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The Galois group of a stable homotopy theory

Akhil Mathew

Department of Mathematics, Harvard University, Cambridge, MA 02138, United States

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A B S T R A C T

To a "stable homotopy theory" (a presentable, symmetric monoidal stable ∞-category), we naturally associate a category of finite étale algebra objects and, using Grothendieck's categorical machine, a profinite group that we call the Galois group. We then calculate the Galois groups in several examples. For instance, we show that the Galois group of the periodic $\mathbf{E}_\infty$-algebra of topological modular forms is trivial and that the Galois group of $K(n)$-local stable homotopy theory is an extended version of the Morava stabilizer group. We also describe the Galois group of the stable module category of a finite group. A fundamental idea throughout is the purely categorical notion of a "descendable" algebra object and an associated analog of faithfully flat descent in this context.

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E-mail address: amathew@math.harvard.edu.

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1. Introduction

Let X be a connected scheme. One of the basic arithmetic invariants that one can extract from X is the *étale fundamental group* $\pi_1(X, \overline{x})$ relative to a "basepoint" $\overline{x} \to X$ (where $\overline{x}$ is the spectrum of a separably closed field). The fundamental group was defined by Grothendieck [30] in terms of the category of finite, étale covers of X. It provides an analog of the usual fundamental group of a topological space (or rather, its profinite completion), and plays an important role in algebraic geometry and number theory, as a precursor to the theory of étale cohomology. From a categorical point of view, it unifies the classical Galois theory of fields and covering space theory via a single framework.

In this paper, we will define an analog of the étale fundamental group, and construct a form of the Galois correspondence, in stable homotopy theory. For example, while the classical theory of [30] enables one to define the fundamental (or Galois) group of a commutative ring, we will define the fundamental group of the homotopy-theoretic analog: an $\mathbf{E}_\infty$-ring spectrum.

The idea of a type of Galois theory applicable to structured ring spectra begins with Rognes's work in [71], where, for a finite group G, the notion of a *G-Galois extension* of $\mathbf{E}_\infty$-ring spectra $A \to B$ was introduced (and more generally, E-local G-Galois extensions for a spectrum E). Rognes's definition is an analog of the notion of a finite G-torsor of commutative rings in the setting of "brave new" algebra, and it includes many highly non-algebraic examples in stable homotopy theory. For instance, the "complexification" map $KO \to KU$ from real to complex K-theory is a fundamental example of a $\mathbb{Z}/2$-Galois extension. Rognes has also explored the more general theory of *Hopf–Galois* extensions, intended as a topological version of the idea of a torsor over a group *scheme* in algebraic geometry, as has Hess in [32]. More recently, the PhD thesis of Pauwels [65] has studied Galois theory in tensor-triangulated categories.

In this paper, we will take the setup of an *axiomatic stable homotopy theory.* For us, this will mean:

Definition 1.1. An **axiomatic stable homotopy theory** is a presentable, symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbf{1})$ where the tensor product commutes with all colimits.

An axiomatic stable homotopy theory defines, at the level of homotopy categories, a *tensor-triangulated category.* Such axiomatic stable homotopy theories arise not only from stable homotopy theory itself, but also from representation theory and algebra, and we will discuss many examples below. We will associate, to every axiomatic stable homotopy theory $\mathcal{C}$, a profinite group (or, in general, groupoid) which we call the *Galois group* $\pi_1(\mathcal{C})$. In order to do this, we will give a definition of a *finite cover* generalizing the notion of a Galois extension, and, using heavily ideas from descent theory, show that these can naturally be arranged into a Galois category in the sense of Grothendieck. We will actually define two flavors of the fundamental group, one of which depends only on the structure of the dualizable objects in $\mathcal{C}$ and is appropriate to the study of "small" symmetric monoidal ∞-categories.

Our thesis is that the Galois group of a stable homotopy theory is a natural invariant that one can attach to it; some of the (better studied) others include the algebraic K-theory (of the compact objects, say), the lattice of thick subcategories, and the Picard group. We will discuss several examples. The classical fundamental group in algebraic geometry can be recovered as the Galois group of the derived category of quasi-coherent sheaves. Rognes's Galois theory (or rather, *faithful* Galois theory) is the case of $\mathcal{C} = \mathrm{Mod}(R)$ for R an $\mathbf{E}_\infty$-algebra.

Given a stable homotopy theory $(\mathcal{C}, \otimes, \mathbf{1})$, the collection of all homotopy classes of maps $\mathbf{1} \to \mathbf{1}$ is naturally a commutative ring $R_{\mathcal{C}}$. In general, there is always a surjection of profinite groups

$$\pi_1 \mathcal{C} \twoheadrightarrow \pi_1^{\mathrm{et}} \operatorname{Spec} R_{\mathcal{C}}. \tag{1}$$

The étale fundamental group of $\operatorname{Spec} R_{\mathcal{C}}$ represents the "algebraic" part of the Galois theory of $\mathcal{C}$. For example, if $\mathcal{C} = \mathrm{Mod}(R)$ for R an $\mathbf{E}_\infty$-algebra, then the "algebraic" part of the Galois theory of $\mathcal{C}$ corresponds to those $\mathbf{E}_\infty$-algebras under R which are finite étale at the level of homotopy groups. It is an insight of Rognes that, in general, the Galois group contains a topological component as well: the map (1) is generally not an isomorphism. The remaining Galois extensions (which behave much differently on the level of homotopy groups) can be quite useful computationally.

In the rest of the paper, we will describe several computations of these Galois groups in various settings. Our basic tool is the following result, which is a refinement of (a natural generalization of) the main result of [14].

Theorem 1.2. *If R is an even periodic $\mathbf{E}_\infty$-ring with $\pi_0 R$ regular noetherian, then the Galois group of R is that of the discrete ring $\pi_0 R$: that is, (1) is an isomorphism.*

Using various techniques of descent theory, and a version of van Kampen's theorem, we are able to compute Galois groups in several other examples of stable homotopy theories "built" from $\mathrm{Mod}(R)$ where R is an even periodic $\mathbf{E}_\infty$-ring; these include in particular many arising from both chromatic stable homotopy theory and modular representation theory. In particular, we prove the following three theorems.

Theorem 1.3. *The Galois group of the ∞-category $L_{K(n)}\mathrm{Sp}$ of $K(n)$-local spectra is the extended Morava stabilizer group.*

Theorem 1.4. *The Galois group of the $\mathbf{E}_\infty$-algebra* TMF *of (periodic) topological modular forms is trivial.*

Theorem 1.5. *Given a finite group G and a separably closed field k of characteristic p, the Galois group of the stable module ∞-category of $k[G]$ is the profinite completion of the nerve of the category of G-sets of the form $\{G/A\}$ where $A \subset G$ is a nontrivial elementary abelian p-subgroup.*

These results suggest a number of other settings in which the computation of Galois groups may be feasible, for example, in stable module ∞-categories for finite group *schemes*. We hope that these results and ideas will, in addition, shed light on some of the other invariants of $\mathbf{E}_\infty$-ring spectra and stable homotopy theories.

2. Axiomatic stable homotopy theory

As mentioned earlier, the goal of this paper is to extract a Galois group(oid) from a *stable homotopy theory*. Once again, we restate the definition.

Definition 2.1. A **stable homotopy theory** is a presentable, symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbf{1})$ where the tensor product commutes with all colimits.

In this section, intended mostly as background, we will describe several general features of the setting of stable homotopy theories. We will discuss a number of examples, and then construct a basic class of commutative algebra objects in any such $\mathcal{C}$ (the so-called "étale algebras") whose associated corepresentable functors can be described very easily. The homotopy categories of stable homotopy theories, which acquire both a tensor structure and a compatible triangulated structure, have been described at length in the memoir [36]. In addition, their invariants have been studied in detail in the program of tensor triangular geometry of Balmer (cf. [4] for a survey).

2.1. Stable ∞-categories

Let $\mathcal{C}$ be a stable ∞-category in the sense of [50, Ch. 1]. Recall that stability is a *condition* on an ∞-category, rather than extra data, in the same manner that, in ordinary category theory, being an abelian category is a property. The homotopy category of a stable ∞-category is canonically *triangulated,* so that stable ∞-categories may be viewed as enhancements of triangulated categories; however, as opposed to traditional DG-enhancements, stable ∞-categories can be used to model phenomena in stable homotopy theory (such as the ∞-category of spectra, or the ∞-category of modules over a structured ring spectrum).

Here we will describe some general features of stable ∞-categories, and in particular the constructions one can perform with them. Most of this is folklore (in the setting of triangulated or DG-categories) or in [50].

Definition 2.2. Let Cat_∞ be the ∞-category of (small) ∞-categories. Given ∞-categories $\mathcal{C}, \mathcal{D}$, the mapping space $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D})$ is the maximal ∞-groupoid contained in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors $\mathcal{C} \to \mathcal{D}$.

Definition 2.3. We define an ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ of (small) stable ∞-categories where:

1. The objects of $\mathrm{Cat}_\infty^{\mathrm{st}}$ are the stable ∞-categories which are idempotent complete.[1]

[1] This can be removed, but will be assumed for convenience.

2. Given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{st}}$, the mapping space $\mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(\mathcal{C}, \mathcal{D})$ is the union of connected components in $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite limits (or, equivalently, colimits). Such functors are called *exact.*

The ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ has all limits, and the forgetful functor $\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty$ commutes with limits. For example, given a diagram in $\mathrm{Cat}_\infty^{\mathrm{st}}$

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D} \arrow[r, "G"] & \mathcal{E} \end{tikzcd},$$

we can form a pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ consisting of triples (X, Y, f) where $X \in \mathcal{C}, Y \in \mathcal{D}$, and $f\colon F(X) \simeq G(Y)$ is an equivalence. This pullback is automatically stable.

Although the construction is more complicated, $\mathrm{Cat}_\infty^{\mathrm{st}}$ is also cocomplete. For example, the colimit (in Cat_∞) of a *filtered* diagram of stable ∞-categories and exact functors is automatically stable, so that the inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \subset \mathrm{Cat}_\infty$ preserves filtered colimits. In general, one has:

Proposition 2.4. $\mathrm{Cat}_\infty^{\mathrm{st}}$ *is a presentable ∞-category.*

To understand this, it is convenient to work with the (big) ∞-category Pr^L.

Definition 2.5. (See [44, 5.5.3].) Pr^L is the ∞-category of presentable ∞-categories and colimit-preserving (or left adjoint) functors.

The ∞-category Pr^L is known to have all colimits (cf. [44, 5.5.3]). We briefly review this here. Given a diagram $F\colon I \to \mathrm{Pr}^L$, we can form the dual I^{op}-indexed diagram in the ∞-category Pr^R of presentable ∞-categories and *right* adjoints between them. Now we can form a *limit* in Pr^R at the level of underlying ∞-categories; by duality between $\mathrm{Pr}^L, \mathrm{Pr}^R$ in the form $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$, this can be identified with the colimit $\varinjlim_I F$ in Pr^L.

In other words, for each map $f\colon i \to i'$ in I, consider the induced adjunction of ∞-categories $L_f, R_f\colon F(i) \rightleftarrows F(i')$. Then an object x in $\varinjlim_I F$ is the data of:

1. For each $i \in I$, an object $x_i \in F(i)$.
2. For each $f\colon i \to i'$, an isomorphism $x_i \simeq R_f(x_{i'})$.
3. Higher homotopies and coherences.

For each i, we get a natural functor in Pr^L, $F(i) \to \varinjlim_I F$. We have a tautological description of the *right adjoint*, which to an object x in $\varinjlim_I F$ as above returns $x_i \in F(i)$.

Example 2.6. Let $\mathcal{S}_*$ be the ∞-category of pointed spaces and pointed maps between them. We have an endofunctor $\Sigma\colon \mathcal{S}_* \to \mathcal{S}_*$ given by suspension, whose right adjoint is the loop functor $\Omega\colon \mathcal{S}_* \to \mathcal{S}_*$. The filtered colimit in Pr^L of the diagram

$$\mathcal{S}_* \stackrel{\Sigma}{\to} \mathcal{S}_* \stackrel{\Sigma}{\to} \dots,$$

can be identified, by this description, as the ∞-category of sequences of pointed spaces $(X_0, X_1, X_2, \dots,)$ together with equivalences $X_n \simeq \Omega X_{n+1}$ for $n \geq 0$: in other words, one recovers the ∞-category of spectra.

Proposition 2.7. *Suppose $F\colon I \to \mathrm{Pr}^L$ is a diagram where, for each $i \in I$, the ∞-category $F(i)$ is compactly generated; and where, for each $i \to i'$, the left adjoint $F(i) \to F(i')$ preserves compact objects.*[2] *Then each $F(i) \to \varinjlim_I F$ preserves compact objects, and $\varinjlim_I F$ is compactly generated.*

Proof. It follows from the explicit description of $\varinjlim_I F$, in fact, that the right adjoints to $F(i) \to \varinjlim_I F$ preserve filtered colimits; this is dual to the statement that the left adjoints preserve compact objects. Moreover, the images of each compact object in each $F(i)$ in $\varinjlim_I F$ can be taken as compact generators, since they are seen to detect equivalences. $\square$

Definition 2.8. $\mathrm{Pr}^{L,\omega}$ is the ∞-category of compactly generated, presentable ∞-categories and colimit-preserving functors which preserve compact objects.

It is fundamental that $\mathrm{Pr}^{L,\omega}$ is equivalent to the ∞-category of idempotent complete, finitely cocomplete ∞-categories and finitely cocontinuous functors, under the construction $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ starting from the latter and ending with the former (and the dual construction that takes an object in $\mathrm{Pr}^{L,\omega}$ to its subcategory of compact objects). Proposition 2.7 implies that colimits exist in $\mathrm{Pr}^{L,\omega}$ and the inclusion $\mathrm{Pr}^{L,\omega} \to \mathrm{Pr}^L$ preserves them.

Corollary 2.9. $\mathrm{Pr}^{L,\omega}$ *is a presentable ∞-category.*

Proof. It suffices to show that any idempotent complete, finitely cocomplete ∞-category is a filtered colimit of such of bounded cardinality (when modeled via quasi-categories, for instance). For simplicity, we will sketch the argument for finitely cocomplete quasi-categories. The idempotent complete case can be handled similarly by replacing filtered colimits with $\aleph_1$-filtered colimits.

To see this, let $\mathcal{C}$ be such a quasi-category. Consider any countable simplicial subset $\mathcal{D}$ of $\mathcal{C}$ which is a quasi-category. We will show that $\mathcal{D}$ is contained in a bigger countable

[2] This is equivalent to the condition that the *right adjoints* preserve filtered colimits.

simplicial subset $\overline{\mathcal{D}}$ of $\mathcal{C}$ which is a finitely cocomplete quasi-category such that $\overline{\mathcal{D}} \to \mathcal{C}$ preserves finite colimits. This will show that $\mathcal{C}$ is the filtered union of such subsets $\overline{\mathcal{D}}$ (ordered by set-theoretic inclusion) and will thus complete the proof.

Thus, fix $\mathcal{D} \subset \mathcal{C}$ countable. For each finite simplicial set K, and each map $K \to \mathcal{D}$, by definition there is an extension $K^{\triangleright} \to \mathcal{C}$ which is a colimit diagram. We can find a countable simplicial set $\mathcal{D}'$ such that $\mathcal{D} \subset \mathcal{D}' \subset \mathcal{C}$ such that every diagram $K \to \mathcal{D}$ extends over a diagram $K^{\triangleright} \to \mathcal{D}'$ such that the composite $K^{\triangleright} \to \mathcal{D}' \to \mathcal{C}$ is a colimit diagram in $\mathcal{C}$. Applying the small object argument (countably many times), we can find a countable quasi-category $\mathcal{D}_1$ with $\mathcal{D} \subset \mathcal{D}_1 \subset \mathcal{C}$ such that any diagram $K \to \mathcal{D}_1$ extends over a diagram $K^{\triangleright} \to \mathcal{D}_1$ such that the composite $K^{\triangleright} \to \mathcal{D}_1 \to \mathcal{C}$ is a colimit diagram. It follows thus that any countable simplicial subset $\mathcal{D}$ of $\mathcal{C}$ containing all the vertices is contained in such a (countable) $\mathcal{D}_1$. (At each stage in the small object argument, we also have to add in fillers to all inner horns.)

Thus, consider any countable simplicial subset $\mathcal{D} \subset \mathcal{C}$ which is a quasi-category containing all the vertices of $\mathcal{C}$, and such that any diagram $K \to \mathcal{D}$ (for K finite) extends over a diagram $K^{\triangleright} \to \mathcal{D}$ such that the composite $K^{\triangleright} \to \mathcal{C}$ is a colimit diagram. We have just shown that $\mathcal{C}$ is a (filtered) union of such. Of course, $\mathcal{D}$ may not have all the colimits we want. Consider the (countable) collection $S_{\mathcal{D}}$ of all diagrams $f\colon K^{\triangleright} \to \mathcal{D}$ whose composite $K^{\triangleright} \xrightarrow{f} \mathcal{D} \to \mathcal{C}$ is a colimit. We want to enlarge $\mathcal{D}$ so that each of these becomes a colimit, but not too much; we want $\mathcal{D}$ to remain countable.

For each $f \in S_{\mathcal{D}}$, consider $\mathcal{D}_{K/} \subset \mathcal{C}_{K/}$. By construction, we have an object in $\mathcal{D}_{K/}$ which is initial in $\mathcal{C}_{K/}$. By adding a countable number of simplices to $\mathcal{D}$, though, we can make this initial in $\mathcal{D}_{K/}$ too; that is, there exists a $\mathcal{D}' \subset \mathcal{D}$ with the same properties such that the object defined is initial in $\mathcal{D}'_{K/}$. Iterating this process (via the small object argument), we can construct a countable simplicial subset $\overline{\mathcal{D}} \subset \mathcal{C}$, containing $\mathcal{D}$, which is a quasi-category and such that any diagram $K \to \overline{\mathcal{D}}$ extends over a diagram $K^{\triangleright} \to \overline{\mathcal{D}}$ which is a colimit preserved under $\overline{\mathcal{D}} \to \mathcal{C}$. This completes the proof. $\square$

We can use this to describe $\mathrm{Cat}_\infty^{\mathrm{st}}$. We have a *fully faithful* functor

$$\mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Pr}^{L,\omega},$$

which sends a stable ∞-category $\mathcal{C}$ to the *compactly generated*, presentable stable ∞-category $\mathrm{Ind}(\mathcal{C})$. In fact, $\mathrm{Cat}_\infty^{\mathrm{st}}$ can be identified with the ∞-category of stable, presentable, and compactly generated ∞-categories, and colimit-preserving functors between them that also preserve compact objects, so that $\mathrm{Cat}_\infty^{\mathrm{st}} \subset \mathrm{Pr}^{L,\omega}$ as a full subcategory.

