What precisely are E_∞ ring spaces and E_∞ ring spectra?

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E_∞ ring spectra were defined in 1972, but the term has since acquired several alternative meanings. The same is true of several related terms. The new formulations are not always known to be equivalent to the old ones and even when they are, the notion of *equivalence* needs discussion: Quillen equivalent categories can be quite seriously inequivalent. Part of the confusion stems from a gap in the modern resurgence of interest in E_∞ structures. E_∞ ring spaces were also defined in 1972 and have never been redefined. They were central to the early applications and they tie in implicitly to modern applications. We summarize the relationships between the old notions and various new ones, explaining what is and is not known. We take the opportunity to rework and modernize many of the early results. New proofs and perspectives are sprinkled throughout.

Introduction

In the early 1970's, the theory of E_∞ rings was intrinsically intertwined with a host of constructions and calculations that centered around the relationship between E_∞ ring spectra and E_∞ ring spaces (see Cohen, Lada and May [10] and May [27]). The two notions were regarded as being on essentially the same footing, and it was understood that the homotopy categories of ringlike E_∞ ring spaces (π_0 is a ring and not just a semi-ring) and of connective E_∞ ring spectra are equivalent.

In the mid 1990's, modern closed symmetric monoidal categories of spectra were introduced, allowing one to define a commutative ring spectrum to be a commutative monoid in any such good category of spectra. The study of such rings is now central to stable homotopy theory. Work of several people, especially Schwede and Shipley, shows that, up to zigzags of Quillen equivalences, the resulting categories of commutative ring spectra are all equivalent. In one of these good categories, commutative ring spectra are equivalent to E_∞ ring spectra. The terms E_∞ ring spectra and commutative ring spectra have therefore been used as synonyms in recent years. A variant notion of E_∞ ring spectrum that can be defined in any such good category of spectra has also been given the same name.

From the point of view of stable homotopy theory, this is perfectly acceptable, since these notions are tied together by a web of Quillen equivalences. From the point of view of homotopy theory as a whole, including both space and spectrum level structures, it is not acceptable. Some of the Quillen equivalences in sight necessarily lose space level information, and in particular lose the original connection between E_∞ ring spectra and E_∞ ring spaces. Since some modern applications, especially those connected with cohomological orientations and spectra of units, are best understood in terms of that connection, it seems to me that it might be helpful to offer a thorough survey of the structures in this general area of mathematics.

This will raise some questions. As we shall see, some new constructions are not at present known to be equivalent, in any sense, to older constructions of objects with the same name, and one certainly cannot deduce comparisons formally. It should also correct some misconceptions. In some cases, an old name has been reappropriated for a definitely inequivalent concept.

The paper divides conceptually into two parts. First, in Sections 1–10, we describe and modernize additive and multiplicative infinite loop space theory. Second, in Sections 11–13, we explain how this early 1970's work fits into the modern framework of symmetric monoidal categories of spectra. There will be two sequels (see May [32; 33]). In the first, we recall how to construct E_∞ ring spaces from bipermutative categories. In the second, we review some of the early applications of E_∞ ring spaces. The following list of sections may help guide the reader.

1. The definition of E_∞ ring spaces
2. $\mathscr{I}$–spaces and the linear isometries operad
3. The canonical E_∞ operad pair
4. The monadic interpretation of the definitions
5. The definition of E_∞ ring prespectra and E_∞ ring spectra
6. The monadic interpretation of E_∞ ring spectra
7. The relationship between E_∞ ring spaces and E_∞ ring spectra
8. A categorical overview of the recognition principle
9. The additive and multiplicative infinite loop space machine
10. Localizations of the special unit spectrum $sl_1 R$
11. E_∞ ring spectra and commutative S–algebras
12. The comparison with commutative diagram ring spectra
13. Naive E_∞ ring spectra
14. Appendix A. Monadicity of functors and comparisons of monads
15. Appendix B. Loop spaces of E_∞ spaces and the recognition principle

We begin by defining E_∞ ring spaces. As we shall see in Section 1, this is really quite easy. The hard part is to produce examples, and that problem will be addressed in [32]. The definition requires a pair $(\mathscr{C}, \mathscr{G})$ of E_∞ operads, with $\mathscr{G}$ acting in a suitable way on $\mathscr{C}$, and E_∞ ring spaces might better be called $(\mathscr{C}, \mathscr{G})$–spaces. It is a truism taken for granted since May [25] that all E_∞ operads are suitably equivalent. However, for E_∞ ring theory, that is quite false. The precise geometry matters, and we must insist that not all E_∞ operads are alike. The operad $\mathscr{C}$ is thought of as additive, and the operad $\mathscr{G}$ is thought of as multiplicative.[1]

There is a standard canonical multiplicative operad $\mathscr{L}$, namely the linear isometries operad. We recall it and related structures that were the starting point of this area of mathematics in Section 2. In our original theory, we often replaced $\mathscr{L}$ by an operad $\mathscr{O} \times \mathscr{L}$, and we prefer to use a generic letter $\mathscr{G}$ for an operad thought of as appropriate to the multiplicative role in the definition of E_∞ ring spaces. The original definition of E_∞ ring spaces was obscured because the canonical additive operad $\mathscr{C}$ that was needed for a clean description was only discovered later, by Steiner [43]. We recall its definition and its relationship to $\mathscr{L}$ in Section 3. This gives us the canonical E_∞ operad pair $(\mathscr{C}, \mathscr{L})$.

Actions by operads are equivalent to actions by an associated monad. As we explain in Section 4, that remains true for actions by operad pairs. That is, just as E_∞ spaces can be described as algebras over a monad, so also E_∞ ring spaces can be described as algebras over a monad. In fact, the monadic version of the definition fits into a beautiful categorical way of thinking about distributivity that was first discovered by Beck [5]. This helps make the definition feel definitively right.

As we also explain in Section 4, *different* monads can have *the same* categories of algebras. This has been known for years, but it is a new observation that this fact can be used to substantially simplify the mathematics. In the sequel [32], we will use this idea to give an elementary construction of E_∞ ring spaces from bipermutative categories (and more general input data). We elaborate on this categorical observation and related facts about maps of monads in Appendix A, which is written jointly with Michael Shulman.

The early 1970's definition in May [27] of an E_∞ ring spectrum was also obscure, this time because the notion of *twisted half-smash product* that allows a clean description was only introduced later, in [18]. The latter notion encapsulates operadically parametrized internalizations of external smash products. As we recall in Section 5, E_∞ ring spectra are spectra in the sense of Lewis et al [18] and May [24], which we shall sometimes

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[1]As we recall in Sections 2 and 9, in many applications of additive infinite loop space theory, we must actually start with $\mathscr{G}$, thinking of it as additive, and convert $\mathscr{G}$–spaces to $\mathscr{C}$–spaces before proceeding.

call *LMS spectra* for definiteness, with additional structure. Just as E_∞ spaces can be described in several ways as algebras over a monad, so also E_∞ ring spectra can be described in several ways as algebras over a monad. We explain this and relate the space and spectrum level monads in Section 6.

There is a 0th space functor Ω^∞ from spectra to spaces,[2] which is right adjoint to the suspension spectrum functor Σ^∞. A central feature of the definitions, both conceptually and calculationally, is that the 0th space R_0 of an E_∞ ring spectrum R is an E_∞ ring space. Moreover, the space $GL_1 R$ of unit components in R_0 and the component $SL_1 R$ of the identity are E_∞–spaces, specifically $\mathcal{L}$–spaces.[3] We shall say more about these spaces in Sections 7, 9 and 10.

There is also a functor from E_∞ ring spaces to E_∞ ring spectra. This is the point of multiplicative infinite loop space theory (see May [27; 29]). Together with the 0th space functor, it gives the claimed equivalence between the homotopy categories of ringlike E_∞ ring spaces and of connective E_∞ ring spectra. We recall this in Section 9.

The state of the art around 1975 was summarized in May [28], and it may help orient Sections 1–10 of this paper to reproduce the diagram that survey focused on. (See Figure 1.) Many of the applications alluded to above are also summarized in [28]. The abbreviations at the top of the diagram refer to permutative categories and bipermutative categories. We will recall and rework how the latter fit into multiplicative infinite loop space theory in the sequel [32].

Passage through the black box is the subject of additive infinite loop space theory on the left and multiplicative infinite loop space theory on the right. These provide functors from E_∞ spaces to spectra and from E_∞ ring spaces to E_∞ ring spectra. We have written a single black box because the multiplicative functor is an enriched specialization of the additive one. The black box gives a recognition principle: it tells us how to recognize spectrum level objects on the space level.

We give a modernized description of these functors in Section 9. My early 1970's work was then viewed as "too categorical" by older algebraic topologists.[4] In retrospect, it was not nearly categorical enough for intuitive conceptual understanding. In the

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[2]Unfortunately for current readability, in [27] the notation Σ^∞ was used for the suspension *prespectrum* functor, the notation Ω^∞ was used for the spectrification functor that has been denoted by L ever since [18], and the notation $Q_\infty = \Omega^\infty \Sigma^\infty$ was used for the current Σ^∞.

[3]$GL_1 R$ and $SL_1 R$ were called FR and SFR when they were introduced in [27]. These spaces played a major role in that book, as we will explain in the second sequel [33]. As we also explain there, F and $GL_1 S$ are both tautologically the same and very different. The currently popular notations follow Waldhausen's later introduction [45] of the higher analogues $GL_n(R)$.

[4]Sad to say, nearly all of the older people active then are now retired or dead.
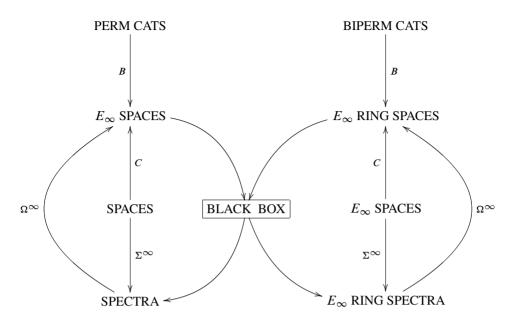
Figure 1: Diagram from May [28]

expectation that I am addressing a more categorically sophisticated modern audience, I explain in Section 8 how the theory is based on an analogy with the Beck monadicity theorem. One key result, a commutation relation between taking loops and applying the additive infinite loop space machine, was obscure in my earlier work, and I'll give a new proof in Appendix B.

The diagram above obscures an essential technical point. The two entries "E_∞ spaces" are different. The one on the upper left refers to spaces with actions by the additive E_∞ operad $\mathscr{C}$, and spaces there mean based spaces with basepoint the unit for the additive operadic product. The one on the right refers to spaces with actions by the multiplicative E_∞ operad $\mathscr{G}$, and spaces there mean spaces with an operadic unit point 1 and a disjoint added basepoint 0. The functor C is the free $\mathscr{C}$–space functor, and it takes $\mathscr{G}$–spaces with 0 to E_∞ ring spaces. This is a key to understanding the various adjunctions hidden in the diagram. The functors labelled C and Σ^∞ are left adjoints.

The unit E_∞ spaces $GL_1 R$ and $SL_1 R$ of an E_∞ ring spectrum R can be fed into the additive infinite loop space machine to produce associated spectra $gl_1 R$ and $sl_1 R$. There is much current interest in understanding their structure. As we recall in Section 10, one can exploit the interrelationship between the additive and multiplicative structures to obtain a general theorem that describes the localizations of $sl_1 R$ at sets of primes in terms of purely multiplicative structure. The calculational force of the result comes from applications to spectra arising from bipermutative categories, as we recall

and illustrate in the second sequel [33]. The reader may prefer to skip this section on a first reading, since it is not essential to the main line of development, but it gives a good illustration of information about spectra of current interest that only makes sense in terms of E_∞ ring spaces.

Turning to the second part, we now jump ahead more than twenty years. In the 1990's, several categories of spectra that are symmetric monoidal under their smash product were introduced. This allows the definition of commutative ring spectra as commutative monoids in a symmetric monoidal category of spectra. Anybody who has read this far knows that the resulting theory of "stable commutative topological rings" has become one of the central areas of study in modern algebraic topology. No matter how such a modern category of spectra is constructed, the essential point is that there is some kind of external smash product in sight, which is commutative and associative in an external sense, and the problem that must be resolved is to figure out how to internalize it without losing commutativity and associativity.

Starting from twisted half-smash products, this internalization was carried out by Elmendorf, Kriz, Mandell and May (EKMM) in [13], where the symmetric monoidal category of S–modules is constructed. We summarize some of the relevant theory in Section 11. Because the construction there starts with twisted half-smash products, the resulting commutative ring spectra are almost the same as E_∞ ring spectra. The *almost* is an important caveat. We didn't mention the unit condition in the previous paragraph, and that plays an important and subtle role in [13] and in the comparisons we shall make. As Lewis noted [17] and we will rephrase, one cannot have a symmetric monoidal category of spectra that is as nicely related to spaces as one would ideally like. The reason this is so stems from an old result of Moore, which says that a connected commutative topological monoid is a product of Eilenberg–Mac Lane spaces.

In diagram spectra, in particular symmetric spectra and orthogonal spectra (see Hovey, Shipley and Smith [15] and Mandell et al [23]), the internalization is entirely different. Application of the elementary categorical notion of left Kan extension replaces the introduction of the twisted half-smash product, and there is no use of operads. However, there is a series of papers by Mandell, May, Schwede and Shipley [22; 23; 38; 39; 41] that lead to the striking conclusion that all reasonable categories of spectra that are symmetric monoidal and have sensible Quillen model structures are Quillen equivalent. Moreover, if one restricts to the commutative monoids, alias commutative ring spectra, in these categories, we again obtain Quillen equivalent model categories.

Nevertheless, as we try to make clear in Section 12, these last Quillen equivalences *lose essential information*. On the diagram spectrum side, one must throw away any information about 0th spaces in order to obtain the Quillen equivalence with EKMM-style commutative ring spectra. In effect, this means that diagram ring spectra do not

know about E_∞ ring spaces and cannot be used to recover the original space level results that were based on implications of that structure.

Philosophically, one conclusion is that fundamentally important homotopical information can be accessible to one and inaccessible to the other of a pair of Quillen equivalent model categories, contrary to current received wisdom. The homotopy categories of connective commutative symmetric ring spectra and of ringlike E_∞ ring spaces are equivalent, but it seems impossible to know this without going through the homotopy category of E_∞ ring spectra, as originally defined.

We hasten to be clear. It does not follow that S–modules are "better" than symmetric or orthogonal spectra. There is by now a huge literature manifesting just how convenient, and in some contexts essential, diagram spectra are.[5] Rather, it does follow that to have access to the full panoply of information and techniques this subject affords, one simply must be eclectic. To use either approach alone is to approach modern stable homotopy theory with blinders on.

A little parenthetically, there is also a quite different alternative notion of a "naive E_∞ ring spectrum" (that is meant as a technical term, not a pejorative). For that, one starts with internal iterated smash products and uses tensors with spaces to define actions by an E_∞ operad. This makes sense in any good modern category of spectra, and the geometric distinction between different choices of E_∞ operad is irrelevant. Most such categories of spectra do not know the difference between symmetric powers $E^{(j)}/\Sigma_j$ and homotopy symmetric powers $(E\Sigma_j)_+ \wedge_{\Sigma_j} E^{(j)}$, and naive E_∞ ring spectra in such a good modern category of spectra are naturally equivalent to commutative ring spectra in that category, as we explain in Section 13.

This summary raises some important compatibility questions. For example, there is a construction, due to Schlichtkrull [36], of unit spectra associated to commutative symmetric ring spectra. It is based on the use of certain diagrams of spaces that are implicit in the structure of symmetric spectra. It is unclear that these unit spectra are equivalent to those that we obtain from the 0th space of an "equivalent" E_∞ ring spectrum. Thus we now have two constructions, not known to be equivalent,[6] of objects bearing the same name. Similarly, there is a construction of (naive) E_∞ symmetric ring spectra associated to op-lax bipermutative categories (which are not equivalent to bipermutative categories as originally defined) that is due to Elmendorf and Mandell [14]. It is again not known whether or not their construction (at least when specialized to genuine bipermutative categories) gives symmetric ring spectra that are "equivalent"

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[5]I've contributed to this in collaboration with Mandell, Schwede, Shipley, and Sigurdsson; see [22; 23; 34].

[6]Since I wrote that, John Lind (at Chicago) has obtained an illuminating proof that they are.

to the E_∞ ring spectra that are constructed from bipermutative categories via our black box. Again, we have two constructions that are not known to be equivalent, both thought of as giving the K–theory commutative ring spectra associated to bipermutative categories.

Answers to such questions are important if one wants to make consistent use of the alternative constructions, especially since the earlier constructions are part of a web of calculations that appear to be inaccessible with the newer constructions. The constructions of [36] and [14] bear no relationship to E_∞ ring spaces as they stand and therefore cannot be used to retrieve the earlier applications or to achieve analogous future applications. However, the new constructions have significant advantages as well as significant disadvantages. Rigorous comparisons are needed. We must be consistent as well as eclectic. There is work to be done!

For background, Thomason and I proved in [35], that any two infinite loop space machines (the additive black box in Diagram 1) are equivalent. The proof broke into two quite different steps. In the first, we compared input data. We showed that Segal's input data (special Γ–spaces) and the operadic input data of Boardman and Vogt and myself (E_∞ spaces) are each equivalent to a more general kind of input data, namely an action of the category of operators $\widehat{\mathscr{C}}$ associated to any chosen E_∞ operad $\mathscr{C}$. We then showed that any two functors $\mathbb{E}$ from $\widehat{\mathscr{C}}$–spaces to spectra that satisfy a group completion property on the 0th space level are equivalent. This property says that there is a natural group completion map $\eta\colon X \longrightarrow \mathbb{E}_0 X$, and we will sketch how that property appears in one infinite loop space machine in Section 9.

No such uniqueness result is known for multiplicative infinite loop space theory. As we explain in [32], variant notions of bipermutative categories give possible choices of input data that are definitely inequivalent. There are also equivalent but inequivalent choices of output data, as I hope the discussion above makes clear. We might take as target any good modern category of commutative ring spectra and then, thinking purely stably, all choices are equivalent. However, the essential feature of [35] was the compatibility statement on the 0th space level. There were no problematic choices since the correct homotopical notion of spectrum is unambiguous, as is the correct homotopical relationship between spectra and their 0th spaces (of fibrant approximations model categorically). As we have indicated, understanding multiplicative infinite loop space theory on the 0th space level depends heavily on choosing the appropriate target category.[7]

With the black box that makes sense of Diagram 1, there are stronger comparisons of input data and 0th spaces than the axiomatization prescribes. Modulo the inversion

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[7]In fact, this point was already emphasized in the introduction of [35].

of a natural homotopy equivalence, the map η is a map of E_∞ spaces in the additive theory and a map of E_∞ ring spaces in the multiplicative theory. This property is central to all of the applications of [10; 27]. For example, it is crucial to the analysis of localizations of spectra of units in Section 10.

This paper contains relatively little that is technically new, although there are many new perspectives and many improved arguments. It is intended to give an overview of the global structure of this general area of mathematics, explaining the ideas in a context uncluttered by technical details. It is a real pleasure to see how many terrific young mathematicians are interested in the theory of structured ring spectra, and my primary purpose is to help explain to them what was known in the early stages of the theory and how it relates to the current state of the art, in hopes that this might help them see connections with things they are working on now. Such a retelling of an old story seems especially needed since notations, definitions, and emphases have drifted over the years and their are some current gaps in our understanding.

Another reason for writing this is that I plan to rework complete details of the analogous equivariant story, a tale known decades ago but never written down. Without a more up-to-date nonequivariant blueprint, that story would likely be quite unreadable. The equivariant version will (or, less optimistically, may) give full details that will supersede those in the 1970's sources.

I'd like to thank the organizers of the Banff conference, Andy Baker and Birgit Richter, who are entirely to blame for the existence of this paper and its sequels. They scheduled me for an open-ended evening closing talk, asking me to talk about the early theory. They then provided the audience with enough to drink to help alleviate the resulting boredom. This work began with preparation for that talk. I'd also like to thank John Lind and an eagle-eyed anonymous referee for catching numerous misprints and thereby sparing the reader much possible confusion.

1 The definition of E_∞ ring spaces

We outline the definition of E_∞ spaces and E_∞ ring spaces. We will be careful about basepoints throughout, since that is a key tricky point and the understanding here will lead to a streamlined passage from alternative inputs, such as bipermutative categories, to E_∞ ring spaces in [32]. Aside from that, we focus on the intuition and refer the reader to [25; 27; 29] for the combinatorial details. Let $\mathscr{U}$ denote the category of (compactly generated) unbased spaces and $\mathscr{T}$ denote the category of based spaces. We tacitly assume that based spaces X have nondegenerate basepoints, or are well-based, which means that $* \longrightarrow X$ is a cofibration.

We assume that the reader is familiar with the definition of an operad. The original definition, and our focus here, is on operads in $\mathscr{U}$, as in [25, pages 1–3], but the definition applies equally well to define operads in any symmetric monoidal category (see May [30; 31]). As in [25], we insist that the operad $\mathscr{O}$ be reduced, in the sense that $\mathscr{O}(0)$ is a point $*$. This is important to the handling of basepoints. Recall that there is an element id $\in \mathscr{O}(1)$ that corresponds to the identity operation[8] and that the j th space $\mathscr{O}(j)$ has a right action of the symmetric group Σ_j. There are structure maps

$$\gamma \colon \mathscr{O}(k) \times \mathscr{O}(j_1) \times \cdots \times \mathscr{O}(j_k) \longrightarrow \mathscr{O}(j_1 + \cdots + j_k)$$

that are suitably equivariant, unital, and associative. We say that $\mathscr{O}$ is an E_∞ operad if $\mathscr{O}(j)$ is contractible and Σ_j acts freely.

The precise details of the definition are dictated by looking at the structure present on the endomorphism operad End_X of a based space X. This actually has two variants, $\mathrm{End}_X^{\mathscr{T}}$ and $\mathrm{End}_X^{\mathscr{U}}$, depending on whether or not we restrict to based maps. The default will be $\mathrm{End}_X = \mathrm{End}_X^{\mathscr{T}}$. The j th space $\mathrm{End}_X(j)$ is the space of (based) maps $X^j \longrightarrow X$ and

$$\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \times \cdots \times f_k).$$

We interpret X^0 to be a point,[9] and $\mathrm{End}_X(0)$ is the map given by the inclusion of the basepoint. Of course, $\mathrm{End}_X^{\mathscr{U}}(0) = X$, so the operad $\mathrm{End}_X^{\mathscr{U}}$ is not reduced.

An action θ of $\mathscr{O}$ on X is a map of operads $\mathscr{O} \longrightarrow \mathrm{End}_X$. Adjointly, it is given by suitably equivariant, unital, and associative action maps

$$\theta \colon \mathscr{O}(j) \times X^j \longrightarrow X.$$

We think of $\mathscr{O}(j)$ as parametrizing a j–fold product operation on X. The basepoint of X must be $\theta(*)$, and there are two ways of thinking about it. We can start with an unbased space X, and then $\theta(*)$ gives it a basepoint, fixing a point in $\mathrm{End}_X^{\mathscr{U}}(0)$, or we can think of the basepoint as preassigned, and then $\theta(*) = *$ is required. With $j_1 = \cdots = j_k = 0$, the compatibility of θ with the structure maps γ ensures that a map of operads $\mathscr{O} \longrightarrow \mathrm{End}_X^{\mathscr{U}}$ necessarily lands in $\mathrm{End}_X = \mathrm{End}_X^{\mathscr{T}}$.

Now consider a pair $(\mathscr{C}, \mathscr{G})$ of operads. Write $\mathscr{C}(0) = \{0\}$ and $\mathscr{G}(0) = \{1\}$. An action of $\mathscr{G}$ on $\mathscr{C}$ consists of maps

$$\lambda \colon \mathscr{G}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 \cdots j_k)$$

for $k \geq 0$ and $j_i \geq 0$ that satisfy certain equivariance, unit, and distributivity properties; see [27, pages 142–144], [29, pages 8–9], or the sequel [32, 4.2]. We will give an

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[8]The notation 1 is standard in the literature, but that would obscure things for us later.

