# INTRODUCTION TO EQUIVARIANT HOMOTOPY THEORY

DENIS NARDIN

In this brief note we will present the basics definitions of stable $G$-equivariant homotopy theory for $G$ a topological group, relative to its finite subgroup. We will not cover the similar but more subtle case of $G$-equivariant homotopy theory for $G$ a compact Lie group, relative to all of its subgroups. Standard references for the subject are [7] (but be wary of the model for $G$-spectra used there) and [5]. Good introductory expositions can be found in [10] and [3].

Contents



## 1. $G$-spaces

Let $G$ a topological group. If $X, Y$ are topological spaces with an action of $G$ we are interested in studying the homotopy classes of equivariant maps between $X$ and $Y$. But what do we mean by homotopies of equivariant maps? There are two choices:

- A homotopy between $f$ and $g$ is just a homotopy $H : X \times [0,1] \to Y$, without further conditions;
- A homotopy between $f$ and $g$ is a $G$-equivariant map $H : X \times [0,1] \to Y$, where $[0,1]$ is given the trivial action, such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$.

We are interested in the latter case.

A $G$-CW-structure on a topological space $X$ with an action of $G$ is given by a CW-structure where the cells are permuted by the action of $G$. That is to say, the $n$-skeleton is obtained from the $(n-1)$-skeleton by attaching $I \times D^n$ along

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$I \times S^{n-1}$, where $I$ is a $G$-set (and the action of $G$ on $D^n$ is trivial). Then the following theorem is true

**Theorem 1** (Bredon). *A map $f : X \to Y$ between $G$-CW-complexes is a $G$-equivariant homotopy equivalence iff $f^H : X^H \to Y^H$ is a homotopy equivalence for all subgroups $H$.*

*Proof.* This is [2, Cor. II.5.5]. □

Conforted by this example, we define $f$ to be a $G$-weak equivalence iff $f^H : X^H \to Y^H$ is a weak equivalence for all $H$. Then the homotopy theory of $G$-topological spaces up to $G$-weak equivalence has a very pleasant description. Let $\mathbf{O}_G$ be the **orbit $\infty$-category** of $G$, that is to say the $\infty$-category associated to the topological category of transitive $G$-spaces.

**Theorem 2** (Elmendorf). *The functor*

$$\mathrm{Top}_G \to \mathrm{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathrm{Space})$$

*sending $X \mapsto \mathrm{Map}_G(-, X)$ is a localization. That is, it induces an equivalence of the right hand side with the $\infty$-category obtained by inverting the $G$-weak equivalences on the left hand side.*

*Proof.* See for example [4, Pr. 3.15]. □

We will call the right hand side the **$\infty$-category of $G$-spaces** and we will write $\mathrm{Space}_G$ for it. If $X$ is a $G$-space, we will often write $X^H$ for $X(G/H)$ (guided by this equivalence).

Now, this whole story works also for all topological groups, provided you define $\mathbf{O}_G$ to be the $\infty$-category corresponding to the topological category of orbits. However, we will need a slight variation of it. If $G$ is a topological group, we say that a map is a $(G, \mathcal{F})$-weak equivalence if it induces a weak equivalence on all fixed points for **finite** subgroups. Then, if $\mathbf{O}_{G,\mathcal{F}}$ is the subcategory of $\mathbf{O}_G$ corresponding to orbits with finite stabilizers, the corresponding variant of Elmendorf's theorem is still true: the functor

$$\mathrm{Top}_G \to \mathrm{Fun}(\mathbf{O}_{G,\mathcal{F}}^{\mathrm{op}}, \mathrm{Space})$$

is a localization at the $(G, \mathcal{F})$-weak equivalences.

Note that if $G' \subseteq G$ is a subgroup, there's a functor

$$\mathrm{res}_{G'}^G : \mathrm{Space}_{G,\mathcal{F}} \to \mathrm{Space}_{G',\mathcal{F}}$$

simply given by the precomposition with the functor $G \times_{G'} - : \mathbf{O}_{G'} \to \mathbf{O}_G$. That is, this is defined so that

$$(\mathrm{res}_{G'}^G X)^H = X^H$$

for all $H < G'$. This functor has a left and right adjoint, called **induction** and **coinduction** respectively that are given by left and right Kan extension. If $G$ and $G'$ are finite groups we can write them as

$$(\mathrm{ind}_{G'}^G X)^H = \coprod_{g \in G/G', g^{-1}Hg \subseteq G'} X^{g^{-1}Hg}$$

$$(\mathrm{coind}_{G'}^G X)^H = \prod_{g \in G' \backslash G/H} X$$

(morally: $\mathrm{ind}_{G'}^G X = G \times_{G'} X$ and $\mathrm{coind}_{G'}^G X = \mathrm{Map}_{G'}(G, X)$).

Finally, note that $\mathrm{Space}_G$ and $\mathrm{Space}_{G,*}$ have a symmetric monoidal structure given by the pointwise cartesian product and smash product respectively.

The following theorem will often be useful

**Theorem 3** (Illman). *Let $G$ be a compact Lie group. Then every $G$-manifold has a $G$-CW-complex structure. In particular, every compact $G$-manifold has a finite $G$-CW-complex structure.*

*Proof.* This is [6, Cor. 7.2]. □

## 2. The definition of $G$-spectra

Recall that we can define the $\infty$-category of spectra as

$$\mathrm{Sp} = \lim(\mathrm{Space}_* \xleftarrow{\Omega} \mathrm{Space}_* \leftarrow \Omega \cdots)\,.$$

In fact we can use a slightly different definition of spectra: this is the definition of "coordinate-free" spectra. Let $\mathcal{I}$ be the $\infty$-category associated to the topological category of finite dimensional real inner product spaces and isometric embeddings. Then there is a functor

$$\mathcal{I} \to \mathrm{Pr}^L$$

sending every object $V$ to $\mathrm{Space}_*$ and an isometric embedding $f : V \to W$ to the functor $S^{fV^\perp} \wedge -$ (where $S^W$ is the one-point compactification of the real vector space $W$).

