anodyne. \square

We obtain as a consequence the following criterion for an edge to be an isomorphism, which we will use later.

33.16. Proposition. *Let C be a quasicategory, and $f: \Delta^1 \rightarrow C$ a map representing a morphism in C . Then there exists $f': N(\text{Iso}) \rightarrow C$ such that $f'u = f$ if and only if f represents an isomorphism in C .*

Proof. (\implies) Clear: consider induced maps on homotopy categories. (\impliedby) If f represents an isomorphism then it factors through $\Delta^1 \rightarrow C^{\text{core}} \subseteq C$. Since the core is a quasigroupoid, and hence a Kan complex, an extension along the anodyne map u to a map $N\text{Iso} \rightarrow C^{\text{core}} \subseteq C$ exists. \square

33.17. *Remark.* Let $X \subset N\text{Iso}$ be the subcomplex which is the union of the images of 2-dimensional elements 010 and 101²⁸. The inclusion $v: \Delta^1 \rightarrow X$ representing the edge 01 has the same property described in (33.16): $f: \Delta^1 \rightarrow C$ represents an isomorphism if and only if it extends along v . The proof is easy: an extension of f to a map $f': X \rightarrow C$ exactly encodes a choice of morphism g in C (i.e., $f'(\langle 10 \rangle)$) together with explicit homotopies $gf \sim_r 1$ and $fg \sim_\ell 1$, (i.e., $f'(\langle 010 \rangle)$ and $f'(\langle 101 \rangle)$).

However, it turns out that $\Delta^1 \rightarrow X$ is *not* a weak equivalence (and therefore that $X \rightarrow N\text{Iso}$ is not a categorical equivalence). In particular, a map $X \rightarrow C$ to a quasicategory can fail to extend along $X \subset N\text{Iso}$.

33.18. *Exercise.* Show that $X \rightarrow N\text{Iso}$ is not anodyne, by constructing a map $X \rightarrow K(\mathbb{Z}, 2)$ which does not extend over $N\text{Iso}$. (See (8.12).)

33.19. *Exercise.* Let Z be the complex of (20.6), and let $F: \Delta^1 = \Delta^{\{1,2\}} \rightarrow Z$ be the map representing the edge $f \in Z_1$. Show that F is anodyne, and state and prove an analogue of (33.16) with Z in place of $N\text{Iso}$.

33.20. **Covering homotopy extension property.** This is the special case of $\overline{\text{Horn}}\square\overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ which we will use several times in the next few sections: for any inclusion $K \subseteq L$, the maps

$$(K \times \Delta^1) \cup_{K \times \{0\}} (L \times \{0\}) \longrightarrow L \times \Delta^1 \longleftarrow (K \times \Delta^1) \cup_{K \times \{1\}} (L \times \{1\})$$

are anodyne. This amounts to saying that for $j \in \{0, 1\}$ a commutative square

$$\begin{array}{ccc} (K \times \Delta^1) \cup (L \times \{j\}) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ L \times \Delta^1 & \longrightarrow & Y \end{array}$$

has a lift whenever p is a Kan fibration. This is sometimes called the “covering homotopy extension property”. It may be helpful to think of this in the equivalent form, which asserts a lifting in

$$\begin{array}{ccc} \{j\} & \longrightarrow & \text{Map}(L, X) \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta^1 & \longrightarrow & \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y) \end{array}$$

when $K \subseteq L$ and $p: X \rightarrow Y$ a Kan fibration. This gets used the following way: to demonstrate $(K \subseteq L) \boxtimes p$ for a given Kan fibration p and inclusion $K \subseteq L$, we “deform” a given lifting problem of this type along a “path” in the space $\text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$ of all such commutative squares to a different lifting problem of the same type for which we know a lift exists.

33.21. **Fundamental theorem for Kan complexes: reduction to Kan fibrations.** We are going to show the following

33.22. **Theorem.** *A map $f: X \rightarrow Y$ between Kan complexes is a weak equivalence if and only if it is fully faithful and essentially surjective.*

Equivalently: a map between quasigroupoids is a categorical equivalence if and only if it is fully faithful and essentially surjective.

We will prove this by reducing to Kan fibrations.

²⁸This is isomorphic to the complex Z' of (20.7).

33.23. Lemma. *To prove (33.22), it suffices to prove: If $f: X \rightarrow Y$ is a Kan fibration between Kan complexes which is fully faithful and essentially surjective, then f is a weak equivalence.*

Proof. The (\implies) direction of (33.22) has already been established (32.2).

(\impliedby) Given f between Kan complexes which is fully faithful and essentially surjective, use the small object argument (13.10) applied to Horn to factor it as

$$X \xrightarrow{j} V \xrightarrow{p} Y,$$

where j is anodyne and p is a Kan fibration. It follows that V is a Kan complex, j is a weak equivalence (33.9) between Kan complexes, and so is fully faithful and essentially surjective. Since the class of fully faithful and essentially surjective maps satisfy 2-out-of-3 (32.9), it follows that p also has this property. If we can use this to show p is a weak equivalence, it follows that f is a weak equivalence, since weak equivalences also satisfy 2-out-of-3 (33.3). \square

We will prove the needed special case (that for Kan fibrations between Kan complexes, fully faithful and essentially surjective implies weak equivalence) in the next couple of sections, after analyzing Kan fibrations in more detail.

34. KAN FIBRATIONS BETWEEN KAN COMPLEXES

In the next few sections, we are going to be considering various properties of Kan fibrations, with particular interest in Kan fibrations between Kan complexes.

In particular, we are going to show that for a Kan fibration $p: X \rightarrow Y$ where X and Y are Kan complexes, all of the following are equivalent ((34.5), (34.8), (35.2), (36.1)):

- (1) p is a trivial fibration;
- (2) p is a weak equivalence;
- (3) p is a fiberwise deformation retraction;
- (4) p has contractible fibers;
- (5) p is fully faithful and essentially surjective.

The equivalence of (2) and (5) will complete the proof of the fundamental theorem for Kan complexes (33.22). (In fact, (1)–(4) are equivalent without the hypothesis that the objects are Kan complexes, though we will not prove this in all cases.)

34.1. Fiberwise deformation retraction. A map $p: X \rightarrow Y$ is said to be a **fiberwise deformation retraction** if there exists

- $s: Y \rightarrow X$ such that $ps = \text{id}_Y$, and
- $k: X \times \Delta^1 \rightarrow X$ such that $k|_{X \times \{0\}} = \text{id}_X$, $k|_{X \times \{1\}} = sp$, and $pk = p\pi$, where $\pi: X \times \Delta^1 \rightarrow X$ is projection; that is, the diagram

$$\begin{array}{ccc} X \times \{0, 1\} & \xrightarrow{(\text{id}_X, sp)} & X \\ \downarrow & \nearrow k & \downarrow p \\ X \times \Delta^1 & \xrightarrow{\pi} X \xrightarrow{p} & Y \end{array}$$

commutes.

Any fiberwise deformation retraction is a weak equivalence: s is a simplicial homotopy inverse to p (33.6).

34.2. Exercise. Show that fiberwise deformation retraction can be reformulated in terms of the relative function complex of (20.9): a map $p: X \rightarrow Y$ is a fiberwise deformation retraction if there exists (i) a vertex $s \in \text{Map}_Y(Y, X)_0$ and (ii) an edge $k \in \text{Map}_Y(X, X)_1$ with associated vertices $k_0 = \text{id}_X$ and $k_1 = sp$ in $\text{Map}_Y(X, X)_0$.

34.3. *Exercise.* Show that the term “fiberwise” is justified: for each $y \in Y_0$, the projection $p^{-1}(y) \rightarrow \{y\}$ of a fiber to its image is a simplicial homotopy equivalence.

34.4. *Exercise.* Show that if $p: X \rightarrow Y$ is a fiberwise deformation retraction as above, then any base change of p is also part of a fiberwise deformation retraction.

Fiberwise deformation retractions of Kan fibrations are always trivial fibrations, as can be shown with the covering homotopy extension property.

34.5. **Lemma.** *Let $p: X \rightarrow Y$ be a Kan fibration between simplicial sets. Then p is a fiberwise deformation retraction if and only if p is a trivial fibration.*

Proof. [JT08, Prop. 3.2.5]. (\implies) The idea is that a fiberwise deformation retraction provides a “formula” which solves every lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow i & \nearrow u & \downarrow p \\ B & \xrightarrow{b} & Y \end{array}$$

where i is a monomorphism. Using the data s and k of the fiberwise deformation retraction we obtain a commutative square

$$\begin{array}{ccc} (A \times \Delta^1) \cup (B \times \{1\}) & \xrightarrow{(k(a \times \text{id}_{\Delta^1}), sb)} & X \\ \downarrow j & \nearrow t & \downarrow p \\ B \times \Delta^1 & \xrightarrow{\pi} B & \xrightarrow{b} Y \end{array}$$

Because p is a Kan fibration and j is anodyne by (33.8), a lift t exists. Then $u := t|_{B \times \{0\}}$ is the desired lift.

(\impliedby) Left as an exercise. □

34.6. *Exercise.* Show that any trivial fibration is a fiberwise deformation retraction.

34.7. **Trivial fibrations between Kan complexes and weak equivalences.** We know that trivial fibrations are always categorical equivalences. We now show that any Kan fibration between Kan complexes which is also a categorical equivalence (and hence a weak equivalence) is a trivial fibration.

34.8. **Proposition.** *A map $p: X \rightarrow Y$ between Kan complexes is a trivial fibration if and only if it is a Kan fibration and a weak equivalence.*

Proof. [JT08, Prop. 3.2.6] (\implies) Clearly trivial fibrations are Kan fibrations since $\text{Horn} \subseteq \overline{\text{Cell}}$. We have already shown that trivial fibrations between quasicategories are always categorical equivalences (20.10), which implies that they are weak equivalences if between Kan complexes (33.2).

(\impliedby) On the other hand, suppose p is a Kan fibration and a weak equivalence. Being a weak equivalence between Kan complexes and hence a categorical equivalence, p admits a categorical inverse: there exists $f: Y \rightarrow X$ and maps $u: X \times \Delta^1 \rightarrow X$ and $v: Y \times \Delta^1 \rightarrow Y$ which give natural isomorphisms $u: fp \rightarrow \text{id}_X$ and $v: pf \rightarrow \text{id}_Y$. We will “deform” this data to a pair (s, k) which exhibits p as a fiberwise deformation retraction.

Step 1. Since $Y \times \{0\} \subset Y \times \Delta^1$ is anodyne by (33.8), a lift α exists in

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f} & X \\ \downarrow & \nearrow \alpha & \downarrow p \\ Y \times \Delta^1 & \xrightarrow{v} & Y \end{array}$$

Let $s := \alpha|_{Y \times \{1\}}$, so $ps = \text{id}_Y$. The map α exhibits a natural isomorphism $\alpha: f \rightarrow s$ of functors $Y \rightarrow X$. Since fp is naturally isomorphic to id_X , we have $\text{id}_X \approx fp \approx sp$, i.e., there exists a natural isomorphism $w: sp \rightarrow \text{id}_X$.

Step 2. Consider the natural isomorphism $(sp)w: sp = spsp \rightarrow sp$ of functors $X \rightarrow X$. We have a commutative diagram

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{a} & \text{Map}(X, X) \\ \downarrow & \nearrow t & \downarrow \text{Map}(X, p) \\ \Delta^2 & \xrightarrow{b} & \text{Map}(X, Y) \end{array}$$

where $a_{01} = w$, $a_{02} = spw$, and b is the degeneracy $b = (pw)_{011}$; this commutes since $pssp = pw$. Since p is a Kan fibration, so is $\text{Map}(X, p)$ by $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ (33.8), and therefore a lift t exists. Let $k = t|_{\Delta^{\{1,2\}}}: \Delta^1 \rightarrow \text{Map}(X, X)$. It is clear that this is a natural isomorphism $k: \text{id}_Y \rightarrow sp$, and that this is “fiberwise”, i.e., $pk = b_{12} = p\pi$ as maps $X \times \Delta^1 \rightarrow Y$.

Thus, we have exhibited p as a fiberwise deformation retraction, so p is a trivial fibration by (34.5). \square

34.9. Contractible Kan complexes.

34.10. Corollary. *Let X be a simplicial set. The following are equivalent.*

- (1) X is a quasicategory which is categorically equivalent to Δ^0 .
- (2) $X \rightarrow \Delta^0$ is a trivial fibration.
- (3) Every $\partial\Delta^n \rightarrow X$ extends over $\partial\Delta^n \subset \Delta^n$.

Such an X is necessarily a Kan complex.

Proof. We have (2) \Leftrightarrow (3) by definition, and we know that (2) \Rightarrow (1). Given (1), we have that X is a quasigroupoid, and hence a Kan complex (29.2), and (2) follows by the previous proposition (34.8). \square

We say that an X satisfying these conditions is a **contractible Kan complex**.

34.11. Monomorphisms which are weak equivalences. We can now characterize the monomorphisms which are weak equivalences, in terms of maps into Kan complexes.

34.12. Proposition. *Let $j: A \rightarrow B$ be a monomorphism of simplicial sets. Then j is a weak equivalence if and only if $\text{Map}(j, X): \text{Map}(B, X) \rightarrow \text{Map}(A, X)$ is a trivial fibration for all Kan complexes X .*

Proof. Assume X is an arbitrary Kan complex. We know that $\text{Map}(j, X)$ is always a Kan fibration between Kan complexes using $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ (33.8). We have by definition that j is a weak equivalence iff all $\text{Map}(j, X)$ are weak equivalences, which holds iff all $\text{Map}(j, X)$ are trivial fibrations since for a Kan fibration between Kan complexes “trivial fibration” and “weak equivalence” mean the same thing by (34.8). \square

34.13. Remark. The class $\text{WkEq} \cap \overline{\text{Cell}}$ of monomorphisms which are weak equivalences is a weakly saturated class. In fact, using (34.12) it is easy to show that it is the left complement of the class of maps of the form $p \square^j$, where $p: X \rightarrow \Delta^0$ is a projection from a Kan complex X , and $j: \partial\Delta^n \rightarrow \Delta^n$ is a cell inclusion. Furthermore, $\text{WkEq} \cap \overline{\text{Cell}}$ contains the weakly saturated class $\overline{\text{Horn}}$ of anodyne maps.

It turns out that $\overline{\text{Horn}} = \text{WkEq} \cap \overline{\text{Cell}}$, i.e., the injective weak equivalences are *precisely the same* as the anodyne maps. This is a fairly non-trivial fact, and we will address it again later. **(Maybe I will, maybe I won't.)** As a consequence, it will follow that (34.8) and (34.16) hold without the condition that the objects be Kan complexes.

34.14. **Enriched lifting for Kan fibrations between Kan complexes.**

34.15. **Proposition.** *If $j: A \rightarrow B$ is a monomorphism and a weak equivalence of simplicial sets, and $p: X \rightarrow Y$ is a Kan fibration between Kan complexes, then the pullback-hom map*

$$p^{\square i}: \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a trivial fibration.

Proof. The pullback-hom map $p^{\square i}$ is a Kan fibration between Kan complexes, using $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ (33.8). Consider the diagram

$$\begin{array}{ccc} \text{Map}(B, X) & \xrightarrow{p^{\square i}} & \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y) & \xrightarrow{q'} & \text{Map}(A, X) \\ & & \downarrow & & \downarrow \\ & & \text{Map}(B, Y) & \xrightarrow{q} & \text{Map}(A, Y) \end{array}$$

in which the square is a pullback. By (34.12) the maps q and $q'(p^{\square i})$ are trivial fibrations. The pullback q' of q is also a trivial fibration, and so $p^{\square i}$ is a weak equivalence by 2-out-of-3 (33.3), and therefore a trivial fibration since it is a Kan fibration between Kan complexes (34.8). \square

We also obtain another characterization of Kan fibrations between Kan complexes.

34.16. **Corollary.** *A map $p: X \rightarrow Y$ between Kan complexes is a Kan fibration if and only if $j \square p$ for all j which are monomorphisms and weak equivalences.*

Proof. (\Leftarrow) Straightforward, since inner horn inclusions are monomorphisms and weak equivalences. (\Rightarrow) Immediate from the previous proposition. \square

35. THE FIBERWISE CRITERION FOR TRIVIAL FIBRATIONS

We give another criterion for Kan fibration to be a trivial fibration, in terms of its fibers.

35.1. **Fiberwise criterion for trivial fibrations.** The fiber $p^{-1}(y)$ of a map $p: X \rightarrow Y$ over a vertex $y \in Y_0$ is the pullback of p along $\{y\} \rightarrow Y$.

If $p: X \rightarrow Y$ is a trivial fibration, then since $\text{TrivFib} = \text{Horn}^{\square}$ we see immediately that every projection $p^{-1}(y) \rightarrow *$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes.

35.2. **Proposition.** *Let $p: X \rightarrow Y$ be a Kan fibration. Then p is a trivial fibration if and only if every fiber of p is a contractible Kan complex.*

Proof. We have just observed (\Rightarrow), so we prove (\Leftarrow). So suppose p is a Kan fibration whose fibers are contractible Kan complexes, and consider a lifting problem

$$\begin{array}{ccc} \partial \Delta^n & \xrightarrow{a} & X \\ \downarrow & \nearrow t & \downarrow p \\ \Delta^n & \xrightarrow{b} & Y \end{array}$$

We will construct a lift t by using the covering homotopy extension property to “deform” this to a lifting problem involving a single fiber of p , which admits a solution by the hypothesis on the fibers of p .

Let $\gamma: \Delta^n \times \Delta^1 \rightarrow \Delta^n$ be the unique map given on vertices by $\gamma(i, 0) = i$ and $\gamma(i, 1) = n$. Thus $\gamma|_{\Delta^n \times \{0\}} = \text{id}$, while $\gamma|_{\Delta^n \times \{1\}}$ factors through the vertex $\{n\} \subseteq \Delta^n$. Since $(\partial\Delta^n \times \{0\} \subseteq \partial\Delta^n \times \Delta^1)$ is anodyne by $\overline{\text{Horn}} \square \text{Cell} \subseteq \overline{\text{Horn}}$ and p is a Kan fibration, a lift exists in

$$\begin{array}{ccc} \partial\Delta^n \times \{0\} & \xrightarrow{a} & X \\ \downarrow & \nearrow c & \downarrow p \\ \partial\Delta^n \times \Delta^1 & \xrightarrow{\gamma} & \Delta^n \times \Delta^1 \xrightarrow{b} \Delta^n \rightarrow Y \end{array}$$

Since $b\gamma|_{\Delta^n \times \{1\}}$ factors through a vertex $y \in Y_0$, we have a lift d in

$$\begin{array}{ccc} \partial\Delta^n \times \{1\} & \xrightarrow{c|_{\partial\Delta^n \times \{1\}}} p^{-1}(y) \xrightarrow{j} & X \\ \downarrow & \nearrow d & \downarrow \\ \Delta^n \times \{1\} & \xrightarrow{\quad} \{y\} \xrightarrow{y} & Y \end{array}$$

since by hypothesis $p^{-1}(y)$ is a contractible Kan complex. Putting this together we obtain a commutative square

$$\begin{array}{ccc} (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(c,j,d)} & X \\ \downarrow & \nearrow s & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y \end{array}$$

which admits a lift s since $(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\})$ is anodyne and p is a Kan fibration. The restriction $t := s|_{\Delta^n \times \{0\}}$ provides a solution to the original lifting problem, since $t|_{\partial\Delta^n \times \{0\}} = c|_{\partial\Delta^n \times \{0\}} = a$ and $pt = (b\gamma)|_{\Delta^n \times \{0\}} = b$. \square

We often apply the above result in the following way.

35.3. Corollary. *Suppose*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a pullback square such that (1) p is a Kan fibration and (2) $g_0: Y'_0 \rightarrow Y_0$ is surjective. Then p is a trivial fibration if and only if p' is a trivial fibration. Furthermore, if all objects are Kan complexes, then p is a weak equivalence if and only if p' is a weak equivalence.

Proof. The fibers of p all appear as fibers of p' by (2). Use the fiberwise criterion (35.2) and (34.8). \square

35.4. Remark. The proof of (35.2) actually shows a little more: If $p: X \rightarrow Y$ is a Kan fibration, then for any fixed $n \geq 0$ we have that $(\partial\Delta^n \times \Delta^1) \square p$ if and only if $(\partial\Delta^n \times \Delta^1) \square p^{-1}(y)$ for all $y \in Y_0$.

36. FUNDAMENTAL THEOREM FOR KAN COMPLEXES

In this section, we will prove quasigroupoid version of the fundamental theorem (33.22), i.e., that fully faithful and essentially surjective maps between quasigroupoids (=Kan complexes) are weak equivalences. We note that we have already reduced (33.23) to the case of Kan fibrations, which follows from the following.

36.1. Proposition. *Let $p: X \rightarrow Y$ be a Kan fibration between Kan complexes. Then*

- (1) p is essentially surjective if and only if $(\partial\Delta^0 \subset \Delta^0) \boxtimes p$, and
- (2) p is fully faithful if and only if $(\partial\Delta^n \subset \Delta^n) \boxtimes p$ for all $n \geq 1$.

Thus, p is both essentially surjective and fully faithful if and only if it is a trivial fibration.

Proof of (36.1)(1). The property $(\partial\Delta^0 \subset \Delta^0) \boxtimes p$ means exactly that $X_0 \rightarrow Y_0$ is surjective. If this holds then clearly p is essentially surjective.

Conversely, if p is essentially surjective and $y \in Y_0$, we may choose $x \in X_0$ and an isomorphism $f: p(x) \rightarrow y$ in Y . Then (x, f) is the data of a lifting problem of $(\{0\} \subset \Delta^1)$ against p , which has a solution $s: \Delta^1 \rightarrow X$ since p is a Kan fibration. The vertex $s_1 \in X_0$ satisfies $p(s_1) = y$, so we have proved that p is surjective on objects. \square

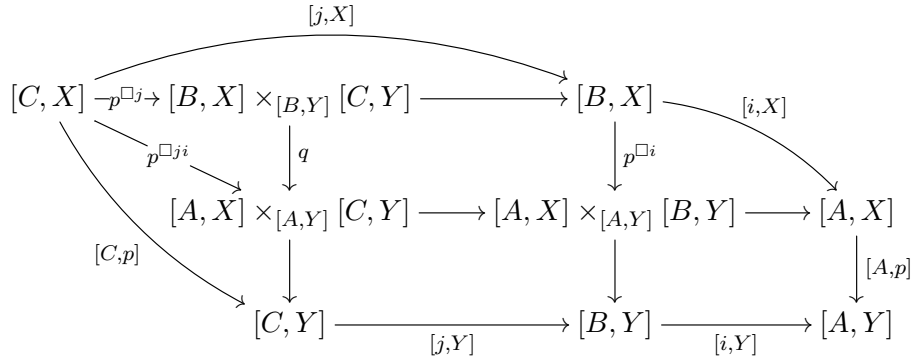
To prove the second part of (36.1), we will need the following result which relates the pullback-homs and composition of maps. You can think of it as an “enriched” version of the fact that $i \boxtimes p$ and $j \boxtimes p$ imply $ji \boxtimes p$.

36.2. Proposition (Transitivity triangle for pullback-homs). *Let $A \xrightarrow{i} B \xrightarrow{j} C$ and $p: X \rightarrow Y$ be maps. Then there is a factorization*

$$p^{\square(j \circ i)} = p^{\square j} \circ q$$

where q is a base-change of $p^{\square i}$.

Proof. I use “[A, X]” as a shorthand for “ $\text{Map}(A, X)$ ”. Form the commutative diagram



in which all three squares are pullbacks. \square

36.3. Remark. I used a special case of (36.2) in the proof of (34.15).

36.4. Exercise. Prove the following transitivity-triangles:

- $(i \circ j) \square f = k \circ (i \square f)$ where k is a cobase-change of $j \square f$.
- $(q \circ p)^{\square i} = r \circ p^{\square i}$ where r is a base-change of $q^{\square i}$.

Given $p: X \rightarrow Y$, define a class \mathcal{C}_p of morphisms of simplicial sets by

$$\mathcal{C}_p := \{ i \in \overline{\text{Cell}} \mid p^{\square i} \in \text{TrivFib} \} = \overline{\text{Cell}} \cap \overline{\square}(p^{\square \text{Cell}}).$$

The equality is because $p^{\square i} \in \text{TrivFib}$ iff $\text{Cell} \boxtimes p^{\square i}$ iff $i \boxtimes p^{\square \text{Cell}}$. It is clear that \mathcal{C}_p is a weakly saturated class, and that $\text{InnHorn} \subseteq \mathcal{C}_p$ since $\text{InnHorn} \boxtimes \text{Cell} \subseteq \text{InnHorn}$.

36.5. Remark. Any $i \in \mathcal{C}_p$ automatically satisfies $i \boxtimes p$, since $p^{\square i} \in \text{TrivFib}$ implies that $p^{\square i}$ is surjective on vertices. That is, $\mathcal{C}_p \subseteq \overline{\square}\{p\}$. Elements of \mathcal{C}_p can be thought of as monomorphisms which satisfy an “enriched” refinement of the lifting property $i \boxtimes p$.

We also have the following “precancellation” (or “right cancellation”) property of \mathcal{C}_p , which is ultimately a consequence of (34.8).

36.6. Lemma. *Let $p: X \rightarrow Y$ be a Kan fibration between Kan complexes. Suppose $A \xrightarrow{i} B \xrightarrow{j} C$ are monomorphisms such that $i, ji \in \mathcal{C}_p$. Then $j \in \mathcal{C}_p$.*

Proof. If $i: A \rightarrow B$ is any monomorphism, then $p^{\square i}: \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$ is a Kan fibration between Kan complexes, using (33.8). Thus $p^{\square i}$, $p^{\square j}$, and $p^{\square ji}$ are Kan fibrations between Kan complexes.

If $i, ji \in \mathcal{C}_p$ then $p^{\square i}$ and $p^{\square ji}$ are trivial fibrations. The transitivity triangle (36.2) gives $p^{\square ji} = p^{\square j} \circ q$, where q is a base change of $p^{\square i}$ whence q is a trivial fibration. Since trivial fibrations are weak equivalences we have that q and $p^{\square ji}$ are weak equivalences, whence $p^{\square j}$ is a weak equivalence by the 2-out-of-3 property (33.3). Therefore $p^{\square j}$ is also a trivial fibration since it is a map between Kan complexes which is a Kan fibration and weak equivalence (34.8). Thus we have proved that $j \in \mathcal{C}_p$. \square

Proof of (36.1)(2). Fix $p: X \rightarrow Y$ a Kan fibration between Kan complexes. For any $n \geq 1$, we can form a diagram

$$\begin{array}{ccccc}
 \coprod_{x_0, \dots, x_n \in X_0^{\times n+1}} \text{map}_X(x_0, \dots, x_n) & \longrightarrow & \text{Map}(\Delta^n, X) & & \\
 \downarrow q_{x_0, \dots, x_n} & & \downarrow q & & \\
 \coprod_{x_0, \dots, x_n \in X_0^{\times n+1}} \text{map}_Y(fx_0, \dots, fx_n) & \xrightarrow{j} & X^{\times n+1} \times_{Y^{\times n+1}} \text{Map}(\Delta^n, Y) & \longrightarrow & \text{Map}(\Delta^n, Y) \\
 \downarrow & & \downarrow & & \downarrow \\
 X_0^{\times n+1} & \xrightarrow{i} & X^{\times n+1} & \xrightarrow{p^{\times n+1}} & Y^{\times n+1}
 \end{array}$$

in which each square is a pullback. Here

- the map q is the pullback-hom $p^{\square(\text{Sk}_0 \Delta^n \subset \Delta^n)}$, so
- q is a Kan fibration between Kan complexes (34.15),
- $q_{\underline{x}}$ is a Kan fibration between Kan complexes for all $\underline{x} \in X_0^{\times n+1}$,
- $\text{map}_X(x_0, \dots, x_n)$ and $\text{map}_Y(fx_0, \dots, fx_n)$ are extended mapping spaces (31.6),
- the set X_0 of vertices of X is identified with the discrete subcomplex of X , so that i is the evident inclusion, and
- j is surjective on vertices since i is so.