[9]This is reasonable since the product of the empty set of objects is the terminal object.

alternative perspective in Section 4 that dictates the details. To deal with basepoints, we interpret the empty product of numbers to be 1 and, with $k = 0$, we require that $\lambda(1) = \mathrm{id} \in \mathscr{C}(1)$. We think of $\mathscr{C}$ as parametrizing addition and $\mathscr{G}$ as parametrizing multiplication. For example, we have an operad $\mathscr{N}$ such that $\mathscr{N}(j) = *$ for all j. An $\mathscr{N}$–space is precisely a commutative monoid. There is one and only one way $\mathscr{N}$ can act on itself, and an $(\mathscr{N}, \mathscr{N})$ space is precisely a commutative topological semi-ring or *rig space*, a ring without negatives. We say that $(\mathscr{C}, \mathscr{G})$ is an E_∞ operad pair if $\mathscr{C}$ and $\mathscr{G}$ are both E_∞ operads. We give a canonical example in Section 3.

Of course, a rig space X must have additive and multiplicative unit elements 0 and 1, and they must be different for nontriviality. It is convenient to think of S^0 as $\{0, 1\}$, so that these points are encoded by a map $e \colon S^0 \longrightarrow X$. In [27; 29], we thought of both of these points as *basepoints*. Here we only think of 0 as a basepoint. This sounds like a trivial distinction, but it leads to a significant change of perspective when we pass from operads to monads in Section 4. We let $\mathscr{T}_e$ denote the category of spaces X together with a map $e \colon S^0 \longrightarrow X$. That is, it is the category of spaces under S^0.

One would like to say that we have an endomorphism operad pair such that an action of an operad pair is a map of operad pairs, but that is not quite how things work. Rather, an action of $(\mathscr{C}, \mathscr{G})$ on X consists of an action θ of $\mathscr{C}$ on $(X, 0)$ and an action ξ of $\mathscr{G}$ on $(X, 1)$ for which 0 is a strict zero, so that $\xi(g; y) = 0$ if any coordinate of y is 0, and for which the parametrized version of the left distributivity law holds. In a rig space X, for variables $(x_{r,1}, \ldots, x_{r,j_r}) \in X^{j_r}$, $1 \le r \le k$, we set $z_r = x_{r,1} + \cdots + x_{r,j_r}$ and find that

$$z_1 \cdots z_k = \sum_Q x_{1,q_1} \cdots x_{k,q_k},$$

where the sum runs over the set of sequences $Q = (q_1, \ldots, q_k)$ such that $1 \le q_r \le j_r$, ordered lexicographically. The parametrized version required of a $(\mathscr{C}, \mathscr{G})$–space is obtained by first defining maps

$$(1\text{-}1) \qquad \xi \colon \mathscr{G}(k) \times \mathscr{C}(j_1) \times X^{j_1} \times \cdots \times \mathscr{C}(j_k) \times X^{j_k} \longrightarrow \mathscr{C}(j_1 \cdots j_k) \times X^{j_1 \cdots j_k}$$

and then requiring the following diagram to commute.

$$(1\text{-}2) \qquad
\begin{array}{ccc}
\mathscr{G}(k) \times \mathscr{C}(j_1) \times X^{j_1} \times \cdots \times \mathscr{C}(j_k) \times X^{j_k} & \xrightarrow{\mathrm{id} \times \theta^k} & \mathscr{G}(k) \times X^k \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
\mathscr{C}(j_1 \cdots j_k) \times X^{j_1 \cdots j_k} & \xrightarrow{\quad \theta \quad} & X
\end{array}$$

The promised map ξ on the left is defined by

$$(1\text{-}3) \qquad \xi(g; c_1, y_1, \ldots, c_k, y_k) = (\lambda(g; c_1, \ldots, c_k); \prod_Q \xi(g; y_Q))$$

where $g \in \mathscr{G}(k)$, $c_r \in \mathscr{C}(j_r)$, $y_r = (x_{r,1}, \ldots, x_{r,j_r})$, the product is taken over the lexicographically ordered set of sequences Q, as above, and $y_Q = (x_{1,q_1}, \ldots, x_{k,q_k})$.

The following observation is trivial, but it will lead in the sequel [32] to significant technical simplifications of [29].

Remark 1.4 All basepoint conditions, including the strict zero condition, are in fact redundant. We have seen that the conditions $\mathscr{C}(0) = 0$ and $\mathscr{G}(0) = 1$ imply that the additive and multiplicative operad actions specify the points 0 and 1 in X. If any $j_r = 0$, then $j_1 \cdots j_r = 0$, we have no coordinates x_{r,i_r}, and we must interpret ξ in (1-1) to be the unique map to the point $\mathscr{C}(0) \times X^0$. Then (1-2) asserts that the right vertical arrow ξ takes the value 0. With all but one $j_r = 1$ and the remaining $j_r = 0$, this forces 0 to be a strict zero for ξ.

2 $\mathscr{I}$–spaces and the linear isometries operad

The canonical multiplicative operad is the linear isometries operad $\mathscr{L}$, which was introduced by Boardman and Vogt [6; 7]; see also [27, Section I.1]. It is an E_∞ operad that enjoys several very special geometric properties. In this brief section, we recall its definition and that of related structures that give rise to $\mathscr{L}$–spaces and $\mathscr{L}$–spectra.

Let $\mathscr{I}$ denote the topological category of finite dimensional real inner product spaces and linear isometric isomorphisms and let $\mathscr{I}_c$ denote the category of finite or countably infinite dimensional real inner product spaces and linear isometries.[10] For the latter, we topologize inner product spaces as the colimits of their finite dimensional subspaces and use the function space topologies on the $\mathscr{I}_c(V, W)$. These are contractible spaces when W is infinite dimensional.

When V is finite dimensional and Y is a based space, we let S^V denote the one-point compactification of V and let $\Omega^V Y = F(S^V, Y)$ denote the V–fold loop space of Y. In general $F(X, Y)$ denotes the space of based maps $X \longrightarrow Y$. Let $U = \mathbb{R}^\infty$ with its standard inner product. Define $\mathscr{L}(j) = \mathscr{I}_c(U^j, U)$, where U^j is the sum of j copies of U, with $U^0 = \{0\}$. The element $\mathrm{id} \in \mathscr{L}(1)$ is the identity isometry, the left

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[10]The notations $\mathscr{I}_*$ and $\mathscr{I}$ were used for our $\mathscr{I}$ and $\mathscr{I}_c$ in [27]. We are following [34].

action of Σ_j on U^j induces the right action of Σ_j on $\mathscr{L}(j)$, and the structure maps γ are defined by

$$\gamma(g; f_1, \ldots, f_j) = g \circ (f_1 \oplus \cdots \oplus f_j).$$

Notice that $\mathscr{L}$ is a suboperad of the endomorphism operad of U.

For use in the next section and in the second sequel [33], we recall some related formal notions from [27, Section I.1]. These reappeared and were given new names and applications in May and Sigurdsson [34, Section 23.3], whose notations we follow. An $\mathscr{I}$–space is a continuous functor $F: \mathscr{I} \longrightarrow \mathscr{U}$. An $\mathscr{I}$–FCP (functor with cartesian product) F is an $\mathscr{I}$–space that is a lax symmetric monoidal functor, where $\mathscr{I}$ is symmetric monoidal under $\oplus$ and $\mathscr{U}$ is symmetric monoidal under cartesian product. This means that there are maps

$$\omega: F(V) \times F(W) \longrightarrow F(V \oplus W)$$

that give a natural transformation $\times \circ (F, F) \longrightarrow F \circ \oplus$ which is associative and commutative up to coherent natural isomorphism. We require that $F(0) = *$ and that ω be the evident identification when $V = 0$ or $W = 0$. When F takes values in based spaces, we require the maps $FV \longrightarrow F(V \oplus W)$ that send x to $\omega(x, *)$ to be closed inclusions. We say that (F, ω) is monoid-valued if F takes values in the cartesian monoidal category of monoids in $\mathscr{U}$ and ω is given by maps of monoids. We then give the monoids $F(V)$ their unit elements as basepoints and insist on the closed inclusion property. All of the classical groups (and *String*) give examples. Since the classifying space functor is product preserving, the spaces $BF(V)$ then give an $\mathscr{I}$–FCP BF.

We can define analogous structures with $\mathscr{I}$ replaced throughout by $\mathscr{I}_c$. Clearly $\mathscr{I}_c$–FCP's restrict to $\mathscr{I}$–FCP's. Conversely, our closed inclusion requirement allows us to pass to colimits over inclusions $V \subset V'$ of subspaces in any given countably infinite dimensional inner product space to obtain $\mathscr{I}_c$–FCP's from $\mathscr{I}$–FCP's. Formally, we have an equivalence between the category of $\mathscr{I}$–FCP's and the category of $\mathscr{I}_c$–FCP's. Details are given in [27, Sections I.1 and VII.2] and [34, Section 23.6], and we will illustrate the argument by example in the next section. When we evaluate an $\mathscr{I}_c$–FCP F on U, we obtain an $\mathscr{L}$–space $F(U)$, often abbreviated to F when there is no danger of confusion. The structure maps

$$\theta: \mathscr{L}(j) \times F(U)^j \longrightarrow F(U)$$

are obtained by first using ω to define $F(U)^j \longrightarrow F(U^j)$ and then using the evaluation maps

$$\mathscr{I}_c(U^j, U) \times F(U^j) \longrightarrow F(U)$$

of the functor F. This simple source of E_∞–spaces is fundamental to the geometric applications, as we recall in Section 10 and the second sequel [33]. We can feed these examples into the additive infinite loop space machine to obtain spectra.

There is a closely related notion of an $\mathscr{I}$–FSP (functor with smash product).[11] For this, we again start with an $\mathscr{I}$–space $T: \mathscr{I} \longrightarrow \mathscr{T}$, but we now regard $\mathscr{T}$ as symmetric monoidal under the smash product rather than the cartesian product. The sphere functor S is specified by $S(V) = S^V$ and is strong symmetric monoidal: $S(0) = S^0$ and $S^V \wedge S^W \cong S^{V \oplus W}$. An $\mathscr{I}$–FSP is a lax symmetric monoidal functor T together with a unit map $S \longrightarrow T$. This structure is given by maps

$$\omega: T(V) \wedge T(W) \longrightarrow T(V \oplus W)$$

and $\eta: S^V \longrightarrow T(V)$. When $W = 0$, we require $\omega \circ (\mathrm{id} \wedge \eta)$ to be the obvious identification

$$T(V) \wedge S^0 \cong T(V \oplus 0).$$

The Thom spaces $TO(V)$ of the universal $O(V)$–bundles give the Thom $\mathscr{I}$–FSP TO, and the other classical groups give analogous Thom $\mathscr{I}$–FSP's. A full understanding of the relationship between $\mathscr{I}$–FCP's and $\mathscr{I}$–FSP's requires the notion of parametrized $\mathscr{I}$–FSP's, as defined and illustrated by examples in [34, 23.2], but we shall say no more about that here.

We shall define E_∞ ring prespectra, or $\mathscr{L}$–prespectra, in Section 5. The definition codifies structure that is implicit in the notion of an $\mathscr{I}$–FSP, so these give examples. That is, we have a functor from $\mathscr{I}$–FSP's to $\mathscr{L}$–prespectra. The simple observation that the classical Thom prespectra arise in nature from $\mathscr{I}$–FSP's is the starting point of E_∞ ring theory and thus of this whole area of mathematics. We shall also define E_∞ ring spectra, or $\mathscr{L}$–spectra, in Section 5, and we shall describe a spectrification functor from $\mathscr{L}$–prespectra to $\mathscr{L}$–spectra. Up to language and clarification of details, these constructions date from 1972 and are given in [27]. It was noticed over twenty-five years later that $\mathscr{I}$–FSP's are exactly equivalent to (commutative) orthogonal ring spectra. This gives an alternative way of processing the simple input data of $\mathscr{I}$–FSP's, as we shall explain. However, we next return to E_∞ ring spaces and explain the canonical operad pair that acts on the 0th spaces of $\mathscr{L}$–spectra, such as Thom spectra.

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[11]These were called $\mathscr{I}_*$–prefunctors when they were first defined in [27, Section IV.2]; the more sensible name FSP was introduced later by Bökstedt [8]. For simplicity, we restrict attention to commutative $\mathscr{I}$–FSP's in this paper. In analogy with $\mathscr{I}$–FCP's, the definition in [27, Section IV.2] required a technically convenient inclusion condition, but it is best not to insist on that.

3 The canonical E_∞ operad pair

The canonical additive E_∞ operad is much less obvious than $\mathcal{L}$. We first recall the little cubes operads $\mathscr{C}_n$, which were also introduced by Boardman and Vogt [6; 7], and the little discs operads $\mathscr{D}_V$. We then explain why neither is adequate for our purposes.

For an open subspace X of a finite dimensional inner product space V, define the embeddings operad Emb_X as follows. Let E_X denote the space of (topological) embeddings $X \longrightarrow X$. Let $\mathrm{Emb}_X(j) \subset E_X^j$ be the space of j–tuples of embeddings with disjoint images. Regard such a j–tuple as an embedding $^j X \longrightarrow X$, where $^j X$ denotes the disjoint union of j copies of X (where $^0 X$ is the empty space). The element $\mathrm{id} \in \mathrm{Emb}_X(1)$ is the identity embedding, the group Σ_j acts on $\mathrm{Emb}_X(j)$ by permuting embeddings, and the structure maps

$$(3\text{-}1) \qquad \gamma\colon \mathrm{Emb}_X(k) \times \mathrm{Emb}_X(j_1) \times \cdots \times \mathrm{Emb}_X(j_k) \longrightarrow \mathrm{Emb}_X(j_1 + \cdots + j_k)$$

are defined as follows. Let $g = (g_1, \dots, g_k) \in \mathrm{Emb}_X(k)$ and $f_r = (f_{r,1}, \dots, f_{r,j_r}) \in \mathrm{Emb}_X(j_r)$, $1 \le r \le k$. Then the rth block of j_r embeddings in $\gamma(g; f_1, \dots, f_j)$ is given by the composites $g_r \circ f_{r,s}$, $1 \le s \le j_r$.

Taking $X = (0,1)^n \subset \mathbb{R}^n$, we obtain a suboperad $\mathscr{C}_n$ of Emb_X by restricting to the little n–cubes, namely those embeddings $f\colon X \longrightarrow X$ such that $f = \ell_1 \times \cdots \times \ell_n$, where $\ell_i(t) = a_i t + b_i$ for real numbers $a_i > 0$ and $b_i \ge 0$.

For a general V, taking X to be the open unit disc $D(V) \subset V$, we obtain a suboperad $\mathscr{D}_V$ of Emb_V by restricting to the little V–discs, namely those embeddings $f\colon D(V) \longrightarrow D(V)$ such that $f(v) = av + b$ for some real number $a > 0$ and some element $b \in D(V)$.

It is easily checked that these definitions do give well-defined suboperads. Let $F(X, j)$ denote the configuration space of j–tuples of distinct elements of X, with its permutation action by Σ_j. These spaces do not fit together to form an operad, and $\mathscr{C}_n$ and $\mathscr{D}_V$ specify fattened up equivalents that do form operads. By restricting little n–cubes or little V–discs to their values at the center point of $(0,1)^n$ or $D(V)$, we obtain Σ_j–equivariant deformation retractions

$$\mathscr{C}_n(j) \longrightarrow F((0,1)^n, j) \cong F(\mathbb{R}^n, j) \quad \text{and} \quad \mathscr{D}_V(j) \longrightarrow F(D(V), j) \cong F(V, j).$$

This gives control over homotopy types.

For a little n–cube f, the product $f \times \mathrm{id}$ gives a little $(n+1)$–cube. Applying this to all little n–cubes gives a *suspension* map of operads $\mathscr{C}_n \longrightarrow \mathscr{C}_{n+1}$. We can pass to colimits over n to construct the infinite little cubes operad $\mathscr{C}_\infty$, and it is an E_∞ operad.

However, little n–cubes are clearly too square to allow an action by the orthogonal group $O(n)$, and we cannot define an action of $\mathscr{L}$ on $\mathscr{C}_\infty$.

For a little V–disc f and an element $g \in O(V)$, we obtain another little V–disc gfg^{-1}. Thus the group $O(V)$ acts on the operad $\mathscr{D}_V$. However, for $V \subset W$, so that $W = V \oplus (W - V)$ where $W - V$ is the orthogonal complement of V in W, little V–discs f are clearly too round for the product $f \times \mathrm{id}$ to be a little W–disc. We can send a little V–disc $v \mapsto av + b$ to the little W–disc $w \mapsto aw + b$, but that is not compatible with the decomposition $S^W \cong S^V \wedge S^{W-V}$ used to identify $\Omega^W Y$ with $\Omega^{W-V} \Omega^V Y$. Therefore we cannot suspend.

In [27], the solution was to introduce the little convex bodies partial operads. They weren't actually operads because the structure maps γ were only defined on subspaces of the expected domain spaces. The use of partial operads introduced quite unpleasant complications. Steiner [43] found a remarkable construction of operads $\mathscr{K}_V$ which combine all of the good properties of the $\mathscr{C}_n$ and the $\mathscr{D}_V$. His operads, which we call the Steiner operads, are defined in terms of paths of embeddings rather than just embeddings.

Define $R_V \subset E_V = \mathrm{Emb}_V(1)$ to be the subspace of distance reducing embeddings $f \colon V \longrightarrow V$. This means that $|f(v) - f(w)| \leq |v - w|$ for all $v, w \in V$. Define a Steiner path to be a map $h \colon I \longrightarrow R_V$ such that $h(1) = \mathrm{id}$ and let P_V be the space of Steiner paths. Define $\pi \colon P_V \longrightarrow R_V$ by evaluation at 0, $\pi(h) = h(0)$. Define $\mathscr{K}_V(j)$ to be the space of j–tuples $(h_1, \ldots, h_j)$ of Steiner paths such that the $\pi(h_r)$ have disjoint images. The element $\mathrm{id} \in \mathscr{K}_V(1)$ is the constant path at the identity embedding, the group Σ_j acts on $\mathscr{K}_V(j)$ by permutations, and the structure maps γ are defined pointwise in the same way as those of Emb_V. That is, for $g = (g_1, \ldots, g_k) \in \mathscr{K}_V(k)$ and $f_r = (f_{r,1}, \ldots, f_{r,j_r}) \in \mathscr{K}(j_r)$, $\gamma(g; f_1, \ldots, f_j)$ is given in by the embeddings $g_r(t) \circ f_{r,s}(t)$, in order. This gives well defined operads, and application of π to Steiner paths gives a map of operads $\pi \colon \mathscr{K}_V \longrightarrow \mathrm{Emb}_V$.

By pullback along π, any space with an action by Emb_V inherits an action by $\mathscr{K}_V$. As in [25, Section 5] or [27, Section VII.2], Emb_V acts naturally on $\Omega^V Y$. A j–tuple of embeddings $V \longrightarrow V$ with disjoint images determines a map from S^V to the wedge of j copies of S^V by collapsing points of S^V not in any of the images to the point at infinity and using the inverses of the given embeddings to blow up their images to full size. A j–tuple of based maps $S^V \longrightarrow Y$ then gives a map from the wedge of the S^V to Y. Thus the resulting action θ_V of $\mathscr{K}_V$ on $\Omega^V Y$ is given by composites

$$\mathscr{K}_V(j) \times (\Omega^V Y)^j \xrightarrow{\pi \times \mathrm{id}} \mathrm{Emb}_V(j) \times (\Omega^V Y)^j \longrightarrow \Omega^V({}^j S^V) \times (\Omega^V Y)^j \longrightarrow \Omega^V Y.$$

Evaluation of embeddings at $0 \in V$ gives maps $\zeta \colon \mathrm{Emb}_V(j) \longrightarrow F(V, j)$. Steiner determines the homotopy types of the $\mathcal{K}_V(j)$ by proving that the composite maps $\zeta \circ \pi \colon \mathcal{K}_V(j) \longrightarrow F(V, j)$ are Σ_j–equivariant deformation retractions.

The operads $\mathcal{K}_V$ have extra geometric structure that make them ideally suited for our purposes. Rewriting $F(V) = F_V$, we see that E, R, and P above are monoid-valued $\mathscr{I}$–FCP's. The monoid products are given (or induced pointwise) by composition of embeddings, and the maps ω are given by cartesian products of embeddings. For the functoriality, if $f \colon V \longrightarrow V$ is an embedding and $g \colon V \longrightarrow W$ is a linear isometric isomorphism, then we obtain an embedding $g f g^{-1} \colon W \longrightarrow W$ which is distance reducing if f is. We have an inclusion $R \subset E$ of monoid-valued $\mathscr{I}$–FCP's, and evaluation at 0 gives a map $\pi \colon P \longrightarrow R \subset E$ of monoid-valued $\mathscr{I}$–FCP's. The operad structure maps of Emb_V and $\mathcal{K}_V$ are induced by the monoid products, as is clear from the specification of γ after (3-1).

The essential point is that, in analogy with (3-1), we have maps
(3-2)
$$\lambda \colon \mathscr{I}(V_1 \oplus \cdots \oplus V_k, W) \times \mathrm{Emb}_{V_1}(j_1) \times \cdots \times \mathrm{Emb}_{V_k}(j_k) \longrightarrow \mathrm{Emb}_W(j_1 + \cdots + j_k)$$

defined as follows. Let $g \colon V_1 \oplus \cdots \oplus V_k \longrightarrow W$ be a linear isometric isomorphism and let $f_r = (f_{r,1}, \ldots, f_{r,j_r}) \in \mathrm{Emb}_{V_r}(j_r)$, $1 \le r \le k$. Again consider the set of sequences $Q = (q_1, \ldots, q_k)$, $1 \le q_r \le j_r$, ordered lexicographically. Identifying direct sums with direct products, the Qth embedding of $\lambda(g; f_1, \ldots, f_k)$ is the composite $g f_Q g^{-1}$, where $f_Q = f_{1,q_1} \times \cdots \times f_{k,q_k}$. Restricting to distance reducing embeddings $f_{r,s}$ and applying the result pointwise to Steiner paths, there result maps

(3-3) $\lambda \colon \mathscr{I}(V_1 \oplus \cdots \oplus V_k, W) \times \mathcal{K}_{V_1}(j_1) \times \cdots \times \mathcal{K}_{V_k}(j_k) \longrightarrow \mathcal{K}_W(j_1 + \cdots + j_k).$

Passing to colimits over inclusions $V \subset V'$ of subspaces in any given countably infinite dimensional inner product space, such as U, we obtain structure exactly like that just described, but now defined on all of $\mathscr{I}_c$, rather than just on $\mathscr{I}$. (Compare [27, Sections I.1 and VII.2] and [34, Section 23.6]). For example, suppose that the V_r and W in (3-3) are infinite dimensional. Since the spaces $\mathrm{Emb}_{V_r}(1)$ are obtained by passage to colimits over the finite dimensional subspaces of the V_r, for each of the embeddings $f_{r,s}$, there is a finite dimensional subspace $A_{r,s}$ such that $f_{r,s}$ is the identity on the orthogonal complement $V_r - A_{r,s}$. Therefore, all of the f_Q are the identity on $V_1 \oplus \cdots \oplus V_k - B$ for a sufficiently large B. On finite dimensional subspaces $gV \subset W$, we define λ as before, using the maps $g f_Q g^{-1}$. On the orthogonal complement $W - gV$ for V large enough to contain B, we can and must define the Qth embedding to be the identity map for each Q.