Now, using $\mathrm{Pr}^L \cong (\mathrm{Pr}^R)^{\mathrm{op}}$ we get a functor $\mathcal{I}^{\mathrm{op}} \to \mathrm{Pr}^R$. The functor $\mathbb{N} \to \mathcal{I}$ sending $n$ to $\mathbb{R}^n$ is cofinal[1], so we can write

$$\mathrm{Sp} = \lim_{\mathcal{I}^{\mathrm{op}}} \mathrm{Space}_*$$

Let $\mathrm{Rep}_G$ be the category of orthogonal $G$-representations and isometric embeddings. Then we define

$$\mathrm{Sp}^G_{\mathcal{F}} = \lim_{V \in \mathrm{Rep}_G^{\mathrm{op}}} \mathrm{Space}_{G,\mathcal{F},*}$$

where the isometric embedding $f : V \to W$ induces the functor $\mathrm{Map}(S^{fV^\perp}, -) : \mathrm{Space}_{G,\mathcal{F},*} \to \mathrm{Space}_{G,\mathcal{F},*}$ (see in the appendix for a rigorous construction of this functor). We will write $\Omega^\infty : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Space}_{G,*}$ for the projection on the component corresponding to the 0-representation and $\Sigma^\infty : \mathrm{Space}_{G,*} \to \mathrm{Sp}^G_{\mathcal{F}}$ for its left adjoint.

If $G$ is a finite group, we can find a cofinal functor $\mathbb{N} \to \mathrm{Rep}_G$ by sending $n$ to $n\rho$, where $\rho$ is the regular representation of $G$, so

$$\mathrm{Sp}^G = \lim\left(\mathrm{Space}_{G,*} \xleftarrow{\Omega^\rho} \mathrm{Space}_{G,*} \xleftarrow{\Omega^\rho} \cdots\right)$$

Hence by corollary 2.22 in [9] we know that $\mathrm{Sp}^G$ is the presentable symmetric monoidal $\infty$-category obtained by $\mathrm{Space}_{G,*}$ by inverting $S^\rho$. In particular $S^V$ is invertible for any $G$-representation $V$ (since there is $W$ and $n$ such that $S^V \wedge S^W \cong S^{n\rho}$). Moreover every $G$-spectrum $X$ has the canonical presentation

$$X \cong \operatorname*{colim}_n \mathbb{S}^{-n\rho} \wedge \Omega^\infty(\mathbb{S}^{n\rho} \wedge X)\,.$$

Note that $\Sigma^\infty : \mathrm{Space}_{G,*} \to \mathrm{Sp}^G$ has a natural symmetric monoidal structure.

[1] This is just the contractibility of the infinite Stiefel manifolds, since the weak homotopy type of $\mathbb{N} \times_{\mathcal{I}} \mathcal{I}_{V/}$ is just $\mathrm{colim}_n \mathrm{Map}_{\mathcal{I}}(V, \mathbb{R}^n)$.

If $G' \subseteq G$ is a closed subgroup, there is a restriction functor

$$\mathrm{res}^G_{G'} : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}^{G'}_{\mathcal{F}}$$

given by

$$Sp^G_{\mathcal{F}} = \lim_{\mathrm{Rep}_G} \mathrm{Space}_{G,\mathcal{F},*} \xrightarrow{\lim \mathrm{res}^G_{G'}} \lim_{\mathrm{Rep}_G} \mathrm{Space}_{G',\mathcal{F},*} \cong \lim_{\mathrm{Rep}'_G} \mathrm{Space}_{G',\mathcal{F},*} = \mathrm{Sp}^{G'}_{\mathcal{F}} \,,$$

where we used that the restriction functor $\mathrm{Rep}_G \to \mathrm{Rep}'_G$ is cofinal. This functor has a levelwise right adjoint obtained by applying levelwise $\mathrm{coind}^G_{G'}$, so it has a right adjoint that we will still denote $\mathrm{coind}^G_{G'}$. Moreover, since the whole diagram lives in $\mathrm{Pr}^R$, the functor $\mathrm{res}^G_{G'}$ has a left adjoint that we will denote $\mathrm{ind}^G_{G'}$. We can give an explicit formula for $\mathrm{ind}^G_{G'}$ given by

$$(\mathrm{ind}^G_{G'} X)_V = \operatorname*{colim}_{W \in \mathrm{Rep}_G} \mathrm{Map}(S^W, \mathrm{ind}^G_{G'} X_{V \oplus W}) \,.$$

We would like to use our understanding of $\mathrm{Sp}^G$ for a finite group $G$, to better understand $\mathrm{Sp}^G_{\mathcal{F}}$ for a compact Lie group $G$.

**Proposition 4.** *Let $G$ be a compact Lie group. Then the restriction maps induce natural equivalences*

$$\mathrm{Space}_{G,\mathcal{F}} \cong \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Space}_H \,,$$

$$\mathrm{Sp}^G_{\mathcal{F}} \cong \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Sp}^H \ .$$

*In particular, since the restriction functors are canonically symmetric monoidal functors the left hand side inherits a presentable symmetric monoidal structure where the spheres are invertible.*

*Proof.* Let us do the first equivalence. If $G/H \in \mathbf{O}_{G,\mathcal{F}}$, we claim that there is a canonical equivalence $\mathbf{O}_H \cong (\mathbf{O}_{G,\mathcal{F}})_{/G/H}$, sending an $H$-orbit $O$ to the $G$-orbit $G \times_H O \to G \times_H * \cong G/H$. The inverse is given by taking the fiber over $eH \in G/H$. Then, using the equivalence $I \cong \mathrm{colim}_{i \in I^{\mathrm{op}}} I_{i/}$ applied to the $\infty$-category $\mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}$ we have

$$\mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}} \cong \operatorname*{colim}_{G/H \in \mathbf{O}_{G,\mathcal{F}}} (\mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}})_{G/H\,/} \cong \operatorname*{colim}_{G/H \in \mathbf{O}_{G,\mathcal{F}}} \mathbf{O}^{\mathrm{op}}_H \,.$$

So, by applying $\mathrm{Fun}(-, \mathrm{Space})$ to the diagram

$$\mathrm{Space}_{G,\mathcal{F}} = \mathrm{Fun}(\mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}, \mathrm{Space}) \cong \mathrm{Fun}(\operatorname*{colim}_{G/H \in \mathbf{O}_{G,\mathcal{F}}} \mathbf{O}^{\mathrm{op}}_H, \mathrm{Space}) \cong \lim_{G/h \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Fun}(\mathbf{O}^{\mathrm{op}}_H, \mathrm{Space}) \,.$$

Now, using that for every finite subgroup $H < G$ the funcor $\mathrm{Rep}_G \to \mathrm{Rep}_H$ is cofinal, we have

$$\begin{aligned}\mathrm{Sp}^G_{\mathcal{F}} &\cong \lim_{\mathrm{Rep}^{\mathrm{op}}_G} \mathrm{Space}_{G,\mathcal{F},*} \cong \lim_{\mathrm{Rep}^{\mathrm{op}}_G} \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Space}_{H,*} \cong \\ &\cong \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \lim_{\mathrm{Rep}^{\mathrm{op}}_G} \mathrm{Space}_{H,*} \cong \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Sp}^H \ .\end{aligned}$$

□

If $G$ is a general topological group, we will turn the above proposition into a definition, and let

$$\mathrm{Sp}^G_{\mathcal{F}} := \lim_{G/H \in \mathbf{O}^{\mathrm{op}}_{G,\mathcal{F}}} \mathrm{Sp}^H \ .$$

This has a symmetric monoidal structure and $\Sigma^\infty : \mathrm{Space}_{G,\mathcal{F},*} \to \mathrm{Sp}^G_{\mathcal{F}}$ is a symmetric monoidal functor.