Note that p is fully faithful if and only if each $q_{\underline{x}}$ is a weak equivalence, and thus if and only if each $q_{\underline{x}} \in \text{TrivFib}$ by (34.8). By the fiberwise criterion for trivial fibrations (35.3), this is equivalent to $q = p^{\square(\text{Sk}_0 \Delta^n \subset \Delta^n)} \in \text{TrivFib}$. Thus,

$$(p \text{ is fully faithful}) \iff \{ \text{Sk}_0 \Delta^n \subset \Delta^n \mid n \geq 1 \} \subseteq \mathcal{C}_p,$$

where $\mathcal{C}_p = \{ i \in \overline{\text{Cell}} \mid p^{\square i} \in \text{TrivFib} \}$ is the weakly saturated class we defined earlier.

Let $\overline{\text{Cell}}_{\geq 1} = \{ \partial \Delta^n \subset \Delta^n \mid n \geq 1 \}$. The weak saturation of $\overline{\text{Cell}}_{\geq 1}$ is the class of monomorphisms which are bijective on vertices. A straightforward argument shows that $\overline{\text{Cell}}_{\geq 1} \square \overline{\text{Cell}} = \overline{\text{Cell}}_{\geq 1}$, from which it follows that

$$\overline{\text{Cell}}_{\geq 1} \boxtimes p \iff \overline{\text{Cell}}_{\geq 1} \subseteq \mathcal{C}_p.$$

Thus, the proof of the proposition amounts to showing that

$$\{ \text{Sk}_0 \Delta^n \subset \Delta^n \mid n \geq 1 \} \subseteq \mathcal{C}_p \iff \overline{\text{Cell}}_{\geq 1} \subseteq \mathcal{C}_p.$$

For (\Leftarrow), note that

$$\{ \text{Sk}_0 \Delta^n \subset \Delta^n \mid n \geq 1 \} \subseteq \overline{\text{Cell}}_{\geq 1} \subseteq \mathcal{C}_p$$

since \mathcal{C}_p is weakly saturated and any $\text{Sk}_0 K \subset K$ is a monomorphism which is a bijection on vertices.

To prove (\implies) suppose $\{\text{Sk}_0 \Delta^n \subset \Delta^n \mid n \geq 1\} \subseteq \mathcal{C}_p$. We will show by induction on $n \geq 1$ that $(\partial\Delta^k \subset \Delta^k) \in \mathcal{C}_p$ for all $1 \leq k \leq n$. Thus, to show $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ consider the monomorphisms

$$\text{Sk}_0 \Delta^n \xrightarrow{i} \partial\Delta^n \xrightarrow{j} \Delta^n.$$

By assumption we have $ji \in \mathcal{C}_p$. Since the non-degenerate elements in $\partial\Delta^n \setminus \text{Sk}_0 \Delta^n$ occur only in dimensions $1, \dots, n-1$, we see from the skeletal filtration (15.23) that $\text{Sk}_0 \Delta^n \subseteq \partial\Delta^n$ is contained in the weak saturation of $\{\partial\Delta^k \subset \Delta^k \mid 1 \leq k < n\}$. By the inductive hypothesis these cells inclusions are contained in \mathcal{C}_p , and thus $i \in \mathcal{C}_p$. Therefore $j \in \mathcal{C}_p$ as desired by (36.6). \square

Better argument: show that $(\partial\Delta^1 \subset \Delta^1) \in \mathcal{C}_p$ implies $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for all $n \geq 1$, by induction on n . Use that $(I^n \subseteq \Delta^n) \in \mathcal{C}_p$ since inner anodyne, and inductively that $(I^n \subseteq \partial\Delta^n) \in \mathcal{C}_p$ by skeleton argument.

37. ISOFIBRATIONS

In this section, we return to isofibrations, which were defined in (28.6). The moral is that isofibrations between quasicategories play a role analogous to Kan fibrations between Kan complexes.

37.1. Characterizations of isofibrations. Recall that a functor $f: C \rightarrow D$ between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every

$$\begin{array}{ccc} \{j\} & \longrightarrow & C \\ \downarrow & \nearrow g & \downarrow p \\ \Delta^1 & \xrightarrow{f} & D \end{array}$$

with $j = 0$ such that f represents an isomorphism admits a lift g which is also represents isomorphism. Furthermore, it is equivalent to require (2') instead of (2), where (2') is the same statement with $j = 1$.

Note that $C \rightarrow *$ is an isofibration for any quasicategory C (because identity maps are isomorphisms).

We have the following ‘‘lifting criterion’’ for isofibrations.

37.2. Proposition. *An map p between quasicategories is an isofibration iff (1) it is an inner fibration and (2) $(\{0\} \subset N(\text{Iso})) \boxtimes p$.*

Proof. (\Leftarrow) Straightforward, using the fact (33.16) that every $f: \Delta^1 \rightarrow D$ representing an isomorphism factors through a map $N(\text{Iso}) \rightarrow D$.

(\implies) Let p be an isofibration. To solve the lifting problem

$$\begin{array}{ccc} \{0\} & \longrightarrow & C \\ \downarrow & \nearrow s & \downarrow p \\ N(\text{Iso}) & \longrightarrow & D \end{array}$$

recall from the proof of (33.15) that $N(\text{Iso}) = \bigcup F_k$ where F_k is obtained from F_{k-1} by gluing along $\Lambda_0^k \subset \Delta^k$; we construct lifts $s_k: F_k \rightarrow C$ inductively. A lift $s_1: F_1 = \Delta^1 \rightarrow C$ exists by the definition of isofibration, and we may assume that s_1 is an isomorphism in C . Then the Joyal lifting theorem (28.13) provides lifts s_k for $k \geq 2$. \square

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in $(\text{InnHorn} \cup \{\{0\} \subset N(\text{Iso})\})^{\boxtimes}$. In particular, the pullback of an isofibration along a map from a quasicategory is also a quasicategory.

37.3. *Remark.* We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called “categorical fibration”, and will be discussed later.

Here is another characterization of isofibrations in terms of cores (=maximal sub-quasigroupoids, defined in (10.7)). Remember that any functor $p: C \rightarrow D$ restricts to a functor $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$ between Kan complexes.

37.4. **Proposition.** *A map $p: C \rightarrow D$ between quasicategories is a isofibration if and only if (1) it is an inner fibration, and (2) $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$ is a Kan fibration.*

Proof. (\implies) Let p be an isofibration. Then p^{core} is also an isofibration, by an elementary argument. (The point is that the relevant lifting problems for p^{core} clearly have lifts s with target C , since p is an isofibration; an easy argument shows that the image of such lifts s must actually land in C^{core} .)

Thus have reduced to showing that any isofibration between Kan complexes is a Kan fibration, which is a straightforward exercise using Joyal lifting (33.12).

(\impliedby) If p^{core} is a Kan fibration, then it is immediate that property (2) of an isofibration holds. \square

In particular, isofibrations between Kan complexes are precisely Kan fibrations. (This can be proved directly using Joyal lifting, as in (33.12).)

37.5. **Lifting properties for isofibrations.** We are now ready to prove the following proposition, which will be the key tool in what follows.

37.6. **Proposition.** *Let $p: C \rightarrow D$ be an isofibration between quasicategories, and $i: K \rightarrow L$ any monomorphism of simplicial sets. Then the induced pullback-hom map*

$$p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$$

is an isofibration.

Proof. Fix a map $p: C \rightarrow D$ between quasicategories. We first note that the class maps $i: K \rightarrow L$ such that $p^{\square i}$ is an isofibration is weakly saturated. In fact, let $S := \text{InnHorn} \cup \{\{0\} \subset N(\text{Iso})\}$. Two applications of (16.4) show that $S \square (p^{\square i})$ iff $(S \square i) \square p$ iff $i \square (p^{\square S})$. Therefore, (37.2) implies that the class $\mathcal{C} := \{i \mid p^{\square i} \in \text{IsoFib}\}$ is the left complement of $p^{\square S}$, and thus weakly saturated.

Therefore, to show that \mathcal{C} contains all monomorphisms, it suffices show that it contains $i = (\partial\Delta^n \subset \Delta^n)$ for $n \geq 0$.

Note that we will certainly have that $p^{\square i}$ is an inner fibration, using using $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.7).

If $n = 0$, then $p^{\square i} = p$ so the claim is trivial²⁹.

Now assume $n \geq 1$. Since $p^{\square i}$ is an inner fibration between quasicategories, it suffices to solve the lifting problem

$$\begin{array}{ccc} \{0\} & \longrightarrow & \text{Fun}(\Delta^n, C) \\ \downarrow & \dashrightarrow & \downarrow p^{\square i} \\ \Delta^1 & \xrightarrow{f} & \text{Fun}(\partial\Delta^n, C) \times_{\text{Fun}(\partial\Delta^n, D)} \text{Fun}(\Delta^n, D) \end{array} \iff \begin{array}{ccc} (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) & \xrightarrow{g} & C \\ \downarrow & \dashrightarrow & \downarrow p \\ \Delta^1 \times \Delta^n & \longrightarrow & D \end{array}$$

where f represents an isomorphism in the target.

The edge $f' := (g|_{\Delta^1 \times \Delta^{\{0\}}})$ is the same as the composite

$$\Delta^1 \xrightarrow{f'} \text{Fun}(\partial\Delta^n, C) \times_{\text{Fun}(\partial\Delta^n, D)} \text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(\partial\Delta^n, C) \rightarrow \text{Fun}(\{0\}, C).$$

²⁹This step is the only place in the proof where we actually use the fact that p is an isofibration, and not merely an inner fibration! In fact, if p is merely an inner fibration, but $K_0 = L_0$, then $p^{\square i}$ is a isofibration. This proof is closely related to that of (30.6).

Since f is an isomorphism in the fiber product, it follows that f' is an isomorphism in C . Therefore a lift exists by the pushout-product version of Joyal lifting (30.3), since $n \geq 1$. \square

One consequence of the above is that if p is an isofibration and i is a monomorphisms, then

$$(p^{\square i})^{\text{core}}: \text{Fun}(L, C)^{\text{core}} \rightarrow (\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D))^{\text{core}}$$

is a Kan fibration (37.4). It turns out we can replace the target (the core of a pullback) with a pullback of cores.

37.7. Corollary. *If $p: C \rightarrow D$ is any functor between quasicategories and $K \subseteq L$, then*

$$(37.8) \quad (\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D))^{\text{core}} = \text{Fun}(K, C)^{\text{core}} \times_{\text{Fun}(K, D)^{\text{core}}} \text{Fun}(L, D)^{\text{core}}.$$

Proof. Both sides of (37.8) can be regarded as subobjects of $\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$, which we note is a quasicategory since $r: \text{Fun}(L, D) \rightarrow \text{Fun}(K, D)$ is an inner fibration and $\text{Fun}(K, C)$ is a quasicategory. The left-hand side is clearly contained in the right-hand side, since any functor between quasicategories takes isomorphisms to isomorphisms. By (37.6), $r: \text{Fun}(L, D) \rightarrow \text{Fun}(K, D)$ is actually a isofibration, and so r^{core} is a Kan fibration (37.4). Thus the right-hand side of (37.8) is a pullback of Kan complexes along a Kan fibration, and thus is a Kan complex, which is necessarily contained in the left-hand side of (37.8). \square

37.9. Trivial fibrations between quasicategories. Now we can prove a generalization of (34.8), which identified trivial fibrations between Kan complexes as the Kan fibrations which are weak equivalences. In the generalization, the role of Kan fibrations is replaced with isofibrations.

37.10. Proposition. *Let $p: C \rightarrow D$ be a map between quasicategories. Then p is a trivial fibration if and only if it is an isofibration and a categorical equivalence.*

Proof. [Joy08a, Theorem 5.15]. (\implies) If p is a trivial fibration, it is an inner fibration and $(\{0\} \subset N(\text{Iso})) \sqsupset p$, so it is an isofibration (37.2). We have already shown that p is a categorical equivalence (20.10).

(\impliedby) Conversely, suppose p is isofibration and categorical equivalence. It suffices to show that for any inclusion $i: K \subseteq L$, the map

$$g := (p^{\square i})^{\text{core}}: \text{Fun}(L, C)^{\text{core}} \rightarrow \text{Fun}(K, C)^{\text{core}} \times_{\text{Fun}(K, D)^{\text{core}}} \text{Fun}(L, D)^{\text{core}}$$

is surjective on 0-simplicies. In fact, we will show that g is a trivial fibration, which implies what we want. By (37.6), the map $p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ is an isofibration; by (37.4), the restriction $(p^{\square i})^{\text{core}}$ of this map to cores is a Kan fibration; by (37.7), this restriction is precisely g . Thus g is a Kan fibration between Kan complexes. It will therefore suffice by (34.8) to show that g is also a categorical equivalence.

The maps

$$\text{Fun}(L, C) \rightarrow \text{Fun}(L, D), \quad \text{Fun}(K, C) \rightarrow \text{Fun}(K, D)$$

are categorical equivalences since p is (**proof?**), and so induce categorical equivalences on cores. These maps are also isofibrations by (37.6), and therefore the restrictions to cores are Kan fibrations. Thus $\text{Fun}(L, C)^{\text{core}} \rightarrow \text{Fun}(L, D)^{\text{core}}$ and $\text{Fun}(K, C)^{\text{core}} \rightarrow \text{Fun}(K, D)^{\text{core}}$ are Kan fibrations and weak equivalences between Kan complexes, and thus are trivial fibrations (34.8). Thus in

$$\text{Fun}(L, C)^{\text{core}} \xrightarrow{g} \text{Fun}(K, C)^{\text{core}} \times_{\text{Fun}(K, D)^{\text{core}}} \text{Fun}(L, D)^{\text{core}} \rightarrow \text{Fun}(L, D)^{\text{core}}$$

the second map is a trivial fibration being a pullback of $\text{Fun}(K, p)^{\text{core}}$, and the composite is a weak equivalence. It follows that g is a weak equivalence by the 2-out-of-3 property for weak equivalences (33.3), and the result is proved. \square

This proof made essential use of our characterization of trivial fibrations between Kan complexes, which is why we had to prove that special case first.

37.11. *Exercise.* Let C be a quasicategory and let $\pi: C \rightarrow hC$ be the tautological map to its homotopy category. Show that

- (1) π is an isofibration, and
- (2) $(\partial\Delta^n \subset \Delta^n) \boxtimes \pi$ for $n = 0, 1, 2$.

Conclude that π is a categorical equivalence if and only if $(\partial\Delta^n \subset \Delta^n) \boxtimes \pi$ for all $n \geq 3$.

37.12. Monomorphisms which are categorical equivalences lift against isofibrations.

Now we show that isofibrations have liftings with respect to all monomorphisms which are categorical equivalences.

37.13. Proposition. *A map $p: C \rightarrow D$ with D a quasicategory is an isofibration if and only if $j \boxtimes p$ for every $j: K \rightarrow L$ which is both a monomorphism and a categorical equivalence.*

Proof. (\Leftarrow) Immediate from the characterization of isofibrations as maps between quasicategories in the right complement of $\text{InnHorn} \cup \{\{0\} \subset N\text{Iso}\}$ (37.2).

(\Rightarrow) Suppose p is an isofibration. It suffices to show that for $j \in \overline{\text{Cell}} \cap \text{CatEq}$,

$$q: \text{Fun}(K, C) \rightarrow \text{Fun}(L, C) \times_{\text{Fun}(L, D)} \text{Fun}(K, D)$$

is a trivial fibration, whence it is surjective on vertices and thus $j \boxtimes p$. Since j is a monomorphism, (37.6) says q is an isofibration between quasicategories, so by the criterion for an isofibration to be a trivial fibration (37.10) it suffices to show that q is a categorical equivalence. Since j is a categorical equivalence both $\text{Fun}(j, C)$ and $\text{Fun}(j, D)$ are categorical equivalences. The map $\text{Fun}(j, D)$ is also an isofibration by (37.6), so it is a trivial fibration by (37.10). Therefore the basechange $\text{Fun}(L, C) \times_{\text{Fun}(L, D)} \text{Fun}(K, D) \rightarrow \text{Fun}(L, C)$ of $\text{Fun}(j, D)$ is a trivial fibration, and the result follows using the 2-out-of-3 property of categorical equivalences (22.10). \square

37.14. Monomorphisms which are categorical equivalences. We can now prove a generalization of (34.12), which characterized the injective weak equivalences.

37.15. Proposition. *Let $j: K \rightarrow L$ be a monomorphism of simplicial sets. Then j is a categorical equivalence if and only if $\text{Map}(j, C): \text{Map}(L, C) \rightarrow \text{Map}(K, C)$ is a trivial fibration for all quasicategories C .*

Proof. Straightforward using the fact that $\text{Map}(j, C)$ is an isofibration for any inclusion (37.6), and that isofibrations which are categorical equivalences are trivial fibrations (37.10). \square

37.16. *Remark.* As a consequence of (37.15), we see that the class $\overline{\text{Cell}} \cap \text{CatEq}$ of injective categorical equivalences between simplicial sets is a weakly saturated class: it is the left complement of the class of maps of the form $p^{\square i}$ where $p: C \rightarrow \Delta^0$ is a projection from a quasicategory, and i is a cell inclusion. Furthermore, this class contains the class of inner anodyne maps.

This class is *not the same* as $\overline{\text{InnHorn}}$. For instance, every inner anodyne map is a bijection on vertices, but $\{0\} \rightarrow N\text{Iso}$ which is not bijective on vertices is an injective categorical equivalence. Neither is it the same as the weak saturation of $\text{InnHorn} \cup \{\{0\} \subset N\text{Iso}\}$. (**Proof?**) This is a significant way in which the theory of quasicategories is not entirely parallel with the theory of Kan complexes.

38. LOCALIZATION OF QUASICATEGORIES

38.1. Quasigroupoidification. Let C be a quasicategory, and X a simplicial set. Let

$$\text{Fun}^{(X)}(X, C) \subseteq \text{Fun}(X, C)$$

denote the full subcategory spanned by objects which are functors $f: X \rightarrow C$ with the property that $f(X) \subseteq C^{\text{core}}$.

Note that $\text{Fun}^{(X)}(X, C)$ is a quasicategory, but not necessarily a quasigroupoid: for instance, morphisms are functors $f: X \times \Delta^1 \rightarrow C$ such that $f(X \times \{0\}) \subseteq C^{\text{core}}$ and $f(X \times \{1\}) \subseteq C^{\text{core}}$, but need not satisfy $f(X \times \Delta^1) \subseteq C^{\text{core}}$.

38.2. *Example.* In (32.4) we considered the full subcategory $\text{Fun}^{(\Delta^1)}(\Delta^1, C) \subseteq \text{Fun}(\Delta^1, C)$, which we there denoted \widehat{C} .

38.3. **Lemma.** *Consider $\Delta^1 \subset N(\text{Iso})$ representing the map $0 \rightarrow 1$ in Iso . The restriction map $\text{Fun}(N(\text{Iso}), C) \rightarrow \text{Fun}(\Delta^1, C)$ factors through a trivial fibration*

$$\text{Fun}(N(\text{Iso}), C) \rightarrow \text{Fun}^{(\Delta^1)}(\Delta^1, C).$$

Proof. We've already proved this is surjective on 0-dimensional elements (33.16). In that proof, we noted that $N(\text{Iso}) = \bigcup F_k$ with $F_k = F_{k-1} \cup_{\Lambda_0^k} \Delta^k$ and $F_1 = \Delta^1$, and the result was immediate using the Joyal extension theorem.

To prove this lemma, it suffices to show that each $\text{Fun}^{(\Delta^1)}(F_k, C) \rightarrow \text{Fun}^{(\Delta^1)}(F_{k-1}, C)$ is a trivial fibration for $k \geq 2$. There are pullback squares

$$\begin{array}{ccc} \text{Fun}^{(\Delta^1)}(F_k, C) & \longrightarrow & \text{Fun}^{(\Delta^{\{0,1\}})}(\Delta^k, C) \\ \downarrow & & \downarrow \\ \text{Fun}^{(\Delta^1)}(F_{k-1}, C) & \longrightarrow & \text{Fun}^{(\Delta^{\{0,1\}})}(\Lambda_0^k, C) \end{array}$$

so it suffices to solve the lifting problems $(\partial\Delta^n \subset \Delta^n) \sqsupset (\text{Fun}^{(\Delta^{\{0,1\}})}(\Delta^k, C) \rightarrow \text{Fun}^{(\Delta^{\{0,1\}})}(\Lambda_0^k, C))$ when $n \geq 0$ and $k \geq 0$. This adjoints to the lifting problem

$$\begin{array}{ccc} & f & \\ \Delta^{\{0,1\}} & \xrightarrow{\quad} & \text{Fun}(\Delta^n, C) \\ & \downarrow & \downarrow \\ & \Delta^k & \text{Fun}(\partial\Delta^n, C) \end{array}$$

where f is such that for each $j \in [n]$ the composite $\Delta^{\{0,1\}} \rightarrow \text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(\{j\}, C) \approx C$ represents an isomorphism in C . By the objectwise criterion for natural isomorphisms (C) we see that f itself represents an isomorphism in $\text{Fun}(\Delta^n, C)$, so the desired lift exists by Joyal lifting (28.13). \square

38.4. **Proposition.** *Let $i: X \rightarrow X'$ be any anodyne map to a Kan complex X' . Then the restriction map $\text{Fun}(i, C): \text{Fun}(X', C) \rightarrow \text{Fun}(X, C)$ factors through a trivial fibration*

$$q: \text{Fun}(X', C) \rightarrow \text{Fun}^{(X)}(X, C).$$

Proof. Every edge in X maps to an isomorphism in X' . Therefore $\text{Fun}(i, C)$ must factor through a map q into the subcomplex $\text{Fun}^{(X)}(X, C) \subseteq \text{Fun}(X, C)$. We also know that $\text{Fun}(i, C)$ is an isofibration by (37.6), and therefore the map q is also an isofibration. Therefore by (37.10) it is enough to show that for each such i the map q is a categorical equivalence.

Given any two anodyne maps $i: X \rightarrow X'$ and $i': X \rightarrow X''$ to Kan complexes, there exists a categorical equivalence $f: X' \rightarrow X''$ such that $fi = i'$ (because $\text{Fun}(X', X'') \rightarrow \text{Fun}(X, X'')$ is a trivial fibration by (33.8)). Therefore in the commutative diagram

$$\begin{array}{ccc} \text{Fun}(X', C) & \xrightarrow{\text{Fun}(f, C)} & \text{Fun}(X'', C) \\ & \searrow q & \swarrow q' \\ & \text{Fun}^{(X)}(X, C) & \end{array}$$

the map $\text{Fun}(f, C)$ is a categorical equivalence, so q is a categorical equivalence iff q' is.

Thus, given a simplicial set X , to show that *any* anodyne $i: X \rightarrow X'$ to a Kan complex induces a categorical equivalence $q: \text{Fun}(X', C) \rightarrow \text{Fun}^{(X)}(X, C)$, it suffices to do so for a *single choice* of such an i .

Now given any simplicial sets X , construct maps f and g of the form

$$X \xrightarrow{f} Y := X \cup_{\coprod \Delta^1} \coprod N(\text{Iso}) \xrightarrow{g} X',$$

where the coproducts defining Y are taken over all maps $\Delta^1 \rightarrow X$, and g is any inner anodyne map to a quasicategory. By construction f and g are anodyne, and therefore $i = gf$ is anodyne.

I claim that the quasicategory X' is actually a Kan complex, i.e., a quasigroupoid, i.e., a quasicategory such that $h(X')$ is a groupoid. To see this, note that every edge in Y factors through a map $\text{Iso} \rightarrow Y$ by construction; from this, we see that every edge in the simplicial set Y must be a preisomorphism, whence its fundamental category hY is a groupoid. Since g is inner anodyne, $hY \rightarrow hX'$ is an equivalence, so hX' is a groupoid as desired.

Now $\text{Fun}(i, C)$ is the composite of

$$\text{Fun}(X', C) \xrightarrow{\text{Fun}(g, C)} \text{Fun}(Y, C) \xrightarrow{p} \text{Fun}^{(X)}(X, C) \subseteq \text{Fun}(X, C).$$

Since g is inner anodyne, $\text{Fun}(g, C)$ is a trivial fibration by (37.6). The map p is a pullback of maps $\text{Fun}(N(\text{Iso}), C) \rightarrow \text{Fun}^{(\Delta^1)}(\Delta^1, C)$, which are also trivial fibrations by (38.3). The result is proved. \square

We write $X \rightarrow X_{\text{Kan}}$ for any choice of anodyne map to a Kan complex, and call it a **quasi-groupoidification** of X . What we have shown is that for any quasicategory C we get a categorical equivalence

$$\text{Fun}(X_{\text{Kan}}, C) \approx \text{Fun}^{(X)}(X, C).$$

We can apply this construction when X is a quasicategory, or even when X is the nerve of an ordinary category, and obtain interesting new Kan complexes.

38.5. Example. It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact to the nerve of some poset [Tho80]. Thus, for every Kan complex K , there exists an ordinary category A and a weak equivalence $NA \rightarrow K$, where therefore induces categorical equivalences $\text{Fun}(K, C) \approx \text{Fun}^{(NA)}(NA, C)$ for every quasicategory C .

We note that there is also a classical groupoidification construction, which given an ordinary category A produces an ordinary groupoid A_{Gpd} by “formally inverting all maps”. We have that $h((NA)_{\text{Kan}}) \approx N(A_{\text{Gpd}})$, but in general $(NA)_{\text{Kan}}$ is not weakly equivalent to $N(A_{\text{Gpd}})$.

38.6. Exercise. Let A be the poset of *proper and non-empty* subsets of $\{0, 1, 2, 3\}$. Show that A_{Gpd} is equivalent to the one-object category, but that $(NA)_{\text{Kan}}$ is not equivalent to the one-object category. (In the second case, the idea is that the geometric realization of NA is a 2-sphere. Explicitly, you can prove non-equivalence by showing $\pi_0 \text{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of §8.9.)

38.7. Localization of quasicategories. There is a more general construction, which applies to a simplicial set X equipped with a subcomplex $W \subseteq X$. Let

$$\text{Fun}^{(W)}(X, C) \subseteq \text{Fun}(X, C)$$

denote the full subcategory spanned by objects $f: X \rightarrow C$ such that $f(W) \subseteq C^{\text{core}}$. Note that this really only depends on the edges in W .

38.8. Proposition. *Given an inclusion $W \subseteq X$, choose an anodyne map $W \rightarrow W_{\text{Kan}}$ to a Kan complex, and then choose an inner anodyne map $g: Y := X \cup_W W_{\text{Kan}} \rightarrow X'$ to a quasicategory.*

Then for any quasicategory C , the restriction map $\text{Fun}(X', C) \rightarrow \text{Fun}(X, C)$ factors through a trivial fibration

$$\text{Fun}(X', C) \rightarrow \text{Fun}^{(W)}(X, C).$$

Proof. We have

$$\begin{array}{ccccc} \text{Fun}(X', C) & \xrightarrow{\text{Fun}(g, C)} & \text{Fun}(Y, C) & \longrightarrow & \text{Fun}^{(W)}(X, C) & \twoheadrightarrow & \text{Fun}(X, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{Fun}(W_{\text{Kan}}, C) & \xrightarrow{p} & \text{Fun}^{(W)}(W, C) & \twoheadrightarrow & \text{Fun}(W, C) \end{array}$$

in which both squares are pullbacks. The map $\text{Fun}(g, C)$ is a trivial fibration since g is inner anodyne, while p is a trivial fibration as we have shown (38.4). \square

We sometimes write $X \rightarrow X_{(W)}$ for any map $X \rightarrow X'$ obtained as in the proposition. Note that any $X \rightarrow X_{(X)}$ is an example of $X \rightarrow X_{\text{Kan}}$. The observation is that for any quasicategory C , we have a categorical equivalence

$$\text{Fun}^{(W)}(X, C) \approx \text{Fun}(X_{(W)}, C).$$

38.9. Quasicategories from relative categories. A **relative category** is a pair $W \subseteq C$ consisting of an *ordinary category* C and a subcategory W containing all the objects of C . The above construction gives, for any relative category, a map

$$C \rightarrow C_{(W)},$$

unique up to categorical equivalence. We may call $C_{(W)}$ the **localization** of C with respect to W . It turns out that many quasicategories of interest arise as such localizations.

