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CLASSIFYING SPACES AND FIBRATIONS

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Abstract

The basic theory of fibrations is generalized to a context in which fibres, and maps on fibres, are constrained to lie in any preassigned category of spaces $\mathcal{F}$. Then axioms are placed on $\mathcal{F}$ to allow the development of a theory of associated principal fibrations and, under several choices of additional hypotheses on $\mathcal{F}$, a classification theorem is proven for such fibrations. The same proof applies to the classification of bundles and generalizes to give a classification theorem for fibrations or bundles with additional structure, such as a reduction of the structural monoid, or a trivialization with respect to a coarser type of fibration, or an orientation with respect to an extraordinary cohomology theory. The proofs are constructive and are based on use of the two-sided geometric bar construction, the topological and homological properties of which are analyzed in detail. Related topics studied include the classification of fibrations by transports, the Eilenberg-Moore and Serre spectral sequences, and the group completion theorem.

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Introduction

"The theory of fibrations is thus fairly complete and well worked out on a conceptual level; the rest should be applications and computations." So ended Stasheff's 1970 survey article [35] on the classification of fibrations.

At the time, the conclusion seemed not unreasonable. The basic outlines of a complete theory were visible, and this theory did seem adequate for most applications. Even in Stasheff's very clear summary, the theory appeared technically to be extremely complicated, but this was felt to be intrinsic to the subject.

However, recent developments make this sanguine view of the adequacy of the theory untenable and, in the process of obtaining a theory which is adequate for the new applications, we shall also see how to avoid most of the previous technical complications.

A brief account of the existing classification theorems will be necessary in order to place our contribution in perspective.

The simplest and most conceptual method of classification is based on the observation that if $\mathcal{C}$ is a small topological category and if a space $\mathcal{B}\mathcal{C}$ is appropriately constructed from the associated simplicial space (technically, by use of face but not degeneracy operators in forming the geometric realization), then $\mathcal{B}\mathcal{C}$ classifies the functor defined on paracompact spaces X as the quotient obtained from the cohomology set $H^1(X; \mathcal{C})$ by identifying homotopic cohomology classes. This method is due to Segal [30], and an exposition has also been given by Stasheff [2, p. 86-94]. $\mathcal{B}\mathcal{C}$ is a generalization of Milnor's classifying space for topological groups [24], and this method of classification is a generalization of one found for bundles by tom Dieck [6]. It is particularly appropriate to the study of foliations via the classification of Haefliger structures (e.g. [2]). While this approach is very general, it is only useful when the structures one wishes to study are obtained by patching together local coordinates by means of cocycles with values in some category $\mathcal{C}$. In practice, this means that the morphisms of $\mathcal{C}$ must at least be homeomorphisms, so that $\mathcal{C}$ is a topological groupoid, and this approach is inapplicable to the classification of fibrations or of bundles with globally defined additional structure.

A second conceptual method of classification is based on appeal to Brown's representability theorem [4]. It has two defects. First, as applied to fibrations, there is no completely rigorous treatment in the literature. The point is that if one wishes to represent a set-valued functor, then one must first verify that one's proposed functor does indeed take well-defined sets as values. This is by no means obvious for the functors of interest in the theory of fibrations, and this set-theoretical question has been totally ignored in the literature. Second, and probably more fundamental, the basic purpose of a classification theorem is to enable one to calculate the represented functor, or at least to calculate invariants of the structures under study. A space constructed by appeal to Brown's theorem can generally be studied only by reverting to analysis of the originally given functor and is therefore of very limited use for purposes of computation.

The bulk of Stasheff's survey is devoted to the various alternative methods of classification, and we shall not give references here. In contrast to the general methods described in the previous two paragraphs, these alternative methods appear to be specific to particular types of fibration, or at least to require considerable reworking to be made applicable to varying types. Technically, with one exception, each such method involves at least one of the theory of simplicial sets, a combinatorial theory of cellular monoids, careful local pasting arguments, or the use of higher homotopies. The exception is Stasheff's original proof [32] of the classification theorem for fibrations with fibres of the homotopy type of a finite CW-complex.

Why are these results not adequate? First, fibrations with localized or completed spheres as fibres play a key role in Sullivan's beautiful proof of the Adams conjecture [39]. Such spaces are not finite CW-complexes, and one source of technical difficulty in Sullivan's argument is the absence of a good model for the relevant classifying spaces. Our Corollary 9.5 will rectify this, and we shall return to this point in [21] where a new theory of localization and completion of topological spaces will be given.

Second, spherical fibrations and bundles oriented with respect to an extraordinary cohomology theory are central to many applications, and there is no proof of a classification theorem for such structures in the literature. As we shall show in [19] and [20], Theorem 11.1 implies such a classification theorem, and its use allows easy derivations of some of the results of Adams on vector bundles and of Sullivan on topological sphere bundles.

Third, for the study of orientations, it is very convenient to have a variant of Stasheff's theorem in which fibrations are given with a cross-section which is a cofibration. Corollary 9.8 will give such a result.

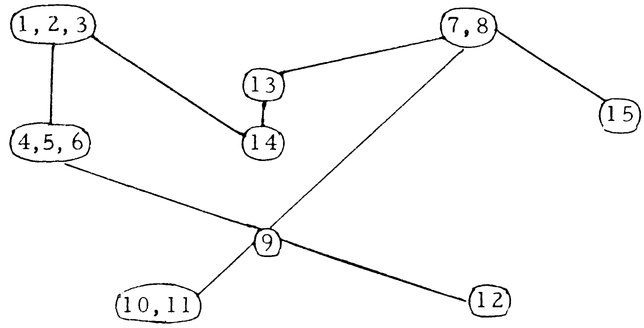
Beyond these explicit applications, there is an evident need for a single coherent theory of fibrations and their classification which will simultaneously yield the various classification theorems desired in practice as special cases of one general result, or at least as consequences of one general pattern of proof. Moreover, such a theory should if possible avoid techniques, such as those listed a few paragraphs earlier, which, however great their interest within the theory of fibrations, are irrelevant to the actual computations based on the theory. Needless to say, our theory does meet these criteria.

We should say a bit more about two of the techniques we avoid. Much has been written about the inevitability of the appearance of higher homotopies in any complete theory of fibrations, and we freely admit that they are indeed implicitly present. Nevertheless, at each place where it is generally felt they ought to appear, we shall find that some conceptual trick leads to an equivalent solution with no such notion visible. Thus we shall classify principal G-fibrations for arbitrary grouplike topological monoids

G in Corollary 9.4, we shall show the independence of the choice of fibre in the classification of fibrations with fibres of a given homotopy type in section 12, and we shall classify such fibrations by use of associative transports (which are actions on fibres by the Moore loop space of the base) in section 14. In each case, the actual details are very much simpler than would be the case if higher homotopies were explicitly introduced.

Similarly, we have chosen to work with Hurewicz, rather than with Dold (or weak) fibrations, throughout. We freely admit that Dold fibrations have important technical advantages and are implicit in the notion of fibre homotopy equivalence. However, since the local pasting arguments for which they are essential are unnecessary in our work, their use would introduce considerable additional complexity while adding nothing of significance to our theory. Although the most important results concerning the local nature of fibrations are valid in our general context, local considerations will only play a role in those instances of our classification theorems which involve bundle theory.

The paper consists of fifteen sections, with logical interrelationships as indicated in the following chart:



Introduction

The first three sections are devoted to a redevelopment of the theory of fibrations, including the basic theorems of Dold [7, 3.3 and 6.3] and Hurewicz [11], for fibrations with fibres constrained to lie in any pre-assigned category $\mathcal{F}$. Technically, the main point here is that the section extension property which Dold takes as fundamental does not generalize to our context, hence we have been forced to find alternative proofs. While these are still based on the ideas of Dold, they are shorter and may seem simpler even in the classical case.

The special properties required of $\mathcal{F}$ in order to classify $\mathcal{F}$-fibrations are discussed in sections 4 and 5, and examples of categories which satisfy these properties are given in section 6. The relevant properties ensure that associated principal fibrations can be constructed and that quasifibrations can be replaced by fibrations; for the latter, the point of interest is that the standard procedure is inadequate for the study of fibrations with cross-section.

In sections 7 and 8, we summarize the topological properties of the two-sided geometric bar construction. Most of the proofs have already been given, in a more general setting, in [17, §9-11 (which are independent of §1-8)] or [18, Appendix]. This construction is a straightforward generalization, implicit in Stasheff's paper [34], of the standard Milgram-Steenrod [23, 38] classifying space functor. The generalization, despite its simplicity, transforms the bar construction from an invariant of topological monoids to an extremely flexible tool in the theory of fibrations and their classification.

Introduction

We reach our basic classification theorems for fibrations and bundles in section 9. The method of proof is to write down an explicit universal fibration (or bundle) and to verify that it classifies by explicitly constructing a classifying map for any given fibration (or bundle). In section 12, we generalize the standard Segal [30] classifying space functor on small topological categories to a two-sided bar construction (technically, using both face and degeneracy operators). This generalization allows us to rework our basic theory, in favorable cases, so as not to give any particular choice of fibre a privileged role.

A general notion of additional global structure on a fibration or bundle is introduced in section 10. Special cases include reductions of the structural monoid, trivializations with respect to a coarser type of fibration, and orientations (of spherical fibrations or bundles) with respect to an extraordinary cohomology theory. We demonstrate in section 11 that the proof of our classification theorems directly generalizes to a proof of classification theorems for such fibrations or bundles with additional structure.

The last three sections are primarily concerned with homological properties of the geometric bar construction. In section 13, after generalizing results of Milgram [23] and Steenrod [38] concerning the cellular properties of classifying spaces, we obtain a technical result (Theorem 13.9) which relates the two-sided algebraic and geometric bar constructions by mixed use of singular and cellular chain groups. This result, which should be regarded as a generalization of a special case of a result of Stasheff [31], gives the Eilenberg-Moore and Rothenberg-Steenrod spectral sequences [26, 29], with their products, and shows that the two are in fact the same. In section 14, we introduce the notion of a transport, use it to prove a classification theorem for fibrations (suggested by a result due to Stasheff [33]), and combine it with Theorem 13.9 to give a novel derivation of the Serre spectral sequence, with its products. Finally, in section 15, we give a brief proof of the "group completion theorem", due to Barratt-Priddy [1] and Quillen [28], which analyzes the homological behavior of the natural map $G \to \Omega BG$ for appropriate non-connected topological monoids G. This result plays a fundamental role in the theory of infinite loop spaces and its application to algebraic K-theory [18].

Added, December, 1979. Parts of sections 5, 6, and 9 admit some improvement, and several papers applying and extending this theory have appeared in the interim. These developments are briefly summarized at the end of the paper, beginning on page 99.

Contents

1. $\mathcal{F}$-spaces and $\mathcal{F}$-maps

We take the position that types of fibrations ought to be specified by assigning structure to the fibres and that this is most sensibly done by specifying a category in which the fibres must lie. We here develop a framework in which to define such fibrations and generalize to this framework a theorem of Dold [7, 3.3] to the effect that a local fibre homotopy equivalence is a fibre homotopy equivalence.

We shall work in the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces [37; 22, §2]; thus products, function spaces, etc. are always to be given the compactly generated topology. Throughout the first five sections $\mathcal{F}$ will denote a category with a faithful "underlying space" functor $\mathcal{F} \to \mathcal{U}$. Thus each object of $\mathcal{F}$ is a space and the set $\mathcal{F}(F, F')$ of morphisms $F \to F'$ in $\mathcal{F}$ is a subset of $\mathcal{U}(F, F')$. We agree either to insist that $\mathcal{F}$ contain with each $F \in \mathcal{F}$ the spaces $F \times *$ and $* \times F$ and the evident homeomorphisms between these spaces and F or to identify these spaces with F, where $*$ is any one-point space.

<u>Definition 1.1</u>. An $\mathcal{F}$-space is a map $\pi: E \to B$ in $\mathcal{U}$ such that $\pi^{-1}(b) \in \mathcal{F}$ for each $b \in B$; B and E are the base space and total space of π. An $\mathcal{F}$-map $(g, f): \nu \to \pi$ is a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\ g\ } & E \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

in $\mathcal{U}$ such that $g\colon \nu^{-1}(a) \to \pi^{-1}(f(a))$ is in $\mathcal{F}$ for each $a \in A$; if $A = B$ and f is the identity map, then g is said to be an $\mathcal{F}$-map over B. An $\mathcal{F}$-homotopy is an $\mathcal{F}$-map (H, h) of the form

$$\begin{array}{ccc} D \times I & \xrightarrow{\ H\ } & E \\ {\scriptstyle \nu \times 1}\downarrow & & \downarrow{\scriptstyle \pi} \\ A \times I & \xrightarrow{\ h\ } & B \end{array}$$

Thus it is required that each (H_s, h_s) be an $\mathcal{F}$-map, $H_s(d) = H(d, s)$; if $A = B$ and $h_s(b) = b$, then H is said to be an $\mathcal{F}$-homotopy over B. An $\mathcal{F}$-map $g\colon D \to E$ over B is an $\mathcal{F}$-homotopy equivalence if there is an $\mathcal{F}$-map $g'\colon E \to D$ over B such that $g'g$ and gg' are $\mathcal{F}$-homotopic over B to the respective identity maps. An $\mathcal{F}$-space $\pi\colon E \to B$ is said to be $\mathcal{F}$-homotopy trivial if it is $\mathcal{F}$-homotopy equivalent to the projection $\pi_1\colon B \times F \to B$ for some $F \in \mathcal{F}$.

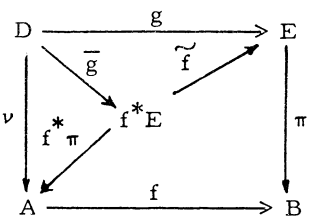
By restriction to one-point base spaces, the definition specializes to define $\mathcal{F}$-homotopies and $\mathcal{F}$-homotopy equivalences between spaces in $\mathcal{F}$.

We can form induced $\mathcal{F}$-spaces precisely as usual. The following lemma fixes notations.

Lemma 1.2. Let $\pi\colon E \to B$ be an $\mathcal{F}$-space and let $f\colon A \to B$ be a map in $\mathcal{U}$. Define a space f^*E and maps $f^*\pi\colon f^*E \to A$ and $\tilde{f}\colon f^*E \to E$ by

$$f^*E = \{(a, e) \mid f(a) = \pi(e)\} \subset A \times E, \quad f^*\pi(a, e) = a, \text{ and } \tilde{f}(a, e) = e.$$

Then $f^*\pi$ is an $\mathcal{J}$-space and $(\tilde{f}, f)$ is an $\mathcal{J}$-map. Moreover, if $\nu : D \to A$ is an $\mathcal{J}$-space and $(g, f): \nu \to \pi$ is an $\mathcal{J}$-map, then the unique map $\overline{g}: D \to f^*E$ which makes the following diagram commutative, namely $\overline{g}(d) = (\nu(d), g(d))$, is an $\mathcal{J}$-map over A:



The remainder of this section will be devoted to the promised generalization of Dold's theorem. We assume given $\mathcal{J}$-spaces $\nu : D \to B$ and $\pi: E \to B$ and an $\mathcal{J}$-map $g: D \to E$ over B. We require some notations and a lemma.

Notations 1.3. Let G denote the subspace of $D \times E^I$ consisting of all pairs (d, ε) such that $g(d) = \varepsilon(0)$ and $\varepsilon(I) \subset \pi^{-1}(\nu(d))$. Define maps $\alpha: G \to D$ and $\beta_s: G \to E$ for $s \in I$ by $\alpha(d, \varepsilon) = d$ and $\beta_s(d, \varepsilon) = \varepsilon(s)$. By an $\mathcal{J}$-section of g, we understand a map $\sigma : E \to G$ such that $\beta_1 \circ \sigma = 1$ on E and such that, for all $b \in B$, each of the maps $\alpha\sigma : \pi^{-1}(b) \to \nu^{-1}(b)$ and $\beta_s \circ \sigma : \pi^{-1}(b) \to \pi^{-1}(b)$ is in $\mathcal{J}$. Thus an $\mathcal{J}$-section of g consists of an $\mathcal{J}$-map $g' = \alpha \circ \sigma : E \to D$ over B together with an $\mathcal{J}$-homotopy $H_s = \beta_s \circ \sigma$ over B from gg' to the identity map of E.

If A is a subspace of B, we let E_A, G_A, etc., denote the part of E, G, etc., over A; explicitly, $G_A = \{(d,\varepsilon) \mid \nu(d) \in A\} \subset G$.

Lemma 1.4. Assume that g is an $\mathcal{J}$-homotopy equivalence. Let $\phi: B \to I$ be a map, let $A = \phi^{-1}(1)$, and let $V = \phi^{-1}(0,1]$. Then, for any $\mathcal{J}$-section $\sigma: E_V \to G_V$ of g, there exists an $\mathcal{J}$-section $\rho: E \to G$ of g such that $\rho = \sigma$ on E_A.

Proof. Let $g': E \to D$ be an $\mathcal{J}$-homotopy inverse of g and let $H: gg' \simeq 1_E$ and $H': g'g \simeq 1_D$ be $\mathcal{J}$-homotopies over B. Define an $\mathcal{J}$-section $\tau: E \to G$ of g by $\tau(e) = (g'(e), H(e))$, where $H(e)(s) = H(e,s)$. Define a homotopy $L: G \times I \to G$ from the composite $\tau \circ \beta_1$ to the identity map of G by the formula

$$L((d,\varepsilon),t) = \begin{cases} (g'\varepsilon(1-2t),\ H\varepsilon(1-2t)) & \text{if } t \leq 1/2 \\ (H'(d,2t-1), J(d,\varepsilon,2t-1)) & \text{if } 1/2 \leq t \end{cases}$$

where $J(d,\varepsilon,t)(s) = jk(s,t)$ for some chosen retraction

$$k: I \times I \to (I \times 0) \cup (0 \times I) \cup (I \times 1)$$

and where $j(s,0) = H(\varepsilon(0),s)$, $j(0,t) = gH'(d,t)$, and $j(s,1) = \varepsilon(s)$. The desired $\mathcal{J}$-section ρ of g is then defined by the formula

$$\rho(e) = \begin{cases} \tau(e) & \text{if } \phi\pi(e) \leq 1/2 \\ L(\sigma(e), 2\phi\pi(e)-1) & \text{if } 1/2 \leq \phi\pi(e) \end{cases}$$

Recall that a cover $\mathcal{C}$ of a space B is said to be numerable if it is locally finite and if for each $U \in \mathcal{C}$ there is a map $\lambda_U : B \to I$ such that $U = \lambda_U^{-1}(0, 1]$. Recall too that a space $B \in \mathcal{U}$ is paracompact if and only if every open cover of B admits a numerable refinement and that any CW-complex is paracompact.

Theorem 1.5. Let $\nu : D \to B$ and $\pi : E \to B$ be $\mathcal{J}$-spaces. Let $g : D \to E$ be an $\mathcal{J}$-map over B such that g restricts to an $\mathcal{J}$-homotopy equivalence over each set of a numerable cover $\mathcal{C}$ of B. Then g is an $\mathcal{J}$-homotopy equivalence.

Proof. It suffices to construct an $\mathcal{J}$-section $\sigma : E \to G$ of g. Indeed, this will give a right $\mathcal{J}$-homotopy inverse g' of g. g' will restrict to an $\mathcal{J}$-homotopy equivalence over each $U \in \mathcal{C}$ since if f_U is an $\mathcal{J}$-homotopy inverse to g_U, then $f_U \simeq f_U g_U g'_U \simeq g'_U$ (where $\simeq$ means "is $\mathcal{J}$-homotopic to"). Therefore g' will itself have a right $\mathcal{J}$-homotopy inverse g", and $g \simeq gg'g'' \simeq g''$. For $U \in \mathcal{C}$, choose a map $\lambda_U : B \to I$ such that $U = \lambda_U^{-1}(0, 1]$. For a union $V = \bigcup_{j \in J} U_j$ of sets $U_j \in \mathcal{C}$ define $\lambda_V = \sum_{j \in J} \lambda_{U_j}$, so that $V = \{x \mid \lambda_V(x) > 0\}$. We assume that $\mathcal{C}$ is irredundant, and then $V \subset W$ if and only if $\lambda_V \leq \lambda_W$. Let $\mathcal{A}$ denote the set of pairs (V, σ) such that V is a union of sets in $\mathcal{C}$ and $\sigma : E_V \to G_V$ is an $\mathcal{J}$-section of g. Partial order $\mathcal{A}$ by $(V,\sigma) < (W, \tau)$ if $V \subset W$ and $\sigma(e) = \tau(e)$ for all $e \in \pi^{-1}V$ such that $\lambda_V \pi(e) = \lambda_W \pi(e)$ (thus, $\sigma(e) \neq \tau(e)$

implies $\pi(e) \in U$ for some $U \in \mathcal{C}$ such that $U \subset W$ but $U \not\subset V$). Any totally ordered subset $\{(V_k, \sigma_k) \mid k \in K\}$ of $\mathcal{Q}$ has the upper bound (V, σ) defined by $V = \bigcup_{k \in K} V_k$ and $\sigma(e) = \sigma_k(e)$ for $e \in \pi^{-1}V$ and all sufficiently large k. Here σ is well-defined and continuous and $(V, \sigma) > (V_k, \sigma_k)$ for all $k \in K$ since if $b \in V$ and $V(b)$ denotes the union of those $U \in \mathcal{C}$ such that $b \in U \subset V$, then $V(b) \subset V_{k(b)}$ for some $k(b) \in K$ (because $\mathcal{C}$ is locally finite) and therefore $\sigma_k = \sigma_{k(b)}$ over $V(b)$ for all $k > k(b)$. By Zorn's lemma, $\mathcal{Q}$ contains a maximal element (V, σ). We claim that $V = B$. Suppose not; choose $U \in \mathcal{C}$ such that $U \not\subset V$ and let $W = U \cup V$. Define $\phi: W \to I$ by

$$\phi(b) = \begin{cases} 1 & \text{if } \lambda_U(b) \le \lambda_V(b) \quad (\text{hence } \lambda_V(b) > 0) \\ \lambda_V(b)/\lambda_U(b) & \text{if } \lambda_U(b) \ge \lambda_V(b) \quad (\text{hence } \lambda_U(b) > 0) \end{cases}$$

$\phi(b) > 0$ if and only if $\lambda_V(b) > 0$, hence σ is defined over $\phi^{-1}(0, 1]$. By the lemma, there is an $\mathcal{J}$-section $\rho: E_U \to G_U$ of g such that $\rho = \sigma$ over $\phi^{-1}(1) \cap U$. Define $\tau: E_W \to G_W$ by

$$\tau(e) = \begin{cases} \sigma(e) & \text{if } \lambda_U\pi(e) \le \lambda_V\pi(e) \\ \rho(e) & \text{if } \lambda_U\pi(e) \ge \lambda_V\tau(e) \end{cases}$$

Clearly $(W, \tau) > (V, \sigma)$, which is the desired contradiction.

2. $\mathcal{J}$-fibrations

Definition 2.1. An $\mathcal{J}$-space $\pi: E \to B$ is an $\mathcal{J}$-fibration if it satisfies the following $\mathcal{J}$-covering homotopy property (abbreviated $\mathcal{J}$-CHP): for every $\mathcal{J}$-space $\nu: D \to A$ and $\mathcal{J}$-map $(g, f): \nu \to \pi$ and every homotopy $h: A \times I \to B$ of f, there exists a homotopy $H: D \times I \to E$ of g such that the pair (H, h) is an $\mathcal{J}$-homotopy.