## 3. Dualities

If $M$ is a $G$-space equipped with an equivariant vector bundle $V$, we can define the Thom spectrum $M^V$ as the suspension spectrum of the 1-point compactification of $V$ (or, equivalently, the cofiber of $\Sigma^\infty_+ S(V) \to \Sigma^\infty_+ M$). As in the nonequivariant case, we can always realize $V$ as a summand of a constant bundle[2] and so we can define Thom spectra for virtual bundles.

**Theorem 5** (Atiyah duality)**.** *Let $G$ be a compact Lie group and $M$ be a $G$-manifold. Then $\Sigma^\infty_+ M$ is a dualizable $G$-spectrum, whose dual is given by $M^{-TM}$. Precisely, if $\nu$ is an equivariant vector bundle over $M$ such that $TM \oplus \nu \cong V \times M$ for some $G$-representation $V$, the composition*

$$M^\nu \times M_+ \to S^V \wedge M_+ \to S^V ,$$

*where the first map is the Pontryagin-Thom collapse map associated to the diagonal embedding $M \to \nu \times M$, is a perfect pairing.*

*Proof.* The same proof as in the non-equivariant case works here, assuming known that every $G$-manifold embeds equivariantly in a $G$-representation. For a detailed treatment in the equivariant case, see section III.5 in [7]. □

If $G$ is a finite group it is worth to analize the case $M = G/H$ where $H$ is a subgroup. By choosing as $\nu$ the 0-vector bundle, we obtain that $\Sigma^\infty_+ G/H$ is self-dual and the pairing is given by

$$\Sigma^\infty_+ G/H \wedge \Sigma^\infty_+ G/H \cong \Sigma^\infty_+ (G/H \times G/H) \to \Sigma^\infty_+ G/H \to S^0$$

induced by the map of pointed $G$-spaces $(G/H \times G/H)_+ \to G/H_+$ which is the identity on the diagonal and collapses everything else on the basepoint.

## 4. Genuine fixed points

Let $G$ be a compact Lie group and $N < G$ be a normal finite subgroup. Then there is a functor

$$(-)^N : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}^{G/N}_{\mathcal{F}}$$

obtained as the composition

$$\mathrm{Sp}^G_{\mathcal{F}} = \lim_{\mathrm{Rep}^{\mathrm{op}}_G} \mathrm{Space}_{G,\mathcal{F},*} \to \lim_{\mathrm{Rep}^{\mathrm{op}}_{G/N}} \mathrm{Space}_{G,\mathcal{F},*} \to \lim_{\mathrm{Rep}^{\mathrm{op}}_{G/N}} \mathrm{Space}_{G/N,\mathcal{F},*}$$

where the first functor is the projection onto the subcategory $\mathrm{Rep}_{G/N} \subseteq \mathrm{Rep}_G$ of those representations where $N$ acts trivially, and the second functor is given by taking $N$-fixed points levelwise.

If $H < G$ is a finite, not necessarily normal, subgroup we can define

$$E^H = \left(\mathrm{res}^G_{N_G H} E\right)^h \in \mathrm{Sp}^{W_G H}$$

where $W_G H = N_G H/H$ is the Weyl group of $H$. We will denote by $E^H$ also the "underlying spectrum" (restriction to $1 < W_G H$) of $E^H$.

**Lemma 6.** *The functor $\mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}$ given by $X \mapsto X^H$ is corepresented by $\Sigma^\infty_+ G/H$ and the $W_G H$-action on the spectrum is given by the right action on $\Sigma^\infty_+ G/H$.*

[2]i.e. an equivariant bundle of the form $M \times W \to M$ for $W$ a $G$-representation.

*Proof.* Unwrapping the definitions $(-)^H$ sends a $G$-spectrum $X$ to the spectrum $\{(\Omega^\infty\Sigma^n X)^H\}_{n\geq 0}$, but this is easy to see to be exactly $\mathrm{map}_G(\Sigma^\infty_+ G/H, -)$. $\square$

Note that the functor $(-)^H : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}$ commutes with both limits and colimits: limits because it comes from a diagram in $\mathrm{Pr}^R$ and colimits because it is corepresented. In particular $\Sigma^\infty_+ G/H$ is compact for each finite $H$.

**Theorem 7.** *The functor $(-)^H : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}$ commutes with both limits and colimits. Moreover, the collection of functors $\{(-)^H\}_{H<G \text{ finite}}$ is jointly conservative.*

*Proof.* The functor $(-)^H$ is obtained as the limit of a diagram of functors in $\mathrm{Pr}^R_\omega$, the $\infty$-category of compactly generated $\infty$-categories and filtered colimits preserving right adjoint functors, so it is a filtered colimit preserving right adjoint functor. In particular it commutes with limits and filtered colimits. But $\mathrm{Sp}^G_{\mathcal{F}}$ is a stable $\infty$-category, and so $(-)^H$ is also exact, hence it commutes with all colimits.

To prove that the collection $\{(-)^H\}_{H<G}$ is jointly conservative, we can restrict to the case of $G$ finite. Indeed if $\mathrm{res}^G_H X = 0$ for all finite subgroups $H$, then $X = 0$ by proposition 4 we must have $X = 0$. Now, let $C$ denote the subcategory of $\mathrm{Sp}^G$ generated by suspension spectra of orbits under colimits and desuspensions. We claim that $C = \mathrm{Sp}^G$.