39. THE PATH FIBRATION

To prove the fundamental theorem of quasicategories for a general map between quasicategories, we will reduce to the special case of isofibrations. We do this by means of the ‘‘path fibration’’ construction, which provides a factorization of a map into a categorical fibration followed by an isofibration.

39.1. The path fibration for quasicategories. Let D be a quasicategory. Recall (32.4) the path category $\widehat{D} \subseteq \text{Fun}(\Delta^1, D)$ of D , defined to be the full subcategory spanned by functors which represent isomorphisms in D .

For a functor $f: C \rightarrow D$ between quasicategories, we define a factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ by means of the following diagram.

$$\begin{array}{ccccc} C & \xrightarrow{j} & P(f) & \xrightarrow{t} & \widehat{D} & \xrightarrow{r_1} & D \\ & \searrow & \downarrow s_0 & & \downarrow r_0 & & \\ & & C & \xrightarrow{f} & D & & \end{array}$$

p (curved arrow from $P(f)$ to D)

Here the square is a pullback square. The map j is the unique one so that $s_0 j = \text{id}_C$, and $t j = \tilde{\pi} f$ where $\tilde{\pi}: D \rightarrow \widehat{D} \subseteq \text{Fun}(\Delta^1, D)$ is adjoint to the projection $D \times \Delta^1 \rightarrow D$. The maps $r_i: \text{Fun}^{(\Delta^1)}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D) \rightarrow D$ are induced by restriction along $\{i\} \subset \Delta^1$.

The factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ is called the **path fibration** of f , because of the following.

39.2. Proposition. *The simplicial set $P(f)$ is a quasicategory, the map j is a categorical equivalence, and p is an isofibration.*

In particular, the objects of $P(f)$ are pairs (c, α) consisting of an object $c \in C_0$ and an isomorphism $\alpha: f(c) \rightarrow d$ in D . The map j sends an object c to $(c, 1_{f(c)})$, while p sends (c, α) to d .

39.3. *Exercise.* Show that if $f: C \rightarrow D$ is a functor between ordinary categories, then $P(f)$ is also an ordinary category.

Proof. From (32.5) we know that both r_0 and r_1 are trivial fibrations. Therefore the base change s_0 of r_0 is a trivial fibration, and hence an inner fibration, which implies that $P(f)$ is a quasicategory.

Since s_0 is a trivial fibration it is a categorical equivalence (20.10), and thus j is a categorical equivalence by 2-out-of-3 (22.10).

To show that p is an isofibration, observe that there is actually a pullback square of the form

$$\begin{array}{ccc} P(f) & \xrightarrow{t} & \widehat{D} \\ s=(s_0,p) \downarrow & & \downarrow r=(r_0,r_1) \\ C \times D & \xrightarrow{f \times \text{id}_D} & D \times D \end{array}$$

(To see this, use patching of pullback squares where we regard $C \times D$ as a pullback of $C \xrightarrow{f} D \leftarrow D \times D$.) We will prove below (39.5) that r is an isofibration, whence its base-change s is also an isofibration, and since the projection $\pi: C \times D \rightarrow D$ is an isofibration the composite $p = \pi s$ is an isofibration as desired. (See exercise (39.4) below for the relevant facts about isofibrations used here.) \square

39.4. *Exercise.* Show that the following facts about isofibrations.

- (1) Any base-change of an isofibration $p: C \rightarrow D$ along a map $D' \rightarrow D$ from a quasicategory is also an isofibration.
- (2) For any quasicategory C , the projection $C \rightarrow *$ is an isofibration, and thus for any quasicategory D the projection $C \times D \rightarrow D$ is an isofibration.
- (3) The composite of two isofibrations is an isofibration.

39.5. **Lemma.** *If D is a quasicategory, then the map $r = (r_0, r_1): \widehat{D} \subseteq \text{Fun}(\Delta^1, D) \rightarrow D \times D$ from the path category induced by restriction along $\partial\Delta^1 \subset \Delta^1$ is an isofibration.*

Proof. We need to produce a lift in a diagram of the form

$$\begin{array}{ccccc} \{0\} & \xrightarrow{g} & \widehat{D} & \longrightarrow & \text{Fun}(\Delta^1, D) \\ \downarrow & \nearrow s & \downarrow r & & \downarrow \\ \Delta^1 & \xrightarrow{f=(f_0,f_1)} & D \times D & \xlongequal{\quad} & D \times D \end{array}$$

where f represents an isomorphism in $D \times D$, or equivalently f_0 and f_1 represent isomorphisms in D . By the usual lifting-adjunction arguments, to produce a lifting $s: \Delta^1 \rightarrow \text{Fun}(\Delta^1, D)$ it is equivalent to producing a lifting \tilde{s} in

$$\begin{array}{ccc} (\{0\} \times \Delta^1) \cup_{\{0\} \times \partial\Delta^1} (\Delta^1 \times \partial\Delta^1) & \xrightarrow{(\tilde{g}, \tilde{f})} & D \\ \downarrow & \nearrow \tilde{s} & \\ \Delta^1 \times \Delta^1 & & \end{array}$$

where \tilde{g} and \tilde{f} are adjoint to g and f . The restriction of (\tilde{g}, \tilde{f}) to the edge $\Delta^1 \times \{0\}$ is the map $f_0: \Delta^1 \rightarrow D$, which represents an isomorphism in D , and thus a lift \tilde{s} exists by the easiest case of the box-product version of Joyal lifting (30.3).

It remains to show that the adjoint $s: \Delta^1 \rightarrow \text{Fun}(\Delta^1, D)$ to \tilde{s} lands in the subobject \tilde{D} , i.e., that the restrictions to each the vertices of Δ^1 represent isomorphisms in D , or equivalently (by the objectwise criterion for natural isomorphisms) that $\tilde{s}\{j\} \times \Delta^1: \{j\} \times \Delta^1 \rightarrow D$ represent isomorphisms in D for $j = 0, 1$.

We know that $\tilde{g} = \tilde{s}\{0\} \times \Delta^1$ represents an isomorphism in D , since it is assumed to correspond to a vertex g of \tilde{D} . Let $\tilde{h} = \tilde{s}\{1\} \times \Delta^1$. Then going around the “boundary” of the square $\Delta^1 \times \Delta^1$, we see that the map \tilde{s} induces a relation $[\tilde{h}][f_0] = [f_1][\tilde{g}]$ in the homotopy category hD . Since the other three of these are known to be isomorphisms, so is $[\tilde{h}]$, as desired. \square

Old proof below.

Proof. For $i \in \{0, 1\}$ the inclusion $\{i\} \subset N(\text{Iso})$ is an equivalence of ordinary categories, hence a categorical equivalence. Therefore the composite map

$$\text{Fun}(N(\text{Iso}), D) \xrightarrow{q} \text{Fun}(\Delta^1)(\Delta^1, D) \xrightarrow{r_i} D$$

induced by restriction along $\{i\} \subset N(\text{Iso})$ is a categorical equivalence. We have shown (38.3) that q is a trivial fibration, and therefore the r_i are categorical equivalences by 2-out-of-3 (22.10).

The restriction map $\text{Fun}(\Delta^1, D) \rightarrow \text{Fun}(\partial\Delta^1, D) = D \times D$ is an isofibration (37.6). We claim that $r: \text{Fun}(\Delta^1)(\Delta^1, D) \rightarrow D \times D$ is also an isofibration. It easily seen that r is an inner fibration, since $\text{Fun}(\Delta^1)(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D)$ is full. To show that r is also isofibration is then straightforward (e.g., if $\alpha: \alpha_0 \rightarrow \alpha_1$ is a natural isomorphism of functors $\Delta^1 \rightarrow D$, and $\alpha_0(\Delta^1) \subseteq D^{\text{core}}$, then $\alpha_1(\Delta^1) \subseteq D^{\text{core}}$).

Since the projections $D \times D \rightarrow D$ are isofibrations, it follows that the r_i are isofibrations, and hence trivial fibrations since they are categorical equivalences (37.10).

Therefore s_0 , being a pullback of r_0 , is a trivial fibration. Therefore $P(f)$ is a quasicategory and j is a categorical equivalence.

In the commutative diagram

$$\begin{array}{ccccc} & & P(f) & \longrightarrow & \text{Fun}(\Delta^1)(\Delta^1, D) \\ & \swarrow p & \downarrow s & & \downarrow r=(r_0, r_1) \\ D & \xleftarrow{\pi} & C \times D & \xrightarrow{f \times \text{id}_D} & D \times D \end{array}$$

the square is a pullback. We have shown that r is an isofibration, and hence so is its pullback s . Therefore $p = \pi s$ is an isofibration, as desired. \square

39.6. The path fibration for Kan complexes. If $f: C \rightarrow D$ is a functor between quasicategories, and D is a Kan complex, then $\hat{D} = \text{Fun}(\Delta^1, D)$, and the diagram defining the path fibration takes the form

$$\begin{array}{ccccc} C & \xrightarrow{j} & P(f) & \xrightarrow{\quad} & \text{Fun}(\Delta^1, D) & \xrightarrow{r_1} & D \\ & \searrow & \downarrow s_0 & & \downarrow r_0 & & \\ & & C & \xrightarrow{f} & D & & \end{array}$$

39.7. Proposition. *If $f: C \rightarrow D$ is a functor between Kan complexes, then j is a weak equivalence and p is a Kan fibration.*

Proof. Immediate. \square

40. PROOF OF THE FUNDAMENTAL THEOREM

We are ready to finish the proof of (E), The Fundamental Theorem of Quasicategories.

40.1. Proposition. *If $f: C \rightarrow D$ is a fully faithful and essentially surjective functor between quasicategories, then f is a categorical equivalence.*

We will prove this below. First note that the quasigroupoid version of this result (which we proved earlier) gives us the following.

40.2. Lemma. *If $f: C \rightarrow D$ is a fully faithful and essentially surjective functor between quasicategories, then $f: C^{\text{core}} \rightarrow D^{\text{core}}$ is a categorical equivalence (in fact, a weak equivalence).*

Proof. Note that if f is fully faithful and essentially surjective, so is f^{core} . Apply (33.22). \square

Note that the path fibration construction gives us a factorization of f into $C \xrightarrow{j} P(f) \xrightarrow{p} D$, where j is a categorical equivalence and p is an isofibration. Because both categorical equivalences and (fully faithful + essentially surjective) are classes which satisfy 2-out-of-3 (22.10), (32.9), proving the fundamental theorem reduces to the case that f is itself an isofibration.

To prove the isofibration case of (40.1), we will deduce it from the following.

40.3. Proposition. *If $p: C \rightarrow D$ is an isofibration which is fully faithful and essentially surjective, then $q = (p^{\square i})^{\text{core}}: \text{Fun}(L, C)^{\text{core}} \rightarrow \text{Fun}(K, C)^{\text{core}} \times_{\text{Fun}(K, D)^{\text{core}}} \text{Fun}(L, D)^{\text{core}}$ is a trivial fibration for every monomorphism $i: K \rightarrow L$.*

Proof that (40.3) implies (40.1). As noted above, it is enough to consider isofibrations $p: C \rightarrow D$ which are fully faithful and essentially surjective. By (40.3), for any monomorphism $i: K \rightarrow L$ the map $q = (p^{\square i})^{\text{core}}$ is a trivial fibration, and therefore surjective on vertices. The core of a quasicategory has all its objects, and thus the box power map $p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ is surjective on vertices. Thus, we have $i \boxtimes p$ for every monomorphism i , whence p is a trivial fibration, and thus a categorical equivalence. \square

State “ $p \in \text{TrivFib}$ iff $(p^{\square i})^{\text{core}} \in \text{TrivFib}$ ” as proposition in earlier section.

40.4. Proof of (40.3). We start with the following lemma, which says that isofibrations between quasicategories which are trivial fibrations on cores are characterized by a lifting property.

40.5. Lemma. *There exists a set of maps S such that for any isofibration $q: C \rightarrow D$ between quasicategories, we have $S \boxtimes q$ iff $q^{\text{core}} \in \text{TrivFib}$.*

Proof. Given an inclusion $K \subseteq L$ of simplicial sets, we can use two applications of the small object argument (13.10) to construct

$$\begin{array}{ccccc} K & \longrightarrow & K_{\text{Kan}} & & \\ \downarrow & & \downarrow & \searrow & \\ L & \longrightarrow & L' & \longrightarrow & L_{\text{Kan}} \end{array}$$

in which the square is pushout, the horizontal maps are anodyne, and the objects K_{Kan} and L_{Kan} are Kan complexes. (So $K \rightarrow K_{\text{Kan}}$ and $L \rightarrow L_{\text{Kan}}$ are examples of quasigroupoidification as in (38.1).)

We will show that $(K \subseteq L) \boxtimes q^{\text{core}}$ iff $(K_{\text{Kan}} \subseteq L_{\text{Kan}}) \boxtimes q$. The lemma will follow immediately by taking $S = \{(\partial \Delta^n)_{\text{Kan}} \subset (\Delta^n)_{\text{Kan}} \mid n \geq 0\}$.

(\implies) Suppose $(K \subseteq L) \boxtimes q^{\text{core}}$. Since K_{Kan} and L_{Kan} are Kan complexes, any maps from them to quasicategories must factor through cores. Thus it suffices to find a lift in the right-hand square

of

$$\begin{array}{ccccccc}
 K & \longrightarrow & K_{\text{Kan}} & \xlongequal{\quad} & K_{\text{Kan}} & \longrightarrow & C^{\text{core}} \\
 \downarrow & & \downarrow & & \downarrow & \nearrow s & \downarrow q^{\text{core}} \\
 L & \dashrightarrow & L' & \longrightarrow & L_{\text{Kan}} & \longrightarrow & D^{\text{core}}
 \end{array}$$

By hypothesis, a lift s exists, and therefore a lift s' since the left-hand square is a pushout. Because $L' \rightarrow L_{\text{Kan}}$ is anodyne and q^{core} is a Kan fibration (37.4), an extension to a lift s'' exists.

(\Leftarrow) Suppose $(K_{\text{Kan}} \subseteq L_{\text{Kan}}) \square q$. Consider a lifting problem

$$\begin{array}{ccc}
 K & \xrightarrow{a} & C^{\text{core}} \\
 \downarrow & \nearrow & \downarrow q^{\text{core}} \\
 L & \longrightarrow & D^{\text{core}}
 \end{array}$$

Because C^{core} is a Kan complex and $K \rightarrow K_{\text{Kan}}$ is anodyne, the map a factors through some $a': K_{\text{Kan}} \rightarrow C^{\text{core}}$, and there is a unique compatible map $b': L' \rightarrow D^{\text{core}}$ from the pushout along $K \subseteq L$. Again, b' factors through $b'': L_{\text{Kan}} \rightarrow D^{\text{core}}$. Thus we have extended the original square to a diagram

$$\begin{array}{ccccccc}
 K & \longrightarrow & K_{\text{Kan}} & \xlongequal{\quad} & K_{\text{Kan}} & \xrightarrow{a'} & C^{\text{core}} & \longrightarrow & C \\
 \downarrow & & \downarrow & & \downarrow & \nearrow t' & \downarrow & \nearrow & \downarrow q \\
 L & \longrightarrow & L' & \longrightarrow & L_{\text{Kan}} & \xrightarrow{b''} & D^{\text{core}} & \longrightarrow & D
 \end{array}$$

A lift t exists by hypothesis, and since L_{Kan} is a Kan complex it factors through a unique lift t' (using that $C^{\text{core}} \rightarrow C$ and $D^{\text{core}} \rightarrow D$ are monomorphisms). The composite $L \rightarrow L_{\text{Kan}} \rightarrow C^{\text{core}}$ is the desired lift. \square

Fix an isofibration $p: C \rightarrow D$ between quasicategories which is fully faithful and essentially surjective. Consider the class

$$\mathcal{C}_p := \{ i \mid i \text{ is a monomorphism and } (p^{\square i})^{\text{core}} \in \text{TrivFib} \}.$$

The statement of (40.3) amounts to showing that \mathcal{C}_p contains every monomorphism.

40.6. Lemma. *The class \mathcal{C}_p is weakly saturated.*

Proof. First note that for any monomorphism i , the map $p^{\square i}$ is an isofibration since p is (37.6). Using the set of maps S provided by the previous lemma (40.5), for a monomorphism i we have that $(p^{\square i})^{\text{core}} \subseteq \text{TrivFib}$ iff $S \square (p^{\square i})$ iff $i \square (p^{\square S})$. Thus \mathcal{C}_p is the intersection of $\square (p^{\square S})$ with $\overline{\text{Cell}}$, and so is weakly saturated. \square

40.7. Lemma. *Let $p: C \rightarrow D$ be an isofibration between quasicategories. If $K \subseteq L$, and if $(\emptyset \subset K)$ and $(\emptyset \subset L)$ are elements of \mathcal{C}_p , then $(K \subseteq L) \in \mathcal{C}_p$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
 & \xrightarrow{\text{Map}(L,p)^{\text{core}}} & \\
 \text{Map}(L,C)^{\text{core}} & \xrightarrow{(p^{\square i})^{\text{core}}} & (\text{Map}(K,C) \times_{\text{Map}(K,D)} \text{Map}(L,D))^{\text{core}} \xrightarrow{q} \text{Map}(L,D)^{\text{core}} \\
 & \downarrow & \downarrow \\
 & \text{Map}(K,C)^{\text{core}} & \xrightarrow{\text{Map}(K,p)^{\text{core}}} \text{Map}(K,D)^{\text{core}}
 \end{array}$$

where the square is a pullback (37.7). By hypothesis, both $\text{Map}(K, p)^{\text{core}}$ and $\text{Map}(L, p)^{\text{core}}$ are trivial fibrations, whence q is also a trivial fibration. By 2-out-of-3, we have that $(p^{\square i})^{\text{core}}$ is a categorical equivalence. Since p is an isofibration, we have that $(p^{\square i})^{\text{core}}$ is a Kan fibration (37.6), (37.7) between Kan complexes, and therefore is a trivial fibration (34.8) as desired. \square

Next we observe that $(\emptyset \subseteq \Delta^n) \in \mathcal{C}_p$ if p is fully faithful and essentially surjective.

40.8. Proposition. *If $p: C \rightarrow D$ is an isofibration which is fully faithful and essentially surjective, then $\text{Fun}(\Delta^n, C)^{\text{core}} \rightarrow \text{Fun}(\Delta^n, D)^{\text{core}}$ is a trivial fibration for all $n \geq 0$.*

Proof. In the case $n = 0$, this means showing that $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$ is a trivial fibration, which we have already observed (40.2). (**Not quite what it says, fix this.**)

For $n \geq 1$, consider the diagram

$$\begin{array}{ccccc}
 \coprod_{c_0, \dots, c_n} \text{map}_C(c_0, \dots, c_n) & \xrightarrow{\quad} & \text{Fun}(\Delta^n, C)^{\text{core}} & & \\
 \downarrow \coprod q_{c_0, \dots, c_n} & & \downarrow q & \searrow \text{Fun}(\Delta^n, p)^{\text{core}} & \\
 \coprod_{c_0, \dots, c_n} \text{map}_D(pc_0, \dots, pc_n) & \xrightarrow{j} & (C^{\times n+1})^{\text{core}} \times_{(D^{\times n+1})^{\text{core}}} \text{Fun}(\Delta^n, D)^{\text{core}} & \xrightarrow{r} & \text{Fun}(\Delta^n, D)^{\text{core}} \\
 \downarrow & & \downarrow & & \downarrow \\
 C_0^{\times n+1} & \xrightarrow{\quad} & (C^{\times n+1})^{\text{core}} & \xrightarrow{(p^{\times n+1})^{\text{core}}} & (D^{\times n+1})^{\text{core}}
 \end{array}$$

in which all the squares are pullbacks. The map r is a base change of $(p^{\times n+1})^{\text{core}}: (C^{\times n+1})^{\text{core}} \rightarrow (D^{\times n+1})^{\text{core}}$, which is isomorphic to the $(n+1)$ -fold product of $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$, which we have just noted is a trivial fibration. Thus, r is a trivial fibration, so it will suffice to show that q is a trivial fibration, for which we will use the fiberwise criterion (35.3).

We know that $p^{\square(\text{Sk}_0 \Delta^n \subset \Delta^n)}$ is an isofibration (37.6), and thus $q = (p^{\square(\text{Sk}_0 \Delta^n \subset \Delta^n)})^{\text{core}}$ is a Kan fibration (37.4). Because p is an essentially surjective isofibration, p^{core} is surjective on vertices, and thus j is surjective on vertices. Thus by the fiberwise criterion (35.3) we need to show that $\coprod q_{c_0, \dots, c_n}$ is a trivial fibration. Since coproducts of trivial fibrations are trivial fibrations (20.2) we thus reduce to showing that each q_{c_0, \dots, c_n} (which is a Kan fibration between Kan complexes being a pullback of q) is a weak equivalence, and hence a trivial fibration (34.8). This is immediate from the fact that p is fully faithful and induces maps compatible with the weak equivalences $\text{map}_C(c_0, \dots, c_n) \rightarrow \text{map}_C(c_0, c_1) \times \dots \times \text{map}_C(c_{n-1}, c_n)$ and $\text{map}_D(pc_0, \dots, pc_n) \rightarrow \text{map}_D(pc_0, pc_1) \times \dots \times \text{map}_D(pc_{n-1}, pc_n)$. \square

Proof of (40.3). As we have already noted it suffices to show that

$$\overline{\text{Cell}} \subseteq \mathcal{C}_p = \{ i \mid i \in \overline{\text{Cell}}, (p^{\square i})^{\text{core}} \in \text{TrivFib} \}$$

for any fully faithful and essentially surjective isofibration $p: C \rightarrow D$. As \mathcal{C}_p is weakly saturated (40.6), it is enough to show that $(\partial \Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for all $n \geq 0$. We will do this by induction on n .

For the case $n = 0$, this is immediate from (40.8). For $n \geq 1$, suppose we have $(\partial \Delta^k \subset \Delta^k) \in \mathcal{C}_p$ for $k < n$. Since $\partial \Delta^n$ is equal to its own $(n-1)$ -skeleton, skeletal filtration (15.23) gives that $(\emptyset \subset \partial \Delta^n) \in \mathcal{C}_p$. Then use (40.7) and (40.8) to conclude that $(\partial \Delta^n \subset \Delta^n) \in \mathcal{C}_p$ as desired. \square

Part 5. Model categories

41. CATEGORICAL FIBRATIONS

A map $p: X \rightarrow Y$ of simplicial sets is a **categorical fibration** if and only if $j \boxtimes p$ for all j which are monomorphisms and categorical equivalences. I'll write CatFib for the class of categorical fibrations.

Categorical fibrations generalize isofibrations. In fact, a map $p: C \rightarrow D$ with D a quasicategory is a categorical fibration if and only if it is an isofibration, as we proved in (37.13).

41.1. Proposition. *A map $p: X \rightarrow Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

Proof. (\implies) We have already proved this: (??) and (??). (\impliedby) If p is a categorical fibration and a categorical equivalence, factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ with j a monomorphism and q a trivial fibration. Then the usual argument shows that p is a retract of q , using the fact that $j \boxtimes p$ since j is a categorical equivalence by 2-of-3. \square

41.2. Proposition. *If $p: X \rightarrow Y$ is a categorical fibration and $j: K \rightarrow L$ is a monomorphism, then*

$$q: \text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is a categorical fibration. Furthermore, if either j or p is also a categorical equivalence, then so is q .

Proof. For the first, let $i: A \rightarrow B$ be a monomorphism which is a categorical equivalence. We have $i \boxtimes q$ iff $(i \boxtimes j) \boxtimes p$. By definition of categorical fibration, it suffices to show that $i \boxtimes j$ is a categorical equivalence, i.e., to show $\text{Map}(i \boxtimes j, C)$ is a categorical equivalence for every quasicategory C . In fact, $\text{Map}(i, C)$ is an isofibration and a categorical equivalence, hence a trivial fibration, and therefore $j \boxtimes \text{Map}(i, C)$.

If p is also a categorical equivalence, then it is a trivial fibration, and the result follows.

If j is also a categorical equivalence, then for any monomorphism i , we have $i \boxtimes q$ iff $(i \boxtimes j) \boxtimes p$ iff $j \boxtimes (p^{\square i})$. But $p^{\square i}$ is a categorical fibration by what we have just proved, so the result holds. \square

41.3. Categorical fibrations and the small object argument. Clearly, $\text{CatFib} = (\overline{\text{Cell}} \cap \text{CatEq})^{\square}$ is a right complement to a class of maps. We would like to know that CatFib is the right complement to a *set* of maps; then we could use the small object argument to factor any map into an injective categorical equivalence followed by a categorical fibration.

Unfortunately, it's apparently not known how to write down an explicit set of maps S so that $S^{\square} = \text{CatFib}$. What is known is that such a set *exists*.

41.4. Proposition. *There exists a set S of maps of simplicial sets such that $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$, whence $S^{\square} = \text{CatFib}$.*

In the rest of this section we will sketch a proof. The idea is to show that CatFib is the right complement of the class of all injective categorical equivalences $K \rightarrow L$ for which the number of elements in K and L is bounded by some explicit cardinal κ . We obtain S by choosing one representative for each isomorphism class in this class; then S is a set because of the cardinality bound.

We will define a **detection functor** $F: \text{Fun}([1], s\text{Set}) \rightarrow \text{Fun}([1], \text{Set})$ on categories of morphisms. This will have the following properties:

- For each map $f: X \rightarrow Y$, the map $F(f)$ is a monomorphism of sets.
- A map $f: X \rightarrow Y$ is a categorical equivalence if and only if $F(f)$ is a bijection.
- The functor F commutes with κ -filtered colimits for some regular cardinal κ .
- The functor F takes κ -small simplicial sets to κ -small sets.

We define F as the composite of several intermediate steps.

Step 1: Recall that the small object argument gives a *functorial* way to factor a map f as $f = pi$, with $i \in \overline{S}$ and $p \in S^{\square}$. “Functorial factorization” means that we get a section of the functor $\text{Fun}([2], s\text{Set}) \rightarrow \text{Fun}([1], s\text{Set})$ defining composition.

We can apply this using $S = \text{InnHorn}$. Thus, given any simplicial set, we *functorially* obtain an inner anodyne map $X \rightarrow X_{\text{qCat}}$ to a quasicategory X_{qCat} . As a result, we have a

functor $f \mapsto f_{\text{qCat}}: \text{Fun}([1], \text{sSet}) \rightarrow \text{Fun}([1], \text{sSet})$, with the property that f is a categorical equivalence if and only if f_{qCat} is, and both source and target of f_{qCat} are quasicategories.

Step 2: Form the path fibration $Q(f): P(f_{\text{qCat}}) \rightarrow Y_{\text{qCat}}$ of f_{qCat} . The map $Q(f)$ is thus an isofibration between quasicategories, and is a trivial fibration if and only if f is a categorical equivalence.

Step 3: Write $Q(f): X' \rightarrow Y'$. Define $E(f)$ to be the map of sets

$$E(f): \coprod_n \text{Hom}(\Delta^n, X') \rightarrow \coprod_n \text{Hom}(\partial\Delta^n, X') \times_{\text{Hom}(\partial\Delta^n, Y')} \text{Hom}(\Delta^n, Y').$$

Thus, f is a categorical equivalence if and only if $E(f)$ is *surjective*.

Step 4: Write $E(f): E_0(f) \rightarrow E_1(f)$, and define $F(f)$ by

$$F(f): \text{colim}[E_0(f) \times_{E_1(f)} E_0(f) \rightrightarrows E_0(f)] \rightarrow E_1(f).$$

In other words, $F(f)$ is the map from the image of $E(f)$ to $E_1(f)$. Thus, $F(f)$ is always a monomorphism, and f is a categorical equivalence if and only if $F(f)$ is a bijection.

There exists a regular cardinal κ such that F commutes with κ -filtered colimits, and takes κ -small simplicial sets to κ -small sets. (In fact, we can take $\kappa = \omega^+$, the successor to the countable cardinal).