Finally, we define the canonical additive E_∞ operad, denoted $\mathscr{C}$, to be the Steiner operad $\mathscr{K}_U$. Taking $V_1 = \cdots = V_k = U$, we have the required maps

$$\lambda\colon \mathscr{L}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 \cdots j_k).$$

They make the (unspecified) distributivity diagrams commute, and we next explain the significance of those diagrams.

4 The monadic interpretation of the definitions

We assume that the reader has seen the definition of a monad. Fixing a ground category $\mathscr{V}$, a quick definition is that a monad (C, μ, η) on $\mathscr{V}$ is a monoid in the category of functors $\mathscr{V} \longrightarrow \mathscr{V}$. Thus $C\colon \mathscr{V} \longrightarrow \mathscr{V}$ is a functor, $\mu\colon CC \longrightarrow C$ and $\eta\colon \mathrm{Id} \longrightarrow C$ are natural transformations, and μ is associative with η as a two-sided unit. A C–algebra is an object $X \in \mathscr{V}$ with a unital and associative action map $\xi\colon CX \longrightarrow X$. We let $C[\mathscr{V}]$ denote the category of C–algebras in $\mathscr{V}$.

Similarly, when $\mathscr{O}$ is an operad in $\mathscr{V}$, we write $\mathscr{O}[\mathscr{V}]$ for the category of $\mathscr{O}$–algebras in $\mathscr{V}$. We note an important philosophical distinction. Monads are very general, and their algebras in principle depend only on $\mathscr{V}$, without reference to any further structure that category might have. In contrast, $\mathscr{O}$–algebras depend on a choice of symmetric monoidal structure on $\mathscr{V}$, and that might vary. We sometimes write $\mathscr{O}[\mathscr{V}, \otimes]$ to emphasize this dependence.

For an operad $\mathscr{O}$ of unbased spaces with $\mathscr{O}(0) = *$, as before, there are two monads in sight, both of which have the same algebras as the operad $\mathscr{O}$.[12] Viewing operads as acting on unbased spaces, we obtain a monad $O_+^{\mathscr{U}}$ on $\mathscr{U}$ with

$$(4\text{-}1) \qquad\qquad O_+^{\mathscr{U}} X = \coprod_{j \geq 0} \mathscr{O}(j) \times_{\Sigma_j} X^j.$$

Here $\eta(x) = (1, x) \in \mathscr{O}(1) \times X$, and μ is obtained by taking coproducts of maps on orbits induced by the structure maps γ. If X has an action θ by $\mathscr{O}$, then $\xi\colon O_+^{\mathscr{U}} X \longrightarrow X$ is given by the action maps $\theta\colon \mathscr{O}(j) \times_{\Sigma_j} X^j \longrightarrow X$, and conversely. The subscript $+$ on the monad is intended to indicate that it is *augmented*, rather than *reduced*. The superscript $\mathscr{U}$ is intended to indicate that the operad from which the monad is constructed is an operad of unbased spaces.

Viewing operads as acting on spaces with preassigned basepoints, we construct a reduced monad $O = O^{\mathscr{U}}$ on $\mathscr{T}$ by quotienting $O_+^{\mathscr{U}} X$ by basepoint identifications.

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[12]Further categorical perspective on the material of this section is given in Appendix A.

There are degeneracy operations $\sigma_i \colon \mathscr{O}(j) \longrightarrow \mathscr{O}(j-1)$ given by

$$\sigma_i(c) = \gamma(c; (\mathrm{id})^{i-1}, *, (\mathrm{id})^{j-i})$$

for $1 \leq i \leq j$, and there are maps $s_i \colon X^{j-1} \longrightarrow X^j$ obtained by inserting the basepoint in the i th position. We set

$$(4\text{-}2) \qquad\qquad OX \equiv O^{\mathscr{U}} X = O_+^{\mathscr{U}} X/(\sim),$$

where $(c, s_i y) \sim (\sigma_i c, y)$ for $c \in \mathscr{O}(j)$ and $y \in X^{j-1}$. Observations in explain why these two operads have the same algebras. The reduced monad $O = O^{\mathscr{U}}$ is more general than the augmented monad $O_+^{\mathscr{U}}$ since the latter can be obtained by applying the former to spaces of the form X_+. That is, for unbased spaces X,

$$(4\text{-}3) \qquad\qquad O^{\mathscr{U}}(X_+) \cong O_+^{\mathscr{U}} X$$

as $\mathscr{O}$–spaces. To keep track of adjunctions later, we note that the functor $(-)_+$ is left adjoint to the functor $i \colon \mathscr{T} \longrightarrow \mathscr{U}$ that is obtained by forgetting basepoints.

The reduced monad $O = O^{\mathscr{U}}$ on $\mathscr{T}$ is of primary topological interest, but the idea that there is a choice will simplify the multiplicative theory. Here we diverge from the original sources [27; 29].[13] Summarizing, we have the following result.

Proposition 4.4 *The following categories are isomorphic.*

(i) *The category $\mathscr{O}[\mathscr{U}, \times] = \mathscr{O}[\mathscr{T}, \times]$ of $\mathscr{O}$–spaces.*

(ii) *The category $O_+^{\mathscr{U}}[\mathscr{U}]$ of $O_+^{\mathscr{U}}$–algebras in $\mathscr{U}$.*

(iii) *The category $O[\mathscr{T}] \equiv O^{\mathscr{U}}[\mathscr{T}]$ of $\mathscr{O}$–algebras in $\mathscr{T}$.*

We have another pair of monads associated to an operad $\mathscr{O}$. Recall again that operads and operad actions make sense in any symmetric monoidal category $\mathscr{V}$. Above, in (i), we are thinking of $\mathscr{T}$ as cartesian monoidal, and we are entitled to use the alternative notations $\mathscr{O}[\mathscr{U}]$ and $\mathscr{O}[\mathscr{T}]$ since $\mathscr{O}$–algebras in $\mathscr{U}$ can equally well be regarded as $\mathscr{O}$–algebras in $\mathscr{T}$. The algebras in Proposition 4.4 have parametrized products $X^j \longrightarrow X$ that are defined on cartesian powers X^j.

However, we can change ground categories to the symmetric monoidal category $\mathscr{T}$ under its smash product, with unit S^0. We write $X^{(j)}$ for the j–fold smash power of a space (or, later, spectrum) X, with $X^{(0)} = S^0$. Remembering that $X_+ \wedge Y_+ \cong (X \times Y)_+$, we can adjoin disjoint basepoints to the spaces $\mathscr{O}(j)$ to obtain an operad

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[13]I am indebted to helpful conversations with Bob Thomason and Tyler Lawson, some twenty-five years apart, for the changed perspective.

$\mathcal{O}_+$ with spaces $\mathcal{O}_+(j) = \mathcal{O}(j)_+$ in $\mathcal{T}$; in particular, $\mathcal{O}_+(0) = S^0$. The actions of $\mathcal{O}_+$ parametrize j–fold products $X^{(j)} \longrightarrow X$, and we have the category $\mathcal{O}_+[\mathcal{T}]$ of $\mathcal{O}_+$–spaces in $\mathcal{T}$.

Now recall that $\mathcal{T}_e$ denotes the category of spaces X under S^0, with given map $e\colon S^0 \longrightarrow X$. In [27; 29], we defined an $\mathcal{O}$–space with zero, or $\mathcal{O}_0$ space, to be an $\mathcal{O}$–space (X, ξ) in $\mathcal{T}_e$ such that 0 acts as a strict zero, so that $\xi(f; x_1, \ldots, x_j) = 0$ if any $x_i = 0$. That is exactly the same structure as an $\mathcal{O}_+$–space in $\mathcal{T}$. The only difference is that now we think of $\xi\colon S^0 = \mathcal{O}_+(0) \longrightarrow X$ as building in the map $e\colon S^0 \longrightarrow X$, which is no longer preassigned. We are entitled to use the alternative notation $\mathcal{O}_+[\mathcal{T}_e]$ for $\mathcal{O}_+[\mathcal{T}]$.

We construct an (augmented) monad $O_+ = O_+^{\mathcal{T}}$ on $\mathcal{T}$ such that an $\mathcal{O}_+$–space in $\mathcal{T}$ is the same as an O_+–algebra in $\mathcal{T}$ by setting

$$(4\text{-}5) \qquad O_+ X \equiv O_+^{\mathcal{T}} X = \bigvee_{j \geq 0} \mathcal{O}(j)_+ \wedge_{\Sigma_j} X^{(j)}.$$

This and (4-1) are special cases of a general definition that applies to operads in any cocomplete symmetric monoidal category, as is discussed in detail in [30; 31].

As a digression, thinking homotopically and letting $E\Sigma_j$ be any contractible free Σ_j–space, one defines the extended j–fold smash power $D_j X$ of a based space X by

$$(4\text{-}6) \qquad D_j X = (E\Sigma_j)_+ \wedge_{\Sigma_j} X^{(j)}.$$

These spaces have many applications. Homotopically, when $\mathcal{O}$ is an E_∞ operad, $O_+ X$ is a model for the wedge over j of the spaces $D_j X$.

Here we have not viewed the element 1 of a space under S^0 as a basepoint. However, we can use basepoint identifications to take account of the unit properties of 1 in an action by $\mathcal{O}_+$. We then obtain a reduced monad $O^{\mathcal{T}}$ on $\mathcal{T}_e$ with the same algebras as the monad $O_+^{\mathcal{T}}$ on $\mathcal{T}$. It is again more general than the monad O_+. For a based space X, $S^0 \vee X$ is the space under S^0 obtained from the based space X by adjoining a point 1 (not regarded as a basepoint). This gives the left adjoint to the inclusion $i\colon \mathcal{T}_e \longrightarrow \mathcal{T}$ that is obtained by forgetting the point 1, and

$$(4\text{-}7) \qquad O^{\mathcal{T}}(S^0 \vee X) \cong O_+^{\mathcal{T}}(X)$$

as $\mathcal{O}_+$–spaces. We summarize again.

Proposition 4.8 *The following categories are isomorphic.*

(i) *The category $\mathcal{O}_+[\mathcal{T}, \wedge] = \mathcal{O}_+[\mathcal{T}_e, \wedge]$ of $\mathcal{O}_+$–spaces.*

(ii)   *The category $O_+[\mathscr{T}] \equiv O_+^{\mathscr{T}}[\mathscr{T}]$ of O_+–algebras in $\mathscr{T}$.*

(iii)   *The category $O^{\mathscr{T}}[\mathscr{T}_e]$ of $O^{\mathscr{T}}$–algebras in $\mathscr{T}_e$.*

In [27; 29], we viewed the multiplicative structure of E_∞ ring spaces as defined on the base category $\mathscr{T}_e$, and we used the monad $G^{\mathscr{T}}$ on $\mathscr{T}_e$ instead of the monad G_+ on $\mathscr{T}$. That unnecessarily complicated the details in [29], where different kinds of input data to multiplicative infinite loop space theory were compared, and we now prefer to use G_+. That is convenient both here and in the sequel [32].

With this as preamble, consider an operad pair $(\mathscr{C}, \mathscr{G})$, such as the canonical one $(\mathscr{C}, \mathscr{L})$ from the previous section. We have several monads in sight whose algebras are the $(\mathscr{C}, \mathscr{G})$–spaces X. We single out the one most convenient for the comparison of E_∞ ring spaces and E_∞ ring spectra by focusing on the *additive monad C* on $\mathscr{T}$, where the basepoint is denoted 0 and is the unit for the operadic product, and the *multiplicative monad G_+*, which is also defined on $\mathscr{T}$. A different choice will be more convenient in the sequel [32].

The diagrams that we omitted in our outline definition of an action of $\mathscr{G}$ on $\mathscr{C}$ are exactly those that are needed to make the following result true.

Proposition 4.9 *The monad C on $\mathscr{T}$ restricts to a monad on the category $\mathscr{G}_+[\mathscr{T}]$ of $\mathscr{G}_+$–spaces in $\mathscr{T}$. A $(\mathscr{C}, \mathscr{G})$–space is the same structure as a C–algebra in $\mathscr{G}_+[\mathscr{T}]$.*

Sketch proof   The details of the definition of an action of $\mathscr{G}$ on $\mathscr{C}$ are designed to ensure that, for a $\mathscr{G}_+$–space (X, ξ), the maps ξ of (1-3) induce maps

$$\xi \colon \mathscr{G}(k)_+ \wedge (CX)^{(k)} \longrightarrow CX$$

that give CX a structure of $\mathscr{G}_+$–space in $\mathscr{T}$ such that

$$\mu \colon CCX \longrightarrow CX \quad \text{and} \quad \eta \colon X \longrightarrow CX$$

are maps of $\mathscr{G}_+$–spaces in $\mathscr{T}$. Then the diagram (1-2) asserts that a $(\mathscr{C}, \mathscr{G})$–space is the same as a C–algebra in $\mathscr{G}_+[\mathscr{T}]$. Details are in [27, Section VI.1].   □

We have the two monads $(C, \mu_\oplus, \eta_\oplus)$ and $(G_+, \mu_\otimes, \eta_\otimes)$, both defined on $\mathscr{T}$, such that C restricts to a monad on $\mathscr{G}_+[\mathscr{T}]$. This puts things in a general categorical context that was studied by Beck [5]. A summary of his results is given in [29, 5.6] and in [32, Appendix B]. He gives several equivalent characterizations of a pair (C, G_+) of monads related in this fashion. One is that the composite functor CG_+ is itself a monad with certain properties. Another is that there is a natural interchange transformation $G_+C \longrightarrow CG_+$ such that appropriate diagrams commute. Category theorists know

well that this is definitively the right context in which to study generalized distributivity phenomena. While I defined E_∞ ring spaces before I knew of Beck's work, his context makes it clear that this is a very natural and conceptual definition.

5 The definition of E_∞ ring prespectra and E_∞ ring spectra

We first recall the categories of (LMS) prespectra and spectra from Lewis et al [18]. As before, we let $U = \mathbb{R}^\infty$ with its standard inner product. Define an indexing space to be a finite dimensional subspace of U with the induced inner product. A (coordinate free) prespectrum T consists of based spaces TV and based maps $\tilde{\sigma}\colon TV \longrightarrow \Omega^{W-V}TW$ for $V \subset W$, where $W - V$ is the orthogonal complement of V and S^{W-V} is its one point compactification; $\tilde{\sigma}$ must be the identity when $V = W$, and the obvious transitivity condition must hold when $V \subset W \subset Z$. A spectrum is a prespectrum such that each map $\tilde{\sigma}$ is a homeomorphism; we then usually write E rather than T.

We let $\mathscr{P}$ and $\mathscr{S}$ denote the categories of prespectra and spectra. Then $\mathscr{S}$ is a full subcategory of $\mathscr{P}$, with inclusion $\ell\colon \mathscr{S} \longrightarrow \mathscr{P}$, and there is a left adjoint spectrification functor $L\colon \mathscr{P} \longrightarrow \mathscr{S}$. When T is an inclusion prespectrum, meaning that each $\tilde{\sigma}$ is an inclusion, $(LT)(V) = \operatorname{colim}_{V \subset W} \Omega^{W-V}TW$.

We may restrict attention to any cofinal set of indexing spaces $\mathscr{V}$ in U; we require 0 to be in $\mathscr{V}$ and we require the union of the V in $\mathscr{V}$ to be all of U. Up to isomorphism, the category $\mathscr{S}$ is independent of the choice of $\mathscr{V}$. The default is $\mathscr{V} = \mathscr{All}$. We can define prespectra and spectra in the same way in any countably infinite dimensional real inner product space U, and we write $\mathscr{P}(U)$ when we wish to remember the universe. The default is $U = \mathbb{R}^\infty$.

For $X \in \mathscr{T}$, we define QX to be $\operatorname{colim} \Omega^V \Sigma^V X$, and we let $\eta\colon X \longrightarrow QX$ be the natural inclusion. We define $(\Sigma^\infty X)(V) = Q\Sigma^V X$. Since $S^W \cong S^V \wedge S^{W-V}$, we have identifications $Q\Sigma^V X \cong \Omega^{W-V} Q\Sigma^W X$, so that $\Sigma^\infty X$ is a spectrum. For a spectrum E, we define $\Omega^\infty E = E(0)$; we often write it as E_0. The functors Σ^∞ and Ω^∞ are adjoint, QX is $\Omega^\infty \Sigma^\infty X$, and η is the unit of the adjunction. We let $\varepsilon\colon \Sigma^\infty \Omega^\infty E \longrightarrow E$ be the counit of the adjunction.

As holds for any adjoint pair of functors, we have a monad (Q, μ, η) on $\mathscr{T}$, where $\mu\colon QQ \longrightarrow Q$ is $\Omega^\infty \varepsilon \Sigma^\infty$, and the functor Ω^∞ takes values in Q–algebras via the action map $\Omega^\infty \varepsilon\colon \Omega^\infty \Sigma^\infty \Omega^\infty E \longrightarrow \Omega^\infty E$. These observations are categorical trivialities, but they are central to the theory and will be exploited heavily. We will see later that this adjunction restricts to an adjunction in $\mathscr{L}_+[\mathscr{T}]$. That fact will lead to the proof that the 0th space of an E_∞ ring spectrum is an E_∞ ring space.

As already said, the starting point of this area of mathematics was the observation that the classical Thom prespectra, such as TU, appear in nature as $\mathscr{I}$–FSP's and therefore as E_∞ ring prespectra in the sense we are about to describe. To distinguish, we will write MU for the corresponding spectrum. Quite recently, Strickland [44, Appendix] has observed similarly that the periodic Thom spectrum, $PMU = MU[x, x^{-1}]$, also arises in nature as the spectrum associated to an E_∞ ring prespectrum. To define this concept, we must consider smash products and change of universe.

We have an external smash product $T \barwedge T'$ that takes a pair of prespectra indexed on $\mathscr{All}$ in U to a prespectrum indexed on the indexing spaces $V \oplus V'$ in $U \oplus U$. It is specified by

$$(T \barwedge T')(V, V') = TV \wedge T'V'$$

with evident structure maps induced by those of T and T'. This product is commutative and associative in an evident sense; for the commutativity, we must take into account the interchange isomorphism $\tau\colon U \oplus U \longrightarrow U \oplus U$. If we think of spaces as prespectra indexed on 0, then the space S^0 is a unit object. Formally, taking the disjoint union over $j \geq 0$ of the categories $\mathscr{P}(U^j)$, we obtain a perfectly good symmetric monoidal category. This was understood in the 1960's. A well-structured treatment of spectra from this external point of view was given by Elmendorf [12].

For a linear isometry $f\colon U \longrightarrow U'$, we have an obvious change of universe functor $f^*\colon \mathscr{P}(U') \longrightarrow \mathscr{P}(U)$ specified by $(f^*T')(V) = T'(fV)$, with evident structure maps. It restricts to a change of universe functor $f^*\colon \mathscr{S}(U') \longrightarrow \mathscr{S}(U)$. These functors have left adjoints f_*. When f is an isomorphism, $f_* = (f^{-1})^*$. For a general f, it is not hard to work out an explicit definition; see [18, Section II.1]. The left adjoint on the spectrum level is $Lf_*\ell$. This is the way one constructs a spectrum level left adjoint from any prespectrum level left adjoint. The external smash product of spectra is defined similarly, $E \barwedge E' = L(\ell E \barwedge \ell E')$.

The reader should have in mind the Thom spaces $TO(V)$ or, using complex inner product spaces, $TU(V)$ associated to well chosen universal V–plane bundles. In contrast to the original definitions of [27; 18], we restrict attention to the linear isometries operad $\mathscr{L}$. There seems to be no useful purpose in allowing more general operads in this exposition.

We agree to write $T^{[j]}$ for external j–fold smash powers. An E_∞ prespectrum, or $\mathscr{L}$–prespectrum, T has an action of $\mathscr{L}$ that is specified by maps of prespectra

$$\xi_j(f)\colon T^{[j]} \longrightarrow f^*T$$

or, equivalently, $f_*T^{[j]} \longrightarrow T$, for all $f \in \mathscr{L}(j)$ that are suitably continuous and are compatible with the operad structure on $\mathscr{L}$. The compatibility conditions are

that $\xi_j(f\tau) = \xi_j(f) \circ \tau_*$ for $\tau \in \Sigma_j$ (where τ is thought of as a linear isomorphism $U^j \longrightarrow U^j$), $\xi_1(\mathrm{id}) = \mathrm{id}$, and

$$\xi_{j_1 + \cdots + j_k}(\gamma(g; f_1, \ldots, f_k)) = \xi_k(g) \circ (\xi_{j_1}(f_1) \overline{\wedge} \cdots \overline{\wedge} \xi_{j_k}(f_k))$$

for $g \in \mathscr{L}(k)$ and $f_r \in \mathscr{L}(j_r)$.

For the continuity condition, let $V = V_1 \oplus \cdots \oplus V_j$ and let $A(V, W) \subset \mathscr{L}(j)$ be the subspace of linear isometries f such that $f(V) \subset W$, where the V_r and W are indexing spaces. We have a map $A(V, W) \times V \longrightarrow A(V, W) \times W$ of bundles over $A(V, W)$ that sends (f, v) to $(f, f(v))$. Its image is a subbundle, and we let $T(V, W)$ be the Thom complex of its complementary bundle. Define a function

$$\zeta = \zeta(V, W) \colon T(V, W) \wedge T V_1 \wedge \cdots \wedge T V_j \longrightarrow T W$$

by

$$\zeta((f, w), y_1, \ldots, y_j) = \sigma(\xi_j(f)(y_1 \wedge \cdots \wedge y_j) \wedge w)$$

for $f \in A(V, W)$, $w \in W - f(V)$, and $y_r \in T V_r$; on the right, σ is the structure map $T(fV) \wedge S^{W - fV} \longrightarrow T W$. The functions $\zeta(V, W)$ must all be continuous.

This is a very simple notion. As already noted in Section 2, it is easy to see that $\mathscr{I}$–FSP's and thus Thom prespectra give examples. However, the continuity condition requires a more conceptual description. We want to think of the maps $\xi_j(f)$ as j–fold products parametrized by $\mathscr{L}(j)$, and we want to collect them together into a single global map. The intuition is that we should be able to construct a "twisted half-smash product prespectrum" $\mathscr{L}(j) \ltimes T^{[j]}$ indexed on U by setting

$$(\mathscr{L}(j) \ltimes T^{[j]})(W) = T(V, W) \wedge T V_1 \wedge \cdots \wedge T V_j.$$

This doesn't quite make sense because of the various possible choices for the V_r, but it does work in exactly this way if we choose appropriate cofinal sequences of indexing spaces in U^j and U. The resulting construction $L(-)\ell$ on the spectrum level is independent of choices. Another intuition is that we are suitably topologizing the disjoint union over $f \in \mathscr{L}(j)$ of the prespectra $f_* T^{[j]}$ indexed on U.

These intuitions are made precise in [18, Chapter VI] and, more conceptually but perhaps less intuitively, [13, Appendix]. The construction of twisted half-smash products of spectra is the starting point of the EKMM approach to the stable homotopy category [13], but for now we are focusing on earlier work. With this construction in place, we have an equivalent definition of an $\mathscr{L}$–prespectrum in terms of action maps

$$\xi_j \colon \mathscr{L}(j) \ltimes T^{[j]} \longrightarrow T$$

such that equivariance, unit, and associativity diagrams commute. The diagrams are exactly like those in the original definition of an action of an operad on a space.

In more detail, and focusing on the spectrum level, the construction of twisted half-smash products actually gives functors

$$A \ltimes E_1 \,\overline{\wedge}\, \cdots \,\overline{\wedge}\, E_j \tag{5-1}$$

for any spectra E_r indexed on U and any map $A \longrightarrow \mathscr{L}(j)$. There are many relations among these functors as j and A vary. In particular there are canonical maps

$$\mathscr{L}(k) \ltimes (\mathscr{L}(j_1) \ltimes E^{[j_1]} \,\overline{\wedge}\, \cdots \,\overline{\wedge}\, \mathscr{L}(j_k) \ltimes E^{[j_k]}) \tag{5-2}$$

$$\downarrow \cong$$

$$(\mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k)) \ltimes E^{[j]}$$

$$\downarrow \gamma \ltimes \mathrm{id}$$

$$\mathscr{L}(j) \ltimes E^{[j]}$$

where $j = j_1 + \cdots + j_k$. Using such maps we can make sense out of the definition of an E_∞ ring spectrum in terms of an action by the operad $\mathscr{L}$.