Since $\mathrm{Space}_{G,*}$ is generated by $G/H_+$ under colimits and $\Sigma^\infty$ is a left adjoint functor, it is clear that suspension spectra are contained in $C$. Now we claim that $C$ is closed under smash products. In fact, for all $H$, the collection of $X$ such that $\Sigma^\infty_+ G/H \wedge X \in C$ is closed under colimits, desuspensions and contains all orbits, so if $X \in C$, $\Sigma^\infty_+ G/H \wedge X \in C$. Similarly, if we fix $X \in C$, the collection of $Y$ such that $Y \wedge X \in C$ is closed under colimits, desuspensions and contains all orbits, so it contains $C$. Finally, since $\Sigma^\infty_+ G/H$ are self-dual, the dual of $\Sigma^\infty X$ for $X$ a pointed finite $G$-CW-complex is also in $C$. In particular $\mathbb{S}^{-n\rho} = D(\Sigma^\infty S^{n\rho})$ is contained in $C$ for all $n$. But for every $G$-spectrum $X$ we can write it as

$$X = \operatorname*{colim}_n \mathbb{S}^{-n\rho} \wedge \Sigma^\infty \Omega^\infty (\mathbb{S}^{n\rho} \wedge X)$$

so $X$ is also contained in $C$. $\square$

**Corollary 8.** *The collection $\{G/H\}_{H<G}$ for $H$ varying across all finite subgroups is a set of compact generators for $\mathrm{Sp}^G_{\mathcal{F}}$. In particular, if $G$ is a compact Lie group, $\mathrm{Sp}^G_{\mathcal{F}}$ is rigidly-compactly generated in the sense of [1].*

## 5. The Wirthmüller isomorphism

Let $G$ be a finite group and $H < G$ a subgroup. Then, we have a natural isomorphism

$$\mathrm{res}^G_H \mathrm{coind}^G_H X = F(\Sigma^\infty_+ G/H, X)$$

(remember that both functors are computed pointwise). Note that here $\Sigma^\infty_+ G/H$ is seen as a $H$**-space** and not as a $G$-space. In particular we have a splitting $G/H \cong \{eH\} \amalg I$ and so there is a canonical map $G/H_+ \to \mathbb{S}$ sending only $eH$ to the non-basepoint. This gives us a natural transformation

$$X \to F(\Sigma^\infty_+ G/H, X) \cong \mathrm{res}_H \mathrm{coind}^G_H X\,.$$

By adjunction, this provides us with a map

$$W : \mathrm{ind}^G_H X \to \mathrm{coind}^G_H X\,.$$

**Theorem 9** (Wirthmüller isomorphism)**.** *The map $W$ is a natural equivalence.*

*Proof.* Since $\mathrm{Sp}^H$ and $\mathrm{Sp}^G$ are rigidly compactly generated, we can apply theorem 3.3 in [1] to $f^* = \mathrm{res}^G_H$ and observe that both sides commute with colimits. Hence it suffices to prove that this map is an equivalence when $X = \Sigma^\infty_+ H/K$. But, from 3.3.(3.13) in [1] it follows that

$$\mathrm{coind}^G_H F(\Sigma^\infty_+ H/K, \mathbb{S}) \cong F(\mathrm{ind}^G_H \Sigma^\infty_+ H/K, \mathbb{S}) \cong F(\Sigma^\infty_+ G/K, \mathbb{S})$$

and a careful study of the definitions proves that the map

$$\mathrm{ind}^G_H \Sigma^\infty_+ H/K \cong \Sigma^\infty_+ G/K \to \mathrm{coind}^G_H \Sigma^\infty_+ H/K \cong \mathrm{coind}^G_H F(\Sigma^\infty_+ H/K, \mathbb{S}) \cong F(\Sigma^\infty_+ G/K, \mathbb{S})\,,$$

is precisely the map which is an equivalence by Atiyah duality. □

## 6. Borel $G$-spectra

Let us consider the functor $u : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}^{BG}$ given by $\mathrm{map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ G/e, -)$ ("underlying spectrum"), where the action is induced by the right action of $G$ on $\Sigma^\infty_+ G/e$. We know it preserves limits and colimits, so it has a left and right adjoint, which we will denote with $f$ and $b$ respectively. In fact we know what $f$ is already: by Morita theory

$$fE = \Sigma^\infty_+ G/e \otimes_{\mathbb{S}[G]} E\,.$$

**Proposition 10.** *The functors $f$ and $b$ are fully faithful.*

*Proof.* To prove $f$ is fully faitful, we need to prove the counit $uf \to 1$ is an equivalence. But, both $uf$ and 1 commute with colimits, so it suffices to check it for $\mathbb{S}[G]$, where it is clear since $u\Sigma^\infty_+ G/e = \mathbb{S}[G]$. But then, to prove that $b$ is fully faithful, we need to check that the unit $1 \to ub$ is an equivalence. But this is precisely the adjoint natural transformation of $uf \to 1$ and so it is an equivalence. □

Our goal is going to be to describe the essential image of $f$ and $b$. We start by $b$.

Let $EG$ be the $G$-space given by

$$EG^H = \begin{cases} * & \text{if } H = 1 \\ \varnothing & \text{otherwise} \end{cases}\,.$$

Note that for any $G' < G$ subgroup, $\mathrm{res}^G_{G'} EG \cong EG'$.

**Proposition 11.** *For a $G$-spectrum $X$ the following properties are equivalent:*

- *The map $X \to F(\Sigma^\infty_+ EG, X)$ is an equivalence;*
- *For each $H < G$ finite the natural map $X^H \to X^{hH}$ is an equivalence;*
- *$X$ is in the essential image of $b$, that is $X \to buX$ is an equivalence.*

*We call such a spectrum a **Borel complete** $G$-spectrum or simply a **Borel** $G$-spectrum.*

To prove the proposition we will need an important property of the space $EG$. Note that a $G$-space $X$ has a map to $EG$ if and only if $X^H = \varnothing$ for $H \neq 1$ and in that case the space of maps is contractible.

**Lemma 12.** *Let $G$ act on the right on $G/e$ in $\mathrm{Space}_{G,\mathcal{F}}$. Then the canonical map $(G/e)_{hG} \to EG$ is an equivalence in $\mathrm{Space}_{G,\mathcal{F}}$.*

*Proof.* Since $(G/e)^H = \varnothing$ if $H \neq 1$, to prove the lemma we just need to show that the underlying space of $(G/e)_{hG}$ is contractible. But this follows from a standard extra degeneracy lemma. $\square$

We can now proceed to the proof of the proposition.

*Proof.* Let us prove (1) implies (2). Up to restricting to $H$ we can assume $G = H$. Then the map induced on fixed points

$$X^G \to F(\Sigma^\infty_+ EG, X)^G \cong \mathrm{map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ (G/e)_{hG}, X) \cong X^{hG}$$

is exactly the map we wanted to prove is an equivalence.