Proof of Proposition 2.4. We need to show that $\mathrm{Cat}_\infty^{\mathrm{st}}$ has all colimits. Using the explicit construction of a colimit of presentable ∞-categories, however, it follows that a colimit of presentable, *stable* ∞-categories is stable. In particular, $\mathrm{Cat}_\infty^{\mathrm{st}}$ has colimits and they are computed in $\mathrm{Pr}^{L,\omega}$.

Finally, we need to show that any object in $\mathrm{Cat}_\infty^{\mathrm{st}}$ is a filtered union of objects in $\mathrm{Cat}_\infty^{\mathrm{st}}$ of bounded cardinality. This can be argued similarly as above (we just need to add stability into the mix). □

Compare also the treatment of stable ∞-categories in [8], which shows (cf. [8, Th. 4.22]) that $\mathrm{Cat}_\infty^{\mathrm{st}}$ can be obtained as an accessible localization of the ∞-category associated to a combinatorial model category and indeed shows that $\mathrm{Cat}_\infty^{\mathrm{st}}$ is compactly generated [8, Cor. 4.25].

We will need some examples of limits and colimits in $\mathrm{Cat}_\infty^{\mathrm{st}}$. Compare [8, sec. 5] for a detailed treatment.

Definition 2.10. Let $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{st}}$ and let $\mathcal{D} \subset \mathcal{C}$ be a full, stable idempotent complete subcategory. We define the **Verdier quotient** $\mathcal{C}/\mathcal{D}$ to be the pushout in $\mathrm{Cat}_\infty^{\mathrm{st}}$

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}/\mathcal{D} \end{tikzcd}.$$

Fix $\mathcal{E} \in \mathrm{Cat}_\infty^{\mathrm{st}}$. By definition, to give an exact functor $\mathcal{C}/\mathcal{D} \to \mathcal{E}$ is equivalent to giving an exact functor $\mathcal{C} \to \mathcal{E}$ which sends every object in $\mathcal{D}$ to a zero object; note that this is a *condition* rather than extra data. The Verdier quotient can be described very explicitly. Namely, consider the inclusion $\mathrm{Ind}(\mathcal{D}) \subset \mathrm{Ind}(\mathcal{C})$ of stable ∞-categories. For any $X \in \mathrm{Ind}(\mathcal{C})$, there is a natural cofiber sequence

$$M_{\mathcal{D}}X \to X \to L_{\mathcal{D}}X,$$

where:

1. $M_{\mathcal{D}}X$ is in the full stable subcategory of $\mathrm{Ind}(\mathcal{C})$ generated under colimits by $\mathcal{D}$ (i.e., $\mathrm{Ind}(\mathcal{D})$).
2. For any $D \in \mathcal{D}$, $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(D, L_{\mathcal{D}}X)$ is contractible.

One can construct this sequence by taking $M_{\mathcal{D}}$ to be the right adjoint to the inclusion functor $\mathrm{Ind}(\mathcal{D}) \subset \mathrm{Ind}(\mathcal{C})$.

We say that an object $X \in \mathrm{Ind}(\mathcal{C})$ is $\mathcal{D}^\perp$-*local* if $M_{\mathcal{D}}X$ is contractible. The full subcategory $\mathcal{D}^\perp \subset \mathrm{Ind}(\mathcal{C})$ of $\mathcal{D}^\perp$-local objects is a localization of $\mathrm{Ind}(\mathcal{C})$, with localization functor given by $L_{\mathcal{D}}$. We have an adjunction

$$\mathrm{Ind}(\mathcal{C}) \rightleftarrows \mathcal{D}^\perp,$$

where the right adjoint, the inclusion $\mathcal{D}^\perp \subset \mathcal{C}$, is fully faithful. The inclusion $\mathcal{D}^\perp \subset \mathrm{Ind}(\mathcal{C})$ preserves filtered colimits since $\mathcal{D} \subset \mathrm{Ind}(\mathcal{C})$ consists of compact objects, so that the localization $L_{\mathcal{D}}$ preserves compact objects. Now, the Verdier quotient can be described as the subcategory of $\mathcal{D}^{\perp}$ spanned by compact objects (in $\mathcal{D}^{\perp}$); it is generated under finite colimits and retracts by the image of objects in $\mathcal{C}$. Moreover, $\mathrm{Ind}(\mathcal{C}/\mathcal{D})$ is precisely $\mathcal{D}^{\perp} \subset \mathrm{Ind}(\mathcal{C})$.

Remark 2.11. The pushout diagram defining the Verdier quotient is also a pullback.

Remark 2.12. A version of this construction makes sense in the world of presentable, stable ∞-categories (which need not be compactly generated).

These Verdier quotients have been considered, for example, in [56] under the name *finite localizations.*

2.2. Stable homotopy theories and 2-rings

In this paper, our goal is to describe an invariant of *symmetric monoidal* stable ∞-categories. For our purposes, we can think of them as *commutative algebra objects* with respect to a certain *tensor product* on $\mathrm{Cat}_\infty^{\mathrm{st}}$. We begin by reviewing this and some basic properties of stable homotopy theories, which are the "big" versions of these.

Definition 2.13. (See [50, 4.8], [17].) Given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{st}}$, we define the *tensor product* $\mathcal{C} \boxtimes \mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{st}}$ via the universal property

$$\mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E}), \tag{2}$$

where $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ consists of those functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve finite colimits in each variable separately.

It is known (see [50, 4.8]) that this defines a symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{st}}$. The commutative algebra objects are *precisely* the symmetric monoidal, stable ∞-categories $(\mathcal{C}, \otimes, \mathbf{1})$ such that the tensor product preserves finite colimits in each variable.

Definition 2.14. We let $2\text{-Ring} = \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})$ be the ∞-category of commutative algebra objects in $\mathrm{Cat}_\infty^{\mathrm{st}}$. We will also write $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ for the ∞-category of stable homotopy theories (i.e., presentable stable symmetric monoidal ∞-categories with bicontinuous tensor product); this is the "big" version of 2-Ring.

The tensor product $\boxtimes\colon \mathrm{Cat}_\infty^{\mathrm{st}} \times \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ preserves filtered colimits in each variable; this follows from (2). In particular, since $\mathrm{Cat}_\infty^{\mathrm{st}}$ is a presentable ∞-category, it follows that 2-Ring is a presentable ∞-category.

In this paper, we will define a functor

$$\pi_{\leq 1}\colon 2\text{-Ring} \to \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}},$$

where we will specify what the latter means below, called the Galois groupoid. The Galois groupoid will parametrize certain very special commutative algebra objects in a given 2-ring. Given a stable homotopy theory $(\mathcal{C}, \otimes, \mathbf{1})$ (in the sense of Definition 2.1), the invariant we will define will depend only on the small subcategory $\mathcal{C}^{\mathrm{dual}}$ of *dualizable* objects in $\mathcal{C}$.

We will also define a slightly larger version of the Galois groupoid that will see more of the "infinitary" structure of the stable homotopy theory, which will make a difference in settings where the unit is not compact (such as $K(n)$-local stable homotopy theory). In this case, it will not be sufficient to work with 2-Ring. However, the interplay between 2-Ring and the theory of (large) stable homotopy theories will be crucial in the following.

Definition 2.15. (Cf. [50, 4.6.1].) In a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbf{1})$, an object X is **dualizable** if there exist an object Y and maps

$$\mathbf{1} \xrightarrow{\mathrm{coev}} Y \otimes X, \quad X \otimes Y \xrightarrow{\mathrm{ev}} \mathbf{1},$$

such that the composites

$$X \simeq X \otimes \mathbf{1} \xrightarrow{1_X \otimes \mathrm{coev}} X \otimes Y \otimes X \xrightarrow{\mathrm{ev} \otimes 1_X} X, \quad Y \simeq \mathbf{1} \otimes Y \xrightarrow{\mathrm{coev} \otimes 1_Y} Y \otimes X \otimes Y \xrightarrow{1_Y \otimes \mathrm{ev}} Y$$

are homotopic to the respective identities. In other words, X is dualizable if and only if it is dualizable in the homotopy category with its induced symmetric monoidal structure.

These definitions force natural homotopy equivalences

$$\mathrm{Hom}_{\mathcal{C}}(Z, Z' \otimes X) \simeq \mathrm{Hom}_{\mathcal{C}}(Z \otimes Y, Z'), \quad Z, Z' \in \mathcal{C}. \tag{3}$$

Now let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. The collection of all dualizable objects in $\mathcal{C}$ (cf. also [36, sec. 2.1]) is a *stable* and idempotent complete subcategory, which is closed under the monoidal product. Moreover, suppose that $\mathbf{1}$ is κ-compact for some regular cardinal κ. Then (3) with $Z = \mathbf{1}$ forces any dualizable object Y to be κ-compact as well. In particular, it follows that the subcategory of $\mathcal{C}$ spanned by the dualizable objects is (essentially) small and belongs to 2-Ring. (By contrast, no amount of compactness is sufficient to imply dualizability.)

We thus have the two constructions:

1. Given a stable homotopy theory, take the symmetric monoidal, stable ∞-category of dualizable objects, which is a 2-ring.
2. Given an object $\mathcal{C} \in$ 2-Ring, $\mathrm{Ind}(\mathcal{C})$ is a stable homotopy theory.

These two constructions are generally not inverse to one another. However, the "finitary" version of the Galois group we will define will be unable to see the difference.

Next, we will describe some basic constructions in 2-Ring. The ∞-category 2-Ring has all limits, and these may be computed at the level of the underlying ∞-categories. As such, these homotopy limit constructions can be used to build new examples of 2-rings from old ones. These constructions will also apply to stable homotopy theories. To start with, we discuss Verdier quotients.

Definition 2.16. Let $(\mathcal{C}, \otimes, \mathbf{1}) \in$ 2-Ring and let $\mathcal{I} \subset \mathcal{C}$ be a full stable, idempotent complete subcategory. We say that $\mathcal{I}$ is an **ideal** or **$\otimes$-ideal** if whenever $X \in \mathcal{C}, Y \in \mathcal{I}$, the tensor product $X \otimes Y \in \mathcal{C}$ actually belongs to $\mathcal{I}$.

If $\mathcal{I} \subset \mathcal{C}$ is an ideal, then the Verdier quotient $\mathcal{C}/\mathcal{I}$ naturally inherits the structure of an object in 2-Ring. This follows naturally from [50, Proposition 2.2.1.9] and the explicit construction of the Verdier quotient. By definition, $\mathrm{Ind}(\mathcal{C}/\mathcal{I})$ consists of the objects $X \in \mathrm{Ind}(\mathcal{C})$ which have the property that $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(I, X)$ is contractible when $I \in \mathcal{I}$. We can describe this as the localization of $\mathrm{Ind}(\mathcal{C})$ at the collection of maps $f\colon X \to Y$ whose cofiber belongs to $\mathrm{Ind}(\mathcal{I})$. These maps, however, form an ideal since $\mathcal{I}$ is an ideal. As before, given $\mathcal{D} \in$ 2-Ring, we have a natural fully faithful inclusion

$$\mathrm{Hom}_{\text{2-Ring}}(\mathcal{C}/\mathcal{I}, \mathcal{D}) \subset \mathrm{Hom}_{\text{2-Ring}}(\mathcal{C}, \mathcal{D}),$$

where the image of the map consists of all symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ which take every object in $\mathcal{I}$ to a zero object.

Finally, we describe some *free* constructions. Let Sp be the ∞-category of spectra, and let $\mathcal{C}$ be a small symmetric monoidal ∞-category. Then the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is a stable homotopy theory under the *Day convolution product* [50, 4.8.1]. Consider the collection of compact objects in here, which we will write as the "monoid algebra" $\mathrm{Sp}^{\omega}[\mathcal{C}]$. One has the universal property

$$\mathrm{Hom}_{\text{2-Ring}}(\mathrm{Sp}^{\omega}[\mathcal{C}], \mathcal{D}) \simeq \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D}),$$

i.e., an equivalence between functors of 2-rings $\mathrm{Sp}[\mathcal{C}] \to \mathcal{D}$ and symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$. We can also define the free stable homotopy theory on $\mathcal{C}$ as the Ind-completion of this 2-ring, or equivalently as $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$.

Example 2.17. The free symmetric monoidal ∞-category on a single object is the disjoint union $\bigsqcup_{n \geq 0} B\Sigma_n$, or the groupoid of finite sets and isomorphisms between them, with $\sqcup$ as the symmetric monoidal product. Using this, we can describe the "free stable homotopy theory" on a single object. As above, an object in this stable homotopy theory consists of giving a spectrum X_n with a Σ_n-action for each n; the tensor structure comes from a convolution product. If we consider the compact objects in here, we obtain the free 2-ring on a given object.

Finally, we will need to discuss a bit of algebra internal to $\mathcal{C}$.

Definition 2.18. There is a natural ∞-category of *commutative algebra objects* in $\mathcal{C}$ (cf. [50, Ch. 2]) which we will denote by $\mathrm{CAlg}(\mathcal{C})$. When $\mathcal{C} = \mathrm{Sp}$ is the ∞-category, we will just write CAlg for the ∞-category of $\mathbf{E}_\infty$-ring spectra.

Recall that a commutative algebra object in $\mathcal{C}$ consists of an object $X \in \mathcal{C}$ together with a multiplication map $m\colon X \otimes X \to X$ and a unit map $\mathbf{1} \to X$, which satisfy the classical axioms of a commutative algebra object up to coherent homotopy; for instance, when $\mathcal{C} = \mathrm{Sp}$, one obtains the classical notion of an $\mathbf{E}_\infty$-ring. The amount of homotopy coherence is sufficient to produce the following:

Definition 2.19. (See [50, Sec. 4.5].) Let $\mathcal{C}$ be a stable homotopy theory. Given $A \in \mathrm{CAlg}(\mathcal{C})$, there is a natural ∞-category $\mathrm{Mod}_{\mathcal{C}}(A)$ of A-module objects internal to $\mathcal{C}$. The ∞-category $\mathrm{Mod}_{\mathcal{C}}(A)$ acquires the structure of a stable homotopy theory with the relative A-linear tensor product.

The relative A-linear tensor product requires the formation of geometric realizations, so we need infinite colimits to exist in $\mathcal{C}$ for the above construction to make sense in general.

2.3. Examples

Stable homotopy theories and 2-rings occur widely in "nature," and in this section, we describe a few basic classes of such widely occurring examples. We begin with two of the most fundamental ones.

Example 2.20 *(Derived categories).* The derived ∞-category $D(R)$ of a commutative ring R (cf. [50, Sec. 1.3]) with the derived tensor product is a stable homotopy theory.

Example 2.21 *(Modules over an $\mathbf{E}_\infty$-ring).* As a more general example, the ∞-category $\mathrm{Mod}(R)$ of modules over an $\mathbf{E}_\infty$-ring spectrum R with the relative smash product is a stable homotopy theory. For instance, taking $R = S^0$, we get the ∞-category Sp of spectra. This is the primary example (together with E-localized versions) considered in [71].

Example 2.22 *(Quasi-coherent sheaves).* Let X be a scheme (or algebraic stack, or even prestack). To X, one can associate a stable homotopy theory $\mathrm{QCoh}(X)$ of *quasi-coherent complexes* on X. By definition, $\mathrm{QCoh}(X)$ is the homotopy limit of the derived ∞-categories $D(R)$ where $\operatorname{Spec} R \to X$ ranges over all maps from affine schemes to X. For more discussion, see [17].

Example 2.23. Consider a cartesian diagram of $\mathbf{E}_\infty$-rings

$$\begin{tikzcd} A \times_{A''} A' \arrow[r] \arrow[d] & A \arrow[d] \\ A' \arrow[r] & A'' \end{tikzcd}.$$

We obtain a diagram of stable homotopy theories

$$\begin{tikzcd} \mathrm{Mod}(A \times_{A''} A') \arrow[r] \arrow[d] & \mathrm{Mod}(A) \arrow[d] \\ \mathrm{Mod}(A') \arrow[r] & \mathrm{Mod}(A'') \end{tikzcd},$$

and in particular a symmetric monoidal functor

$$\mathrm{Mod}(A \times_{A''} A') \to \mathrm{Mod}(A) \times_{\mathrm{Mod}(A'')} \mathrm{Mod}(A').$$

This functor is generally not an equivalence in 2-Ring.

This functor is *always* fully faithful. However, if A, A', A'' are *connective* and $A \to A''$, $A' \to A''$ induce surjections on π_0, then it is proved in [45, Theorem 7.2] that the functor induces an equivalence on the *connective* objects or, more generally, on the k-connective objects for any $k \in \mathbb{Z}$. In particular, if we let Mod^ω denote perfect modules, we have an equivalence of 2-rings

$$\mathrm{Mod}^\omega(A \times_{A''} A') \simeq \mathrm{Mod}^\omega(A) \times_{\mathrm{Mod}^\omega(A'')} \mathrm{Mod}^\omega(A'),$$

since an $A \times_{A''} A'$-module is perfect if and only if its base-changes to A, A' are. However, the Ind-construction generally does not commute even with finite limits.

Example 2.24 *(Functor categories).* As another example of a (weak) 2-limit, we consider any ∞-category K and a stable homotopy theory $\mathcal{C}$; then $\mathrm{Fun}(K, \mathcal{C})$ is naturally a stable homotopy theory under the "pointwise" tensor product. If $K = BG$ for a group G, then this example endows the ∞-category of objects in $\mathcal{C}$ with a G-action with the structure of a stable homotopy theory.