Definition 5.3 An E_∞ ring spectrum, or $\mathscr{L}$–spectrum, is a spectrum R with an action of $\mathscr{L}$ given by an equivariant, unital, and associative system of maps

$$\mathscr{L}(j) \ltimes R^{[j]} \longrightarrow R. \tag{5-4}$$

Lemma 5.5 *If T is an $\mathscr{L}$–prespectrum, then LT is an $\mathscr{L}$–spectrum.*

This holds since the spectrification functor $L\colon \mathscr{P} \longrightarrow \mathscr{S}$ satisfies

$$L(\mathscr{L}(j) \ltimes T^{[j]}) \cong \mathscr{L}(j) \ltimes (LT)^{[j]}.$$

6 The monadic interpretation of E_∞ ring spectra

At this point, we face an unfortunate clash of notations, and for the moment the reader is asked to forget all about prespectra and the spectrification functor $L\colon \mathscr{P} \longrightarrow \mathscr{S}$. We focus solely on spectra in this section.

In analogy with Section 4, we explain that there are two monads in sight, both of whose algebras coincide with E_∞ ring spectra.[14] One is in [18], but the one more relevant to our current explanations is new.

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[14]Again, further categorical perspective is supplied in Appendix A.

In thinking about external smash products, we took spectra indexed on zero to be spaces. Since $\mathcal{L}(0)$ is the inclusion $0 \longrightarrow U$, the change of universe functor $\mathcal{L}(0) \ltimes (-)$ can and must be interpreted as the functor $\Sigma^{\infty} \colon \mathcal{T} \longrightarrow \mathcal{S}$. Similarly, the zero fold external smash power $E^{[0]}$ should be interpreted as the space S^0. Since $\Sigma^{\infty} S^0$ is the sphere spectrum S, the 0th structure map in (5-4) is a map $e \colon S \longrightarrow R$. We have the same dichotomy as in Section 4. We can either think of the map e as preassigned, in which case we think of our ground category as the category $\mathcal{S}_e$ of spectra under S, or we can think of $e = \xi_0$ as part of the structure of an E_{∞} ring spectrum, in which case we think of our ground category as $\mathcal{S}$.

In analogy with the space level monad L_+ we define a monad L_+ on the category of spectra by letting

$$L_+ E = \bigvee_{j \geq 0} \mathcal{L}(j) \ltimes_{\Sigma_j} E^{[j]}.$$

The 0th term is S, and $\eta \colon S \longrightarrow L_+ E$ is the inclusion. The product μ is induced by passage to orbits and wedges from the canonical maps (5-2).

We also have a reduced monad L defined on the category $\mathcal{S}_e$. The monad L on $\mathcal{T}_e$ is obtained from the monad L_+ on $\mathcal{T}$ by basepoint identifications. The construction can be formalized in terms of coequalizer diagrams. We obtain the analogous monad L on $\mathcal{S}_e$ by "base sphere" identifications that are formalized by precisely analogous coequalizer diagrams [18, Section VII.3]. In analogy with (4-7), the spectrum level monads L and L_+ are related by a natural isomorphism

(6-1) $L(S \vee E) \cong L_+ E.$

This isomorphism, like (4-7), is monadic. This means that the isomorphisms are compatible with the structure maps of the two monads, as is made precise in Definition 14.1. The algebras of L are the same as those of L_+, and we have the following analogue of Propositions 4.4 and 4.8.

Proposition 6.2 *The following categories are isomorphic.*

(i)   *The category $\mathcal{L}[\mathcal{S}] = \mathcal{L}[\mathcal{S}_e]$ of $\mathcal{L}$–spectra.*

(ii) *The category $L_+[\mathcal{S}]$ of L_+–algebras in $\mathcal{S}$.*

(iii) *The category $L[\mathcal{S}_e]$ of L–algebras in $\mathcal{S}_e$.*

A central feature of twisted half-smash products is that there is a natural untwisting isomorphism

(6-3) $A \ltimes (\Sigma^{\infty} X_1 \,\overline{\wedge}\, \cdots \overline{\wedge}\, \Sigma^{\infty} X_j) \cong \Sigma^{\infty}(A_+ \wedge X_1 \wedge \cdots X_j).$

for any space A over $\mathcal{L}(j)$. Using this, we obtain a monadic natural isomorphism

(6-4) $$L_+ \Sigma^\infty X \cong \Sigma^\infty L_+ X$$

relating the space and spectrum level monads L_+.

It has become usual to compress notation by defining

(6-5) $$\Sigma_+^\infty X = \Sigma^\infty (X_+)$$

for a space X, ignoring its given basepoint if it has one. When X has a (nondegenerate) basepoint, the equivalence $\Sigma(S^0 \vee X) \simeq S^1 \vee \Sigma X$ implies an equivalence[15]

(6-6) $$\Sigma_+^\infty X \simeq S \vee \Sigma^\infty X$$

under S. With the notation of (6-5), the relationship between the space and spectrum level monads L is given by a monadic natural isomorphism

(6-7) $$L \Sigma_+^\infty X \cong \Sigma_+^\infty L X.$$

(See [18, VII.3.5]). Remarkably, as we shall show in Appendix A, the commutation relations (6-1), (6-4), and (6-7) are formal, independent of calculational understanding of the monads in question.

We digress to recall a calculationally important splitting theorem that is implicit in these formalities. Using nothing but (6-1) and (6-3)–(6-7), we find

$$S \vee \Sigma^\infty L X \simeq \Sigma_+^\infty L X$$

$$\cong L \Sigma_+^\infty X$$

$$\simeq L(S \vee \Sigma^\infty X)$$

$$\cong L_+ \Sigma^\infty X$$

$$= \bigvee_{j \geq 0} \mathcal{L}(j) \ltimes_{\Sigma_j} (\Sigma^\infty X)^{[j]}$$

$$\cong S \vee \bigvee_{j \geq 1} \Sigma^\infty (\mathcal{L}(j)_+ \wedge_{\Sigma_j} X^{(j)}).$$

Quotienting out S and recalling (4-6), we obtain a natural equivalence

(6-8) $$\Sigma^\infty L X \simeq \bigvee_{j \geq 1} \Sigma^\infty D_j X.$$

---

[15]In contrast with the isomorphisms appearing in this discussion, this equivalence plays no role in our formal theory; we only recall it for use in a digression that we are about to insert.

We may replace LX by CX for any other E_∞ operad $\mathscr{C}$, such as the Steiner operad, and then $CX \simeq QX$ when X is connected. Thus, if X is connected,

$$(6\text{-}9) \qquad \Sigma^\infty QX \simeq \bigvee_{j \geq 1} \Sigma^\infty D_j X.$$

This beautiful argument was discovered by Ralph Cohen [11]. As explained in [18, Section VII.5], we can exploit the projection $\mathscr{O} \times \mathscr{L} \longrightarrow \mathscr{L}$ to obtain a splitting theorem for OX analogous to (6-8), where $\mathscr{O}$ is any operad, not necessarily E_∞.

7 The relationship between E_∞ ring spaces and E_∞ ring spectra

We show that the 0th space of an E_∞ ring spectrum is an E_∞ ring space. This observation is at the conceptual heart of what we want to convey. It was central to the 1970's applications, but it seems to have dropped off the radar screen, and this has led to some confusion in the modern literature. One reason may be that the only proof ever published is in the original source [27], which preceded the definition of twisted half-smash products and the full understanding of the category of spectra. Since this is the part of [27] that is perhaps most obscured by later changes of notations and definitions, we give a cleaner updated treatment that gives more explicit information.

Recall that $\mathscr{L}_+[\mathscr{T}] \cong L_+[\mathscr{T}]$ and $\mathscr{L}[\mathscr{S}] \cong L_+[\mathscr{S}]$ denote the categories of $\mathscr{L}_+$–spaces, or $\mathscr{L}$–spaces with zero, and of $\mathscr{L}$–spectra, thought of as identified with the categories of L_+–algebras in $\mathscr{T}$ and of L_+–algebras in $\mathscr{S}$. To distinguish, we write X for based spaces, Z for $\mathscr{L}_+$–spaces, E for spectra, and R for $\mathscr{L}$–spectra.

Proposition 7.1 *The (topological) adjunction*

$$\mathscr{S}(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E)$$

induces a (topological) adjunction

$$L_+[\mathscr{S}](\Sigma^\infty Z, R) \cong L_+[\mathscr{T}](Z, \Omega^\infty R).$$

Therefore the monad Q on $\mathscr{T}$ restricts to a monad Q on $L_+[\mathscr{T}]$ and, when restricted to $\mathscr{L}$–spectra, the functor Ω^∞ takes values in algebras over $L_+[\mathscr{T}]$.

Proof This is a formal consequence of the fact that the isomorphism (6-4) is monadic, as is explained in general categorical terms in Proposition 14.4. If (Z, ξ) is an L_+–algebra, then $\Sigma^\infty Z$ is an L_+–algebra with structure map

$$L_+ \Sigma^\infty Z \xrightarrow{\ \cong\ } \Sigma^\infty L_+ Z \xrightarrow{\ \Sigma^\infty \xi\ } \Sigma^\infty Z.$$

The isomorphism (6-4) and the adjunction give a natural composite δ:

$$L_+\Omega^\infty E\xrightarrow{\eta}\Omega^\infty\Sigma^\infty L_+\Omega^\infty E\xrightarrow{\cong}\Omega^\infty L_+\Sigma^\infty\Omega^\infty E\xrightarrow{\Omega^\infty L_+\varepsilon}\Omega^\infty L_+E.$$

If (R,ξ) is an L_+–algebra, then $\Omega^\infty R$ is an L_+–algebra with structure map

$$L_+\Omega^\infty R\xrightarrow{\delta}\Omega^\infty L_+R\xrightarrow{\Omega^\infty\xi}\Omega^\infty R.$$

Diagram chases show that the unit η and counit ε of the adjunction are maps of L_+–algebras when Z and R are L_+–algebras. The last statement is another instance of an already cited categorical triviality about the monad associated to an adjunction, and we shall return to the relevant categorical observations in the next section. $\qquad\square$

In the notation of algebras over operads, this has the following consequence.

Corollary 7.2 *The adjunction of Proposition 7.1 induces an adjunction*

$$\mathcal{L}[\mathcal{S}](\Sigma_+^\infty Y, R) \cong \mathcal{L}[\mathcal{U}](Y, \Omega^\infty R)$$

between the category of $\mathcal{L}$–spaces and the category of $\mathcal{L}$–spectra.

Proof Recall that the functor $i\colon \mathcal{T} \longrightarrow \mathcal{U}$ given by forgetting the basepoint induces an isomorphism $\mathcal{L}[\mathcal{T}]\cong\mathcal{L}[\mathcal{U}]$ since maps of $\mathcal{L}$–spaces must preserve the basepoints given by the operad action. Also, since adjoining a disjoint basepoint 0 to an $\mathcal{L}$–space Y gives an $\mathcal{L}$–space with zero, or $\mathcal{L}_+$–space in $\mathcal{T}$, while forgetting the basepoint 0 of an $\mathcal{L}_+$–space in $\mathcal{T}$ gives an $\mathcal{L}$–space in $\mathcal{U}$, the evident adjunction

$$\mathcal{T}(U_+, X) \cong \mathcal{U}(U, i X)$$

for based spaces X and unbased spaces U induces an adjunction

$$\mathcal{L}_+[\mathcal{T}](Y_+, Z) \cong \mathcal{L}[\mathcal{U}](Y, Z)$$

for $\mathcal{L}$–spaces Y and $\mathcal{L}_+$–spaces Z. Taking $Z = \Omega^\infty R$, the result follows by composing with the adjunction of Proposition 7.1. $\qquad\square$

Now let us bring the Steiner operads into play. For an indexing space $V\subset U$, $\mathcal{K}_V$ acts naturally on V–fold loop spaces $\Omega^V Y$. These actions are compatible for $V\subset W$, and by passage to colimits we obtain a natural action θ of the Steiner operad $\mathcal{C}=\mathcal{K}_U$ on $\Omega^\infty E$ for all spectra E. For spaces X, we define $\alpha\colon CX \longrightarrow QX$ to be the composite

$$(7\text{-}3) \qquad CX\xrightarrow{C\eta}C\Omega^\infty\Sigma^\infty X\xrightarrow{\theta}\Omega^\infty\Sigma^\infty X = QX.$$

Another purely formal argument shows that α is a map of monads in $\mathscr{T}$ [25, 5.2]. This observation is central to the entire theory.

We have seen in Propositions 4.9 and 7.1 that C and Q also define monads on $\mathscr{L}_+[\mathscr{T}]$, and we have the following crucial compatibility.

Proposition 7.4 *The map $\alpha\colon C \longrightarrow Q$ of monads on $\mathscr{T}$ restricts to a map of monads on $\mathscr{L}_+[\mathscr{T}]$.*

Sketch proof We must show that $\alpha\colon CX \longrightarrow QX$ is a map of $\mathscr{L}_+$–spaces when X is an $\mathscr{L}_+$–space. Since it is clear by naturality that $C\eta\colon CX \longrightarrow CQX$ is a map of $\mathscr{L}_+$–spaces, it suffices to show that $C\theta\colon CQX \longrightarrow QX$ is a map of $\mathscr{L}_+$–spaces. We may use embeddings operads rather than Steiner operads since the action of $\mathscr{K}_V$ on $\Omega^V Y$ is obtained by pullback of the action of Emb_V. The argument given for the little convex bodies operad in [27, VII.2.4, page 179] applies verbatim here. It is a passage to colimits argument similar to that sketched at the end of Section 3. $\square$

Recall that $(\mathscr{C}, \mathscr{L})$–spaces are the same as C–algebras in $\mathscr{L}_+[\mathscr{T}]$, and these are our E_∞ ring spaces. Similarly, our E_∞ ring spectra are the same as $\mathscr{L}$–spectra, and the 0th space functor takes $\mathscr{L}$–spectra to Q–algebras in $\mathscr{L}_+[\mathscr{T}]$. By pullback along α, this gives the following promised conclusion.

Corollary 7.5 *The 0th spaces of E_∞ ring spectra are naturally E_∞ ring spaces.*

In particular, the 0th space of an E_∞ ring spectrum is both a $\mathscr{C}$–space and an $\mathscr{L}$–space with 0. The interplay between the Dyer–Lashof homology operations constructed from the two operad actions is essential to the caculational applications, and for that we must use all of the components. However, to apply infinite loop space theory using the multiplicative operad $\mathscr{L}$, we must at least delete the component of 0, and it makes sense to also delete the other nonunit components.

Definition 7.6 The 0th space R_0 of an (up to homotopy) ring spectrum R is an (up to homotopy) ring space, and $\pi_0(\Omega^\infty R)$ is a ring. Define $GL_1 R$ to be the subspace of R_0 that consists of the components of the unit elements. Define $SL_1 R$ to be the component of the identity element.[16] For a space X, $[X_+, GL_1 R]$ is the group of units in the (unreduced) cohomology ring $R^0(X)$.

Corollary 7.7 *If R is an E_∞ ring spectrum, then the unit spaces $GL_1 R$ and $SL_1 R$ are $\mathscr{L}$–spaces.*

Again, we emphasize how simply and naturally these definitions fit together.

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[16]To repeat, these spaces were introduced in [27], where they were denoted FR and SFR.

8 A categorical overview of the recognition principle

The passage from space level to spectrum level information through the black box of 1 admits a simple conceptual outline. We explain it here.[17]

We consider two categories, $\mathscr{V}$ and $\mathscr{W}$, with an adjoint pair of functors (Ξ, Λ) between them.[18] We write

$$\eta\colon \mathrm{Id} \longrightarrow \Lambda\,\Xi \quad \text{and} \quad \varepsilon\colon \Xi\,\Lambda \longrightarrow \mathrm{Id}$$

for the unit and counit of the adjunction. The reader should be thinking of (Σ^n, Ω^n), where $\mathscr{V} = \mathscr{T}$ and $\mathscr{W}$ is the category of n–fold loop sequences $\{\Omega^i Y \,|\, 0 \le i \le n\}$ and maps $\{\Omega^i f \,|\, 0 \le i \le n\}$. This is a copy of $\mathscr{T}$, but we want to remember how it encodes n–fold loop spaces. It is analogous and more relevant to the present theory to think of $(\Sigma^\infty, \Omega^\infty)$, where $\mathscr{V} = \mathscr{T}$ and $\mathscr{W} = \mathscr{S}$.

As we have already noted several times, we have the monad

$$(\Lambda\,\Xi, \mu, \eta)$$

on $\mathscr{V}$, where $\mu = \Lambda \varepsilon \Xi$. We also have a right action of the monad $\Lambda\,\Xi$ on the functor Ξ and a left action of $\Lambda\,\Xi$ on Λ. These are given by the natural maps

$$\varepsilon\,\Xi\colon \Xi\,\Lambda\,\Xi \longrightarrow \Xi \quad \text{and} \quad \Lambda\varepsilon\colon \Lambda\,\Xi\,\Lambda \longrightarrow \Lambda.$$

Actions of monads on functors satisfy unit and associativity diagrams just like those that define the action of a monad on an object. If we think of an object $X \in \mathscr{V}$ as a functor from the trivial one object category $*$ to $\mathscr{V}$, then an action of a monad on the object X is the same as a left action of the monad on the functor X.

Now suppose we also have some monad C on $\mathscr{V}$ and a map of monads $\alpha\colon C \longrightarrow \Lambda\,\Xi$. By pullback, we inherit a right action of C on Ξ and a left action of C on Λ, which we denote by ρ and λ. Thus

$$\rho = \varepsilon\,\Xi \circ \Xi\alpha\colon \Xi C \longrightarrow \Xi \quad \text{and} \quad \lambda = \Lambda\varepsilon \circ \alpha\Lambda\colon C\Lambda \longrightarrow \Lambda.$$

For a C–algebra X in $\mathscr{V}$, we seek an object $\mathbb{E}X$ in $\mathscr{W}$ such that X is weakly equivalent to $\Lambda\mathbb{E}X$ as a C–algebra. This is the general situation addressed by our black box, and we first remind ourselves of how we would approach the problem if were looking for a categorical analogue. We will reprove a special case of Beck's monadicity theorem (see Mac Lane [21], Section VI.7), but in a way that suggests how to proceed homotopically.

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[17]I am indebted to Saunders Mac Lane, Gaunce Lewis, and Matt Ando for ideas I learned from them some thirty-five years apart.

[18] Ξ is the capital Greek letter Xi; Ξ and Λ are meant to look a little like Σ and Ω.

Assume that $\mathscr{W}$ is cocomplete. For any right C–functor $\Psi\colon \mathscr{V} \longrightarrow \mathscr{W}$ with right action ρ and left C–functor $\Phi\colon \mathscr{U} \longrightarrow \mathscr{V}$ with left action λ, where $\mathscr{U}$ is any other category, we have the monadic tensor product $\Psi \otimes_C \Phi\colon \mathscr{U} \longrightarrow \mathscr{W}$ that is defined on objects $U \in \mathscr{U}$ as the coequalizer displayed in the diagram

$$(8\text{-}1) \qquad\qquad (\Psi C \Phi)(U) \underset{\Psi\lambda}{\overset{\rho\Phi}{\rightrightarrows}} (\Psi\Phi)(U) \longrightarrow (\Psi \otimes_C \Phi)(U).$$

We are interested primarily in the case when $\mathscr{U} = *$ and $\Phi = X$ for a C–algebra (X, ξ) in $\mathscr{V}$, and we then write $\Psi \otimes_C X$. Specializing to the case $\Psi = \Xi$, this is the coequalizer in $\mathscr{W}$ displayed in the diagram

$$(8\text{-}2) \qquad\qquad \Xi CX \underset{\Xi\xi}{\overset{\rho}{\rightrightarrows}} \Xi X \longrightarrow \Xi \otimes_C X.$$

For comparison, we have the canonical split coequalizer

$$(8\text{-}3) \qquad\qquad CCX \underset{C\xi}{\overset{\mu}{\rightrightarrows}} CX \overset{\xi}{\longrightarrow} X$$

in $\mathscr{V}$, which is split by $\eta_{CX}\colon CX \longrightarrow CCX$ and $\eta_X\colon X \longrightarrow CX$.

Let $C[\mathscr{V}]$ denote the category of C–algebras in $\mathscr{V}$. Then $\Xi \otimes_C (-)$ is a functor $C[\mathscr{V}] \longrightarrow \mathscr{W}$, and our original adjunction restricts to an adjunction

$$(8\text{-}4) \qquad\qquad \mathscr{W}(\Xi \otimes_C X, Y) \cong C[\mathscr{V}](X, \Lambda Y).$$

Indeed, consider a map $f\colon X \longrightarrow \Lambda Y$ of C–algebras, so that $f \circ \xi = \lambda \circ Cf$. Taking its adjoint $\tilde{f}\colon \Xi X \longrightarrow Y$, we see by a little diagram chase that it equalizes the pair of maps $\Xi\xi$ and ρ and therefore induces the required adjoint map $\Xi \otimes_C X \longrightarrow Y$.

This construction applies in particular to the monad $\Lambda\Xi$, and for the moment we take $C = \Lambda\Xi$. The Beck monadicity theorem says that the adjunction (8-4) is then an adjoint equivalence under appropriate hypotheses, which we now explain.

Consider those parallel pairs of arrows (f, g) in the diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{h}{\dashrightarrow} Z$$

in $\mathscr{W}$ such that there exists a split coequalizer diagram

$$\Lambda X \underset{\Lambda g}{\overset{\Lambda f}{\rightrightarrows}} \Lambda Y \overset{j}{\longrightarrow} V$$

in $\mathscr{V}$. Assume that Λ preserves and reflects coequalizers of such pairs (f, g) of parallel arrows. Preservation means that if h is a coequalizer of f and g, then $\Lambda h \colon \Lambda Y \longrightarrow \Lambda Z$ is a coequalizer of Λf and Λg. It follows that there is a unique isomorphism $i \colon V \longrightarrow \Lambda Z$ such that $i \circ j = \Lambda h$. Reflection means that there is a coequalizer h of f and g and an isomorphism $i \colon V \longrightarrow \Lambda Z$ such that $i \circ j = \Lambda h$.

By preservation, if (X, ξ) is a C–algebra, then $\eta \colon X \longrightarrow \Lambda(\Xi \otimes_C X)$ must be an isomorphism because application of Λ to the arrows $\rho = \varepsilon \Xi$ and $\Xi \xi$ of (8-2) gives the pair of maps that have the split coequalizer (8-3). By reflection, if Y is an object of $\mathscr{W}$, then $\varepsilon \colon \Xi \otimes_C \Lambda Y \longrightarrow Y$ must be an isomorphism since if we apply Λ to the coequalizer (8-2) with $X = \Lambda Y$ we obtain the split coequalizer (8-3) for the C–algebra ΛY. This proves that the adjunction (8-4) is an adjoint equivalence.