Now we will prove (2) implies (3). In fact the map on $H$-fixed poitns

$$X^H \to (buX)^H \cong \mathrm{map}_{\mathrm{Sp}^G_{\mathcal{F}}}(\Sigma^\infty_+ G/H, buX) \cong \mathrm{map}_{\mathrm{Sp}^{BG}}(\mathbb{S}[G/H], uX) \cong X^{hH} ,$$

is the map we claimed it was an equivalence.

Finally we will prove (3) implies (1). In fact, if $X = bY$, the map $bY \to F(\Sigma^\infty_+ EG, bY)$ induces for every $Z \in \mathrm{Sp}^G_{\mathcal{F}}$

$$\mathrm{map}_{\mathrm{Sp}^G_{\mathcal{F}}}(Z, bY) \cong \mathrm{map}_{\mathrm{Sp}^{BG}}(uZ, Y) \to \mathrm{map}_{\mathrm{Sp}^G_{\mathcal{F}}}(Z, F(\Sigma^\infty_+ EG, bY)) \cong \mathrm{map}_{\mathrm{Sp}^{BG}}(u(Z \wedge \Sigma^\infty_+ EG), Y)$$

but the projection map $\Sigma^\infty_+ EG \wedge Z \to Z$ is an equivalence on underlying spectra (this is true for suspension spectra and both directions commute with colimits). $\square$

**Lemma 13.** *Let $G$ be a finite group. Then there is a natural equivalence of spectra*

$$(\Sigma^\infty_+ EG \wedge X)^G \cong X_{hG} .$$

*In particular, for any topological group $G$, the functor $\Sigma^\infty_+ EG \wedge -$ sends equivalences of underlying spectra to equivalences.*

*Proof.*

$$(\Sigma^\infty_+ EG \wedge X)^G \cong ((\Sigma^\infty_+ G/e)_{hG} \wedge X)^G \cong (\Sigma^\infty_+ G/e \wedge X)^G_{hG} \cong X_{hG}$$

(where in the last step we used Atiyah duality; in the case of compact Lie group a shift whould appear). $\square$

Finally, to describe the essential image of $f$, we need to describe the natural transformation $fu \to 1$.

**Proposition 14.** *For $X \in \mathrm{Sp}^G_{\mathcal{F}}$ the following are equivalent*

- *$X$ is in the stable subcategory generated under colimits by $\Sigma^\infty_+ G/e$;*
- *$X$ is in the essential image of $f$.*
- *$\Sigma^\infty_+ EG \wedge X \to X$ is an equivalence*

*In particular, $(fX)^H \cong X_{hH}$. Such spectra are called **free spectra** (because they are built out of free cells).*

*Proof.* We know (1) and (2) are equivalent because $f$ is fully faithful, its essential image is closed under colimits and $\mathrm{Sp}^{BG}$ is generated by $\mathbb{S}[G]$, which is such that $f(\mathbb{S}[G]) = \Sigma^\infty_+ G/e$.

To prove (1) $\Rightarrow$ (3) it suffices to prove it when $X = \Sigma^\infty_+ G/e$, which is obvious.

To prove (3) $\Rightarrow$ (2) we consider the square

$$\begin{tikzcd} \Sigma^\infty_+ EG \wedge fu(X) \arrow[r] \arrow[d] & fu(X) \arrow[d] \\ \Sigma^\infty_+ EG \wedge X \arrow[r] & X \end{tikzcd} .$$

We know that both horizontal arrows are equivalences and the left vertical arrow is an equivalence because $\Sigma^\infty_+ EG \wedge -$ sends equivalences on the underlying spectra to equivalences. So the right vertical arrow is an equivalence too. $\square$

## 7. The tom Dieck splitting

For this section, let $G$ be a finite group and let $X$ be a pointed $G$-space. Then, for any finite subgroup $H$ of $G$ we can construct a natural map

$$\Sigma^\infty X^H \cong \Sigma^\infty \mathrm{Map}_{\mathrm{Space}_{G,*}}(G/H_+, X) \to \mathrm{map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ G/H, \Sigma^\infty X) = (\Sigma^\infty X)^H$$

adjoint to the map

$$\mathrm{Map}_{\mathrm{Space}_{G,*}}(G/H_+, X) \to \mathrm{Map}_{\mathrm{Sp}^G_{\mathcal{F}}}(\Sigma^\infty_+ G/H, \Sigma^\infty X) \cong \Omega^\infty \mathrm{map}_{\mathrm{Sp}^G}(\Sigma^\infty_+ G/H, \Sigma^\infty X)$$

induced by $\Sigma^\infty$. Now, thanks to Atiyah duality, there is an equivalence

$$(\Sigma^\infty X)^H \cong F(\Sigma^\infty_+ G/H, \Sigma^\infty X)^G \cong (\Sigma^\infty_+ G/H \wedge \Sigma^\infty X)^G .$$

So

$$(\Sigma^\infty X)^H_{hWH} \cong (\Sigma^\infty_+ G/H \wedge \Sigma^\infty X)^G_{hWH} \cong \left(\Sigma^\infty_+ G/H_{hWH} \wedge \Sigma^\infty X\right)^G .$$

For brevity, let $D_H = (G/H)_{hWH}$. Then, the map $D_H \to *$ induces a map

$$(\Sigma^\infty X)^H_{hWH} \cong (\Sigma^\infty_+ D_H \wedge \Sigma^\infty X)^G \to (\Sigma^\infty X)^G .$$

We can put all of them together to get a map

$$\zeta_X : \bigoplus_{(H)} \Sigma^\infty X^H_{hWH} \to \bigoplus_{(H)} (\Sigma^\infty X)^H_{hWH} \to (\Sigma^\infty X)^G$$

where the sum is indexed by the conjugacy classes of subgroups of $G$.

**Theorem 15** (tom Dieck splitting)**.** *Let $G$ be a finite group. Then for any $G$-space $X$, the natural transformation $\zeta_X$ is an equivalence of spectra.*

**Remark 16.** *This theorem should be thought of as an equivariant version of Segal's formulation of the Barrat-Priddy-Quillen theorem. That is saying that for a space $X$, $\Sigma^\infty_+ X$ is the connective spectrum corresponding to the group completion of the groupoid of finite sets with a map to $X$. This version is essentially saying that, for a $G$-space $X$, $(\Sigma^\infty_+ X)^G$ is the connective spectrum corresponding to the group-completion of the groupoid of finite $G$-sets with a map to $X$.*

We will prove the theorem in various steps.