Finally, we list several other miscellaneous examples of stable homotopy theories.

Example 2.25 *(Hopf algebras).* Let A be a finite-dimensional cocommutative Hopf algebra over the field k. In this case, the (ordinary) category $\mathcal{A}$ of discrete A-modules has a natural symmetric monoidal structure via the k-linear tensor product. In particular, its *derived* ∞-category $D(\mathcal{A})$ is naturally symmetric monoidal, and is thus a stable homotopy theory. Stated more algebro-geometrically, $\mathrm{Spec}\, A^\vee$ is a group scheme G over the field k, and $D(\mathcal{A})$ is the ∞-category of quasi-coherent sheaves of complexes on the classifying stack BG.

Example 2.26 *(Stable module ∞-categories).* Let A be a finite-dimensional cocommutative Hopf algebra over the field k. Consider the subcategory $D(\mathcal{A})^\omega \subset D(\mathcal{A})$ (where $\mathcal{A}$ is the abelian category of A-modules, as in Example 2.25) of A-module spectra which are perfect as k-module spectra. Inside $D(\mathcal{A})^\omega$ is the subcategory $\mathcal{I}$ of those objects which are perfect as A-module spectra. This subcategory is stable, and is an *ideal* by the observation (a projection formula of sorts) that the k-linear tensor product of A with any A-module is free as an A-module.

Definition 2.27. The **stable module ∞-category** $\mathrm{St}_A = \mathrm{Ind}(D(\mathcal{A})^\omega/\mathcal{I})$ is the Ind-completion of the Verdier quotient $D(\mathcal{A})^\omega/\mathcal{I}$. If $A = k[G]$ is the group algebra of a finite group G, we write $\mathrm{St}_G(k)$ for $\mathrm{St}_{k[G]}$.

The stable module ∞-categories of finite-dimensional Hopf algebras (especially group algebras) and their various invariants (such as the Picard groups and the thick subcategories) have been studied extensively in the modular representation theory literature. For a recent survey, see [10].

Example 2.28 *(Bousfield localizations).* Let $\mathcal{C}$ be a stable homotopy theory, and let $E \in \mathcal{C}$. In this case, there is a naturally associated stable homotopy theory $L_E\mathcal{C}$ of *E-local objects*. By definition, $L_E\mathcal{C}$ is a full subcategory of $\mathcal{C}$; an object $X \in \mathcal{C}$ belongs to $L_E\mathcal{C}$ if and only if whenever $Y \in \mathcal{C}$ satisfies $Y \otimes E \simeq 0$, the spectrum $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is contractible. The ∞-category $L_E\mathcal{C}$ is symmetric monoidal under the *E-localized tensor product*: since the tensor product of two E-local objects need not be E-local, one needs to localize further. For example, the unit object in $L_E\mathcal{C}$ is $L_E\mathbf{1}$.

There is a natural adjunction

$$\mathcal{C} \rightleftarrows L_E\mathcal{C},$$

where the (symmetric monoidal) left adjoint sends an object to its E-localization, and where the (lax symmetric monoidal) right adjoint is the inclusion.

2.4. Morita theory

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. In general, there is a very useful criterion for recognizing when $\mathcal{C}$ is equivalent (as a stable homotopy theory) to the ∞-category of modules over an $\mathbf{E}_\infty$-ring.

Note first that if R is an $\mathbf{E}_\infty$-ring, then the unit object of $\mathrm{Mod}(R)$ is a compact generator. The following result, which for stable ∞-categories (without the symmetric monoidal structure) is due to Schwede and Shipley [74] (preceded by ideas of Rickard and others on tilting theory), asserts the converse.

Theorem 2.29. *(See [50, Proposition 8.1.2.7].) Let $\mathcal{C}$ be a stable homotopy theory where $\mathbf{1}$ is a compact generator. Then there is a natural symmetric monoidal equivalence*

$$\mathrm{Mod}(R) \simeq \mathcal{C},$$

where $R \simeq \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is naturally an $\mathbf{E}_\infty$-ring.

In general, given a symmetric monoidal stable ∞-category $\mathcal{C}$, the endomorphism ring $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is *always* naturally an $\mathbf{E}_\infty$-ring, and one has a natural adjunction

$$\mathrm{Mod}(R) \rightleftarrows \mathcal{C},$$

where the left adjoint "tensors up" an R-module with $\mathbf{1} \in \mathcal{C}$, and the right adjoint sends $X \in \mathcal{C}$ to the mapping spectrum $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$, which naturally acquires the structure of an R-module. The left adjoint is symmetric monoidal, and the right adjoint is *lax* symmetric monoidal. In general, one does not expect the right adjoint to preserve filtered colimits: it does so if and only if $\mathbf{1}$ is compact. In this case, if $\mathbf{1}$ is compact, we get a fully faithful inclusion

$$\mathrm{Mod}(R) \subset \mathcal{C},$$

which exhibits $\mathrm{Mod}(R)$ as a *colocalization* of $\mathcal{C}$. If $\mathbf{1}$ is not compact, we at least get a fully faithful inclusion of the *perfect* R-modules into $\mathcal{C}$.

For example, let G be a finite p-group and k be a field of characteristic p. In this case, every finite-dimensional G-representation on a k-vector space is unipotent: any such has a finite filtration whose subquotients are isomorphic to the trivial representation. From this, one might suspect that one has an equivalence of stable homotopy theories $\mathrm{Fun}(BG, \mathrm{Mod}(k)) \simeq \mathrm{Mod}(k^{hG})$, where k^{hG} is the $\mathbf{E}_\infty$-ring of endomorphisms of the unit object k, but this fails because the unit object of $\mathrm{Mod}(k[G])$ fails to be compact: taking G-homotopy fixed points does not commute with homotopy colimits. However, by fixing this reasoning, one obtains an equivalence

$$\mathrm{Fun}(BG, \mathrm{Mod}^\omega(k)) \simeq \mathrm{Mod}^\omega(k^{hG}), \tag{4}$$

between perfect k-module spectra with a G-action and perfect k^{hG}-modules. If one works with stable module ∞-categories, then the unit object *is* compact (more or less by fiat) and one has:

Theorem 2.30. *(See Keller [43].) Let G be a finite p-group and k a field of characteristic p. Then we have an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}(k^{tG}) \simeq \mathrm{St}_G(k),$$

between the ∞-category of modules over the Tate $\mathbf{E}_\infty$-ring k^{tG} and the stable module ∞-category of G-representations over k.

The *Tate construction* k^{tG}, for our purposes, can be *defined* as the endomorphism $\mathbf{E}_\infty$-ring of the unit object in the stable module ∞-category $\mathrm{St}_G(k)$. As a k-module spectrum, it can also be obtained as the cofiber of the *norm map* $k_{hG} \to k^{hG}$. We also refer to [53, sec. 2] for further discussion on this point.

2.5. Étale algebras

Let R be an $\mathbf{E}_\infty$-ring spectrum. Given an $\mathbf{E}_\infty$-R-algebra R', recall that the homotopy groups $\pi_* R'$ form a graded-commutative $\pi_* R$-algebra. In general, there is no reason for a given graded-commutative $\pi_* R$-algebra to be realizable as the homotopy groups in this way, although one often has various obstruction theories (see for instance [69,68,27] for examples of obstruction theories in different contexts) to attack such questions. There is, however, always one case in which the obstruction theories degenerate completely.

Definition 2.31. An $\mathbf{E}_\infty$-R-algebra R' is **étale** if:

1. The map $\pi_0 R \to \pi_0 R'$ is étale (in the sense of ordinary commutative algebra).
2. The natural map $\pi_0 R' \otimes_{\pi_0 R} \pi_* R \to \pi_* R'$ is an isomorphism.

The basic result in this setting is that the theory of étale algebras is entirely algebraic: the obstructions to existence and uniqueness all vanish.

Theorem 2.32. *(See [50, Theorem 7.5.4.2].) Let R be an $\mathbf{E}_\infty$-ring. Then the ∞-category of étale R-algebras is equivalent (under π_0) to the ordinary category of étale $\pi_0 R$-algebras.*

One can show more, in fact: given an étale R-algebra R', then for any $\mathbf{E}_\infty$ R-algebra R'', the natural map

$$\mathrm{Hom}_{R/}(R', R'') \to \mathrm{Hom}_{\pi_0 R/}(\pi_0 R', \pi_0 R'')$$

is a homotopy equivalence. Using an adjoint functor theorem approach (and the infinitesimal criterion for étaleness), one may even *define* R' in terms of $\pi_0 R'$ in this manner, although checking that it has the desired homotopy groups takes additional work. In particular, note that étale R-algebras are *0-cotruncated* objects of the ∞-category $\mathrm{CAlg}_{R/}$: that is, the space of maps out of any such is always homotopy discrete. The finite covers that we shall consider in this paper will also have this property.

Example 2.33. This implies that one can adjoin nth roots of unity to the sphere spectrum S^0 once n is inverted. An argument of Hopkins implies that the inversion of n is necessary: for $p > 2$, one cannot adjoin a pth root of unity to p-adic K-theory, as one sees by considering the θ-operator on $K(1)$-local $\mathbf{E}_\infty$-rings under K-theory (cf. [35]) which satisfies $x^p = \psi(x) - p\theta(x)$ where ψ is a homomorphism on π_0. If one could adjoin ζ_p to

p-adic K-theory, then one would have $-p\theta(\zeta_p) = 1 - \zeta_p^a$ for some unit $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, but p does not divide $1 - \zeta_p^a$ in $\mathbb{Z}_p[\zeta_p]$.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. We will now attempt to do the above in $\mathcal{C}$ itself. We will obtain some of the simplest classes of objects in $\mathrm{CAlg}(\mathcal{C})$. The following notation will be convenient.

Definition 2.34. Given a stable homotopy theory $(\mathcal{C}, \otimes, \mathbf{1})$, we will write

$$\pi_* X \simeq \pi_* \operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, X). \tag{5}$$

In particular, $\pi_*\mathbf{1} \simeq \pi_*\mathrm{End}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is a graded-commutative ring, and for any $X \in \mathcal{C}$, $\pi_* X$ is naturally a $\pi_*\mathbf{1}$-module.

Remark 2.35. Of course, π_* does not commute with infinite direct sums unless $\mathbf{1}$ is compact. For example, π_* fails to commute with direct sums in $L_{K(n)}\,\mathrm{Sp}$ (which is actually compactly generated, albeit not by the unit object).

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. As in the previous section, we have an adjunction of symmetric monoidal ∞-categories

$$(\cdot \otimes_R \mathbf{1}, \operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)) : \ \mathrm{Mod}(R) \rightleftarrows \mathcal{C},$$

where $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is an $\mathbf{E}_\infty$-ring. Given an étale $\pi_0 R \simeq \pi_0\mathbf{1}$-algebra R_0', we can thus construct an étale R-algebra R' and an associated object $R' \otimes_R \mathbf{1} \in \mathrm{CAlg}(\mathcal{C})$. The object $R' \otimes_R \mathbf{1}$ naturally acquires the structure of a commutative algebra, and, by playing again with adjunctions, we find that

$$\operatorname{Hom}_{\mathrm{CAlg}(\mathcal{C})}(R' \otimes_R \mathbf{1}, T) \simeq \operatorname{Hom}_{\pi_0\mathbf{1}}(R_0', \pi_0 T), \quad T \in \mathrm{CAlg}(\mathcal{C}).$$

Definition 2.36. The objects of $\mathrm{CAlg}(\mathcal{C})$ obtained in this manner are called **classically étale.**

The classically étale objects in $\mathrm{CAlg}(\mathcal{C})$ span a subcategory of $\mathrm{CAlg}(\mathcal{C})$. In general, this is not equivalent to the category of étale $\pi_0 R$-algebras if $\mathbf{1}$ is not compact (for example, $\mathrm{Mod}(R) \to \mathcal{C}$ need not be conservative; take $\mathcal{C} = L_{K(n)}\,\mathrm{Sp}$ and $L_{K(n)}S^0 \otimes \mathbb{Q}$). However, note that the functor

$$\mathrm{Mod}^\omega(R) \to \mathcal{C},$$

from the ∞-category $\mathrm{Mod}^\omega(R)$ of perfect R-modules into $\mathcal{C}$, is *always* fully faithful. It follows that there is a full subcategory of $\mathrm{CAlg}(\mathcal{C})$ *equivalent* to the category of *finite* étale $\pi_0 R$-algebras. This subcategory will give us the “algebraic” part of the Galois group of $\mathcal{C}$.

We now specialize to the case of *idempotents*. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory, and $A \in \mathrm{CAlg}(\mathcal{C})$ a commutative algebra object, so that $\pi_0 A$ is a commutative ring.

Definition 2.37. An **idempotent** of A is an idempotent of the commutative ring $\pi_0 A$. We will denote the set of idempotents of A by $\mathrm{Idem}(A)$.

The set $\mathrm{Idem}(A)$ acquires some additional structure; as the set of idempotents in a commutative ring, it is naturally a *Boolean algebra* under the multiplication in $\pi_0 A$ and the addition that takes idempotents e, e' and forms $e + e' - ee'$. For future reference, recall the following:

Definition 2.38. A **Boolean algebra** is a commutative ring B such that $x^2 = x$ for every $x \in B$. The collection of all Boolean algebras forms a full subcategory Bool of the category of commutative rings.

Suppose given an idempotent e of A, so that $1 - e$ is also an idempotent. In this case, we can obtain a *splitting*

$$A \simeq A[e^{-1}] \times A[(1-e)^{-1}]$$

as a product of two objects in $\mathrm{CAlg}(\mathcal{C})$, as observed in [54]. To see this, we may reduce to the case when $A = \mathbf{1}$, by replacing $\mathcal{C}$ by $\mathrm{Mod}_{\mathcal{C}}(A)$. In this case, we obtain the splitting from the discussion above in Definition 2.36: $A[e^{-1}]$ and $A[(1-e)^{-1}]$ are both classically étale and in the thick subcategory generated by A. Conversely, given such a splitting, we obtain corresponding idempotents, e.g., reducing to the case of an $\mathbf{E}_\infty$-ring.

Suppose the unit object $\mathbf{1} \in \mathcal{C}$ decomposes as a product $\mathbf{1}_1 \times \mathbf{1}_2 \in \mathrm{CAlg}(\mathcal{C})$. In this case, we have a decomposition at the level of stable homotopy theories

$$\mathcal{C} \simeq \mathrm{Mod}_{\mathcal{C}}(\mathbf{1}_1) \times \mathrm{Mod}_{\mathcal{C}}(\mathbf{1}_2),$$

so in practice, most stable homotopy theories that in practice we will be interested in will have no such nontrivial idempotents. However, the theory of idempotents will be very important for us in this paper.

For example, using the theory of idempotents, we can describe maps *out of* a product of commutative algebras.

Proposition 2.39. *Let $A, B \in \mathrm{CAlg}(\mathcal{C})$. Then if $C \in \mathrm{CAlg}(\mathcal{C})$, then we have a homotopy equivalence*

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(A \times B, C) \simeq \bigsqcup_{C \simeq C_1 \times C_2} \mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(A, C_1) \times \mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(B, C_2),$$

where the disjoint union is taken over all decompositions $C \simeq C_1 \times C_2$ in $\mathrm{CAlg}(\mathcal{C})$ (i.e., over idempotents in C).

Proof. Starting with a map $A \times B \to C$, we get a decomposition of C into two factors coming from the two natural idempotents in $A \times B$, whose images in C give two orthogonal idempotents summing to 1. Conversely, starting with something in the right-hand-side, given via maps $A \to C_1$ and $B \to C_2$ and an equivalence $C \simeq C_1 \times C_2$, we can take the product of the two maps to get $A \times B \to C$. The equivalence follows from the universal property of localization. $\square$

For example, consider the case of $A, B = \mathbf{1}$. In this case, we find that, if $C \in \mathrm{CAlg}(\mathcal{C})$, then

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(\mathbf{1} \times \mathbf{1}, C)$$

is homotopy discrete, and consists of the *set* of idempotents in C. We could have obtained this from the theory of “classically étale” objects earlier. Using this description as a corepresentable functor, we find:

Corollary 2.40. *The functor $A \mapsto \mathrm{Idem}(A)$, $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Bool}$, commutes with limits.*

Remark 2.41. Corollary 2.40 can also be proved directly. Since π_* commutes with arbitrary products in $\mathcal{C}$, it follows that $A \mapsto \mathrm{Idem}(A)$ commutes with arbitrary products. It thus suffices to show that if we have a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd},$$

in $\mathrm{CAlg}(\mathcal{C})$, then the induced diagram of Boolean algebras

$$\begin{tikzcd} \mathrm{Idem}(A) \arrow[r] \arrow[d] & \mathrm{Idem}(B) \arrow[d] \\ \mathrm{Idem}(C) \arrow[r] & \mathrm{Idem}(D) \end{tikzcd}$$

is also cartesian. In fact, we have a surjective map of commutative rings $\pi_0(A) \to \pi_0(B) \times_{\pi_0(D)} \pi_0(C)$ whose kernel is the image of the connecting homomorphism $\pi_1(D) \to \pi_0(A)$. It thus suffices to show that the product of two elements in the image of this connecting homomorphism vanishes, since square-zero ideals do not affect idempotents.

Equivalently, we claim that if $x, y \in \pi_0(A)$ map to zero in $\pi_0(B)$ and $\pi_0(C)$, then $xy = 0$. In fact, x and y define maps $A \to A$ and, in fact, endomorphisms of the exact triangle

$$A \to B \oplus C \to D,$$

and each is nullhomotopic on $B \oplus C$ and on D. A diagram chase with exact triangles now shows that xy defines the *zero map* $A \to A$, as desired.