Using the detection functor, we can prove the following key lemma.

41.5. Lemma. *Let $f: X \subseteq Y$ be an inclusion which is a categorical equivalence. Every κ -small subcomplex $A \subseteq Y$ is contained in a κ -small subcomplex $B \subseteq Y$ with the property that $B \cap X \subseteq B$ is a categorical equivalence.*

Proof. For a subcomplex $A \subseteq Y$ let f_A denote the inclusion $A \cap X \subseteq A$. The collection of all κ -small subcomplexes of Y is κ -filtered. Thus

$$\text{colim}_{\kappa\text{-small } A \subseteq Y} F(f_A) = F(f),$$

which we have assumed is an isomorphism. Thus for any κ -small $A \subseteq$ there must exist a κ -small $A' \supset A$ such that a lift exists in

$$\begin{array}{ccc} F_0(f_A) & \longrightarrow & F_0(f_{A'}) \\ \downarrow & \nearrow & \downarrow \\ F_1(f_A) & \longrightarrow & F_1(f_{A'}) \end{array}$$

This is because $F_1(f_A)$ is a κ -small set, so any lift $F_1(f_A) \rightarrow F_0(F)$ factors through some stage of the κ -filtered colimit.

We use transfinite induction to obtain a sequence $\{A_i\}$ indexed by $i < \kappa$, where at limit ordinals we take a colimit. Set $B := \text{colim } A_i$. Because κ is regular $|B| < \kappa$, and we have that $F(f_B)$ is an isomorphism by construction. \square

Consider the collection of monomorphisms $i: A \rightarrow B$ such that i is a categorical equivalence and $|B| < \kappa$. Choose a set S of such spanning all isomorphism classes of such maps; this is a set because of the cardinality bound. Clearly $S \subseteq \overline{\text{Cell}} \cap \text{CatEq}$.

41.6. Proposition. *We have $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$.*

Proof. [Joy08a, D.2.16]. Given an injective categorical equivalence $X \subseteq Y$, we consider the following poset \mathcal{P} . The objects of \mathcal{P} are subobjects $P \subseteq Y$ such that $X \subseteq P$ so that the inclusion $X \rightarrow P$ is contained in \overline{S} . The morphisms of \mathcal{P} are inclusions $P \rightarrow Q$ of subobjects of Y which are contained in \overline{S} . Because \overline{S} is weakly saturated, the hypotheses of Zorn's lemma apply to give a maximal element M of \mathcal{P} . Since $X \subseteq Y$ is assumed to be a categorical equivalence, 2-out-of-3 gives that $M \subseteq Y$ is a categorical equivalence.

If $M = Y$ we are done, so suppose $M \neq Y$. Then there exists a κ -small $A \subseteq Y$ not contained in M , which by the above lemma can be chosen so that $A \cap M \subseteq A$ is a categorical equivalence,

and thus an element of S . The pushout $M \subseteq A \cup M$ of this map is thus in \overline{S} contradicting the maximality of M . \square

In particular, we learn that every map can be factored into an injective categorical equivalence followed by a categorical fibration.

42. THE JOYAL MODEL STRUCTURE ON SIMPLICIAL SETS

42.1. Model categories. A **model category** (in the sense of Quillen) is a category \mathcal{M} with three classes of maps: W , Cof , Fib , conventionally called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying the following axioms.

- \mathcal{M} has all small limits and colimits.
- W satisfies the 2-out-of-3 property.
- $(Cof \cap W, Fib)$ and $(Cof, Fib \cap W)$ are weak factorization systems (13.12).

Conventionally, an object X is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration.

42.2. Remark. The third axiom implies that Cof , $Cof \cap W$, Fib , and $Fib \cap W$ are closed under retracts.

42.3. Exercise. Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* construct a factorization of f which is itself a retract of a factorization of g ³⁰.

42.4. Exercise (Slice model categories). Let \mathcal{M} be a model category, and let X be an object of \mathcal{M} . Show that the slice categories $\mathcal{M}_{X/}$ and $\mathcal{M}_{/X}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \rightarrow \mathcal{M}$ or $\mathcal{M}_{X/} \rightarrow \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in \mathcal{M} .

42.5. The Joyal model category.

42.6. Theorem (Joyal). *The category of simplicial sets admits a model structure, in which*

- $W = \text{categorical equivalences (CatEq)}$,
- $Cof = \text{monomorphisms } (\overline{\text{Cell}})$,
- $Fib = \text{categorical fibrations (CatFib)}$.

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

Proof. Categorical equivalences satisfy 2-out-of-3 by (22.10). We have that

- $Cof = \overline{\text{Cell}}$ by definition,
- $Fib \cap W = \text{TFib} = \text{Cell}^{\square}$ by (41.1),
- $Cof \cap W = \overline{S}$ for some set S (41.4),
- $Fib = \text{CatFib} = (Cof \cap W)^{\square} = S^{\square}$ by definition,

so both $(Cof \cap W, Fib)$ and $(Cof, Fib \cap W)$ are weak factorization systems via the small object argument (13.10). Thus, we get a model category.

We have shown (37.13) that the categorical fibrations $p: C \rightarrow D$ with D a quasicategory are precisely the isofibrations. Applied when $D = *$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations. \square

42.7. Remark. It is a standard fact that a model category structure is uniquely determined by its cofibrations and fibrant objects. Thus, the Joyal model structure is the unique model structure on simplicial sets with $Cof = \text{monomorphisms}$ and with fibrant objects the quasicategories.

³⁰In many formulations of model categories, closure of weak equivalences under retracts is taken as one of the axioms. The formulation we use is described in Riehl, “A concise definition of a model category” [Rie09], which gives a solution to this exercise.

42.8. **Cartesian model categories.** Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed, with the following properties. Suppose $i: A \rightarrow B$ and $j: K \rightarrow L$ are cofibrations and $p: X \rightarrow Y$ is a fibration. Then

•

$$i \square j: (A \times L) \cup_{A \times K} (B \times K) \rightarrow B \times L$$

is a cofibration, and is in addition a weak equivalence if either i or j is also a weak equivalence, and

•

$$p \square j: \text{Map}(L, X) \rightarrow \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$$

is a fibration, and is in addition a weak equivalence if either j or p is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

42.9. **Proposition.** *The Joyal model structure is cartesian.*

Proof. This is just (41.2). □

43. THE KAN-QUILLEN MODEL STRUCTURE ON SIMPLICIAL SETS

A map $p: X \rightarrow Y$ is a **groupoidal fibration** if and only if $j \square p$ for all j which are monomorphisms and weak equivalences. I write GpdFib for the class of categorical fibrations.

43.1. **The Kan-Quillen model structure.**

43.2. **Theorem** (Cisinski). *The category of simplicial sets admits a model structure, in which*

- $W =$ weak equivalences (WkEq),
- $\text{Cof} =$ monomorphisms ($\overline{\text{Cell}}$),
- $\text{Fib} =$ groupoidal fibrations (GpdFib).

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.

Proof. This goes very much the same way as the Joyal model structure, and I won't spell it out in detail. First build a detecting functor F so that a map f is a weak equivalence iff $F(f)$ is a bijection; this is just as in the categorical equivalence case, except that we make use of functorial replacement $X \mapsto X_{\text{Kan}}$ by Kan complexes, rather than by quasicategories. Using this, we can show that $\overline{\text{Cell}} \cap \text{WkEq} = \overline{S}$ and $\text{GpdFib} = S^{\square}$ for some set S , giving the factorization system $(\text{Cof} \cap W, \text{Fib})$.

We know that trivial fibrations are weak equivalences and are certainly groupoidal fibrations. The converse is proved just as in the categorical fibration case (41.1). This gives the other factorization system $(\text{Cof}, \text{Fib} \cap W)$.

We have already proved that Kan fibrations between Kan complexes have the lifting property of groupoidal fibrations (34.16), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case. □

43.3. **Proposition.** *The Quillen model structure is cartesian.*

Proof. We must show that $p \square j$ is a groupoidal fibration if j is a monomorphism and p a groupoidal fibration, and also that it is a weak equivalence if either j or p is. This is proved by an argument nearly identical to the proof of (41.2). □

43.4. **Kan fibrations are groupoidal fibrations.** The above model structure was actually first produced by Quillen. In Quillen’s formulation, the fibrations were taken to be the *Kan fibrations*. In fact, this is the same model structure, by

43.5. **Proposition** (Quillen). $\text{KanFib} = \text{GpdFib}$.

We will not give a proof of this here. The non-trivial part is to show that $\text{KanFib} \subseteq \text{GpdFib}$; note that we already know that a Kan fibration *between Kan complexes* is a groupoidal fibration by (34.16). This proposition is usually proved via an argument (due to Quillen) based on the theory of *minimal fibrations*. See for instance Quillen’s original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that KanFib is the weak cosaturation of the class of *Kan fibrations between Kan complexes*. In fact one can even show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

44. MODEL CATEGORIES AND HOMOTOPY COLIMITS

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of equivalence.

44.1. **Creating new model categories.** Given a model category \mathcal{M} , many other categories related to it can also be equipped with model category structures, such as functor categories $\text{Fun}(C, \mathcal{M})$ where C is a small category. We won’t consider general formulations of this here, but we will consider some special cases.

As an example, we consider the case of $C = [1] = \{0 \xrightarrow{01} 1\}$.

44.2. **Proposition.** *There exists a model structure on $\mathcal{N} := \text{Fun}([1], \mathcal{M})$ in which a map $\alpha: X \rightarrow X'$ is*

- a weak equivalence if $\alpha(i): X(i) \rightarrow X'(i)$ is a weak equivalence in \mathcal{M} for $i = 0, 1$
- a cofibration if $\alpha(i)$ is a cofibration in \mathcal{M} for $i = 0, 1$, and
- a fibration in \mathcal{M} if both $\alpha(1)$ and the map $(X(01), \alpha(0)): X(0) \rightarrow X(1) \times_{X'(1)} X'(0)$ are fibrations in \mathcal{M} .

Proof. It is clear that \mathcal{N} has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that $(\text{Cof} \cap W, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap W)$ are weak factorization systems.

Let $\alpha: X \rightarrow X'$ be a map in \mathcal{N} . Choose

- a factorization $X(1) \rightarrow Y(1) \rightarrow X'(1)$ of $\alpha(1)$ of the form $\text{Fib} \circ \text{Cof} \cap W$, and
- a factorization $X(0) \rightarrow Y(0) \rightarrow X'(0) \times_{X'(1)} Y(1) \dots$

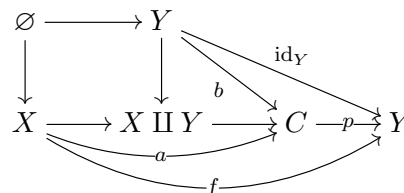
...

□

44.3. **Reedy lemma.**

44.4. **Proposition** (Reedy lemma). *Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a functor between model categories which takes trivial cofibrations to weak equivalences. Then F takes weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f: X \rightarrow Y$ be a weak equivalence between cofibrant objects in \mathcal{M} . Form the commutative diagram



where the square is a pushout, and we have chosen a factorization $X \amalg Y \rightarrow C \rightarrow Y$ into a cofibration followed by a weak equivalence (e.g., a trivial fibration). Because X and Y are cofibrant, the maps $X \rightarrow X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-of-3 property, we see that a and b are trivial cofibrations. Applying F gives

$$\begin{array}{ccccc}
 & & F(Y) & & \\
 & & \downarrow & \searrow \text{id} & \\
 & & F(b) & & \\
 & & \downarrow & & \\
 F(X) & \xrightarrow{F(a)} & F(C) & \xrightarrow{F(p)} & F(Y)
 \end{array}$$

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(b)$ is a weak equivalence by 2-of-3, and therefore $F(f) = F(p)F(b)$ is a weak equivalence, as desired. \square

The opposite of a model category is also a model category, by switching the roles of fibrations and cofibrations. Thus, there is a dual formulation of the Reedy lemma.

44.5. Proposition (Reedy lemma, dual form). *Let $G: \mathcal{N} \rightarrow \mathcal{M}$ be a functor between model categories which takes trivial fibrations to weak equivalences. Then G takes weak equivalences between fibrant objects to weak equivalences.*

44.6. Quillen pairs. Given an adjoint pair of functors $F: \mathcal{M} \rightleftarrows \mathcal{N} : G$ between model categories, we see from the properties of weak factorization systems that

- F preserves cofibrations if and only if G preserves trivial fibrations, and
- F preserves trivial cofibrations if and only if G preserves fibrations.

If both of these are true, we say that (F, G) is a **Quillen pair**.

Note that if (F, G) is a Quillen pair, then the Reedy lemma (44.4) applies to F , and the dual form of the Reedy lemma (44.5) applies to G .

44.7. Good colimits. We can apply the above to certain examples of colimit functors, which we will refer to generically as “good colimits”. There are three types of these: arbitrary coproducts of cofibrant objects, countable sequential colimits of cofibrant objects along cofibrations, and pushouts of cofibrant objects along a cofibration. We will show that “good colimits are weak equivalence invariant”.

44.8. Exercise. Let S be a small discrete category (i.e., all maps are identities). Show that if \mathcal{M} is a model category, then $\text{Fun}(S, \mathcal{M})$ is a model category in which $\alpha: X \rightarrow X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s: X_s \rightarrow X'_s$ is one in \mathcal{M} .

Then show that $\text{colim}: \text{Fun}(S, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$ is a Quillen pair, and use this to prove the next proposition.

44.9. Proposition. *Given a collection $f_s: X_s \rightarrow X'_s$ of weak equivalences between cofibrant objects in \mathcal{M} , the induced map $\amalg f_s: \amalg X_s \rightarrow \amalg X'_s$ is a weak equivalence.*

44.10. Exercise. Let V be the category

$$0 \xleftarrow{10} 1 \xrightarrow{12} 2.$$

Show that if \mathcal{M} is a model category, then $\text{Fun}(V, \mathcal{M})$ is a model category in which $\alpha: X \rightarrow X'$ is

- a weak equivalence if $\alpha(i): X(i) \rightarrow X'(i)$ is a weak equivalence for $i = 0, 1, 2$ (i.e., an **objectwise weak equivalence**), and is
- a cofibration if $\alpha(0)$ and $\alpha(1)$ are cofibrations, and the evident map $X(2) \cup_{X(1)} X'(1) \rightarrow X'(2)$ is a cofibration.

Then show that $\text{colim}: \text{Fun}(V, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$ is a Quillen pair, and use this to prove the next proposition. (Hint: determine what the fibrations in $\text{Fun}(V, \mathcal{M})$ must be according to the lifting property.)

44.11. Proposition. *Given a natural transformation $\alpha: X \rightarrow X'$ of functors $V \rightarrow \mathcal{M}$, i.e., a diagram*

$$\begin{array}{ccccc} X(0) & \longleftarrow & X(1) & \xrightarrow{X(12)} & X(2) \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ X'(0) & \longleftarrow & X'(1) & \xrightarrow{X'(12)} & X'(2) \end{array}$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(12)$ and $X'(12)$ are cofibrations, the induced map $\text{colim}_V X \rightarrow \text{colim}_V X'$ is a weak equivalence.

44.12. Exercise. Let ω be the category

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$$

with objects indexed by natural numbers. Show that if \mathcal{M} is a model category, then $\text{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha: X \rightarrow X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence,
- a cofibration if (i) $\alpha(0)$ is a cofibration, and $X'(i) \cup_{X(i)} X(i+1) \rightarrow X'(i+1)$ is a cofibration for all $i \geq 0$.

Then show that $\text{colim}: \text{Fun}(\omega, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$ is a Quillen pair, and use this to prove the next proposition.

44.13. Proposition. *Give a natural transformation $\alpha: X \rightarrow X'$ of functors $\omega \rightarrow \mathcal{M}$ such that all maps $\alpha(i): X(i) \rightarrow X'(i)$ are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(i) \rightarrow X(i+1)$ and $X'(i) \rightarrow X'(i+1)$ are cofibrations, the induced map $\text{colim}_\omega X \rightarrow \text{colim}_\omega X'$ is a weak equivalence.*

In the Joyal and Quillen model structures, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (44.9), (44.11), (44.13) a **good colimit**. Thus, we see that good colimits are homotopy invariant. These “good colimits” are examples of what are called *homotopy colimits*.

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant.

44.14. Exercise. State and prove the dual versions of all the results in this section.

45. A SIMPLICIAL SET IS WEAKLY EQUIVALENT TO ITS OPPOSITE

Recall the opposite construction $X \mapsto X^{\text{op}}$ on general simplicial sets. We will now prove the following, which is a kind of a generalization of the fact that every groupoid is isomorphic to its own opposite.

45.1. Proposition. *Every simplicial set X is weakly equivalent to its opposite, in the sense that there exists a zig-zag of weak equivalences connecting X and X^{op} .*

Note that for any ordinary groupoid G , it is straightforward to construct an isomorphism $G \rightarrow G^{\text{op}}$, by sending each map to its inverse. Unfortunately, this proof cannot be replicated in our setting, even if we assume that X is a quasigroupoid.

45.2. *Remark.* Here is one possible proof (in some sense, the most natural proof). Note that there is a *homeomorphism* of geometric realizations $|X| \approx |X^{\text{op}}|$. Then use the fact that geometric realization induces an equivalence $h(\text{sSet}, \text{WkEq}) \approx h(\text{Top}, \text{WkEq})$. Of course, we haven't actually proved this fact about homotopy categories yet.

I'll give some different proofs which are internal to simplicial sets. I do this in part because it's interesting to see how this is done, but also because it allows me to set up some technology which will be useful later.

45.3. **General “singular” and “realization” functors.** Consider a functor $C: \Delta \rightarrow \mathcal{A}$ into some category \mathcal{A} , i.e., a **cosimplicial object** in \mathcal{A} . Often we write $C^n \in \mathcal{A}$ instead of $C([n])$ for the values of this functor, and so write C^\bullet for the whole functor. In most of our examples, we will actually have $\mathcal{A} = \text{sSet}$, so C^\bullet will be a cosimplicial simplicial set.

Given such a C^\bullet , **realization** and **singular** functors are an adjoint pair

$$\text{Re} = \text{Re}_{C^\bullet}: \text{sSet} \rightleftarrows \mathcal{A} : \text{Si}_{C^\bullet} = \text{Si}$$

associated to C^\bullet . The singular functor Si is always defined, and is defined by

$$(\text{Si } X)_n := \text{Hom}_{\text{sSet}}(C([n]), X),$$

with simplicial operators induced by the fact that C is a functor on Δ . The left adjoint Re is defined if \mathcal{A} is cocomplete³¹.

45.4. *Exercise.* Show that if \mathcal{A} is cocomplete, and $X \in \text{sSet}$, then

$$\text{Re } X \approx \text{Cok} \left[\coprod_{f: [m] \rightarrow [n]} \coprod_{x \in X_n} C^m \rightrightarrows \coprod_{[p]} \coprod_{x \in X_p} C^p \right].$$

(Part of the exercise is to figure out what the two maps are.)

45.5. *Example.* For the cosimplicial space $\Delta_{\text{top}}^\bullet: \Delta \rightarrow \text{Top}$ taking $[n]$ to the topological n -simplex, $\text{Re}_{\Delta_{\text{top}}^\bullet}$ and $\text{Si}_{\Delta_{\text{top}}^\bullet}$ are just the usual geometric realization and singular complex functors.

45.6. *Example.* For the tautological functor $\Delta^\bullet: \Delta \rightarrow \text{sSet}$ sending $[n] \mapsto \Delta^n$, the functors $\text{Re}_{\Delta^\bullet}$ and $\text{Si}_{\Delta^\bullet}$ are isomorphic to the identity functor on simplicial sets.

45.7. *Example.* For the functor $(\Delta^\bullet)^{\text{op}} = \Delta^\bullet \circ \text{op}: \Delta \rightarrow \text{sSet}$, the functors $\text{Re}_{\Delta^\bullet \circ \text{op}}$ and $\text{Si}_{\Delta^\bullet \circ \text{op}}$ are both isomorphic to the *opposite* functor $X \mapsto X^{\text{op}}$.

45.8. **The cosimplicial object Iso^\bullet .** Consider $\text{Iso}^\bullet: \Delta \rightarrow \text{sSet}$, where Iso^n is the (nerve of the) groupoid with objects $\{0, 1, \dots, n\}$ and unique isomorphisms between every object. This has corresponding realization and singular functors

$$\text{Re} = \text{Re}_{\text{Iso}^\bullet}: \text{sSet} \rightarrow \text{sSet} : \text{Si}_{\text{Iso}^\bullet} = \text{Si}.$$

There is an evident natural transformation $\alpha: \Delta^\bullet \rightarrow \text{Iso}^\bullet$, by the unique functors $[k] \rightarrow \text{Iso}^k$ sending object j to j . This gives an adjoint-related pair of natural transformations

$$\eta_X: X \rightarrow \text{Re}_{\text{Iso}^\bullet} X, \quad \epsilon_X: \text{Si}_{\text{Iso}^\bullet} X \rightarrow X.$$

If we precompose with op , we obtain a cosimplicial object $\text{Iso}^\bullet \circ \text{op} = (\text{Iso}^\bullet)^{\text{op}}$. One sees that there are natural isomorphisms

$$(45.9) \quad \text{Re}_{\text{Iso}^\bullet \circ \text{op}} \approx (\text{Re}_{\text{Iso}^\bullet})^{\text{op}}, \quad \text{Si}_{\text{Iso}^\bullet \circ \text{op}} \approx (\text{Si}_{\text{Iso}^\bullet})^{\text{op}},$$

and that the evident natural transformation $\alpha \circ \text{op}: \Delta^\bullet \circ \text{op} \rightarrow \text{Iso}^\bullet \circ \text{op}$ gives rise to an adjoint-related pair of natural transformations

$$\eta'_X: X^{\text{op}} \rightarrow \text{Re}_{\text{Iso}^\bullet \circ \text{op}} X, \quad \epsilon'_X: \text{Si}_{\text{Iso}^\bullet \circ \text{op}} X \rightarrow X^{\text{op}}.$$

³¹This means that \mathcal{A} has colimits for all functors $F: B \rightarrow \mathcal{A}$ from small categories B .

Furthermore, under the isomorphisms of (45.9) the maps η'_X and ϵ'_X are identified with $\eta_{X^{\text{op}}}$ and $\epsilon_{X^{\text{op}}}$.

45.10. *Exercise.* Prove the statements of the previous paragraph.

Although Δ^\bullet and $\Delta^\bullet \circ \text{op}$ are not isomorphic as functors $\Delta \rightarrow \text{sSet}$, it is the case that $\text{Iso}^\bullet \approx \text{Iso}^\bullet \circ \text{op}$, using the isomorphisms of categories $\text{Iso}^n \rightarrow \text{Iso}^n$ given on objects by $x \mapsto n - x$. Putting all this together, we obtain natural transformations

$$X \xrightarrow{\eta_X} \text{Re}_{\text{Iso}^\bullet} X \approx \text{Re}_{\text{Iso}^\bullet} X^{\text{op}} \xleftarrow{\eta_{X^{\text{op}}}} X^{\text{op}}, \quad X \xleftarrow{\epsilon_X} \text{Si}_{\text{Iso}^\bullet} X \approx \text{Si}_{\text{Iso}^\bullet} X^{\text{op}} \xrightarrow{\epsilon_{X^{\text{op}}}} X^{\text{op}}.$$

We'll show that that η_X , and hence $\eta_{X^{\text{op}}}$, are always weak equivalences, and that ϵ_X , and hence $\epsilon_{X^{\text{op}}}$, are weak equivalences whenever X is a Kan complex. In the following, $\text{Re} = \text{Re}_{\text{Iso}^\bullet}$ and $\text{Si} = \text{Si}_{\text{Iso}^\bullet}$.

45.11. **Lemma.** *For each monomorphism $K \rightarrow L$, the induced map $(\text{Re } K) \amalg_K L \rightarrow \text{Re } L$ is a monomorphism. In particular,*

- *Re preserves monomorphisms and Si preserves trivial fibrations, and*
- *$\eta_L: L \rightarrow \text{Re } L$ is always a monomorphism.*

Proof. Formally, it is enough to check the case of $\partial\Delta^n \subset \Delta^n$. To see this, check that the lifting problems

$$\begin{array}{ccc} (\text{Re } K) \amalg_K L & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \text{Re } L & \longrightarrow & Y \end{array} \iff \begin{array}{ccc} K & \longrightarrow & \text{Si } X \\ \downarrow & \nearrow & \downarrow \\ L & \longrightarrow & (\text{Si } Y) \times_Y X \end{array}$$

are equivalent. This means we need to show that all monomorphisms are contained in the weakly saturated class $\square\mathcal{C}$, where \mathcal{C} is the class of all the maps $(\text{Si } p, \epsilon_X): \text{Si } X \rightarrow (\text{Si } Y) \times_Y X$ such that $p \in \text{TrivFib}$, which means we only need to show that Cell is contained in it.

This is a calculation: $\text{Re}(\partial\Delta^n) \rightarrow \text{Re}(\Delta^n) = N(\text{Iso}^n)$ is isomorphic to inclusion of the subcomplex $K \subseteq N(\text{Iso}^n)$ whose k -dimensional elements are sequences $x_0 \rightarrow \dots \rightarrow x_k$ such that $\{x_0, \dots, x_k\} \neq \{0, \dots, n\}$. To show this, use the fact that Δ^n is a colimit of its $(n - 1)$ -dimensional faces along their intersections, and that Re preserves colimits. The image of the element $(0, 1, \dots, n)$ in $\text{Re}(\Delta^n)$ intersects K exactly in its boundary, so $(\text{Re } \partial\Delta^n) \cup_{\partial\Delta^n} \Delta^n \rightarrow \text{Re } \Delta^n$ is a monomorphism as desired. \square

45.12. *Remark.* Given any natural transformation $\lambda: F \rightarrow G$ of functors, and map $f: X \rightarrow Y$, we get induced maps

$$F(Y) \cup_{F(X)} G(X) \rightarrow G(Y), \quad F(X) \rightarrow F(Y) \times_{G(Y)} G(X).$$

These can be thought of as a variant of the “box” construction we’ve considered elsewhere (26.5), but associated to the “evaluation pairing” $\text{Fun}(\text{sSet}, \text{sSet}) \times \text{sSet} \rightarrow \text{sSet}$ rather than a functor $\text{sSet} \times \text{sSet} \rightarrow \text{sSet}$.

45.13. **Skeletal induction.** The next step is to show that $K \rightarrow \text{Re } K$ is a weak equivalence for every simplicial set K . To do this, we will use the following bit of machinery.

45.14. **Proposition** (Skeletal induction). *Let \mathcal{C} be a class of simplicial sets with the following properties.*

- (1) *If $X \in \mathcal{C}$, then every object isomorphic to X is in \mathcal{C} .*
- (2) *Every $\Delta^n \in \mathcal{C}$.*
- (3) *The class \mathcal{C} is closed under good colimits. That is:*
 - (a) *any coproduct of objects of \mathcal{C} is in \mathcal{C} ;*

- (b) any pushout of a diagram $X_0 \leftarrow X_1 \rightarrow X_2$ of objects in \mathcal{C} along a monomorphism $X_1 \rightarrow X_2$ is in \mathcal{C} ;
- (c) any colimit of a countable sequence $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ of objects in \mathcal{C} , such that each $X_k \rightarrow X_{k+1}$ is a monomorphism, is in \mathcal{C} .

Then \mathcal{C} is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (15.23). To show $X \in \mathcal{C}$, it suffices to show each $\text{Sk}_n X \in \mathcal{C}$ by (3c). So we show that all n -skeleta are in \mathcal{C} by induction on n , with base case $n = -1$ (the empty simplicial set), which is really a special case of (3a). Since $\text{Sk}_{n-1} X \subseteq \text{Sk}_n X$ is a pushout along a coproduct of maps $\partial\Delta^n = \text{Sk}_{n-1} \Delta^n \rightarrow \Delta^n$, this follows using (2), (3a), (3b), and the inductive hypothesis. \square

45.15. Proposition. *For every simplicial set X , the map $X \rightarrow \text{Re } X$ is a weak equivalence.*

Proof. Let \mathcal{C} be the class of X such that $\eta: X \rightarrow \text{Re } X$ is a weak equivalence. We verify the hypotheses of the above proposition. Property (1) is obvious.