A $\mathcal{U}$-fibration is clearly just an ordinary (Hurewicz) fibration. We here generalize the elementary theory of fibrations to $\mathcal{J}$-fibrations and generalize Dold's theorem [7 , 6.3] to the effect that a map of fibrations is a fibre homotopy equivalence if it restricts to a homotopy equivalence on each fibre. We observe first that the $\mathcal{J}$-homotopy (H, h) asserted to exist by the $\mathcal{J}$-CHP is itself unique up to $\mathcal{J}$-homotopy.

Lemma 2.2. Let $(H, h), (H', h')$, and (J, j) be $\mathcal{J}$-homotopies with domain $\nu \times 1: D \times I \to A \times I$ and range $\pi: E \to B$, where ν is an $\mathcal{J}$-space and π is an $\mathcal{J}$-fibration. Assume that (J, j) is an $\mathcal{J}$-homotopy from (H_0, h_0) to (H'_0, h'_0) and assume given $k: A \times I \times I \to B$ such that

$$k(a, s, 0) = h(a, s), \ k(a, s, 1) = h'(a, s), \text{ and } k(a, 0, t) = j(a, t).$$

Let $C = (I \times 0) \cup (I \times 1) \cup (0 \times I) \subset I \times I$ and define $g: D \times C \to E$ by

$$g(d, s, 0) = H(d, s), \ g(d, s, 1) = H'(d, s), \text{ and } g(d, 0, t) = J(d, t).$$

Then there exists $K: D \times I \times I \to E$ such that $K|D \times C = g$ and the pair (K, k) is an $\mathcal{J}$-homotopy.

Proof. (g, f) is an $\mathcal{J}$-map, where $f = k|A \times C$. Since the pairs $(I \times I, C)$ and $(I \times I, I \times 0)$ are homeomorphic, the conclusion follows directly from the $\mathcal{J}$-CHP.

The following result is an easy consequence of the lemma.

Proposition 2.3. An $\mathcal{J}$-fibration $\pi: E \to B$ determines a functor L from the fundamental groupoid of B to the homotopy category of $\mathcal{J}$ by $L(b) = \pi^{-1}(b)$ for $b \in B$ and $L[h] = [H_1]$ for a path $h: I \to B$, where $H: \pi^{-1}h(0) \times I \to E$ is any homotopy of the inclusion $\pi^{-1}h(0) \to E$ such that (H, h) is an $\mathcal{J}$-homotopy. In particular, if B is connected, then any two fibres of π have the same $\mathcal{J}$-homotopy type.

We shall find it convenient to compose paths in the reverse of the usual order; with this convention, the functor L is covariant.

We show next that induced $\mathcal{J}$-fibrations behave properly.

Lemma 2.4. Let $\pi: E \to B \times I$ be an $\mathcal{J}$-fibration and let $\pi^s: E^s \to B$ denote the part of π over $B \times \{s\}$. Then π^0 and π^1 are $\mathcal{J}$-homotopy equivalent.

Proof. Define $h: B \times I \times I \times I \to B \times I$ by $h(b, r, s, t) = (b, (1-t)r+ts)$. By the $\mathcal{J}$-CHP, there exists $H: E \times I \times I \to E$ such that $H(e, s, 0) = e$ and (H, h) is an $\mathcal{J}$-homotopy. Define $K: E \times I \to E$ by $K(e, s) = H(e, s, 1)$. Observe that if $\pi': E \to B$ and $\pi'': E \to I$ are defined by $\pi(e) = (\pi'(e), \pi''(e))$, then $\pi K(e,s) = (\pi'(e), s)$ and traversal of $H(e, \pi''(e), t)$, $0 \le t \le 1$, gives an $\mathcal{J}$-homotopy over $B \times I$ from the identity of E to the $\mathcal{J}$-map $k: E \to E$ over $B \times I$ defined by $k(e) = K(e, \pi''(e))$. Define $k^1: E^0 \to E^1$ and $k^0: E^1 \to E^0$ by $k^1(x) = K(x, 1)$ and $k^0(y) = K(y, 0)$. Via the homotopies $K(K(y,s),1)$ and $K(K(x,1-s),0)$, $0 \le s \le 1$, the maps k^1k^0 and k^0k^1 are $\mathcal{J}$-homotopic over B to $kk|E^1$ and $kk|E^0$, respectively. Therefore k^1 and k^0 are inverse $\mathcal{J}$-homotopy equivalences.

Proposition 2.5. Let $\pi: E \to B$ be an $\mathcal{J}$-fibration. Then $f^*\pi: f^*E \to A$ is an $\mathcal{J}$-fibration for any map $f: A \to B$ and homotopic maps $A \to B$ induce $\mathcal{J}$-homotopy equivalent $\mathcal{J}$-fibrations over A. In particular, any $\mathcal{J}$-fibration over a contractible base space is $\mathcal{J}$-homotopy trivial.

Proof. The first half follows from Definition 2.1 and Lemma 1.2, and the second half follows by application of the previous lemma to $h^*\pi: h^*E \to A$ for any homotopy $h: A \times I \to B$.

Dold's theorem [7 , 6.3] now generalizes readily to our context.

Theorem 2.6. Let $\nu: D \to B$ and $\pi: E \to B$ be $\mathcal{J}$-fibrations. Let $g: D \to E$ be an $\mathcal{J}$-map over B such that $g: \nu^{-1}(b) \to \pi^{-1}(b)$ is an $\mathcal{J}$-homotopy equivalence for each $b \in B$. Assume that B admits a numerable cover $\mathcal{C}$ such that the inclusion map $U \to B$ is null-homotopic for each $U \in \mathcal{C}$. Then g is an $\mathcal{J}$-homotopy equivalence.

Proof. Let $U \in \mathcal{C}$ and let $h: U \times I \to B$ be a null-homotopy, $h_0(u) = u$ and $h_1(u) = b$. Define $\bar{g}: h^*D \to h^*E$ by the universal property of

h^*E and let $\overline{g}^s$ denote the restriction of $\overline{g}$ to the part of h^*D over $U \times \{s\}$. Then $\overline{g}^0 = g\colon \nu^{-1}U \to \pi^{-1}U$ and $\overline{g}^1 = 1 \times g\colon U \times \nu^{-1}(b) \to U \times \pi^{-1}(b)$. Construct maps K, k, k^1 and k^0 for $h^*\pi$ and J, j, j^1 and j^0 for $h^*\nu$ by the proof of Lemma 2.4. Then $\overline{g}^0$ is $\mathcal{J}$-homotopic over $U \times \{0\}$ to $k\overline{g}^0 j$ and, via the homotopy $K(\overline{g}^s J(x,s), 0)$, $0 \le s \le 1$, $k\overline{g}^0 j$ is $\mathcal{J}$-homotopic over $U \times \{0\}$ to the composite of $\mathcal{J}$-homotopy equivalences $k^0 \overline{g}^1 j^1$. Thus g is an $\mathcal{J}$-homotopy equivalence over each $U \in \mathcal{C}$, and the result follows from Theorem 1.5.

Observe that the assumption on B is invariant under homotopy equivalence and is satisfied by spaces, such as CW-complexes, which are paracompact and locally contractible. In [7 , 6.7], Dold has given a direct construction of a cover of the required type for any CW-complex B. Of course, if B is connected, the assumption on $g\colon \nu^{-1}(b) \to \pi^{-1}(b)$ will be satisfied for all $b \in B$ if it is satisfied for any one $b \in B$.

3. $\mathcal{J}$-lifting functions

We here generalize to our context the relationship between fibrations and lifting functions and Hurewicz's theorem [11] to the effect that a local fibration is a fibration. We first fix notations for Moore paths; use of such paths will simplify proofs here and will be essential in later sections.

Notations 3.1. For $B \in \mathcal{U}$, let ΠB denote the set of paths (β, s) $\beta: [0, s] \to B$. When convenient, we let $\beta(t) = \beta(s)$ for $t \geq s$; a point $(\beta, s) \in \Pi B$ is then specified by $\beta: [0, \infty] \to B$ and $s \in [0, \infty)$, and ΠB is topologized as a subspace of $\mathcal{U}([0, \infty], B) \times [0, \infty)$. Define the composite $(\alpha\beta, r+s)$ of paths (α, r) and (β, s) such that $\alpha(0) = \beta(s)$ by

$$(\alpha\beta)(t) = \beta(t) \text{ if } 0 \leq t \leq s \text{ and } (\alpha\beta)(t) = \alpha(t-s) \text{ if } s \leq t \leq r+s.$$

We shall abbreviate $\beta = (\beta, s)$, and we shall write $\ell(\beta) = s$ and $p(\beta) = \beta(s)$ for the length map and end-point projection. We shall consider B to be contained in ΠB as the subspace consisting of all paths of length zero.

Definition 3.2. Let $\pi: E \to B$ be an $\mathcal{J}$-space. Define a space ΓE and maps $\Gamma\pi : \Gamma E \to B$, $\eta: E \to \Gamma E$, and $\mu: \Gamma\Gamma E \to \Gamma E$ by

$$\Gamma E = \{(\beta, e) \mid \beta(0) = \pi(e)\} \subset \Pi B \times E \text{ and } \Gamma\pi(\beta, e) = p(\beta),$$

$$\eta(e) = (\pi(e), e) \quad \text{and} \quad \mu(\alpha, (\beta, e)) = (\alpha\beta, e).$$

An $\mathcal{J}$-lifting function ξ for π is a map $\xi: \Gamma E \to E$ such that $\pi \circ \xi = \Gamma\pi$, $\xi \circ \eta = 1$, and the map $\xi \circ \beta: \pi^{-1}\beta(0) \to \pi^{-1}p(\beta)$ is in $\mathcal{J}$ for each $\beta \in \Pi B$, where $\tilde{\beta}: \pi^{-1}\beta(0) \to (\Gamma\pi)^{-1}p(\beta)$ is given by $\tilde{\beta}(e) = (\beta, e)$. ξ is said to be transitive if, whenever $\beta(0) = \pi(e)$ and $\alpha(0) = p(\beta)$,

$$\xi(\alpha, \xi(\beta, e)) = \xi(\alpha\beta, e).$$

For example, μ is a transitive $\mathcal{U}$-lifting function for $\Gamma\pi: \Gamma E \to B$.

Lemma 3.3. If ξ is a (transitive) $\mathcal{J}$-lifting function for an $\mathcal{J}$-space $\pi: E \to B$ and $f: A \to B$ is a map, then $f^*\xi$ is a (transitive) $\mathcal{J}$-lifting function for $f^*\pi: f^*E \to B$, where $f^*\xi$ is defined by

$$(f^*\xi)(\alpha, (a, e)) = (p(\alpha), \xi(f \circ \alpha, e))$$

for $\alpha \in \Pi A$, $a \in A$, and $e \in E$ such that $\alpha(0) = a$ and $f(a) = \pi(e)$.

Proposition 3.4. An $\mathcal{J}$-space $\pi: E \to B$ is an $\mathcal{J}$-fibration if and only if π has an $\mathcal{J}$-lifting function ξ.

Proof. If $p_0: \Pi B \to B$ is the initial projection, then $\Gamma E = p_0^* E$ as a space and $(\tilde{p}_0, p_0): p_0^*\pi \to \pi$ is an $\mathcal{J}$-map. Define a homotopy $h: \Pi B \times [0, \infty] \to B$ of p_0 by $h(\beta, t) = \beta(t)$. If π is an $\mathcal{J}$-fibration, there is a homotopy $H: \Gamma E \times [0, \infty] \to E$ of $\tilde{p}_0$ such that (H, h) is an $\mathcal{J}$-homotopy, and an $\mathcal{J}$-lifting function ξ is then given by $\xi(\beta, e) = H((\beta, e), \ell(\beta))$. Conversely, assume given ξ. For an $\mathcal{J}$-map $(g, f): \nu \to \pi$, $\nu: D \to A$, and a homotopy h of f, let $h_t(a)$ denote the path of length t in B given by $h_t(a)(u) = h(a, u)$, $a \in A$, and define $H(d, t) = \xi(h_t \nu(d), g(d))$. Clearly H is a homotopy of g such that (H, h) is an $\mathcal{J}$-homotopy.

The following immediate consequence should be noted.

Corollary 3.5. Let $\mathcal{J} \to \mathcal{J}'$ be a functor over $\mathcal{U}$ (that is, the underlying space functor $\mathcal{J} \to \mathcal{U}$ is the composite $\mathcal{J} \to \mathcal{J}' \to \mathcal{U}$). Then an $\mathcal{J}$-fibration π is an $\mathcal{J}'$-fibration; in particular, π is a fibration.

When $\mathcal{J} = \mathcal{U}$, Definition 3.2 describes the standard procedure for replacing a map by a fibration. We note the following facts about this process.

Remarks 3.6. Let $\pi: E \to B$ be a map (that is, a $\mathcal{U}$-space).

(i) If ξ is a lifting function for π, then $1 \simeq \eta\xi$ over B via the homotopy $\gamma_t(\beta, e) = (\beta'_t, \xi(\beta_t, e))$, where

$$\ell(\beta_t) = t\ell(\beta) \quad \text{and} \quad \beta_t(u) = \beta(u)$$

and

$$\ell(\beta'_t) = (1-t)\ell(\beta) \quad \text{and} \quad \beta'_t(u) = \beta(u + t\ell(\beta)) .$$

Therefore η and ξ are inverse fibre homotopy equivalences.

(ii) $\eta(E)$ is a strong deformation retract of ΓE via the homotopy $h_t(\beta, e) = (\beta_t, e)$. Thus η restricts to a weak homotopy equivalence on each fibre if π is a quasi-fibration (so that $\pi_*: \pi_i(E, \pi^{-1}b, e) \to \pi_i(B, b)$ is a bijection, $i \geq 1$, and $\pi_0(\pi^{-1}b, e) \to \pi_0(E, e) \to \pi_0(B, b) \to *$ is exact for all $b \in B$ and $e \in \pi^{-1}b$).

Remarks 3.7. For a $\mathcal{U}$-map $(g, f): \nu \to \pi$, $\nu: D \to A$ and $\pi: E \to B$, define a $\mathcal{U}$-map $\Gamma(g, f): \Gamma\nu \to \Gamma\pi$ by $\Gamma(g, f) = (\Gamma g, f)$, where $\Gamma g: \Gamma D \to \Gamma E$ is given by $\Gamma g(\alpha, d) = (f \circ \alpha, g(d))$. Then Γ is a functor from the category of $\mathcal{U}$-spaces to itself. The $\mathcal{U}$-maps $\eta: E \to \Gamma E$ and $\mu: \Gamma\Gamma E \to \Gamma E$ over B define natural transformations $\eta: 1 \to \Gamma$ and $\mu: \Gamma\Gamma \to \Gamma$ such that the following diagrams of $\mathcal{U}$-maps over B are commutative for each $\pi: E \to B$:

$$\begin{array}{ccccc} \Gamma E & \xrightarrow{\Gamma\eta} & \Gamma\Gamma E & \xleftarrow{\eta} & \Gamma E \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle\mu} & \diagup\!\!\diagup & \\ & & \Gamma E & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma\Gamma\Gamma E & \xrightarrow{\mu} & \Gamma\Gamma E \\ {\scriptstyle\Gamma\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \Gamma\Gamma E & \xrightarrow{\mu} & \Gamma E \end{array}$$

In categorical language, (Γ, μ, η) is a monad in the category of $\mathcal{U}$-spaces [17, 2.1]. A transitive lifting function ξ for π is a $\mathcal{U}$-map $\xi: \Gamma E \to E$ over B such that the following diagrams of $\mathcal{U}$-maps over B are commutative:

$$\begin{array}{ccc} E & \xrightarrow{\eta} & \Gamma E \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle\xi} \\ & & E \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma\Gamma E & \xrightarrow{\mu} & \Gamma E \\ {\scriptstyle\Gamma\xi}\downarrow & & \downarrow{\scriptstyle\xi} \\ \Gamma E & \xrightarrow{\xi} & E \end{array}$$

Thus the pair (π, ξ) is a Γ-algebra in the sense of [17, 2.2]. Moreover, by [17, 2.9], for any $\mathcal{U}$-space π, $(\Gamma\pi, \mu)$ is the free Γ-algebra generated by π.

The observation above was also noted by Malraison [15].

The rest of this section is devoted to the proof of the following generalization of Hurewicz's theorem [11].

<u>Theorem 3.8.</u> Let $\pi: E \to B$ be an $\mathcal{F}$-space and assume that B admits a numerable cover $\mathcal{C}$ such that $\pi: E_U \to U$ is an $\mathcal{F}$-fibration for each $U \in \mathcal{C}$. Then π is an $\mathcal{F}$-fibration. Therefore an $\mathcal{F}$-space

over a paracompact base space is an $\mathcal{J}$-fibration if and only if it is a local $\mathcal{J}$-fibration.

Proof. Our argument is a corrected version of Brown's modification [3] of Hurewicz's original proof. We shall construct an $\mathcal{J}$-lifting function ξ for π. For any finite ordered set $s = \{U_1, \ldots, U_n\}$ of not necessarily distinct sets in $\mathcal{C}$, define

$$W_s = \{\beta \mid \beta(t) \in U_i \text{ if } (i-1)\ell(\beta)/n \leq t \leq i\ell(\beta)/n\} \subset \Pi B.$$

By hypothesis, there exists an $\mathcal{J}$-lifting function $\xi_i: \Gamma E_{U_i} \to E_{U_i}$ for each i. For $0 \leq u < v \leq 1$ and a path $\beta \in \Pi B$, define the sub-path $\beta[u,v]$ of β by $\ell\,\beta[u,v] = (v-u)\ell(\beta)$ and

$$\beta[u,v](t) = \beta(t + u\ell(\beta)) .$$

Let $(i-1)/n \leq u < i/n$ and $(j-1)/n < v \leq j/n$ for integers $0 \leq i \leq j \leq n$. For $e \in \pi^{-1}\beta[u,v](0)$ and $\beta \in W_s$, define

$$(1) \qquad \xi_s(\beta[u,v], e) = \xi_j(\beta[\tfrac{j-1}{n}, v], \xi_{j-1}(\beta[\tfrac{j-2}{n}, \tfrac{j-1}{n}], \ldots$$

$$\ldots\ \xi_{i+1}(\beta[\tfrac{i}{n}, \tfrac{i+1}{n}], \xi_i(\beta[u, \tfrac{i}{n}], e)) \ \ldots\ )) .$$

Let $\lambda_i: B \to I$ be such that $\lambda_i^{-1}(0,1] = U_i$ and define $\lambda_s: \Pi B \to I$ by

$$\lambda_s(\beta) = \inf\{\lambda_i\beta(t) \mid (i-1)\ell(\beta)/n \leq t \leq i\ell(\beta)/n \text{ and } 1 \leq i \leq n\}.$$

Then $W_s = \lambda_s^{-1}(0,1]$. $\{W_s\}$ is a cover of ΠB, but it is not locally finite. Let $c(s) = n$ if s has n elements and note that $\{W_s \mid c(s) < n\}$ is a locally finite set for each fixed n. Define θ_n on ΠB by $\theta_n(\beta) = \sum_{c(s)<n} \lambda_s(\beta)$. Then define γ_s on ΠB by

$$\gamma_s(\beta) = \max(0, \lambda_s(\beta) - n\theta_n(\beta)) \quad \text{if} \quad c(s) = n.$$

Define $V_s = \{\beta \mid \gamma_s(\beta) > 0\} \subset W_s$. It is easily verified that $\{V_s\}$ is a locally finite cover of ΠB. Total order the set $\mathcal{S}$ of finite ordered sets of sets in $\mathcal{C}$. For $(\beta, e) \in \Gamma E$, define

(2) $$\xi(\beta, e) = \xi_{s_q}(\beta[t_{q-1}, t_q], \xi_{s_{q-1}}(\beta[t_{q-2}, t_{q-1}], \ldots, \xi_{s_1}(\beta[t_0, t_1], e)\ldots)),$$

where $s_1 < \ldots < s_q$ are all elements $s \in \mathcal{S}$ such that $\beta \in V_s$ and where $t_j = \sum_{i=1}^{j} \gamma_{s_i}(\beta) / \sum_{i=1}^{q} \gamma_{s_i}(\beta)$.

ξ is the desired lifting function. It is clear from (1) and (2) that ξ restricts to a finite composite of maps in $\mathcal{F}$ for each fixed β.

4. Categories of fibres

In order to classify $\mathcal{F}$-fibrations, we must of course place severe restrictions on the category $\mathcal{F}$. We here define the notion of a "category of fibres," which is essentially a category with just enough structure to allow the development of a theory of associated principal fibrations. Such a theory is essential in our approach to classification theorems and is an obvious desideratum of any general theory of fibrations.

We topologize $\mathcal{F}(X, X')$ as a subspace of the function space $\mathcal{U}(X,X')$, with the (compactly generated) compact-open topology.

Definition 4.1. Let $\mathcal{F}$ have a distinguished object F. Then $(\mathcal{F}, F)$ is said to be a category of fibres if every map in $\mathcal{F}$ is a weak homotopy equivalence, $\mathcal{F}(F, X)$ is non-empty for each $X \in \mathcal{F}$, and composition with ϕ

$$\mathcal{F}(1, \phi): \mathcal{F}(F,F) \rightarrow \mathcal{F}(F, X)$$

is a weak homotopy equivalence for each $\phi \in \mathcal{F}(F, X)$. $\mathcal{F}$ is said to be a homogeneous category of fibres if $(\mathcal{F}, F)$ is a category of fibres for every object F.

Recall that a topological monoid is an associative H-space G with a two-sided identity element e and that a left G-space is a space X with an associative and unital action map $G \times X \rightarrow X$. G is said to be grouplike if $\pi_0 G$ is a group under the product induced by that of G. This holds if each right translation map $g: G \rightarrow G$ induces an isomorphism on $\pi_0 G$, and then translation by g on any (left or right) G-space is necessarily a homotopy equivalence.

Definition 4.2. A category of fibres $(\mathcal{G}, G)$ is said to be principal if G is a grouplike topological monoid, each object $Y \in \mathcal{G}$ is a (non-empty) right G-space, and the space $\mathcal{G}(Y, Y')$ coincides with the space of right G-maps from Y to Y'. Identify the space $\mathcal{G}(G, Y)$ with Y via $\phi \Longleftrightarrow \phi(e)$ and note that $\mathcal{G}(1, \phi): G \to Y$ is given by $g \to \phi(e)g$ and is required to be a weak homotopy equivalence. This condition already implies that all maps in $\mathcal{G}$ are weak homotopy equivalences (as one sees by composing a map $Y \to Y'$ in $\mathcal{G}$ with any map $G \to Y$ in $\mathcal{G}$).

Observe that to specify a principal category of fibres, we need only specify an appropriate collection of right G-spaces.

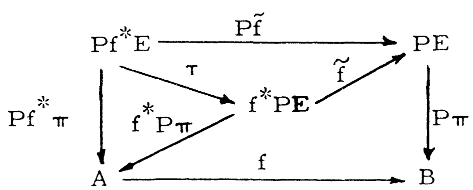
Definition 4.3. Let $(\mathcal{F}, F)$ be a category of fibres. Define the associated principal category of fibres $(\mathcal{G}, G)$ by letting $\mathcal{G}$ have objects $\mathcal{F}(F, X)$ for $X \in \mathcal{F}$, with $G = \mathcal{F}(F, F)$; the product on G and the action of G on $\mathcal{F}(F, X)$ are given by composition. For an $\mathcal{F}$-space $\pi: E \to B$, define a $\mathcal{G}$-space $P\pi: PE \to B$ by letting PE be the subspace of $\mathcal{U}(F, E)$ which consists of those maps $\psi: F \to E$ such that $\psi(F) \subset \pi^{-1}(b)$ for some $b \in B$ and $\psi: F \to \pi^{-1}(b)$ is a map in $\mathcal{F}$ and by letting $(P\pi)(\psi) = \pi\psi(F)$. For an $\mathcal{F}$-map $(g, f): \nu \to \pi$, $\nu: D \to A$, define a $\mathcal{G}$-map $P(g, f): P\nu \to P\pi$ by $P(g, f) = (Pg, f)$, where $(Pg)(\psi) = g \circ \psi$ for $\psi \in PD$. Then P is a functor from the category of $\mathcal{F}$-spaces to the category of $\mathcal{G}$-spaces.