3. Descent theory

Let $A \to B$ be a faithfully flat map of discrete commutative rings. Grothendieck's theory of *faithfully flat descent* (cf. [30, Exp. VIII]) can be used to describe the category $\mathrm{Mod}^{\mathrm{disc}}(A)$ of (discrete, or classical) A-modules in terms of the three categories $\mathrm{Mod}^{\mathrm{disc}}(B), \mathrm{Mod}^{\mathrm{disc}}(B \otimes_A B), \mathrm{Mod}^{\mathrm{disc}}(B \otimes_A B \otimes_A B)$. Namely, it identifies the category $\mathrm{Mod}^{\mathrm{disc}}(A)$ with the category of B-modules with *descent data*, or states that the diagram

$$\mathrm{Mod}^{\mathrm{disc}}(A) \to \mathrm{Mod}^{\mathrm{disc}}(B) \rightrightarrows \mathrm{Mod}^{\mathrm{disc}}(B \otimes_A B) \substack{\to\\\to\\\to} \mathrm{Mod}^{\mathrm{disc}}(B \otimes_A B \otimes_A B),$$

is a limit diagram in the 2-category of categories. This diagram of categories comes from the *cobar construction* on $A \to B$, which is the augmented cosimplicial commutative ring

$$A \to B \rightrightarrows B \otimes_A B \substack{\to\\\to\\\to} \cdots .$$

Grothendieck's theorem can be proved via the *Barr–Beck theorem,* by showing that if $A \to B$ is faithfully flat, the natural tensor-forgetful adjunction $\mathrm{Mod}^{\mathrm{disc}}(A) \rightleftarrows \mathrm{Mod}^{\mathrm{disc}}(B)$ is comonadic. Such results are extremely useful in practice, for instance because the category of B-modules may be much easier to study. From another point of view, these results imply that any A-module M can be expressed as an equalizer of B-modules (and maps of A-modules), via

$$M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B,$$

where the two maps are $m \otimes b \mapsto m \otimes b \otimes 1$ and $m \otimes b \mapsto m \otimes 1 \otimes b$.

In the setting of "brave new" algebra, descent theory for maps of $\mathbf{E}_\infty$ (or weaker) algebras has been extensively considered in the papers [48,46]. In this setting, one has a map of $\mathbf{E}_\infty$-rings $A \to B$, and one wishes to describe the stable ∞-category $\mathrm{Mod}(A)$ in terms of the stable ∞-categories $\mathrm{Mod}(B), \mathrm{Mod}(B \otimes_A B), \dots$. A sample result would run along the following lines.

Theorem 3.1. *(See [46, Theorem 6.1].) Let $A \to B$ be a map of $\mathbf{E}_\infty$-rings such that $\pi_0(A) \to \pi_0(B)$ is faithfully flat and the map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism. Then the adjunction $\mathrm{Mod}(A) \rightleftarrows \mathrm{Mod}(B)$ is comonadic, so that $\mathrm{Mod}(A)$ can be recovered as the totalization of the cosimplicial ∞-category*

$$\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \substack{\to\\\to\\\to} \cdots .$$

In practice, the condition of faithful flatness on $\pi_*(A) \to \pi_*(B)$ can be weakened significantly; there are numerous examples of morphisms of $\mathbf{E}_\infty$-rings which do not behave well on the level of π_0 but under which one does have a good theory of descent (e.g., the conclusion of Theorem 3.1 holds). For instance, there is a good theory of descent along $KO \to KU$, and this can be used to describe features of the ∞-category $\mathrm{Mod}(KO)$ in terms of the ∞-category $\mathrm{Mod}(KU)$. One advantage of considering descent in this more general setting is that KU is *much simpler* algebraically: its homotopy groups are given by $\pi_*(KU) \simeq \mathbb{Z}[\beta^{\pm}]$, which is a regular ring, even one-dimensional (if one pays attention to the grading), while $\pi_*(KO)$ is of infinite homological dimension. There are many additional tricks one has when working with modules over a more tractable $\mathbf{E}_\infty$-ring such as KU; we shall see a couple of them below in the proof of Theorem 6.29.

Remark 3.2. For some applications of these ideas to computations, see the paper [52] (for descriptions of thick subcategories) and [28,62,34] (for calculations of certain Picard groups).

In this section, we will describe a class of maps of $\mathbf{E}_\infty$-rings $A \to B$ that have an *especially good* theory of descent. We will actually work in more generality, and fix a stable homotopy theory $(\mathcal{C}, \otimes, \mathbf{1})$, and isolate a class of commutative algebra objects for which the analogous theory of descent (internal to $\mathcal{C}$) works especially well (so well, in fact, that it will be tautologically preserved by any morphism of stable homotopy theories). Namely, we will define $A \in \mathrm{CAlg}(\mathcal{C})$ to be *descendable* if the thick $\otimes$-ideal that A generates contains the unit object $\mathbf{1} \in \mathcal{C}$. This definition, which is motivated by the *nilpotence technology* of Devinatz, Hopkins, Smith, and Ravenel [39,22] (one part of which states that the map $L_n S^0 \to E_n$ from the E_n-local sphere to Morava E-theory E_n satisfies this property), is enough to imply that the conclusion of Theorem 3.1 holds, and has the virtue of being purely diagrammatic. The definition has also been recently and independently considered by Balmer [3] (under the name “nil-faithfulness”) in the setting of tensor-triangulated categories.

In the rest of the section, we will give several examples of descendable morphisms, and describe in Section 3.7 an application to descent for *2-modules* (or linear ∞-categories), which has applications to the study of the *Brauer group*. This provides a slight strengthening of the descent results in [47,48].

3.1. Comonads and descent

The language of ∞-categories gives very powerful tools for proving descent theorems such as Theorem 3.1 as well as its generalizations; specifically, the Barr–Beck–Lurie theorem of [50] gives a criterion to check when an adjunction is comonadic (in the ∞-categorical sense), although the result is usually stated in its equivalent form for monadic adjunctions. This result has recently been reproved from the point of view of weighted (co)limits by Riehl and Verity [73].

Theorem 3.3. *(See Barr, Beck and Lurie [50, Section 4.7].) Let $F, G\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction between ∞-categories. Then the adjunction is comonadic if and only if:*

1. *F is conservative.*
2. *Given a cosimplicial object $X^\bullet$ in $\mathcal{C}$ such that $F(X^\bullet)$ admits a splitting, then $\mathrm{Tot}(X^\bullet)$ exists in $\mathcal{C}$ and the map $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}F(X^\bullet)$ is an equivalence.*

In practice, we will be working with presentable ∞-categories, so the existence of totalizations will be assured. The conditions of the Barr–Beck–Lurie theorem are thus automatically satisfied if F preserves *all* totalizations (as sometimes happens) and is conservative.

Example 3.4. Let $A \to B$ be a morphism of $\mathbf{E}_\infty$-rings. The forgetful functor $\mathrm{Mod}(B) \to \mathrm{Mod}(A)$ is conservative and preserves all limits and *colimits*. By the adjoint functor theorem, it is a left adjoint. (The right adjoint to this functor sends an A-module M to the B-module $\mathrm{Hom}_A(B, M)$.) By the Barr–Beck–Lurie theorem, this adjunction is comonadic.

However, we will need to consider the more general case. Given a comonadic adjunction as above, one can recover any object $C \in \mathcal{C}$ as the homotopy limit of the *cobar construction*

$$C \to \left(TC \rightrightarrows T^2C \substack{\to\\\to\\\to} \cdots\right), \tag{6}$$

where $T = GF$ is the induced comonad on $\mathcal{C}$. The cobar construction is a cosimplicial diagram in $\mathcal{C}$ consisting of objects which are in the image of G.

Here a fundamental distinction between ∞-category theory and 1-category theory appears. In 1-category theory, the limit of a cosimplicial diagram can be computed as a (reflexive) *equalizer*; only the first zeroth and first stage of the cosimplicial diagram are relevant. In n-category theory (i.e., $(n, 1)$-category theory), one only needs to work with the n-truncation of a cosimplicial object. But in an ∞-category $\mathcal{C}$, given a cosimplicial diagram $X^\bullet\colon \Delta \to \mathcal{C}$, one obtains a *tower* of partial totalizations

$$\cdots \to \mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet) \to \cdots \to \mathrm{Tot}_1(X^\bullet) \to \mathrm{Tot}_0(X^\bullet),$$

whose homotopy inverse limit is the totalization or inverse limit $\mathrm{Tot}(X^\bullet)$. By definition, $\mathrm{Tot}_n(X^\bullet)$ is the inverse limit of the n-truncation of $X^\bullet$.

In an n-category, the above tower stabilizes at a finite stage: that is, the successive maps $\mathrm{Tot}_m(X^\bullet) \to \mathrm{Tot}_{m-1}(X^\bullet)$ become equivalences for m large (in fact, $m > n$). In ∞-category theory, this is almost never expected. For example, it will never hold for the cobar constructions that we obtain from descent along maps of $\mathbf{E}_\infty$-rings except in trivial cases. In particular, (6) is an infinite homotopy limit rather than a finite one.

Nonetheless, there are certain types of towers that exhibit a weaker form of stabilization, and behave close to finite homotopy limits if one is willing to include retracts. Even with ∞-categories, there are several instances where this weaker form of stabilization occurs, and it is the purpose of this section to discuss that.

3.2. Pro-objects

Consider the following two towers of abelian groups:

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ \mathbb{Z} \arrow[d, "2"] & \mathbb{Z} \arrow[d, "0"] \\ \mathbb{Z} \arrow[d, "2"] & \mathbb{Z} \arrow[d, "0"] \\ \mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

Both of these have inverse limit zero. However, there is an essential difference between the two. The second inverse system has inverse limit zero for essentially "diagrammatic" reasons. In particular, the inverse limit would remain zero if we applied any additive functor whatsoever. The first inverse system has inverse limit zero for a more "accidental" reason: that there are no integers infinitely divisible by two. If we tensored this inverse system with $\mathbb{Z}[1/2]$, the inverse limit would be $\mathbb{Z}[1/2]$.

The essential difference can be described efficiently using the theory of *pro-objects*: the second inverse system is actually *pro-zero*, while the first inverse system is a more complicated pro-object. The theory of pro-objects (and, in particular, constant pro-objects) in ∞-categories will be integral to our discussion of descent, so we spend the present subsection reviewing it.

We begin by describing the construction that associates to a given ∞-category an ∞-category of pro-objects. Although we have already used freely the (dual) Ind-construction, we review it formally for convenience.

Definition 3.5. (See [44, Section 5.3].) Let $\mathcal{C}$ be an ∞-category with finite limits. Then the ∞-category $\mathrm{Pro}(\mathcal{C})$ is an ∞-category with *all* limits, receiving a map $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ with the following properties:

1. $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ respects finite limits.
2. Given an ∞-category $\mathcal{D}$ with all limits, restriction induces an equivalence of ∞-categories

$$\mathrm{Fun}^R(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{D})$$

between the ∞-category $\mathrm{Fun}^R(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ of limit-preserving functors $\mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ and the ∞-category $\mathrm{Fun}^\omega(\mathcal{C}, \mathcal{D})$ of functors $\mathcal{C} \to \mathcal{D}$ which preserve finite limits.

There are several situations in which the ∞-categories of pro-objects can be explicitly described. We refer to [9, Sec. 3.2] for a detailed discussion.

Example 3.6. (Cf. [44, 7.1.6].) The ∞-category $\mathrm{Pro}(\mathcal{S})$ (where $\mathcal{S}$, as usual, is the ∞-category of spaces) can be described via

$$\mathrm{Pro}(\mathcal{S}) \simeq \mathrm{Fun}_{\mathrm{acc}}^{\omega-\mathrm{ct}}(\mathcal{S}, \mathcal{S})^{\mathrm{op}};$$

that is, $\mathrm{Pro}(\mathcal{S})$ is anti-equivalent to the ∞-category of accessible[3] functors $\mathcal{S} \to \mathcal{S}$ which respect finite limits. This association sends a given space X to the functor $\mathrm{Hom}(X, \cdot)$ and sends formal cofiltered limits to filtered colimits of functors.

Example 3.7. Similarly, one can describe the ∞-category $\mathrm{Pro}(\mathrm{Sp})$ of *pro-spectra* as the opposite to the ∞-category of accessible, exact functors $\mathrm{Sp} \to \mathrm{Sp}$ (a spectrum X is sent to $\mathrm{Hom}_{\mathrm{Sp}}(X, \cdot)$ via the co-Yoneda embedding).

By construction, any object in $\mathrm{Pro}(\mathcal{C})$ can be written as a "formal" filtered inverse limit of objects in $\mathcal{C}$: that is, $\mathcal{C}$ generates $\mathrm{Pro}(\mathcal{C})$ under cofiltered limits. Moreover, $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C})$ as a full subcategory. If $\mathcal{C}$ is idempotent complete, then $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C})$ consists of the cocompact objects.

Remark 3.8. If $\mathcal{C}$ is an ordinary category, then $\mathrm{Pro}(\mathcal{C})$ is a discrete category (the usual pro-category) too.

We now discuss the inclusion $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C})$, where $\mathcal{C}$ is an ∞-category with finite limits.

Definition 3.9. An object in $\mathrm{Pro}(\mathcal{C})$ is **constant** if it is equivalent to an object in the image of $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$.

Proposition 3.10. *Let $\mathcal{C}$ have finite limits. A cofiltered diagram $F \colon I \to \mathcal{C}$ defines a constant pro-object if and only if the following two conditions are satisfied:*

1. *F admits a limit in $\mathcal{C}$.*
2. *Given any functor $G \colon \mathcal{C} \to \mathcal{D}$ preserving finite limits, the inverse limit of F is preserved under G.*

In other words, the inverse limit of F is required to exist for essentially "diagrammatic reasons."

[3] In other words, commuting with sufficiently filtered colimits.

Proof. One direction of this is easy to see (take $\mathcal{D} = \mathrm{Pro}(\mathcal{C})$). Conversely, if F defines a constant pro-object, then given $\mathcal{C} \to \mathcal{D}$, we consider the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathrm{Pro}(\mathcal{C}) \arrow[r, "\widetilde{G}"] & \mathrm{Pro}(\mathcal{D}) \end{tikzcd}.$$

The functor $F\colon I \to \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ has an inverse limit, which actually lands inside the full subcategory $\mathcal{C} \subset \mathrm{Pro}(\mathcal{C})$. Since $\widetilde{G}\colon \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ preserves all limits, it follows formally that $\widetilde{G} \circ F$ has an inverse limit lying inside $\mathcal{D} \subset \mathrm{Pro}(\mathcal{D})$ and that G preserves the inverse limit. □

Example 3.11 *(Split cosimplicial objects).* Let $\mathcal{C}$ be an ∞-category with finite limits. Let $X^\bullet$ be a cosimplicial object of $\mathcal{C}$. Suppose $X^\bullet$ extends to a *split, augmented cosimplicial object.* In this case, the pro-object associated to the Tot tower of $X^\bullet$ (i.e., the tower $\{\mathrm{Tot}_n X^\bullet\}$) is constant.

In fact, let $\mathcal{D}$ be any ∞-category, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Let $\overline{X}\colon \Delta^+ \to \mathcal{C}$ be the augmented cosimplicial object extending $X^\bullet$ that can be split. Then, by [50, Section 4.7.3], the composite diagram

$$\Delta_+ \stackrel{\overline{X}}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{D},$$

is a limit diagram: that is, $F(\overline{X}^{-1}) \simeq \mathrm{Tot} F(X^\bullet)$, and in particular $\mathrm{Tot} F(X^\bullet)$ exists.

Suppose $\mathcal{D}$ admits finite limits and F preserves finite limits. Then $F(\mathrm{Tot}_n X^\bullet) \simeq \mathrm{Tot}_n F(X^\bullet)$, since F preserves finite limits, so that

$$F(\overline{X}^{-1}) \simeq \mathrm{holim}_n\, \mathrm{Tot}_n F(X^\bullet) \simeq \mathrm{holim}_n\, F(\mathrm{Tot}_n X^\bullet),$$

in $\mathcal{D}$. In particular, the tower $F(\mathrm{Tot}_n X^\bullet)$ converges to $F(\overline{X}^{-1})$. By Proposition 3.10, this proves constancy as desired.

Example 3.12 *(Idempotent towers).* Let $X \in \mathcal{C}$ and let $e\colon X \to X$ be an *idempotent* self-map; this means not only that $e^2 \simeq e$, but a choice of coherent homotopies, which can be expressed by the condition that one has an *action* of the monoid $\{1, x\}$ with two elements (where $x^2 = x$) on X. In this case, the tower

$$\cdots \to X \stackrel{e}{\to} X \stackrel{e}{\to} X,$$

is pro-constant if it admits a homotopy limit (e.g., if $\mathcal{C}$ is idempotent complete). This holds for the same reasons: the image of an idempotent is always a *universal* limit (see [44, Section 4.4.5]).

Conversely, the fact that a pro-object indexed by a cofiltered diagram $F: I \to \mathcal{C}$ is constant has many useful implications coming from the fact that the inverse limit of F is "universal."

Example 3.13. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. Given a cofiltered diagram $F: I \to \mathcal{C}$, it follows that if the induced pro-object is constant, then for any $X \in \mathcal{C}$, the natural map

$$(\varprojlim_I F(i)) \otimes X \to \varprojlim_I (F(i) \otimes X),$$

is an equivalence. See Lemma 3.39 below for a partial converse.

Next, we show that in a finite diagram of ∞-categories, a pro-object is constant if and only if it is constant at each stage.

Let K be a finite simplicial set, and let $F: K \to \mathrm{Cat}_\infty$ be a functor into the ∞-category Cat_∞ of ∞-categories. Suppose that each $F(k)$ has finite limits and each edge in K is taken to a functor which respects finite limits. In this case, we obtain a natural functor

$$\mathrm{Pro}\left(\varprojlim_K F(k)\right) \to \varprojlim_K \mathrm{Pro}(F(k)), \tag{7}$$

which respects all limits.