Returning to our original map of monads $\alpha \colon C \longrightarrow \Lambda \Xi$, but thinking homotopically, one might hope that $\Lambda(\Xi \otimes_C X)$ is equivalent to X under reasonable hypotheses. However, since coequalizers usually behave badly homotopically, we need a homotopical approximation. Here thinking model categorically seems more of a hindrance than a help, and we instead use the two-sided monadic bar construction of [25]. It can be defined in the generality of (8-1) as a functor

$$B(\Psi, C, \Theta) \colon \mathscr{U} \longrightarrow \mathscr{W},$$

but we restrict attention to the case of a C–algebra X, where it is

$$B(\Psi, C, X).$$

We have a simplicial object $B_*(\Psi, C, X)$ in $\mathscr{W}$ whose object of q–simplices is

$$B_q(\Psi, C, X) = \Psi C^q X.$$

The right action $\Psi C \longrightarrow \Psi$ induces the 0th face map, the map $C^{i-1}\mu$, $1 \leq i < q$, induces the ith face map, and the action $CX \longrightarrow X$ induces the qth face map. The maps $C^i \eta$, $0 \leq i \leq q$ induce the degeneracy maps. We need to realize simplicial objects Y_* in $\mathscr{V}$ and $\mathscr{W}$ to objects of $\mathscr{V}$ and $\mathscr{W}$, and we need to do so compatibly. For that, we need a covariant standard simplex functor $\Delta^* \colon \Delta \longrightarrow \mathscr{V}$, where Δ is the standard category of finite sets n and monotonic maps.[19] We compose with Ξ to obtain a standard simplex functor $\Delta^* \colon \Delta \longrightarrow \mathscr{W}$. We define

$$|X_*|_\mathscr{V} = X_* \otimes_\Delta \Delta^*$$

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[19]Equivalently, n is the ordered set $0 < 1 < \cdots < n$, and maps preserve order.

for simplicial objects X_* in $\mathscr{V}$, and similarly for $\mathscr{W}$. In the cases of interest, realization is a left adjoint. We define

$$B(\Psi, C, X) = |B_*(\Psi, C, X)|.$$

Commuting the left adjoint C past realization, we find that

(8-5) $$|CX_*|_{\mathscr{V}} \cong C|X_*|_{\mathscr{V}}$$

and conclude that the realization of a simplicial C–algebra is a C–algebra. The iterated action map $\xi^{q+1}\colon C^{q+1}X \longrightarrow X$ gives a map ε_* from $B_*(C, C, X)$ to the constant simplicial object at X. Passing to realizations, ε_* gives a natural map of C–algebras $\varepsilon\colon B(C, C, X) \longrightarrow X$. Forgetting the C–action, ε_* is a simplicial homotopy equivalence in the category of simplicial objects in $\mathscr{W}$, in the combinatorial sense that is defined for simplicial objects in any category. In reasonable situations, for example categories tensored over spaces, passage to realization converts this to a homotopy equivalence in $\mathscr{W}$. Commuting coequalizers past realization, we find

(8-6) $$B(\Psi, C, X) \cong \Psi \otimes_C B(C, C, X).$$

This has the same flavor as applying cofibrant approximation to X and then applying $\Psi \otimes_C (-)$, but the two-sided bar construction has considerable advantages. For example, in our topological situations, it is a continuous functor, whereas cofibrant approximations generally are not. Also, starting from a general object X, one could not expect something as strong as an underlying homotopy equivalence from a cofibrant approximation $X' \longrightarrow X$. More fundamentally, the functoriality in all three variables is central to the arguments. Presumably, for good model categories $\mathscr{V}$, if one restricts X to be cofibrant in $\mathscr{V}$, then $B(C, C, X)$ is cofibrant in $\mathscr{V}[\mathscr{T}]$, at least up to homotopy equivalence, and thus can be viewed as an especially nice substitute for cofibrant approximation, but I've never gone back to work out such model categorical details.[20]

Now the black box works as follows. We take $\mathscr{V}$ and $\mathscr{W}$ to be categories with a reasonable homotopy theory, such as model categories. Our candidate for $\mathbb{E}X$ is $B(\Xi, C, X)$. There are three steps that are needed to obtain an equivalence between a suitable C–algebra X and $\Lambda\mathbb{E}X$. The fundamental one is to prove an approximation theorem to the effect that C is a homotopical approximation to $\Lambda\Xi$. This step has nothing to do with monads, depending only on the homotopical properties of the comparison map α.

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[20]This exercise has recently been carried out in [1]. It is to be emphasized that nothing in the outline we are giving simplifies in the slightest. Rather, the details sketched here are reinterpreted model-theoretically. As anticipated, the essential point is to observe that $B(C, C, X)$ is of the homotopy type of a cofibrant object when X is cofibrant.

Step 8.7 For suitable objects $X \in \mathcal{V}$, $\alpha \colon CX \longrightarrow \Lambda \Xi X$ is a weak equivalence.

The second step is a general homotopical property of realization that also has nothing to do with the monadic framework. It implies that the good homotopical behavior of α is preserved by various maps between bar constructions.

Step 8.8 For suitable simplicial objects Y_* and Y'_* in $\mathcal{W}$, if $f_* \colon Y_* \longrightarrow Y'_*$ is a map such that each f_q is a weak equivalence, then $|f_*|$ is a weak equivalence in $\mathcal{W}$, and similarly for $\mathcal{V}$.

In the space level cases of interest, the weaker property of being a group completion will generalize being a weak equivalence in Steps 8.7 and 8.8, but we defer discussion of that until the next section.

The third step is an analogue of commuting Λ past coequalizers in the categorical argument we are mimicking. It has two parts, one homotopical and the other monadic.

Step 8.9 For suitable simplicial objects Y_* in $\mathcal{W}$, the canonical natural map

$$\zeta \colon |\Lambda Y_*|_{\mathcal{V}} \longrightarrow \Lambda |Y_*|_{\mathcal{W}}$$

is both a weak equivalence and a map of C–algebras.

Here ΛY_* is obtained by applying Λ levelwise to simplicial objects. To construct the canonical map ζ, we first obtain a natural isomorphism

(8-10) $$\Xi |X_*|_{\mathcal{V}} \cong |\Xi X_*|_{\mathcal{W}}$$

by commuting left adjoints, where X_* is a simplicial object in $\mathcal{V}$. Applying this with X_* replaced by ΛY_* we obtain

$$|\varepsilon|_{\mathcal{W}} \colon \Xi |\Lambda Y_*|_{\mathcal{V}} \cong |\Xi \Lambda Y_*|_{\mathcal{W}} \longrightarrow |Y_*|_{\mathcal{W}}.$$

Its adjoint is the required natural map $\zeta \colon |\Lambda Y_*|_{\mathcal{V}} \longrightarrow \Lambda |Y_*|_{\mathcal{W}}$.

Assuming that these steps have been taken, the black box works as follows to relate the homotopy categories of $C[\mathcal{V}]$ and $\mathcal{W}$. For a C–algebra X in $\mathcal{V}$, we have the diagram of maps of C–algebras

$$X \xleftarrow{\ \varepsilon\ } B(C, C, X) \xrightarrow{\ B(\alpha, \mathrm{id}, \mathrm{id})\ } B(\Lambda \Xi, C, X) \xrightarrow{\ \zeta\ } \Lambda B(\Xi, C, X) = \Lambda \mathbb{E} X$$

in which all maps are weak equivalences (or group completions). On the left, we have a map, but not a C–map, $\eta \colon X \longrightarrow B(C, C, X)$ which is an inverse homotopy

equivalence to ε. We also write η for the resulting composite $X \longrightarrow \Lambda \mathbb{E} X$. This is the analogue of the map $\eta \colon X \longrightarrow \Lambda(\Xi \otimes_C X)$ in our categorical sketch.

For an object Y in $\mathscr{W}$, observe that

$$B_q(\Xi, \Lambda\Xi, \Lambda Y) = (\Xi\Lambda)^{q+1}Y$$

and the maps $(\Xi\Lambda)^{q+1} \longrightarrow Y$ obtained by iterating ε give a map of simplicial objects from $B_*(\Xi, \Lambda\Xi, \Lambda Y)$ to the constant simplicial object at Y. On passage to realization, we obtain a composite natural map

$$\mathbb{E}\Lambda Y = B(\Xi, C, \Lambda Y) \xrightarrow{B(\mathrm{id},\alpha,\mathrm{id})} B(\Xi, \Lambda\Xi, \Lambda Y) \xrightarrow{\varepsilon} Y,$$

which we also write as ε. This is the analogue of $\varepsilon \colon \Xi \otimes_C \Lambda Y \longrightarrow Y$ in our categorical sketch. We have the following commutative diagram in which all maps except the top two are weak equivalences, hence they are too.

$$
\begin{array}{ccccc}
\Lambda\mathbb{E}\Lambda Y = \Lambda B(\Xi, C, \Lambda Y) & \xrightarrow{\Lambda B(\mathrm{id},\alpha,\mathrm{id})} & \Lambda B(\Xi, \Lambda\Xi, \Lambda Y) & \xrightarrow{\Lambda\varepsilon} & \Lambda Y \\
\ \ \uparrow{\zeta} & & \ \ \uparrow{\zeta} & \nearrow{\varepsilon} & \\
B(\Lambda\Xi, C, \Lambda Y) & \xrightarrow{B(\mathrm{id},\alpha,\mathrm{id})} & B(\Lambda\Xi, \Lambda\Xi, \Lambda Y) & &
\end{array}
$$

We do not expect Λ to reflect weak equivalences in general, so we do not expect $\mathbb{E}\Lambda Y \simeq Y$ in general, but we do expect this on suitably restricted Y. We conclude that the adjunction (8-4) induces an adjoint equivalence of suitably restricted homotopy categories.

9   The additive and multiplicative infinite loop space machine

The original use of this approach in [25] took (Ξ, Λ) to be (Σ^n, Ω^n) and C to be the monad associated to the little n–cubes operad $\mathscr{C}_n$. It took α to be the map of monads given by the composites $\theta \circ C_n\eta \colon C_n X \longrightarrow \Omega^n \Sigma^n X$. For connected $\mathscr{C}_n$–spaces X, details of all steps may be found in [25].

For the nonconnected case, we say that an H–monoid[21] is grouplike if $\pi_0(X)$ is a group under the induced multiplication. We say that a map $f \colon X \longrightarrow Y$ between homotopy commutative H–monoids is a group completion if Y is grouplike and two things hold. First, $\pi_0(Y)$ is the group completion of $\pi_0(X)$ in the sense that any map of monoids from $\pi_0(X)$ to a group G factors uniquely through a group

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[21]This is just a convenient abbreviated way of writing homotopy associative H–space.

homomorphism $\pi_0(Y) \longrightarrow G$. Second, for any commutative ring of coefficients, or equivalently any field of coefficients, the homomorphism $f_*\colon H_*(X) \longrightarrow H_*(Y)$ of graded commutative rings is a localization (in the classical algebraic sense) at the submonoid $\pi_0(X)$ of $H_0(X)$. That is, $H_*(Y)$ is $H_*(X)[\pi_0(X)^{-1}]$.

For $n \geq 2$, α_n is a group completion for all $\mathscr{C}_n$–spaces X by calculations of Fred Cohen [10] or by an argument due to Segal [40]. Thus α_n is a weak equivalence for all grouplike X. This gives Step 8.7, and Steps 8.8 and 8.9 are dealt with in May [26].

As Gaunce Lewis pointed out to me many years ago and Ando et al [1] rediscovered, in the stable case $n = \infty$ we can compare spaces and spectra directly. We take $\mathscr{V} = \mathscr{T}$ and $\mathscr{W} = \mathscr{S}$, we take (Ξ, Λ) to be $(\Sigma^\infty, \Omega^\infty)$, and we take C to be the monad associated to the Steiner operad (for $U = \mathbb{R}^\infty$); in the additive theory, we could equally well use the infinite little cubes operad $\mathscr{C}_\infty$. As recalled in (7-3), we have a map of monads $\alpha\colon C \longrightarrow Q$. The calculations needed to prove that α is a group completion preceded those in the deeper case of finite n [10], and we have Step 8.7. Step 8.8 is given for spaces in [25, Chapter 11] and [26, Appendix] and for spectra in [13, Chapter X]; see [13, X.1.3 and X.2.4].

We need to say a little about Step 8.9. For simplicial spaces X_* such that each X_q is connected (which holds when we apply this step) and which satisfy the usual (Reedy) cofibrancy condition (which follows in our examples from the assumed nondegeneracy of basepoints), the map $\zeta\colon |\Omega X_*| \longrightarrow \Omega |X_*|$ is a weak equivalence by [25, 12.3]. That is the hardest thing in [25]. Moreover, the n–fold iterate $\zeta^n\colon |\Omega^n X_*| \longrightarrow \Omega^n |X_*|$ is a map of $\mathscr{C}_n$–spaces by [25, 12.4]. The latter argument works equally well with $\mathscr{C}_n$ replaced by the Steiner operad $\mathscr{K}_{\mathbb{R}^n}$, so we have Step 8.9 for each Ω^n. For sufficiently nice simplicial spectra E_*, such as those relevant here, the canonical map

$$(9\text{-}1) \qquad \zeta\colon |\Omega^\infty E_*|_\mathscr{T} \longrightarrow \Omega^\infty |E_*|_\mathscr{S}$$

can be identified with the colimit of the iterated canonical maps

$$(9\text{-}2) \qquad \zeta^n\colon |\Omega^n (E_n)_*|_\mathscr{T} \longrightarrow \Omega^n |(E_n)_*|_\mathscr{T},$$

and Step 8.9 for Ω^∞ follows by passage to colimits from Step 8.9 for the Ω^n, applied to simplicial $(n-1)$–connected spaces. Here, for simplicity of comparison with [25], we have indexed spectra sequentially, that is on the cofinal sequence $\mathbb{R}^n$ in U. The nth spaces $(E_n)_*$ of the simplicial spectrum E_* give a simplicial space and the $\Omega^n (E_n)_*$ are compatibly isomorphic to $(E_0)_*$. Thus, on the left side, the colimit is $|\Omega^\infty E_*|_\mathscr{T}$. When $E_* = LT_*$ for a simplicial inclusion prespectrum T_*, the right side can be computed as $L|T_*|_\mathscr{P}$, where the prespectrum level realization is defined levelwise. One checks that $|T_*|_\mathscr{P}$ is again an inclusion prespectrum, and the identification of the colimit on the right with $\Omega^\infty |E_*|_\mathscr{S}$ follows.

Granting these details, we have the following additive infinite loop space machine. Recall that a spectrum is connective if its negative homotopy groups are zero and that a map f of connective spectra is a weak equivalence if and only if $\Omega^\infty f$ is a weak equivalence.

Theorem 9.3 *For a $\mathscr{C}$–space, define $\mathbb{E}X = B(\Sigma^\infty, C, X)$. Then $\mathbb{E}X$ is connective and there is a natural diagram of maps of C–spaces*

$$X \xleftarrow{\ \varepsilon\ } B(C, C, X) \xrightarrow{\ B(\alpha,\mathrm{id},\mathrm{id})\ } B(Q, C, X) \xrightarrow{\ \zeta\ } \Omega^\infty \mathbb{E}X$$

in which ε is a homotopy equivalence with natural homotopy inverse η, ζ is a weak equivalence, and $B(\alpha, \mathrm{id}, \mathrm{id})$ is a group completion. The composite $\eta\colon X \longrightarrow \Omega^\infty \mathbb{E}X$ is therefore a group completion and thus a weak equivalence if X is grouplike. For a spectrum Y, there is a composite natural map of spectra

$$\varepsilon\colon \mathbb{E}\Omega^\infty Y \xrightarrow{\ B(\mathrm{id},\alpha,\mathrm{id})\ } B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\ \varepsilon\ } Y,$$

and the induced maps of $\mathscr{C}$–spaces

$$\Omega^\infty \varepsilon\colon \Omega^\infty \mathbb{E}\Omega^\infty Y \xrightarrow{\ \Omega^\infty B(\mathrm{id},\alpha,\mathrm{id})\ } \Omega^\infty B(\Sigma^\infty, Q, \Omega^\infty Y) \xrightarrow{\ \Omega^\infty \varepsilon\ } \Omega^\infty Y$$

are weak equivalences. Therefore $\mathbb{E}$ and Ω^∞ induce an adjoint equivalence between the homotopy category of grouplike E_∞ spaces and the homotopy category of connective spectra.

The previous theorem refers only to the Steiner operad $\mathscr{C}$, for canonicity, but we can apply it equally well to any other E_∞ operad $\mathscr{O}$. We can form the product operad $\mathscr{P} = \mathscr{C} \times \mathscr{O}$, and the jth levels of its projections $\pi_1\colon \mathscr{P} \longrightarrow \mathscr{C}$ and $\pi_2\colon \mathscr{P} \longrightarrow \mathscr{O}$ are Σ_j–equivariant homotopy equivalences. While the monad P associated to $\mathscr{P}$ is not a product, the induced projections of monads $P \longrightarrow C$ and $P \longrightarrow O$ are natural weak equivalences. This allows us to replace C by P in the previous theorem. If X is an O–space, then it is a P–space by pullback along π_2, and Σ^∞ is a right P–functor by pullback along π_1.

There is another way to think about this trick that I now find preferable. Instead of repeating Theorem 9.3 with C replaced by P, one can first change input data and then apply Theorem 9.3 as it stands. Here we again use the two-sided bar construction. For $\mathscr{O}$–spaces X regarded by pullback as $\mathscr{P}$–spaces, we have a pair of natural weak equivalences of $\mathscr{P}$–spaces

(9-4) $$X \xleftarrow{\ \varepsilon\ } B(P, P, X) \xrightarrow{\ B(\pi_1,\mathrm{id},\mathrm{id})\ } B(C, P, X),$$

where $B(C, P, X)$ is a C–space regarded as a $\mathscr{P}$–space by pullback along π_1. The same maps show that if X is a $\mathscr{P}$–space, then it is weakly equivalent as a $\mathscr{P}$–space to $B(C, P, X)$. Thus the categories of $\mathscr{O}$–spaces and $\mathscr{P}$–spaces can be used interchangeably. Reversing the roles of $\mathscr{C}$ and $\mathscr{O}$, the categories of $\mathscr{C}$–spaces and $\mathscr{P}$–spaces can also be used interchangeably. We conclude that $\mathscr{O}$–spaces for any E_∞ operad $\mathscr{O}$ can be used as input to the additive infinite loop space machine. We have the following conclusion.

Corollary 9.5 *For any* E_∞ *operad* $\mathscr{O}$*, the additive infinite loop space machine* $\mathbb{E}$ *and the* 0 *th space functor* Ω^∞ *induce an adjoint equivalence between the homotopy category of grouplike* $\mathscr{O}$*–spaces and the homotopy category of connective spectra.*

In particular, many of the interesting examples are $\mathscr{L}$–spaces. We can apply the additive infinite loop space machine to them, ignoring the special role of $\mathscr{L}$ in the multiplicative theory. As we recall in the second sequel [33], examples include various stable classifying spaces and homogeneous spaces of geometric interest. By Corollary 7.7, they also include the unit spaces $GL_1 R$ and $SL_1 R$ of an E_∞ ring spectrum R. The importance of these spaces in geometric topology is explained in [33]. The following definitions and results highlight their importance in stable homotopy theory and play a significant role in [1].[22] We start with a reinterpretation of the adjunction of Corollary 7.2 for grouplike $\mathscr{L}$–spaces Y.

Lemma 9.6 *If* Y *is a grouplike* $\mathscr{L}$*–space, then*

$$\mathscr{L}[\mathscr{S}](\Sigma_+^\infty Y, R) \cong \mathscr{L}[\mathscr{U}](Y, GL_1 R)$$

Proof A map of $\mathscr{L}$–spaces $Y \longrightarrow \Omega^\infty R$ must take values in $GL_1 R$ since the group $\pi_0 Y$ must map to the group of units of the ring $\pi_0 \Omega^\infty R$. $\square$

The notations of the following definition have recently become standard, although the definition itself dates back to [27].

Definition 9.7 Let R be an $\mathscr{L}$–spectrum. Using the operad $\mathscr{L}$ in the additive infinite loop space machine, define $gl_1 R$ and $sl_1 R$ to be the spectra obtained from the $\mathscr{L}$–spaces $GL_1 R$ and $SL_1 R$, so that $\Omega^\infty gl_1 R \simeq GL_1 R$ and $\Omega^\infty sl_1 R \simeq SL_1 R$.

Corollary 9.8 *On homotopy categories, the functor* gl_1 *from* E_∞ *ring spectra to spectra is right adjoint to the functor* $\Sigma_+^\infty \Omega^\infty$ *from spectra to* E_∞ *ring spectra.*

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[22]That paper reads in part like a sequel to this one. However, aside from a very brief remark that merely acknowledges their existence, E_∞ ring spaces are deliberately avoided there.

Proof Here we implicitly replace the C–space $\Omega^\infty R$ by a weakly equivalent $\mathscr{L}$–space, as above. Using this replacement, we can view the functor $\Sigma^\infty_+\Omega^\infty$ as taking values in $\mathscr{L}$–spectra. Now the conclusion is obtained by composing the equivalence of Corollary 9.5 with the adjunction of Lemma 9.6 and using that, in the homotopy category, maps from a spectrum E to a connective spectrum F, such as $gl_1 R$, are the same as maps from the connective cover cE of E into F, while $\Omega^\infty cE \simeq \Omega^\infty E$. $\square$

Note that we replaced $\mathscr{L}$ by $\mathscr{C}$ to define $GL_1 R$ as a $\mathscr{C}$–space valued functor before applying $\mathbb{E}$, and we replaced $\mathscr{C}$ by $\mathscr{L}$ to define Ω^∞ as an $\mathscr{L}$–space valued functor. Another important example of an E_∞ operad should also be mentioned.

Remark 9.9 There is a categorical operad, denoted $\mathscr{D}$, that is obtained by applying the classifying space functor to the translation categories of the groups Σ_j. This operad acts on the classifying spaces of permutative categories, as we recall from [26, Section 4] in [32]. Another construction of the same E_∞ operad is obtained by applying a certain product-preserving functor from spaces to contractible spaces to the operad $\mathscr{M}$ that defines monoids; see [25, page 161] and [26, 4.8]. The second construction shows that $\mathscr{M}$ is a suboperad, so that a $\mathscr{D}$–space has a canonical product that makes it a topological monoid. The simplicial version of $\mathscr{D}$ is called the Barratt–Eccles operad in view of Barratt and Eccles' use of it in [2; 3; 4].

Remark 9.10 In the applications, one often uses the consistency statement that, for an E_∞ space X, there is a natural map of spectra $\Sigma\mathbb{E}\Omega X \longrightarrow \mathbb{E}X$ that is an equivalence if X is connected. This is proven in [25, Section 14], with improvements in [26, Section 3] and [27, VI.3.4]. The result is considerably less obvious than it seems, and the cited proofs are rather impenetrable, even to me. I found a considerably simpler conceptual proof while writing this paper. Since this is irrelevant to our multiplicative story, I'll avoid interrupting the flow here by deferring the new proof to Appendix B.

Now we add in the multiplicative structure, and we find that it is startlingly easy to do so. Let us say that a rig space X is ringlike if it is grouplike under its additive H–monoid structure. A map of rig spaces $f\colon X \longrightarrow Y$ is a ring completion if Y is ringlike and f is a group completion of the additive structure. Replacing $\mathscr{T}$ and $\mathscr{S}$ by $L_+[\mathscr{T}]$ and $L_+[\mathscr{S}]$ and using that $\alpha\colon C \longrightarrow Q$ is a map of monads on $L_+[\mathscr{T}]$, the formal structure of the previous section still applies verbatim, and the homotopy properties depend only on the additive structure. The only point that needs mentioning is that, for the monadic part of Step 8.9, we now identify the map ζ of (9-1) with the colimit over V of the maps

(9-11)
$$\zeta^V\colon |\Omega^V (EV)_*|_{\mathscr{T}} \longrightarrow \Omega^V |(EV)_*|_{\mathscr{T}}$$

and see that ζ is a map of $\mathscr{L}_+$–spaces because of the naturality of the colimit system (9-11) with respect to linear isometries. Therefore the additive infinite loop space machine specializes to a multiplicative infinite loop space machine.