The definition of P is based on ideas of Dold and Lashof [8], who called P Prin.

Remark 4.4. A principal category of fibres $(\mathcal{G}, G)$ may be identified with its associated principal category of fibres. Let $\pi: E \to B$ be a $\mathcal{G}$-space. Define a bijection of sets $\alpha: E \to PE$ by $\alpha(x)(g) = xg$ for $x \in E$ and $g \in G$; $\alpha^{-1}(\psi) = \psi(e)$ for $\psi: G \to E$ in PE. α^{-1} is always continuous and α is continuous provided that the given right actions of G on the fibres of π define a continuous function $E \times G \to E$. Henceforward, by a $\mathcal{G}$-space, we understand one for which $E \times G \to E$ is continuous; this is a reasonable restriction since, in the contrary case, we can retopologize E by requiring α to be a homeomorphism and so make the action continuous. With this convention, we can identify E and PE via α and regard P as the identity functor on $\mathcal{G}$-spaces.

The following pair of lemmas record obvious properties of the "associated principal $\mathcal{G}$-space" functor P for a fixed category of fibres $(\mathcal{F}, F)$.

Lemma 4.5. If $\pi: E \to B$ is an $\mathcal{F}$-space and $f: A \to B$ is a map, then there is a unique $\mathcal{G}$-map $\tau: Pf^*E \to f^*PE$ over A such that the following diagram is commutative:



<u>Lemma 4.6.</u> Let $\pi: E \to B$ be an $\mathcal{J}$-fibration with (transitive) $\mathcal{J}$-lifting function $\xi: \Gamma E \to E$. Then $P\pi: PE \to B$ is a $\mathcal{G}$-fibration with (transitive) $\mathcal{G}$-lifting function $P\xi: \Gamma PE \to PE$ defined by

$$(P\xi)(\beta, \psi)(x) = \xi(\beta, \psi(x)) \text{ , hence } (P\xi)(\beta, \psi) = \xi \circ \tilde{\beta} \circ \psi : F \to \pi^{-1} p(\beta) \text{ ,}$$

for $x \in F$, $\beta \in \Pi B$, and $\psi \in PE$ such that $\beta(0) = (P\pi)(\psi)$.

5. $\mathcal{F}$-quasifibrations and based fibres

Our explicit classifying space constructions will yield universal quasifibrations. Since pullbacks of quasifibrations need not be quasifibrations, we shall sometimes have to use the functor Γ to replace $\mathcal{F}$-quasifibrations, by which we understand $\mathcal{F}$-spaces $\pi: E \to B$ such that π is a quasifibration, by $\mathcal{F}$-fibrations. The following definition records the minimum amount of information that will suffice for this purpose.

Definition 5.1. A category of fibres $(\mathcal{F}, F)$ is Γ-**complete in a full** subcategory $\mathcal{Q}$ of $\mathcal{U}$ if $\mathcal{F} \subset \mathcal{Q}$, $\mathcal{H} \subset \mathcal{Q}$, and the following statements are valid for $\mathcal{F}$-quasifibrations $\pi: E \to B$ with B and E in $\mathcal{Q}$.

(1) $\Gamma\pi: \Gamma E \to B$ is a $\mathcal{F}$-fibration with $\mathcal{F}$-lifting function μ.

(2) $\eta: E \to \Gamma E$ is an $\mathcal{F}$-map over B.

(3) Γ takes $\mathcal{F}$-maps between $\mathcal{F}$-quasifibrations in $\mathcal{Q}$ to $\mathcal{F}$-maps.

Let $\mathcal{T}$ denote the category of nondegenerately based spaces in $\mathcal{U}$ and basepoint preserving maps. In some very important examples, the functor $\mathcal{F} \to \mathcal{U}$ factors through $\mathcal{T}$. In the definition just given, there is clearly no way to give the fibres of $\Gamma\pi$ basepoints such that each $\mu\tilde{\beta}: (\Gamma\pi)^{-1}\beta(0) \to (\Gamma\pi)^{-1}p(\beta)$ is basepoint preserving (since $\mu\tilde{\beta}(\alpha, e) = (\beta\alpha, e)$). We need a few more definitions in order to circumvent this difficulty.

Definition 5.2. When given a category $\mathcal{F}$ with a faithful functor $\mathcal{F} \to \mathcal{T}$, redefine an $\mathcal{F}$-space to be a map $\pi: E \to B$ such that not only is $\pi^{-1}(b)$ in $\mathcal{F}$ for all $b \in B$ but also the function $\sigma: B \to E$ specified by sending b to the basepoint of $\pi^{-1}(b)$ is continuous and is a fibrewise cofibration, in the sense that there exists a representation (h, u) of $(E,\sigma B)$ as an NDR-pair [17, A.1] such that $h: E \times I \to E$ is a $\mathcal{J}$-homotopy over B (so that (h, u) restricts to a representation of $(\pi^{-1}(b), \sigma b)$ as an NDR-pair). All other definitions and results obtained so far in this paper apply verbatim to these $\mathcal{F}$-spaces with a canonical cross-section (because $\mathcal{F}$-maps are automatically section preserving). We refer to a category of fibres $(\mathcal{F}, F)$ such that $\mathcal{F}$ maps faithfully to $\mathcal{J}$ as a category of based fibres.

Definition 5.3. Let $\pi: E \to B$ be a $\mathcal{J}$-space. Define a $\mathcal{J}$-space $\Gamma'\pi: \Gamma'E \to B$ and maps $\eta': E \to \Gamma'E$ and $\mu': \Gamma\Gamma'E \to \Gamma'E$ as follows. $\Gamma'E$ is obtained by growing a long whisker on each fibre of $\Gamma\pi$. Formally, $\Gamma'E$ is the quotient space obtained from the disjoint union of ΓE and $B \times [0, \infty]$ by identifying $(b, \sigma b) \in \Gamma E$ with $(b, 0) \in B \times [0, \infty]$ for each $b \in B$. $\Gamma'\pi$ coincides with $\Gamma\pi$ on ΓE and with the projection to the first coordinate on $B \times [0, \infty]$. The cross-section $\Gamma\sigma: B \to \Gamma'E$ is defined by $(\Gamma\sigma)(b) = (b, \infty)$ and is clearly a fibrewise cofibration. With (h, u) as in the previous definition, define η' by

$$\eta'(e) = \begin{cases} (\pi(e), 1/u(e) - 2) & \text{if } 0 \leq u(e) \leq 1/2 \\ (\pi(e), h(e, 2 - 2u(e)) & \text{if } 1/2 \leq u(e) \leq 1 \end{cases}$$

Then η' is a $\mathcal{J}$-map over B and a homotopy equivalence. Define μ' by $\mu' = \mu$ on $\Gamma\Gamma E \subset \Gamma\Gamma'E$ and by

$$\mu'(\beta, (b, t)) = \begin{cases} (\beta'_t, \sigma\beta'_t(0)) & \text{if } 0 \leq t \leq \ell(\beta) \\ (p(\beta), t - \ell(\beta)) & \text{if } \ell(\beta) \leq t \leq \infty \end{cases}$$

where $\beta(0) = b$, $\ell(\beta_t') = \ell(\beta) - t$, and $\beta_t'(s) = \beta(s+t)$. Then μ' is easily verified to be a $\mathcal{J}$-lifting function for $\Gamma'\pi$. With the evident definition on $\mathcal{J}$-maps, Γ' becomes a functor from $\mathcal{J}$-spaces to $\mathcal{J}$-fibrations.

Definition 5.4. A category of based fibres $(\mathcal{F}, F)$ is Γ'-complete in a full subcategory $\mathcal{Q}$ of $\mathcal{U}$ if $\mathcal{F} \subset \mathcal{Q}$, $\mathcal{H} \subset \mathcal{Q}$, and the following statements are valid for $\mathcal{F}$-quasifibrations $\pi: E \to B$ with B and E in $\mathcal{Q}$.

(1) $\Gamma'\pi: \Gamma'E \to B$ is an $\mathcal{F}$-fibration with $\mathcal{F}$-lifting function μ'.

(2) $\eta': E \to \Gamma'E$ is an $\mathcal{F}$-map over B.

(3) Γ' takes $\mathcal{F}$-maps between $\mathcal{F}$-quasifibrations in $\mathcal{Q}$ to $\mathcal{F}$-maps.

A five lemma argument gives the following observation.

Lemma 5.5. Let $(\mathcal{F}, F)$ be Γ-complete (or Γ'-complete) in $\mathcal{Q}$ and let $\pi: E \to B$ be an $\mathcal{F}$-quasifibration with B and E in $\mathcal{Q}$. If $P\pi: PE \to B$ is again a quasifibration, then the $\mathcal{H}$-map $P\eta: PE \to P\Gamma E$ (or $P\eta': PE \to P\Gamma'E$) over B is a weak homotopy equivalence.

We record the following remarks for use in [19].

Remarks 5.6. Let $\nu: D \to A$ and $\pi: E \to B$ be $\mathcal{J}$-spaces. Define a $\mathcal{J}$-space $\nu \wedge \pi: D \wedge E \to A \times B$, the fibrewise smash product of ν and π, as follows. Let $D \wedge E = D \times E/(\approx)$, where the equivalence identifies the wedge $(\sigma a, \pi^{-1}b) \vee (\nu^{-1}a, \sigma b)$ to the point $(\sigma a, \sigma b)$ for each $(a, b) \in A \times B$, and let $\nu \wedge \pi$ be induced from $\nu \times \pi$; the cross-section of $\nu \wedge \pi$ is induced from $\sigma \times \sigma$. If ν and π are $\mathcal{J}$-fibrations, then so is $\nu \wedge \pi$ since $\nu \wedge \pi$ clearly inherits the $\mathcal{J}$-CHP from ν and π. There is a natural $\mathcal{J}$-map $g: \Gamma'D \wedge \Gamma'E \to \Gamma'(D \wedge E)$ over $A \times B$ specified for $(\alpha, d) \in \Gamma D$,

$(\beta, e) \in \Gamma E$, $(a, s) \in A \times [0, \infty]$ and $(b, t) \in B \times [0, \infty]$ by

$$g((\alpha, d) \wedge (\beta, e)) = (\alpha \times \beta, d \wedge e)$$

$$g((a, s) \wedge (b, t)) = (a \times b, \max(s, t))$$

$$g((\alpha, d) \wedge (b, t)) = \begin{cases} (\alpha'_t \times b, \sigma\alpha'_t(0) \wedge \sigma b) & \text{if } t \leq \ell(\alpha) \\ (p\alpha \times b, t - \ell(\alpha)) & \text{if } \ell(\alpha) \leq t \end{cases}$$

and

$$g((a, s) \wedge (\beta, e)) = \begin{cases} (a \times \beta'_s, \sigma a \wedge \sigma\beta'_s(0)) & \text{if } s \leq \ell(\beta) \\ (a \times p\beta, s - \ell(\beta)) & \text{if } \ell(\beta) \leq s \end{cases}$$

where $\ell(\alpha'_t) = \ell(\alpha) - t$, $\alpha'_t(u) = \alpha(t+u)$, $\ell(\beta'_s) = \ell(\beta) - s$, and $\beta'_s(u) = \beta(s+u)$. It is not hard to see that g restricts to a weak homotopy equivalence on each fibre if ν and π are quasifibrations with connected fibres.

6. Examples of categories of fibres

We here define the functors to which our classification theorem will apply (in favorable cases) and discuss various examples of categories of fibres.

Definition 6.1. Let $A \in \mathcal{U}$. Define $\mathcal{E}\mathcal{J}(A)$ to be the collection (assumed to be a set) of equivalence classes of $\mathcal{J}$-fibrations over A under the equivalence relation generated by the $\mathcal{J}$-maps over A. For a map $f: A \to A'$, define $f^*: \mathcal{E}\mathcal{J}(A') \to \mathcal{E}\mathcal{J}(A)$ by $f^*\{\nu\} = \{f^*\nu\}$, where $\{\nu\}$ denotes the equivalence class of ν. By Proposition 2.5, $\mathcal{E}\mathcal{J}$ is a contravariant functor from the homotopy category of $\mathcal{U}$ to the category of sets. By Lemmas 4.5 and 4.6, P induces a natural transformation $\mathcal{E}\mathcal{J} \to \mathcal{E}\mathcal{B}$ when $(\mathcal{J}, F)$ is a category of fibres. By Corollary 3.5, any functor $\mathcal{J} \to \mathcal{J}'$ over $\mathcal{U}$ induces a natural transformation $\mathcal{E}\mathcal{J} \to \mathcal{E}\mathcal{J}'$.

Note that the assumption that our equivalence relation leads to a set of equivalence classes is non-trivial. It will hold in our classification theorem because our constructive proof will display a set of $\mathcal{J}$-fibrations over A such that any given $\mathcal{J}$-fibration over A is equivalent to an element of the displayed set.

At first sight, our choice of equivalence relation may seem less natural than the obvious (and more restrictive) one of $\mathcal{J}$-homotopy equivalence. In the classical examples, every map in $\mathcal{J}$ is an $\mathcal{J}$-homotopy equivalence. In such cases, Theorem 2.6 ensures that our equivalence relation coincides with $\mathcal{J}$-homotopy equivalence (over good base spaces).

In the contrary case, it is very hard to verify that a given $\mathcal{F}$-map is in fact an $\mathcal{F}$-homotopy equivalence. Our equivalence relation allows us to ignore this problem and to freely use arbitrary $\mathcal{F}$-maps over A. It is this freedom which enables us to avoid both local pasting arguments and higher homotopies.

We now turn to examples. We first consider the principal case.

Examples 6.2. Let G be a grouplike topological monoid. Specify four successively smaller categories $\mathcal{Y}$ such that $(\mathcal{Y}, G)$ is a principal category of fibres by letting a right G-space Y be an object of $\mathcal{Y}$ if and only if the maps $\tilde{y}: G \rightarrow Y$ defined for $y \in Y$ by $\tilde{y}(g) = yg$ are all

(i) weak homotopy equivalences; write $(\mathcal{Y}, G) = G\mathcal{U}$.

(ii) homotopy equivalences; write $(\mathcal{Y}, G) = G\mathcal{W}$.

(iii) G-equivariant homotopy equivalences.

(iv) homeomorphisms, where G is a topological group.

Let $\mathcal{W}$ denote the full subcategory of $\mathcal{U}$ of spaces having the homotopy type of CW-complexes. Clearly, (i) and (ii) are appropriate to $\mathcal{U}$ and $\mathcal{W}$, respectively, but are conceptually similar. Case (iii) is most refractory, and we shall not study it. The point is that, in general, there is no effective way of telling when a G-equivariant map which is a homotopy equivalence is a G-equivariant homotopy equivalence. In particular, we have no analog of the following lemma.

Lemma 6.3. Let G be a grouplike topological monoid. Then $G\mathcal{U}$ is Γ-complete in $\mathcal{U}$ and, if $G \in \mathcal{W}$, $G\mathcal{W}$ is Γ-complete in $\mathcal{W}$.

Proof. Write $\mathcal{G}$ for $G\mathcal{U}$ or $G\mathcal{W}$ and let $\pi: E \to B$ be a $\mathcal{G}$-quasifibration, with B and E in $\mathcal{U}$ or $\mathcal{W}$. Via $(\beta, e)g = (\beta, eg)$, the right action of G on E induces a right action of G on ΓE such that $\eta: E \to \Gamma E$, $\mu: \Gamma\Gamma E \to \Gamma E$, and the $\tilde{\beta}$ are all maps of right G-spaces. We must show that $(\Gamma\pi)^{-1}(b) \in \mathcal{G}$ for all $b \in B$. Let $y = (\beta, x)$ be a typical point in $(\Gamma\pi)^{-1}(b)$, so that $\beta(0) = \pi(x)$ and $\rho(\beta) = b$. We must verify that the map $\tilde{y}: G \to (\Gamma\pi)^{-1}(b)$ is a weak homotopy equivalence; when $\mathcal{G} = G\mathcal{W}$ and $G \in \mathcal{W}$, the Whitehead theorem and the theorem of Stasheff [32] quoted just below will imply that $(\Gamma\pi)^{-1}(b) \in \mathcal{W}$ and will thus complete the proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \pi^{-1}\beta(0) & & \\
 & \overset{\tilde{x}'}{\nearrow} & & \overset{\eta}{\searrow} & \\
G & & \Big\downarrow \tilde{\beta} & & (\Gamma\pi)^{-1}\beta(0), \\
 & \underset{\tilde{y}}{\searrow} & & \underset{\bar{\beta}}{\swarrow} & \\
 & & (\Gamma\pi)^{-1}(b) & &
\end{array}$$

where $\bar{\beta}(\gamma, w) = (\beta\gamma, w)$ for $\gamma \in \Pi B$ and $w \in E$ such that $\gamma(0) = \pi(w)$ and $p\gamma = \beta(0)$. Since $\eta, \bar{\beta}$, and $\tilde{x}$ are weak homotopy equivalences by Remarks 3.6, Proposition 2.3, and hypothesis, $\tilde{\beta}$ and $\tilde{y}$ are also weak homotopy equivalences.

The required result of Stasheff[1] can be stated as follows.

Theorem 6.4. Let $\nu : D \to A$ be a fibration with $A \in \mathcal{W}$. Then

(i) $D \in \mathcal{W}$ if and only if $\nu^{-1}(a) \in \mathcal{W}$ for all $a \in A$; and

(ii) If $A' \in \mathcal{W}$, $f: A' \to A$ is a map, and $D \in \mathcal{W}$, then $f^*D \in \mathcal{W}$.

1. The proof in [32] is not correct, but can be patched.

For the next example, recall the following standard results about the relationship between $\mathcal{U}$ and $\mathcal{W}$ (e.g. [21, 2.9 and 3.10]). Let [X, Y] denote the set of homotopy classes of maps $X \to Y$.

Theorem 6.5. (i) If $\phi: Y \to Z$ is a weak homotopy equivalence and $X \in \mathcal{W}$, then $\phi_*: [X, Y] \to [X, Z]$ is an isomorphism.

(ii) There are a functor and natural transformation on the homotopy category of $\mathcal{U}$ which assign a CW-complex X' and a homotopy class of weak homotopy equivalences $X' \to X$ to a space X.

Example 6.6. Let $F \in \mathcal{W}$. Define two categories of fibres $F\mathcal{U}$ and $F\mathcal{W}$ with distinguished object F as follows.

(i) $X \in F\mathcal{U}$ if X is of the same weak homotopy type as F; the maps in $F\mathcal{U}$ are the weak homotopy equivalences $X \to X'$.

(ii) $X \in F\mathcal{W}$ if X is of the same homotopy type as F; the maps in $F\mathcal{W}$ are the homotopy equivalences $X \to X'$; thus $F\mathcal{W} = \mathcal{W} \cap F\mathcal{U}$.

In (i), $F\mathcal{U}(F, X)$ is non-empty by Theorem 6.5(ii) and the fact that $F \in \mathcal{W}$. For $\phi: F \to X$ in $F\mathcal{U}$ and any CW-complex K,

$$\phi_*: [K \times F, F] \to [K \times F, X]$$

is an isomorphism, by Theorem 6.5(i), and therefore

$$\mathcal{U}(1, \phi)_*: [K, \mathcal{U}(F, F)] \to [K, \mathcal{U}(F, X)]$$

is an isomorphism. Since $F\mathcal{U}(F, F)$ and $F\mathcal{U}(F, X)$ are unions of components of $\mathcal{U}(F, F)$ and $\mathcal{U}(F, X)$, because a map homotopic to a weak homotopy equivalence is a weak homotopy equivalence, it follows that $F\mathcal{U}(1, \phi)$ is a weak homotopy equivalence and thus that (i) and (ii) do indeed define categories of fibres.

Recall the following result of Milnor [25].

Theorem 6.7. If $X \in \mathcal{W}$ and $C \in \mathcal{U}$ is compact, then $\mathcal{U}(C, X) \in \mathcal{W}$.

For this reason, case (ii) is adequate when F is compact. When F is not compact, for example when F is a localization or completion of the n-sphere at a set of primes, we shall have to use fibres not in $\mathcal{W}$, as allowed in case (i).

Theorem 6.4 and a proof similar to, but simpler than, that of Lemma 6.3 give the following result.

Lemma 6.8. Let $F \in \mathcal{W}$. Then $F\mathcal{U}$ is Γ-complete in $\mathcal{U}$ and, if F is compact, $F\mathcal{W}$ is Γ-complete in $\mathcal{W}$.

There is a based variant of the preceding example. Let $\mathcal{V} = \mathcal{T} \cap \mathcal{W}$, the category of nondegenerately based spaces in $\mathcal{W}$, and recall that Theorems 6.5 and 6.7 remain valid if all maps and homotopies in sight are required to preserve basepoints.

Example 6.9. Let $F \in \mathcal{V}$. Define two categories of based fibres $F\mathcal{T}$ and $F\mathcal{V}$ with distinguished object F as follows.

(i) $X \in F\mathcal{T}$ if $X \in \mathcal{T}$ is of the same based weak homotopy type as F; the maps in $F\mathcal{T}$ are the based weak homotopy equivalences.

(ii) $X \in F\mathcal{V}$ if $X \in \mathcal{V}$ is of the same based homotopy type as F; the maps in $F\mathcal{V}$ are the based homotopy equivalences; thus $F\mathcal{V} = \mathcal{V} \cap F\mathcal{T}$.

Lemma 6.10. Let $F \in \mathcal{V}$. Then $F\mathcal{J}$ is Γ'-complete in $\mathcal{U}$ and, if F is compact, $F\mathcal{V}$ is Γ'-complete in $\mathcal{N}$.

Note that we impose basepoints only on fibres, not on base spaces or total spaces.

Example 6.11. Let G be a topological group and let F be a left G-space on which G acts effectively. Define a category $\mathcal{F}$ as follows. Let $\mathcal{F}$ have objects all pairs (X, x) such that X is a left G-space and $x: F \to X$ is a homeomorphism of left G-spaces. Let the set of morphisms from (X, x) to (X', x') be $\{x'gx^{-1} | g \in G\}$, with the evident operation of composition. $\mathcal{F}$ has the distinguished object (F, 1), and we call $(\mathcal{F}, F)$ a category of bundle fibres. If $(\mathcal{G}, G)$ is the associated principal category of fibres, then G is the given group retopologized with its possibly coarser topology as a subspace of $\mathcal{U}(F, F)$; we insist that G, so topologized, again be a topological group. Of course, in practice, the two topologies usually agree. By Theorem 3.8, a Steenrod fibre bundle with group G (with either topology) and fibre F which is trivial over each set of a numerable cover of its base space is an $\mathcal{F}$-fibration. Following Dold [7], we say that such a bundle is numerable.