To prove property (2) recall that $\eta_{\Delta^1}: \Delta^1 \rightarrow \text{Iso}^1$ is anodyne (33.15). We can identify η_{Δ^n} as a retract of $(\eta_{\Delta^1})^{\times n}: (\Delta^1)^{\times n} \rightarrow (\text{Iso}^1)^{\times n}$, which is necessarily anodyne since a product of an anodyne map with any identity map is anodyne. The retractions $\Delta^n \xrightarrow{f} (\Delta^1)^{\times n} \xrightarrow{g} \Delta^n$ and $\text{Iso}^n \xrightarrow{f} (\text{Iso}^1)^{\times n} \xrightarrow{g} \text{Iso}^n$ are maps which are given on vertices by

$$f(k) = (\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{n-k}), \quad g(k_1, \dots, k_n) = \max \{ j \mid k_j = 1 \}.$$

Property (3) involves colimits, which in every case are *good colimits*. In each case, we need to show that a map $\text{colim } X_i \rightarrow \text{Re } \text{colim } X_i$ is a weak equivalence when each $X_i \rightarrow \text{Re } X_i$ is. The functor Re preserves colimits and monomorphisms (45.11), so in every case we are comparing good colimits, so the result follows from (44.9), (44.11), (44.13). \square

We thus obtain the desired result.

45.16. Corollary. *Every simplicial set is weakly equivalent to its opposite X^{op} .*

Proof. Both maps in $X \xrightarrow{\eta_X} \text{Re } X \approx \text{Re } X^{\text{op}} \xleftarrow{\eta_{X^{\text{op}}}} X^{\text{op}}$ are weak equivalences (45.15). \square

45.17. Proposition. *For each monomorphism $K \rightarrow L$, the induced map $(\text{Re } K) \amalg_K L \rightarrow \text{Re } L$ is a monomorphism and a weak equivalence.*

Proof. Both squares

$$\begin{array}{ccc} K & \xrightarrow{\eta} & \text{Re } K \\ \downarrow & & \downarrow \\ L & \xrightarrow{\eta} & (\text{Re } K) \amalg_K L \end{array} \quad \begin{array}{ccc} \text{Re } K & \xrightarrow{\text{id}} & \text{Re } K \\ \downarrow & & \downarrow \\ \text{Re } L & \xrightarrow{\text{id}} & \text{Re } L \end{array}$$

are good pushouts, using (45.11). The evident map from the left square to the right square is a weak equivalence at the upper left, upper right, and lower left corners (45.15), so the result follows from the invariance of good pushouts (44.11). \square

45.18. Corollary. *If $p: X \rightarrow Y$ is a Kan fibration, then $\text{Si } X \rightarrow \text{Si } Y \times_Y X$ is a trivial fibration. In particular, if X is a Kan complex, then $\text{Si } X \rightarrow X$ is a trivial fibration.*

In particular, for any Kan complex X , both maps in $X \xleftarrow{\epsilon_X} \text{Si } X \approx \text{Si } X^{\text{op}} \xrightarrow{\epsilon_{X^{\text{op}}}} X^{\text{op}}$ are trivial fibrations.

Proof. Straightforward, using (45.17). \square

45.19. Remark. The object $\text{Re } X$ is not generally categorically equivalent to X .

It can be shown that if C is a quasicategory, then $\text{Si } C$ is categorically equivalent to C^{core} .

46. INITIAL AND TERMINAL OBJECTS, REVISITED

Recall the definition of initial and terminal objects in a quasicategory. One characterization was: x is an initial object of C iff the left fibration $p: C_{x/} \rightarrow C$ is a trivial fibration, and a terminal object iff the right fibration $p': C_{/x} \rightarrow C$ is a trivial fibration.

When C is the nerve of an ordinary category, these reduce to the usual definitions of initial and terminal object. In this case, there is an equivalent characterization: x is initial if and only if $\text{Hom}_C(x, y)$ is a singleton set for all objects y of C , and terminal if and only if $\text{Hom}_C(y, x)$ is a singleton set for all y .

We would like to generalize this to the case of quasicategories.

F. Deferred Proposition. *An object x of a quasicategory is initial if and only if $\text{map}_C(x, c)$ is contractible for all objects c of C , and terminal if and only if $\text{map}_C(c, x)$ is contractible for all objects c of C .*

To prove this, you need to be able to relate mapping spaces of a quasicategory to the join/slice constructions that we used to define initial and terminal. We will establish such a relation in the next few sections.

46.1. Right and left mapping spaces. Let x, y be objects of a quasicategory C . We define the **right mapping space** $\text{map}_C^R(x, y)$ and **left mapping space** $\text{map}_C^L(x, y)$ by pullback diagrams

$$\begin{array}{ccc} \text{map}_C^R(x, y) & \longrightarrow & C_{x/} \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow{y} & C \end{array} \qquad \begin{array}{ccc} \text{map}_C^L(x, y) & \longrightarrow & C_{/y} \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow{x} & C \end{array}$$

where the maps labelled π are the evident forgetful maps.

For instance, an n -dimensional element of $\text{map}_C^R(x, y)$ is precisely a map $a: \Delta^{n+1} \rightarrow C$ such that $a|\Delta^{\{0, \dots, n\}}$ represents the vertex x , and $a|\Delta^{\{n+1\}} = y$. In particular, a vertex of $\text{map}_C^R(x, y)$ is a morphism $x \rightarrow y$ in C , while an edge of $\text{map}_C^R(x, y)$ is a 2-dimensional element in C exhibiting the \sim_r relation between two maps, which we used to define the homotopy category in §9.

Recall (26.15) that when C is a quasicategory, the maps $C_{x/} \rightarrow C$ and $C_{/y} \rightarrow C$ are left fibrations and right fibrations respectively. Thus both $\text{map}_C^R(x, y)$ and $\text{map}_C^L(x, y)$ are Kan complexes, by the following.

46.2. Exercise. Show that if $X \rightarrow \Delta^0$ is a left fibration or a right fibration, then X is a Kan complex. (Hint: Joyal lifting.)

Furthermore, by the above remarks relating edges in the right and left mapping spaces to the homotopy relation, we have that $\pi_0 \text{map}_C^R(x, y) \approx \pi_0 \text{map}_C^L(x, y) \approx \text{Hom}_{hC}(x, y)$.

We will show below that both $\text{map}_C^R(x, y)$ and $\text{map}_C^L(x, y)$ are actually *weakly equivalent* to the standard mapping space $\text{map}_C(x, y)$.

46.3. Box products and right and left anodyne maps. Recall that $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.7) and $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$. We have an analogous fact for left or right anodyne maps.

46.4. Proposition. *We have that $\overline{\text{LHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{LHorn}}$ and $\overline{\text{RHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{RHorn}}$.*

Proof. See **appendix**. □

46.5. Fiberwise criterion for trivial fibrations, revisited. Recall the fiberwise criterion for trivial fibrations (35.2): a Kan fibration p is a trivial fibration if and only if the fibers of p are contractible Kan complexes. In fact, this still holds if we only know p is a left or right fibration.

46.6. Proposition. *Suppose $p: X \rightarrow Y$ is a right fibration or left fibration of simplicial sets. Then p is a trivial fibration if and only if it has contractible fibers.*

Proof. [Lur09, 2.1.3.4]. Let's consider the case of $p: X \rightarrow Y$ a left fibration. The direction (\implies) is immediate, so we only need to prove (\impliedby).

We attempt to carry out the argument of the proof of (35.2), and show that $(\partial\Delta^n \subset \Delta^n) \boxtimes p$ for all $n \geq 0$. The case of $n = 0$ is immediate, since the fibers of p must be non-empty, since they are contractible, so we can assume $n \geq 1$.

Examining that proof of (35.2), we see that we used only the hypothesis that p is a Kan fibration in order to have that

$$(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1) \boxtimes p, \quad ((\partial\Delta^n \subset \Delta^n) \square (\{1\} \subset \Delta^1)) \boxtimes p.$$

In the first case, the inclusion $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is left anodyne by (46.4), so the lifting problem still has a solution when p is only a left fibration.

In the second case, we need to argue a little differently. In the proof of (35.2) this lifting problem appears when producing a lifting (for $n \geq 1$) in a diagram of the form

$$\begin{array}{ccc} (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(c,jd)} & X \\ \downarrow & \dashrightarrow^s & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y \end{array}$$

Pulling back along the factorization of the bottom map $b\gamma$, we obtain a diagram

$$\begin{array}{ccccc} (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{\quad} & C & \longrightarrow & X \\ \downarrow & \dashrightarrow^{s'} & \downarrow p' & & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{\quad \gamma \quad} & \Delta^n & \xrightarrow{b} & Y \end{array}$$

where the right-hand square is a pullback. Observe that (i) p' is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) the map γ (as defined in the proof of (35.2)) sends the edge $\{n\} \times \Delta^1$ to the degenerate edge $\langle nn \rangle$ in Δ^n . Therefore we can apply the pushout-product version of Joyal lifting (30.3) to produce a lift s' . \square

46.7. Corollary. *An object x of a quasicategory C is initial if and only if $\text{map}_C^R(x, c)$ is contractible for all objects c of C , and is final if and only if $\text{map}_C^L(c, x)$ is contractible for all objects c of C .*

Proof. The fibers of the left fibration $C_{x/} \rightarrow C$ are precisely the right mapping spaces $\text{map}_C^R(x, c)$. By what we just proved (46.6) these fibers are all contractible if and only if $C_{x/} \rightarrow C$ is a trivial fibration, which we have noted (25.5) is equivalent to x being initial in C . \square

47. THE ALTERNATE JOIN AND SLICE

We now want to compare the right and left mapping spaces, which are fibers of the projections $C_{x/} \rightarrow C$ and $C_{/x} \rightarrow C$, to the ordinary mapping spaces, which are fibers of $\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\partial\Delta^1, C)$. We do this using constructions called the ‘‘alternate join’’ and ‘‘alternate slice’’ [Lur09, §4.2.1].

Given an object x in C , consider the map

$$q: \text{Fun}(\Delta^1, C) \times_{\text{Fun}(\{0\}, C)} \{x\} \rightarrow \text{Fun}(\{1\}, C) = C$$

induced by restriction along $\{0\} \subset \Delta^1$. Note that the fiber of q over some object c of C is precisely the quasigroupoid $\text{map}_C(x, c)$. The domain of q is an example of what we will call the ‘‘alternate slice’’ construction, for which we will use the (unmemorable) notation $C^{x/}$.

47.1. *Exercise.* Show that if C is an ordinary category, then $C^{x/}$ is *isomorphic* to the usual slice category $C_{x/}$, and q is *isomorphic* to the usual projection $p: C_{x/} \rightarrow C$.

For a general quasicategory, q is *not* isomorphic p . What is true is that there is a commutative diagram

$$\begin{array}{ccc} C_{x/} & \xrightarrow{f} & C^{x/} = \text{Fun}(\Delta^1, C) \times_{\text{Fun}(\{0\}, C)} \{x\} \\ & \searrow p & \nearrow q \\ & & C \end{array}$$

The map f sends an element $a: \Delta^k \rightarrow C_{/x}$, which corresponds to $\tilde{a}: \Delta^{k+1} \rightarrow C$ such that $\tilde{a}_0 = x$, to an element in $C^{x/}$ corresponding to $\tilde{a}r: \Delta^k \times \Delta^1 \rightarrow C$, where $r: \Delta^k \times \Delta^1 \rightarrow \Delta^{k+1}$ is the unique map given on vertices by $r(i, 0) = 0$, $r(i, 1) = i + 1$.

The characterization (F) of initial objects in terms of contractible mapping spaces thus amounts to the claim that p is a trivial fibration if and only if q has contractible fibers. In fact, we'll prove that

- both p and q are left fibrations,
- f is a categorical equivalence.

Because p and q are left fibrations, they are trivial fibrations iff their fibers are contractible (46.6). Because f is a categorical equivalence, p is a categorical equivalence if and only if q is by 2-out-of-3 (22.10). The result follows because p and q are in particular isofibrations (28.10), and an isofibration is a trivial fibration if and only if it is a categorical equivalence (37.10).

In other words, we can regard $C^{x/}$ as an alternate version of the slice construction, so we call it the “alternate slice”. It is related to an alternate version of the join, denoted $X \diamond Y$ and called the “alternate join”, which we define first.

47.2. **The alternate join.** Given simplicial sets X and Y , define the **alternate join** by the pushout diagram

$$\begin{array}{ccc} (X \times \{0\} \times Y) \amalg (X \times \{1\} \times Y) & \xrightarrow{\quad} & X \times \Delta^1 \times Y \\ \downarrow & & \downarrow \\ (X \times \{0\} \times \Delta^0) \amalg (\Delta^0 \times \{1\} \times Y) & \xrightarrow{\quad} & X \diamond Y \end{array}$$

where the maps on top and left are induced by the evident inclusion and projection maps.

The alternate join comes with a natural comparison map

$$X \diamond Y \rightarrow X \star Y,$$

defined as follows. Using the recipe of (23.12) for constructing maps to a join, we get a map $X \times \Delta^1 \times Y \rightarrow X \star Y$ corresponding to the triple $(g, f^{(0)}, f^{(1)})$, where $g: X \times \Delta^1 \times Y \rightarrow \Delta^1$, $f^{(0)}: X \times \{0\} \times Y \rightarrow X$, and $f^{(1)}: X \times \{1\} \times Y \rightarrow Y$ are the evident projections. A similar procedure produces compatible maps to $X \star Y$ from the other vertices of the pushout square defining $X \diamond Y$. Note that the comparison map induces a bijection on vertices.

47.3. *Example.* We have

$$X \diamond \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \quad \Delta^0 \diamond Y \approx (\Delta^1 \times Y)/(\{0\} \times Y),$$

simplicial sets obtained by collapsing subcomplexes to a single point. These come with evident maps $X \diamond \Delta^0 \rightarrow X^\triangleright$ and $\Delta^0 \diamond Y \rightarrow Y^\triangleleft$.

Like the true join, $X \diamond \emptyset \approx X \approx \emptyset \diamond X$, and the functors $X \diamond -: \mathbf{sSet} \rightarrow \mathbf{sSet}_{X/}$ and $- \diamond Y: \mathbf{sSet} \rightarrow \mathbf{sSet}_{Y/}$ commute with colimits.

Warning: when X and Y are non-empty, $X \times \Delta^1 \times Y \rightarrow X \diamond Y$ is surjective, but this is not the case when either X or Y are empty.

Unlike the true join, the alternate join is not monoidal: $(X \diamond Y) \diamond Z \not\approx X \diamond (Y \diamond Z)$ in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

The alternate join is a categorically invariant construction.

47.4. Proposition. *The alternate join \diamond preserves categorical equivalences in either variable. That is, if $Y \rightarrow Y'$ is a categorical equivalence, then so are $X \diamond Y \rightarrow X \diamond Y'$ and $Y \diamond Z \rightarrow Y' \diamond Z$.*

Proof. The \diamond product is constructed using a “good” pushout, i.e., a pushout along a cofibration. The result follows because both products and good pushouts preserve categorical equivalences (44.11). \square

47.5. Alternate slice. Given $p: S \rightarrow X$ and $q: T \rightarrow X$, we define the **alternate slices** $X^{p/}$ and $X^{/q}$ via the bijective correspondences

$$\left\{ \begin{array}{ccc} S \diamond \emptyset & & \\ \downarrow & \searrow p & \\ S \diamond K & \dashrightarrow & X \end{array} \right\} \iff \{K \dashrightarrow X^{p/}\}, \quad \left\{ \begin{array}{ccc} \emptyset \diamond T & & \\ \downarrow & \searrow q & \\ K \diamond T & \dashrightarrow & X \end{array} \right\} \iff \{K \dashrightarrow X^{/q}\}.$$

just as we defined ordinary slices using joins. These constructions give right adjoints to the alternate join functors:

$$S \diamond (-): s\text{Set} \rightleftarrows s\text{Set}_{S/} : (p \mapsto X^{p/}), \quad (-) \diamond T: s\text{Set} \rightleftarrows s\text{Set}_{T/} : (q \mapsto X^{/q}).$$

Alternate slices are “functorial” in exactly the sense that ordinary slices are (24.13): a sequence of maps $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$ induces $X^{p/} \rightarrow Y^{fpj/}$ and $X^{/p} \rightarrow Y^{/fpj}$.

47.6. Exercise. Show that there are pullback squares of the form

$$\begin{array}{ccc} X^{p/} & \longrightarrow & \text{Map}(S \times \Delta^1, X) \\ \downarrow & & \downarrow \\ X & \xrightarrow{(\tilde{p}, c)} & \text{Map}(S \times \{0\}, X) \times \text{Map}(S \times \{1\}, X) \end{array} \quad \begin{array}{ccc} X^{/p} & \longrightarrow & \text{Map}(\Delta^1 \times S, X) \\ \downarrow & & \downarrow \\ X & \xrightarrow{(c, \tilde{p})} & \text{Map}(\{0\} \times S, X) \times \text{Map}(\{1\} \times S, X) \end{array}$$

where $\tilde{p}: X \rightarrow \text{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\text{proj}} S \xrightarrow{p} X$, and $c: X \rightarrow \text{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\text{proj}} X \xrightarrow{\text{id}} X$.

Using the adjunction relation between joins and slices, and alternate joins and slices, the natural comparison map $X \diamond Y \rightarrow X \star Y$ induces natural comparison maps on alternate slices. That is, given $p: S \rightarrow X$ and $q: T \rightarrow Y$ we have natural comparison maps

$$X_{p/} \rightarrow X^{p/} \quad \text{and} \quad Y_{/q} \rightarrow Y^{/q}.$$

47.7. Joins, slices, and function complexes. Recall the function complex $\text{Map}(X, Y) \in s\text{Set}$, defined for any pair of simplicial sets X, Y . Recall also (20.15) the relative function complex under S , which for objects $p: S \rightarrow X$ and $q: S \rightarrow Y$ in $s\text{Set}_{S/}$ is a simplicial set

$$\text{Map}_{S/}(X, Y) := \text{Map}(X, Y) \times_{\text{Map}(S, Y)} \{q\}$$

with bijective correspondences

$$\left\{ K \dashrightarrow \text{Map}_{S/}(X, Y) \right\} \iff \left\{ \begin{array}{ccc} K \times S & \xrightarrow{\text{proj}} & S \\ \text{id} \times p \downarrow & & \downarrow q \\ K \times X & \dashrightarrow & Y \end{array} \right\}$$

natural in the simplicial set K . The set of vertices of $\text{Map}_{S/}(X, Y)$ is precisely the set $\text{Hom}_{s\text{Set}_{S/}}(X, Y)$ of morphisms in the category $s\text{Set}_{S/}$.

Given X and $p: S \rightarrow Y$, we have join/slice adjunctions

$$\text{Hom}_{s\text{Set}}(X, Y_{p/}) \approx \text{Hom}_{s\text{Set}_{S/}}(S \star X, Y), \quad \text{Hom}_{s\text{Set}}(X, Y_{/p}) \approx \text{Hom}_{s\text{Set}_{S/}}(X \star S, Y).$$

We now construct maps

$$\text{Map}(X, Y_{p/}) \rightarrow \text{Map}_{S/}(S \star X, Y), \quad \text{Map}(X, Y_{/p}) \rightarrow \text{Map}_{S/}(X \star S, Y).$$

which are natural in both X and p , and which on vertices are exactly the join/slice adjunctions. We will call these the **enriched adjunction maps** for join/slice; they are not isomorphisms in general.

I write this out in the case of slice-over, by constructing a transformation

$$\left\{ K \dashrightarrow \text{Map}(X, Y_{p/}) \right\} \Longrightarrow \left\{ K \dashrightarrow \text{Map}_{S/}(S \star X, Y) \right\}$$

natural in the simplicial set K . Applying the product/function complex adjunction, and the join/slice adjunction, this amounts to defining natural maps

$$\left\{ \begin{array}{ccc} S \star \emptyset & & \\ \downarrow & \searrow p & \\ S \star (K \times X) & \dashrightarrow & Y \end{array} \right\} \Longrightarrow \left\{ \begin{array}{ccc} K \times (S \star \emptyset) & \xrightarrow{\text{proj}} & S \\ \downarrow & & \downarrow p \\ K \times (S \star X) & \dashrightarrow & Y \end{array} \right\}$$

Thus it suffices to produce natural maps

$$K \times (S \star X) \rightarrow S \star (K \times X)$$

which in the case that $X = \emptyset$ reduce to the projection map $K \times S \rightarrow S$. We take this to be the map corresponding by (23.12) to the triple $(g, f_{\{0\}}, f_{\{1\}})$ so that g is the composite

$$K \times (S \star X) \rightarrow K \times (\Delta^0 \star \Delta^0) \rightarrow \Delta^0 \star \Delta^0 = \Delta^1,$$

and

$$f^{(0)} = \text{proj}: K \times (S \star \emptyset) \rightarrow S, \quad f^{(1)} = \text{id}: K \times (\emptyset \star X) \rightarrow K \times X.$$

It is now straightforward to derive explicit formulas for the desired transformation (by specializing to $K = \Delta^n$), and to show that it is natural.

47.8. *Exercise.* Construct a natural ‘‘distributivity’’ map $K \times (X \star Y) \rightarrow (K \times X) \star (K \times Y)$.

47.9. **Alternate joins, alternate slices, and function complexes.** We can carry out the same procedure for alternate joins and slices, to obtain maps

$$\text{Map}(X, Y^{p/}) \rightarrow \text{Map}_{S/}(S \diamond X, Y), \quad \text{Map}(X, Y^{/p}) \rightarrow \text{Map}_{S/}(X \diamond S, Y)$$

which are natural in both X and p , and which on vertices are exactly the alternate join/slice adjunctions. We will call these the **enriched adjunction maps** for alternate join/slice.

Tracing through the same steps as in the previous section, we see that (in the first case) we need natural maps

$$K \times (S \diamond X) \rightarrow S \diamond (K \times X)$$

which when $X = \emptyset$ reduce to the projection map $K \times S \rightarrow S$. In this case it is entirely straightforward to construct such a map, since both objects are naturally quotients of the product $K \times S \times \Delta^1 \times X \approx S \times \Delta^1 \times K \times X$. In fact, examination of the constructions shows that the evident diagram

$$\begin{array}{ccc} K \times S & \xrightarrow{\text{proj}} & S \\ \downarrow & & \downarrow \\ K \times (S \diamond X) & \longrightarrow & S \diamond (K \times X) \end{array}$$

is a pushout square. (*Exercise:* prove this.) Given this consideration, we see that we have actually defined natural *isomorphisms*

$$\mathrm{Map}(X, Y^{p/}) \xrightarrow{\cong} \mathrm{Map}_{S/}(S \diamond X, Y), \quad \mathrm{Map}(X, Y^{/p}) \xrightarrow{\cong} \mathrm{Map}_{S/}(X \diamond S, Y).$$

Furthermore, these natural isomorphisms are compatible with the transformations for join/slice.

47.10. Proposition. *The evident diagrams*

$$\begin{array}{ccc} \mathrm{Map}(X, Y_{p/}) & \longrightarrow & \mathrm{Map}_{S/}(S \star X, Y) & & \mathrm{Map}(X, Y_{/p}) & \longrightarrow & \mathrm{Map}_{S/}(X \star S, Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(X, Y^{p/}) & \xrightarrow{\cong} & \mathrm{Map}_{S/}(S \diamond X, Y) & & \mathrm{Map}(X, Y^{/p}) & \xrightarrow{\cong} & \mathrm{Map}_{S/}(X \diamond S, Y) \end{array}$$

commute.

Proof. This amounts to showing that the evident diagram

$$\begin{array}{ccc} K \times (S \diamond X) & \longrightarrow & S \diamond (K \times X) \\ \downarrow & & \downarrow \\ K \times (S \star X) & \longrightarrow & S \star (K \times X) \end{array}$$

commutes, which we leave to the reader. \square

Below we will show that if $Y = C$ is a quasicategory, then all of the maps in these diagrams are categorical equivalences. As a consequence, we will obtain categorical equivalences $C_{p/} \rightarrow C^{p/}$ and $C_{/p} \rightarrow C^{/p}$.

48. EQUIVALENCE OF THE TWO JOIN AND SLICE CONSTRUCTIONS

48.1. The enriched adjunction map for joins/slices preserves isomorphism classes of objects. We now consider the natural maps

$$\mathrm{Map}(X, C_{p/}) \rightarrow \mathrm{Map}_{S/}(S \star X, C), \quad \mathrm{Map}(X, C_{/p}) \rightarrow \mathrm{Map}_{S/}(X \star S, C)$$

in the case when $p: S \rightarrow C$ is a map to a quasicategory C . In this case both sources and targets of the natural maps in question are themselves quasicategories, and both induce bijections on sets of objects. Eventually we will show that these functors are categorical equivalences. Right now we will just prove that these functors induce bijections on *isomorphism classes of objects*.

48.2. Proposition. *For X a simplicial set and $p: S \rightarrow C$ a map to a quasicategory, the enriched adjunction map for join/slice induces bijections*

$$\pi_0(\mathrm{Map}(X, C_{p/})^{\mathrm{core}}) \xrightarrow{\cong} \pi_0(\mathrm{Map}_{S/}(S \star X, C)^{\mathrm{core}}), \quad \pi_0(\mathrm{Map}(X, C_{/p})^{\mathrm{core}}) \xrightarrow{\cong} \pi_0(\mathrm{Map}_{S/}(X \star S, C)^{\mathrm{core}}),$$

Proof. We give the proof in the slice-over case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes.

Let $f_0, f_1: X \rightarrow C_{p/}$ be objects of $\mathrm{Map}(X, C_{p/})$, which correspond to objects $\tilde{f}_0, \tilde{f}_1: S \star X \rightarrow C$ of $\mathrm{Map}_{S/}(S \star X, C)$, with $\tilde{f}_j|_S = p$. If \tilde{f}_0 and \tilde{f}_1 are isomorphic objects, then there exists a map $N\mathrm{Iso} \rightarrow \mathrm{Map}_{S/}(S \star X, C)$ representing such an isomorphism (33.16). The data of such a map amounts to a an arrow \tilde{f} fitting in the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{p} & C & \longrightarrow & \mathrm{Map}(N\mathrm{Iso}, C) \\ \downarrow & & & \nearrow \tilde{f} & \downarrow \\ S \star X & \xrightarrow{(\tilde{f}_0, \tilde{f}_1)} & C \times C & & C \times C \end{array}$$

where $C \rightarrow \text{Map}(N\text{Iso}, C) \rightarrow C \times C$ are induced by restriction along $* \leftarrow N\text{Iso} \leftarrow \{0, 1\}$. Write $D = \text{Map}(N\text{Iso}, C)$ and $\bar{p}: S \rightarrow D$ for the map along the top of the rectangle. Applying the join/slice adjunction, we see that we have a diagram

$$\begin{array}{ccc} & & D_{\bar{p}/} \\ & \nearrow f & \downarrow \\ X & \xrightarrow{(f_0, f_1)} & C_{p/} \times C_{/p} \end{array}$$

That is, we have produced an object f in $\text{Map}(X, D_{\bar{p}/})$ which under the two evident projections $\pi_0, \pi_1: \text{Map}(X, D_{\bar{p}/}) \rightarrow \text{Map}(X, C)$ is sent to f_0 and f_1 respectively.