$\boxed{\textbf{Step 1}}$ The natural transformation $\zeta_X$ is an equivalence for all $X$ such that $X^H = *$ if $H \neq G$.

Note that almost all terms vanish, and we are reduced to proving that the map

$$\Sigma^\infty(X^G) \to (\Sigma^\infty X)^G$$

is an equivalence. Using the definition of $\Sigma^\infty$ and of $(-)^G$ we see that we need to prove

$$\operatorname*{colim}_{V\in\mathcal{J}} \mathrm{Map}_*(S^V, S^V \wedge X^G) \to \operatorname*{colim}_{V\in\mathrm{Rep}_G} \mathrm{Map}_{\mathrm{Space}_{G,\mathcal{F},*}}(S^V, S^V \wedge X)$$

is an equivalence (where the map is induced by seeing an inner product space $V$ as a $G$-representation with trivial action). We will prove this by providing an explicit inverse

$$\operatorname*{colim}_{V\in\mathrm{Rep}_G} \mathrm{Map}_{\mathrm{Space}_{G,\mathcal{F},*}}(S^V, S^V \wedge X) \to \operatorname*{colim}_{V\in\mathcal{J}} \mathrm{Map}_*(S^V, S^V \wedge X^G)$$

sending a map of $G$-spaces $f : S^V \to S^V \wedge X$ to $f^G : S^{V^G} \to S^{V^G} \wedge X^G$. it is obvious that it is a left inverse, so we only need to show this new map is an equivalence. This is provided by the following lemma. Note that since the map $\mathrm{Rep}_G \to \mathcal{J}$ sending $V$ to $V^G$ is cofinal, we only need to prove it is an equivalence levelwise.

**Lemma 17.** *Let $X$ be a pointed $G$-space such that $X^H = *$ for $H \neq G$. Then for every $G$-space $B$ the map*

$$\mathrm{Map}_G(B, X) \to \mathrm{Map}_*(B^N, X^G)$$

*is an equivalence.*

*Proof.* Since the category of those $B$ satisfying the thesis is closed under colimits, it suffices to prove it for $B = G/H$. There are two cases

- $H = G$. Then the map is $X^H \to X^H$, which is obviously an equivalence.
- $H \neq G$. Then $(G/H)^G = \varnothing$, so the right hand side is contractible. The left hand side is $X^H$, which is also contractible by hypothesis.

□

$\boxed{\textbf{Step 2}}$ Let $K < G$ be a subgroup of $G$. Then the natural transformation $\zeta_X$ is an equivalence for all $X$ such that $X^H = *$ for $H \neq K$.

As before, all terms vanish except for the $H = K$ term, so we need to prove the map $\Sigma^\infty X^K_{hWK} \to (\Sigma^\infty X)^G$ is an equivalence. This is the composition of two maps

$$\Sigma^\infty X^K_{hWK} \to (\Sigma^\infty X)^K_{hWK} \cong \Sigma^\infty((D_K)_+ \wedge X)^G \to \left(\Sigma^\infty_+ X\right)^G$$

The first map is an equivalence by step 1, so it suffices to prove the second map is an equivalence. We claim

$$(D_K)_+ \wedge X \to X$$

is an equivalence of $G$-spaces. In fact it is obviously an equivalence on $H$-fixed points for $H \neq K$ (since in that case both sides are contractible) so it suffices to check that $D^K_K = *$. But

$$(D_K)^K = (G/K)^K_{hWK} = (NK/K)_{hWK} = * .$$

$\boxed{\textbf{Step 3}}$ The general case.

For any finite subgroup $H < G$ let $EG/H$ be the $G$-space defined by

$$(EG/H)^K = \begin{cases} * & \text{if } K \subseteq gHg^{-1} \\ \varnothing & \text{otherwise} \end{cases} .$$

Note that the assignment $G/H \to EG/H$ forms a functor $\mathbf{O}_G^{\mathrm{op}} \to \mathrm{Space}_G$ whose colimit is the point. Both sides of tom Dieck splitting commute with colimits in $X$, so it suffices to prove it holds for $X \wedge EG/H_+$ for all $H$. We want to prove that if the tom Dieck splitting is true for $X \wedge EG/K_+$ for all $K < H$, then it is true for $X \wedge EG/H_+$. From there the general case will follow from an easy induction on $\#H$ (the inductive hypothesis is vacously true for $H = e$). Let us consider the cofiber sequence

$$\operatorname*{colim}_{\substack{H/K \in \mathbf{O}_H^{\mathrm{op}} \\ H/K \neq *}} X \wedge EG/K_+ \to X \wedge EG/H_+ \to X \wedge C_H \,.$$

Then we know the tom Dieck splitting holds for the first term in the sequence (since it is a colimit of $G$-spaces for which the tom Dieck splitting holds). If we knew that the tom Dieck splitting held for $X \wedge C_H$ we would be done. But since colimits in $\mathrm{Space}_{G,*}$ are computed levelwise on fixed points

$$C_H^K = \begin{cases} S^0 & \text{if } K = H \\ * & \text{otherwise} \end{cases} .$$

Hence $X \wedge C_H$ is one of the spaces considered on step 2.

## 8. Geometric fixed points

The genuine fixed point functor is a nice fixed point functor, but it has some puzzling behaviour, that differs from what we'd like. For example it is only lax symmetric monoidal (since it is corepresented by a coalgebra object $\Sigma^\infty_+ G/e$) but not symmetric monoidal. In this section we'll see a different fixed point functor that will fix this.

Let $G$ be a topological group and $N \lhd G$ be a finite normal subgroup. The we can define the $G$-space $E\mathcal{P}_N$ by

$$(E\mathcal{P}_N)^H = \begin{cases} \varnothing & \text{if } N < K \\ \varnothing & \text{otherwise} \end{cases} .$$

Let $\widetilde{E\mathcal{P}_N}$ be the undreduced suspension of $E\mathcal{P}_N$, i.e. the cofiber in pointed spaces of

$$(E\mathcal{P}_N)_+ \to S^0 \to \widetilde{E\mathcal{P}_N}$$

Then we define the **geometric fixed points functor**

$$\Phi^N : \mathrm{Sp}^G \to \mathrm{Sp}^{G/N} \qquad X \mapsto (X \wedge \Sigma^\infty \widetilde{E\mathcal{P}_N})^N \,.$$