7. The geometric bar construction

We here review the definition and properties of the two-sided geometric bar construction introduced in [17, §9-11]. Let G be a topological monoid such that its identity element e is a strongly nondegenerate base-point (in the sense that (G, e) is a strong NDR-pair [17, A.1]). Let X and Y be left and right G-spaces. Define a simplicial topological space $B_*(Y, G, X)$ by letting the space of j-simplices be $Y \times G^j \times X$, with typical elements written in the form $y[g_1, \ldots, g_j]x$, and letting the face and degeneracy operators be given by

$$\partial_i(y[g_1, \ldots, g_j]x) = \begin{cases} yg_1[g_2, \ldots, g_j]x & \text{if } i = 0 \\ y[g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_j]x & \text{if } 1 \leq i < j \\ y[g_1, \ldots, g_{j-1}]g_jx & \text{if } i = j \end{cases}$$

and $$s_i(y[g_1, \ldots, g_j]x) = y[g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_j]x \quad .$$

Let B(Y, G, X) denote the geometric realization of $B_*(Y, G, X)$, as defined in [17, 11.1]. Then B is a functor to $\mathcal{U}$ from the category $\mathcal{A}(\mathcal{U})$ of triples (Y, G, X); the morphisms of $\mathcal{A}(\mathcal{U})$ are triples $(k, f, j): (Y, G, X) \to (Y', G', X')$ where $f: G \to G'$ is a map of topological monoids and $j: X \to X'$ and $k: Y \to Y'$ are f-equivariant maps, $j(gx) = f(g)j(x)$ and $k(yg) = k(y)f(g)$. The functor B was first defined (implicitly) by Stasheff [34]. Let * denote the one-point G-space and define

$$BG = B(*, G, *) \quad \text{and} \quad EG = B(*, G, G).$$

BG is the standard classifying space of G, namely the normalized version of the Dold-Lashof [8] construction, as defined by Stasheff [31, p. 289], exploited by Milgram [23], and analyzed in detail by Steenrod [38].

Many of the results of this section and the next are due to the authors cited above, but our explicit use of simplicial spaces simplifies nearly all of the proofs by reducing them to trivial verifications on the level of simplicial spaces followed by quotations of general results about geometric realization. The following series of propositions give the basic facts about the topological behavior of the functor B.

Proposition 7.1. $B_*(Y, G, X)$ is a proper simplicial space. $B(Y, G, X)$ is n-connected if G is (n-1)-connected and X and Y are n-connected.

Proof. The first statement means that $(Y, \emptyset) \times (G, e)^j \times (X, \emptyset)$ is a strong NDR-pair (where $\emptyset$ is the empty set) and holds by [17, A.3]. The second statement follows by [17, 11.12] (its extra hypothesis of strict propriety being unnecessary by [18, A.5]).

Now [18, A.6 and A.4] imply the following two results.

Proposition 7.2. If Y, G, and X are in $\mathcal{W}$, then so is $B(Y, G, X)$.

Proposition 7.3. Let $(k, f, j): (Y, G, X) \to (Y', G', X')$ be a morphism in $\mathcal{A}(\mathcal{U})$.

(i) If k, f, and j induce isomorphisms on integral homology, then so does $B(k, f, j)$.

(ii) If k, f, and j are homotopy equivalences, then so is B(k, f, j).

Note in (ii) that no equivariance conditions are required of the given homotopy inverses and homotopies.

Since B_* preserves products by [17, 10.1] and geometric realization preserves products by [17, 11.5], the following result holds.

Proposition 7.4. For (Y, G, X) and (Y', G', X') in $\mathcal{Q}(\mathcal{U})$, the projections define a natural homeomorphism

$$B(Y \times Y', G \times G', X \times X') \to B(Y, G, X) \times B(Y', G', X').$$

We shall often write

$$\tau = \tau(\rho): Z \to B(Y, G, X) \quad \text{and} \quad \varepsilon = \varepsilon(\lambda): B(Y, G, X) \to Z$$

for the maps induced via [17, 9.2 and 11.8] from a map $\rho: Z \to Y \times X$ and from a map $\lambda: Y \times X \to Z$ such that $\lambda(yg, x) = \lambda(y, gx)$; the intended choice of ρ and λ should be clear from the context. Clearly ε factors through $Y \times_G X$, the quotient of $Y \times X$ by the equivalence relation generated by $(yg, x) \approx (y, gx)$. Note that B(G, G, X) is a left G-space (again, because realization preserves products). The following result is a consequence of [17, 9.8, 9.9, and 11.10].

Proposition 7.5. $\varepsilon: B(G, G, X) \to X$ is a map of left G-spaces and a strong deformation retraction (with right inverse τ). The symmetric conclusion holds for $\varepsilon: B(Y, G, G) \to Y$.

We shall always write

$$p: B(Y,G,X) \to B(Y,G,*) \quad \text{and} \quad q: B(Y,G,X) \to B(*,G,X)$$

for the maps induced from the trivial G-maps $X \to *$ and $Y \to *$.

Theorem 7.6. If G is grouplike, then p and q are quasi-fibrations.

Proof. Consider p, the case q being handled symmetrically. As realizations of simplicial spaces, $B(Y,G,*)$ and $B(Y,G,X)$ are filtered spaces [17, 11.1] and $F_jB(Y,G,X) = p^{-1}F_jB(Y,G,*)$. Visibly,

$$F_0B(Y,G,X) = Y \times X \text{ and, if } j > 0,$$

$$F_j(B(Y,G,X) - F_{j-1}B(Y,G,X) = (F_jB(Y,G,*) - F_{j-1}B(Y,G,*)) \times X.$$

By [17, A.3 and A.4], any representations of (G,e) and $(\Delta_j, \partial\Delta_j)$ as strong NDR-pairs determine a representation (k,v) of $(G,e)^j \times (\Delta_j, \partial\Delta_j)$ as a strong NDR-pair. Together with the obvious representations of $(X, \emptyset)$ and $(Y, \emptyset)$ as strong NDR-pairs (namely, the constant homotopies and the trivial maps onto $\{1\} \subset I$), (k,v) determines representations (h,u) and (H, up) of

$$(F_jB(Y,G,*), F_{j-1}B(Y,G,*) \text{ and } (F_jB(Y,G,X), F_{j-1}B(Y,G,X))$$

as strong NDR-pairs such that H covers h. Let $U = u^{-1}[0,1)$. Then h restricts to a deformation of U onto $F_{j-1}B(Y,G,*)$. (It is for this that strong NDR-pairs, rather than just NDR-pairs, are needed.) By results of Dold and Thom [9] (as formulated in [17, 7.2]), it suffices to verify that, for all $z \in U$, $H_1: p^{-1}(z) \to p^{-1}h_1(z)$ is a weak homotopy equivalence. If $z \in F_{j-1}B(Y,G,*)$, H_1 is the identity. Thus let $z = |y[g_1, \ldots, g_j], a| \in U$, where $g_k \in G - \{e\}$ and $a \in \Delta_j - \partial\Delta_j$, and let $|y'[g'_1, \ldots, g'_i], a'|$, $i < j$, be the non-degenerate representative for $h_1(z)$ [17, 11.3]. Since G is grouplike, it suffices to show that there exists $g \in G$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\iota} & & \downarrow{\iota'} \\ p^{-1}(z) & \xrightarrow{H_1} & p^{-1}h_1(z) \end{array}$$

commutes, where ι and ι' are the homeomorphisms

$$\iota(x) = |y[g_1, \ldots, g_j]x, a| \quad \text{and} \quad \iota'(x) = |y'[g_1', \ldots, g_i']x, a'|.$$

Let $k_1(g_1, \ldots, g_j, a) = (g_1'', \ldots, g_j'', a'')$. The reduction of the point $H_1\iota(x) = |y[g_1'', \ldots, g_j'']x, a''|$ to non-degenerate form by use of [17, 11.3] will yield a point $\iota'(gx)$, where g results from last face operators and is independent of x since the particular face and degeneracy operators required for the reduction are independent of x.

$p: EG \to BG$ should be thought of as the universal $G\mathcal{U}$-quasifibration. The following corollary asserts its essential uniqueness.

Corollary 7.7. Let G be grouplike and let $p': E' \to B'$ be a $G\mathcal{U}$-quasifibration such that E' is aspherical. Then the maps ε and q are weak homotopy equivalences in the following commutative diagram:

$$\begin{array}{ccccc} E' & \xleftarrow{\varepsilon} & B(E', G, G) & \xrightarrow{q} & EG \\ \downarrow{p'} & & \downarrow{p} & & \downarrow{p} \\ B' & \xleftarrow{\varepsilon} & B(E', G, *) & \xrightarrow{q} & BG \end{array}$$

$p: B(Y,G,X) \to B(Y,G,*)$ should be thought of as the quasifibration with fibre X associated to the principal quasifibration $p: B(Y,G,G) \to B(Y,G,*)$. According to the following result, it can be thought of as classified by q.

<u>Proposition 7.8.</u> Let $(k,f,1): (Z,H,X) \to (Y,G,X)$ be a morphism in $\mathcal{Q}(\mathcal{U})$. Then the following diagrams are pullbacks:

$$\begin{array}{ccc} B(Z,H,X) & \xrightarrow{B(k,f,1)} & B(Y,G,X) \\ \downarrow p & & \downarrow p \\ B(Z,H,*) & \xrightarrow{B(k,f,1)} & B(Y,G,*) \end{array} \quad \text{and} \quad \begin{array}{ccc} B(Y,G,X) & \xrightarrow{q} & B(*,G,X) \\ \downarrow p & & \downarrow p \\ B(Y,G,*) & \xrightarrow{q} & BG \end{array}$$

<u>Proof.</u> The second diagram is the case $k: Y \to *$ and $f = 1$ of the first. Since geometric realization preserves pullbacks [17, 11.6], the result follows from the observation that the diagrams

$$\begin{array}{ccc} Z \times H^j \times X & \xrightarrow{k \times f^j \times 1} & Y \times G^j \times X \\ \downarrow p & & \downarrow p \\ Z \times H^j & \xrightarrow{k \times f^j} & Y \times G^j \end{array}$$

are pullbacks for all $j \geq 0$.

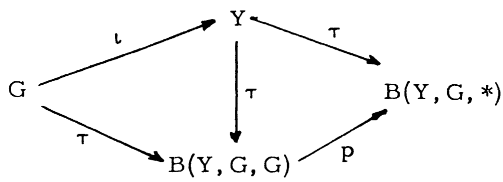
<u>Proposition 7.9.</u> If G is grouplike and Y is a right G-space, then

$$G \xrightarrow{\iota} Y \xrightarrow{\tau} B(Y,G,*) \xrightarrow{q} BG$$

is a quasifibration sequence, where $\iota(g) = y_0 g$ for any chosen $y_0 \in Y$.

<u>Proof.</u> We must show that τ is equivalent to a quasifibration, with ι equivalent to the inclusion of the fibre. Consider the following diagram,

in which the maps τ with range $B(Y, G, G)$ are induced from the maps $G \to Y \times G$ and $Y \to Y \times G$ specified by $g \to (y_0, g)$ and $y \to (y, e)$:



The right triangle commutes and the left triangle homotopy commutes via the homotopy $h(g, t) = |y_0[g]e, (t, 1-t)|$.

Our final result strengthens the analogy with bundle theory.

Proposition 7.10. Let $(\mathcal{F}, F)$ be a category of fibres with associated principal category of fibres $(\mathcal{H}, G)$. Let $f: H \to G$ be a map of topological monoids and let Y be a right H-space. Then there is a homeomorphism

$$\alpha: B(Y, H, G) \to PB(Y, H, F)$$

of $\mathcal{G}$-spaces over $B(Y, H, *)$. In particular, if H is grouplike, $Pp: PB(Y, H, F) \to B(Y, H, *)$ is a quasifibration.

Proof. $\alpha|y[h_1, \ldots, h_j]g, a|(f) = |y[h_1, \ldots, h_j]g(f), a|$ for $y \in Y$, $h_i \in H$, $g \in G = \mathcal{F}(F, F)$, $a \in \Delta_j$, and $f \in F$. α^{-1} is given by

$$\alpha^{-1}(\psi) = |y[h_1, \ldots, h_j]g, a|,$$

where $(Pp)(\psi) = |y[h_1, \ldots, h_j], a|$, in non-degenerate form, and where $g: F \to F$ is defined by $\psi(f) = |y[h_1, \ldots, h_j]g(f), a|$.

8. Groups, homogeneous spaces, and Abelian monoids

We here give special properties of BG and EG when G is a topological group or Abelian topological monoid. We also develop a generalized concept of "homogeneous space" for use in section 10.

The following theorem is due to Steenrod [38].

Theorem 8.1. Let G be a topological group. Then EG admits a natural structure of topological group such that the following statements hold.

(i) G is a closed subgroup of EG and the action $EG \times G \to EG$ agrees with the product in EG.

(ii) BG is the homogeneous space EG/G of right cosets and $p: EG \to BG$ is the natural projection.

(iii) The natural homeomorphism $E(G \times G') \to EG \times EG'$ is an isomorphism of groups when G and G' are groups.

Proof. A product-preserving functor D_* from spaces to simplicial spaces was constructed in [17, 10.2]; of course, D_* necessarily takes topological groups to simplicial topological groups. A homeomorphism $\alpha_*: E_*G \to D_*G$ of simplicial right G-spaces was defined in [17, 10.3]. Thus, by [17, 11.7], EG inherits a natural structure of topological group from $|D_*G|$. $G = D_0G$, and (i) and (ii) hold by inspection of [17, 10.2 and 10.3]. Part (iii) is clear.

Steenrod's construction of a group structure on EG is rather different from ours, and I have not tried to compare definitions. The following theorem is an improvement due to McCord [22, § 4] of a result due to Milgram and Steenrod [23, 38].

Theorem 8.2. Let G be a topological group (with identity element a nondegenerate basepoint). Then $p: EG \to BG$ is a numerable principal G-bundle.

Proof. Write $E = EG$ and $E_n = F_n EG$. The representation of (E_j, E_{j-1}) as an NDR-pair defined in the proof of Theorem 7.6 is G-equivariant. As observed by Steenrod [38, 4.2], [37, 7.1 and 9.4] imply that each (E, E_n) also admits a representation, (h_n, u_n) say, as a G-NDR pair. For $n \geq 0$, define a G-map $\rho_n: E \to I$ (where G acts trivially on I) by

$$\rho_0(x) = 1 - u_0(x) \text{ and, if } n > 0, \quad \rho_n(x) = (1-u_n(x))u_{n-1}(h_n(x, 1)).$$

Let $r_i: E \to E$ be the G-map defined by $r_i(x) = h_i(x, 1)$. Then

$$E_0 \subset \rho_0^{-1}(0, 1] \subset r_0^{-1} E_0 \text{ and, if } n > 0, \ E_n - E_{n-1} \subset \rho_n^{-1}(0, 1] \subset r_n^{-1}(E_n - E_{n-1}).$$

Define further G-maps $\pi_n: E \to I$, $n \geq 0$, by

$$\pi_n(x) = \max\,(0, \rho_n(x) - n \sum_{i=0}^{n-1} \rho_i(x))$$

and define $W_n = \pi_n^{-1}(0, 1]$ and $V_n = pW_n \subset BG$. Then $\{V_n\}$ is a numerable open cover of BG. We have a G-homeomorphism

$$E_0 \cong F_0 BG \times G \text{ or, if } n > 0, \quad E_n - E_{n-1} \cong (F_n BG - F_{n-1} BG) \times G,$$

and we define $\gamma_n: W_n \to G$ to be the composite of $r_0: W_0 \to E_0$ or, if $n > 0$, $r_n: W_n \to E_n - E_{n-1}$ and the second coordinate of this homeomorphism. Define $\xi_n: W_n \times G \to W_n$ by

$$\xi_n(y, g) = y\gamma_n(y)^{-1} g\,.$$

Then ξ_n induces a map $\zeta_n : V_n \times G \to W_n$, and ζ_n is a homeomorphism with inverse $p \times \gamma_n$ by direct calculation. Now $p : EG \to BG$ is a principal G-bundle by [36, 7.4], since the product structure on W_0 gives a local cross-section of G in EG, and the result is proven.

The theorem and Proposition 7.8 give the following result.

Corollary 8.3. For any right G-space Y, $p : B(Y, G, G) \to B(Y, G, *)$ is a principal G-bundle classified by $q : B(Y, G, *) \to BG$.

We can see the following complement in two ways.

Corollary 8.4. For any right G-space Y and left G-space F on which G acts effectively, $p : B(Y, G, F) \to B(Y, G, *)$ is the G-bundle with fibre F associated to $p : B(Y, G, G) \to B(Y, G, *)$.

Proof. On the one hand, it is evident that

$$B(Y, G, F) = B(Y, G, G) \times_G F .$$

On the other hand, if $H = G$ and $PB(Y, G, F)$ is retopologized as the classical associated principal bundle in Proposition 7.10, then the map α displayed there is an equivalence of principal G-bundles.

In a sense, every bundle arises in this fashion.

Proposition 8.5. Let G act principally from the right on Y and effectively from the left on F. Then the following diagram is a pullback in which the maps ε are weak homotopy equivalences (and $\delta : F \to *$ is the trivial map):

$$\begin{array}{ccc} B(Y,G,F) & \xrightarrow{\varepsilon} & Y \times_G F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1 \times_G \delta} \\ B(Y,G,*) & \xrightarrow{\varepsilon} & Y \times_G * \end{array}$$

Proof. The bottom map ε is a weak homotopy equivalence by the case $F = G$ of the diagram and Proposition 7.5, and the diagram implies that the top map ε is also a weak homotopy equivalence. By [17, 9.2, 11.8, and 11.6], it remains to verify that the following diagram is a pullback for $j \geq 0$:

$$\begin{array}{ccc} Y \times G^j \times F & \xrightarrow{\varepsilon} & Y \times_G F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1 \times_G \delta} \\ Y \times G^j & \xrightarrow{\varepsilon} & Y \times_G * \end{array}$$

Write $\{y, x\}$ for the image of $(y, x) \in Y \times F$ in $Y \times_G F$. The map from $Y \times G^j \times F$ into the fibred product of the bottom map ε and $1 \times_G \delta$ specified by

$$(y, g_1, \ldots, g_j, x) \to ((y, g_1, \ldots, g_j), \{y, g_1 \ldots g_j x\})$$

is a homeomorphism with inverse specified by

$$((y, g_1, \ldots, g_j), \{y', x'\}) \rightarrow (y, g_1, \ldots, g_j, g_j^{-1} \ldots g_1^{-1} g x'),$$

where g is the unique element of G such that $y' = yg$.

When $Y = G'$, where G is a closed subgroup with a local cross-section in G', $G' \times_G *$ is the homogeneous space of right cosets of G in G'. With Stasheff [34], we define generalized homogeneous spaces as follows.

Definition 8.6. Let $f: H \rightarrow G$ be any map of topological monoids. Define

$$G/H = B(G, H, *) \quad \text{and} \quad H\backslash G = B(*, H, G),$$

where H acts on G (from the left and right) through f.

We shall compare G/H to the fibre of Bf, but we must first insert the standard comparison of G to ΩBG (where BG has the basepoint $* = |[\ ], (1)| = F_0 BG$). Write $\mathcal{J}(I, X)$ for the path space of a based space X and write $p: \mathcal{J}(I, X) \rightarrow X$ for the endpoint projection. Write χ for the standard inverse map $\Omega X \rightarrow \Omega X$. For a based map $k: Y \rightarrow X$, write Fk for the homotopy theoretic fibre of k,

$$Fk = \{(\beta, y) \mid \beta \in \mathcal{J}(I, X),\ y \in Y,\ \beta(1) = k(y)\},$$

and write $\iota: \Omega X \rightarrow Fk$ and $\pi: Fk \rightarrow Y$ for the natural inclusion and projection, $\iota(\beta) = (\beta, *)$ and $\pi(\beta, y) = y$. With these notations, the proofs of the following two propositions are straightforward verifications from the definition, [17, 11.1], of geometric realization and the form of the face and degeneracy operators on the relevant simplicial spaces.

<u>Proposition 8.7</u>. For a topological monoid G, define $\tilde{\zeta}: EG \to \mathcal{J}(I, BG)$ by

$$\tilde{\zeta}|[g_1, \ldots, g_j]g_{j+1}, a|(t) = |[g_1, \ldots, g_{j+1}], (ta, 1-t)|$$

for $g_i \in G$, $a \in \Delta_j$, and $t \in I$. Define $\zeta: G \to \Omega BG$ by

$$\zeta(g)(t) = |[g], (t, 1-t)|.$$

Then the following diagram is commutative, hence ζ is a weak homotopy equivalence if G is grouplike:

$$\begin{array}{ccccc} G & \xrightarrow{\tau} & EG & \xrightarrow{p} & BG \\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\tilde{\zeta}} & & \| \\ \Omega BG & \xrightarrow{\subset} & \mathcal{J}(I, BG) & \xrightarrow{p} & BG \end{array}$$

The behavior of ζ when G is not grouplike will be studied in section 15.

<u>Proposition 8.8</u>. Let $f: H \to G$ be a map of topological monoids. Define $\psi: G/H \to FBf$ by $\psi(x) = (\beta(x), q(x))$, where

$$\beta|g[h_1, \ldots, h_j], a|(t) = |[g, f(h_1), \ldots, f(h_j)], (1-t, ta)|$$

for $g \in G$, $h_i \in H$, $a \in \Delta_j$, and $t \in I$. Then the following diagram is commutative, hence ψ is a weak homotopy equivalence if H and G are both grouplike:

$$\begin{array}{ccccccccc} H & \xrightarrow{f} & G & \xrightarrow{\tau} & G/H & \xrightarrow{q} & BH & \xrightarrow{Bf} & BG \\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\psi} & & \| & & \| \\ \Omega BH & \xrightarrow{\Omega Bf} & \Omega BG & \xrightarrow{\iota\chi} & FBf & \xrightarrow{\pi} & BH & \xrightarrow{Bf} & BG \end{array}$$

By symmetry, an analogous result is valid for $H\backslash G$, and it follows that G/H and $H\backslash G$ are weakly homotopy equivalent when H and G are grouplike. We shall use the following observations in section 10.

Remarks 8.9. Let $f: H \to G$ be a morphism of monoids. Then the following two diagrams are commutative:

$$\begin{array}{ccccc} G & \xrightarrow{\tau} & B(H\backslash G, G, G) & \xrightarrow{p} & B(H\backslash G, G, *) \\ \| & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon(p)} \\ G & \xrightarrow{\tau} & H\backslash G & \xrightarrow{p} & BH \end{array}$$

and

$$\begin{array}{ccccc} BH & \xleftarrow{\varepsilon(p)} & B(H\backslash G, G, *) & \xrightarrow{q} & BG \\ {\scriptstyle Bf}\downarrow & & \downarrow{\scriptstyle Bf} & & \| \\ BG & \xleftarrow{\varepsilon(p)} & B(G\backslash G, G, *) & \xrightarrow{q} & BG \end{array}$$

where, in the middle, Bf is short for $B(B(1,f,1),1,1)$. Let H and G be grouplike. Then, by Proposition 7.5 and Theorem 7.6, the first diagram shows that $\varepsilon(p): B(H\backslash G, G, *) \to BH$ is a weak homotopy equivalence. In the second diagram, $G\backslash G = EG$ and the bottom maps $\varepsilon(p)$ and q are weak homotopy equivalences by Corollary 7.7. The last step of the proof of Theorem 9.2 below will give that, for $A \in \mathcal{W}$, the automorphism $q_* \varepsilon(p)_*^{-1}$ of $[A, BG]$ is the identity. We conclude that, from the point of view of representable (or rather, represented) functors on $h\mathcal{W}$, the maps $q: B(H\backslash G, G, *) \to BG$ and $Bf: BH \to BG$ can be used interchangeably.