Theorem 9.12 *For a $(\mathscr{C}, \mathscr{L})$–space X, define $\mathbb{E}X = B(\Sigma^\infty, C, X)$. Then $\mathbb{E}X$ is a connective $\mathscr{L}$–spectrum and all maps in the diagram*

$$X \xleftarrow{\ \varepsilon\ } B(C, C, X) \xrightarrow{\ B(\alpha, \mathrm{id}, \mathrm{id})\ } B(Q, C, X) \xrightarrow{\ \zeta\ } \Omega^\infty \mathbb{E}X$$

of the additive infinite loop space machine are maps of $(\mathscr{C}, \mathscr{L})$–spaces. Therefore the composite $\eta\colon X \longrightarrow \Omega^\infty \mathbb{E}X$ is a ring completion. For an $\mathscr{L}$–spectrum R, the maps

$$\varepsilon\colon \mathbb{E}\Omega^\infty R \xrightarrow{\ B(\mathrm{id}, \alpha, \mathrm{id})\ } B(\Sigma^\infty, Q, \Omega^\infty R) \xrightarrow{\ \varepsilon\ } R$$

are maps of $\mathscr{L}$–spectra and the maps

$$\Omega^\infty \varepsilon\colon \Omega^\infty \mathbb{E}\Omega^\infty R \xrightarrow{\ \Omega^\infty B(\mathrm{id}, \alpha, \mathrm{id})\ } \Omega^\infty B(\Sigma^\infty, Q, \Omega^\infty R) \xrightarrow{\ \Omega^\infty \varepsilon\ } \Omega^\infty R$$

are maps of $(\mathscr{C}, \mathscr{L})$–spaces. Therefore $\mathbb{E}$ and Ω^∞ induce an adjoint equivalence between the homotopy category of ringlike E_∞ ring spaces and the homotopy category of connective E_∞ ring spectra.

Again, we emphasize how simply and naturally these structures fit together. However, here we face an embarrassment. We would like to apply this machine to construct new E_∞ ring spectra, and the problem is that the only operad pairs we have in sight are $(\mathscr{C}, \mathscr{L})$ and $(\mathscr{N}, \mathscr{N})$. We could apply the product of operads trick to operad pairs if we only had examples to which to apply it. We return to this point in the sequel [32], where we show how to convert such naturally occurring data as bipermutative categories to E_∞ ring spaces, but the theory in this paper is independent of that problem.

10   Localizations of the special unit spectrum $sl_1 R$

The Barratt–Priddy–Quillen theorem tells us how to construct the sphere spectrum from symmetric groups. This result is built into the additive infinite loop space machine. A multiplicative elaboration is also built into the infinite loop space machine, as we explain here. For a $\mathscr{C}$–space or $(\mathscr{C}, \mathscr{L})$–space X, we abbreviate notation by writing $\Gamma X = \Omega^\infty \mathbb{E}X$, using notations like $\Gamma_1 X$ to indicate components. We write $\eta\colon X \longrightarrow \Gamma X$ for the group completion map of Theorem 9.3.[23] Since it is the composite

---

[23]The letter Γ is chosen as a reminder of the group completion property.

of E_∞ maps or, multiplicatively, E_∞ ring maps and the homotopy inverse of such a map, we may think of it as an E_∞ or E_∞ ring map.

Theorem 10.1 *For a based space Y, α and the left map η are group completions and $\Gamma\alpha$ and the right map η are equivalences in the commutative diagram*

$$\begin{array}{ccc} CY & \xrightarrow{\ \alpha\ } & QY \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \Gamma CY & \xrightarrow{\ \Gamma\alpha\ } & \Gamma QY \end{array}$$

If Y is an $\mathcal{L}_+$–space, then this is a diagram of E_∞ ring spaces.

Replacing Y by Y_+, we see by inspection that $C(Y_+)$ is the disjoint union over $j \geq 0$ of the spaces $\mathscr{C}(j) \times_{\Sigma_j} Y^j$, and of course $\mathscr{C}(j)$ is a model for $E\Sigma_j$. When $Y = BG$ for a topological group G, these are classifying spaces $B(\Sigma_j \int G)$. When $Y = *$ and thus $Y_+ = S^0$, they are classifying spaces $B\Sigma_j$, and we see that the 0th space of the sphere spectrum is the group completion of the H–monoid $\amalg_{j \geq 0} B\Sigma_j$. This is one version of the Barratt–Quillen theorem.

For an E_∞ space X with a map $S^0 \longrightarrow X$, there is a natural map of monoids from the additive monoid $\mathbb{Z}_{\geq 0}$ of nonnegative integers to the monoid $\pi_0(X)$. It is obtained by passage to π_0 from the composite $CS^0 \longrightarrow CX \longrightarrow X$. We assume that it is a monomorphism, as holds in the interesting cases. Write X_m for the mth component. Translation by an element in X_n (using the H–space structure induced by the operad action) induces a map $n\colon X_m \longrightarrow X_{m+n}$. We have the homotopy commutative ladder

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{\ 1\ } & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \xrightarrow{\ 1\ } & X_n & \longrightarrow & \cdots \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} & & & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \\ \Gamma_0 X & \xrightarrow{\ 1\ } & \Gamma_1 X & \longrightarrow & \cdots & \longrightarrow & \Gamma_{n-1} X & \xrightarrow{\ 1\ } & \Gamma_n X & \longrightarrow & \cdots \end{array}$$

Write $\bar{X}$ for the telescope of the top row. The maps on the bottom row are homotopy equivalences, so the ladder induces a map $\bar\eta\colon \bar{X} \longrightarrow \Gamma_0 X$. Since η is a group completion, $\bar\eta$ induces an isomorphism on homology. Taking $X = CS^0$, it follows that $\bar\eta\colon B\Sigma_\infty \longrightarrow Q_0 S^0$ is a homology isomorphism and therefore that $Q_0 S^0$ is the plus construction on $B\Sigma_\infty$. This is another version of the Barratt–Quillen theorem.

We describe a multiplicative analogue of this argument and result, due to Tornehave in the case of QS^0 and generalized in [27, VII.5.3], where full details may be found. Recall that we write $gl_1 R$ and $sl_1 R$ for the spectra $\mathbb{E}GL_1 R$ and $\mathbb{E}SL_1 R$ that the

black box associates to the E_∞ spaces $SL_1 R \subset GL_1 R$, where R is an E_∞ ring spectrum. The map $sl_1 R \longrightarrow gl_1 R$ is a connected cover. It is usually not an easy matter to identify $sl_1 R$ explicitly. The cited result gives a general step in this direction. The point to emphasize is that the result intrinsically concerns the relationship between the additive and multiplicative E_∞ space structures on an E_∞ ring space. Even if one's focus is solely on understanding the spectrum $sl_1 R$ associated to the E_∞ ring spectra R, one cannot see a result like this without introducing E_∞ ring spaces.

As in [27, Section VII.5], we start with an E_∞ ring space X and we assume that the canonical map of rigs from the rig $\mathbb{Z}_{\geq 0}$ of nonnegative integers to $\pi_0(X)$ is a monomorphism. When $X = \Omega^\infty R$, $\mathbb{E}X$ is equivalent as an E_∞ ring spectrum to the connective cover of R and $GL_1 \mathbb{E}(X)$ is equivalent as an E_∞ space to $GL_1 R$. The general case is especially interesting when X is the classifying space $B\mathscr{A}$ of a bipermutative category $\mathscr{A}$ (as defined in [27, Section VI.3]; see the sequel [32]).

Let M be a multiplicative submonoid of $\mathbb{Z}_{\geq 0}$ that does not contain zero. For example, M might be $\{p^i\}$ for a prime p, or it might be the set of positive integers prime to p. Let $\mathbb{Z}_M = \mathbb{Z}[M^{-1}]$ denote the localization of $\mathbb{Z}$ at M; thus $\mathbb{Z}_M = \mathbb{Z}[p^{-1}]$ in our first example and $\mathbb{Z}_M = \mathbb{Z}_{(p)}$ in the second. Let X_M denote the disjoint union of the components X_m with $m \in M$. Often, especially when $X = B\mathscr{A}$, we have a good understanding of X_M.

Clearly X_M is a sub-$\mathscr{L}$–space of X. Converting it to a $\mathscr{C}$–space and applying the additive infinite loop space machine or, equivalently, applying the additive infinite loop space machine constructed starting with $\mathscr{C} \times \mathscr{L}$, we obtain a connective spectrum $\mathbb{E}(X_M) = \mathbb{E}(X_M, \xi)$. The alternative notation highlights that the spectrum comes from the multiplicative operad action on X. This gives us the infinite loop space $\Gamma_1(X_M, \xi) = \Omega_1^\infty \mathbb{E}(X_M, \xi)$, which depends only on X_M.

We shall relate this to $SL_1 \mathbb{E}(X) = \Omega_1^\infty \mathbb{E}(X, \theta)$. The alternative notation highlights that $\mathbb{E}(X, \theta)$ is constructed from the additive operad action on X and has multiplicative structure inherited from the multiplicative operad action.

A key example to have in mind is $X = B\mathscr{GL}(R)$, where $\mathscr{GL}(R)$ is the general linear bipermutative category of a commutative ring R; its objects are the $n \geq 0$ and its morphisms are the general linear groups $GL(n, R)$. In that case $\mathbb{E}X = KR$ is the algebraic K–theory E_∞ ring spectrum of R. The construction still makes sense when R is a topological ring. We can take $R = \mathbb{R}$ or $R = \mathbb{C}$, and we can restrict to orthogonal or unitary matrices without changing the homotopy type. Then $K\mathbb{R} = kO$ and $K\mathbb{C} = kU$ are the real and complex connective topological K–theory spectra with "special linear" spaces $BO_\otimes$ and $BU_\otimes$.

To establish the desired relation between $SL_1\mathbb{E}(X)$ and $\Gamma_1(X_M,\xi)$, we need a mild homological hypothesis on X, namely that X is convergent at M. It always holds when X is ringlike, when $X = CY$ for an $\mathscr{L}_+$–space Y, and when $X = B\mathscr{A}$ for the usual bipermutative categories $\mathscr{A}$; see [27, VII.5.2]. We specify it during the sketch proof of the following result.

Theorem 10.2 *If X is convergent at M, then, as an E_∞ space, the localization of $\Omega_1^\infty\mathbb{E}(X,\theta)$ at M is equivalent to the basepoint component $\Omega_1^\infty\mathbb{E}(X_M,\xi)$.*

Thus, although $\mathbb{E}(X,\theta)$ is constructed using the $\mathscr{C}$–space structure on X, the localizations of $sl_1\mathbb{E}(X)$ depend only on X as an $\mathscr{L}$–space. When $X = \Omega^\infty R$, $\mathbb{E}(X,\theta)$ is equivalent to the connective cover of R and $sl_1\mathbb{E}X$ is equivalent to sl_1R.

Corollary 10.3 *For an E_∞ ring spectrum R, the localization $(sl_1R)_M$ is equivalent to the connected cover of $\mathbb{E}((\Omega^\infty R)_M,\xi)$.*

Sketch proof of Theorem 10.2 We repeat the key diagram from [27, page 196]:
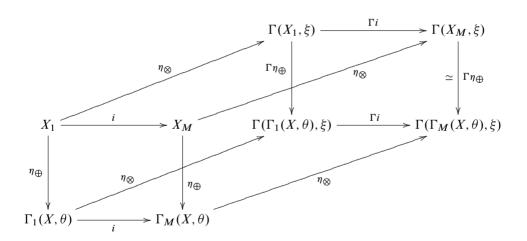

Figure 2: Diagram of $\mathscr{L}$–spaces and homotopy inverses of equivalences that are maps of $\mathscr{L}$–spaces

Remember that $\Gamma X = \Omega^\infty\mathbb{E}X$; $\Gamma_1 X$ and $\Gamma_M X$ denote the component of 1 and the disjoint union of the components $\Gamma_m X$ for $m \in M$. We have distinguished applications of the infinite loop space machine with respect to actions θ of $\mathscr{C}$ and ξ of $\mathscr{L}$, and we write $\eta_\oplus$ and $\eta_\otimes$ for the corresponding group completions. The letter i always denotes an inclusion. Figure 2 is a diagram of $\mathscr{L}$–spaces and homotopy inverses of

equivalences that are maps of $\mathcal{L}$–spaces. Note that the spaces on the left face are connected, so that their images on the right face lie in the respective components of 1. There are two steps.

(i) If X is ringlike, then the composite $\Gamma i \circ \eta_\otimes \colon X_1 \longrightarrow \Gamma_1(X_M, \xi)$ on the top face is a localization of X_1 at M.

(ii) If X is convergent, then the vertical map $\Gamma \eta_\oplus$ labelled $\simeq$ at the top right is a weak equivalence.

Applying (i) to the bottom face, as we may, we see that the (zigzag) composite from $\Gamma_1(X, \theta)$ at the bottom left to $\Gamma_1(X_M, \xi)$ at the top right is a localization at M.

To prove (i) and (ii), write the elements of M in increasing order, $\{1, m_1, m_2, \ldots\}$, let $n_i = m_1 \cdots m_i$, and consider the following homotopy commutative ladder.

$$
\begin{array}{ccccccccc}
X_1 & \xrightarrow{\ m_1\ } & X_{n_1} & \longrightarrow & \cdots & \longrightarrow & X_{n_{i-1}} & \xrightarrow{\ m_i\ } & X_{n_i} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \eta_\otimes} & & \downarrow{\scriptstyle \eta_\otimes} & & & & \downarrow{\scriptstyle \eta_\otimes} & & \downarrow{\scriptstyle \eta_\otimes} & & \\
\Gamma_1(X_M, \xi) & \xrightarrow{\ m_1\ } & \Gamma_{n_1}(X_M, \xi) & \longrightarrow & \cdots & \longrightarrow & \Gamma_{n_{i-1}}(X_M, \xi) & \xrightarrow{\ m_i\ } & \Gamma_{n_i}(X_M, \xi) & \longrightarrow & \cdots
\end{array}
$$

Here the translations by m_i mean multiplication (using the H–space structure induced by the action ξ) by an element of X_{m_i}.

Let $\bar{X}_M$ be the telescope of the top row. Take homology with coefficients in a commutative ring. The m_i in the bottom row are equivalences since $\Gamma(X_M, \xi)$ is grouplike, so the diagram gives a map $\bar{X} \longrightarrow \Gamma_1(X_M, \xi)$. The homological definition of a group completion, applied to $\eta_\otimes$, implies that this map is a homology isomorphism. Note that the leftmost arrow $\eta_\otimes$ factors through $\Gamma(X_1, \xi) = \Gamma_1(X_1, \xi)$. Exploiting formulas that relate the additive and multiplicative Pontryagin products on $H_*(X)$, we can check that $\tilde{H}_*(\bar{X}; \mathbb{F}_p) = 0$ if p divides an element $m \in M$. The point is that m is the sum of m copies of 1, and there is a distributivity formula for $x \cdot m$ in terms of the additive H–space structure $*$. This implies that the space $\Gamma_1(X_M, \xi)$ is M–local.

For (i), write $*(-n)\colon X_n \longrightarrow X_0$ for the equivalence given by using $*$ to send x to $x * y_{-n}$ for a point $y_{-n} \in X_{-n}$. Formulas in the definition of a $(\mathcal{C}, \mathcal{L})$–space imply that the following ladder is homotopy commutative.

$$
\begin{array}{ccccccccc}
X_1 & \xrightarrow{\ m_1\ } & X_{n_1} & \longrightarrow & \cdots & \longrightarrow & X_{n_{i-1}} & \xrightarrow{\ m_i\ } & X_{n_i} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle *(-1)} & & \downarrow{\scriptstyle *(-n_1)} & & & & \downarrow{\scriptstyle *(-n_{i-1})} & & \downarrow{\scriptstyle *(-n_i)} & & \\
X_0 & \xrightarrow{\ m_1\ } & X_0 & \longrightarrow & \cdots & \longrightarrow & X_0 & \xrightarrow{\ m_i\ } & X_0 & \longrightarrow & \cdots
\end{array}
$$

A standard construction of localizations of H–spaces gives that the telescope of the bottom row is a localization $X_0 \longrightarrow (X_0)[M^{-1}]$, hence so is the telescope $X_1 \longrightarrow \bar{X}_M$ of the top row, hence so is its composite with $\bar{X}_M \longrightarrow \Gamma_1(X_M, \xi)$. This proves (i).

For (ii), we consider the additive analogue of our first ladder:

$$
\begin{array}{ccccccccc}
X_1 & \xrightarrow{m_1} & X_{n_1} & \longrightarrow & \cdots & \longrightarrow & X_{n_{i-1}} & \xrightarrow{m_i} & X_{n_i} & \longrightarrow \cdots \\
\eta_\oplus \downarrow & & \eta_\oplus \downarrow & & & & \eta_\oplus \downarrow & & \eta_\oplus \downarrow & \\
\Gamma_1(X, \theta) & \xrightarrow{m_1} & \Gamma_{n_1}(X, \theta) & \longrightarrow & \cdots & \longrightarrow & \Gamma_{n_{i-1}}(X, \theta) & \xrightarrow{m_i} & \Gamma_{n_i}(X, \theta) & \longrightarrow \cdots
\end{array}
$$

We say that X is convergent at M if, for each prime p which does not divide any element of M, there is an eventually increasing sequence $n_i(p)$ such that

$$(\eta_\oplus)_* \colon H_j(X_i; \mathbb{F}_p) \longrightarrow H_j(\Gamma_i(X, \theta); \mathbb{F}_p)$$

is an isomorphism for all $j \leq n_i(p)$. With this condition, the induced map of telescopes is a mod p homology isomorphism for such primes p. This implies the same statement for the map

$$\Gamma_1 \eta_\oplus \colon \Gamma_1(X_M, \xi) \longrightarrow \Gamma_1(\Gamma_M(X, \theta), \xi).$$

Since this is a map between M–local spaces, it is an equivalence. This proves (ii) on components of 1, and it follows on other components.   □

For a general example, consider CY for an L_+–space Y.

Corollary 10.4 *There is a natural commutative diagram of E_∞ spaces*

$$
\begin{array}{ccccc}
\Gamma_1(CY, \theta) & \xrightarrow{\Gamma_1 \alpha} & \Gamma_1(QY, \theta) & \xleftarrow{\eta_\oplus} & Q_1 Y \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_1(C_M Y, \xi) & \xrightarrow{\Gamma_1 \alpha} & \Gamma_1(Q_M Y, \xi) & = & \Gamma_1(Q_M Y, \xi)
\end{array}
$$

in which the horizontal arrows are weak equivalences and the vertical arrows are localizations at M.

Now specialize to the case $Y = S^0$. Then $Q_1 S^0$, the unit component of the 0th space of the sphere spectrum, is the space $SL_1 S = SF$ of degree 1 stable homotopy equivalences of spheres. We see that its localization at M is the infinite loop space constructed from the $\mathcal{L}$–space $C_M S^0$. The latter space is the disjoint union of the Eilenberg–Mac Lane spaces $\mathscr{C}(m)/\Sigma_m = K(\Sigma_m, 1)$, given an E_∞ space structure that realizes the products $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{mn}$ determined by lexicographically ordering

the products of sets of m and n elements for $m, n \in M$. Thus the localizations of SF can be recovered from symmetric groups in a way that captures their infinite loop structures.

11 E_∞ ring spectra and commutative S–algebras

Jumping ahead over twenty years, we here review the basic definitions of Elmendorf et al [13], leaving all details to that source. However, to establish context, let us first recall the following result of Lewis [17].

Theorem 11.1 *Let $\mathscr{S}$ be any category that is enriched in based topological spaces and satisfies the following three properties.*

(i) *$\mathscr{S}$ is closed symmetric monoidal under continuous smash product and function spectra functors $\wedge$ and F that satisfy the topological adjunction*

$$\mathscr{S}(E \wedge E', E'') \cong \mathscr{S}(E, F(E', E'')).$$

(ii) *There are continuous functors Σ^∞ and Ω^∞ between spaces and spectra that satisfy the topological adjunction*

$$\mathscr{S}(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E).$$

(iii) *The unit for the smash product in $\mathscr{S}$ is $S \equiv \Sigma^\infty S^0$.*

Then, for any commutative monoid R in $\mathscr{S}$, such as S itself, the component $SL_1(R)$ of the identity element in $\Omega^\infty R$ is a product of Eilenberg–Mac Lane spaces.

Proof The enrichment of the adjunctions means that the displayed isomorphisms are homeomorphisms. By [17, 3.4], the hypotheses imply that $\mathscr{S}$ is tensored over $\mathscr{T}$. In turn, by [17, 3.2], this implies that the functor Ω^∞ is lax symmetric monoidal with respect to the unit $\eta\colon S^0 \longrightarrow \Omega^\infty \Sigma^\infty S^0 = \Omega^\infty S$ of the adjunction and a natural transformation

$$\phi\colon \Omega^\infty D \wedge \Omega^\infty E \longrightarrow \Omega^\infty(D \wedge E).$$

Now let $D = E = R$ with product μ and unit $\eta\colon S \longrightarrow R$. The adjoint of η is a map $S^0 \longrightarrow \Omega^\infty R$, and we let $1 \in \Omega^\infty R$ be the image of $1 \in S^0$. The composite
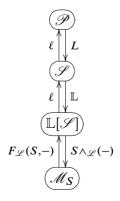
$$\Omega^\infty E \times \Omega^\infty R \longrightarrow \Omega^\infty R \wedge \Omega^\infty R \xrightarrow{\phi} \Omega^\infty(R \wedge R) \xrightarrow{\Omega^\infty \mu} \Omega^\infty R$$

gives $\Omega^\infty R$ a structure of commutative topological monoid with unit 1. Restricting to the component $SL_1 R$ of 1, we have a connected commutative topological monoid, and Moore's theorem (eg [25, 3.6]) gives the conclusion. $\qquad\square$

As Lewis goes on to say, if $\Omega^\infty \Sigma^\infty X$ is homeomorphic under X to QX, as we have seen holds for the category $\mathscr{S}$ of (LMS) spectra, and if (i)–(iii) hold, then we can conclude in particular that $SF = SL_1 S$ is a product of Eilenberg–Mac Lane spaces, which is false. We interpolate a model theoretic variant of this contradiction.

Remark 11.2 The sphere spectrum S is a commutative ring spectrum in any symmetric monoidal category of spectra $\mathscr{S}$ with unit object S. Suppose that S is cofibrant in some model structure on $\mathscr{S}$ whose homotopy category is equivalent to the stable homotopy category and whose fibrant objects are Ω–prespectra. More precisely, we require an underlying prespectrum functor $U \colon \mathscr{S} \longrightarrow \mathscr{P}$ such that UE is an Ω–prespectrum if E is fibrant, and we also require the resulting 0th space functor U_0 to be lax symmetric monoidal. Then we cannot construct a model category of commutative ring spectra by letting the weak equivalences and fibrations be the maps that are weak equivalences and fibrations in $\mathscr{S}$. If we could, a fibrant approximation of S as a commutative ring spectrum would be an Ω–spectrum whose 0th space is equivalent to QS^0. Its 1–component would be a connected commutative monoid equivalent to SF.

All good modern categories of spectra satisfy (i) and (iii) (or their simplicially enriched analogues) and therefore cannot satisfy (ii). However, as our summary so far should make clear, one must not let go of (ii) lightly. One needs something like it to avoid severing the relationship between spectrum and space level homotopy theory. Our summaries of modern definitions will focus on the relationship between spectra and spaces. Since we are now switching towards a focus on stable homotopy theory, we start to keep track of model structures. Returning to our fixed category $\mathscr{S}$ of (LMS) spectra, we shall describe a sequence of Quillen equivalences, in which the right adjoints labelled ℓ are both inclusions of subcategories.

The category $\mathscr{P}$ of prespectra has a level model structure whose weak equivalences and fibrations are defined levelwise, and it has a stable model structure whose weak

equivalences are the maps that induce isomorphisms of (stabilized) homotopy groups and whose cofibrations are the level cofibrations; its fibrant objects are the Ω–prespectra. The category $\mathscr{S}$ of spectra is a model category whose level model structure and stable model structures coincide. That is, the weak equivalences and fibrations are defined levelwise, and these are already the correct stable weak equivalences because the colimits that define the homotopy groups of a spectrum run over a system of isomorphisms. The spectrification functor L and inclusion ℓ give a Quillen equivalence between $\mathscr{P}$ and $\mathscr{S}$.