Proposition 3.14. *The functor* $\mathrm{Pro}\left(\varprojlim_K F(k)\right) \to \varprojlim_K \mathrm{Pro}(F(k))$ *is fully faithful.*

Proof. In fact, the functors $F(k) \to \mathrm{Pro}(F(k))$ are fully faithful for each $k \in K$, so that

$$\varprojlim_K F(k) \to \varprojlim_K \mathrm{Pro}(F(k))$$

is fully faithful and respects finite limits. In order for the right Kan extension (7) to be fully faithful, it follows by [44, Section 5.3] that it suffices for the embedding $\varprojlim_K F(k) \to \varprojlim_K \mathrm{Pro}(F(k))$ to land in the *cocompact* objects. However, over a finite diagram of ∞-categories, an object is cocompact if and only if it is cocompact pointwise, because finite limits commute with filtered colimits in spaces. □

Corollary 3.15. *Let K be a finite simplicial set and let $F: K \to \mathrm{Cat}_\infty$ be a functor as above. Then a pro-object in $\varprojlim_K F(k)$ is constant if and only if its evaluation in $\mathrm{Pro}(F(k))$ is constant for each vertex $k \in K$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \varprojlim_K F(k) \arrow[r, "\simeq"] \arrow[d] & \varprojlim_K F(k) \arrow[d] \\ \mathrm{Pro}(\varprojlim_K F(k)) \arrow[r] & \varprojlim_K \mathrm{Pro}(F(k)) \end{tikzcd},$$

where the bottom arrow is fully faithful. Given an object in $\mathrm{Pro}(\varprojlim_K F(k))$, it is constant if and only if the image in $\varprojlim_K \mathrm{Pro}(F(k))$ belongs to $\varprojlim_K F(k)$. Since each $F(k) \to \mathrm{Pro}(F(k))$ is fully faithful, this can be checked pointwise. $\square$

Remark 3.16. The functor (7) is usually not essentially surjective; consider (with Ind-objects) for instance the failure of essential surjectivity in Example 2.23.

3.3. Descendable algebra objects

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a 2-ring or a stable homotopy theory. In this subsection, we will describe a definition of a commutative algebra object in $\mathcal{C}$ which “admits descent” in a very strong sense, and prove some basic properties.

We start by recalling a basic definition.

Definition 3.17. If $\mathcal{E}$ is a stable ∞-category, we will say that a full subcategory $\mathcal{D} \subset \mathcal{E}$ is **thick** if $\mathcal{D}$ is closed under finite limits and colimits and under retracts. In particular, $\mathcal{D}$ is stable. Further, if $\mathcal{E}$ is given a symmetric monoidal structure, then $\mathcal{D}$ is a **thick $\otimes$-ideal** if in addition it is a $\otimes$-ideal.

Given a collection of objects in $\mathcal{E}$, the thick subcategory (resp. thick $\otimes$-ideal) that they **generate** is defined to be the smallest thick subcategory (resp. thick $\otimes$-ideal) containing that collection.

The theory of thick subcategories, introduced in [22,39], has played an important role in making “descent” arguments in proving the basic structural results of chromatic homotopy theory. Thus, it is not too surprising that the following definition might be useful. This notion has been independently studied under the name *nil-faithfulness* by Balmer [3].

Definition 3.18. Given $A \in \mathrm{CAlg}(\mathcal{C})$, we will say that A **admits descent** or is **descendable** if the thick $\otimes$-ideal generated by A is all of $\mathcal{C}$.

More generally, in a stable homotopy theory $(\mathcal{C}, \otimes, \mathbf{1})$, we will say that a morphism $A \to B$ in $\mathrm{CAlg}(\mathcal{C})$ **admits descent** if B, considered as a commutative algebra object in $\mathrm{Mod}_{\mathcal{C}}(A)$, admits descent in the above sense.

We now prove a few basic properties of the property of "admitting descent," for instance the (evidently desirable) claim that an analog of Theorem 3.1 goes through. Here is the first observation.

Proposition 3.19. *If $A \in \mathrm{CAlg}(\mathcal{C})$ admits descent, then A is faithful: if $M \in \mathcal{C}$, and $M \otimes A \simeq 0$, then M is contractible.*

Proof. Consider the collection of all objects $N \in \mathcal{C}$ such that $M \otimes N \simeq 0$. This is clearly a thick $\otimes$-ideal. Since it contains A, it must contain $\mathbf{1}$, so that M is contractible. □

Given $A \in \mathrm{CAlg}(\mathcal{C})$, one can form the *cobar resolution*

$$A \rightrightarrows A \otimes A \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots ,$$

which is a cosimplicial object in $\mathrm{CAlg}(\mathcal{C})$, receiving an augmentation from $\mathbf{1}$. Call this cosimplicial object $\mathrm{CB}^\bullet(A)$ and the augmented version $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$.

Proposition 3.20. *Given $A \in \mathrm{CAlg}(\mathcal{C})$, A admits descent if and only if the cosimplicial diagram $\mathrm{CB}^\bullet(A)$ defines a constant pro-object on the level of towers $\{\mathrm{Tot}_n \mathrm{CB}^\bullet(A)\}_{n\geq 0}$ which converges to $\mathbf{1}$ (i.e., $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ is a limit diagram).*

Proof. Suppose A admits descent. Consider the collection $\mathcal{C}_{\mathrm{good}}$ of $M \in \mathcal{C}$ such that the augmented cosimplicial diagram $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \otimes M$ is a limit diagram, and such that the induced Tot tower converging to M defines a constant pro-object. Our goal is to show that $\mathbf{1} \in \mathcal{C}_{\mathrm{good}}$.

Note first that $A \in \mathcal{C}_{\mathrm{good}}$: in fact, the augmented cosimplicial diagram $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \otimes A$ is *split* and so is a limit diagram and defines a constant pro-object (Example 3.11). Moreover, $\mathcal{C}_{\mathrm{good}}$ is a thick $\otimes$-ideal. The collection of pro-objects which are constant is thick, and the tensor product of a constant pro-object with any object of $\mathcal{C}$ is constant (and the limit commutes with the tensor product). Since $A \in \mathcal{C}_{\mathrm{good}}$, it follows that $\mathbf{1} \in \mathcal{C}_{\mathrm{good}}$, which completes the proof in one direction.

Conversely, if $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ is a limit diagram, and $\mathrm{CB}^\bullet(A)$ defines a constant pro-object, it follows that $\mathbf{1}$ is a retract of $\mathrm{Tot}_n \mathrm{CB}^\bullet(A)$, for $n \gg 0$. However, $\mathrm{Tot}_n \mathrm{CB}^\bullet(A)$ clearly lives in the thick $\otimes$-ideal generated by A, which shows that A admits descent. □

In other words, thanks to Proposition 3.20, A admits descent if and only if the unit object $\mathbf{1}$ can be obtained as a retract of a finite colimit of a diagram in $\mathcal{C}$ consisting of objects, each of which admits the structure of a module over A.

One advantage of the purely categorical (and finitistic) definition of admitting descent is that it is preserved under base change. The next result follows from Proposition 3.20.

Corollary 3.21. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor between symmetric monoidal, stable ∞-categories. Given $A \in \mathrm{CAlg}(\mathcal{C})$, if A admits descent, then $F(A)$ does as well.*

Proposition 3.22. *Let $\mathcal{C}$ be a stable homotopy theory. Let $A \in \mathrm{CAlg}(\mathcal{C})$ admit descent. Then the adjunction*

$$\mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A),$$

given by tensoring with A and forgetting, is comonadic. In particular, the natural functor from $\mathcal{C}$ to the totalization

$$\mathcal{C} \to \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \substack{\to\\\to\\\to} \cdots\right)$$

is an equivalence.

Proof. We need to check that the hypotheses of the Barr–Beck–Lurie theorem go through. We refer to [50, Th. 4.7.6.2] for the connection between comonadicity and the totalization of ∞-categories considered above, which is an ∞-categorical generalization of the classical Beck–Bénabou–Roubaud theorem [12].

By Proposition 3.19, tensoring with A is conservative. Now, fix a cosimplicial object $X^\bullet \colon \Delta \to \mathcal{C}$ such that $A \otimes X^\bullet$ is split. We need to show that the map

$$A \otimes \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(A \otimes X^\bullet)$$

is an equivalence. This will follow if the pro-object defined by $X^\bullet$ (i.e., by the Tot tower) is constant. To see that, consider the collection of objects $M \in \mathcal{C}$ such that $M \otimes X^\bullet$ defines a constant pro-object. By assumption (and Example 3.11), this collection contains A, and it is a thick ⊗-ideal. It follows that $X^\bullet$ itself defines a constant pro-object, so we are done. □

Remark 3.23. We have used the fact that we have a symmetric monoidal functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$, which embeds $\mathcal{C}$ as a full subcategory of $\mathrm{Pro}(\mathcal{C})$: in particular, the tensor product of two constant pro-objects in $\mathrm{Pro}(\mathcal{C})$ is constant.

Finally, we prove a few basic permanence properties for admitting descent.

Proposition 3.24. *Suppose $\mathcal{C}$ is a stable homotopy theory. Let $A \to B \to C$ be maps in* $\mathrm{CAlg}(\mathcal{C})$.

1. *If $A \to B$ and $B \to C$ admit descent, so does $A \to C$.*
2. *If $A \to C$ admits descent, so does $A \to B$.*

Proof. Consider the first claim. Suppose $A \to B$ and $B \to C$ admit descent. Then, via the cobar construction, we find that B belongs to the thick subcategory of $\mathrm{Mod}_{\mathcal{C}}(B)$ generated by the C-modules. It follows that B belongs to the thick subcategory of $\mathrm{Mod}_{\mathcal{C}}(A)$ generated by the C-modules, and therefore every B-module belongs to the thick ⊗-ideal in $\mathrm{Mod}_{\mathcal{C}}(A)$ generated by C. Since $A \to B$ admits descent, we find that the thick $\otimes$-ideal that C generates in $\mathrm{Mod}_{\mathcal{C}}(A)$ contains A.

For the second claim, we note simply that a C-module is in particular a B-module: the thick $\otimes$-ideal that B generates contains any B-module, for instance C. $\square$

Proposition 3.25. *Let K be a finite simplicial set and let $p\colon K \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ be a diagram. Then a commutative algebra object $A \in \mathrm{CAlg}(\varprojlim_K p)$ admits descent if and only if its "evaluations" in $\mathrm{CAlg}(p(k))$ admit descent for each $k \in K$.*

Proof. Admitting descent is preserved under symmetric monoidal, exact functors, so one direction is evident. For the other, if $A \in \mathrm{CAlg}(\varprojlim_K p)$ has the property that its image in each $\mathrm{CAlg}(p(k))$ admits descent, then consider the cobar construction $\mathrm{CB}^\bullet(A)$. It defines a constant pro-object after evaluating at each $k \in K$, and therefore, by Corollary 3.15, it defines a constant pro-object in $\varprojlim_K p$ too. The inverse limit is necessarily the unit (since this is true at each vertex), so A admits descent. $\square$

3.4. Nilpotence

In this subsection, we present a slightly different formulation of the definition of admitting descent, which makes clear the connection with nilpotence.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory and let $A \in \mathcal{C}$ be any object. Given a map $f\colon X \to Y$ in $\mathcal{C}$, we say that f is *A-zero* if $A \otimes X \xrightarrow{1_A \otimes f} A \otimes Y$ is nullhomotopic (as a morphism in $\mathcal{C}$).

The collection of all A-zero maps forms what is classically called a *tensor ideal* in the triangulated category $\mathrm{Ho}(\mathcal{C})$. The main result of this subsection is that a *commutative algebra* object A admits descent if and only if this ideal is nilpotent, in a natural sense.

Definition 3.26. A collection $\mathcal{I}$ of maps in $\mathrm{Ho}(\mathcal{C})$ is a **tensor ideal** if the following hold:

1. For each X, Y, the collection of homotopy classes of maps $X \to Y$ that belong to $\mathcal{I}$ is a subgroup.
2. Given $f\colon X \to Y, g\colon Y \to Z, h\colon Z \to W$, then if $g \in \mathcal{I}$, we have $h \circ g \circ f \in \mathcal{I}$.
3. Given $g\colon Y \to Z$ in $\mathcal{I}$ and any other object $T \in \mathcal{C}$, the tensor product $g \otimes 1_T\colon Y \otimes T \to Z \otimes T$ belongs to $\mathcal{I}$.

For any $A \in \mathcal{C}$, the collection of A-zero maps is clearly a tensor ideal $\mathcal{I}_A$. Given two tensor ideals $\mathcal{I}, \mathcal{J}$, we will define the product $\mathcal{I}\mathcal{J}$ to be the smallest tensor ideal containing all composites $g \circ f$ where $f \in \mathcal{J}$ and $g \in \mathcal{I}$.

Proposition 3.27. *Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object. Then the following are equivalent:*

1. *There exists $s \in \mathbb{N}$ such that the composite of s consecutive A-zero maps is zero.*
2. $\mathcal{I}_A^s = 0$ *for some* $s \in \mathbb{Z}_{\geq 0}$.
3. A *admits descent.*

This result is essentially [3, Proposition 3.15].

Proof. Suppose first A admits descent. We want to show that $\mathcal{I}_A^s = 0$ for some $s \gg 0$. Now, $\mathcal{I}_{\mathbf{1}} = 0$, so our strategy is to use a thick subcategory argument.

We make the following three claims:

1. If $M, N \in \mathcal{C}$, then $\mathcal{I}_M \subset \mathcal{I}_{M \otimes N}$.
2. If N is a retract of M, then $\mathcal{I}_M \subset \mathcal{I}_N$.
3. Given a cofiber sequence

$$M' \to M \to M''$$

in $\mathcal{C}$, we have

$$\mathcal{I}_{M'} \mathcal{I}_{M''} \subset \mathcal{I}_M.$$

Of these, the first and second are obvious. For the third, it suffices to show that the composite of an M'-null map and an M''-null map is M-null. Suppose $f \colon X \to Y$ is M''-null and $g \colon Y \to Z$ is M'-null. We want to show that $g \circ f$ is M-null. We have a diagram

$$\begin{tikzcd}
X \otimes M' \arrow[r] \arrow[d] & Y \otimes M' \arrow[r] \arrow[d] & Z \otimes M' \arrow[d] \\
X \otimes M \arrow[r] \arrow[d] & Y \otimes M \arrow[r] \arrow[d] & Z \otimes M \arrow[d] \\
X \otimes M'' \arrow[r] & Y \otimes M'' \arrow[r] & Z \otimes M''
\end{tikzcd}.$$

Here the vertical arrows are cofiber sequences. Chasing through this diagram, we find that $X \otimes M \to Y \otimes M$ factors through $X \otimes M \to Y \otimes M'$, so that the composite $X \otimes M \to Z \otimes M$ factors through $X \otimes M \to Y \otimes M' \xrightarrow{0} Z \otimes M' \to Z \otimes M$ and is thus nullhomotopic.

It thus follows (from the above three items) that if $M \in \mathcal{C}$ is arbitrary, then for any $\overline{M} \in \mathcal{C}$ belonging to the thick $\otimes$-ideal generated by M, we have

$$\mathcal{I}_M^s \subset \mathcal{I}_{\overline{M}},$$

for some integer $s \gg 0$. If $\mathbf{1} \in \mathcal{C}$ belongs to this thick $\otimes$-ideal, that forces $\mathcal{I}_M$ to be nilpotent.

Conversely, suppose there exists $s \in \mathbb{Z}_{\geq 0}$ such that the composite of s consecutive A-zero maps is zero. We will show that A admits descent. Given an object $M \in \mathcal{C}$, we want to show that M belongs to the thick $\otimes$-ideal generated by A. For this, consider the functor

$$F_1(X) = \mathrm{fib}(X \to X \otimes A);$$

we have a natural map $F_1(X) \to X$, which is A-zero, and whose cofiber belongs to the thick $\otimes$-ideal generated by A. Iteratively define $F_n(X) = F_1(F_{n-1}(X))$ for $n > 0$. We get a tower

$$\cdots \to F_n(M) \to F_{n-1}(M) \to \cdots \to F_1(M) \to M,$$

where all the successive cofibers of $F_i(M) \to F_{i-1}(M)$ belong to the thick $\otimes$-ideal generated by A. By chasing cofiber sequences, this means that the cofiber of each $F_i(M) \to M$ belongs to the thick $\otimes$-ideal generated by A.

Moreover, each of the maps in this tower is A-zero. It follows that $F_s(M) \to M$ is zero. Thus the cofiber of $F_s(M) \to M$ is $M \oplus \Sigma F_s(M)$, which belongs to the thick $\otimes$-ideal generated by A. Therefore, M belongs to this thick $\otimes$-ideal, and we are done. □

3.5. Local properties of modules

In classical algebra, many properties of modules are local for the étale (or flat) topology. These statements can be generalized to the setting of $\mathbf{E}_\infty$-ring spectra, where one considers morphisms $R \to R'$ of $\mathbf{E}_\infty$-rings that are étale (or flat, etc.) on the level of π_0 and such that the natural map $\pi_0 R' \otimes_{\pi_0 R} \pi_* R \to \pi_* R'$ is an isomorphism.

Our next goal is to prove a couple of basic results in our setting for descendable morphisms.