The following pair of remarks summarize properties of BG and EG when G is an Abelian topological monoid and relate the functors B and E to the infinite symmetric product . These results are due to Milgram [23].

Remarks 8.10. If G is Abelian, its product is a morphism of monoids and therefore EG and BG are Abelian topological monoids by Proposition 7.4 and naturality. If, in addition, G is a topological group, then its inverse map is also a morphism of monoids and EG and BG are topological groups by naturality. The group structure so defined on EG coincides with that obtained in Theorem 8.1 since a trivial verification shows that this is true on the level of simplicial spaces. Of course, BG is the quotient group EG/G when G is an Abelian group.

Remarks 8.11. Let NX denote the infinite symmetric product of a space $X \in \mathcal{T}$ and let $\eta : X \to NX$ denote the natural inclusion [9 or 17, §3]. Then there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota} & CX & \xrightarrow{\pi} & \Sigma X\ , \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\eta}} & & \downarrow{\scriptstyle \bar{\eta}} \\ NX & \xrightarrow{\tau} & ENX & \xrightarrow{p} & BNX \end{array}$$

where CX and ΣX are the (reduced) cone and suspension on X, ι and π are the natural inclusion and quotient map, $\bar{\eta}$ is determined by commutativity of the diagram, and

$$\tilde{\eta}(x, t) = |[\eta(x)]e, (t, 1-t)|$$

for $x \in X$ and $t \in I$; here the left square commutes since

$$\tau\eta(x) = |[\ \ ]\eta(x),(1)| = |[\eta(x)]e,(1,0)| = \tilde{\eta}\iota(x).$$

Since NX is the free Abelian topological monoid generated by X, there result maps $\phi(\tilde{\eta})$ and $\phi(\bar{\eta})$ of topological monoids such that the following diagram is commutative:

$$\begin{array}{ccccc}
NX & \xrightarrow{N\iota} & NCX & \xrightarrow{N\pi} & N\Sigma X \\
\| & & \downarrow{\scriptstyle \phi(\tilde{\eta})} & & \downarrow{\scriptstyle \phi(\bar{\eta})} \\
NX & \xrightarrow{\tau} & ENX & \xrightarrow{p} & BNX
\end{array}$$

As noted by Milgram [23 , p. 245], $\phi(\tilde{\eta})$ and $\phi(\bar{\eta})$ are in fact homeomorphisms.

9. The classification theorems

It is now an easy matter to use the bar construction to prove a general classification theorem for fibrations, and another for bundles. We shall only classify over base spaces in $\mathcal{W}$; greater generality would be useless for purposes of calculation. Nevertheless, for some important examples, we cannot insist that all spaces in sight be in $\mathcal{W}$; in such cases, we shall rely on the following consequences of the Whitehead theorem.

Remarks 9.1. Let $f: B \to A$ be a weak homotopy equivalence, where $A \in \mathcal{W}$. Since $f_*: [A, B] \to [A, A]$ is an isomorphism, there exists one and, up to homotopy, only one map $g: A \to B$ such that $fg \simeq 1$. Moreover, g is natural in the sense that, given a homotopy commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle j} \\ B' & \xrightarrow{f'} & A' \end{array}$$

in which $A, A' \in \mathcal{W}$ and f and f' are weak homotopy equivalences, the following diagram is also homotopy commutative:

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ A' & \xrightarrow{g'} & B' \end{array}$$

(since $f'g'j \simeq j \simeq jfg \simeq f'kg$ and f'_* is an isomorphism).

To avoid cluttering up the statement and proof of the following theorem with minor technicalities, we tacitly assume that the identity elements of all monoids are (strongly) nondegenerate basepoints and that all cross-sections are fibrewise cofibrations. These assumptions will be discussed in Remarks 9.3 and 9.7 below.

Theorem 9.2. Assume one of the following hypotheses.

(a) $(\mathcal{F}, F)$ is a category of fibres which is either

(i) Γ-complete in $\mathcal{U}$ or

(ii) Γ-complete in $\mathcal{W}$.

(b) $(\mathcal{F}, F)$ is a category of based fibres which is either

(i) Γ'-complete in $\mathcal{U}$ or

(ii) Γ'-complete in $\mathcal{W}$.

(c) $(\mathcal{F}, F)$ is a category of bundle fibres.

Let $(\mathcal{G}, G)$ be the associated principal category of fibres of $(\mathcal{F}, F)$. Then, for $A \in \mathcal{W}$, the set $\mathcal{E}\mathcal{F}(A)$ of equivalence classes of $\mathcal{F}$-fibrations over A is naturally isomorphic to $[A, BG]$.

Proof. In the cases (ii), Theorems 6.4 and 6.7 and Proposition 7.2 will ensure that all spaces in sight are in $\mathcal{W}$. By abuse, let us agree to write (Γ, η) for (Γ', η') in case (b) and to write (Γ, η) for the identity functor and identity natural transformation in case (c). With this uniform notation, let

$$\pi = \Gamma p : \Gamma B(*, G, F) \to BG$$

in all cases. By Definitions 5.1 and 5.4 and Theorem 7.6 in cases (a) and (b) and by Example 6.11 and Theorem 8.2 in case (c), π is an $\mathcal{J}$-fibration. Define

$$\Psi : [A, BG] \to \mathcal{E}\mathcal{J}(A)$$

by $\Psi[f] = \{f^*\pi\}$. Ψ is well-defined and natural by Proposition 2.5. In the other direction, define

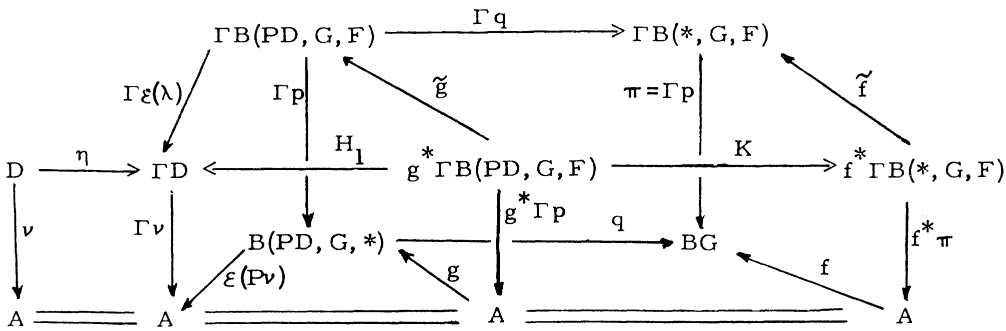
$$\Phi : \mathcal{E}\mathcal{J}(A) \to [A, BG]$$

as follows. Given an $\mathcal{J}$-fibration $\nu : D \to A$, consider the following commutative diagram, where $\gamma : PD \times G \to PD$ is given by composition:

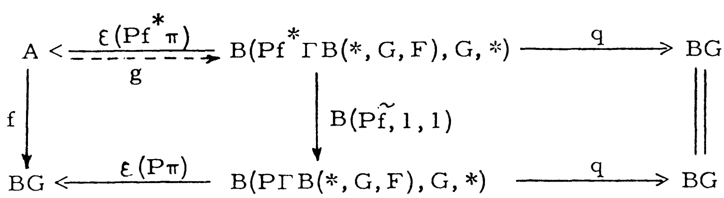
$$\begin{array}{ccccc}
PD & \xleftarrow{\varepsilon(\gamma)} & B(PD, G, G) & \xrightarrow{q} & EG \\
\downarrow{\scriptstyle P\nu} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
A & \underset{g}{\overset{\varepsilon(P\nu)}{\leftrightarrows}} & B(PD, G, *) & \xrightarrow{q} & BG
\end{array}$$

$\varepsilon(\gamma)$ is a homotopy equivalence by Proposition 7.5 and it restricts to a weak homotopy equivalence on each fibre since $(\varepsilon(\gamma), \varepsilon(P\nu))$ is a $\mathcal{G}$-map. Since p and $P\nu$ are quasifibrations, $\varepsilon(P\nu)$ is a weak homotopy equivalence by the five lemma. Let g be a right homotopy inverse to $\varepsilon(P\nu)$ and define $\Phi\{\nu\} = [qg]$. Φ is well-defined and natural by the evident naturality of the diagram above, before insertion of g, and by Remarks 9.1. $\Psi\Phi$ is the identity transformation on $\mathcal{E}\mathcal{J}(A)$. Indeed, with the notations above, the following diagram displays a chain of $\mathcal{J}$-maps over A which connects ν to

$f^*\pi$, where $f = qg$, and thus displays an equivalence (in the sense of Definition 6.1) between these $\mathcal{J}$-fibrations over A:



Here $\lambda: PD \times F \to D$ is the evaluation map, H_1 is obtained by application of the $\mathcal{J}$-CHP to the $\mathcal{J}$-map $(\Gamma\varepsilon(\lambda)\circ\tilde{g}, \varepsilon(P\nu)\circ g)$ and any homotopy $h: A \times I \to A$ from $\varepsilon(P\nu)\circ g$ to the identity, and K is given by the universal property of $f^*\pi$. Finally, to analyze $\Phi\Psi$, assume given $f: A \to BG$ and consider the following diagram:



The argument used to define Φ demonstrates that $\varepsilon(P\pi)$ is a weak homotopy equivalence (although it need not have a right inverse since BG need not be in $\mathcal{W}$). By Lemma 5.5 and Propositions 7.5 and 7.10, $P\Gamma B(*,G,F)$ is of the same weak homotopy type as EG and is thus aspherical. Since the bottom map q is a quasifibration with aspherical fibre, it is a weak homotopy equivalence. By Theorem 6.5 and the diagram, $\Phi\Psi$ is an automorphism of $[A, BG]$. But then Ψ is a bijection and $\Psi = (\Psi\Phi)\Psi = \Psi(\Phi\Psi)$, hence $\Phi\Psi$ is the identity transformation.

Remarks 9.3. In case (c), the requirement that (G, e) be an NDR-pair appears to be an essential hypothesis. In cases (a) and (b), such an hypothesis can be eliminated as follows. Let G' be the monoid obtained from G by growing a whisker from e [17,A.8]. Via the retraction $G' \to G$ (which is a homotopy equivalence) any left or right G-space is also a G'-space. Replace each $B(Y, G, X)$ in the statement and proof of the theorem by $B(Y, G', X)$. Then, with trivial modifications, the argument goes through to give $\mathcal{E}\mathcal{H}(A) \cong [A, BG']$. Note in particular that $B(Y, G', G')$ is homotopy equivalent to $B(Y, G', G)$ by Proposition 7.3 and that $B(Y, G', G)$ is homeomorphic to $PB(Y, G', F)$ by Proposition 7.10.

In all of the following corollaries, we agree to read BG' for BG if the basepoint of G happens to be degenerate.

In view of Example 6.2 and Lemma 6.3, we have the following classification theorem for principal fibrations.

Corollary 9.4. Let G be a grouplike topological monoid and let $A \in \mathcal{W}$.

(i) $\mathcal{E}G\mathcal{U}(A)$ is naturally isomorphic to $[A, BG]$.

(ii) If $G \in \mathcal{W}$, $\mathcal{E}G\mathcal{W}(A)$ is naturally isomorphic to $[A, BG]$, hence the natural map $\iota : \mathcal{E}G\mathcal{W}(A) \to \mathcal{E}G\mathcal{U}(A)$ is a bijection.

Similarly, in view of Example 6.6, Theorem 6.7, and Lemma 6.8, we have the following generalization of Stasheff's classification theorem [32] for fibrations with fibres of the homotopy type of a given finite CW-complex.

Corollary 9.5. Let $F \in \mathcal{W}$, let HF denote the topological monoid of homotopy equivalences of F, and let $A \in \mathcal{W}$.

(i) $\mathcal{E}F\mathcal{U}(A)$ is naturally isomorphic to $[A, BHF]$.

(ii) If F is compact, $\mathcal{E}F\mathcal{W}(A)$ is naturally isomorphic to $[A, BHF]$, hence the natural map $\iota: \mathcal{E}F\mathcal{W}(A) \to \mathcal{E}F\mathcal{U}(A)$ is a bijection.

Of course, in (ii), the equivalence relation used to define $\mathcal{E}F\mathcal{W}(A)$ coincides with fibre homotopy equivalence.

The compatibility of the previous two corollaries is immediate from our construction of classifying maps. Thus we have the following result.

Corollary 9.6. For $F \in \mathcal{W}$ and $A \in \mathcal{W}$, $P: \mathcal{E}F\mathcal{U}(A) \to \mathcal{E}HF\mathcal{U}(A)$ and, if F is compact, $P: \mathcal{E}F\mathcal{W}(A) \to \mathcal{E}HF\mathcal{W}(A)$ are bijections of sets.

Remarks 9.7. In case (b), we assumed in the proof of Theorem 9.2 that the cross-section of $p: B(Y,G,F) \to B(Y,G,*)$ is a cofibration for certain Y. Let F' be the G-space obtained from F by growing a whisker from the given basepoint and letting G act trivially on the whisker. The basepoint $1 \in F'$ is the endpoint of the whisker, and $(F', 1)$ is a G-equivariant NDR-pair. The cross-section of $p: B(Y,G,F') \to B(Y,G,*)$ is thus a fibrewise cofibration. Provided that $F' \in \mathcal{J}$ and the retraction $F' \to F$ is a map in $\mathcal{J}$, the proof of Theorem 9.2 goes through, with trivial modifications, with F replaced by F'.

In view of Example 6.9, Theorem 6.7, and Lemma 6.10, we have the following new variant of Stasheff's theorem. In [19], this result will play a key role in the study of E-oriented spherical fibrations for a commutative ring spectrum E.

Corollary 9.8. Let $F \in \mathcal{V}$, let JF denote the topological monoid of based homotopy equivalences of F, and let $A \in \mathcal{W}$.

(i) $\mathcal{E}F\mathcal{T}(A)$ is naturally isomorphic to [A, BJF].

(ii) If F is compact, $\mathcal{E}F\mathcal{V}(A)$ is naturally isomorphic to [A, BJF], hence the natural map $\iota : \mathcal{E}F\mathcal{V}(A) \to \mathcal{E}F\mathcal{T}(A)$ is a bijection.

In (ii), the equivalence relation used to define $\mathcal{E}\mathcal{J}\mathcal{V}(A)$ coincides with section preserving fibre homotopy equivalence, where homotopies are required to be section preserving for each parameter value $t \in I$.

Corollary 9.9. For $F \in \mathcal{V}$ and $A \in \mathcal{W}$, $P: \mathcal{E}F\mathcal{J}(A) \to \mathcal{E}JF\mathcal{U}(A)$ and, if F is compact, $P: \mathcal{E}F\mathcal{V}(A) \to \mathcal{E}JF\mathcal{W}(A)$ are bijections of sets.

Theorem 9.10. Let a topological group G act effectively from the left on a space F and let $\mathcal{B}\mathcal{J}(A)$ denote the set of equivalence classes of numerable G-bundles with fibre F over A. Assume that (G, e) is an NDR-pair. Then, for $A \in \mathcal{W}$, $\mathcal{B}\mathcal{J}(A)$ is naturally isomorphic to [A, BG].

Proof. Numerable Steenrod fibre bundles are known to satisfy the bundle-CHP (which is formulated precisely as was the $\mathcal{F}$-CHP but with $\mathcal{F}$-maps replaced by bundle maps) and the obvious analog of Lemma 2.4 [36,§11 and 7]. With P the classical associated principal bundle functor, the proof is formally identical to that of case (c) of Theorem 9.2.

Corollary 9.11. Let a topological group G act effectively from the left on a space F and let $(\mathcal{F}, F)$ denote the corresponding category of bundle fibres. For spaces $A \in \mathcal{W}$, let $\rho : \mathcal{B}\mathcal{F}(A) \to \mathcal{E}\mathcal{F}(A)$ denote the natural transformation obtained by regarding a G-bundle with fibre F as an $\mathcal{F}$-fibration. Then ρ is a bijection of sets provided that the identity map from G, with its given topology, to $\mathcal{F}(F,F)$, with the compact-open topology, is a weak homotopy equivalence and (G, e) is an NDR-pair in both topologies.

Proof. By our construction of classifying maps, the following diagram of natural transformations is commutative:

$$\begin{array}{ccc} \mathcal{B}\mathcal{F}(A) & \xrightarrow{\rho} & \mathcal{E}\mathcal{F}(A) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ [A, BG] & \xrightarrow{B(1)_*} & [A, B\mathcal{F}(F,F)] \end{array}$$

The conclusion follows, since B(1) is a weak homotopy equivalence if 1 is (by a comparison of quasifibrations).

We thus have a precise comparison between bundle theory and fibration theory.

10. The definition and examples of Y-structures

Until otherwise specified, let $(\mathcal{F}, F)$ be a category of fibres which satisfies one of the hypotheses of Theorem 9.2, let $(\mathcal{G}, G)$ be its associated principal category of fibres, and let Y be any right G-space. Consider $q: B(Y, G, *) \to BG$. $B(Y, G, *)$ can be thought of as the classifying space for $\mathcal{F}$-fibrations together with a "Y-structure". In many important special cases, Y-structures can be described intrinsically, without reference to the classification theorem, and can then be proven to be classified by $B(Y, G, *)$. We give a general intrinsic definition and several examples in this section and prove such a classification theorem in the next. The motivating example of E-oriented spherical fibrations will be treated in [19]; it will in fact be a special case of Example 10.6 below.

Definition 10.1. Assume given an auxiliary space Z and an inclusion of Y in the function space $\mathcal{U}(F, Z)$ such that the right action of G on Y is induced by restriction from the action of $G = \mathcal{F}(F, F)$ on $\mathcal{U}(F, Z)$ given by composition. Define a Y-structure θ on an $\mathcal{F}$-space $\nu: D \to A$ to be a map $\theta: D \to Z$ such that the composite $\theta \circ \psi: F \to Z$ is an element of Y for every element $\psi: F \to D$ of PD. Define an $\mathcal{F}$-map $(\nu, \theta) \to (\nu', \theta')$ of $\mathcal{F}$-spaces with Y-structure to be an $\mathcal{F}$-map $(g, f): \nu \to \nu'$ such that $\theta' g$ is homotopic to θ via a homotopy $h: D \times I \to Z$ such that $h_t \psi: F \to Z$ is an element of Y for every $\psi \in PD$ and $t \in I$ (that is, via a homotopy through Y-structures). Define $\mathcal{E}\mathcal{F}(A; Y)$ to be the set of equivalence classes of

$\mathcal{J}$-fibrations with Y-structure under the equivalence relation generated by the $\mathcal{J}$-maps over A.

Our notions of $\mathcal{J}$-maps and of equivalence suggest that a Y-structure on a given $\mathcal{J}$-space should be reinterpreted as a homotopy class of Y-structures, and we adopt this terminology henceforward. Although the definition may seem artificial, at first sight, we shall see that it does satisfactorily account for the most important types of additional structure on $\mathcal{J}$-fibrations.

When $(\mathcal{J}, F)$ satisfies hypothesis (a) or (b) of Theorem 9.2, we shall need further conditions on Y and Z in order to ensure that $\mathcal{J}$-quasifibrations with Y-structure in $\mathcal{L}$ ($\mathcal{L} = \mathcal{U}$ or $\mathcal{L} = \mathcal{W}$) can be replaced functorially by $\mathcal{J}$-fibrations with Y-structure in $\mathcal{L}$. As in the proof of Theorem 9.2, we agree to write (Γ, η) for $(\Gamma, \eta), (\Gamma', \eta')$, or the identity functor and identity natural transformation according to whether $(\mathcal{J}, F)$ satisfies hypothesis (a), (b), or (c) of that theorem. The following definition should be compared with Definitions 5.1 and 5.4.

<u>Definition 10.2.</u> Let $(\mathcal{J}, F)$ be Γ-complete in $\mathcal{L}$ [that is, Γ or Γ' complete], and let Y be a sub right G-space of $\mathcal{U}(F, Z)$. The pair (Y, Z) will be said to be admissible if $Y \in \mathcal{L}$ and the following statements are valid for $\mathcal{J}$-quasifibrations $\pi: E \to B$ in $\mathcal{L}$ with (homotopy class of) Y-structure $\theta: E \to Z$.

(1) $\Gamma\pi: \Gamma E \to \Gamma B$ admits a Y-structure $\Gamma\theta: \Gamma E \to Z$.

(2) $\eta: E \to \Gamma E$ defines an $\mathcal{J}$-map $(\pi, \theta) \to (\Gamma\pi, \Gamma\theta)$ over B.

(3) Γ takes $\mathcal{F}$-maps $(\pi, \theta) \to (\pi', \theta')$ to $\mathcal{F}$-maps $(\Gamma\pi, \Gamma\theta) \to (\Gamma\pi', \Gamma\theta')$.

If $(\mathcal{F}, F)$ is a category of bundle fibres, any pair (Y, Z) such that Y is a sub right G-space of $\mathcal{U}(F, Z)$ will be said to be admissible.

The following two examples give generalized versions of familiar types of Y-structures.

Example 10.3. Let $(\mathcal{F}', F)$ be a second category of fibres, with associated principal category $(\mathcal{G}', G')$, and let $j: \mathcal{F} \to \mathcal{F}'$ be a functor over $\mathcal{U}$ (or over $\mathcal{J}$ if $\mathcal{F}$ and $\mathcal{F}'$ are based). Then j defines a morphism of monoids $G \to G' = \mathcal{F}'(F, F)$. In Definition 10.1, set $Y = G'$ and $Z = F$. Then a G'-structure $\theta: D \to F$ on an $\mathcal{F}$-fibration $\nu: D \to A$ is just the second coordinate of an $\mathcal{F}'$-map $D \to A \times F$ over A (at least if $\alpha\beta \in \mathcal{F}'$ and $\beta \in \mathcal{F}'$ implies $\alpha \in \mathcal{F}'$). In other words, a G'-structure is precisely an $\mathcal{F}'$-trivialization of the $\mathcal{F}$-fibration ν. Of course, $B(G', G, *)$ is the generalized homogeneous space G'/G of Definition 8.6. In the interesting applications, $(\mathcal{F}, F)$ will be a category of bundle fibres, hence the question of admissibility will not arise.