Of course, $\mathscr{S}$ satisfies (ii) but not (i) and (iii). We take the main step towards the latter properties by introducing the category $\mathbb{L}[\mathscr{S}]$ of $\mathbb{L}$–spectra.

The space $\mathscr{L}(1)$ is a monoid under composition, and we have the notion of an action of $\mathscr{L}(1)$ on a spectrum E. It is given by a map

$$\xi\colon \mathbb{L}E = \mathscr{L}(1) \ltimes E \longrightarrow E$$

that is unital and associative in the evident sense. Since $\mathscr{L}(1)$ is contractible, the unit condition implies that ξ must be a weak equivalence. Moreover, $\mathbb{L}E$ is an $\mathbb{L}$–spectrum for any spectrum E, and the action map $\xi\colon \mathbb{L}E \longrightarrow E$ is a map of $\mathbb{L}$–spectra for any $\mathbb{L}$–spectrum E. The inclusion $\ell\colon \mathbb{L}[\mathscr{S}] \longrightarrow \mathscr{S}$ forgets the action maps. It is right adjoint to the free $\mathbb{L}$–spectrum functor $\mathbb{L}\colon \mathscr{S} \longrightarrow \mathbb{L}[\mathscr{S}]$. Define the weak equivalences and fibrations of $\mathbb{L}$–spectra to be the maps f such that ℓf is a weak equivalence or fibration. Then $\mathbb{L}[\mathscr{S}]$ is a model category and $(\mathbb{L}, \ell)$ is a Quillen equivalence between $\mathscr{S}$ and $\mathbb{L}[\mathscr{S}]$. Indeed, the unit $\eta\colon E \longrightarrow \ell\mathbb{L}E$ and counit $\xi\colon \mathbb{L}\ell E \longrightarrow E$ of the adjunction are weak equivalences, and every object in both categories is fibrant, a very convenient property.

Using the untwisting isomorphism (6-3) and the projection $\mathscr{L}(1)_+ \longrightarrow S^0$, we see that the spectra $\Sigma^\infty X$ are naturally $\mathbb{L}$–spectra. However $S = \Sigma^\infty S^0$, which is cofibrant in $\mathscr{S}$, is not cofibrant in $\mathbb{L}[\mathscr{S}]$; rather, $\mathbb{L}S$ is a cofibrant approximation.

We have a commutative and associative[24] smash product $E \wedge_{\mathscr{L}} E'$ in $\mathbb{L}[\mathscr{S}]$; we write τ for the commutativity isomorphism $E \wedge_{\mathscr{L}} E' \longrightarrow E' \wedge_{\mathscr{L}} E$. The smash product is defined as a coequalizer $\mathscr{L}(2) \ltimes_{\mathscr{L}(1) \times \mathscr{L}(1)} E \wedge E'$, but we refer the reader to [13] for details. There is a natural unit map $\lambda\colon S \wedge_{\mathscr{L}} E \longrightarrow E$. It is a weak equivalence for all $\mathbb{L}$–spectra E, but it is not in general an isomorphism. That is, S is only a weak unit.

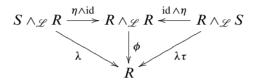Moreover, there is a natural isomorphism of $\mathbb{L}$–spectra

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge_{\mathscr{L}} \Sigma^\infty Y.$$

---

[24]This crucial property is a consequence of a remarkable motivating observation, due to Mike Hopkins, about special properties of the structure maps of the linear isometries operad.

Note, however, that the $(\Sigma^\infty, \Omega^\infty)$ adjunction must now change. We may reasonably continue to write Ω^∞ for the 0th space functor $\Omega^\infty \circ \ell$, but its left adjoint is now the composite $\mathbb{L} \circ \Sigma^\infty$.

While Lewis' contradictory desiderata do not hold, we are not too far off since we still have a sensible 0th space functor. We are also very close to a description of E_∞ ring spectra as commutative monoids in a symmetric monoidal category.

Definition 11.3 A commutative monoid in $\mathbb{L}[\mathscr{S}]$ is an $\mathbb{L}$–spectrum R with a unit map $\eta\colon S \longrightarrow R$ and a commutative and associative product $\phi\colon R \wedge_{\mathscr{L}} R \longrightarrow R$ such that the following unit diagram is commutative

$$
\begin{array}{ccccc}
S \wedge_{\mathscr{L}} R & \xrightarrow{\eta \wedge \mathrm{id}} & R \wedge_{\mathscr{L}} R & \xleftarrow{\mathrm{id} \wedge \eta} & R \wedge_{\mathscr{L}} S \\
 & \searrow^{\lambda} & \downarrow^{\phi} & \swarrow^{\lambda\tau} & \\
 & & R & &
\end{array}
$$

The only difference from an honest commutative monoid is that the diagonal unit arrows are weak equivalences rather than isomorphisms. Thinking of the unit maps $e\colon S \longrightarrow R$ of E_∞ ring spectra as preassigned, we can specify a product $*$ in the category $\mathbb{L}[\mathscr{S}]_e$ of $\mathbb{L}$–spectra under S that gives that category a symmetric monoidal structure, and then an E_∞ ring spectrum is an honest commutative monoid in that category [13, XIII.1.16]. We prefer to keep to the perspective of Definition 11.3, and [13, II.4.6] gives the following result.

Theorem 11.4 *The category of commutative monoids in $\mathbb{L}[\mathscr{S}]$ is isomorphic to the category of E_∞ ring spectra.*

Obviously, we have not lost the connection with E_∞ ring spaces. Since we are used to working in symmetric monoidal categories and want to work in a category of spectra rather than of spectra under S, we take one further step. The unit map $\lambda\colon S \wedge_{\mathscr{L}} E \longrightarrow E$ is often an isomorphism. This holds when $E = \Sigma^\infty X$ and when $E = S \wedge_{\mathscr{L}} E'$ for another $\mathbb{L}$–spectrum E'. We define an S–module to be an $\mathbb{L}$–spectrum E for which λ is an isomorphism, and we let $\mathscr{M}_S$ be the category of S–modules. It is symmetric monoidal with unit S under the smash product $E \wedge_S E' = E \wedge_{\mathscr{L}} E'$ that is inherited from $\mathbb{L}[\mathscr{S}]$, and it also inherits a natural isomorphism of S–modules

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge_S \Sigma^\infty Y.$$

Commutative monoids in $\mathscr{M}_S$ are called commutative S–algebras. They are those commutative monoids in $\mathbb{L}[\mathscr{S}]$ whose unit maps are isomorphisms. Thus they are especially nice E_∞ ring spectra. For any E_∞ ring spectrum R, $S \wedge_{\mathscr{L}} R$ is a commutative

S–algebra and the unit equivalence $S \wedge_{\mathscr{L}} R \longrightarrow R$ is a map of E_∞ ring spectra. Thus there is no real loss of generality in restricting attention to the commutative S–algebras. Their 0th spaces are still E_∞ ring spaces.

However the 0th space functor $\Omega^\infty \colon \mathscr{M}_S \longrightarrow \mathscr{T}$ is not a right adjoint. The functor $S \wedge_{\mathscr{L}} (-) \colon \mathbb{L}[\mathscr{S}] \longrightarrow \mathscr{M}_S$ is *right* adjoint to the inclusion $\ell \colon \mathscr{M}_S \longrightarrow \mathbb{L}[\mathscr{S}]$, and it has a right adjoint $F_{\mathscr{L}}(S, -)$. Thus, for $D \in \mathbb{L}[\mathscr{S}]$ and $E \in \mathscr{M}_S$, we have

$$\mathscr{M}_S(E, S \wedge_{\mathscr{L}} D) \cong \mathbb{L}[\mathscr{S}](\ell E, D)$$

and

$$\mathscr{M}_S(S \wedge_{\mathscr{L}} D, E) \cong \mathbb{L}[\mathscr{S}](D, F_{\mathscr{L}}(S, E)).$$

Letting the weak equivalences and fibrations in $\mathscr{M}_S$ be created by $F_{\mathscr{L}}(S, -)$, the second adjunction gives a Quillen equivalence between $\mathbb{L}[\mathscr{S}]$ and $\mathscr{M}_S$. Since there is a natural weak equivalence $\tilde{\lambda} \colon \ell E \longrightarrow F_{\mathscr{L}}(S, E)$, the weak equivalences, but not the fibrations, are also created by ℓ. On the 0th space level, $\tilde{\lambda}$ induces a natural weak equivalence

$$\Omega^\infty \ell E \simeq \Omega^\infty F_{\mathscr{L}}(S, E).$$

We conclude that we have lost no 0th space information beyond that which would lead to a contradiction to Theorem 11.1 in our passage from $\mathscr{S}$ to $\mathscr{M}_S$.

As explained in [13, Section II.2], there is actually a *mirror image* category $\mathscr{M}^S$ that is equivalent to $\mathscr{M}_S$ and whose 0th space functor is equivalent, rather than just weakly equivalent, to the right adjoint of a functor $\mathscr{S} \longrightarrow \mathscr{M}^S$. It is the subcategory of objects in $\mathbb{L}[\mathscr{S}]$ whose counit maps $\tilde{\lambda} \colon E \longrightarrow F_{\mathscr{L}}(S, E)$ are isomorphisms. It has adjunctions that are mirror image to those of $\mathscr{M}_S$, switching left and right. Writing $r \colon \mathscr{M}^S \longrightarrow \mathbb{L}[\mathscr{S}]$ for the inclusion and taking $D \in \mathbb{L}[\mathscr{S}]$ and $E \in \mathscr{M}^S$, we have

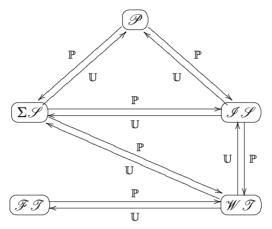$$\mathscr{M}^S(F_{\mathscr{L}}(S, D), E) \cong \mathbb{L}[\mathscr{S}](D, rE)$$

and

$$\mathscr{M}^S(E, F_{\mathscr{L}}(S, D)) \cong \mathbb{L}[\mathscr{S}](\ell(S \wedge_{\mathscr{L}} rE), D).$$

The following theorem from [13, Section VII.4] is more central to our story and should be compared with Remark 11.2.

Theorem 11.5 *The category of E_∞ ring spectra is a Quillen model category with fibrations and weak equivalences created by the forgetful functor to $\mathbb{L}[\mathscr{S}]$. The category of commutative S–algebras is a Quillen model category with fibrations and weak equivalences created by the forgetful functor to $\mathscr{M}_S$.*

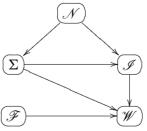# 12 The comparison with commutative diagram ring spectra

For purposes of comparison and to give some completeness to this survey, we copy the following schematic diagram of Quillen equivalences[25] from [23].

We have a lexicon:

(i) $\mathscr{P}$ is the category of $\mathscr{N}$–spectra, or (coordinatized) prespectra.

(ii) $\Sigma\mathscr{S}$ is the category of Σ–spectra, or symmetric spectra.

(iii) $\mathscr{I}\mathscr{S}$ is the category of $\mathscr{I}$–spectra, or orthogonal spectra.

(iv) $\mathscr{F}\mathscr{T}$ is the category of $\mathscr{F}$–spaces, or Γ–spaces.

(v) $\mathscr{W}\mathscr{T}$ is the category of $\mathscr{W}$–spaces.

These categories all start with some small (topological) category $\mathscr{D}$ and the category $\mathscr{D}\mathscr{T}$ of (continuous) covariant functors $\mathscr{D} \longrightarrow \mathscr{T}$, which are called $\mathscr{D}$–spaces. The domain categories have inclusions among them, as indicated in the following diagram of domain categories $\mathscr{D}$.

[25]There is a caveat in that $\mathscr{F}\mathscr{T}$ only models connective spectra.

To go from $\mathscr{D}$–spaces to $\mathscr{D}$–spectra, we start with a sphere space functor $S\colon \mathscr{D} \to \mathscr{T}$ with smash products. It makes sense to define a module over S, and the S–modules are the $\mathscr{D}$–spectra. Alternatively but equivalently, one can use S to build a new (topological) domain category $\mathscr{D}_S$ such that a $\mathscr{D}_S$–space is a $\mathscr{D}$–spectrum. Either way, we obtain the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$–spectra. When $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$, there is no distinction between $\mathscr{D}$–spaces and $\mathscr{D}$–spectra and $\mathscr{D}\mathscr{T} = \mathscr{D}\mathscr{S}$.

In the previous diagram, $\mathscr{N}$ is the category of nonnegative integers, Σ is the category of symmetric groups, $\mathscr{I}$ is the category of linear isometric isomorphisms as before, $\mathscr{F}$ is the category of finite based sets, which is the opposite category of Segal's category Γ, and $\mathscr{W}$ is the category of based spaces that are homeomorphic to finite CW complexes. The functors $\mathbb{U}$ in the first diagram are forgetful functors associated to these inclusions of domain categories, and the functors $\mathbb{P}$ are prolongation functors left adjoint to the $\mathbb{U}$. All of these categories except $\mathscr{P}$ are symmetric monoidal. The reason is that the functor $S\colon \mathscr{D} \longrightarrow \mathscr{T}$ is symmetric monoidal in the other cases, but not in the case of $\mathscr{N}$. The functors $\mathbb{U}$ between symmetric monoidal categories are lax symmetric monoidal, the functors $\mathbb{P}$ between symmetric monoidal categories are strong symmetric monoidal, and the functors $\mathbb{P}$ and $\mathbb{U}$ restrict to adjoint pairs relating the various categories of rings, commutative rings, and modules over rings.

We are working with spaces but, except that orthogonal spectra should be omitted, we have an analogous diagram of categories of spectra that are based on simplicial sets (see Bousfield and Friedlander [9], Hovey, Shipley and Smith [15], Lydakis [20; 19] and Schwede [37]). That diagram compares to ours via the usual adjunction between simplicial sets and topological spaces. Each of these categories of spectra has intrinsic interest, and they have various advantages and disadvantages. We focus implicitly on symmetric and orthogonal spectra in what follows; up to a point, $\mathscr{W}$–spaces and $\mathscr{F}$–spaces work similarly. Full details are in Mandell et al [23] and the references just cited.

We emphasize that no nontrivial symmetric (or orthogonal) spectrum E can also be an LMS spectrum. If it were, its 0th space E_0, with trivial Σ_2–action, would be homeomorphic as a Σ_2–space to the nontrivial Σ_2–space $\Omega^2 E_2$.

We recall briefly how smash products are defined in diagram categories. There are two equivalent ways. Fix a symmetric monoidal domain category $\mathscr{D}$ with product denoted $\oplus$. For $\mathscr{D}$–spaces T and T', there is an external smash product $T \barwedge T'$, which is a $\mathscr{D} \times \mathscr{D}$–space. It is specified by

$$(T \barwedge T')(d, e) = Td \wedge T'e.$$

Applying left Kan extension along $\oplus$, one obtains a $\mathscr{D}$–space $T \wedge T'$. This construction is characterized by an adjunction

$$((\mathscr{D} \times \mathscr{D})\mathscr{T})(T \bar{\wedge} T', V \circ \oplus) \cong \mathscr{D}\mathscr{T}(T \wedge T', V)$$

for $\mathscr{D}$–spaces V. When T and T' are S–modules, one can construct a "tensor product" $T \wedge_S T'$ by mimicking the coequalizer description of the tensor product of modules over a commutative ring. That gives the required internal smash product of S–modules. Alternatively and equivalently, one can observe that $\mathscr{D}_S$ is a symmetric monoidal category when $S\colon \mathscr{D} \longrightarrow \mathscr{T}$ is a symmetric monoidal functor, and one can then apply left Kan extension directly, with $\mathscr{D}$ replaced by $\mathscr{D}_S$. Either way, $\mathscr{D}\mathscr{S}$ becomes a symmetric monoidal category with unit S.

In view of the use of left Kan extension, monoids R in $\mathscr{D}\mathscr{S}$ have an external equivalent defined in terms of maps $R(d) \wedge R(e) \longrightarrow R(d \oplus e)$. These are called $\mathscr{D}$–FSP's. As we have already recalled, Thom spectra give naturally occurring examples of $\mathscr{I}$–FSP's.

We also recall briefly how the model structures are defined. We begin with the evident level model structure. Its weak equivalences and fibrations are defined levelwise. We then define stable weak equivalences and use them and the cofibrations of the level model structure to construct the stable model structure. The resulting fibrant objects are the Ω–spectra. In all of these categories except that of symmetric spectra, the stable weak equivalences are the maps whose underlying maps of prespectra induce isomorphisms of stabilized homotopy groups.[26] It turns out that a map f of symmetric spectra is a stable weak equivalence if and only if $\mathbb{P}f$ is a stable weak equivalence of orthogonal spectra in the sense just defined. There are other model structures here, as we shall see. The thing to notice is that, in the model structures just specified, the sphere spectra S are cofibrant. Compare Remark 11.2 and Theorem 11.5.

These model structures are compared in [23]. Later work of Schwede and Shipley [38; 39; 41] gives $\Sigma\mathscr{S}$ a privileged role. Given any other sufficiently good stable model category whose homotopy category is correct, in the sense that it is equivalent to Boardman's original stable homotopy category, there is a left Quillen equivalence from $\Sigma\mathscr{S}$ to that category.

However, this is not always the best way to compare two models for the stable homotopy category. If one has models $\mathscr{S}_1$ and $\mathscr{S}_2$ and compares both to $\Sigma\mathscr{S}$, then $\mathscr{S}_1$ and $\mathscr{S}_2$ are compared by a zigzag of Quillen equivalences. It is preferable to avoid composing left and right Quillen adjoints, since such composites do not preserve structure. For example, using a necessary modification of the model structure on $\Sigma\mathscr{S}$ to be explained

---

[26]The *underlying prespectrum* of an $\mathscr{F}$–space is obtained by first prolonging it to a $\mathscr{W}$–space and then taking the underlying prespectrum of that, and we are suppressing some details.

shortly, Schwede gives a left Quillen equivalence $\Sigma\mathscr{S} \longrightarrow \mathscr{M}_S$ [38], and [23] shows that the prolongation functor $\mathbb{P}\colon \Sigma\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$ is a left Quillen equivalence. This gives a zigzag of Quillen equivalences between $\mathscr{M}_S$ and $\mathscr{I}\mathscr{S}$. These categories are both defined using $\mathscr{I}$, albeit in quite different ways, and it is more natural and useful to construct a left Quillen equivalence $\mathbb{N}\colon \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}_S$. Using a similar necessary modification of the model structure on $\mathscr{I}\mathscr{S}$, this is done in Mandell and May [22, Chapter I]; Schwede's left Quillen equivalence is the composite $\mathbb{N} \circ \mathbb{P}$.

In any case, there is a web of explicit Quillen equivalences relating all good known models for the stable homotopy category, and these equivalences even preserve the symmetric monoidal structure and so preserve rings, modules, and algebras [22; 23; 38; 41]. Thus, as long as one focuses on stable homotopy theory, any convenient model can be used, and information can easily be transferred from one to another. More precisely, if one focuses on criteria (i) and (iii) of Theorem 11.1, one encounters no problems. However, our focus is on (ii), the relationship between spectra and spaces, and here there are significant problems.

For a start, it is clear that we cannot have a symmetric monoidal Quillen left adjoint from $\Sigma\mathscr{S}$ or $\mathscr{I}\mathscr{S}$ to $\mathscr{M}_S$ with the model structures that we have specified since the sphere spectra in $\Sigma\mathscr{S}$ and $\mathscr{I}\mathscr{S}$ are cofibrant and the sphere spectrum in $\mathscr{M}_S$ is not. For the comparison, one must use different model structures on $\Sigma\mathscr{S}$ and $\mathscr{I}\mathscr{S}$, namely the positive stable model structures. These are obtained just as above but starting with the level model structures whose weak equivalences and fibrations are defined using only the positive levels, not the 0th space level. This does not change the stable weak equivalences, and the resulting positive stable model structures are Quillen equivalent to the original stable model structures.

However, the fibrant spectra are now the positive Ω–spectra, for which the structure maps $\tilde{\sigma}\colon T_n \longrightarrow \Omega T_{n+1}$ of the underlying prespectrum are weak equivalences only for $n > 0$. This in principle throws away all information about the 0th space, even after fibrant approximation. The analogue of Theorem 11.5 reads as follows. Actually, a significant technical improvement of the positive stable model structure has been obtained by Shipley [42], but her improvement does not effect the discussion here: one still must use the positive variant.

Theorem 12.1 *The categories of commutative symmetric ring spectra and commutative orthogonal ring spectra have Quillen model structures whose weak equivalences and fibrations are created by the forgetful functors to the categories of symmetric spectra and orthogonal spectra with their positive stable model structures.*

Parenthetically, as far as I know it is unclear whether or not there is an analogue of this result for $\mathscr{W}$–spaces. The results of [22; 23; 38; 41] already referred to give

the following comparisons, provided that we use the positive model structures on the diagram spectrum level.

Theorem 12.2 *There are Quillen equivalences from the category of commutative symmetric ring spectra to the category of commutative orthogonal ring spectra and from the latter to the category of commutative S–algebras.*

Thus we have comparison functors

<div align="center">

Commutative symmetric ring spectra

$\downarrow \mathbb{P}$

Commutative orthogonal ring spectra

$\downarrow \mathbb{N}$

Commutative S–algebras

$\uparrow S \wedge_{\mathscr{L}} (-)$

E_∞ ring spectra

$\downarrow \Omega^\infty$

E_∞ ring spaces

</div>

The functors $\mathbb{P}$, $\mathbb{N}$, and $S \wedge_{\mathscr{L}} (-)$ are *left* Quillen equivalences. The functor Ω^∞ is a *right* adjoint. The composite is not homotopically meaningful since, after fibrant approximation, commutative symmetric ring spectra do not have meaningful 0th spaces; in fact, their 0th spaces are then just S^0. If one only uses diagram spectra, the original E_∞ ring theory relating spaces and spectra is lost.

13 Naive E_∞ ring spectra

Again recall that we can define operads and operad actions in any symmetric monoidal category. If a symmetric monoidal category $\mathscr{W}$ is tensored over a symmetric monoidal category $\mathscr{V}$, then we can just as well define actions of operads in $\mathscr{V}$ on objects of $\mathscr{W}$. All good modern categories of spectra are tensored over based spaces (or simplicial sets). We can therefore define an action of an operad $\mathscr{O}_+$ in $\mathscr{T}$ on a spectrum in any such category. Continuing to write $\wedge$ for the tensor of a space and a spectrum, such an action on a spectrum R is given by maps of spectra

$$\mathscr{O}(j)_+ \wedge R^{(j)} \longrightarrow R.$$

Taking $\mathcal{O}$ to be an E_∞ operad, we call such $\mathcal{O}$–spectra naive E_∞ ring spectra.[27] They are defined in terms of the already constructed internal smash product and thus have nothing to do with the internalization of an external smash product that is intrinsic to the original definition of E_∞ ring spectra.

They are of interest because some natural constructions land in naive E_∞ spectra (of one kind or another). In some cases, such as $\mathcal{W}$–spaces, where we do not know of a model structure on commutative ring spectra, naive E_∞ ring spectra provide an adequate stopgap. In other cases, including symmetric spectra, orthogonal spectra, and S–modules, we can convert naive E_∞ ring spectra to equivalent commutative ring spectra, as we noted without proof in [23, 0.14]. The reason is the following result, which deserves considerable emphasis. See [13, III.5.5], [23, 15.5], and, more recently and efficiently, [42, 3.3] for the proof.

Proposition 13.1 *For a positive cofibrant symmetric or orthogonal spectrum or for a cofibrant S–module E, the projection*

$$\pi\colon (E\Sigma_j)_+ \wedge_{\Sigma_j} E^{(j)} \longrightarrow E^{(j)}/\Sigma_j$$

(induced by $E\Sigma_j \longrightarrow *$) is a weak equivalence.