From the tom Dieck splitting it is easy to see that

$$\Phi^N \Sigma^\infty X \cong \Sigma^\infty (X^N) \,.$$

We'd like to establish a few properties of this functor

**Lemma 18.** *Let $X \in \mathrm{Sp}^G_{\mathcal{F}}$. Then the following are equivalent*

- *The map $X \to X \wedge \Sigma^\infty \widetilde{E\mathcal{P}_N}$ is an equivalence;*
- *For all $H$ finite subgroups of $G$ that do not contain $N$, $X^H = 0$.*

*Proof.* (1) $\Rightarrow$ (2) follows because

$$\mathrm{res}^G_H (X \wedge \Sigma^\infty \widetilde{E\mathcal{P}_N}) \cong \mathrm{res}^G_H X \wedge \Sigma^\infty \mathrm{res}^G_H \widetilde{E\mathcal{P}_N} = 0 \,.$$

Let us now prove (2) $\Rightarrow$ (1). First let us show that we can assume $G$ finite. In fact it is enough to show that, for every finite subgroup $K$ of $G$, the map

$$\mathrm{res}^G_K X \to \mathrm{res}^G_K X \wedge \Sigma^\infty \widetilde{E\mathcal{P}_N}$$

is an equivalence. But if $K$ does not contain $N$ both sides are contractible (since $\mathrm{res}^G_K X = 0$), so it suffices to do the case $G = K$ finite.

Now assume $G$ finite and $X^H = 0$ for all $H$ not containing $N$. Then it suffices to show $X \wedge \Sigma^\infty_+ E\mathcal{P}_N = 0$. But $E\mathcal{P}_N$ is[3] a colimit of orbits of the form $G/H$ where $H$ does not contain $N$. Hence the thesis will follow from the more general statment $X \wedge Y = 0$ for all $Y$ in the stable subcategory generated under colimits by $\Sigma^\infty_+ G/H$ for $H$ not containing $N$. Since smashing by $X$ commutes with colimits it suffices to prove

$$X \wedge \Sigma^\infty_+ G/H = 0$$

for all $H$ not containing $N$. But for all subgroups $K \subseteq G$, using Atiyah duality for finite $G$,

$$(X \wedge \Sigma^\infty_+ G/H)^K \cong \mathrm{map}_G(\Sigma^\infty_+(G/H \times G/K)X) = \bigoplus_{G/H' \subseteq G/H \times G/K} X^{H'},$$

but since $N$ is not contained in the stabilizer of any point of $G/H \times G/K$, we are done. ∎

Now, note that $\mathbb{S} \to \Sigma^\infty \widetilde{E\mathcal{P}_n}$ is an idempotent: the spectra described in the lemma are the local objects for a smashing localization. Moreover $\Phi^N$ is symmetric monoidal: it is the composite of two lax symmetric monoidal functors, so it has a lax symmetric monoidal structure and the map $\mathbb{S} \to \Phi^N \mathbb{S}$ is an equivalence. Finally the set of those $X, Y$ such that the map $\Phi^N X \wedge \Phi^N Y \to \Phi^N(X \wedge Y)$ is an equivalence is closed under colimits and it contains the suspension spectra.

**Theorem 19.** *The functor $(-)^N$ restricts to an equivalence between the subcategory of $\mathrm{Sp}^G_{\mathcal{F}}$ described in the lemma and $\mathrm{Sp}^{G/N}_{\mathcal{F}}$.*

*Proof.* Let $C$ be the subcategory of those spectra satisfying the equivalent conditions of the lemma. Since $\Phi^N$ coincides with $(-)^N$ on $C$, it commutes with all limits and colimits. Moreover it is evidently conservative. So, if we show that its left adjoint is fully faithful we have concluded (since conservativity implies that if the unit is an equivalence, so is the counit).

Let us denote the left adjoint by $L_N$. Then we need to prove that $1 \to \Phi^N L_N$ is an equivalence. Since both sides are colimit-preserving functors it is enough to show that it is an equivalence for the orbits of the form $\Sigma^\infty_+ G/K \in \mathrm{Sp}^{G/N}_{\mathcal{F}}$. But $L_N \Sigma^\infty_+ G/K$ is the object of $C$ that corepresents the functor $X \mapsto X^K$, that is $\Sigma^\infty \widetilde{E\mathcal{P}_N} \wedge \Sigma^\infty_+ G/K$, and its geometric fixed points are evidently $\Sigma^\infty_+ G/K$ again, by the description of the geometric fixed points of a suspension spectrum. ∎

**Corollary 20.** *The functor $\Phi^N : \mathrm{Sp}^G_{\mathcal{F}} \to \mathrm{Sp}^{G/N}_{\mathcal{F}}$ has a fully faithful right adjoint, with essential image those spectra $X$ such that $X^H = 0$ for all $H$ not containing $N$.*

[3]Since every presheaf is a colimit indexed by its category of points

Let us now consider the case of $G = C_{p^\infty}$. Then there is a diagram

$$\begin{tikzcd}
X \wedge \Sigma^\infty_+ EG \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \wedge \Sigma^\infty \widetilde{EG} \arrow[d] \\
F(\Sigma^\infty_+ EG, X) \wedge \Sigma^\infty_+ EG \arrow[r] & F(\Sigma^\infty_+ EG, X) \arrow[r] & F(\Sigma^\infty_+ EG, X) \wedge \Sigma^\infty \widetilde{EG}
\end{tikzcd}$$

The left vertical arrow is an equivalence since $- \wedge \Sigma^\infty_+ EG$ sends equivalences of underlying spectra to equivalences. Moreover, if we let $n \geq 1$ and we take $C_{p^n}$-fixed points we obtain

$$\begin{tikzcd}
X_{hC_{p^n}} \arrow[r] \arrow[d, equal] & X^{C_{p^n}} \arrow[r] \arrow[d] & \left(\Phi^{C_p} X\right)^{C_{p^n}/C_p} \arrow[d] \\
X_{hC_{p^n}} \arrow[r] & X^{hC_{p^n}} \arrow[r] & X^{tC_{p^n}}
\end{tikzcd}.$$

For $X$ a $C_{p^\infty}$-spectrum, we will denote the map $X^{C_{p^n}} \to \left(\Phi^{C_p} X\right)^{C_{p^n}/C_p}$ by $R$ and call it the **restriction** map. Moreover, we will call the inclusion of fixed points **Frobenius** and denote it by $F$.[4]

## 9. Cyclotomic spectra

Let $p$ be a prime. Then a **genuine $p$-cyclotomic spectrum** is a $C_{p^\infty}$-spectrum $X$ together with an equivalence

$$X \xrightarrow{\sim} \Phi^{C_p} X$$

as $C_{p^\infty}$-spectra. There is a canonical functor from genuine $p$-cyclotomic spectra to $p$-cyclotomic spectra obtained by postcomposing the equivalence $X \to \Phi^{C_p} X$ with the natural map $\Phi^{C_p} X \to X^{tC_p}$.