Example 10.4. Let $f: H \to G$ be any morphism of monoids and set $Y = H\backslash G = B(*, H, G)$ and $Z = \Gamma B(*, H, F)$. Y is homeomorphic to $PB(*, H, F)$, by Proposition 7.10, and the inclusion of Y in $\mathcal{U}(F, Z)$ is the composite of this homeomorphism and the inclusion $P\eta$ of $PB(*, H, F)$ in PZ. Let $\nu: D \to A$ be an $\mathcal{F}$-space and let $\theta: D \to Z$ be a Y-structure. Because $\theta\psi$ is in Y for ψ in PD, θ is fibrewise with respect to ν and $\Gamma p: \Gamma B(*, H, F) \to BH$ (at least if every point of D is in the image of some ψ, as always holds in practice). We agree to strengthen the notion of an $H\backslash G$-structure by insisting that the induced function $A \to BH$ be continuous. Then $\Gamma\theta : \Gamma D \to \Gamma Z$ is defined, by Remarks 3.7, and the composite

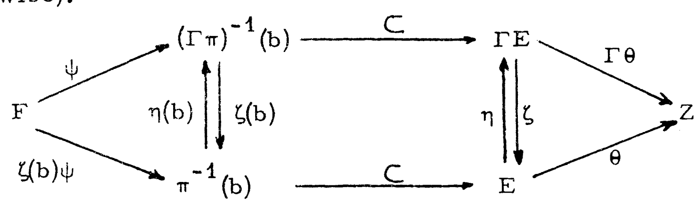
$$\Gamma D \xrightarrow{\Gamma\theta} \Gamma Z = \Gamma\Gamma B(*, H, F) \xrightarrow{\mu} \Gamma B(*, H, F) = Z$$

is an $H\backslash G$-structure for $\Gamma\nu : \Gamma D \to A$ such that $\mu \circ \Gamma\theta \circ \eta = \theta$. Thus the pair (Y, Z) will always be admissible when $(\mathcal{F}, F)$ satisfies hypothesis (a) or (b), provided only that $H \in \mathcal{W}$ if $\mathcal{L} = \mathcal{W}$. We call an $H\backslash G$-structure $\theta : D \to Z$ on an $\mathcal{F}$-fibration $\nu : D \to A$ a reduction of the structural monoid of ν to H. If H is a topological group and ν admits such a reduction θ, then ν is equivalent to the $\mathcal{F}$-fibration induced from $p : B(*, H, F) \to BH$ by the map $A \to BH$ derived from θ; of course, this $\mathcal{F}$-fibration is an H-bundle with fibre F if H acts effectively on F. As explained in Remarks 8.9, $B(H\backslash G, G, *)$ is weakly homotopy equivalent to BH (when H is grouplike) in such a way that the maps $q : B(H\backslash G, G, *) \to BG$ and $Bf : BH \to BG$ are equivalent.

We also have the following generic types of Y-structures, the second of which will be central to [19].

<u>Example 10.5.</u> Let $F \in \mathcal{W}$ be compact, let $Z \in \mathcal{W}$, and let Y be the union of any set of components of $\mathcal{U}(F, Z)$ which is invariant under composition with homotopy equivalences of F. Then (Y, Z) is an admissible pair for $F\mathcal{W}$. Indeed, let $\pi : E \to B$ be an $F\mathcal{W}$-quasifibration with B and E in $\mathcal{W}$ and with Y-structure $\theta : E \to Z$. Choose a homotopy inverse $\zeta : \Gamma E \to E$ to η and define $\Gamma\theta = \theta\zeta : \Gamma E \to Z$. For $\psi : F \to (\Gamma\pi)^{-1}(b)$ in $P\Gamma E$ (that is,

a homotopy equivalence), consider the following diagram, in which $\eta(b)$ denotes the restriction of η to $\pi^{-1}(b)$ and $\zeta(b)$ is a chosen homotopy inverse to $\eta(b)$ (which need not be the restriction of ζ since ζ need not be fibrewise):



Here $\Gamma\theta\circ\psi \simeq \Gamma\theta\circ\eta(b)\circ\zeta(b)\circ\psi = \theta\circ\zeta\circ\eta\circ\zeta(b)\circ\psi \simeq \theta\circ\zeta(b)\circ\psi$ and, since $\zeta(b)\psi \in PD$, it follows that $\Gamma\theta\circ\psi$ is an element of Y. Clearly $\eta:(\pi,\theta)\to(\Gamma\pi,\Gamma\theta)$ is an $F\mathcal{W}$-map over B and Γ is functorial.

Example 10.6. Let $F\in\mathcal{V}$ be compact, let $Z\in\mathcal{V}$, and let Y be the union of any set of components of $\mathcal{J}(F,Z)$ which is invariant under composition with based homotopy equivalences of F. Then (Y,Z) is an admissible pair for $F\mathcal{V}$. Indeed, retaining the notations of the previous example (with (Γ,η) interpreted as (Γ',η')), we note that ζ can be chosen to be section preserving (although not fibre preserving) and that $\zeta\eta$ is then homotopic to the identity via a homotopy through section preserving maps (because the sections of π and $\Gamma\pi$ are cofibrations). These facts, and the nondegeneracy of the basepoints of the fibres of π and $\Gamma\pi$, allow the required use of based maps and homotopies in the verification that $\Gamma\theta\circ\psi$ is in Y for ψ in $P\Gamma E$. Observe that a Y-structure $\theta: D\to Z$ on an $F\mathcal{V}$-fibration $\nu: D\to A$ factors through the "Thom complex" $D/\sigma A$ since

$\theta\psi \in Y$ for $\psi \in PD$ implies that θ carries the basepoint of each fibre of ν to the basepoint of Z.

Of course, Definition 10.1 admits a bundle theoretic analog.

Definition 10.7. Let G be a topological group which acts effectively on a space F and let $\mathcal{F}$ be the derived category of bundle fibres (Example 6.11). Let Y be a right G-space and let $Y \to \mathcal{U}(F, Z)$ be a continuous one-to-one map under which the right action of G on Y agrees (as a function) with the right action of $\mathcal{F}(F, F)$ on $\mathcal{U}(F, Z)$ given by composition; the pair (Y, Z) is then said to be admissible. Define a Y-structure θ on a G-bundle $\nu : D \to A$ with fibre F to be a map $\theta: D \to Z$ such that the composite $\theta\psi: F \to Z$ is an element of Y for every element $\psi: F \to D$ of PD and such that the function $\tilde{\theta}: PF \to Y$ specified by $\tilde{\theta}(\psi) = \theta \circ \psi$ is continuous (where the associated principal bundle PD has its standard topology). Define a bundle map $(\nu, \theta) \to (\nu', \theta')$ of bundles with Y-structure to be a bundle map $(g, f): \nu \to \nu'$ such that $\theta' g$ is homotopic to θ by a homotopy through Y-structures. Define $\mathcal{B}\mathcal{F}(A; Y)$ to be the set of equivalence classes of G-bundles with fibre F and with Y-structure over A.

Example 10.3 applies directly to bundles, with $(\mathcal{F}, F)$ interpreted as the category of bundle fibres derived from F and G. There is also an obvious bundle theoretic analog of Example 10.3 in which G' is taken to be a group which contains G and also acts effectively on F.

In Example 10.4, interpreted bundle theoretically, if H is also a group and if $\nu : D \to A$ is a G-bundle with fibre F and $H\backslash G$-structure $\theta : D \to B(*, H, F)$, then θ determines an equivalence of G-bundles from D to the bundle $E \times_H F$, where E is the principal H-bundle induced from the universal bundle $EH \to BH$ by $A \to BH$. Thus our notion of a reduction of the group of a bundle agrees with the standard one. (Compare Lashof [12, §1], where the term lifting is used to emphasize that $H \to G$ is not assumed to be an inclusion.)

11. The classification of Y-structures

The following fundamental result should be regarded as an elaboration of Theorem 9.2 (to which it reduces when $Y = *$ and $Z = *$). We again tacitly assume that the identity elements of all monoids are strongly nondegenerate basepoints and that all cross-sections are fibrewise cofibrations. The discussions of these points in Remarks 9.3 and 9.7 apply verbatim to the present situation. We shall often abbreviate maps of the bar construction of the form $B(f,1,1)$ to Bf here.

Theorem 11.1. Let $(\mathcal{F}, F)$ be a category of fibres which satisfies one of the hypotheses (a), (b), or (c) of Theorem 9.2 and let $(\mathcal{G}, G)$ be its associated principal category of fibres. Let Y be a sub right G-space of $\mathcal{U}(F, Z)$ such that the pair (Y, Z) is admissible. Then, for $A \in \mathcal{W}$, the set $\mathcal{E}\mathcal{F}(A;Y)$ of equivalence classes of $\mathcal{F}$-fibrations with Y-structure over A is naturally isomorphic to $[A, B(Y,G,*)]$.

Proof. As usual, write (Γ, η) ambiguously for $(\Gamma,\eta), (\Gamma',\eta')$, and the identity functor and identity natural transformation in cases (a), (b), or (c). Let $\lambda: Y \times F \to Z$ be adjoint to the inclusion $Y \to \mathcal{U}(F, Z)$. Then $\varepsilon(\lambda): B(Y,G,F) \to Z$ is a Y-structure on $p: B(Y,G,F) \to B(Y,G,*)$. Define an $\mathcal{F}$-fibration with Y-structure (π, ω) by

$$\pi = \Gamma p: \Gamma B(Y,G,F) \to B(Y,G,*) \quad \text{and} \quad \omega = \Gamma\varepsilon(\lambda): \Gamma B(Y,G,F) \to Z.$$

Now define $\Psi: [A, B(Y, G, *)] \to \mathcal{E}\mathcal{F}(A;Y)$ by $\Psi[f] = \{(f^*\pi, \omega\tilde{f})\}$. If $h: A \times I \to B(Y,G,*)$ is a homotopy from f to f', then the following composite is a homotopy through Y-structures from $\omega\tilde{f}$ to $\omega\tilde{f}'J_1$, where J is an $\mathcal{J}$-homotopy over A which starts at the identity map of $f^*\Gamma B(Y,G,F)$ and is obtained by application of the $\mathcal{J}$-CHP to the identity map of $A\times I$ (regarded as a homotopy):

$$f^*\Gamma B(Y,G,F)\times I \xrightarrow{J} h^*\Gamma B(Y,G,F) \xrightarrow{\tilde{h}} \Gamma B(Y,G,F) \xrightarrow{\omega} Z.$$

Therefore Ψ is well-defined. The same argument shows that $\mathcal{E}\mathcal{J}(A;Y)$ is in fact a functor of A, and Ψ is clearly natural. Define $\Phi: \mathcal{E}\mathcal{J}(A;Y)\to[A, B(Y,G,*)]$ as follows. Let $\nu: D\to A$ be an $\mathcal{J}$-fibration with Y-structure $\theta: D\to Z$ and let $\tilde{\theta}: PD\to Y$ be the map of right G-spaces specified by $\tilde{\theta}(\psi)=\theta\circ\psi$. Consider the maps

$$A \underset{g}{\overset{\varepsilon(P\nu)}{\rightleftarrows}} B(PD,G,*) \xrightarrow{B\tilde{\theta}} B(Y,G,*)\ ,$$

choose a right homotopy inverse g to $\varepsilon(P\nu)$ (as in the proof of Theorem 9.2), and define $\Phi\{(\nu,\theta)\}=[B\tilde{\theta}\circ g]$. Given (ν',θ'), an $\mathcal{J}$-map $k: D\to D'$ over A, and a homotopy $h: D\times I\to Z$ through Y-structures from θ to $\theta'k$, define $Ph: PD\times I\to Y$ by $(Ph)_t(\psi)=h_t\circ\psi$. For $\phi\in G$, $(Ph)_t(\psi\circ\phi)=(Ph)_t(\psi)\circ\phi$, hence Ph is a G-equivariant homotopy. It therefore induces a homotopy from $B\tilde{\theta}$ to $B\tilde{\theta}\circ BPk$. Thus Φ is well-defined, and a similar argument shows that Φ is natural. $\Psi\Phi$ is the identity transformation on $\mathcal{E}\mathcal{J}(A;Y)$. Indeed, with (ν,θ) as in the definition of Φ, set $f=B\tilde{\theta}\circ g$ and replace q by $B\tilde{\theta}$, BG by $B(Y,G,*)$, and $B(*,G,F)$ by $B(Y,G,F)$ in the diagram used for the corresponding step of the proof of Theorem 9.2. Then the resulting diagram displays an equivalence between ν and $f^*\pi$, and it is immediate from an argument like that used to prove that Ψ is well-defined and, in cases (a) and (b), from the functoriality of Γ such that η is an $\mathcal{J}$-map given by Definition 10.2 that the constructed equivalence is one of $\mathcal{J}$-fibrations with Y-structure. For the verification that $\Phi\Psi$ is an automorphism and therefore also the identity, construction of a diagram just like that used for the corresponding step of Theorem 9.2 shows that we need only check that

$$B\tilde{\omega}: B(P\Gamma B(Y,G,F),G,*) \rightarrow B(Y,G,*)$$

is a weak homotopy equivalence. By comparison of the quasifibrations q from the displayed spaces to BG, it suffices to check that $\tilde{\omega}: P\Gamma B(Y,G,F) \rightarrow Y$ is a weak homotopy equivalence, and this follows from Lemma 5.5, Propositions 7.5 and 7.10, and the fact that $\omega \circ \eta \simeq \varepsilon(\lambda)$, so that

$$\tilde{\omega} \circ P\eta \simeq \varepsilon : PB(Y,G,F) \cong B(Y,G,G) \rightarrow Y.$$

Under the hypotheses of the theorem, consider the quasifibration sequence

$$G \xrightarrow{\iota} Y \xrightarrow{\tau} B(Y,G,*) \xrightarrow{q} BG$$

obtained in Proposition 7.9. The following remarks interpret the corresponding sequence of represented functors on $\mathcal{N}$.

Remarks 11.2. (i) $q_*: [A, B(Y,G,*)] \rightarrow [A, BG]$ represents the forgetful transformation $\mathcal{E}\mathcal{J}(A;Y) \rightarrow \mathcal{E}\mathcal{J}(A)$ obtained by sending $\{(\nu,\theta)\}$ to $\{\nu\}$ since there is an $\mathcal{J}$-map j over $B(Y,G,*)$ such that the following diagram commutes and since $\Phi\{\nu\} = [q]\Phi\{(\nu, \theta)\}$ by the proofs of Theorems 9.2 and 11.1:

$$\begin{array}{ccccc} \Gamma B(Y,G,F) & & \xrightarrow{\Gamma q} & & \Gamma B(*,G,F) \\ & \searrow^{j} & & \nearrow^{\tilde{q}} & \\ \pi\downarrow & & q^*\Gamma B(*,G,F) & & \downarrow\pi \\ & \swarrow_{q^*\pi} & & & \\ B(Y,G,*) & & \xrightarrow{q} & & BG \end{array}$$

(ii) $[A, Y]$ is naturally isomorphic to the set of (homotopy classes of) Y-structures on the trivial $\mathcal{F}$-fibration $\varepsilon: A \times F \to A$. Indeed, given $f: A \to Y$, its adjoint $A \times F \to Z$ gives the corresponding Y-structure.

(iii) $\tau_*: [A, Y] \to [A, B(Y, G, *)]$ represents the transformation which sends a Y-structure θ on ε to the equivalence class $\{(\varepsilon, \theta)\}$, by inspection of the proof of Theorem 11.1.

(iv) $[A, G]$ is naturally isomorphic to the set of $\mathcal{F}$-homotopy classes of $\mathcal{F}$-maps over A from ε to itself. Indeed, given $f: A \to G$, its adjoint $A \times F \to F$ gives the second coordinate of the corresponding $\mathcal{F}$-map over A.

(v) $\iota_*: [A, G] \to [A, Y]$ represents the transformation which sends an $\mathcal{F}$-map $g: A \times F \to A \times F$ to the Y-structure $\theta_0 \circ g$, where $\theta_0: A \times F \to Z$ is the Y-structure on ε with adjoint the trivial map $A \to \{y_0\} \in Y$, $y_0 = \iota(e)$.

Observe that if Y happens to admit a delooping, or classifying space, BY and if ι deloops to a map $B\iota: BG \to BY$ with fibre equivalent to $q: B(Y, G, *) \to BG$, then $B\iota$ defines the obstruction to the existence of a Y-structure on an $\mathcal{F}$-fibration ν; that is, ν admits a Y-structure if and only if $(B\iota)_*\Phi\{\nu\}$ is the trivial homotopy class. For example, when $Y = G'$

is as in Example 10.3, the quasifibration sequence above extends to

$$G \xrightarrow{j} G' \xrightarrow{\tau} G'/G \xrightarrow{q} BG \xrightarrow{Bj} BG'$$

by Proposition 8.8, and Bj defines the obstruction to the existence of an $\mathcal{J}'$-trivialization of an $\mathcal{J}$-fibration.

Remark 11.3. In the applications, one is often interested in two (or more) types of structure on $\mathcal{J}$-fibrations. The theorem already handles such situations since, if Y and Y' are right G-spaces, then the square

$$\begin{array}{ccc} B(Y \times Y', G, *) & \xrightarrow{B\pi_1} & B(Y, G, *) \\ \Big\downarrow{\scriptstyle B\pi_2} & & \Big\downarrow{\scriptstyle q} \\ B(Y', G, *) & \xrightarrow{q} & BG \end{array}$$

is a pullback and since, in cases (a) or (b), if (Y, Z) and (Y', Z') are admissible pairs, then so also is $(Y \times Y', Z \times Z')$. When $Y' = H\backslash G$ for some morphism of monoids $f: H \to G$, the pullback above can be used interchangeably with the pullback

$$\begin{array}{ccc} B(Y, H, *) & \xrightarrow{B(1, f, 1)} & B(Y, G, *) \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\ BH & \xrightarrow{Bf} & BG \end{array}$$

in view of Remarks 8.9 and the following commutative diagram, in which all vertical arrows are weak homotopy equivalences:

$$\begin{array}{ccccc}
B(H\backslash G,G,*) & \xrightarrow{\ q\ } & BG & \xleftarrow{\ q\ } & B(Y,G,*) \\
\Big\| & & \Big\uparrow{\scriptstyle q} & & \Big\uparrow{\scriptstyle B\varepsilon} \\
B(H\backslash G,G,*) & \xrightarrow{\ Bf\ } & B(G\backslash G,G,*) & \xleftarrow{\ Bq\ } & B(B(Y,G,G),G,*) \\
{\scriptstyle \varepsilon(p)}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon(p)} & & \Big\downarrow{\scriptstyle \varepsilon(p)} \\
BH & \xrightarrow{\ Bf\ } & BG & \xleftarrow{\ q\ } & B(Y,G,*)
\end{array}$$

The proof of the following bundle theoretic analog is formally identical to the proof of Theorem 11.1. Recall Definition 10.7.

Theorem 11.4. Let a topological group G act effectively from the left on a space F and assume that (G, e) is an NDR-pair. Let the pair (Y, Z) be admissible. Then, for $A \in \mathcal{W}$, the set $\mathcal{B}\mathcal{F}(A;Y)$ is naturally isomorphic to $[A, B(Y,G,*)]$.

Of course, Remarks 11.3 apply verbatim to Theorem 11.4, and the obvious bundle theoretic analogs of Remarks 11.2 are valid.

We have an evident natural transformation $\zeta : \mathcal{B}\mathcal{F}(A;Y) \to \mathcal{E}\mathcal{F}(A;Y)$ where, on the right, Y has its topology as a subspace of $\mathcal{U}(F,Z)$. If the identity map from Y, with its given topology, to Y, with its function space topology, is a weak homotopy equivalence and if the hypotheses of Corollary 9.11 are satisfied, then ζ is a bijection of sets.

12. A categorical generalization of the bar construction

We here introduce an amusing categorical construction which allows us to generalize the material of section 9 to a context in which any set of fibres, rather than just a single fibre, is given a privileged role. This reworking of the theory yields an analysis of the effect of changing the choice of privileged fibre.

Let $\mathcal{O}$ be a fixed set (of objects) regarded for our purposes as a discrete topological space. Define an $\mathcal{O}$-graph to be a space $\mathcal{A}$ (of arrows) together with continuous maps $S: \mathcal{A} \to \mathcal{O}$ and $T: \mathcal{A} \to \mathcal{O}$ (called source and target). Let $\mathcal{O}Gr$ denote the category of $\mathcal{O}$-graphs; its morphisms are continuous maps $f: \mathcal{A} \to \mathcal{A}'$ such that $S \circ f = S$ and $T \circ f = T$. Regard $\mathcal{O}$ itself as that $\mathcal{O}$-graph with arrow space $\mathcal{O}$ and with S and T the identity map. Define the product over $\mathcal{O}$ of $\mathcal{O}$-graphs $\mathcal{A}$ and $\mathcal{A}'$ to be the $\mathcal{O}$-graph $\mathcal{A} \square \mathcal{A}'$ with arrow space $\{(a, a') \mid Sa = Ta'\} \subset \mathcal{A} \times \mathcal{A}'$ and with source and target defined by $S(a, a') = Sa'$ and $T(a, a') = Ta$. Clearly $\square$ is associative, up to the evident natural isomorphism, and is unital with respect to the natural isomorphisms $\lambda: \mathcal{A} \to \mathcal{O} \square \mathcal{A}$ and $\rho: \mathcal{A} \to \mathcal{A} \square \mathcal{O}$ specified by $\lambda(a) = (Ta, a)$ and $\rho(a) = (a, Sa)$ for $a \in \mathcal{A}$.

Thus $\mathcal{O}Gr$ is a monoidal category with product $\square$ and unit $\mathcal{O}$. We can therefore define the notion of a monoid $(\mathcal{B}, C, I)$ in $\mathcal{O}Gr$. Here $C: \mathcal{B} \square \mathcal{B} \to \mathcal{B}$ and $I: \mathcal{O} \to \mathcal{B}$ are maps of $\mathcal{O}$-graphs (called composition and identity) such that C is associative and I is a two-sided unit for C. In other words, $\mathcal{B}$ is just a small topological category with object space $\mathcal{O}$.

So far we have followed Mac Lane [14, p. 10 and 48], but we must now take cognizance of the assymmetry of $\square$. Define a right $\mathcal{O}$-graph to be a space $\mathcal{Y}$ together with a map $S: \mathcal{Y} \to \mathcal{O}$. Similarly, for a left $\mathcal{O}$-graph $\mathcal{X}$, only $T: \mathcal{X} \to \mathcal{O}$ is to be given. Observe that, for an $\mathcal{O}$-graph $\mathcal{A}$, we can define $\mathcal{Y} \square \mathcal{A}$ and $\mathcal{A} \square \mathcal{X}$ as right and left $\mathcal{O}$-graphs, and we can define $\mathcal{Y} \square \mathcal{X}$ as a space. Now let $\mathcal{G}$ be a monoid in $\mathcal{O}$Gr. Define a right $\mathcal{O}$-graph over $\mathcal{G}$ to be a right $\mathcal{O}$-graph $\mathcal{Y}$ together with a map $R: \mathcal{Y} \square \mathcal{G} \to \mathcal{Y}$ of right $\mathcal{O}$-graphs which satisfies the evident associativity and unit formulas $R(1 \square C) = R(R \square 1)$ and $R(1 \square I) = \rho^{-1}$. The notion of a left $\mathcal{O}$-graph over $\mathcal{G}$ is defined by symmetry.

At this point we can generalize the definition of the two-sided geometric bar construction to triples $(\mathcal{Y}, \mathcal{G}, \mathcal{X})$, where $\mathcal{G}$ is a monoid in $\mathcal{O}$Gr and $\mathcal{Y}$ and $\mathcal{X}$ are right and left $\mathcal{O}$-graphs over $\mathcal{G}$. Indeed, we need only replace $\times$ by $\square$ in the definition of section 7 to obtain a simplicial topological space $B_*(\mathcal{Y}, \mathcal{G}, \mathcal{X})$, and we define $B(\mathcal{Y}, \mathcal{G}, \mathcal{X})$ to be its geometric realization. To ensure that the construction has good topological properties, we insist that $(\mathcal{G}, I\mathcal{O})$ be a strong NDR-pair [17, A. 1]. When $\mathcal{O}$ is a singleton set, this two-sided bar construction reduces to that in section 7. The $\mathcal{O}$-graph $\mathcal{O}$ is itself a right and left $\mathcal{O}$-graph over $\mathcal{G}$ via

$$\mathcal{G} \square \mathcal{O} \xrightarrow{\rho^{-1}} \mathcal{G} \xrightarrow{T} \mathcal{O} \quad \text{and} \quad \mathcal{O} \square \mathcal{G} \xrightarrow{\lambda^{-1}} \mathcal{G} \xrightarrow{S} \mathcal{O}$$

$B\mathcal{G} = B(\mathcal{O}, \mathcal{G}, \mathcal{O})$ is the standard classifying space of the category $\mathcal{G}$ (e.g. Segal [30] or [18, §4]), and we write $E\mathcal{G} = B(\mathcal{O}, \mathcal{G}, \mathcal{G})$.