This is analogous to something that is only true in characteristic zero in the setting of differential graded modules over a field. In that context, it implies that E_∞ DGA's can be approximated functorially by quasiisomorphic commutative DGA's (see Kříž and May [16, II.1.5]). The following result is precisely analogous to the cited result and can be proven in much the same way. That way, it is another exercise in the use of the two-sided monadic bar construction. The cofibrancy issues can be handled with the methods of [41], and I have no doubt that in all cases the following result can be upgraded to a Quillen equivalence. The more general result in Elmendorf and Mandell [14, 1.4] shows this to be true for simplicial symmetric spectra and suitable simplicial operads, so I will be purposefully vague and leave details to the interested reader. Let $\mathcal{O}$ be an E_∞ operad and work in one of the categories of spectra cited in Proposition 13.1.

Proposition 13.2 *There is a functor that assigns a weakly equivalent commutative ring spectrum to a (suitably cofibrant) naive $\mathcal{O}$–spectrum. The homotopy categories of naive $\mathcal{O}$–spectra and commutative ring spectra are equivalent.*

---

[27]In recent e-mails, Tyler Lawson has jokingly called these MIT E_∞ ring spectra, to contrast them with the original Chicago variety. He alerted me to the fact that some people working in the area may be unaware of or indifferent to the distinction.

It is immediately clear from this result and the discussion in the previous section that naive E_∞ ring spectra in $\Sigma\mathscr{S}$ and $\mathscr{I}\mathscr{S}$ have nothing to do with E_∞ ring spaces, whereas the 0th spaces of naive E_∞ ring spectra in $\mathscr{M}_S$ are weakly equivalent to E_∞ ring spaces.
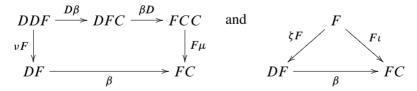
14 Appendix A. Monadicity of functors and comparisons of monads

Change of monad results are well-known to category theorists, but perhaps not as readily accessible in the categorical literature as they might be, so we give some elementary details here.[28] We first make precise two notions of a map relating monads (C, μ, ι) and (D, ν, ζ) in different categories $\mathscr{V}$ and $\mathscr{W}$. We have used both, relying on context to determine which one is intended.

Definition 14.1 Let (C, μ, ι) and (D, ν, ζ) be monads on categories $\mathscr{V}$ and $\mathscr{W}$. An op-lax map (F, α) from C to D is a functor $F \colon \mathscr{V} \longrightarrow \mathscr{W}$ and a natural transformation $\alpha \colon FC \longrightarrow DF$ such that the following diagrams commute.

$$
\begin{array}{ccccc}
FCC & \xrightarrow{\alpha C} & DFC & \xrightarrow{D\alpha} & DDF \\
{\scriptstyle F\mu}\downarrow & & & & \downarrow{\scriptstyle \nu F} \\
FC & & \xrightarrow{\alpha} & & DF
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
 & F & \\
{\scriptstyle F\iota}\swarrow & & \searrow{\scriptstyle \zeta F} \\
FC & \xrightarrow{\alpha} & DF
\end{array}
$$

A lax map (F, β) from C to D is a functor $F \colon \mathscr{V} \longrightarrow \mathscr{W}$ and a natural transformation $\beta \colon DF \longrightarrow FC$ such that the following diagrams commute.

$$
\begin{array}{ccccc}
DDF & \xrightarrow{D\beta} & DFC & \xrightarrow{\beta D} & FCC \\
{\scriptstyle \nu F}\downarrow & & & & \downarrow{\scriptstyle F\mu} \\
DF & & \xrightarrow{\beta} & & FC
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
 & F & \\
{\scriptstyle \zeta F}\swarrow & & \searrow{\scriptstyle F\iota} \\
DF & \xrightarrow{\beta} & FC
\end{array}
$$

If $\alpha \colon FC \longrightarrow DF$ is a natural isomorphism, then (F, α) is an op-lax map $C \longrightarrow D$ if and only if (F, α^{-1}) is a lax map $D \longrightarrow C$. When this holds, we say that α and α^{-1} are monadic natural isomorphisms.

These notions are most familiar when $\mathscr{V} = \mathscr{W}$ and $F = \mathrm{Id}$. In this case, a lax map $D \longrightarrow C$ coincides with an op-lax map $C \longrightarrow D$, and this is the usual notion of a map of monads from C to D. The map $\alpha \colon C \longrightarrow Q$ used in the approximation theorem

---

[28]This appendix is written jointly with Michael Shulman.

and the recognition principle is an example. As we have used extensively, maps (Id, α) lead to pullback of action functors.

Lemma 14.2 *If* (Id, α) *is a map of monads* $C \longrightarrow D$ *on a category* $\mathscr{V}$, *then a left or right action of* D *on a functor induces a left or right action of* C *by pullback of the action along* α. *In particular, if* (Y, χ) *is a* D–*algebra in* $\mathscr{V}$, *then* $(Y, \chi \circ \alpha)$ *is a* C–*algebra in* $\mathscr{V}$.

We have also used pushforward actions when $\mathscr{V}$ and $\mathscr{W}$ vary, and for that we need lax maps.

Lemma 14.3 *If* (F, β) *is a lax map from a monad* C *on* $\mathscr{V}$ *to a monad* D *on* $\mathscr{W}$, *then a left or right action of* C *on a functor induces a left or right action of* D *by pushforward of the action along* (F, β). *In particular, if* (X, ξ) *is a* C–*algebra in* $\mathscr{V}$, *then* $(FX, F\xi \circ \beta)$ *is a* D–*algebra in* $\mathscr{W}$.

Now let F have a right adjoint U. Let $\eta: \text{Id} \longrightarrow UF$ and $\varepsilon: FU \longrightarrow \text{Id}$ be the unit and counit of the adjunction. We have encountered several examples of monadic natural isomorphisms (F, β) relating a monad C in $\mathscr{V}$ to a monad D in $\mathscr{W}$, where F has a left adjoint U. Thus β is a natural isomorphism $DF \longrightarrow FC$. In this situation, we have a natural map $\delta: CU \longrightarrow UD$, namely the composite

$$CU \xrightarrow{\eta CU} UFCU \xrightarrow{U\beta^{-1}U} UDFU \xrightarrow{UD\varepsilon} UD.$$

It is usually not an isomorphism, and in particular is not an isomorphism in our examples. Implicitly or explicitly, we have several times used the following result.

Proposition 14.4 *The pair* (U, δ) *is a lax map from the monad* D *in* $\mathscr{W}$ *to the monad* C *in* $\mathscr{V}$. *Via pushforward along* (F, β) *and* (U, δ), *the adjoint pair* (F, U) *induces an adjoint pair of functors between the categories* $C[\mathscr{V}]$ *and* $D[\mathscr{W}]$ *of* C–*algebras in* $\mathscr{V}$ *and* D–*algebras in* $\mathscr{W}$:
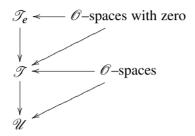
$$D[\mathscr{W}](FX, Y) \cong C[\mathscr{V}](X, UY).$$

Sketch proof The arguments are straightforward diagram chases. The essential point is that, for a C–algebra (X, ξ) and a D–algebra (Y, χ), the map $\eta: X \longrightarrow UFX$ is a map of C–algebras and the map $\varepsilon: FUY \longrightarrow Y$ is a map of D–algebras. $\qquad\square$
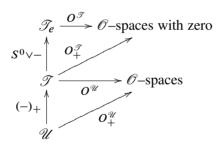
These observations are closely related to the categorical study of monadicity. A functor $U: \mathscr{W} \longrightarrow \mathscr{V}$ is said to be *monadic* if it has a left adjoint F such that U induces an

equivalence from $\mathscr{V}$ to the category of algebras over the monad UF. This is a property of the functor U. If U is monadic, then its left adjoint F and the induced monad UF such that $\mathscr{V}$ is equivalent to the category of UF–algebras are uniquely determined by $\mathscr{V}$, $\mathscr{W}$, and the functor U.

This discussion illuminates the comparisons of monads in Section 4 and Section 6. For the first it is helpful to consider the following diagram of forgetful functors.

$$
\begin{array}{ccc}
\mathscr{T}_e & \longleftarrow & \mathscr{O}\text{–spaces with zero} \\
\downarrow & \swarrow & \\
\mathscr{T} & \longleftarrow & \mathscr{O}\text{–spaces} \\
\downarrow & \swarrow & \\
\mathscr{U} & &
\end{array}
$$

Note that the operadic unit point is 1 in $\mathscr{O}$–spaces with zero but 0 in $\mathscr{O}$–spaces; the diagonal arrows are obtained by forgetting the respective operadic unit points. These forgetful functors are all monadic. If we abuse notation by using the same name for each left adjoint and for the monad induced by the corresponding adjunction, then we have the following diagram of left adjoints.

$$
\begin{array}{ccc}
\mathscr{T}_e & \xrightarrow{\;O^{\mathscr{T}}\;} & \mathscr{O}\text{–spaces with zero} \\
S^0\vee-\uparrow \quad O^{\mathscr{T}}_+ & \nearrow & \\
\mathscr{T} & \xrightarrow{O^{\mathscr{U}}} & \mathscr{O}\text{–spaces} \\
(-)_+\uparrow \quad O^{\mathscr{U}}_+ & \nearrow & \\
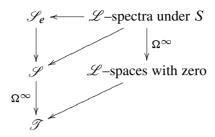\mathscr{U} & &
\end{array}
$$

Since the original diagram of forgetful functors obviously commutes, so does the corresponding diagram of left adjoints. This formally implies the relations
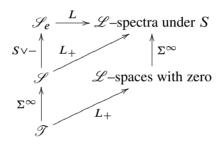
$$
O^{\mathscr{U}}(X_+) \cong O^{\mathscr{U}}_+(X) \quad \text{and} \quad O^{\mathscr{T}}(S^0 \vee X) \cong O^{\mathscr{T}}_+(X)
$$

of (4-3) and (4-7). The explicit descriptions of the four monads $O^{\mathscr{U}}_+$, $O^{\mathscr{U}}$, $O^{\mathscr{T}}_+$, and $O^{\mathscr{T}}$ are, of course, necessary to the applications, but it is helpful conceptually to remember that their definitions are forced on us by knowledge of the corresponding forgetful functors. As an incidental point, it is also important to remember that, unlike the case of adjunctions, the composite of two monadic functors need not be monadic, although it is in many examples, such as those above.

Similarly, for Section 6, it is helpful to consider two commutative diagrams of forgetful functors. In both, all functors other than the Ω^∞ are monadic. The first is

$$
\begin{array}{ccc}
\mathscr{S}_e & \longleftarrow & \mathscr{L}\text{--spectra under } S \\
\downarrow & \swarrow & \downarrow \Omega^\infty \\
& \mathscr{S} & \mathscr{L}\text{--spaces with zero} \\
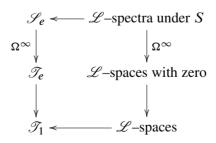\Omega^\infty \downarrow & \swarrow & \\
& \mathscr{T} &
\end{array}
$$

The lower diagonal arrow forgets the action of $\mathscr{L}$ and remembers the basepoint 0. The corresponding diagram of left adjoints is

$$
\begin{array}{ccc}
\mathscr{S}_e & \xrightarrow{\ L\ } & \mathscr{L}\text{--spectra under } S \\
S\vee-\ \uparrow \quad {}^{L_+}\nearrow & & \uparrow \Sigma^\infty \\
& \mathscr{S} & \mathscr{L}\text{--spaces with zero} \\
\Sigma^\infty \uparrow \quad {}^{\ }\nearrow & {}_{L_+} & \\
& \mathscr{T} &
\end{array}
$$

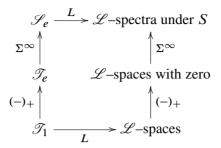Its commutativity implies the relations

$$
L(S \vee E) \cong L_+(E) \quad \text{and} \quad L_+\Sigma^\infty X \cong \Sigma^\infty L_+ X
$$

of (6-1) and (6-4). The second diagram of forgetful functors is

$$
\begin{array}{ccc}
\mathscr{S}_e & \longleftarrow & \mathscr{L}\text{--spectra under } S \\
\Omega^\infty \downarrow & & \downarrow \Omega^\infty \\
\mathscr{T}_e & & \mathscr{L}\text{--spaces with zero} \\
\downarrow & & \downarrow \\
\mathscr{T}_1 & \longleftarrow & \mathscr{L}\text{--spaces}
\end{array}
$$

Here $\mathscr{T}_1$ denotes the category of based spaces with basepoint 1. The lower two vertical arrows forget the basepoint 0 and remember the operadic unit 1 as basepoint. The

corresponding diagram of left adjoints is

$$
\begin{array}{ccc}
\mathscr{S}_e & \xrightarrow{\;\;L\;\;} & \mathscr{L}\text{--spectra under } S \\
\Sigma^\infty \uparrow & & \uparrow \Sigma^\infty \\
\mathscr{T}_e & & \mathscr{L}\text{--spaces with zero} \\
(-)_+ \uparrow & & \uparrow (-)_+ \\
\mathscr{T}_1 & \xrightarrow{\;\;L\;\;} & \mathscr{L}\text{--spaces}
\end{array}
$$

This implies the relation $L\Sigma^\infty(X_+) \cong \Sigma^\infty(LX)_+$ of (6-7), which came as a computational surprise when it was first discovered.
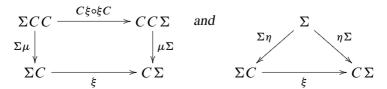
15 Appendix B. Loop spaces of E_∞ spaces and the recognition principle

Let X be an $\mathscr{O}$–space, where $\mathscr{O}$ is an E_∞ operad. Either replacing X by an equivalent $\mathscr{C}$–space or using the additive infinite loop space machine on $\mathscr{C} \times \mathscr{O}$–spaces, we construct a spectrum $\mathbb{E}X$ as in Theorem 9.3. For definiteness, we use notations corresponding to the first choice. As promised in Remark 9.10, we shall reprove the following result. The proof will give more precise information than the statement, and we will recall a consequence that will be relevant to our discussion of orientation theory in the second sequel [33] after giving the proof.

Theorem 15.1 *The space ΩX is an $\mathscr{O}$–space and there is a natural map of spectra $\omega\colon \Sigma\mathbb{E}\Omega X \longrightarrow \mathbb{E}X$ that is a weak equivalence if X is connected. Therefore its adjoint $\tilde{\omega}\colon \mathbb{E}\Omega X \longrightarrow \Omega\mathbb{E}X$ is also a weak equivalence when X is connected.*

We begin with a general result on monads, but stated with notations that suggest our application. It is an elaboration of [25, 5.3]. The proof is easy diagram chasing.

Lemma 15.2 *Let $\mathscr{T}$ be any category, let C be a monad on $\mathscr{T}$, and let (Σ, Ω) be an adjoint pair of endofunctors on $\mathscr{T}$. Let $\xi\colon \Sigma C \longrightarrow C\Sigma$ be a monadic natural isomorphism, so that the following diagrams commute.*

$$
\begin{array}{ccc}
\Sigma C C & \xrightarrow{C\xi \circ \xi C} & C C \Sigma \\
\Sigma\mu \downarrow & & \downarrow \mu\Sigma \\
\Sigma C & \xrightarrow{\;\;\xi\;\;} & C\Sigma
\end{array}
\qquad and \qquad
\begin{array}{ccc}
& \Sigma & \\
\Sigma\eta \swarrow & & \searrow \eta\Sigma \\
\Sigma C & \xrightarrow{\;\;\xi\;\;} & C\Sigma
\end{array}
$$

(i) *The functor* $\Omega C \Sigma$ *is a monad on* $\mathcal{T}$ *with unit and product the composites*

$$\mathrm{Id}\xrightarrow{\;\eta\;}\Omega\Sigma\xrightarrow{\;\Omega\eta\Sigma\;}\Omega C\Sigma$$

and

$$\Omega C\Sigma\Omega C\Sigma\xrightarrow{\;\Omega C\varepsilon C\Sigma\;}\Omega CC\Sigma\xrightarrow{\;\Omega\mu\Sigma\;}\Omega C\Sigma.$$

Moreover, the adjoint $\widetilde{\xi}\colon C\longrightarrow\Omega C\Sigma$ *of* ξ *is a map of monads on* $\mathcal{T}$.

(ii) *If* (X,θ) *is a* C–*algebra, then* ΩX *is an* $\Omega C\Sigma$–*algebra with action map*

$$\Omega C\Sigma\Omega X\xrightarrow{\;\Omega C\varepsilon\;}\Omega CX\xrightarrow{\;\Omega\theta\;}\Omega X,$$

hence ΩX *is a* C–*algebra by pull back along* $\widetilde{\xi}$.

(iii) *If* (F,ν) *is a* C–*functor* $(F\colon\mathcal{T}\longrightarrow\mathcal{V}$ *for some category* $\mathcal{V})$, *then* $F\Sigma$ *is an* $\Omega C\Sigma$–*functor with action transformation*

$$F\Sigma\Omega C\Sigma\xrightarrow{\;F\varepsilon C\Sigma\;}FC\Sigma\xrightarrow{\;\nu\Sigma\;}F\Sigma,$$

hence $F\Sigma$ *is a* C–*functor by pull back along* $\widetilde{\xi}$.

If $\alpha\colon C\longrightarrow C'$ *is a map of monads on* $\mathcal{T}$, *then so is* $\Omega\alpha\Sigma\colon\Omega C\Sigma\longrightarrow\Omega C'\Sigma$.

The relevant examples start with the loop suspension adjunction (Σ,Ω) on $\mathcal{T}$.

Lemma 15.3 *For any (reduced) operad* $\mathscr{C}$ *in* $\mathscr{U}$ *with associated monad* C *on* $\mathcal{T}$, *there is a monadic natural transformation* $\xi\colon\Sigma C\longrightarrow C\Sigma$. *There is also a monadic natural transformation* $\rho\colon\Sigma Q\longrightarrow Q\Sigma$ *such that the following diagram commutes, where* $\mathscr{C}$ *is the Steiner operad (or its product with any other operad).*

$$
\begin{array}{ccc}
\Sigma C & \xrightarrow{\;\xi\;} & C\Sigma \\
{\scriptstyle\Sigma\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha\Sigma} \\
\Sigma Q & \xrightarrow{\;\rho\;} & Q\Sigma
\end{array}
$$

Proof For $c\in\mathscr{C}(j)$, $x_i\in X$, and $t\in I$, we define

$$\xi((c;x_1,\ldots,x_j)\wedge t)=(c;x_1\wedge t,\ldots,x_j\wedge t)$$
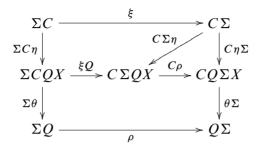
and check monadicity by diagram chases. A point $f\in QX$ can be represented by a map $f\colon S^n\longrightarrow X\wedge S^n$ for n sufficiently large and a point of $Q\Sigma X$ can be represented by a map $g\colon S^n\longrightarrow X\wedge S^1\wedge S^n$. We define

$$\rho(f\wedge t)(y)=x\wedge t\wedge z,$$

where $y \in S^n$ and $f(y) = x \wedge z \in X \wedge S^n$ and check monadicity by somewhat laborious diagram chases. For the diagram, recall that α is the composite

$$CX \xrightarrow{C\eta} CQX \xrightarrow{\theta} QX$$

and expand the required diagram accordingly to get

The top left trapezoid is a naturality diagram and the top right triangle is easily seen to commute by checking before application of C. The bottom rectangle requires going back to the definition of the action θ, but it is easily checked from that. $\square$

For any operad C, the action $\tilde{\theta}$ of C on ΩX induced via Lemma 15.2 (ii) from an action θ of C on X is given by the obvious pointwise formula

$$\tilde{\theta}(c; f_1, \ldots, f_j)(t) = \theta(c; f_1(t), \ldots, f_j(t))$$

for $c \in \mathscr{C}(j)$ and $f_i \in \Omega X$. The conceptual description leads to the following proof.

Proof of Theorem 15.1   For a spectrum E, $(\Omega E)_0 = \Omega(E_0)$, and it follows that we have a natural isomorphism of adjoints $\chi \colon \Sigma\Sigma^\infty \longrightarrow \Sigma^\infty\Sigma$. We claim that this is an isomorphism of C–functors, where the action of C on the right is given by Lemma 15.2 (iii). To see this, we recall that the action of C on Σ^∞ is the composite

$$\Sigma^\infty C \xrightarrow{\Sigma^\infty \alpha} \Sigma^\infty Q = \Sigma^\infty \Omega^\infty \Sigma^\infty \xrightarrow{\varepsilon \Sigma^\infty} \Sigma^\infty$$

and check that the following diagram commutes.

The top left square is a naturality diagram and the top right square is Σ^∞ applied to the diagram of Lemma 15.3. The bottom rectangle is another chase. The functor Σ on spectra commutes with geometric realization, and there results an identification

(15-4) $$\Sigma\mathbb{E}\Omega X = \Sigma B(\Sigma^\infty, C, \Omega X) \cong B(\Sigma^\infty\Sigma, C, \Omega X).$$

The action of C on ΩX is given by Lemma 15.2 (ii), and we have a map

(15-5) $$B(\mathrm{id}, \widetilde{\xi}, \mathrm{id})\colon B(\Sigma^\infty\Sigma, C, \Omega X) \longrightarrow B(\Sigma^\infty\Sigma, \Omega C\Sigma, \Omega X).$$

The target is the geometric realization of a simplicial spectrum with q–simplices

$$\Sigma^\infty\Sigma(\Omega C\Sigma)^q\Omega X = \Sigma^\infty(\Sigma\Omega C)^q\Sigma\Omega X.$$

Applying $\varepsilon\colon \Sigma\Omega \longrightarrow \mathrm{Id}$ in the $q+1$ positions, we obtain maps

$$\Sigma^\infty(\Sigma\Omega C)^q\Sigma\Omega X \longrightarrow \Sigma^\infty C^q X.$$
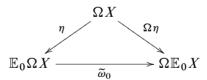
By further diagram chases showing compatibility with faces and degeneracies, these maps specify a map of simplicial spectra. Its geometric realization is a map

(15-6) $$B(\Sigma^\infty\Sigma, \Omega C\Sigma, \Omega X) \longrightarrow B(\Sigma^\infty, C, X) = \mathbb{E}X.$$

Composing (15-4), (15-5), and (15-6), we have the required map of spectra

$$\omega\colon \Sigma\mathbb{E}\Omega X \longrightarrow \mathbb{E}X.$$

Passing to adjoints and 0th spaces, we find that the following diagram commutes.

$$
\begin{array}{ccc}
 & \Omega X & \\
\eta \swarrow & & \searrow \Omega\eta \\
\mathbb{E}_0\Omega X & \xrightarrow[\widetilde{\omega}_0]{} & \Omega\mathbb{E}_0 X
\end{array}
$$

Since η is a group completion in general, both η and $\Omega\eta$ in the diagram are equivalences when X is connected and therefore ω is then an equivalence. □

As was observed in [26, 3.4], if G is a monoid in $\mathscr{O}[\mathscr{T}]$, then BG is an $\mathscr{O}$–space such that $G \longrightarrow \Omega BG$ is a map of $\mathscr{O}$–spaces. Since $(\Omega\mathbb{E}X)_1 = \Omega\mathbb{E}_1 X \cong \mathbb{E}_0 X$, Theorem 15.1 has the following consequence [26, 3.7].

Corollary 15.7 *If G is a monoid in $\mathscr{O}[\mathscr{T}]$, Then $\mathbb{E}G$, $\mathbb{E}\Omega BG$, and $\Omega\mathbb{E}BG$ are naturally equivalent spectra. Therefore the first delooping $\mathbb{E}_1 G$ and the classical classifying space $BG \simeq \mathbb{E}_0 BG$ are equivalent as $\mathscr{O}$–spaces.*

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