Proposition 3.28. *Let $A \to B$ be a descendable morphism of $\mathbf{E}_\infty$-rings. Let M be an A-module such that $B \otimes_A M$ is a perfect B-module. Then M is a perfect A-module.*

Proof. Consider a filtered category $\mathcal{I}$ and a functor $\iota \colon \mathcal{I} \to \mathrm{Mod}(A)$. We then need to show that

$$\varinjlim \mathrm{Hom}_A(M, M_\iota) \to \mathrm{Hom}_A(M, \varinjlim M_\iota),$$

is an equivalence. Consider the collection $\mathcal{U}$ of A-modules N such that

$$\varinjlim \mathrm{Hom}_A(M, M_\iota \otimes_A N) \to \mathrm{Hom}_A(M, \varinjlim M_\iota \otimes_A N),$$

is a weak equivalence; we would like to show that it contains A itself. The collection $\mathcal{U}$ clearly forms a thick subcategory. Observe that it contains $N = B$ using the adjunction relation

$$\mathrm{Hom}_A(P, P' \otimes_A B) \simeq \mathrm{Hom}_B(P \otimes_A B, P' \otimes_A B),$$

valid for $P, P' \in \mathrm{Mod}(A)$, and the assumption that $M \otimes_A B$ is compact in $\mathrm{Mod}(B)$. More generally, this implies that every tensor product $B \otimes_A \cdots \otimes_A B$ belongs to $\mathcal{U}$. Since A is a retract of a finite limit of copies of such A-modules, via the cobar construction, it follows that $A \in \mathcal{C}$ and that M is compact or perfect in $\mathrm{Mod}(A)$. $\square$

Remark 3.29. More generally, the argument of Proposition 3.28 shows that if $\mathcal{C}$ is an A-linear ∞-category, and $M \in \mathcal{C}$ is an object that becomes compact after tensoring with B (as an object of $\mathrm{Mod}_{\mathcal{C}}(B)$), then M was compact to begin with. Proposition 3.28 itself could have also been proved by observing that $\mathrm{Mod}(A)$ is a totalization $\mathrm{Tot}\left(\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}}\right)$ by Proposition 3.22 and an A-module is thus dualizable (equivalently, compact) if and only if its base-change to $\mathrm{Mod}(B)$ is, as dualizability in an inverse limit of symmetric monoidal ∞-categories can be checked vertexwise (cf. [50, Prop. 4.6.1.11]).

Proposition 3.30. *Let $A \to B$ be a descendable morphism of $\mathbf{E}_\infty$-rings. Let M be an A-module. Then M is invertible if and only if $M \otimes_A B$ is invertible.*

Proof. Observe first that $M \otimes_A B$ is perfect (since it is invertible), so M is also perfect via Proposition 3.28. The evaluation map $M \otimes_A M^\vee \to A$ has the property that it becomes an equivalence after tensoring up to B, since the formation of $M \mapsto M^\vee$ commutes with base extension for M perfect. It follows that $M \otimes_A M^\vee \to A$ is itself an equivalence, so that M is invertible. $\square$

Let M be an A-module. If $A \to B$ is a descendable morphism of $\mathbf{E}_\infty$-rings such that $M \otimes_A B$ is a finite direct sum of copies of B, the A-module M itself need not look anything like a free module. (The finite covers explored in this paper are examples.) However, such "locally free" A-modules seem to have interesting and quite restricted properties.

3.6. First examples

In the following section, we will discuss more difficult examples of the phenomenon of admitting descent, and try to give a better feel for it. Here, we describe some relatively "formal" examples of maps which admit descent.

We start by considering the evident faithfully flat case. In general, we *do not know* if a faithfully flat map $A \to B$ of $\mathbf{E}_\infty$-ring spectra (i.e., such that $\pi_0(A) \to \pi_0(B)$ is faithfully flat and such that $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism) necessarily admits descent, even in the case of discrete $\mathbf{E}_\infty$-rings. This would have some implications. For example, if A and B are discrete commutative rings, it would imply that if M is an

A-module and $\gamma \in \mathrm{Ext}^n_A(M, M)$ is a class whose image in $\mathrm{Ext}^n_B(M \otimes_A B, M \otimes_A B)$ vanishes, then γ is nilpotent. Nonetheless, one has:

Proposition 3.31. *Suppose $A \to B$ is a faithfully flat map of $\mathbf{E}_\infty$-rings such that $\pi_*(A)$ is countable. Then $A \to B$ admits descent.*

Proof. We can use the criterion of Proposition 3.27. We claim that we can take $s = 2$. That is, given composable maps $M \to M' \to M''$ of A-modules each of which becomes nullhomotopic after tensoring up to B, the *composite* is nullhomotopic.

To see this, we observe that any B-zero map in $\mathrm{Mod}(A)$ is *phantom*. In other words, if $M \to M'$ is B-zero, then any composite

$$P \to M \to M',$$

where P is a perfect A-module, is already nullhomotopic. To see this, note that $P \to M'$ is B-zero, but to show that it is already nullhomotopic, we can dualize and consider

$$\pi_*(\mathbb{D}P \otimes_A M') \to \pi_*(\mathbb{D}P \otimes_A M' \otimes_A B),$$

which is injective since B is faithfully flat over A on the level of homotopy groups. The injectivity of this map forces any B-zero map $P \to M'$ to be automatically zero to begin with.

Finally, we can conclude if we know that the composite of two phantom maps in $\mathrm{Mod}(A)$ is zero. This claim is [36, Theorem 4.1.8]; we need countability of $\pi_*(A)$ to conclude that homology theories on A-modules are representable (by [36, Theorem 4.1.5]). $\square$

Without the countability hypothesis, the result about phantom maps is known to be false. It is, however, possible to strengthen Proposition 3.31 using more recent techniques of *transfinite* Adams representability [61]. We are grateful to Oriol Raventós for explaining the following to us.

Proposition 3.32. *Let $A \to B$ be a faithfully flat morphism of $\mathbf{E}_\infty$-rings such that $\pi_*(A)$ has cardinality at most $\aleph_k$ for some $k \in \mathbb{N}$. Then $A \to B$ admits descent.*

Proof. As above, it suffices to show that the composite of $k + 2$ phantom maps of A-modules is necessarily nullhomotopic. Consider the class $\mathcal{C} = \mathrm{Perf}(A)$ of perfect A-modules, which at most $\aleph_\kappa$ isomorphism classes of objects.

Consider the category $\mathrm{Mod}(\mathcal{C})$ of functors $\mathcal{C}^{op} \to \mathrm{Ab}$ which preserve finite coproducts. Given any object $X \in \mathrm{Mod}(A)$, the Yoneda lemma gives an object $h_X \in \mathrm{Mod}(\mathcal{C})$. Note that h_X is a filtered colimit of functors representable by objects in $\mathcal{C}$. Taking $\alpha = \aleph_0$, we apply [61, Prop. 2.13], we find that h_X for any $X \in \mathcal{C}$ has projective dimension $\leq k + 1$. By [61, Cor. 6.3.5], we find that X is $(k + 2) - \mathcal{C}$-cellular in the sense of [61, Def. 6.1.5].

Since X was arbitrary, we find by [61, Prop. 6.1.6] that the composite of $(k+2)$ phantom maps is zero. $\square$

Since descendability is preserved under base change, we obtain:

Corollary 3.33. *Let $A \to B$ be a faithfully flat map of $\mathbf{E}_\infty$-rings such that $\pi_0(B)$ has a presentation $\pi_0(A)$-algebra with at most $\aleph_k$ generators and relations for some $k \in \mathbb{N}$. Then $A \to B$ admits descent.*

For example, a finitely presented faithfully flat map of discrete rings is descendable. For a finitely presented map $A \to B$ of noetherian rings, Bhatt and Scholze have shown [16, Th. 5.26] that $A \to B$ is admits descent if and only if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an h-cover, which is significantly weaker.

In addition to faithfully flat maps which are not too large, there are examples of descendable maps of $\mathbf{E}_\infty$-rings which look more like (relatively mild) quotients.

Proposition 3.34. *Suppose A is an $\mathbf{E}_\infty$-ring which is connective and such that $\pi_i A = 0$ for $i \gg 0$. Then the map $A \to \pi_0 A$ admits descent.*

Proof. Given an A-module M such that $\pi_*(M)$ is concentrated in one degree, it admits the structure of a $\pi_0 A$-module (canonically) and thus belongs to the thick $\otimes$-ideal generated by $\pi_0 A$. However, A admits a finite resolution by such A-modules, since one has a finite Postnikov decomposition of A in $\mathrm{Mod}(A)$ whose successive cofibers have a single homotopy group, and therefore belongs to the thick $\otimes$-ideal generated by $\pi_0 A$. $\square$

Proposition 3.35. *Let R be a discrete commutative ring. Let $I \subset R$ be a nilpotent ideal. Then the map $R \to R/I$ of discrete commutative rings, considered as a map of $\mathbf{E}_\infty$-rings, admits descent.*

Proof. For $k \gg 0$, we have a finite filtration of R in the world of discrete R-modules

$$0 = I^k \subset I^{k-1} \subset \cdots \subset I \subset R,$$

whose successive quotients are R/I-modules. This implies that R/I generates all of $\mathrm{Mod}(R)$ as a thick $\otimes$-ideal. $\square$

There are also examples of descendable morphisms where the condition on the thick $\otimes$-ideals follows from a defining limit diagram.

Proposition 3.36. *Let R be an $\mathbf{E}_\infty$-ring and let X be a finite connected CW complex. Then the map $C^*(X; R) \to R$ given by evaluating at a basepoint $* \in X$ admits descent.*

Proof. In fact, the $\mathbf{E}_\infty$-ring $C^*(X; R)$ is a finite limit (indexed by X) of copies of R by definition. That is, $C^*(X; R) \simeq \varprojlim_X R$. $\square$

Proposition 3.37. *Let R be an $\mathbf{E}_\infty$-ring and let $x \in \pi_0 R$. Then the map $R \to R[x^{-1}] \times \widehat{R}_x$ (where $\widehat{R}_x$ is the x-adic completion) admits descent.*

Proof. This follows from the arithmetic square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R[x^{-1}] \arrow[d] \\ \widehat{R}_x \arrow[r] & \widehat{R}_x[x^{-1}] \end{tikzcd} .$$

All three of the terms in the fiber product here are $R[x^{-1}] \times \widehat{R}_x$-modules, so R belongs to the thick subcategory generated by the $R[x^{-1}] \times \widehat{R}_x$-modules and we are done. □

Next we include a deeper result, which will imply (for example) that the faithful Galois extensions considered by [71] admit descent; this will be very important in the rest of the paper. The theory of nilpotence with respect to a dualizable algebra object has been treated in more detail in [59].

Theorem 3.38. *Let $\mathcal{C}$ be a stable homotopy theory. Suppose $\mathbf{1} \in \mathcal{C}$ is compact, and suppose $A \in \mathrm{CAlg}(\mathcal{C})$ is dualizable and faithful (i.e., tensoring with A is conservative). Then A admits descent.*

Proof. Consider the cobar construction $\mathrm{CB}^\bullet(A)$ on A. The first claim is that it converges to $\mathbf{1}$: that is, the augmented cosimplicial construction $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ is a limit diagram. To see this, we can apply the Barr–Beck–Lurie theorem to A. Since A is dualizable, we have for $X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(Y, A \otimes X) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{D}A \otimes Y, X),$$

and in particular tensoring with A commutes with all limits in $\mathcal{C}$. Since tensoring with A is conservative, we find that the hypotheses of the Barr–Beck–Lurie theorem go into effect (cf. also [6, 2.6]). In particular, $\mathrm{CB}^\bullet(A)$ converges to $\mathbf{1}$ and, moreover, for any $M \in \mathcal{C}$, $\mathrm{CB}^\bullet(A) \otimes M$ converges to M. We need to show that the induced pro-object is *constant*, though. This will follow from the next lemma. □

Lemma 3.39. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory where $\mathbf{1}$ is compact. Let I be a cofiltered category, and let $F \colon I \to \mathcal{C}$ be a functor. Suppose that for each $i \in I$, $F(i) \in \mathcal{C}$ is dualizable. Then F defines a constant pro-object (or is* pro-constant*) if and only if the following are satisfied.*

1. *$\varprojlim_I F(i)$ is a dualizable object.*

2. *For each object $C \in \mathcal{C}$, the natural map*

$$(\varprojlim_I F(i)) \otimes C \to \varprojlim_I (F(i) \otimes C) \tag{8}$$

is an equivalence.

Proof. Let $\mathbb{D}$ be the duality functor (of internal hom into $\mathbf{1}$); it induces a contravariant auto-equivalence on the subcategory $\mathcal{C}^{\mathrm{dual}}$ of dualizable objects in $\mathcal{C}$. To say that F defines a constant pro-object in $\mathcal{C}$ (or, equivalently, $\mathcal{C}^{\mathrm{dual}}$) is to say that $\mathbb{D}F$, which is an *ind*-object of $\mathcal{C}^{\mathrm{dual}}$, defines a constant ind-object. In other words, we have a commutative diagram of ∞-categories,

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{dual}} \arrow[r, "\mathbb{D}", "\simeq"'] \arrow[d, "\subset"] & \mathcal{C}^{\mathrm{dual,\,op}} \arrow[d, "\subset"] \\
\mathrm{Pro}(\mathcal{C}^{\mathrm{dual}}) \arrow[r, "\mathbb{D}", "\simeq"'] \arrow[d, "\subset"] & \mathrm{Ind}(\mathcal{C}^{\mathrm{dual}})^{\mathrm{op}} . \\
\mathrm{Pro}(\mathcal{C}) &
\end{tikzcd}$$

Now, since $\mathcal{C}^{\mathrm{dual}} \subset \mathcal{C}$ consists of compact objects (since $\mathbf{1} \in \mathcal{C}$ is compact), we know that there is a fully faithful inclusion $\mathrm{Ind}(\mathcal{C}^{\mathrm{dual}}) \subset \mathcal{C}$, which sends an ind-object to its colimit. If $\mathcal{C}$ is generated by dualizable objects, this is even an equivalence, but we do not need this.

As a result, to show that $\mathbb{D}F \in \mathrm{Ind}(\mathcal{C}^{\mathrm{dual}})$ defines a constant ind-object, it is sufficient to show that its colimit in $\mathcal{C}$ actually belongs to $\mathcal{C}^{\mathrm{dual}}$. Let $X = \varprojlim_I F(i) \in \mathcal{C}$; by hypothesis, this is a dualizable object. We have a natural map (in $\mathcal{C}$)

$$\varinjlim_I \mathbb{D}F(i) \to \mathbb{D}X,$$

and if we can prove that this is an equivalence, we will have shown that $\varinjlim_I \mathbb{D}F(i)$ is a dualizable object and thus the ind-system is constant. In other words, we must show that if $C \in \mathcal{C}$ is arbitrary, then the natural map of spectra

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{D}X, C) \to \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(\mathbb{D}F(i), C)$$

is an equivalence. But this map is precisely $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)$ applied to (8), so we are done. □

Remark 3.40. This result requires $\mathbf{1}$ to be compact. If $\mathcal{C}$ is the stable homotopy theory of p-adically complete chain complexes of abelian groups (i.e., the localization of $D(\mathbb{Z})$ at $\mathbb{Z}/p\mathbb{Z}$), then $\mathbb{Z}/p\mathbb{Z}$ is a dualizable, faithful commutative algebra object, but the associated pro-object is not constant, or the p-adic integers $\mathbb{Z}_p$ would be torsion.

Remark 3.41. One can prove the same results (e.g., Theorem 3.38) if $A \in \mathcal{C}$ is given an *associative* (or $\mathbf{E}_1$) algebra structure, rather than an $\mathbf{E}_\infty$-algebra structure. However, the symmetric monoidal structure on $\mathcal{C}$ itself is crucial throughout.

3.7. Application: descent for linear ∞-categories

In fact, the definition of descent considered here gives a more general result than Proposition 3.22. Let $\mathcal{C}$ be an A-linear ∞-category in the sense of [46]. In other words, $\mathcal{C}$ is a presentable, stable ∞-category which is a *module* in the symmetric monoidal ∞-category Pr^L of presentable, stable ∞-categories over $\mathrm{Mod}(A)$. This means that there is a bifunctor, which preserves colimits in each variable,

$$\otimes_A \colon \mathrm{Mod}(A) \times \mathcal{C} \to \mathcal{C}, \quad (M, C) \mapsto M \otimes_A C$$

together with additional compatibility data: for instance, equivalences $A \otimes_A M \simeq M$ for each $M \in \mathcal{C}$.

Given such a $\mathcal{C}$, one can study, for any A-algebra B, the ∞-category $\mathrm{Mod}_{\mathcal{C}}(B)$ of B-modules *internal to* $\mathcal{C}$: this is the "relative tensor product" in Pr^L

$$\mathrm{Mod}_{\mathcal{C}}(B) = \mathcal{C} \otimes_{\mathrm{Mod}(A)} \mathrm{Mod}(B).$$

Useful references for this, and for the tensor product of presentable ∞-categories, are [25] and [17].

Informally, $\mathrm{Mod}_{\mathcal{C}}(B)$ is the target of an A-bilinear functor

$$\otimes_A \colon \mathcal{C} \times \mathrm{Mod}(B) \to \mathrm{Mod}_{\mathcal{C}}(B), \quad (X, M) \mapsto X \otimes_A M,$$

which is colimit-preserving in each variable, and it is universal for such. As in the case $\mathcal{C} = \mathrm{Mod}(A)$, one has an adjunction

$$\mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(B),$$

given by "tensoring up" and forgetting the B-module structure.

One can then ask whether descent holds in $\mathcal{C}$, just as we studied earlier for A-modules. In other words, we can ask whether $\mathcal{C}$ is equivalent to the ∞-category of B-modules in $\mathcal{C}$ equipped with analogous "descent data": equivalently, whether the "tensoring up" functor $\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(B)$ is comonadic. Stated another way, we are asking whether, for any $\mathrm{Mod}(A)$-module *∞-category* $\mathcal{C}$, we have an equivalence of A-linear ∞-categories

$$\mathcal{C} \simeq \mathrm{Tot}\left(\mathcal{C} \otimes_{\mathrm{Mod}(A)} \mathrm{Mod}(B)^{\otimes(\bullet+1)}\right). \tag{9}$$

In fact, the proof of Proposition 3.22 applies and we get:

Corollary 3.42. *Suppose $A \to B$ is a descendable morphism of $\mathbf{E}_\infty$-rings. Then $A \to B$ satisfies descent for any A-linear ∞-category $\mathcal{C}$ in that the functor from $\mathcal{C}$ to "descent data" is an equivalence.*

Proof. By the Barr–Beck–Lurie theorem, we need to see that tensoring with B defines a conservative functor $\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(B)$ which respects B-split totalizations. Conservativity can be proved as in Proposition 3.19. Given $R \in \mathcal{C}$, the collection of A-modules M such that $M \otimes_A R \simeq 0$ is a thick $\otimes$-ideal in $\mathrm{Mod}(A)$. If B belongs to this thick $\otimes$-ideal, so must A, and R must be zero.