All of the results of section 7 and some of the results of section 8 generalize to $B(\mathcal{Y}, \mathcal{G}, \mathcal{X})$. Indeed, this is apparent from the fact that most of the proofs depended only on general properties of geometric realization and elementary properties of the simplicial bar construction. Note for Proposition 7.4 that the product $\mathcal{A} \times \mathcal{A}'$ of an $\mathcal{O}$-graph $\mathcal{A}$ and an $\mathcal{O}'$-graph $\mathcal{A}'$ is an $\mathcal{O} \times \mathcal{O}'$-graph and that there are evident natural isomorphisms of simplicial spaces

$$B_*(\mathcal{Y} \times \mathcal{Y}', \mathcal{G} \times \mathcal{G}', \mathcal{X} \times \mathcal{X}') \cong B_*(\mathcal{Y}, \mathcal{G}, \mathcal{X}) \times B_*(\mathcal{Y}', \mathcal{G}', \mathcal{X}')$$

for triples $(\mathcal{Y}, \mathcal{G}, \mathcal{X})$ over $\mathcal{O}$ and $(\mathcal{Y}', \mathcal{G}', \mathcal{X}')$ over $\mathcal{O}'$. For Theorem 7.6, we say that $\mathcal{G}$ is grouplike if its homotopy category is a groupoid (so that every homotopy class of maps in $\mathcal{G}$ is invertible); here p and q are induced from $T: \mathcal{X} \to \mathcal{O}$ and $S: \mathcal{Y} \to \mathcal{O}$ and are quasifibrations. Theorem 8.1 and Remarks 8.10 wholly fail to generalize; $E\mathcal{G}$ is clearly not a group (or groupoid) if $\mathcal{G}$ is a groupoid, and commutativity in $\mathcal{G}$ only makes sense on subcategories with one object.

Now suppose given a homogeneous category of fibres. Call it $\mathcal{G}$ rather than $\mathcal{F}$, to accord with our present emphasis on the morphism spaces rather than the object spaces, but continue to speak of $\mathcal{F}$-spaces and $\mathcal{F}$-fibrations. (Note that, in our previous notations, the object spaces of the associated principal category were certain of the morphism spaces of the original category.) Let $\mathcal{O}$ denote the collection of objects of $\mathcal{G}$, assume that $\mathcal{O}$ is a set, and give $\mathcal{O}$ the discrete topology; explicitly, $\mathcal{O}$ is to have one point $\{F\}$ for each object space F. Let $\mathcal{F}$ denote the left $\mathcal{O}$-graph over $\mathcal{G}$ which, as a space, is the disjoint union of the object spaces of $\mathcal{H}$; $T: \mathcal{F} \to \mathcal{O}$ is the map which collapses a space F of the disjoint union to the point $\{F\}$, and the left action $\mathcal{H} \square \mathcal{F} \to \mathcal{F}$ is defined by the evaluation maps $\mathcal{H}(F, F') \times F \to F'$.

Let $\nu: D \to A$ be an $\mathcal{F}$-space. Define a right $\mathcal{O}$-graph $\mathcal{P}D$ over $\mathcal{H}$ as follows. As a space, $\mathcal{P}D$ is the disjoint union over F of the subspaces of $\mathcal{U}(F, D)$ which consist of the maps in $\mathcal{H}$ from F to a fibre of ν. $S: \mathcal{P}D \to \mathcal{O}$ assigns the point $\{F\}$ to all $\psi: F \to D$ in $\mathcal{P}D$, and the right action $\mathcal{P}D \square \mathcal{H} \to \mathcal{H}$ is induced by the composition maps $\mathcal{U}(F', D) \times \mathcal{H}(F, F') \to \mathcal{U}(F, D)$.

We can now formally replace all bar constructions $B(Y, G, X)$ which occur in the proof of Theorem 9.2 by corresponding bar constructions $B(\mathcal{Y}, \mathcal{H}, \mathcal{X})$. In cases (a) and (b), the proof goes through without the slightest change. We must avoid case (c), because Theorem 8.2 is not available, but here all fibres are homeomorphic and the present elaboration would be uninteresting in any case. In practice, we must also avoid case (i) of (a) and (b), since here $\mathcal{F}$ is usually not homogeneous. This leaves case (ii), and here the categories $\mathcal{F}\mathcal{W}$ and $\mathcal{F}\mathcal{V}$ of Examples 6.6 and 6.9 are homogeneous.

We have ignored one difficulty: $\mathcal{O}$ was assumed to be a set, whereas the categories of interest are large. Probably the best solution to this problem is to restrict the constructions above to the various small full subcategories of $\mathcal{H}$. For example, Theorem 9.2 as originally developed is the case when $\mathcal{H}$ is replaced by its full subcategory with one object F.

Thus reinterpret $\mathcal{O}$ above to be a given set of objects of our original category. Now note that the bar construction is functorial in $\mathcal{O}$. Indeed, if $\mathcal{O}$ is a subset of $\mathcal{O}'$ and if $(\mathcal{Y}', \mathcal{H}', \mathcal{X}')$ is defined over $\mathcal{O}'$, then $(\mathcal{Y}, \mathcal{H}, \mathcal{X})$ is defined over $\mathcal{O}$, where

$$\mathcal{Y} = S^{-1}\mathcal{O} \subset \mathcal{Y}' \text{ , } \mathcal{H} = S^{-1}\mathcal{O} \cap T^{-1}\mathcal{O} \subset \mathcal{H}' \text{, and } \mathcal{X} = T^{-1}\mathcal{O} \subset \mathcal{X}'$$

and the inclusions induce a well-defined map

$$B(\mathcal{Y}, \mathcal{H}, \mathcal{X}) \rightarrow B(\mathcal{Y}', \mathcal{H}', \mathcal{X}').$$

In practice, by comparisons of quasifibrations, these maps will all be weak homotopy equivalent to BHX'.

For example, to show that BHX is homotopy equivalences to BHX' when X and X' are spaces of the homotopy type of a given compact space $F \in \mathcal{W}$, we need only map the quasifibrations EHX → BHX and EHX' → BHX' to the quasifibration $E\mathcal{H} \rightarrow B\mathcal{H}$, where $\mathcal{H}$ is the full subcategory of $F\mathcal{W}$ with objects X and X'. Indeed, by use of larger sets of objects, we actually obtain a coherent system of homotopy equivalences connecting the BHX as X ranges through the given homotopy type. Of course, these remarks apply equally well to any other homogeneous category of fibres and can easily be elaborated to a precise comparison of the natural transformations Φ and Ψ (of the proof of Theorem 9.2) obtained by use of different choices of privileged fibre.

13. The algebraic and geometric bar constructions

Let R be a commutative ring and take all homology with coefficients in R throughout the last three sections.

Just as the two-sided geometric bar construction is obtained by composing the simplicial bar construction on appropriate triples of spaces with geometric realization, so the two-sided algebraic bar construction is obtained by composing the simplicial bar construction B_* on appropriate triples of differential R-modules with the condensation functor C from simplicial differential R-modules to differential R-modules. The reader is referred to the Appendix of [10] for the details of the definition. Write B(M, U, N) for the algebraic bar construction, where U is a differential R-algebra and M and N are right and left differential U-modules. Write EU = B(R, U, U) and BU = B(R, U, R), rather than the standard notations of [10], in conformity with the notations for the geometric bar construction.

In view of the similarity between their definitions, it is natural to expect there to be a close relationship between the geometric and algebraic bar constructions. In the case of BG, this relationship was first made precise by Stasheff [31], singularly, and later by Milgram [23], cellularly.

Let $\mathcal{C}$ denote the subcategory of $\mathcal{U}$ which consists of the CW-complexes and cellular maps. There is a derived subcategory $\mathcal{A}(\mathcal{C})$ of $\mathcal{A}(\mathcal{U})$, with objects those triples (Y, G, X) such that $Y, G, X \in \mathcal{C}$ and the product and unit of G and its actions on Y and X are cellular maps. The component maps of morphisms in $\mathcal{A}(\mathcal{C})$ are also required to be cellular.

We shall write $C_\#$ for cellular chains and C_* for normalized singular chains (both with coefficients in R). The product of CW-complexes X and Y is a CW-complex (since we are working in $\mathcal{U}$) with a natural cell structure such that $C_\#(X \times Y)$ can be identified with $C_\# X \otimes C_\# Y$.

In order to get the right signs, we agree to write simplices to the left, $|X| = \coprod \Delta_p \times X_p/(\approx)$, when forming the geometric realization of simplicial spaces.

Proposition 13.1. Let $(Y, G, X) \in \mathcal{A}(\mathcal{G})$. Then $B(Y, G, X)$ is a CW-complex with a natural cell structure such that $C_\# B(Y, G, X)$ is naturally isomorphic to $B(C_\# Y, C_\# G, C_\# X)$.

Proof. Give $B_p(Y, G, X) = Y \times G^p \times X$ the product cell structure. Then, by [17, 11.4], $B(Y, G, X)$ is a CW-complex with one (n+p)-cell $\Delta_p \times \sigma$ for each n-cell $\sigma = \sigma_0 \times \sigma_1 \times \ldots \times \sigma_p \times \sigma_{p+1}$ of $Y \times (G - e)^p \times X$. Let $\Delta_p \times \sigma$ correspond to the element $\sigma_0[\sigma_1, \ldots, \sigma_p]\sigma_{p+1}$ of $B(C_\# Y, C_\# G, C_\# X)$. The pieces $\partial\Delta_p \times \sigma$ and $\Delta_p \times \partial\sigma$ of the boundary of $\Delta_p \times \sigma$ give rise to the simplicial (or external) and internal components of the differential $d = \sum_{i=0}^{p} (-1)^i \partial_i + (-1)^p \partial$ in the bar construction (see [10, A.2]).

We require the following addendum to [17, 11.15] in order to describe the behavior of products under the isomorphism of chains given in the proposition.

Lemma 13.2. Let X and Y be simplicial objects in $\mathcal{C}$. Then the natural homeomorphism $f': |X \times Y| \to |X| \times |Y|$ is naturally homotopic to a cellular map f and the natural homeomorphism $g = (f')^{-1}$ is itself cellular.

Proof. $f = (G_0 \times G_1)f'$, where the G_i are the Alexander-Whitney type homotopy equivalences specified in the proof of [17, 11.15], and the requisite verifications are the same as there.

We also write f and g for the maps of the geometric bar construction derived via the identifications (of [17, 10.1])

$$B_*(Y, G, X) \times B_*(Y', G', X') = B_*(Y \times Y', G \times G', X \times X').$$

In precise analogy, [10, A.3] gives natural chain homotopy equivalences, the Alexander-Whitney and shuffle maps,

$$\xi: C(M \otimes N) \to CM \otimes CN \quad \text{and} \quad \eta: CM \otimes CN \to C(M \otimes N)$$

for simplicial differential R-modules M and N. We also write ξ and η for the maps of the algebraic bar construction derived via the identifications (again, of [17, 10.1])

$$B_*(M, U, N) \otimes B_*(M', U', N') = B_*(M \otimes M', U \otimes U', N \otimes N').$$

Proposition 13.3. Let (Y, G, X) and (Y', G', X') be objects of $\mathcal{A}(\mathcal{C})$. Then, under the isomorphism of Proposition 13.1, $C_\# f$ coincides with ξ and $C_\# g$ coincides with η.

Proof. This is verified by explicit calculations. We omit the details (which really amount only to checks of signs) since the precise definitions of ξ and η were motivated by the present result.

The following results give special properties of BG and EG.

Proposition 13.4. If G is a cellular topological monoid, then BG admits a cellular diagonal approximation with respect to which $C_{\#}BG$ is naturally isomorphic to $BC_{\#}G$ as a differential coalgebra.

Proof. $f' \circ B\Delta = \Delta: BG \to BG \times BG$, and $f \circ B\Delta = (G_0 \times G_1)\Delta$ is the desired diagonal approximation; it is cellular by [17, 11.15]. The chain level statement does not follow by naturality from the previous result, since $B\Delta$ need not be cellular, but instead requires an easy direct calculation from the explicit coproduct

$$D[g_1, \ldots, g_p] = \sum_{i=0}^{p} (-1)^{(p-i)q_i}[g_1, \ldots, g_i] \otimes [g_{i+1}, \ldots, g_p] \; ,$$
$$q_i = \sum_{j=1}^{i} \deg g_j \; ,$$

on $BC_{\#}G$. Note that D is independent of any possible coproduct on $C_{\#}G$.

Proposition 13.5. If G is a cellular topological group, then so is EG.

Proof. This follows by naturality from the identification of EG with $|D_*G|$ in the proof of Theorem 8.1.

Proposition 13.6. If G is a cellular Abelian topological monoid, then so are BG and EG. $C_{\#}BG$ is naturally isomorphic to $BC_{\#}G$ as a differential Hopf algebra.

Proof. The first part is immediate by naturality and the second part follows from Propositions 13.4 and 13.3 since the product on $BC_{\#}G$ is derived from the shuffle map by naturality.

Remarks 13.7. Let π be an Abelian group regarded as a discrete CW-complex. As noted by Milgram [23], if we define $K(\pi, n) = B^n\pi$ by iteration, then $K(\pi, n)$ is a cellular Abelian topological group such that $C_{\#}K(\pi, n) \cong B^n(R\pi)$ as a differential Hopf algebra, where $R\pi$ denotes the group ring of π. This gives an alternative derivation to the classical one (given in detail in [10, Appendix]) of the geometric preliminaries necessary for Cartan's calculations of $H^*K(\pi, n)$.

In order to apply the above results in full generality, we note the following result. Although it surely ought to be well-known, it seems not to appear in the literature. Let S denote the total singular complex functor from spaces to simplicial sets, let T denote the geometric realization functor from simplicial sets to spaces, and let $\Phi : TSX \to X$ denote the natural weak homotopy equivalence, $\Phi|u, f| = f(u)$ for $u \in \Delta_p$ and $f: \Delta_p \to X$ (see e.g. [16]). Of course, T takes values in $\mathcal{C}$.

Proposition 13.8. The normalized singular chains C_*X are naturally isomorphic to the cellular chains $C_{\#}TSX$. The Alexander-Whitney and shuffle maps ξ and η agree under this isomorphism with the respective composites

$$C_\# TS(X\times Y) = C_\# T(SX\times SY) \xrightarrow{C_\# f} C_\#(TSX\times TSY) = C_\# TSX \otimes C_\# TSY$$

and

$$C_\# TSX \otimes C_\# TSY = C_\#(TSX\times TSY) \xrightarrow{C_\# g} C_\# T(SX\times SY) = C_\# TS(X\times Y) .$$

Proof. Let a non-degenerate simplex $f:\Delta_p \to X$ correspond to the cell $\Delta_p \times f \subset TSX$, where the p-simplex f is regarded as a vertex of the discrete CW-complex S_pX. This correspondence sets up the required isomorphism. (Note that, by the paragraph after [10, A.7], ξ and η on singular chains are special cases of the maps ξ and η of [10, A.3].)

Thus the geometry of simplices provides a rigorous construction of the Alexander-Whitney and shuffle maps on singular chains, rather than just the motivation for an algebraic definition.

The following theorem is the main technical result on the relationship between the geometric and algebraic bar constructions.

Theorem 13.9. For $(Y,G,X) \in \mathcal{A}(\mathcal{U})$, $B(TSY, TSG, TSX)$ admits a cellular diagonal approximation with respect to which $C_\# B(TSY, TSG, TSX)$ is naturally isomorphic to $B(C_*Y, C_*G, C_*X)$ as a differential coalgebra. Therefore $H^*B(Y,G,X)$ is naturally isomorphic as an algebra to the homology of the dual of $B(C_*Y, C_*G, C_*X)$.

Proof. On the level of differential R-modules, Propositions 13.1 and 13.8 establish the required isomorphism. For the statement about the diagonal, consider the following diagram (in which the isomorphisms are given by the results just cited):

$$
\begin{array}{ccc}
C_{\#}B(TSY, TSG, TSX) & \cong & B(C_{*}Y, C_{*}G, C_{*}X) \\
\Big\downarrow{\scriptstyle C_{\#}B(TS\Delta, TS\Delta, TS\Delta)} & & \Big\downarrow{\scriptstyle B(C_{*}\Delta, C_{*}\Delta, C_{*}\Delta)} \\
C_{\#}B(TS(Y\times Y), TS(G\times G), TS(X\times X)) & \cong & B(C_{*}(Y\times Y), C_{*}(G\times G), C_{*}(X\times X)) \\
\Big\downarrow{\scriptstyle C_{\#}B(f,f,f)} & & \Big\downarrow{\scriptstyle B(\xi,\xi,\xi)} \\
C_{\#}B(TSY\times TSY, TSG\times TSG, TSX\times TSX) & \cong & B(C_{*}Y\otimes C_{*}Y, C_{*}G\otimes C_{*}G, C_{*}X\otimes C_{*}X) \\
\Big\downarrow{\scriptstyle C_{\#}f} & & \Big\downarrow{\scriptstyle \xi} \\
C_{\#}B(TSY,TSG,TSX)\otimes C_{\#}B(TSY,TSG,TSX) & \cong & B(C_{*}Y,C_{*}G,C_{*}X)\otimes B(C_{*}Y,C_{*}G,C_{*}X)
\end{array}
$$

The fact that (ξ, ξ, ξ) is a morphism of triples in the appropriate category, so that $B(\xi, \xi, \xi)$ is defined, follows from the commutative diagram displayed in [10, A.3]. Since $\xi = C_{\#}f$ and $\eta = C_{\#}g$, the same combinatorial proof shows that the analogous diagram with condensation replaced by geometric realization also commutes, hence that (f, f, f) is a morphism in $\mathfrak{A}(\mathfrak{C})$. The upper two squares of our diagram commute by naturality (from Proposition 13.1), and the bottom square commutes by Proposition 13.3. The map $f \circ \dot{B}(f \circ TS\Delta, f \circ TS\Delta, f \circ TS\Delta)$ is homotopic to the diagonal, and the coproduct on $B(C_{*}Y, C_{*}G, C_{*}X)$ is defined to be $\xi \circ B(\xi \circ C_{*}\Delta, \xi \circ C_{*}\Delta, \xi \circ C_{*}\Delta)$.

The last statement (which is given in cohomology solely in order to avoid flatness hypotheses for coproducts) follows since the map

$$B(\Phi, \Phi, \Phi): B(TSY, TSG, TSX) \to B(Y, G, X)$$

induces an isomorphism on homology by Proposition 7.3.

We complete this section with a discussion of the following theorem.

Theorem 13.10. Let $(Y, G, X) \in \mathcal{A}(\mathcal{U})$ and let R be a field. Then there is a natural spectral sequence of differential coalgebras which converges from the coalgebra $E^2 = \mathrm{Tor}^{H_*G}(H_*Y, H_*X)$ to the coalgebra $H_*B(Y, G, X)$.

R is taken to be a field to avoid awkward flatness hypotheses.

There are two conceptually different proofs. First, following Rothenberg and Steenrod [29], we can construct an exact couple by passage to homology from the filtration by successive cofibrations of the space $B(Y, G, X)$. This approach is analyzed in [17, 11.14], where the E^2-term is computed. Because of its geometric nature, this approach makes it simple to put Steenrod operations into the spectral sequence when $R = Z_p$ and is applicable to any homology theory with an appropriate Kunneth theorem.

Second, following Eilenberg and Moore [26], we can filter the algebraic bar construction $B(M, U, N)$ by

$$F_pB(M, U, N) = \mathrm{Image} \sum_{i=0}^{p} B_i(M, U, N)$$

and observe that $E^1B(M, U, N) = B(HM, HU, HN)$, which is a suitable differential R-module (or R-coalgebra if HU is a Hopf algebra and HM and HN are right and left coalgebras over HU) for the computation of $\mathrm{Tor}^{HU}(HM, HN)$. This approach is analyzed in [10, A.9 and § 1, 2, 5]. Because of its algebraic nature, it gives maximal information on the internal structure of the spectral sequence.

Theorem 13.9 demonstrates the applicability of the second approach to the calculation of $H_*B(Y, G, X)$ and proves that both approaches yield the same spectral sequence. Indeed, the map $B(\Phi, \Phi, \Phi)$ certainly induces an isomorphism of Rothenberg-Steenrod spectral sequences, and the isomorphism between $C_\# B(TSY, TSG, TSX)$ and $B(C_*Y, C_*G, C_*X)$ implies that the Rothenberg-Steenrod spectral sequence for the triple (TSY, TSG, TSX) is isomorphic to the Eilenberg-Moore spectral sequence for the triple (C_*Y, C_*G, C_*X). Note that our chain level isomorphism yields a particularly conceptual proof that the obvious algebraic coproduct in E^2 converges to the correct geometric coproduct in the limit.

14. Transports and the Serre spectral sequence

As in section 1, assume given a category $\mathcal{F}$ with a faithful underlying space functor to $\mathcal{U}$. Recall that $\mathcal{T}$ denotes the category of non-degenerately based spaces in $\mathcal{U}$. For $A \in \mathcal{T}$, let ΛA and PA denote the Moore loop space and path space on A and let $p: PA \to A$ denote the end-point projection. The associated principal fibration functor previously denoted by P will be denoted by Prin here.

Definition 14.1. An $\mathcal{F}$-transport over a space $A \in \mathcal{T}$ is a space $X \in \mathcal{F}$ together with an (associative and unital) action $\tau: \Lambda A \times X \to X$ such that $\tau(\lambda, \ ): X \to X$ is a map in $\mathcal{F}$ for each $\lambda \in \Lambda A$. Define $\mathcal{T}\mathcal{F}(A)$ to be the collection, assumed to be a set, of equivalence classes of $\mathcal{F}$-transports over A under the equivalence relation generated by $\tau \approx \tau'$ if there exists a map $\phi: X \to X'$ in $\mathcal{F}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \Lambda A \times X & \xrightarrow{\tau} & X \\ {\scriptstyle 1\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \Lambda A \times X' & \xrightarrow{\tau'} & X' \end{array}$$

For a map $f: A' \to A$ in $\mathcal{T}$ and an $\mathcal{F}$-transport τ over A, define an $\mathcal{F}$-transport $f^*\tau$ over A' by $(f^*\tau)(\lambda', x) = \tau(f \circ \lambda', x)$ for $\lambda' \in \Lambda A'$ and $x \in X$. Then $\mathcal{T}\mathcal{F}$ is a contravariant functor from $\mathcal{T}$ to sets.

Observe that the adjoint $\Lambda A \to \mathcal{F}(X, X)$ of an $\mathcal{F}$-transport τ is a map of topological monoids and that, conversely, such a map has adjoint an $\mathcal{F}$-transport over A. Clearly $\mathcal{T}\mathcal{F}(A)$ is isomorphic to the set of equivalence classes of maps of monoids $\gamma: \Lambda A \to \mathcal{F}(X, X)$, $X \in \mathcal{F}$, under the equivalence relation generated by $\gamma \approx \gamma'$ if there exists a map $\phi: X \to X'$ in $\mathcal{J}$ such that $\phi(\gamma\lambda)(x) = (\gamma'\lambda)(\phi x)$ for all $\lambda \in \Lambda A$ and $x \in X$.

While the proof of the following theorem works somewhat more generally, we shall restrict attention to the categories of Example 6.6 for simplicity. The idea of the result, and the term "transport", are due to Stasheff [33].