We have that both projections $D \rightarrow C$ are trivial fibrations, whence so are both projections $D_{\bar{p}/} \rightarrow C_{p/}$ (**this needs a proof**), and hence both projections π_0 and π_1 . Considering the induced commutative diagram

$$\begin{array}{ccccc} & & \text{id} & \xrightarrow{\quad} & \text{Map}(X, C_{p/}) \\ & \nearrow & & & \downarrow \pi_0 \\ \text{Map}(X, C_{p/}) & \longrightarrow & \text{Map}(X, D_{\bar{p}/}) & & \text{Map}(X, C_{p/}) \\ & \searrow & & & \downarrow \pi_1 \\ & & \text{id} & \xrightarrow{\quad} & \text{Map}(X, C_{p/}) \end{array}$$

we see that every arrow in this diagram is a categorical equivalence, and therefore both π_0 and π_1 induce *the same bijection* on isomorphism classes on objects. Therefore f_0 and f_1 are isomorphic objects, as desired. \square

48.3. Equivalence of join and alternate join. The key result of this section is the following.

48.4. Proposition. *The canonical comparison map $X \diamond Y \rightarrow X \star Y$ is a categorical equivalence for all simplicial sets X and Y .*

What we have proved implies the categorical invariance of the usual join.

48.5. Corollary. *The join \star preserves categorical equivalences in either variable. That is, if $Y \rightarrow Y'$ is a categorical equivalence, then so are $X \star Y \rightarrow X \star Y'$ and $Y \star Z \rightarrow Y' \star Z$.*

Proof. Immediate using (48.4), the invariance of the alternate join under categorical equivalence (47.4), and the 2-out-of-3 property of categorical equivalences (22.10). \square

The proof is based on the following general strategy.

48.6. Proposition. *Let $\alpha: F \rightarrow F'$ be a natural transformation between functors $s\text{Set} \rightarrow \mathcal{M}$, where \mathcal{M} is some model category. If*

- (1) F and F' preserve colimits,
- (2) F and F' take monomorphisms to cofibrations,
- (3) F and F' take inner anodyne maps to weak equivalences in \mathcal{M} , and
- (4) $\alpha(\Delta^1): F(\Delta^1) \rightarrow F'(\Delta^1)$ is a weak equivalence in \mathcal{M} ,

then $\alpha(X): F(X) \rightarrow F'(X)$ is a weak equivalence in \mathcal{M} for all simplicial sets X .

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets $\mathcal{C} := \{X \mid \alpha(X) \text{ is a weak equivalence}\}$. We use skeletal induction (45.14) to show that \mathcal{C} contains all simplicial sets.

It is clear that \mathcal{C} is closed under isomorphic objects. Because F and F' preserve colimits (1) and cofibrations (2), they take good colimit diagrams in $s\text{Set}$ to good colimit diagrams in \mathcal{M} . Since good colimits are weak equivalence invariant (44.9), (44.11), (44.13), we see that \mathcal{C} is closed under forming good colimits. It remains to show that $\Delta^n \in \mathcal{C}$ for all n .

We have $\Delta^1 \in \mathcal{C}$ by (4). Since Δ^0 is a retract of Δ^1 , we get that $\Delta^0 \in \mathcal{C}$ since weak equivalences in \mathcal{M} are closed under retracts (42.3).

The spines I^n can be built from Δ^0 and Δ^1 by a sequence of good pushouts (glue on one 1-simplex at a time), so the $I^n \in \mathcal{C}$. The inclusions $I^n \subset \Delta^n$ are inner anodyne (12.11), so by (3) and the 2-out-of-3 property of weak equivalences in \mathcal{M} it follows that $\Delta^n \in \mathcal{C}$. \square

We will apply this idea to functors $s\text{Set} \rightarrow s\text{Set}_{X/}$, where the slice category $s\text{Set}_{X/}$ inherits its model structure from the Joyal model structure on $s\text{Set}$ (42.4).

Proof of (48.4). The functors $X \diamond (-), X \star (-), (-) \diamond X, (-) \star X: s\text{Set} \rightarrow s\text{Set}_{X/}$ satisfy the first three properties required of the functors in the previous proposition (48.6). That is, they (1) preserve colimits, (2) take monomorphisms to monomorphisms, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for \diamond follows from (47.4), while condition (3) for \star this follows from (26.12) since $\overline{\text{InnHorn}} \subseteq \overline{\text{LHorn}} \cap \overline{\text{RHorn}}$.

Thus, to show $X \diamond Y \rightarrow X \star Y$ is a categorical equivalence for a fixed X and arbitrary Y , it suffices by the previous proposition to show that $X \diamond \Delta^1 \rightarrow X \star \Delta^1$ is a categorical equivalence. The same argument lets us reduce to the case when $X = \Delta^1$, i.e., to showing that a single map $\bar{f}: \Delta^1 \diamond \Delta^1 \rightarrow \Delta^1 \star \Delta^1$ is a categorical equivalence.

We will show \bar{f} is a categorical equivalence by producing a map $\bar{g}: \Delta^1 \star \Delta^1 \rightarrow \Delta^1 \diamond \Delta^1$ such that $\bar{f}\bar{g} = \text{id}_{\Delta^1 \star \Delta^1}$ and $\bar{g}\bar{f}$ is preisomorphic to the identity map of $\Delta^1 \diamond \Delta^1$, via (20.8).

Since $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, we start with maps involving the cube. I will write vertices in $(\Delta^1)^\times$ as sequences $(a_1 a_2 a_3)$ where $a_i \in \{0, 1\}$. Let

$$f: (\Delta^1)^{\times 3} \rightarrow \Delta^3 = \Delta^1 \star \Delta^1$$

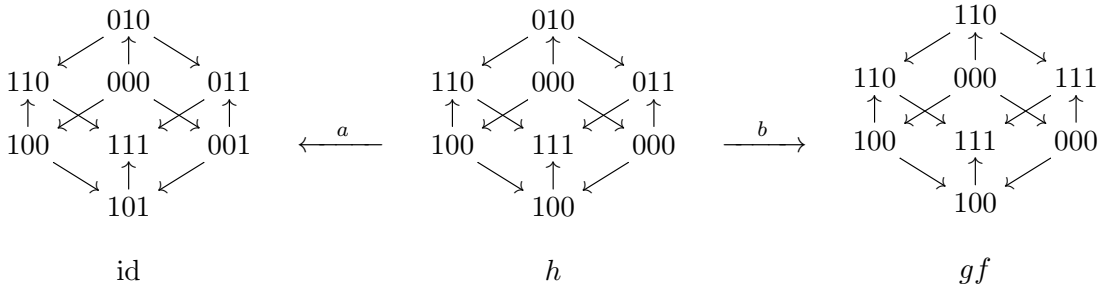
be the map which on vertices sends

$$(a_1 a_2 a_3) \mapsto \sup \{ k \mid a_k = 1 \}.$$

On passage to quotients this gives the comparison map $\bar{f}: \Delta^1 \diamond \Delta^1 \rightarrow \Delta^1 \star \Delta^1$ of the proposition.

Let $g: \Delta^3 \rightarrow (\Delta^1)^{\times 3}$ be the map classifying the element $\langle (000), (100), (110), (111) \rangle$, and let $\bar{g}: \Delta^3 \rightarrow \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \text{id}_{\Delta^3} = \bar{f}\bar{g}$.

Let $h \in \text{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \text{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be as indicated in the following picture.



These pass to elements $\bar{h}, \bar{a}, \bar{b}$ in $\text{Map}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges \bar{a} and \bar{b} are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\bar{a} \text{ or } \bar{b}} \Delta^1 \diamond \Delta^1$ represent degenerate edges. Thus $\bar{f}\bar{g}$ and $\bar{g}\bar{f}$ are preisomorphic to identity maps, and hence \bar{f} is a categorical equivalence as desired. \square

48.7. Equivalence of slice and alternate slice.

48.8. Proposition. *For any quasicategory C and map $p: S \rightarrow C$, the comparison maps $C_{p/} \rightarrow C^{p/}$ and $C_{/p} \rightarrow C^{/p}$ are categorical equivalences.*

Proof. We do the first case. We use the following fact: if $f: A \rightarrow B$ is a functor between quasicategories, then f is a categorical equivalence if and only if the induced maps $\pi_0(\mathrm{Fun}(X, A)^{\mathrm{core}}) \rightarrow (\mathrm{Fun}(X, B)^{\mathrm{core}})$ are bijections for all simplicial sets X . **I probably did this before somewhere.**

Recall (47.10) that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(X, C_{p/}) & \longrightarrow & \mathrm{Map}_{S/}(S \star X, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(X, C^{p/}) & \xrightarrow{\approx} & \mathrm{Map}_{S/}(S \diamond X, C) \end{array}$$

in which the bottom map is an isomorphism. By (48.2) the top map is a bijection on isomorphism classes of objects. By (48.4) $S \diamond X \rightarrow S \star X$ is a categorical equivalence, and thus the right-hand map is a categorical equivalence, and hence a bijection on isomorphism classes of objects. It follows that the left-hand map is a bijection on isomorphism classes of objects, and the proposition is proved. \square

48.9. Corollary. *For any quasicategory C and map $p: S \rightarrow C$, the enriched adjunction maps $\mathrm{Fun}(X, C_{p/}) \rightarrow \mathrm{Map}_{S/}(S \star X, C)$ and $\mathrm{Fun}(X, C_{/p}) \rightarrow \mathrm{Map}_{S/}(X \star S, C)$ are categorical equivalences.*

48.10. Alternate pushout-join. Just as we defined the “pushout-join” \boxtimes , we can define the “alternate pushout-join” \boxcirc : given $f: A \rightarrow B$ and $g: K \rightarrow L$, we obtain

$$f \boxcirc g: (B \diamond K) \cup_{A \diamond K} (A \diamond L) \rightarrow B \diamond L.$$

48.11. Proposition. *We have that*

$$\overline{\mathrm{RHorn}} \boxcirc \overline{\mathrm{Cell}} \cup \overline{\mathrm{Cell}} \boxcirc \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}.$$

Proof. We’ll show that $\overline{\mathrm{RHorn}} \boxcirc \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$. It is straightforward to show that the \boxcirc -product of two monomorphisms is a monomorphism. Thus, it suffices to show that for $f: A \rightarrow B$ right anodyne and any inclusion $g: K \rightarrow L$, the map $f \boxcirc g$ is a categorical equivalence. We know that $\overline{\mathrm{RHorn}} \boxtimes \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq}$ (26.12), so $f \boxtimes g$ is a categorical equivalence. Furthermore, in

$$\begin{array}{ccc} (B \diamond K) \cup_{A \diamond K} (A \diamond L) & \longrightarrow & B \diamond L \\ \downarrow & & \downarrow \\ (B \star K) \cup_{A \star K} (A \star L) & \longrightarrow & B \star L \end{array}$$

the vertical maps are categorical equivalences; this uses the result proved above (48.4), as well as the fact that since f and g are monomorphisms, the domains of $f \boxcirc g$ and $f \boxtimes g$ are constructed from good pushouts. \square

Question: is $\overline{\mathrm{LHorn}} \boxcirc \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$?

48.12. Proposition. *Given $K \xrightarrow{j} L \xrightarrow{p} C$, if C is a quasicategory and j is a monomorphism, then $C^{p/} \rightarrow C^{pj/}$ is a left fibration, and $C^{/p} \rightarrow C^{/pj}$ is a right fibration.*

Proof. Follows from $\overline{\mathrm{LHorn}} \boxcirc \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ and $\overline{\mathrm{Cell}} \boxcirc \overline{\mathrm{RHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$. \square

48.13. Equivalence of various mapping spaces. Finally we can prove our original goal.

48.14. Proposition. *For any quasicategory C and object $x \in C_0$, the natural comparison maps $\mathrm{map}_C^R(x, y) \rightarrow \mathrm{map}_C(x, y) \leftarrow \mathrm{map}_C^L(x, y)$ are weak equivalences.*

Proof. In

$$\begin{array}{ccc} C_{x/} & \xrightarrow{f} & C^{x/} \\ & \searrow p & \swarrow q \\ & & C \end{array}$$

the map f is a categorical equivalence (48.8) and p and q are left fibrations by (??) and (??) respectively, and hence are categorical fibrations (??). It follows that the induced maps on fibers $\text{map}_C^R(x, c) \rightarrow \text{map}_C(x, c)$ are categorical equivalences and hence weak equivalences, since the pullbacks describing the pullbacks are good pullbacks (with respect to the Joyal model structure). \square

48.15. Slices as fibers. Rewrite this in terms of the enriched adjunction maps.

The alternate slice $C^{f/}$ has another convenient characterization: it is the fiber over f of a map between functor categories.

48.16. Proposition. *For a map $f: S \rightarrow X$ of simplicial sets, the alternate slice $X^{f/}$ is isomorphic to the fiber of the restriction map*

$$\text{Map}(S \diamond \Delta^0, X) \rightarrow \text{Map}(S, X).$$

over f .

Proof. Let F be the fiber of the restriction map. There is an evident correspondence

$$K \dashrightarrow F \quad \iff \quad \begin{array}{ccc} S \times K & \xrightarrow{\pi} & S \\ \downarrow & & \downarrow f \\ (S \diamond \Delta^0) \times K & \dashrightarrow & C \end{array}$$

The claim follows by showing that the evident quotient map $S \times \Delta^1 \times K \rightarrow (S \diamond \Delta^0) \times K$ extends to an isomorphism

$$S \diamond K \xrightarrow{\sim} ((S \diamond \Delta^0) \times K) \cup_{S \times K} S$$

compatible with the standard inclusions of S . \square

We can also consider the fiber of the inclusion $S \subset S \star \Delta^0$ into the standard cone. This gives yet another version of the slice.

48.17. Corollary. *Let C be a quasicategory, and let $F(f) :=$ the fiber of $\text{Fun}(S^\triangleright, C) \rightarrow \text{Fun}(S, C)$ over f . Then there is a chain of categorical equivalences*

$$F(f) \rightarrow C^{f/} \leftarrow C_{f/}.$$

Furthermore, $F(f)$ and $C_{f/}$ have the same set of 0-dimensional elements, and both arrows above coincide on 0-dimensional elements.

Proof. The second equivalence is just (48.8). For the first equivalence, note that

$$\begin{array}{ccc} \text{Fun}(S \star \Delta^0, C) & \longrightarrow & \text{Fun}(S \diamond \Delta^0, C) \\ \downarrow & & \downarrow \\ \text{Fun}(S, C) & \xlongequal{\quad} & \text{Fun}(S, C) \end{array}$$

the top horizontal map is a categorical equivalence using (48.4), while the vertical maps are both categorical fibrations. Therefore the induced map on fibers over f is a categorical equivalence, since the pullback squares in question are good.

The vertices of $F(f)$ and $C_{f/}$ are exactly the set $\{S^\triangleright \rightarrow C\}$. Both inclusions $F(f)_0 \rightarrow (C^{f/})_0 \leftarrow (C_{f/})_0$ are induced by restriction along the standard comparison map $S \diamond \Delta^0 \rightarrow S \star \Delta^0$. \square

Part 6. The quasicategory of ∞ -categories

Category theory naturally provides an example of itself. That is, the collection of categories and functors between them is itself a category. In particular, we write Cat for the (large) category whose objects are (small) categories, and whose morphisms are functors between them.

By analogy, one may expect that quasicategory theory provides an example of itself, e.g., that the collection of quasicategories and functors between forms part of the data of a *quasicategory*. In fact, that is the case: we write qCat for the (large) category whose objects are (small) quasicategories, and whose morphisms are functors between them. Being an ordinary category, qCat is *a fortiori* a quasicategory.

This is unsatisfying. We would hope to have a richer object, i.e., a quasicategory which we might name Cat_∞ , whose objects and functors are as in qCat , but which in addition has some kind of non-trivial “higher structure”.

Such “quasicategory of ∞ -categories” Cat_∞ does exist. However, its description is nowhere as evident or natural as Cat . In fact, there are many possible constructions of Cat_∞ which are not isomorphic to each other (though they are categorically equivalent to each other). We will describe a particular construction of Cat_∞ (due to Lurie, and which is in some sense the standard construction) below.

In order to understand the role that Cat_∞ plays, we will start by thinking about ordinary categories. We will see that even in the classical case, what is wanted is not the ordinary category Cat , but rather a certain quasicategorical thickening of it, which we will call Cat_1 .

49. THE QUASICATEGORY OF CATEGORIES

49.1. The category of categories. We write Cat for the category of categories, by which we mean the category of *small* categories (i.e., categories whose collections of objects and morphisms are sets). Sometimes I will need to talk about a larger category CAT of possibly non-small categories (so that Cat is an object of CAT).

To understand Cat (and CAT), let us think about what functors to it look like. Given a category C , we obtain the category $\text{Fun}(C, \text{Cat})$ of functors from C to the category of categories. Explicitly:

- An object of $\text{Fun}(C, \text{Cat})$ is a functor $F: C \rightarrow \text{Cat}$, which assigns
 - to each object $c \in \text{ob } C$ a category $F(c)$, and
 - each morphism $\alpha: c \rightarrow c'$ in C a functor $F(\alpha): F(c) \rightarrow F(c')$, such that
 - $F(\text{id}_c)$ is the identity functor of $F(c)$, and $F(\beta\alpha) = F(\beta) \circ F(\alpha)$ for all composable arrows $c \xrightarrow{\alpha} c' \xrightarrow{\beta} c''$ in C .
- A morphism $\gamma: F \rightarrow F'$ in $\text{Fun}(C, \text{Cat})$ is a natural transformation of functors, which assigns
 - to each object $c \in \text{ob } C$ a functor $\gamma(c): F(c) \rightarrow F'(c)$ such that
 - for each morphism $\alpha: c \rightarrow c'$ in C we have an equality $F'(\alpha)\gamma(c) = \gamma(c')F(\alpha)$ of functors $F(c) \rightarrow F'(c')$.

The same description applies to $\text{Fun}(C, \text{CAT})$.

A functor $C \rightarrow \text{Cat}$ may be thought of as a “family of categories parameterized by C ”.

49.2. Example. Let Ring be the category of associative rings and homomorphisms; it is a large category. We may define a functor $\mathcal{M}: \text{Ring}^{\text{op}} \rightarrow \text{CAT}$ as follows.

- For a ring $R \in \text{ob } \text{Ring}$, let $\mathcal{M}(R) := \text{Mod}_R$ the category of left R -modules.
- For a homomorphism $\alpha: R \rightarrow R'$ of rings, let $\mathcal{M}(\alpha): \text{Mod}_{R'} \rightarrow \text{Mod}_R$ be the restriction-along- α functor, which on objects sends an R' -module M to the R -module with same underlying abelian group and with $rx = \alpha(r)x$ for $r \in R$ and $x \in M$.

It is straightforward to see that this indeed defines a functor $\text{Ring}^{\text{op}} \rightarrow \text{CAT}$.

There however many examples of “families of categories parameterized by a category” which can not be easily described by a functor to Cat or CAT .

49.3. *Example.* As in (49.2) let Ring be the category of associative rings and homomorphisms. We may attempt to define a functor $\mathcal{M}': \text{Ring} \rightarrow \text{CAT}$ as follows.

- For a ring $R \in \text{ob Ring}$, let $\mathcal{M}'(R) := \text{Mod}_R$ the category of left R -modules.
- For a homomorphism $\alpha: R \rightarrow R'$, let $\mathcal{M}'(\alpha): \text{Mod}_R \rightarrow \text{Mod}_{R'}$ be the extension-of-scalars-along- α functor, which on objects sends an R -module M to the R' -module $R' \otimes_R M$.

However, the above data does not define a functor. Given morphisms $R \xrightarrow{\alpha} R' \xrightarrow{\beta} R''$, we need to have an *equality* of functors $\mathcal{M}'(\beta) \circ \mathcal{M}'(\alpha) = \mathcal{M}'(\beta\alpha)$. However, we only have a *natural isomorphism* of functors, given by isomorphisms $\mathcal{M}'(\beta, \alpha): R'' \otimes_{R'} (R' \otimes_R M) \xrightarrow{\sim} R'' \otimes_R M$ which are natural in the R -module M .

The data of (49.3) does not define a functor, but rather a **pseudofunctor**. A pseudofunctor³² from $F: C \rightarrow \text{Cat}$ associates

- to each object $c \in \text{ob } C$ a category $F(c)$,
- to each morphism $\alpha: c \rightarrow c'$ in C a functor $F(\alpha): F(c) \rightarrow F(c')$,
- to each object $c \in \text{ob } C$, a natural isomorphism $\epsilon_c: \text{id}_{F(c)} \xrightarrow{\sim} F(\text{id}_c)$ of functors $F(c) \rightarrow F(c)$,
- to each composable sequence $c \xrightarrow{\alpha} c' \xrightarrow{\beta} c''$ in C , a natural isomorphism $\mu_{\beta, \alpha}: F(\beta) \circ F(\alpha) \rightarrow F(\beta\alpha)$ of functors $F(c) \rightarrow F(c'')$,

such that

-

strictly unitary pseudofunctor ...

Part 7. Old stuff

Note. From this point forward, these notes are not an organize narrative, but rather a collection of bits and pieces that might be worked into something useful at some point.

50. COHERENT NERVE

50.1. **The coherent nerve.** The coherent nerve \mathcal{N} is a construction which turns a simplicially enriched category into a simplicial set, and in particular turns a Kan-enriched category into a quasicategory. It was invented by Cordier [Cor82]. The coherent nerve is constructed as right adjoint of a “realization/singular” pair

$$\mathfrak{C}: \text{sSet} \rightleftarrows \text{sCat} : \mathcal{N}.$$

Given a finite totally ordered set S , define

$$\mathcal{P}(S) := \{ A \subseteq S \mid \{\min, \max\} \subseteq A \subseteq S \}.$$

This is a poset, ordered by set containment; here \min, \max denote the least and greatest elements of S (possibly the same). If S is empty, so is $\mathcal{P}(S)$.

Let $\mathfrak{C}(\Delta^n)$ denote the simplicially enriched category defined as follows.

- The objects are elements of $[n] = \{0, \dots, n\}$.
- For objects $x, y \in [n]$, the function complex is

$$\text{Map}_{\mathfrak{C}(\Delta^n)}(x, y) := N\mathcal{P}([x, y]), \quad [x, y] := \{ t \in [n] \mid x \leq t \leq y \},$$

which is set to be empty if $x > y$.

³²Pseudofunctors may be defined more generally with domain and codomain arbitrary 2-categories, or even arbitrary bicategories.

- Composition is induced by the union operation on subsets:

$$(T, S) \mapsto T \cup S: \mathcal{P}([y, z]) \times \mathcal{P}([x, y]) \rightarrow \mathcal{P}([x, z]).$$

Every $f: [m] \rightarrow [n]$ in Δ gives rise to an enriched functor $\mathfrak{C}(f): \mathfrak{C}(\Delta^m) \rightarrow \mathfrak{C}(\Delta^n)$, which on objects operates as f does on elements of the ordered sets, and is induced on morphisms by

$$S \mapsto f(S): \mathcal{P}([x, y]) \rightarrow \mathcal{P}([fx, fy])$$

We obtain (after identifying Δ with its image in $s\text{Set}$) a functor $\mathfrak{C}: \Delta \rightarrow s\text{Cat}$.

Given a simplicially enriched category C , its **coherent nerve** (or **simplicial nerve**) is the simplicial set $\mathcal{N}C$ defined by

$$(\mathcal{N}C)_n = \text{Hom}_{s\text{Cat}}(\mathfrak{C}(\Delta^n), C).$$

...

50.2. Quasicategories from simplicial nerves.

50.3. Proposition. *If C is a category enriched over Kan complexes, then $\mathcal{N}(C)$ is a quasicategory.*

Proof. □

51. CORRESPONDENCES

A **correspondence** is defined to be an inner fibration $p: M \rightarrow \Delta^1$. A map of correspondences is a morphism in the slice category $s\text{Set}/_{\Delta^1}$.

51.1. Correspondences of ordinary categories. If M is an ordinary category, then any functor $p: M \rightarrow \Delta^1$ is an inner fibration. Given such a functor, we can identify the following data:

- categories $C := p^{-1}(\{0\})$ and $D := p^{-1}(\{1\})$, the preimages of the vertices, and
- for each pair of objects $c \in \text{ob } C$, $d \in \text{ob } D$, a set

$$\mathcal{M}(c, d) := \text{Hom}_M(c, d),$$

which

- fit together to give a functor

$$\mathcal{M}: C^{\text{op}} \times D \rightarrow \text{Set}.$$

Conversely, given the data of categories C and D , and a functor $\mathcal{M}: C^{\text{op}} \times D \rightarrow \text{Set}$, we can construct a category M with functor $p: M \rightarrow \Delta^1$ in the evident way, with

$$\text{ob } M := \text{ob } C \amalg \text{ob } D, \quad \text{mor } M := \text{mor } C \amalg \left(\prod_{c,d} \mathcal{M}(c, d) \right) \amalg \text{mor } D.$$

Under the above, maps $f: M \rightarrow M'$ between correspondences which are categories are sent to data consisting of: functors $u: C \rightarrow C'$ and $v: D \rightarrow D'$, and natural transformations

$$\mathcal{M} \rightarrow \mathcal{M}' \circ (u \times v) \quad \text{of functors} \quad C^{\text{op}} \times D \rightarrow \text{Set}.$$

51.2. Example. If C and D are categories, then the functor $C \star D \rightarrow \Delta^0 \star \Delta^0 \approx \Delta^1$ is an example of a correspondence. The corresponding functor $\mathcal{M}: C^{\text{op}} \times D \rightarrow \text{Set}$ is the one with $\mathcal{M}(c, d) = \{*\}$ for all objects.

51.3. Example. Let $F: C \rightarrow D$ be a functor between categories. Then we get a functor $\mathcal{M}: C^{\text{op}} \times D \rightarrow \text{Set}$ defined by

$$\mathcal{M}(c, d) := \text{Hom}_D(F(c), D),$$

and thus an associated correspondence $p: M \rightarrow \Delta^1$.

Similarly, let $G: D \rightarrow C$ be a functor between categories. Then we get a functor $\mathcal{M}': C^{\text{op}} \times D \rightarrow \text{Set}$ defined by

$$\mathcal{M}'(c, d) := \text{Hom}_C(c, G(d)),$$

and thus an associated correspondence $p': M' \rightarrow \Delta^1$.

51.4. *Example.* Suppose $F: C \rightleftarrows D: G$ is an adjoint pair of functors. If we form \mathcal{M} and \mathcal{M}' as in the previous example, we see that the adjunction gives a natural isomorphism $\mathcal{M} \approx \mathcal{M}'$ of functors $C^{\text{op}} \times D \rightarrow \text{Set}$. The associated correspondences $M \rightarrow \Delta^1$ and $M' \rightarrow \Delta^1$ are isomorphic.

52. CARTESIAN AND COCARTESIAN MORPHISMS

In the following, we fix an inner fibration $p: M \rightarrow S$. We will often assume that S (and thus M) is a quasicategory.

Consider an edge $f: x \rightarrow y$ in M . We say that the edge represented by $f: \Delta^1 \rightarrow M$ is **p -cartesian** if a lift exists in every diagram of the form

$$\begin{array}{ccccc}
 & & f & & \\
 & & \curvearrowright & & \\
 \Delta^{\{n-1,n\}} & \longrightarrow & \Lambda_n^n & \longrightarrow & M \\
 & & \downarrow & \nearrow & \downarrow p \\
 & & \Delta^n & \longrightarrow & S
 \end{array}$$

for all $n \geq 2$.

There is a dual notion of a **p -cocartesian edge**, where Λ_n^n is replaced by Λ_0^n , and we use the leading edge of the simplex instead of the trailing edge.

We have already seen examples of this property.

- Let $p: C \rightarrow *$ where C is a quasicategory. By the Joyal extension theorem (28.2), we have that an edge in C is p -cartesian if and only if it is p -cocartesian if and only if it is an isomorphism.
- Let $p: M \rightarrow S$ be an inner fibration between quasicategories, and suppose $f \in M_1$ is an edge such that $p(f)$ is an isomorphism in S . By the Joyal lifting theorem (28.13), f is p -cartesian if and only if it is p -cocartesian if and only if f is an isomorphism in M .
- If $p: M \rightarrow S$ is a right fibration, then every edge in M is p -cartesian. Likewise, if p is a left fibration, then every edge in M is p -cocartesian.

Thus, Joyal's theorem completely describe cartesian/cocartesian edges over an *isomorphism* in a quasicategory.

We have an equivalent formulation: f is p -cartesian if and only if

$$M_{/f} \rightarrow M_{/y} \times_{S_{/p(y)}} S_{/pf}$$

is a trivial fibration.