Let $X^\bullet \colon \Delta \to \mathcal{C}$ be a cosimplicial object which becomes split after tensoring with B. As in Proposition 3.22, it suffices to show that the pro-object that $X^\bullet$ defines is constant in $\mathcal{C}$. This follows via the same thick subcategory argument: one considers the collection of $M \in \mathrm{Mod}(A)$ such that $X^\bullet \otimes_A M$ defines a constant pro-object, and observes that M is a thick $\otimes$-ideal containing B, thus containing A. Thus $X^\bullet$ defines a constant pro-object. $\square$

We note that the argument via pro-objects yields a mild strengthening of the results in the DAG series. In particular, it shows that if $A \to B$ is a morphism of $\mathbf{E}_\infty$-rings which is faithfully flat and presented by at most $\aleph_k$ generators and relations (for some $k \in \mathbb{N}$), it satisfies descent for any A-linear ∞-category. In the DAG series, this is proved assuming *étaleness* [48, Th. 5.4] or for faithfully flat morphisms assuming existence of a t-structure [46, Th. 6.12]. In fact, this idea of descent via thick subcategories seems to be the right setting for considering the above questions, in view of the following result, which was explained to us by Jacob Lurie:

Proposition 3.43. *Let $A \to B$ be a morphism of $\mathbf{E}_\infty$-rings such that, for any A-linear ∞-category, descent holds, i.e., we have an equivalence (9). Then $A \to B$ admits descent.*

Proof. Suppose $A \to B$ does not admit descent. We will look for a counterexample to (9). We will exhibit an A-linear presentable ∞-category $\mathcal{D}$ and an object $X \in \mathcal{D}$ such that the totalization of the cobar construction $\mathrm{CB}^\bullet(B) \otimes_A X$ is not equivalent to X.

The idea is to take $\mathcal{D} = \mathrm{Pro}(\mathrm{Mod}(A))$ and $X = A$. Consider the cobar construction $B \rightrightarrows B \otimes_A B \substack{\to\\ \to\\ \to} \cdots$. The totalization of the cobar construction in $\mathrm{Pro}(\mathrm{Mod}(A))$ is *precisely* the cobar construction considered as a pro-object via the Tot tower. In particular, if $A \to B$ fails to admit descent, the cobar construction does not converge to A in $\mathrm{Pro}(\mathrm{Mod}(A))$.

In order to make this argument precise, we have to address the fact that $\mathrm{Pro}(\mathrm{Mod}(A))$ is not really an A-linear ∞-category: it is not, for example, presentable. Choose a regular cardinal κ such that B is κ-compact as an A-module. Choose a small subcategory $\mathcal{C} \subset \mathrm{Pro}(\mathrm{Mod}(A))$ which contains the constant object A and is closed under κ-small colimits and countable limits. Then $\mathcal{C}$ is tensored over the ∞-category $\mathrm{Mod}^\kappa(A)$ of κ-compact A-modules, so the presentable ∞-category $\mathcal{D} = \mathrm{Ind}_\kappa(\mathcal{C})$ is tensored over

$\mathrm{Mod}(A)$ in a compatible manner. Moreover, in this ∞-category the totalization of the cobar construction $B \rightrightarrows B \otimes_A B \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$ does not converge to A as that does not happen in $\mathcal{C}$. $\square$

Finally, we note a "categorified" version of descent, which, while likely far from the strongest possible, is already of interest in studying the Brauer group of $\mathbf{E}_\infty$-rings such as TMF. This phenomenon has been extensively studied (under the name "1-affineness") in [26]. We will only consider a very simple and special case of this question.

The idea is that instead of considering descent for modules over a ring spectrum R (possibly internal to a linear ∞-category), we will consider descent for the linear ∞-categories themselves, which we will call *2-modules*, meaning modules *over* the presentable, symmetric monoidal ∞-category $\mathrm{Mod}(R)$.

Definition 3.44. Given an $\mathbf{E}_\infty$-ring R, there is a symmetric monoidal ∞-category $2\text{-}\mathrm{Mod}(R)$ of R-linear ∞-categories with the R-linear tensor product. In other words, $2\text{-}\mathrm{Mod}(R)$ consists of modules (in the symmetric monoidal ∞-category of presentable, stable ∞-categories) over $\mathrm{Mod}(R)$.

For a useful reference, see [25,1]. We now record:

Proposition 3.45. *Let $A \to B$ be a descendable morphism of $\mathbf{E}_\infty$-rings. Then* 2-Mod *satisfies descent along $A \to B$.*

As noted in [26] and [48], this is a formal consequence of descent in linear ∞-categories (that is, Corollary 3.42), but we recall the proof for convenience.

Proof. Recall that we have the adjunction

$$(F, G) = \left(\otimes_{\mathrm{Mod}(A)} \mathrm{Mod}(B),\ \text{forget}\right): \quad 2\text{-}\mathrm{Mod}(A) \rightleftarrows 2\text{-}\mathrm{Mod}(B),$$

where G is the forgetful functor from B-linear ∞-categories to A-linear ∞-categories, and where F is "tensoring up." The assertion of the proposition is that this adjunction is comonadic. By the Barr–Beck–Lurie theorem, it suffices to show now that F is conservative and preserves certain totalizations.

But F is conservative because any $\mathcal{C}$-linear ∞-category can be recovered from its "descent data" after tensoring up to B (Corollary 3.42). Moreover, F commutes with all limits. In fact, F sends an A-linear ∞-category $\mathcal{C}$ to the collection of B-module objects in $\mathcal{C}$, and this procedure is compatible with limits. $\square$

It would be interesting to give conditions under which one could show that a 2-module over R admitted a compact generator if and only if it did so locally on R in some sense. This would yield a type of descent for the *Brauer spectrum* of R (see for instance [1]), whose π_0 consists of equivalence classes of invertible 2-modules that admit a compact generator. Descent for compactly generated R-linear ∞-categories is known to hold in the *usual* étale topology on $\mathbf{E}_\infty$-rings [48, Theorem 6.1], although the proof is long and complex. Descent also holds for the finite covers considered in this paper which are *faithful*. It would be interesting to see if it held for $L_n S^0 \to E_n$, possibly in some $K(n)$-local sense.

4. Nilpotence and Quillen stratification

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object in $\mathcal{C}$. In general, we might hope that (for whatever reason) phenomena in $\mathrm{Mod}_{\mathcal{C}}(A)$ might be simpler to understand than phenomena in $\mathcal{C}$. For example, if $\mathcal{C} = \mathrm{Sp}$, we do not know the homotopy groups of the sphere spectrum, but there are many $\mathbf{E}_\infty$-rings whose homotopy groups we do know completely: for instance, $H\mathbb{F}_p$ and MU. We might then try to use our knowledge of A and some sort of descent to understand phenomena in $\mathcal{C}$. For instance, we might attempt to compute the homotopy groups of an object $M \in \mathcal{C}$ by constructing the cobar resolution

$$M \to \left(M \otimes A \rightrightarrows M \otimes A \otimes A \substack{\to\\\to\\\to} \cdots \right),$$

and hope that it converges to M. This method is essentially the *Adams spectral sequence*, which, in case $\mathcal{C} = \mathrm{Sp}$, is one of the most important tools one has for calculating and understanding the stable homotopy groups of spheres.

In the previous section, we introduced a type of commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ such that, roughly, the above descent method converged very efficiently — much more efficiently, for instance, than the classical Adams or Adams–Novikov spectral sequences. One can see this at the level of descent spectral sequences in the existence of *horizontal vanishing lines* that occur at finite stages. In particular, in this situation, one can understand phenomena in $\mathcal{C}$ from phenomena in $\mathrm{Mod}_{\mathcal{C}}(A)$ and $\mathrm{Mod}_{\mathcal{C}}(A \otimes A)$ "up to (bounded) nilpotence." We began discussing this in Proposition 3.27. The purpose of this section is to continue that discussion and to describe several fundamental (and highly non-trivial) examples of commutative algebra objects that admit descent. These ideas have also been explored in [3], and we learned of the connection with Quillen stratification from there.

4.1. Descent spectral sequences

Let $\mathcal{C}$ be a stable homotopy theory. Let $A \in \mathrm{CAlg}(\mathcal{C})$ and let $M \in \mathcal{C}$. As usual, we can try to study M via the A-module $M \otimes A$ and, more generally, the cobar construction $M \otimes \mathrm{CB}^\bullet(A)$. In this subsection, we will describe the effect of descendability on the resulting spectral sequence.

Definition 4.1. The Tot tower of the cobar construction $M \otimes \mathrm{CB}^\bullet(A)$ is called the **Adams tower** $\{T_n(A, M)\}$ of M. The induced spectral sequence converging to $\pi_* \varprojlim(M \otimes \mathrm{CB}^\bullet(A))$ is called the **Adams spectral sequence** for M (based on A).

The Adams tower has the property that it comes equipped with maps

$$\begin{tikzcd} & \vdots \arrow[d] \\ & T_2(A,M) \arrow[d] \\ & T_1(A,M) \arrow[d] \\ M \arrow[r] \arrow[ur] \arrow[uur] & T_0(A,M) \simeq A \otimes M \end{tikzcd}.$$

In other words, it is equipped with a map from the *constant* tower at M. We let the cofiber of this map of towers be $\{U_n(A, M)\}_{n \geq 0}$.

The tower $\{U_n(A, M)\}$ has the property that the cofiber of any map $U_n(A, M) \to U_{n-1}(A, M)$ admits the structure of an A-module. Moreover, each map $U_n(A, M) \to U_{n-1}(A, M)$ is null after tensoring with A.

Suppose now that A *admits descent*. In this case, the towers we are considering have particularly good properties.

Definition 4.2. (See [37,52].) Let $\mathrm{Tow}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C})$ be the ∞-category of towers in $\mathcal{C}$.

We shall say that a tower $\{X_n\}_{n \geq 0}$ is **nilpotent** if there exists N such that $X_{n+N} \to X_n$ is null for each $n \in \mathbb{Z}_{\geq 0}$. It is shown in [37] that the collection of nilpotent towers is a thick subcategory of $\mathrm{Tow}(\mathcal{C})$. We shall say that a tower is **strongly constant** if it belongs to the thick subcategory of $\mathrm{Tow}(\mathcal{C})$ generated by the nilpotent towers and the constant towers.

Observe that a nilpotent tower is pro-zero, and a strongly constant tower is pro-constant. In general, nilpotence of a tower is *much* stronger than being pro-zero. For example, a tower $\{X_n\}$ is pro-zero if there is a cofinal set of integers i for which the X_i are contractible. This does not imply nilpotence.

We now recall the following fact about strongly constant towers:

Proposition 4.3. *(See [37].) Let $\{X_n\}_{n \geq 0} \in \mathrm{Tow}(\mathcal{C})$ be a strongly constant tower. Then, for $Y \in \mathcal{C}$, the spectral sequence for $\pi_* \mathrm{Hom}(Y, \varprojlim X_n)$ has a horizontal vanishing line at a finite stage.*

In fact, in [37], it is shown that admitting such horizontal vanishing lines is a *generic* property of objects in $\mathrm{Tow}(\mathcal{C})$: that is, the collection of objects with that property is a thick subcategory. Moreover, this property holds for nilpotent towers and for constant towers.

Corollary 4.4. *Let $A \in \mathrm{CAlg}(\mathcal{C})$ admit descent. Then the Adams tower $\{T_n(A, M)\}$ is strongly constant. In particular, the Adams spectral sequence converges with a horizontal vanishing line at a finite stage (independent of M).*

Proof. In fact, by Proposition 3.27, it follows that the tower $\{U_n(A, M)\}$ is nilpotent, since all the successive maps in the tower are A-zero, so the tower $\{T_n(A, M)\}$ is therefore strongly constant. □

It follows from this that we can get a rough global description of the graded-commutative ring $\pi_*\mathbf{1}$ if we have a description of π_*A. This is the description that leads, for instance, to the description of various group cohomology rings "up to nilpotents."

Theorem 4.5. *Let $A \in \mathrm{CAlg}(\mathcal{C})$ admit descent. Let R_* be the equalizer of $\pi_*(A) \rightrightarrows \pi_*(A \otimes A)$. There is a map $\pi_*(\mathbf{1}) \to R_*$ with the following properties:*

1. *The kernel of $\pi_*(\mathbf{1}) \to R_*$ is a nilpotent ideal.*
2. *Given an element $x \in R_*$ with $Nx = 0$, then x^{N^k} belongs to the image of $\pi_*(\mathbf{1}) \to R_*$ for $k \gg 0$ (which can be chosen uniformly in N).*

In the examples arising in practice, one already has a complete or near-complete picture *rationally*, so the torsion information is the most interesting. For example, if p is nilpotent in $\pi_*(\mathbf{1})$, then the map that one gets is classically called a *uniform F-isomorphism.*

Proof. In fact, R_* as written is the zero-line (i.e., $s = 0$) of the E_2-page of the A-based Adams spectral sequence converging to the homotopy groups of $\mathbf{1}$. The map that we have written down is precisely the edge homomorphism in the spectral sequence. We know that anything of positive filtration at E_∞ must be nilpotent of bounded order because of the horizontal vanishing line. That implies the first claim.

For the second claim, let $x \in E_2^{0,t}$ be N-torsion. We want to show that x^{N^k} survives the spectral sequence for some k (which can be chosen independently of x). In fact, x^N can support no d_2 by the Leibnitz rule. Similarly, x^{N^2} can support no d_3 and survives until E_4. Since the spectral sequence collapses at a finite stage, we conclude that some x^{N^k} must survive, and k depends only on the finite stage at which the spectral sequence collapses. □

Remark 4.6. One can obtain an analog of Theorem 4.5 for any commutative algebra object in $\mathcal{C}$ replacing $\mathbf{1}$.

4.2. Quillen stratification for finite groups

Let G be a finite group, and let R be a (discrete) commutative ring. Consider the ∞-category $\mathrm{Mod}_G(R) \simeq \mathrm{Fun}(BG, \mathrm{Mod}(R))$ of R-module spectra with a G-action (equivalently, the ∞-category of module spectra over the *group ring*), which is symmetric monoidal under the R-linear tensor product. Given a subgroup $H \subset G$, we have a natural symmetric monoidal functor

$$\mathrm{Mod}_G(R) \to \mathrm{Mod}_H(R),$$

given by restricting the G-action to H. As in ordinary algebra, we can identify this with a form of tensoring up: we can identify $\mathrm{Mod}_H(R)$ with the ∞-category of modules over the commutative algebra object $\prod_{G/H} R \in \mathrm{Mod}_G(R)$, with G permuting the factors. We state this formally as a proposition (compare [5,7]).

Proposition 4.7. *Consider the commutative algebra object $\prod_{G/H} R \in \mathrm{CAlg}(\mathrm{Mod}_G(R))$, with G-action permuting the factors. Then the forgetful functor identifies $\mathrm{Mod}_H(R)$ with the symmetric monoidal ∞-category of modules in $\mathrm{Mod}_G(R)$ over $\prod_{G/H} R$.*

We can interpret this in the following algebro-geometric manner as well. The ∞-category $\mathrm{Mod}_G(R)$ can be described as the ∞-category of quasi-coherent complexes on the classifying stack BG of the discrete group G, over the base ring R. Similarly, $\mathrm{Mod}_H(R)$ can be described as the ∞-category of quasi-coherent sheaves on BH. One has an affine map $\phi\colon BH \to BG$ (in fact, a finite étale cover), so that quasi-coherent complexes on BH can be identified with quasi-coherent complexes on BG with a module action by $\pi_*(\mathcal{O}_{BH})$, which corresponds to $\prod_{G/H} R$.

In particular, we can attempt to perform "descent" along the restriction functor $\mathrm{Mod}_G(R) \to \mathrm{Mod}_H(R)$, or descent with the commutative algebra object $\prod_{G/H} R$, or descent for quasi-coherent sheaves along the cover $BH \to BG$. If R contains $\mathbb{Q}$ or, more generally, if $|G|/|H|$ is invertible in R, there are never any problems, because the G-equivariant *norm map* $\prod_{G/H} R \to R$ will exhibit R as a retract of the object $\prod_{G/H} R$, so that the commutative algebra object $\prod_{G/H} R$ is descendable.

The question is much more subtle in modular characteristic. For example, given a finite group G and a field k of characteristic p with $p \mid |G|$, the group cohomology $H^*(G; k)$ is always infinite-dimensional, which prevents the commutative algebra object $\prod_G k$ from being descendable. Nonetheless, one has the following result. Recall that a group is called *elementary abelian* if it is of the form $(\mathbb{Z}/p)^n$ for some prime number p.

Theorem 4.8. *(See Carlson [18], Balmer [3].) Let G be a finite group, and let $\mathcal{A}$ be a collection of elementary abelian subgroups of G such that every maximal elementary abelian subgroup of G is conjugate to an element of $\mathcal{A}$. Then the commutative algebra object $\prod_{H \in \mathcal{A}} \prod_{G/H} R$ admits descent in $\mathrm{Mod}_G(R)$.*

In other words, there is a strong theory of descent along the map $\bigsqcup_{A \in \mathcal{A}} BA \to BG$ of stacks. If p is invertible in R and H is an elementary abelian p-group, then $\prod_{G/H} R \in \mathrm{Mod}_G(R)$ is a retract of $\prod_G R$. To translate to our terminology, we note that [18, Theorem 2.1] states that there is a finitely generated $\mathbb{Z}[G]$-module V with the property that there exists a finite filtration $0 = V_0 \subset \cdots \subset V_k = \mathbb{Z} \oplus V$ such that the successive quotients are all *induced* $\mathbb{Z}[G]$-modules from elementary abelian subgroups of G. Given an object of $\mathrm{Mod}_G(\mathbb{Z})$ which is induced from $H \subset G$, we observe that it is naturally a module in $\mathrm{Mod}_G(\mathbb{Z})$ over $\prod_{G/H} \mathbb{Z}$.