Theorem 14.2. Let $F \in \mathcal{W}$ and let $\mathcal{J}$ be $F\mathcal{U}$ or, if F is compact, $F\mathcal{W}$. Then, for $A \in \mathcal{V}$, $\mathcal{E}\mathcal{J}(A)$ is naturally isomorphic to $\mathcal{T}\mathcal{J}(A)$.

Proof. $\mathcal{E}\mathcal{J}(A) \cong [A, BHF]$ is a well-defined set, and it will follow from the rest of the proof that $\mathcal{T}\mathcal{J}(A)$ is also well-defined. Define $\Phi : \mathcal{E}\mathcal{J}(A) \to \mathcal{T}\mathcal{J}(A)$ as follows. Given an $\mathcal{J}$-fibration $\nu : D \to A$, let $F\nu = (\Gamma\nu)^{-1}(*) \subset \Gamma D$ and let $\tau: \Lambda A \times F\nu \to F\nu$ be obtained by restriction from $\mu : \Gamma\Gamma D \to \Gamma D$. Then let $\Phi\{\nu\} = \{\tau\}$. Define $\Psi: \mathcal{T}\mathcal{J}(A) \to \mathcal{E}\mathcal{J}(A)$ as follows. Given an $\mathcal{J}$-transport $\tau: \Lambda A \times X \to X$, use it and the natural right action of ΛA on PA to construct the quotient space $PA \times_{\Lambda A} X$; it is weak Hausdorff because of the nondegeneracy of the basepoint $* \in A$. Define $\pi: PA \times_{\Lambda A} X \to A$ by $\pi(\beta, x) = p(\beta)$ and note that $\pi^{-1}(*) = X$. Define a transitive lifting function

$$\xi : \quad \Gamma(PA \times_{\Lambda A} X) \to PA \times_{\Lambda A} X$$

for π by $\xi(\alpha, (\beta, x)) = (\alpha\beta, x)$. Then define $\Psi\{\tau\} = \{\nu\}$. If τ is derived from $\nu : D \to A$ as in the definition of Φ, then

$$\mu : PA \times F\nu \subset \Gamma\Gamma D \to \Gamma D$$

induces an $\mathcal{J}$-map $PA \times_{\Lambda A} F\nu \to \Gamma D$ over A which restricts on $F\nu$ to the identity map. Since $\{\Gamma\nu\} = \{\nu\}$, it follows that $\Psi\Phi$ is the identity transformation of $\mathcal{E}\mathfrak{J}(A)$. Conversely, if π is derived as above from a $\mathcal{J}$-transport τ, so that

$$F\pi = \{(\alpha, (\beta, x)) \mid \alpha \in \prod A,\ (\beta, x) \in PA \times_{\Lambda A} F,\ \alpha(0) = p(\beta), p(\alpha) = *\},$$

define $\phi: F\pi \to X$ by $\phi(\alpha, (\beta, x)) = \tau(\alpha\beta, x)$. Then the diagram

$$\begin{array}{ccc} \Lambda A \times F\pi & \xrightarrow{\ \mu\ } & F\pi \\ \downarrow{\scriptstyle 1\times\phi} & & \downarrow{\scriptstyle \phi} \\ \Lambda A \times X & \xrightarrow{\ \tau\ } & X \end{array}$$

is commutative, because $\tau(\lambda, \tau(\alpha\beta, x)) = \tau(\lambda\alpha\beta, x)$, and therefore $\Phi\Psi$ is the identity transformation of $\mathcal{J}\mathfrak{J}(A)$.

Since, up to equivalence, a fibration determines and is determined by a transport, it is plausible that the Serre spectral sequence can be derived by use of a differential R-module which depends only on a transport. We shall show that this is the case. We first note the following fact.

Lemma 14.3. If $A \in \mathcal{J}$ is connected, then the diagram

$$A \xleftarrow{\ \varepsilon(p)\ } B(PA, \Lambda A, *) \xrightarrow{\ q\ } B\Lambda A$$

displays a weak homotopy equivalence between A and $B\Lambda A$.

Proof. Since A is connected, the fibration $p: PA \to A$ maps onto A and is thus a quasifibration. The result follows from Corollary 7.7.

Recall that homology and cohomology are to be taken with coefficients in R.

Theorem 14.4. Let $A \in \mathcal{J}$ be connected and let $\nu : D \to A$ be a quasifibration with $F = \nu^{-1}(*)$. Then there is a natural spectral sequence $\{E_r \nu\}$ of differential algebras such that $E_2\nu = H^*(A; \mathcal{H}^*(F))$ as an algebra and $\{E_r \nu\}$ converges to the algebra H^*D.

Proof. The result is stated in cohomology to avoid flatness hypotheses for coproducts, and we shall work in homology. Replacing ν by $\Gamma\nu : \Gamma D \to A$ if necessary, we may assume without loss of generality that ν has a transitive lifting function $\xi : \Gamma D \to D$. By restriction of ξ, we obtain

$$\xi : PA \times F \to D \quad \text{and} \quad \tau : \Lambda A \times F \to F .$$

The following diagram is commutative:

$$\begin{array}{ccccc} D & \xleftarrow{\varepsilon(\xi)} & B(PA, \Lambda A, F) & \xrightarrow{q} & B(*, \Lambda A, F) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ A & \xleftarrow{\varepsilon(p)} & B(PA, \Lambda A, *) & \xrightarrow{q} & B\Lambda A \end{array}$$

$\varepsilon(\xi)$ restricts to the identity map on $F = p^{-1}(*)$, hence, by the lemma and Theorem 7.6, the upper row displays a weak homotopy equivalence between D and $B(*, \Lambda A, F)$. By Theorem 13.9, we conclude that $H_*D \cong HB(R, C_*\Lambda A, C_*F)$. Filter this algebraic bar construction by writing, additively,

$$B(R, C_*\Lambda A, C_*F) = BC_*\Lambda A \otimes C_*F$$

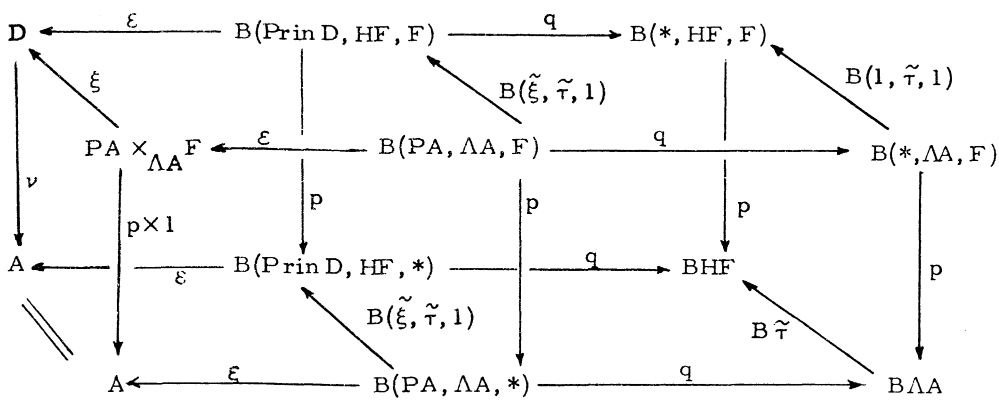
and then defining

$$F_pB(R, C_*\Lambda A, C_*F) = \sum_{i \leq p} B_iC_*\Lambda A \otimes C_*F .$$

Let $\{E^r\nu\}$ denote the derived spectral sequence. The degree i referred to in the filtration is the total degree, hence, visibly, the differential d^0 is just $1 \otimes d$. Thus, additively, $E^1_{pq}\nu = B_pC_*\Lambda A \otimes H_qF$ by the Künneth theorem. We may rewrite $E^1\nu = B(R, C_*\Lambda A, H_*F)$; this equality holds as differential R-modules, the point being that the last face operator depends on the action of $C_*\Lambda A$ on H_*F induced by the transport τ. Since $BC_*\Lambda A \cong C_\# B\Lambda A$, by Theorem 13.9 again, we conclude that $E^1\nu \cong C_\#(B\Lambda A; \mathcal{H}_*F)$. The reader is referred to Steenrod [36, §31] for a thorough treatment of cellular chains with local coefficients. Thus $E^2\nu \cong H_*(A; \mathcal{H}_*F)$. For the coproducts, merely note that Theorem 13.9 implies both that $B(R, C_*\Lambda A, C_*F)$ is a filtered differential coalgebra isomorphic to $C_\# B(*, \Lambda A, F)$ and that $E^1\nu$ is isomorphic to $C_\#(B\Lambda A; \mathcal{H}_*F)$ as a differential coalgebra.

A comparison to the standard construction of the Serre spectral sequence from a filtration on C_*D can easily be obtained by means of a chain level elaboration of the geometric diagram displayed in the proof. Our construction is similar in philosophy, but not in detail, to that given by Brown [3] in terms of twisted tensor products.

There is more than just a formal similarity between the diagram used in the previous argument and those used in the proof of Theorem 9.2. Indeed, with the notations of the proof just given, the following diagram is commutative (where $\tilde{\tau}$ and $\tilde{\xi}$ denote the adjoints of τ and ξ) :



This diagram displays a connection between the classification theorems of Corollary 9.5 and Theorem 14.2. For $A \in \mathcal{V}$, the classifying map $A \to BHF$ for ν is transported to $B\tilde{\tau} : B\Lambda A \to BHF$ via the canonical homotopy equivalence between A and $B\Lambda A$. Of course, since Corollary 9.5 refers to a fixed chosen space $F \in \mathcal{F}$ whereas Theorem 14.2 refers to arbitrary spaces in $\mathcal{F}$, it would be necessary to use the theory of section 12 to make a more systematic comparison between the cited results.

15. The group completion theorem

We say that an H-space X is admissible if it is homotopy associative and if left translation by any given element is homotopic to right translation by the same element; the latter condition certainly holds if X is homotopy commutative.

Let $f: X \rightarrow Y$ be an H-map between admissible H-spaces, where Y is grouplike. We say that f is a group completion if $f_*: H_*X \rightarrow H_*Y$ is a localization of the ring H_*X at its multiplicative submonoid $\pi_0 X$ for every commutative coefficient ring R. This condition will hold if it does so for all prime fields $R = Z_p$ and for $R = Q$ [28 or 18, 1.4], hence we assume that R is a field below.

The purpose of this section is to prove the following result, which is a version due to Quillen [28] of a theorem of Barratt and Priddy [1].

Theorem 15.1. Let G be a topological monoid such that G and ΩBG are admissible H-spaces. Then the natural inclusion $\zeta: G \rightarrow \Omega BG$ is a group completion.

According to Quillen, the admissibility of ΩBG need not be assumed. The argument in [28] seemed unconvincing on this point, and Quillen subsequently obtained a quite different proof [private communication]. However, this hypothesis is usually satisfied in practice and allows the present technically simplified version of Quillen's original argument.

We begin the proof with the following lemma.

Lemma 15.2. $\pi_0\zeta : \pi_0 G \to \pi_0 \Omega BG$ is a group completion; that is, $\pi_0\zeta$ is universal with respect to morphisms of monoids from $\pi_0 G$ to groups.

Proof. $\pi_1 BG = H_1(BG; Z)$ since $\pi_0 \Omega BG$ is commutative by the admissibility of ΩBG. By Theorem 13.9, we may use $BC_* G$ to compute $H_1 BG$. We find that it is the quotient of $H_0 G$ by the image of $H_0 G \otimes H_0 G$ under the map $d = \partial_0 - \partial_1 + \partial_2$, or

$$d(x \otimes y) = \varepsilon(x)y - xy + \varepsilon(y)x ,$$

where $\varepsilon : H_0 G \to Z$ is the augmentation and xy is the Pontryagin product. In other words, $H_1 BG$ is the quotient of the free Abelian group generated by the set $\pi_0 G$ by the subgroup generated by $\{x+y-xy \mid x, y \in \pi_0 G\}$. Clearly $\pi_0\zeta$ agrees with the natural map, and the conclusion follows.

To proceed further, we need Segal's (unpublished) analog for "special" simplicial spaces of the total singular complex. Let $\mathcal{S}\mathcal{J}$ denote the category of proper simplicial based spaces.

Definition 15.3. An object $Y \in \mathcal{S}\mathcal{J}$ is said to be reduced if $Y_0 = *$ and to be special if, in addition, the map $(\delta^0, \ldots, \delta^{p-1}) : Y_p \to Y_1^p$ is a homotopy equivalence for all p, where $\delta^i = \partial_0 \ldots \partial_{i-1}\partial_{i+2} \ldots \partial_p$. For example, $B_* G$ is special for any topological monoid $G \in \mathcal{J}$. Let $\mathcal{S}^+\mathcal{J}$ denote the full subcategory of $\mathcal{S}\mathcal{J}$ whose objects are special. Let $T: \mathcal{S}^+\mathcal{J} \to \mathcal{J}$ denote the geometric realization functor. Define a right adjoint $S: \mathcal{J} \to \mathcal{S}^+\mathcal{J}$ to T by letting $S_p X$ be the space of maps

$(\Delta_p, \Delta_p^0) \to (X, *)$, where Δ_p^0 denotes the set of vertices of Δ_p, and letting the face and degeneracy operators be induced from those of Δ_p just as for the total singular complex [16, p. 2]. Obviously $S_1X = \Omega X$, and it is trivial to verify that SX is special. The adjunction

$$\mathcal{S}^+\mathcal{J}(Y, SX) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{J}(TY, X)$$

is given by $(\phi f)|y, u| = (fy)(u)$ and $(\psi g)(y)(u) = g|y, u|$ for $f: Y \to SX$ and $g: TY \to X$, where $y \in Y_p$ and $u \in \Delta_p$, precisely as in the classical case [16, p. 62]. Write

$$\Phi = \phi(1): TSX \to X \quad\text{and}\quad \Psi = \psi(1): Y \to STY.$$

We shall need the following variant of Lemma 14.3. Its proof requires only trivial verifications and use of Proposition 8.7.

Lemma 15.4. Let $X \in \mathcal{J}$ and define $\xi: B\Lambda X \to X$ by the formula

$$\xi|[\lambda_1, \ldots, \lambda_p], u| = (\lambda_1 \cdots \lambda_p)\Big(\sum_{i=1}^{p} u_i \ell(\lambda_i)\Big)$$

for $\lambda_i \in \Lambda X$ and $u = (t_0, \ldots, t_p) \in \Delta_p$, where $u_i = t_0 + \ldots + t_{i-1}$. Then the following composite is the identity map, hence ξ is a weak homotopy equivalence if X is connected:

$$\Omega X \subset \Lambda X \xrightarrow{\zeta} \Omega B\Lambda X \xrightarrow{\Omega\xi} \Omega X .$$

Proposition 15.5. $\Phi: TSX \to X$ is a weak homotopy equivalence for any connected space $X \in \mathcal{J}$ and $T\Psi: TY \to TSTY$ is a weak homotopy equivalence for any special simplicial space $Y \in \mathcal{S}^+\mathcal{J}$.

Proof. On purely categorical grounds, $\xi : B\Lambda X \to X$ is the composite

$$B\Lambda X = TB_*\Lambda X \xrightarrow{T\psi(\xi)} TSX \xrightarrow{\Phi} X .$$

A glance at the definitions shows that, on 1-simplices, $\psi_1(\xi): \Lambda X \to \Omega X$ agrees with the natural retraction. Since $B_*\Lambda X$ and SX are special, it follows that $\psi_p(\xi): B_p\Lambda X \to S_pX$ is a homotopy equivalence for all p. Therefore $T\psi(\xi)$ is a homotopy equivalence by [18, A.3]. This implies the first clause, and the second clause follows since the composite

$$TY \xrightarrow{T\Psi} TSTY \xrightarrow{\Phi} TY$$

is the identity map and since TY is connected by [17, 11.11].

As a final preliminary, we require the following variant of the comparison theorem (which is similar to Quillen's formulation [27, 3.8]).

Lemma 15.6. Let $\{f^r\}: \{E^r\} \to \{\overline{E}^r\}$ be a morphism of first quadrant spectral sequences. Assume that

(i) $E^\infty = E^0A$, $\overline{E}^\infty = E^0\overline{A}$, and $f^\infty = E^0 f$ for filtered graded Abelian groups A and $\overline{A}$ and a filtration preserving isomorphism $f: A \to \overline{A}$ of graded Abelian groups;

(ii) $E^2_{0q} = 0$ and $\overline{E}^2_{0q} = 0$ for $q > 0$; and

(iii) For a given $n \geq 0$, f^2_{pq} is an isomorphism for $q < n$ and all p.

Then f^2_{1n} is an isomorphism and f^2_{2n} is an epimorphism.

Proof. By the argument in [13, p. 356 and 357], (iii) implies

(iv) f^r_{pq} is a monomorphism if $q < n$ and an isomorphism if $q \leq n+1-r$.

By induction on q (for fixed $p+q$), (iv) and (i) imply

(v) f^{∞}_{pq} is an isomorphism if $q < n$ and an epimorphism if $q = n$. (Use the exact sequences $0 \to F_{p-1}A \to F_pA \to E^{\infty}_{p*} \to 0$.) By downwards induction on r, (ii), (iv), and (v) imply

(vi) f^r_{pq}, $r \geq p$, is an isomorphism if $q < n$ and an epimorphism if $q = n$. (Use the exact sequences $E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{pq} \longrightarrow E^{r+1}_{pq} \longrightarrow 0$.) With $r = p = 2$, the second conclusion follows. Finally, E^r_{1n} is an isomorphism for $r \geq 2$ by an argument just like that at the bottom of [13, p. 357].

We can now use a variant of Quillen's argument [28, §4] to prove the theorem. For $Y \in \mathscr{S}^+\mathscr{J}$, consider the Segal spectral sequence $\{E^rY\}$ [30; 17, 11.14]. $\{E^rY\}$ converges to H_*TY. Since we are taking coefficients in a field R and Y is special, $E^1Y = BH_*Y_1$ and $E^2Y = \mathrm{Tor}^{H_*Y_1}(R,R)$. Clearly $E^2_{00}Y = R$ and $E^2_{0q}Y = 0$ for $q > 0$. From $\Psi: B_*G \to STB_*G = SBG$ we derive $\{E^r\Psi\}: \{E^rB_*G\} \to \{E^rSBG\}$. Since $T\Psi$ is a weak homotopy equivalence, by Proposition 15.5, hypothesis (i) of Lemma 15.6 is satisfied. Definition 15.3 shows that, on 1-simplices, $\Psi_1 = \zeta : G \to \Omega BG$. Therefore $E^2\Psi = \mathrm{Tor}^{\zeta_*}(1,1)$.

For brevity of notation, we agree to write

$$g = \pi_0 G\ ,\ \bar{g} = \pi_0\Omega BG\ ,\ A = H_*G\ ,\ B = H_*\Omega BG\ .$$

Let $\iota : A \to \bar{A}$ denote the localization of A at its submonoid g. Note that $\bar{A}$ is A-flat (as a limit of free A-modules [18, 1.2]) and that $R = R \otimes_A \bar{A}$ (because the augmentation $\varepsilon: \bar{A} \to R$ takes the value one on elements of $\bar{g}$ regarded via Lemma 15.2 as elements of $\bar{A}$). Therefore

$$\mathrm{Tor}^{\iota}(1,1):\ \mathrm{Tor}^{A}(R,R) \to \mathrm{Tor}^{\overline{A}}(R,R)$$

is an isomorphism by [5, VI 4.1.1]. Thus, if $\zeta : \overline{A} \to B$ denotes the unique ring homomorphism such that $\overline{\zeta}\iota = \zeta_*$, then $E^2\Psi$ can be identified with

$$\mathrm{Tor}^{\overline{\zeta}}(1,1): \mathrm{Tor}^{\overline{A}}(R,R) \to \mathrm{Tor}^{B}(R,R)\ .$$

A, hence also $\overline{A}$, and B are Hopf algebras which, as coalgebras, are direct sums of connected coalgebras. Let $\overline{A}_e$ and B_e denote the components of the identity elements of $\overline{A}$ and B. If $R\overline{g}$ denotes the group ring of $\overline{g}$, then $A = \overline{A}_e \otimes R\overline{g}$ and $B = B_e \otimes R\overline{g}$ as R-algebras because $\overline{g}$ is central in both $\overline{A}$ and B by the admissibility of G and ΩBG. Clearly $\overline{\zeta} = \overline{\zeta}_e \otimes 1$, where $\overline{\zeta}_e$ is the restriction of $\overline{\zeta}$ to $\overline{A}_e$. By the Kunneth theorem for torsion products [5, XI 3.1],

$$\mathrm{Tor}^{\overline{A}}(R,R) = \mathrm{Tor}^{\overline{A}_e}(R,R) \otimes \mathrm{Tor}^{R\overline{g}}(R,R)$$

and similarly for B, hence $\mathrm{Tor}^{\overline{\zeta}}(1,1) = \mathrm{Tor}^{\overline{\zeta}_e}(1,1) \otimes 1$. We shall prove the following statement by induction on n.

$$P_n : \overline{\zeta}_e : \overline{A}_e \to B_e \text{ is an isomorphism in degrees } \leq n.$$

P_0 is trivial and we assume P_{n-1}. Then $E^2_{pq}\Psi$ is an isomorphism for $q < n$ and therefore, by Lemma 15.6, $E^2_{1n}\Psi$ is an isomorphism and $E^2_{2n}\Psi$ is an epimorphism. This means (e.g., by [40, § 7]) that $\overline{\zeta}_e$ induces a bijection between minimal sets of generators and a surjection between minimal sets of defining relations in degrees $\leq$ n and thus implies P_n. Alternatively, as in Quillen [28], P_n follows by application of the five lemma to exact sequences given by initial segments of the bar constructions on $\overline{A}_e$ and B_e.

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Addenda: December, 1979

We list some improvements and applications; lettered references are to the bibliography which follows.

Gaunce Lewis [A, B] has found an alternative way to handle the problems with based fibrations mentioned on page 21. Given a map $\pi: E \to B$ with section σ, we can grow whiskers on fibres to replace π by a map whose section is a fibrewise cofibration. The whiskering of fibres in Remarks 9.7 is a special case, and Lewis has given a generally applicable criterion for when the section of $p: B(Y, G, F) \to B(Y, G, *)$ is already a fibrewise cofibration, without whiskering. When the section σ of π is a fibrewise cofibration, Lewis has shown that $\Gamma\pi: \Gamma E \to B$ can be given a new lifting function which, unlike the natural lifting function μ, is actually a $\mathcal{J}$-lifting function. This allows one to avoid introducing Γ'. Direct use of Γ simplifies various arguments, such as those on page 24.

R. Schön [E] has given a very clever and utterly trivial proof of the important theorem of Stasheff quoted on page 27.

In [D, Lemma 1.1], I prove that $\mathcal{E}F\mathcal{U}(A)$ and $\mathcal{E}F\mathcal{J}(A)$ of Corollaries 9.5 and 9.8 are equal to the appropriate sets of fibre homotopy equivalence classes of fibrations over A, even when F is not compact. Thus the technical distinction emphasized on page 25 tends to disappear in the most important applications.

References [19] and [20], which give the promised application of the present theory to the classification of oriented bundles and fibrations, have appeared as two chapters in [C]. The promised application to fibrewise localization and completion will appear in [D].

Stefan Waner [F, G, H] has given an equivariant generalization of the present theory and of its application to oriented fibrations. Here one considers G-fibrations for a compact Lie group G. The depth and complication of the generalization comes from the need to allow non-trivial actions of G on the base space and different actions of the appropriate isotropy subgroups on fibres.

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