52.1. **Cartesian edges and correspondence.** Let $p: M \rightarrow \Delta^1$ be a correspondence, with M an ordinary category. We write $C := p^{-1}(\{0\})$, $D := p^{-1}(\{1\})$, and $\mathcal{M}: C^{\text{op}} \times D \rightarrow \text{Set}$ for the associated functor.

Suppose $f: c \rightarrow d$ is an edge such that $p(f) = \langle 01 \rangle$.

52.2. **Lemma.** *The edge f is p -cartesian if and only if, for each $u: x \rightarrow d$ with $p(u) = \langle 01 \rangle$, there exists a unique $v: x \rightarrow c$ such that $fv = u$.*

In particular, if f is p -cartesian, then composition

$$f_*: \text{Hom}_M(x, c) \rightarrow \text{Hom}_M(x, d)$$

is a bijection for all $x \in \text{ob } C$. Equivalently, the map

$$\text{Hom}_C(x, c) \rightarrow \mathcal{M}(x, d), \quad v \mapsto fv$$

is a bijection, so $\mathcal{M}(-, d): C^{\text{op}} \rightarrow \text{Set}$ is represented by c .

52.3. **Box criterion for cartesian edges.**

52.4. **Proposition.** [Lur09, 2.4.1.8] *Let $p: M \rightarrow S$ be an inner fibration, and $f \in M_1$ an edge. Then f is p -cartesian if and only if a lift exists in every diagram of the form*

$$\begin{array}{ccc}
 \Delta^1 \times \{n\} & \xrightarrow{\quad f \quad} & M \\
 \downarrow & \searrow & \downarrow p \\
 (\{1\} \times \Delta^n) \cup_{\{1\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) & \xrightarrow{\quad} & M \\
 \downarrow & \searrow & \downarrow p \\
 \Delta^1 \times \Delta^n & \xrightarrow{\quad} & S
 \end{array}$$

for all $n \geq 1$.

Proof. The if part is just like the proof of the box version of Joyal lifting. □

We reformulate this criterion. Consider the box power map

$$q := p^{\square(\{1\} \subset \Delta^1)}: \text{Map}(\Delta^1, M) \rightarrow \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M).$$

Then the above proposition says that f is p -cartesian iff a lift exists in every diagram

$$\begin{array}{ccc}
 \partial \Delta^n & \xrightarrow{\quad a \quad} & \text{Map}(\Delta^1, M) \\
 \downarrow & \searrow & \downarrow q \\
 \Delta^n & \xrightarrow{\quad b \quad} & \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)
 \end{array}$$

such that $n \geq 1$ and $a(n) = f \in \text{Map}(\Delta^1, M)_0$.

52.5. **Uniqueness of lifts to Cartesian edges.** Let $U \subseteq \text{Map}(\Delta^1, M)$ be the full subsimplicial set spanned by the vertices which represent p -cartesian edges. Likewise, let $V \subseteq \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)$ denote the essential image of U under q , i.e., the full subsimplicial set spanned by the vertices $q(U_0)$. Obviously, the map q restricts to a map $q': U \rightarrow V$.

Note in particular that V_0 is the subset of $\{(g, y) \in S_1 \times M_0 \mid g_1 = p(y)\}$ such that there *exists* a Cartesian edge $f \in M_1$ with $f_1 = y$ and $p(f) = g$, and the preimage of (g, y) under $q': U \rightarrow V$ is the set of all choices of lifts. The following in particular asserts a kind of uniqueness for choices of lifts.

52.6. **Proposition.** *The map $q': U \rightarrow V$ is a trivial fibration.*

Proof. Consider

$$\begin{array}{ccc}
 \partial \Delta^n & \xrightarrow{\quad a \quad} & U \xrightarrow{\quad} & \text{Map}(\Delta^1, M) \\
 \downarrow & & \downarrow q|_U & \downarrow q \\
 \Delta^n & \xrightarrow{\quad} & V \xrightarrow{\quad} & \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)
 \end{array}$$

If $n \geq 1$, then a lift $s: \Delta^n \rightarrow \text{Map}(\Delta^1, M)$ exists by the previous proposition, since $a(n) \in U_0 \subseteq \text{Map}(\Delta^1, M)_0$ represents a p -cartesian edge. Because $(\partial \Delta^n)_0 = (\Delta^n)_0$ when $n \geq 1$, we see that s maps into the full subcomplex U .

If $n = 0$, this amounts to $U_0 \rightarrow V_0$ being surjective, which holds by definition. □

52.7. **Cartesian fibration.** A **cartesian fibration** is a map $p: M \rightarrow S$ which is an inner fibration, and is such that for all $(g, y) \in S_1 \times M_0$ with $g_1 = p(y)$, there exists a p -cartesian edge f with $p(f) = g$ and $f_1 = y$.

52.8. *Example.* Every left or right fibration is a cartesian fibration, since all edges are cartesian.

By the above, we see that an inner fibration $p: M \rightarrow S$ is a cartesian fibration if and only if $V = \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)$.

52.9. **Cartesian correspondences.** Given a map $p: M \rightarrow S$, for any element $a \in S_k$ write

$$M_a := \text{Map}_{/S}(\Delta^k, M) = \text{Map}_{/S}(a, p).$$

Note that if $a = bf$ for some simplicial operator $f: [k] \rightarrow [l]$, we obtain an induced restriction map

$$f^*: M_b \rightarrow M_a.$$

Given a correspondence $p: M \rightarrow \Delta^1$, we obtain

$$C = M_{\langle 0 \rangle} \xleftarrow{\langle 0 \rangle^*} M_{\langle 01 \rangle} \xrightarrow{\langle 1 \rangle^*} M_{\langle 1 \rangle} = D.$$

Note that these are all quasicategories. The objects of $M_{\langle 01 \rangle}$ are precisely the edges in M lying over $\langle 01 \rangle$.

52.10. **Proposition.** *Let $p: M \rightarrow S$ be a cartesian fibration, and let $M_{\langle 01 \rangle}^{\text{cart}} \subseteq M_{\langle 01 \rangle}$ be the full subcategory spanned by elements corresponding to cartesian edges. Then $M_{\langle 01 \rangle}^{\text{cart}} \rightarrow M_{\langle 1 \rangle}$ is a trivial fibration.*

Proof. Every square in

$$\begin{array}{ccccc} M_{\langle 01 \rangle}^{\text{cart}} & \xrightarrow{\quad} & U & & \\ \downarrow & & \downarrow i & & \\ M_{\langle 01 \rangle} & \xrightarrow{\quad} & \text{Map}(\Delta^1, M) & & \\ \downarrow & & \downarrow q & & \\ M_{\langle 1 \rangle} & \xrightarrow{\quad} & \text{Map}(\Delta^1, \Delta^1) \times_{\text{Map}(\{1\}, \Delta^1)} \text{Map}(\{1\}, M) & \xrightarrow{\quad} & \text{Map}(\{1\}, M) \\ \downarrow & & \downarrow & & \downarrow \\ \{\text{id}_{\Delta^1}\} & \xrightarrow{\quad} & \text{Map}(\Delta^1, \Delta^1) & \xrightarrow{\quad} & \text{Map}(\{1\}, \Delta^1) \end{array}$$

is a pullback. The result follows because $qi = q'$ is a trivial fibration. □

More generally, given an inner fibration $p: M \rightarrow S$ and an element $a \in S_k$, the objects of the quasicategory M_a correspond to k -dimensional elements $b \in M_k$ such that $p(b) = a$. Let $M_a^{\text{cart}} \subseteq M_a$ denote the full subcategory spanned by objects corresponding to $b \in M_k$ such that all edges of b are p -cartesian.

52.11. **Proposition.** *Let $p: M \rightarrow S$ be an inner fibration, and $f \in M_1$ an edge. Consider*

$$\begin{array}{ccc} \Delta^k \times \{n\} & \xrightarrow{\quad f \quad} & M \\ \downarrow & \searrow & \downarrow p \\ (\Lambda_j^k \times \Delta^n) \cup_{\Lambda_j^k \times \partial \Delta^n} (\Delta^k \times \partial \Delta^n) & \xrightarrow{\quad} & S \\ \downarrow & \nearrow & \\ \Delta^1 \times \Delta^n & \xrightarrow{\quad} & S \end{array}$$

where $\Lambda_j^k \subset \Delta^k$ is a right horn inclusion, and f represents a p -cartesian edge. Then a lift exists whenever $n, k \geq 1$, and also when $k \geq 2, n = 0$.

Proof. This should also be like the box version of Joyal lifting. Note that if $k = 0$, we recover the definition of p -cartesian edge. \square

This admits a reformulation: if f is p -cartesian, then if $0 < j \leq k$ and $a(n) = f$ is p -cartesian there is a lift in

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{a} & \text{Map}(\Delta^k, M) \\ \downarrow & \nearrow \text{---} & \downarrow q \\ \Delta^n & \xrightarrow{b} & \text{Map}(\Delta^k, S) \times_{\text{Map}(\Lambda_j^k, S)} \text{Map}(\Lambda_j^k, M) \end{array}$$

when $k \geq 1$ and $n \geq 1$, or for all $n \geq 0$ if $k \geq 2$.

52.12. Cartesian fibrations and right fibrations.

52.13. Proposition. [Lur09, 2.4.2.4] *A map $p: M \rightarrow S$ is a right fibration iff it is a cartesian fibration whose fibers are Kan complexes.*

Proof. We have already seen that a right fibration is a cartesian fibration, and has Kan complexes as fibers.

Now suppose p is cartesian fibration with Kan complex fibers. Let $f: x \rightarrow y$ be an edge in M . Since p is cartesian, there exists a p -cartesian edge $f': x' \rightarrow y$ over $p(f)$. Since p is cartesian fibration and f' a cartesian edge, there exists $a \in M_2$ with $a_{02} = f$ and $a_{12} = f'$ and $p(a) = (p(f))_{001}$. Thus $g := a_{01}$ is an edge in the fiber over $(p(f))_0$, so is an isomorphism in that fiber. \square

52.14. Mapping space criterion for cartesian edges.

52.15. Proposition. [Lur09, 2.4.4.3] *Let $p: C \rightarrow D$ be an inner fibration between quasicategories, and $f: x \rightarrow y$ a morphism in C . The following are equivalent.*

- (1) f is p -cartesian.
- (2) For every $c \in C_0$, the diagram

$$\begin{array}{ccc} \text{map}_C(c, x) & \xrightarrow{f_*} & \text{map}_C(c, y) \\ \downarrow & & \downarrow \\ \text{map}_D(p(c), p(x)) & \xrightarrow{p(f)_*} & \text{map}_D(p(c), p(y)) \end{array}$$

is a homotopy pullback.

53. LIMITS AND COLIMITS AS FUNCTORS

Suppose J and C are categories. We say that C has all J -colimits if every functor $F: J \rightarrow C$ has a colimit in C . It is a standard observation that if F is such a functor, then we can assemble a functor

$$\text{colim}_J: \text{Fun}(J, C) \rightarrow C.$$

In fact, we can regard this functor as a composite of functors

$$\text{Fun}(J, C) \xrightarrow{s} \text{Fun}(J^\triangleright, C) \xrightarrow{\text{eval. at } v} C,$$

where s is some section of the restriction functor $\text{Fun}(J^\triangleright, C) \rightarrow \text{Fun}(J, C)$ which takes values in colimit cones.

Even when C does not have all J -colimits, we can assert the following. Consider the diagram

$$\begin{array}{ccc} \text{Fun}^{\text{colim cone}}(J^\triangleright, C) & \xrightarrow{\quad} & \text{Fun}(J^\triangleright, C) \\ \downarrow p & & \downarrow \\ \text{Fun}^{\exists \text{ colim}}(J, C) & \xrightarrow{\quad} & \text{Fun}(J, C) \end{array}$$

in which the objects on the left are the evident full subcategories of the corresponding objects on the right, i.e., the ones consisting of colimit cones, and of functors which admit colimits. Then p is an equivalence of categories, and in fact is a trivial fibration. Therefore there is a contractible groupoid of sections of p , and any section s gives rise to a colimit functor

$$\text{Fun}^{\exists \text{ colim}}(J, C) \xrightarrow{s} \text{Fun}^{\text{colim cone}}(J^\triangleright, C) \xrightarrow{\text{eval. at } v} C.$$

We want to prove the analogous statement for quasicategories. Thus, given a quasicategory C and a simplicial set S , let $\text{Fun}^{\text{colim cone}}(S^\triangleright, C) \subseteq \text{Fun}(S^\triangleright, C)$ denote the full subcategory spanned by $S^\triangleright \rightarrow C$ which are colimit cones, and let $\text{Fun}^{\exists \text{ colim}}(S, C) \subseteq \text{Fun}(S, C)$ denote the full subcategory spanned by $S \rightarrow C$ for which a colimit exists.

53.1. Proposition. *The induced projection $q: \text{Fun}^{\text{colim cone}}(S^\triangleright, C) \rightarrow \text{Fun}^{\exists \text{ colim}}(S, C)$ is a trivial fibration.*

We refer to this as the *functoriality of colimits*. We will prove it below.

The strategy is to show (1) that q is an isofibration, and (2) that q is fully faithful and essentially surjective. Then (40.1) applies to show that q is a categorical equivalence, and so a trivial fibration by (41.1).

Parts of this are already clear. For instance, q is certainly an inner fibration, since $p: \text{Fun}(S^\triangleright, C) \rightarrow \text{Fun}(S, C)$ is one, and q is the restriction of p to full subcategories. Likewise, q is manifestly essentially surjective.

53.2. Conical maps. In what follows, C will be a quasicategory and S a simplicial set, and we write

$$V = V(S) := \text{Fun}(S^\triangleright, C), \quad U = U(S) := \text{Fun}(S, C).$$

Let $p: V \rightarrow U$ be the evident restriction map.

Let's say that a morphism $\hat{\alpha}: \hat{f} \rightarrow \hat{g}$ in V is **conical** if its evaluation $\hat{\alpha}(v): \hat{f}(v) \rightarrow \hat{g}(v)$ at the cone point of S^\triangleright is an isomorphism in C .

“Conical” here is really equivalent to “ p -Cartesian morphism” where $p: \text{Fun}(S^\triangleright, C) \rightarrow \text{Fun}(S, C)$. This whole section needs to have that observation baked in.

What follows are two propositions involving conical maps. We will prove them soon. The first says that any morphism in U can be lifted to a *conical* morphism in V with prescribed target.

53.3. Proposition. *Fix a quasicategory C and a simplicial set S . Suppose given*

- a functor $\hat{g}: S^\triangleright \rightarrow C$, and
- a natural transformation $\alpha: f \Rightarrow g$ of functors $S \rightarrow C$ such that $g = \hat{g}|_S$.

Then there exists a conical morphism $\hat{\alpha}: \hat{f} \rightarrow \hat{g}$ in V such that $\hat{\alpha}|_S = \alpha$.

$$\begin{array}{ccc} \{1\} & \xrightarrow{\hat{g}} & \text{Fun}(S^\triangleright, C) = V \\ \downarrow & \nearrow \hat{\alpha} & \downarrow \\ \Delta^1 & \xrightarrow{\alpha} & \text{Fun}(S, C) = U \end{array}$$

conical

The second says that morphisms in V can be “transported” along conical maps.

53.4. Proposition. *Fix a quasicategory C , simplicial set S , and a map $\widehat{\alpha}: \widehat{f} \rightarrow \widehat{g}$ in V , and let $\alpha: f \rightarrow g$ denote $\widehat{\alpha}|_S$. For any object \widehat{h} of V , consider the square*

$$\begin{array}{ccc} \mathrm{map}_V(\widehat{h}, \widehat{f}) & \xrightarrow{\widehat{\alpha}_\circ} & \mathrm{map}_V(\widehat{h}, \widehat{g}) \\ \downarrow & & \downarrow \\ \mathrm{map}_U(h, f) & \xrightarrow{\alpha_\circ} & \mathrm{map}_U(h, g) \end{array}$$

where $h = \widehat{h}|_S$, and the horizontal maps are induced by postcomposition with $\widehat{\alpha}$ and α respectively. If $\widehat{\alpha}$ is conical, then the above square is a homotopy pullback square.

We will explain and prove these two propositions soon. For the time being, you should convince yourself that if C is the nerve of an ordinary category, then both propositions are entirely straightforward to prove.

53.5. Proof of functoriality of colimits, using properties of conical maps. Recall that $\widehat{f}: S^\triangleright \rightarrow C$ extending $f: S \rightarrow C$ is a colimit cone if and only if it corresponds to an initial object of $C_{f/}$. Using the categorical equivalences

$$F(f) \rightarrow C^{f/} \leftarrow C_{f/}$$

where $F(f) \subseteq V$ is the fiber of $p: V \rightarrow U$ over f , we see that it is equivalent to say that \widehat{f} is initial in $F(f)$.

The following gives a criterion for being a colimit cone in terms of the whole functor category $V = \mathrm{Fun}(S^\triangleright, C)$, rather than just in terms of the fiber over some f .

53.6. Proposition. *A functor $\widehat{f}: S^\triangleright \rightarrow C$ is a colimit cone if and only if*

$$p': \mathrm{map}_V(\widehat{f}, \widehat{g}) \rightarrow \mathrm{map}_U(f, g)$$

is a weak equivalence for every $\widehat{g}: S^\triangleright \rightarrow C$, $g = \widehat{g}|_S = p(\widehat{g})$.

Proof. Since $p: V \rightarrow U$ is a categorical fibration, the induced maps p' on mapping spaces are Kan fibrations. Thus, p' is a weak equivalence if and only if its fibers are contractible.

(\Leftarrow) Suppose every p' is a weak equivalence. Then in particular p' is a weak equivalence for any $\widehat{g}: S^\triangleright \rightarrow C$ such that $\widehat{g}|_S = f$. In this case, the fiber of p' over $1_f \in \mathrm{map}_U(f, f)$ is precisely the mapping space $\mathrm{map}_{F(f)}(\widehat{f}, \widehat{g})$ in the fiber quasicategory $F(f) \subseteq \mathrm{Fun}(S^\triangleright, C)$, and this fiber is contractible. Therefore, \widehat{f} is an initial object of $F(f)$, and therefore \widehat{f} is initial in $C_{f/}$ by the above discussion. We have shown that \widehat{f} is a colimit cone.

(\Rightarrow) Suppose \widehat{f} is a colimit cone. Therefore for \widehat{f}' such that $\widehat{f}'|_S = f$ the fiber of $\mathrm{map}_V(\widehat{f}, \widehat{f}') \rightarrow \mathrm{map}_U(f, f)$ over 1_f is contractible. We need to show that the fiber of $p': \mathrm{map}_V(\widehat{f}, \widehat{g}) \rightarrow \mathrm{map}_U(f, g)$ over a general $\alpha \in \mathrm{map}_U(f, g)$ is contractible.

Given such an α , choose a conical map $\widehat{\alpha}: \widehat{f}' \rightarrow \widehat{g}$ with $\widehat{\alpha}|_S = \alpha$ (53.3), and consider the resulting square

$$\begin{array}{ccc} \mathrm{map}_V(\widehat{f}, \widehat{f}') & \xrightarrow{\widehat{\alpha}_\circ} & \mathrm{map}_V(\widehat{f}, \widehat{g}) \\ p' \downarrow & & \downarrow p'' \\ \mathrm{map}_U(f, f) & \xrightarrow{\alpha_\circ} & \mathrm{map}_U(f, g) \\ 1_f \longleftarrow & & \longrightarrow \alpha \end{array}$$

Since $\widehat{\alpha}$ is conical, the square is a homotopy pullback square (53.4). Therefore, the fiber of p'' over α is weakly equivalent to the fiber of p' over 1_f , which is contractible since \widehat{f} is a colimit cone. \square

Proof of (53.1). First we show that $q: \text{Fun}^{\text{colim cone}}(S^\triangleright, C) \rightarrow \text{Fun}^{\exists \text{ colim}}(S, C)$ is an isofibration; we have already observed that it is an inner fibration. Given an isomorphism $\alpha: f \rightarrow g$ between objects in $\text{Fun}^{\exists \text{ colim}}(S, C) \subseteq U$ and a choice of colimit cone \widehat{g} over g , choose a conical lift $\widehat{\alpha}: \widehat{f} \rightarrow \widehat{g}$. The arrow $\widehat{\alpha}: S^\triangleright \times \Delta^1 \rightarrow C$ restricts to an isomorphism at each vertex of S^\triangleright , and so is a natural isomorphism by the objectwise criterion for natural isomorphisms. Thus \widehat{f} is also a colimit cone by (53.6), so $\widehat{\alpha}$ is an isomorphism in $\text{Fun}^{\text{colim cone}}(S^\triangleright, C)$.

We have already observed that q is essentially surjective (in fact, it is surjective on vertices). That q is fully faithful is immediate from (53.6). \square

53.7. Proof of properties of conical maps.

Proof of (53.3). Recall the situation: we are given a natural transformation $\alpha: f \Rightarrow g$ of functors $S \rightarrow C$, and a lift $\widehat{g}: S^\triangleright \rightarrow C$ of the target to the cone, and we want to find a *conical* lift of α :

$$\begin{array}{ccc} \{1\} & \xrightarrow{\widehat{g}} & \text{Fun}(S^\triangleright, C) \\ \downarrow & \nearrow \widehat{\alpha} & \downarrow \\ \Delta^1 & \xrightarrow{\alpha} & \text{Fun}(S, C) \end{array}$$

conical

We make use of a natural map

$$\kappa: S^\triangleright \times K \rightarrow (S \times K)^\triangleright.$$

Note that this map sends $\{v\} \times K$ to the cone point $\{v\}$. Consider the composite

$$\lambda: (S \times \Delta^1) \cup_{S \times \{1\}} (S^\triangleright \times \{1\}) \rightarrow S^\triangleright \times \Delta^1 \xrightarrow{\kappa} (S \times \Delta^1)^\triangleright$$

where the first map is the box-product $(S \subset S^\triangleright) \square (\{1\} \subset \Delta^1)$. By inspection, we see that the composite map can be identified with the box-join

$$(S \times \{1\} \subseteq S \times \Delta^1) \boxtimes (\emptyset \subseteq \Delta^0).$$

Since $\text{RHorn} \square \text{Cell} \subseteq \overline{\text{RHorn}}$ (46.4) we have that $(S \times \{1\} \subseteq S \times \Delta^1)$ is right anodyne. Likewise, since $\text{RHorn} \boxtimes \text{Cell} \subseteq \overline{\text{InnHorn}}$ (26.12), we conclude that λ is inner anodyne. Therefore, an extension $\overline{\alpha}$ exists in

$$\begin{array}{ccc} (S \times \Delta^1) \cup_{S \times \{1\}} (S^\triangleright \times \{1\}) & \xrightarrow{(\alpha, \widehat{g})} & C \\ \downarrow & \nearrow \overline{\alpha} & \\ S^\triangleright \times \Delta^1 & & \\ \kappa \downarrow & & \\ (S \times \Delta^1)^\triangleright & & \end{array}$$

We set $\widetilde{\alpha} := \overline{\alpha} \circ \kappa$. It is clear that $\widetilde{\alpha}$ is conical: $\widehat{\alpha}(v)$ is the identity map of $\overline{\alpha}(v)$. \square

For the proof of (53.4), let's first note that, as stated, it actually doesn't make sense! This proposition asserts that for conical $\widehat{\alpha}$, the diagram

$$\begin{array}{ccc} \text{map}_V(\widehat{h}, \widehat{f}) & \xrightarrow{\widehat{\alpha} \circ} & \text{map}_V(\widehat{h}, \widehat{g}) \\ \downarrow & & \downarrow \\ \text{map}_U(h, f) & \xrightarrow{\alpha \circ} & \text{map}_U(h, g) \end{array}$$

is a homotopy pullback. However, the horizontal maps (“postcomposition” with α and $\widehat{\alpha}$) are only defined as a homotopy class of maps in $h\text{Kan}$. For instance, “ $\alpha \circ$ ” is the homotopy class defined by

the zig-zag around the left and top of the diagram

$$\begin{array}{ccccc} \mathrm{map}_U(h, f, g)_\alpha & \longrightarrow & \mathrm{map}_U(h, f, g) & \xrightarrow{\mathrm{comp}} & \mathrm{map}_U(h, g) \\ \sim \downarrow & & \downarrow \sim & & \\ \mathrm{map}_U(h, f) \times \{\alpha\} & \longrightarrow & \mathrm{map}_U(h, f) \times \mathrm{map}(f, g) & & \end{array}$$

where the left-hand square is a pullback. The correct statement of (53.4) is that in

$$\begin{array}{ccccc} \mathrm{map}_V(\widehat{h}, \widehat{f}) & \xleftarrow{\sim} & \mathrm{map}_V(\widehat{h}, \widehat{f}, \widehat{g})_{\widehat{\alpha}} & \longrightarrow & \mathrm{map}_V(\widehat{h}, \widehat{g}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{map}_U(h, f) & \xleftarrow{\sim} & \mathrm{map}_U(h, f, g)_\alpha & \longrightarrow & \mathrm{map}_U(h, g) \end{array}$$

the right-hand square is a homotopy pullback.

We can refine this a little further. Fix a map $e: \Delta^{\{1,2\}} \rightarrow C$. For a simplicial set S , let $K \subseteq S^\triangleright \times \Delta^2$ be a subcomplex containing the edge $\{v\} \times \Delta^{\{1,2\}}$, and define $\mathrm{Map}(K, C)_e$ by the pullback square

$$\begin{array}{ccc} \mathrm{Map}(K, C)_e & \longrightarrow & \mathrm{Map}(K, C) \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \mathrm{Map}(\{v\} \times \Delta^{\{1,2\}}, C) \end{array}$$

To prove our proposition, it suffices to show that for every isomorphism e in C , the map

$$\mathrm{Map}(S^\triangleright \times \Delta^2, C)_e \rightarrow \mathrm{Map}((S^\triangleright \times \Lambda_2^2) \cup_{S \times \Lambda_2^2} (S \times \Delta^2), C)_e$$

is a trivial fibration. Equivalently, we must produce a lift in each diagram of the form

$$\begin{array}{ccccccc} (S^\triangleright \times \partial\Delta^m) \cup_{S \times \partial\Delta^m} (S \times \Delta^m) & \longrightarrow & \mathrm{Map}(\Delta^2, C) & & & & \\ \downarrow & \nearrow \text{---} & \downarrow & & & & \\ \{v\} \times \Delta^{\{1,2\}} & \longrightarrow & S^\triangleright \times \Delta^m & \longrightarrow & \mathrm{Map}(\Lambda_2^2, C) & \longrightarrow & \mathrm{Map}(\Delta^{\{1,2\}}, C) \\ & & & & & & \uparrow \\ & & & & & & \{e\} \end{array}$$

We reduce to producing a lift in

$$\begin{array}{ccc} & \xrightarrow{\quad e \quad} & \\ \{n\} \times \Delta^{\{1,2\}} & \longrightarrow & (\Delta^n \times \Lambda_2^2) \cup_{\partial\Delta^n \times \Lambda_2^2} (\partial\Delta^n \times \Delta^2) \longrightarrow C \\ & & \downarrow \\ & & \Delta^n \times \Delta^2 \end{array}$$

where e is an isomorphism in C . This is precisely the box-version of Joyal extension.

54. MORE STUFF

I'm not sure what this is needed for.

Recall that the join constructions $K \star -$ and $- \star K$ are colimit preserving functors $s\mathrm{Set} \rightarrow s\mathrm{Set}_{K/}$ to the category of simplicial sets *under* K . In particular, viewed as functors $s\mathrm{Set} \rightarrow s\mathrm{Set}$ to plain simplicial sets, they preserve pushouts, and transfinite compositions.