Now, if $X$ is a genuine $p$-cyclotomic spectrum, the diagram of the previous section induces a commutative square

$$\begin{tikzcd}
X^{C_{p^n}} \arrow[r, "R"] \arrow[d, "F"] & (\Phi^{C_p} X)^{C_{p^n}/C_p} \cong X^{C_{p^{n-1}}} \arrow[d, "F"] \\
X^{C_{p^{n-1}}} \arrow[r, "R"] & (\Phi^{C_p} X)^{C_{p^{n-1}}/C_p} \cong X^{C_{p^{n-2}}}
\end{tikzcd}$$

And, passing to the limit in $R$ we have

$$TR(X; p) := \lim_R X^{C_{p^n}}$$

[4]These names originate from the fact that when $X = THH(k)$ for some ring $k$, the maps induced on $\pi_0 THH(k)^{C_{p^n}} = W_{p^n} k$ are the classical restriction and Frobenius map for rings of Witt vectors. Similarly, the map induced on fixed points by the transfer $\Sigma^\infty_+ S^1/C_{p^n} \to \Sigma^\infty_+ S^1/C_{p^{n+1}}$ is called the **Verschiebung**.

equipped with an endomorphism $F$. Finally we define

$$TC(X;p) := TR(X;p)^{hF} = \mathrm{Eq}\left(TR(X;p) \underset{F}{\overset{1}{\rightrightarrows}} TR(X;p)\right)$$

It is possible (although nontrivial) to assemble all endofunctors $\Phi^{C_n} : \mathrm{Sp}_{\mathcal{F}}^{S^1} \to \mathrm{Sp}_{\mathcal{F}}^{S^1}$ into an action of the multiplicative monoid of natural numbers $\mathbb{N}_\times$ on the $\infty$-category $\mathrm{Sp}_{\mathcal{F}}^{S^1}$. A **genuine cyclotomic spectrum** is a homotopy fixed point for this action. As before, there is a functor from genuine cyclotomic spectra to cyclotomic spectra.

## 10. The cyclotomic structure on $THH$

To see that $THH(R)$ has a $C_n$-structure for each $n$, let us consider the map

$$sd_n : \Delta \to \Delta$$

that sends $I$ to $I \times \{1 < \cdots < n\}$ with the lexicographic order. Then we claim

$$sd_n B^{cyc}(R) \cong B^{cyc}(R^{\otimes n}, R^{\otimes n})$$

where $R^{\otimes n}$ is a bimodule with the standard left action, but the twisted right action

$$(x_1 \otimes \cdots \otimes x_n) \cdot (r_1 \otimes \cdots \otimes r_n) = x_n r_1 \otimes x_1 r_2 \otimes \cdots x_{n-1} r_n\,.$$

So if we define the $C_n$-spectrum

$$THH(R) = |B^{cyc}(N^{C_n}R, N^{C_n}R)$$

(where $N^{C_n}R$ is a bimodule over itself with the standard left action and the twisted right action), its underlying spectrum is $THH(R)$. It is possible (although non obvious) to prove that this $C_n$-spectra are compatible with the various restrictions and so induce an element of $\mathrm{Sp}_{\mathcal{F}}^{S^1}$. Moreover, since $\Phi^{C_n}$ commutes with colimits and tensor products, it is clear that there is an equivalence

$$\Phi^{C_n} THH(R) = |B^{cyc}(\Phi^{C_n}N^{C_n}R, \Phi^{C_n}N^{C_n}R))| \cong |B^{cyc}(R,R)| = THH(R)\,.$$

## Appendix A. Constructing $\mathrm{Sp}^G$

Let $G$ be a compact Lie group. In this section we will construct a presentable fibration $\mathcal{E} \to \mathrm{Rep}_G$ such

- For every $V \in \mathrm{Rep}_G$ the fiber over $V$ is equivalent to $\mathrm{Space}_{G,*}$;
- For every $f : V \to W$ is a $G$-equivariant isometric embedding, the pullback functor $f^* : \mathrm{Space}_{G,*} \to \mathrm{Space}_{G,*}$ is given by $\mathrm{Map}(S^{fV^\perp}, -)$.

We will construct $\mathcal{E}$ as the simplicial nerve of a simplicial category $\mathbf{E}$ defined as follows

- Objects are pairs $(V, X)$ where $V$ is an orthogonal $G$-representation and $X$ is a well-pointed $G$-topological space;
- An $n$-simplex in $\mathrm{Map}((X,V),(X',V'))$ is given by a pair $(h, \xi)$ where $h : V \times \Delta^n \to V'$ is a $\Delta^n$-family of $G$-equivariant isometric embeddings (i.e. a continous map such that for each $t \in \Delta^n$, $h|_{V\times\{t\}}$ is a $G$-equivariant isometric embedding) and

$$\xi : Th(\Delta^n) \wedge X \to X'$$

is a $G$-equivariant map, where $Th(\Delta^n)$ is the Thom space for the vector bundle on $\Delta^n$ with total space

$$\{(t,x) \in \Delta^n \times V' \mid x \perp h(V \times \{t\})\} .$$

($\xi$ should be thought of as a continously variying family of maps $S^{V^\perp} \wedge X \to X'$). There is an obvious projection of simplicial categories $\mathbf{E} \to \mathbf{Rep}_G$ that remembers only .

**Theorem 21.** *The simplicial nerve $N^\Delta(-)$ turns the projection $\mathbf{E} \to \mathbf{Rep}_G$ into an inner fibration where an edge $(f : V \to V', \xi : S^{V^\perp} \wedge X \to X')$ is cocartesian if and only if $\xi$ is an equivalence.*

*Proof.* This follows from [8, 2.4.1.10], after we verify that the projection induces a Kan fibration on mapping spaces. But this follows from $Th(\Lambda^n_i) \subseteq Th(\Delta^n)$ being a trivial cofibration in the model structure on $G$-spaces and $X$ being a well-pointed $G$-space. □

Since the fibers of $p : \mathcal{E} \to \mathrm{Rep}_G$ are presentable $\infty$-categories and the push-forward $f_!$ is given by $S^{fV^\perp} \wedge -$, which is evidently left adjoint to the functor $\mathrm{Map}(S^{fV^\perp}, -)$, our construction is concluded.

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