54.1. **Proposition.** *If \mathcal{A} is a class of maps in $s\text{Set}$, then $K \star \overline{\mathcal{A}} \subseteq \overline{K \star \mathcal{A}}$ and $\overline{\mathcal{A}} \star K \subseteq \overline{\mathcal{A} \star K}$.*

Proof. Check that $K \star -: s\text{Set} \rightarrow s\text{Set}$ preserves isomorphisms, transfinite composition, pushouts, and retracts. \square

54.2. *Remark.* Given $f: X \rightarrow Y$ and K , we have a factorization of $K \star f$ as

$$K \star X \rightarrow (K \star X) \amalg_{\emptyset \star X} (\emptyset \star Y) \xrightarrow{(\emptyset \subseteq K) \boxtimes f} K \star Y.$$

54.3. **Proposition.** *We have $\Delta^0 \star \overline{\text{Cell}} \subseteq \overline{\text{LHorn}}$ and $\overline{\text{Cell}} \star \Delta^0 \subseteq \overline{\text{RHorn}}$.*

54.4. **Proposition.** *Let C be a quasicategory and x an object of C . Then x is an initial object iff $\{x\} \rightarrow C$ is left anodyne, and x is a terminal object iff $\{x\} \rightarrow C$ is right anodyne.*

Proof. (\implies) Let x be terminal, and consider $j: \{x\} \rightarrow C$. Since j^\triangleright is right anodyne, it suffices to show that j is a retract of j^\triangleright . To do this, we construct a map r fitting into

$$\begin{array}{ccccc} & & \text{id} & & \\ & & \curvearrowright & & \\ \{x\} & \longrightarrow & \{x\}^\triangleright & \longrightarrow & \{x\} \\ & \downarrow j & \downarrow j^\triangleright & & \downarrow j \\ C & \longrightarrow & C^\triangleright & \xrightarrow{-r-} & C \\ & & \text{id} & & \end{array}$$

This amounts to solving the lifting problem

$$\begin{array}{ccc} C \cup \{x\}^\triangleright & \xrightarrow{(\text{id}, 1_x)} & C \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ C^\triangleright & & C \end{array} \iff \begin{array}{ccc} \{x\} & \xrightarrow{1_x} & C_{x/} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ C & \xlongequal{\quad} & C \end{array}$$

Since x is terminal, $C_{x/} \rightarrow C$ is a trivial fibration (??), so a lift exists.

(\impliedby) Suppose $j: \{x\} \rightarrow C$ is right anodyne. Since $C_{x/} \rightarrow C$ is a right fibration, a lift exists in

$$\begin{array}{ccc} \{x\} & \xrightarrow{1_x} & C_{x/} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ C & \xlongequal{\quad} & C \end{array}$$

which is equivalent to x being terminal. \square

54.5. **Corollary.** *Let $p: D \rightarrow C$ be a right fibration between quasicategories, and let x be an object of C . Then the induced map*

$$\text{Map}(C_{/x}, D) \rightarrow \text{Map}(\{1_x\}, D) \times_{\text{Map}(\{1_x\}, C)} \text{Map}(C_{/x}, C)$$

is a trivial fibration. In particular, the map

$$\text{Map}_C(C_{/x}, D) \rightarrow \text{Map}_C(\{1_x\}, D)$$

induced by restriction over the projection map $(C_{/x} \rightarrow C) \in \text{Map}(C_{/x}, C)$ is a trivial fibration between Kan complexes.

55. STRAIGHTENING AND UNSTRAIGHTENING

Let $\mathfrak{D}_{\Delta^n}: \mathfrak{C}(\Delta^n)^{\text{op}} \rightarrow \text{sSet}$ be the simplicially enriched functor defined as follows.

- For each object $x \in \{0, \dots, n\}$, set

$$\mathfrak{D}_{\Delta^n}(x) := N\mathcal{P}_\ell(x),$$

the nerve of the poset

$$\mathcal{P}_\ell(x) := \{S \mid \{x\} \subseteq S \subseteq [x, n]\}$$

of subsets of the interval $[x, n] = \{x, \dots, n\}$ which contain the left endpoint.

- The structure of enriched functor is induced by the union operation on subsets:

$$(T, S) \mapsto T \cup S: \mathcal{P}(x, y) \times \mathcal{P}_\ell(y) \rightarrow \mathcal{P}_\ell(x).$$

- For each map $\delta: \Delta^m \rightarrow \Delta^n$, we define a natural transformation

$$\mathfrak{D}_\delta: \mathfrak{D}_{\Delta^m} \rightarrow \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\text{op}}$$

of simplicially enriched functors $\mathfrak{C}(\Delta^m)^{\text{op}} \rightarrow \text{sSet}$, which at each object x of $\mathfrak{C}(\Delta^m)^{\text{op}}$ is a map $\mathfrak{D}_{\Delta^m}(x) \rightarrow \mathfrak{D}_{\Delta^n}(\delta x)$ induced by the map of posets

$$S \mapsto \delta(S): \mathcal{P}_\ell(x) \rightarrow \mathcal{P}_\ell(\delta x).$$

55.1. *Remark.* The functor $\mathfrak{D}_{\Delta^n}: \mathfrak{C}(\Delta^n)^{\text{op}} \rightarrow \text{sSet}$ is isomorphic to the representable functor $\text{Map}_{\mathfrak{C}((\Delta^n)^\triangleright)}(-, v)$, where v represents the cone point of $(\Delta^n)^\triangleright$. Likewise, the natural transformation $\mathfrak{D}_\delta: \mathfrak{D}_{\Delta^m} \rightarrow \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\text{op}}$ coincides with the transformation

$$\text{Map}_{\mathfrak{C}((\Delta^m)^\triangleright)}(-, v) \xrightarrow{\mathfrak{C}(\delta^\triangleright)} \text{Map}_{\mathfrak{C}((\Delta^n)^\triangleright)}(\delta(-), v)$$

induced by $\delta^\triangleright: (\Delta^m)^\triangleright \rightarrow (\Delta^n)^\triangleright$.

Fix a simplicial set S , and consider a simplicially enriched functor $F: \mathfrak{C}(S)^{\text{op}} \rightarrow \text{sSet}$. We define a morphism

$$\text{Un}_S(F): X \rightarrow S$$

of simplicial sets, called the **unstraightening** of F over S , as follows.

- An n -dimensional element of $\text{Un}_S F$ is a pair

$$f: \Delta^n \rightarrow S, \quad t: \mathfrak{D}_{\Delta^n} \rightarrow F \circ \mathfrak{C}(f),$$

where f is a map of simplicial sets, and t is a map of simplicially enriched functors $\mathfrak{C}(\Delta^n)^{\text{op}} \rightarrow \text{sSet}$.

- To a map $\delta: \Delta^m \rightarrow \Delta^n$ we have an induced map $(\text{Un}_S F)_n \rightarrow (\text{Un}_S F)_m$, which sends an n -dimensional element (f, t) to the pair

$$\Delta^m \xrightarrow{\delta} \Delta^n \xrightarrow{f} S, \quad \mathfrak{D}_{\Delta^m} \xrightarrow{\mathfrak{D}_\delta} \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\text{op}} \xrightarrow{t \circ \mathfrak{C}(\delta)^{\text{op}}} F \circ \mathfrak{C}(f) \circ \mathfrak{C}(\delta)^{\text{op}}.$$

56. CARTESIAN FIBRATIONS

Let $p: C \rightarrow D$ a functor between ordinary categories. A morphism $f: x' \rightarrow x$ in C is called **p -Cartesian** if for every object c of C the evident commutative square

$$\begin{array}{ccc} \text{Hom}_C(c, x') & \xrightarrow{f \circ} & \text{Hom}_C(c, x) \\ p \downarrow & & \downarrow p \\ \text{Hom}_D(p(c), p(x')) & \xrightarrow{p(f) \circ} & \text{Hom}_D(p(c), p(x)) \end{array}$$

is a pullback square of sets.

Given an object $x \in \text{ob } C$ and a morphism $g: y' \rightarrow p(x)$ in D , a **Cartesian lift of g at x** is a p -Cartesian morphism $f: x' \rightarrow x$ such that $p(f) = g$.

We say that $p: C \rightarrow D$ is a **Cartesian fibration** of categories if every pair $(x \in \text{ob } C, g: y' \rightarrow p(x) \in D)$ admits a Cartesian lift.

Here are some observations, whose verification we leave to the reader. Fix a functor $p: C \rightarrow D$.

- Every isomorphism in C is p -Cartesian.
- Every Cartesian lift of an isomorphism in D is itself an isomorphism.
- If $f: x' \rightarrow x$ is p -Cartesian, then for any $g: x'' \rightarrow x'$ in C , we have that g is p -Cartesian if and only if gf is p -Cartesian.
- Any two Cartesian lifts of g at x are “canonically isomorphic”.

Explicitly, fix $g: y' \rightarrow y$ in D and an object x in C such that $y = p(x)$. If $f_1: x'_1 \rightarrow x$ and $f_2: x'_2 \rightarrow x$ are any two Cartesian lifts of g , then there exists a unique map $u: x'_1 \rightarrow x'_2$ such that $p(u) = 1_{y'}$ and $f_2u = f_1$; the map u is necessarily an isomorphism.

- The map p is a right fibration if and only if it is a Cartesian fibration and every morphism in C is p -Cartesian.

Now suppose that $p: C \rightarrow D$ is a Cartesian fibration. For an object y of D , we write $C_y := p^{-1}(y)$ for the fiber of C over y .

- The map p is an isofibration.
- For each morphism $g: y' \rightarrow y$ in D and object x in C with $p(x) = y$, fix a choice of Cartesian lift \tilde{g}_x of g at x . Using this data, we obtain functors

$$g^!: C_y \rightarrow C'_y$$

so that for morphism $\alpha: x_1 \rightarrow x_2$ in C_y , the map $g^!(\alpha)$ in C'_y is the unique one fitting into

$$\begin{array}{ccc} x'_1 & \xrightarrow{\tilde{g}_{x_1}} & x_1 \\ g^!(\alpha) \downarrow & & \downarrow \alpha \\ x'_2 & \xrightarrow{\tilde{g}_{x_2}} & x_2 \end{array}$$

The functor $g^!$ depends on the choices of Cartesian lifts of g . Any two set of choices of lifts give rise to isomorphic functors.

- For each pair of morphisms $y'' \xrightarrow{h} y' \xrightarrow{g} y$, we obtain a natural isomorphism of functors

$$\gamma: h^! \circ g^! \xrightarrow{\cong} (hg)^!: C_y \rightarrow C_{y''}.$$

This natural transformation is given by the unique maps γ_x in $C_{y''}$ fitting into

$$\begin{array}{ccc} x'' & \xrightarrow{\tilde{h}_{x'}} & x' \\ \gamma_x \downarrow & & \downarrow \tilde{g}_x \\ x''' & \xrightarrow{(\tilde{gh})_x} & x \end{array}$$

Similarly, there is a natural isomorphism $\text{id} \xrightarrow{\cong} (1_y)^!: C_y \rightarrow C_y$. The data of the functors $g^!$ together with these natural isomorphisms define a **pseudofunctor** $D^{\text{op}} \rightarrow \text{Cat}$, which on objects sends $y \mapsto C_y$.

- We can produce an actual functor $F: D^{\text{op}} \rightarrow \text{Cat}$ with $F(y)$ equivalent to C_y as follows. Given functors $p': C' \rightarrow D$ and $p: C \rightarrow D$, let $\text{Fun}_D(C', C)$ denote the category of fiberwise functors and natural transformations; i.e., the fiber of $p \circ -: \text{Fun}(C', C) \rightarrow \text{Fun}(C', D)$ over q .

Let $\text{Fun}_D^+(C', C) \subseteq \text{Fun}_D(C', C)$ denote the full subcategory of functors $f: C' \rightarrow C$ which take p' -Cartesian morphisms to p -Cartesian morphisms.

We obtain a functor $F: D^{\text{op}} \rightarrow \text{Cat}$, given on objects by

$$F(y) := \text{Fun}_D^+(D/y, C).$$

One can show that restriction to $\{1_y\} \subseteq D/y$ defines an equivalence of categories $F(y) \rightarrow C_y$.

- Given D , there is a 2-category \mathcal{F}_D , whose objects are Cartesian fibrations $p: C \rightarrow D$; for any two objects $p: C \rightarrow D$ and $p': C' \rightarrow D$ we take $\text{Fun}_D^+(C', C)$ as the category of morphisms from p' to p . One can show that \mathcal{F}_D is 2-equivalent to the 2-category $\text{Fun}(D^{\text{op}}, \text{Cat})$.

Part 8. Appendices

57. APPENDIX: GENERALIZED HORNS

A **generalized horn**³³ is a subcomplex $\Lambda_S^n \subset \Delta^n$ of the standard n -simplex, where $S \subseteq [n]$ and

$$(\Lambda_S^n)_k := \{f: [k] \rightarrow [n] \mid S \not\subseteq f([k])\}.$$

In other words, a generalized horn is a union of some codimension 1 faces of the n -simplex:

$$\Lambda_S^n = \bigcup_{s \in S} \Delta^{[n] \setminus s}.$$

In particular,

$$\Lambda_{[n]}^n = \partial \Delta^n, \quad \Lambda_{[n] \setminus j}^n = \Lambda_j^n, \quad \Lambda_{\{j\}}^n = \Delta^{[n] \setminus j}, \quad \Lambda_{\emptyset}^n = \emptyset.$$

In general $S \subseteq T$ implies $\Lambda_S^n \subseteq \Lambda_T^n$.

57.1. Proposition (Joyal [Joy08a, Prop. 2.12]). *Let $S \subsetneq [n]$ be a proper subset.*

- (1) $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{Horn}}$ if $S \neq \emptyset$.
- (2) $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{LHorn}}$ if $n \in S$.
- (3) $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{RHorn}}$ if $0 \in S$.
- (4) $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{InnHorn}}$ if S is not an “interval”; i.e., if there exist $a < b < c$ with $a, c \in S$ and $b \notin S$.

Proof. We start with an observation. Consider $S \subsetneq [n]$ and $t \in [n] \setminus S$. Observe the diagram

$$\begin{array}{ccc} \Delta^{[n] \setminus t} \cap \Lambda_S^n & \xrightarrow{\quad} & \Delta^{[n] \setminus t} \\ \downarrow & & \downarrow \\ \Lambda_S^n & \xrightarrow{\quad} & \Lambda_{S \cup t}^n \xrightarrow{\quad} \Delta^n \end{array}$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn $\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t}$. Thus, $(\Lambda_S^n \subset \Delta^n)$ is contained in the weak saturation of any set containing the two inclusions

$$\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t} \quad \text{and} \quad \Lambda_{S \cup t}^n \subset \Delta^n.$$

Each of the statements of the proposition is proved by an evident induction on the size of $[n] \setminus S$, using the above observation. I’ll do case (4), as the other cases are similar. If $S \subset [n]$ is not an interval, there exists some $s < u < s'$ with $s, s' \in S$ and $u \notin S$. If $[n] \setminus S = \{u\}$ then we already have an inner horn. If not, then choose $t \in [n] \setminus (S \cup \{u\})$, in which case $S \cup t$ is not an interval in $[n]$, and S is not an interval in $[n] \setminus t$. Therefore both $\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t}$ and $\Lambda_{S \cup t}^n \subset \Delta^n$ are inner anodyne by the inductive hypothesis. The proofs of the other cases are similar. \square

³³This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our Λ_S^n is Joyal’s $\Lambda^{[n] \setminus S}$. I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like $\Lambda^{n, S}$?

57.2. **Proposition** (Joyal [Joy08a, Prop. 2.13]). *For all $n \geq 2$, we have that $(I^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. We can factor the inclusion spine inclusion as $h_n = g_n f_n$:

$$I^n \xrightarrow{f_n} \Delta^{\{1, \dots, n\}} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on n that $f_n, g_n, h_n \in \overline{\text{InnHorn}}$, noting that the case $n = 2$ is immediate.

To show that $f_n \in \overline{\text{InnHorn}}$, consider the pushout square

$$\begin{array}{ccc} I^{\{1, \dots, n\}} & \xrightarrow{\quad} & \Delta^{\{1, \dots, n\}} \\ \downarrow & & \downarrow \\ I^n & \xrightarrow{f_n} & \Delta^{\{1, \dots, n\}} \cup I^n \end{array}$$

in which the top arrow is isomorphic to h_{n-1} , which is in $\overline{\text{InnHorn}}$ by induction.

To show that $g_n \in \overline{\text{InnHorn}}$, consider the diagram

$$\begin{array}{ccccc} \Delta^{\{1, \dots, n-1\}} \cup I^{\{0, \dots, n-1\}} & \xrightarrow{g_{n-1}} & \Delta^{\{0, \dots, n-1\}} & & \\ \downarrow & & \downarrow & & \\ \Delta^{\{1, \dots, n\}} \cup I^n & \xrightarrow{\quad} & \Delta^{\{1, \dots, n\}} \cup \Delta^{\{0, \dots, n-1\}} & \xrightarrow{\quad} & \Delta^n \end{array}$$

in which the square is a pushout, the top horizontal arrow is isomorphic to g_{n-1} , an element of $\overline{\text{InnHorn}}$ by induction, and the bottom right horizontal arrow is equal to $\Lambda_{\{0, n\}}^n \subset \Delta^n$, which is in $\overline{\text{InnHorn}}$ by (57.1)(4). □

58. APPENDIX: BOX PRODUCT LEMMAS

Here is where I'll prove various statements mentioned in the text.

- $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$ (46.4), proved in (58.1) below.
- $\text{RHorn} \square \text{Cell} \subseteq \overline{\text{RHorn}}$ (46.4), proved in (58.1) below.
- $\text{Horn} \square \text{Cell} \subseteq \overline{\text{Horn}}$, is a consequence of the above, since $\text{Horn} = \text{LHorn} \cup \text{RHorn}$ and $\overline{\text{LHorn}} \cup \overline{\text{RHorn}} \subseteq \overline{\text{Horn}}$.
- $\text{InnHorn} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$ (16.9), proved in (58.3) below.

58.1. **Left and right horns.** We prove the case of $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$ here. Given this $\text{RHorn} \square \text{Cell} \subseteq \overline{\text{RHorn}}$ follows since $\text{op}: s\text{Set} \rightarrow s\text{Set}$ carries LHorn to RHorn and preserves Cell .

Joyal [Joy08a, 2.25]³⁴ observes that $(\Lambda_k^n \subset \Delta^n)$ is a retract of $(\Lambda_k^n \subset \Delta^n) \square (\{0\} \subset \Delta^1)$ when $0 \leq k < n$. The retraction is

$$\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n$$

defined by $s(x) = (x, 1)$ and

$$r(x, 0) = \begin{cases} x & \text{if } x \leq k, \\ k & \text{if } x \geq k, \end{cases} \quad r(x, 1) = x.$$

Note that $r(\Delta^{[n] \setminus j} \times \Delta^1) = \Delta^{[n] \setminus j}$ if $j \neq k$, and $r(\Delta^n \times \{0\}) = \Delta^{\{0, \dots, k\}} \subseteq \Delta^{[n] \setminus (k+1)}$, so this gives the desired retraction.

The existence of the retraction reduces showing $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$ to proving

$$(\{0\} \subset \Delta^1) \square \text{Cell} \subseteq \overline{\text{LHorn}},$$

since $(\Lambda_k^n \subset \Delta^n) \in \text{Cell}$ and thus $(\Lambda_k^n \subset \Delta^n) \square \text{Cell} \subseteq \overline{\text{Cell}}$.

³⁴Lurie [Lur09, 2.1.2.6] states this incorrectly.

58.2. **Lemma.** *We have that $(\{0\} \subset \Delta^1) \square \text{Cell} \subseteq \overline{\text{LHorn}}$.*

Proof. ... Let $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n)$, so that $(\{0\} \subset \Delta^1) \square (\partial\Delta^n \subset \Delta^n)$ is the inclusion $K \rightarrow \Delta^1 \times \Delta^n$. We will show that we can build $\Delta^1 \times \Delta^n$ from K by an explicit sequence of steps, where in each case we attach an $(n+1)$ -sequence along a left horn.

For each $0 \leq a \leq n$ let τ_a be the $(n+1)$ -dimensional element of $\Delta^1 \times \Delta^n$ defined by

$$\tau_a = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, n) \rangle.$$

We obtain an ascending filtration of $\Delta^1 \times \Delta^n$ by starting with K and attaching simplices in the following order:

$$\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0.$$

The τ s range through all non-degenerate $(n+1)$ -dimensional elements of $\Delta^1 \times \Delta^n$, so $K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n$. (Here I am using the same notation for elements $\tau_a \in (\Delta^1 \times \Delta^n)_{n+1}$ and for the corresponding subcomplex of $\Delta^1 \times \Delta^n$ which is isomorphic to Δ^{n+1} .)

The claim is that each attachment is along a specified horn inclusion. More precisely, for $a \in [n]$ the simplex τ_a is attached to $K \cup \bigcup_{k>a} \tau_k$ along the horn at the vertex $(0, a)$ in τ_a , i.e., via a $\Lambda_a^{n+1} \subset \Delta^n$ horn inclusion. Note that if when $a > 0$ this is an inner horn, while when $a = 0$ this is the inclusion $\Lambda_0^{n+1} \subset \Delta^n$; in either case, it is a left horn. Given the claim, it follows that $(\{0\} \subset \Delta^1) \square (\partial\Delta^n \subset \Delta^n) \in \overline{\text{LHorn}}$ as desired.

The proof of the claim amounts to the following list of elementary observations about τ_a :

- Every codimension-one face is contained in $\Delta^1 \times \partial\Delta^n$ except: the face opposite vertex $(0, a)$, and the face opposite vertex $(1, a)$.
- The face opposite vertex $(1, a)$ is contained in $\{0\} \times \Delta^n$ if $a = n$, or is a face of τ_{a+1} if $a < n$.
- The face opposite vertex $(0, a)$ is not contained in $\Delta^1 \times \partial\Delta^n$, nor in $\{0\} \times \Delta^n$. Nor is it contained in any τ_i with $i > a$ (because the vertex $(1, a)$ is in this face but not in τ_i with $i > a$).

Taken together these show that $\tau_a \cap (K \cup \bigcup_{k>a} \tau_k)$ is the a th horn in the $(n+1)$ -simplex τ_a . □

58.3. **Inner horns.** Here is an argument for the key case for inner horns.

Consider $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$, the unique maps which are given on vertices by

$$s(y) = \begin{cases} (0, y) & \text{if } y < j, \\ (1, y) & \text{if } y = j, \\ (2, y) & \text{if } y > j, \end{cases} \quad r(x, y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit $(\Lambda_j^n \subset \Delta^n)$ as a retract of $(\Lambda_1^2 \subset \Delta^2) \square (\Lambda_j^n \subset \Delta^n)$, so

$$\text{InnHorn} \subseteq \{\Lambda_1^2 \subset \Delta^2\} \square \overline{\text{Cell}}.$$

We have (17.5) that $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$, so the above implies that $\text{InnHorn} \square \overline{\text{Cell}} \subseteq \{\Lambda_1^2 \subset \Delta^2\} \square \overline{\text{Cell}}$. Thus the assertions “ $\text{InnHorn} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” and “ $\{\Lambda_0^2 \subset \Delta^2\} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” are equivalent. Thus both assertions follow from the following.

58.4. **Lemma.** *For all $n \geq 0$ we have that $(\Lambda_1^2 \subset \Delta^2) \square (\partial\Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. [Lur09, 2.3.2.1].

For each $0 \leq a \leq b < n$, let σ_{ab} be the $(n+1)$ -simplex of $\Delta^2 \times \Delta^n$ defined by

$$\sigma_{ab} = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, b), (2, b+1), \dots, (2, n) \rangle.$$

For each $0 \leq a \leq b \leq n$, let τ_{ab} be the $(n+2)$ -simplex of $\Delta^2 \times \Delta^n$ defined by

$$\tau_{ab} = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, b), (2, b), \dots, (2, n) \rangle.$$

The set $\{\tau_{ab}\}$ consists of all the non-degenerate $(n+2)$ -dimensional elements. Note that σ_{ab} is a face of τ_{ab} and of $\tau_{a,b+1}$, but not a face of any other τ .

We attach simplices to $K := (\Lambda_1^2 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n)$ in the following order:

$$\sigma_{00}, \sigma_{01}, \sigma_{11}, \sigma_{02}, \sigma_{12}, \sigma_{22}, \dots, \sigma_{0,n-1}, \dots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00}, \tau_{01}, \tau_{11}, \tau_{02}, \tau_{12}, \tau_{22}, \dots, \tau_{0,n}, \dots, \tau_{n,n}.$$

The τ s range through all the non-degenerate $(n+2)$ -dimensional elements of $\Delta^2 \times \Delta^n$, so that $K \cup \bigcup \sigma_{a,b} \cup \bigcup \tau_{a,b} = \Delta^2 \times \Delta^n$.

The claim is that each attachment is along an inner horn inclusion. More precisely, each σ_{ab} gets attached along the horn at the vertex $(1, a)$ in σ_{ab} , i.e., via a $\Lambda_{a+1}^{n+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \leq b < n$. Likewise, each τ_{ab} gets attached along the horn at vertex $(1, a)$ in τ_{ab} , i.e., via a $\Lambda_{a+1}^{n+2} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \leq b \leq n$.

The proof of the claim amounts to the following lists of elementary observations.

For $\sigma_{a,b}$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$, except the following: the face opposite vertex $(0, a)$, and the face opposite vertex $(1, a)$.
- The face opposite vertex $(0, a)$ is either contained in $\Lambda_0^2 \times \Delta^n$ if $a = 0$, or a face of $\sigma_{a-1,b}$ if $a > 0$.
- The face of $\sigma_{a,b}$ opposite vertex $(1, a)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_0^2 \times \Delta^n$, nor in any $\sigma_{i,b}$ with $i < a$ (because of the vertex $(0, a)$), nor in any $\sigma_{i,j}$ with $i \leq j < b$ (because of the vertex $(1, b)$ if $a < b$, or the vertex $(0, a)$ if $a = b$).

For $\tau_{a,b}$ when $a < b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$ except the following: the face opposite vertex $(0, a)$, the face opposite vertex $(1, a)$, the face opposite vertex $(1, b)$, and the face opposite vertex $(2, b)$.
- The face opposite vertex $(2, b)$ is $\sigma_{a,b}$, while the face opposite vertex $(1, b)$ is $\sigma_{a,b-1}$.
- The face opposite vertex $(0, a)$ is either contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or is a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1, a)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(1, b)$ and $(2, b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0, a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(1, b)$).

For $\tau_{a,b}$ when $a = b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$ except the following: the face opposite vertex $(0, a)$, the face opposite vertex $(1, a) = (1, b)$, and the face opposite vertex $(2, b)$.
- The face opposite vertex $(2, b)$ is $\sigma_{a,b}$.
- The face opposite vertex $(0, a)$ is contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or is a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1, a) = (1, b)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(0, a)$ and $(2, b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0, a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(0, a)$).

□

58.5. A pushout-product version of Joyal lifting. We now give a proof of (30.3): we will prove the case of $(i, j) = (0, 0)$, i.e., given $p: C \rightarrow D$ an inner fibration of quasicategories, $n \geq 1$,

and

$$\begin{array}{ccc}
 \Delta^1 \times \{0\} & \xrightarrow{f} & (\{0\} \times \Delta^n) \cup_{\{0\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) \longrightarrow C \\
 & & \downarrow \\
 & & \Delta^1 \times \Delta^n \longrightarrow D \\
 & & \uparrow \text{ (dashed arrow) } \\
 & & C \xrightarrow{p} D
 \end{array}$$

such that f represents an isomorphism in C , we will construct a lift. (Note that if $n = 0$ such a lift does not generally exist.)

We refer to the proof of (58.2), where we observed that we can build $\Delta^1 \times \Delta^n$ from $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial \Delta^n)$ by successively attaching a sequence τ_n, \dots, τ_0 of $(n + 1)$ -simplices along horns; in particular, τ_a is attached to $K \cup \bigcup_{k>a} \tau_k$ along a horn inclusion isomorphic to $\Lambda_a^{n+1} \subset \Delta^{n+1}$.

Given this, we thus construct the desired lift by inductively choosing a lift defined on each τ_a relative to the given lift on its Λ_a^{n+1} -horn. When $a > 0$ such a lift exists because p is an inner fibration and τ_a is attached along an inner horn, while when $a = 0$ a lift exists by Joyal lifting (28.13), as $\Delta^1 \times \{0\}$ is the leading edge of τ_0 .

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