Note moreover that the map

$$\bigsqcup_{A \in \mathcal{A}} BA \to BG, \tag{10}$$

which we have identified as having a good theory of descent, is explicit enough that we can also write down the relative fiber product $(\bigsqcup_{A \in \mathcal{A}} BA) \times_{BG} (\bigsqcup_{A \in \mathcal{A}} BA)$ via a double coset decomposition. Stated another way, the tensor products of commutative algebra objects of the form $\prod_{G/H} R$, which appear in the cobar construction, can be described explicitly.

From this, and Theorem 4.5 (and the immediately following remark), one obtains the following corollary, which is known to modular representation theorists and is a generalization of the description by Quillen [66] of the cohomology ring of a finite group up to F-isomorphism.

Corollary 4.9. *Let R be an $\mathbf{E}_2$-algebra in $\mathrm{Mod}(\mathbb{Z})$ with an action of the finite group G. Suppose p is nilpotent in R. Let $\mathcal{A}$ be the collection of all elementary abelian p-subgroups of G. Then the map*

$$R^{hG} \to \prod_{A \in \mathcal{A}} R^{hA},$$

has nilpotent kernel in π_*. The image, up to uniform F-isomorphism, consists of all families which are compatible under restriction and conjugation.

A family $(a_A \in \pi_* R^{hA})_{A \in \mathcal{A}}$ is compatible under restriction and conjugation if, whenever $g \in G$ conjugates A into A', then the induced map $R^{hA} \simeq R^{hA'}$ carries a_A into $a_{A'}$; and, furthermore, whenever $B \subset A$, then the map $R^{hA} \to R^{hB}$ carries a_A into a_B. These compatible families form the E_2-page of the descent spectral sequence for the cover (10). When $R = \mathbb{F}_p$ (as was considered by Quillen), the above corollary is extremely useful since the cohomology rings of elementary abelian groups are entirely known and easy to work with. Given a connected space X with $\pi_1 X$ finite, one could also apply it to the π_1-action on $C^*(\widetilde{X}; \mathbb{F}_p)$ where $\widetilde{X}$ is the universal cover.

We will use this picture extensively in the future, in particular for the *stable module ∞-categories.* For now, we note a simple example of one of its consequences.

Corollary 4.10. *The inclusion $\mathbb{Z}/p \subset \mathbb{Z}/p^k$ induces a descendable map of $\mathbf{E}_\infty$-rings*

$$\mathbb{F}_p^{h\mathbb{Z}/p^k} \to \mathbb{F}_p^{h\mathbb{Z}/p},$$

for each $k > 0$.

Proof. Consider the ∞-category $\mathrm{Mod}_{\mathbb{Z}/p^k}(\mathbb{F}_p)$ of $\mathbb{F}_p$-module spectra with a $\mathbb{Z}/p^k$-action. Inside here we have the commutative algebra object $\prod_{\mathbb{Z}/p^{k-1}} \mathbb{F}_p$ which, by Theorem 4.8, admits descent.

Note that, as in (4), the subcategory $\mathrm{Mod}^{\omega}_{\mathbb{Z}/p^k}(\mathbb{F}_p)$ of perfect $\mathbb{F}_p$-modules with a $\mathbb{Z}/p^k$-action is symmetric monoidally equivalent to the ∞-category of perfect $\mathbb{F}_p^{h\mathbb{Z}/p^k}$-modules. Thus, if we show that $\prod_{\mathbb{Z}/p^{k-1}} \mathbb{F}_p$ generates the unit $\mathbb{F}_p$ itself as a thick $\otimes$-ideal in $\mathrm{Mod}^{\omega}_{\mathbb{Z}/p^k}(\mathbb{F}_p)$ (rather than the larger ∞-category $\mathrm{Mod}_{\mathbb{Z}/p^k}(\mathbb{F}_p)$), we will be done. But this extra claim comes along for free, since we can use the cobar construction. The cobar construction on $\prod_{\mathbb{Z}/p^{k-1}} \mathbb{F}_p$ is constant as a pro-object either way, and that means that $\mathbb{F}_p$ belongs to the thick $\otimes$-ideal generated by $\prod_{\mathbb{Z}/p^{k-1}} \mathbb{F}_p$ in the smaller ∞-category. $\square$

We refer to [59,58] for many further examples of these phenomena in equivariant homotopy theory (e.g., when R is replaced by a ring spectrum) and analogs of F-isomorphism and induction theorems.

4.3. Stratification for Hopf algebras

Let k be a field of characteristic p, and let A be a finite-dimensional commutative Hopf algebra over k. One may attempt to obtain a similar picture in the derived ∞-category of A-comodules. This has been considered by several authors, for example in [63,76,24]. The case of the previous subsection was $A = \prod_G k$ when G is a finite group, given the coproduct dual to the multiplication in $k[G]$. In this subsection, which will not be used in the sequel, we describe the connection between some of this work and the notion of descent theory considered here. In this subsection, we assume that all Hopf algebras A that occur are *graded connected*, i.e., $A = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A_i$ with $A_0 = k$ and the Hopf algebra structure respects the grading.

The Hopf algebra A defines a *finite group scheme* $G = \mathrm{Spec}\, A$ over k, and we are interested in the ∞-category of quasi-coherent complexes on the classifying stack BG and in understanding descent in here. For every closed subgroup $H \subset G$, we obtain a morphism of stacks

$$f_H \colon BH \to BG,$$

which is *affine*, even finite: in particular, quasi-coherent sheaves on BH can be identified with modules in $\mathrm{QCoh}(BG)$ over $(f_H)_*(\mathcal{O}_{BH}) \in \mathrm{CAlg}(BG)$. One might hope that a certain collection of (proper) subgroup schemes $H \subset G$ would have the property that the commutative algebra objects $(f_H)_*(\mathcal{O}_{BH})$ jointly generate, as a thick $\otimes$-ideal, all of $\mathrm{QCoh}(BG)$.

When G is constant (although this is not covered by our present graded connected setup), then the Quillen stratification theory (i.e., Theorem 4.8) identifies such a collection of subgroups. The key step is to show that if G is not elementary abelian, then the collection of $(f_H)_*(\mathcal{O}_{BH})$ as H ranges over *all* proper subgroups of G jointly satisfy descent. The picture is somewhat more complicated for finite group schemes.

Definition 4.11. (See Palmieri [63].) A group scheme G is **elementary** if it is commutative and satisfies the following condition. Let $\mathcal{O}(G)^\vee$ be the "group algebra," i.e., the dual to the ring $\mathcal{O}(G)$ of functions on G. Then for every x in the augmentation ideal of $\mathcal{O}(G)^\vee$, we have $x^p = 0$. Dualizing, this is equivalent to the condition that the *Verschiebung* should annihilate G.

Remark 4.12. The "group algebra" $\mathcal{O}(G)^\vee$ plays a central role here because $\mathrm{QCoh}(BG)$, if we forget the symmetric monoidal structure, is simply $\mathrm{Mod}(\mathcal{O}(G)^\vee)$; the Hopf algebra structure on $\mathcal{O}(G)^\vee$ gives rise to the symmetric monoidal structure.

In [63], Palmieri also defines a weaker notion of *quasi-elementarity* for finite group schemes G, in terms of the vanishing of certain products of Bocksteins. It is this more complicated condition that actually ends up intervening.

Consider first a group scheme G of rank p over k (which is necessarily commutative), arising as the spectrum of a graded connected Hopf algebra. Then the underlying algebra $\mathcal{O}(G)^\vee$ is isomorphic to $k[x]/x^p$. In particular, there is a basic generating class $\beta \in H^2(BG) \simeq \mathrm{Ext}^2_{\mathcal{O}(G)^\vee}(k,k)$ called the *Bockstein* β_G. The Bockstein, considered as a map $\mathbf{1} \to \Sigma^2\mathbf{1}$ in $\mathrm{QCoh}(BG)$, has the property that the *cofiber* of β belongs to the thick subcategory generated by the "regular representation" $\mathcal{O}(G)^\vee$, in view of the exact sequence of $\mathcal{O}(G)^\vee \simeq k[x]/x^p$-modules

$$0 \to k \to \mathcal{O}(G)^\vee \to \mathcal{O}(G)^\vee \to k \to 0,$$

which exhibits the two-term complex $\mathcal{O}(G)^\vee \to \mathcal{O}(G)^\vee$ as the cofiber of β (up to a shift). Since the map $\mathcal{O}(G)^\vee \to \mathcal{O}(G)^\vee$ is nilpotent (it is given by multiplication by x), it follows that the thick subcategory generated by the cofiber of β is equal to that generated by the standard representation.

Definition 4.13. A group scheme G arising from a graded connected Hopf algebra is **quasi-elementary** if the product $\prod_{\phi\colon G \twoheadrightarrow G'} \phi^*(\beta_{G'})$ for all surjections $\phi\colon G \twoheadrightarrow G'$ for G' a group scheme of rank p (always respecting the grading), is not nilpotent in the cohomology of BG.

Remark 4.14. Let $G = \operatorname{Spec} A$ be a nontrivial group scheme arising from a graded connected Hopf algebra. Then there is always a surjective map $G \twoheadrightarrow G'$ with G' of rank p (respecting the grading). To see this, we observe that there is a nontrivial primitive element $x \in A_i$ for $i > 0$ and, replacing x with a suitable power, we may assume that $x^p = 0$. This defines the map to a graded version of α_p.

For finite groups, it is a classical theorem of Serre that quasi-elementarity is equivalent to being elementary abelian: if G is a finite p-group which is not elementary abelian, then there are nonzero classes $\alpha_1, \dots, \alpha_n \in H^1(G; \mathbb{Z}/p)$ such that the product of the Bocksteins $\prod \beta(\alpha_i)$ vanishes. Serre's result is, as explained in [18,3], at the source of the Quillen stratification theory for finite groups, in particular Theorem 4.8.

Proposition 4.15. *(Cf. [63, Th. 1.2].) Let G be a finite group scheme arising from a graded connected Hopf algebra over k. Then G is not quasi-elementary if and only if the objects $(f_H)_*(\mathcal{O}_{BH}) \in \mathrm{CAlg}(\mathrm{QCoh}(BG))$, for $H \subset G$ a proper normal subgroup scheme (respecting the grading), generate the unit as a thick $\otimes$-ideal.*

Proof. Suppose κ is nilpotent. For each rank p quotient $\phi\colon G \twoheadrightarrow G'$, we have a map $\mathbf{1} \to \Sigma^2\mathbf{1}$ in $\mathrm{QCoh}(BG')$ whose cofiber is in the thick subcategory of $\mathrm{QCoh}(BG')$ generated by the pushforward of the structure sheaf via $* \to BG'$. Pulling back, we get, for each rank p quotient $\phi\colon G \twoheadrightarrow G'$ with kernel H_ϕ, a map $\beta_\phi\colon \mathbf{1} \to \mathbf{1}[2]$ in $\mathrm{QCoh}(BG)$ whose cofiber is in the thick subcategory generated by $(f_{H_\phi})_*(\mathcal{O}_{BH_\phi})$ where $f_{H_\phi}\colon BH_\phi \to BG$ is the natural map. It follows in particular that the cofiber of each β_ϕ belongs to the thick subcategory $\mathcal{C} \subset \mathrm{QCoh}(BG)$ generated by the $\{(f_H)_*(\mathcal{O}_{BH})\}$ for H a proper normal subgroup scheme of G. Therefore, using the octahedral axiom, the cofiber of each *composite* of a finite string of β_ϕ's (e.g., κ and its powers) belongs to $\mathcal{C}$. It follows finally that, by nilpotence of κ, the unit object itself belongs to $\mathcal{C}$.

Conversely, suppose that the $\{(f_H)_*(\mathcal{O}_{BH})\}$ generate the unit as a thick $\otimes$-ideal: that is, descent holds. In this case, we show that the product of Bocksteins $\kappa = \prod_{\phi\colon G \twoheadrightarrow G'} \phi^*(\beta_{G'})$ in Definition 4.13 is forced to be nilpotent. In fact, we observe that for every proper normal subgroup $H \subset G$, there exists a morphism from G/H to a rank p group scheme Q. The pull-back of the Bockstein β_Q to $H^2(BG)$ restricts to zero on H; in particular, κ restricts to zero on each normal subgroup scheme of G. By descent, it follows that κ is nilpotent. $\square$

By induction, one gets:

Corollary 4.16. *Let G be a group scheme over k arising from a graded connected Hopf algebra. Then the commutative algebra objects $(f_H)_*(\mathcal{O}_{BH}) \in \mathrm{CAlg}(\mathrm{QCoh}(BG))$, as $H \subset G$ ranges over all the quasi-elementary subgroup schemes (respecting the grading), admits descent.*

Unfortunately, it is known that quasi-elementarity and elementarity are not equivalent for general finite group schemes [76]. There is, however, one important case when this is known.

Let $p = 2$. Consider the dual Steenrod algebra $\mathcal{A} \simeq \mathbb{F}_2[\xi_1, \xi_2, \dots]$. This is a graded, connected, and commutative (but not cocommutative) Hopf algebra over $\mathbb{F}_2$. The object $\operatorname{Spec} \mathcal{A}$, which is now an (infinite-dimensional) group scheme, admits an elegant algebro-geometric interpretation as the automorphism group scheme of the formal additive group $\widehat{\mathbb{G}_a}$. Let A be a finite-dimensional graded Hopf algebra quotient of the dual Steenrod algebra, so that $G = \operatorname{Spec} A$ is a finite group scheme inside the group scheme of automorphisms of $\widehat{\mathbb{G}_a}$.

Theorem 4.17. *(See Wilkerson [76].) Let A be as above, and let $\mathcal{B}$ range over all the elementary subgroup schemes $H \subset G$ (respecting the grading). Then the map $\bigsqcup_{H \in \mathcal{B}} BH \to BG$ admits descent, in the sense that the commutative algebra object $\prod_{H \in \mathcal{B}} (f_H)_*(\mathcal{O}_{BH}) \in \mathrm{CAlg}(\mathrm{QCoh}(BG))$ does.*

In particular, it is known that for subgroup schemes of $\operatorname{Spec} \mathcal{A}$ (cut out by homogeneous ideals), elementarity and quasi-elementarity are equivalent. Related ideas have been used by Palmieri [64] to give a complete description of the cohomology of such Hopf algebras up to F-isomorphism at the prime 2.

4.4. Chromatic homotopy theory

Thick subcategory ideas were originally introduced in chromatic homotopy theory. Let E_n denote a Morava E-theory of height n; thus $\pi_0(E_n) \simeq W(k)[[v_1, \dots, v_{n-1}]]$ where $W(k)$ denotes the Witt vectors on a perfect field k of characteristic p. Moreover, $\pi_*(E_n) \simeq \pi_0(E_n)[t_2^{\pm 1}]$ and E_n is thus *even periodic*; the associated formal group is the Lubin–Tate universal deformation of a height n formal group over the field k. By a deep theorem of Goerss, Hopkins and Miller, E_n has the (canonical) structure of an $\mathbf{E}_\infty$-ring.

Let L_n denote the functor of localization at E_n. The basic result is the following:

Theorem 4.18. *(See Hopkins and Ravenel [67, Chapter 8].) The map $L_n S^0 \to E_n$ admits descent.*

In other words, the E_n-based Adams–Novikov spectral sequence degenerates with a horizontal vanishing line at a finite stage, for any spectrum. This degeneration does *not* happen at the E_2-page (e.g., for the sphere) and usually implies that a great many differentials are necessary early on. Theorem 4.18, which implies that E_n-localization is *smashing*, is fundamental to the global structure of the stable homotopy category and its localizations. As in the finite group case, one of the advantages of results such as Theorem 4.18 is that E_n is much simpler algebraically than is $L_n S^0$.

The Hopkins–Ravenel result is a basic finiteness property of the E_n-local stable homotopy category. It implies, for instance, that many homotopy limits that one takes (such as the homotopy fixed points for the $\mathbb{Z}/2$-action on KU) behave much more like finite homotopy limits than infinite ones.

Example 4.19. Let R be an $\mathbf{E}_2$-ring spectrum which is L_n-local. Then it follows that the map from $\pi_*(R)$ to the zero-line of the E_2-page of the Adams–Novikov spectral sequence for R is an F-isomorphism. Indeed, we know that the map from $\pi_*(R)$ to the zero-line at E_2 is a rational isomorphism and, moreover, everything above the $s = 0$ line vanishes at E_2. (This comes from the algebraic fact that rationally, the moduli stack of formal groups is a $B\mathbb{G}_m$ and has no higher cohomology.)

Example 4.20. Let R be an L_n-local ring spectrum. Then any class in $\pi_*(R)$ which maps to zero in $(E_n)_*(R)$ is nilpotent. This is a very special case of the general (closely related) nilpotence theorem of [22,39]. For an $\mathbf{E}_\infty$-ring, by playing with power operations, one can actually prove a stronger result [60]: any *torsion* class is automatically nilpotent.

5. Axiomatic Galois theory

Let $(X, *)$ be a pointed, connected topological space. A basic and useful invariant of $(X, *)$ is the *fundamental group* $\pi_1(X, *)$, defined as the group of homotopy classes of paths $\gamma\colon [0, 1] \to X$ with $\gamma(0) = \gamma(1) = *$. This definition has the disadvantage, at least from the point of view of an algebraist, of intrinsically using the unit interval $[0, 1]$ and the topological structure of the real numbers $\mathbb{R}$. However, the fundamental group also has another incarnation that makes no reference to the theory of real numbers, via the theory of *covering spaces*.

Definition 5.1. A map $p\colon Y \to X$ of topological spaces is a **covering space** if, for every $x \in X$, there exists a neighborhood U_x of x such that in the pullback

$$\begin{tikzcd} Y \times_X U_x \arrow[r] \arrow[d] & Y \arrow[d] \\ U_x \arrow[r] & X \end{tikzcd},$$

the map $Y \times_X U_x \to U_x$ has the form $\bigsqcup_S U_x \to U_x$ for a set S.

The theory of covering spaces makes, at least a priori, no clear use of $[0, 1]$. Moreover, understanding the theory of covering spaces of X is essentially equivalent to understanding the group $\pi_1(X, *)$. If X is locally contractible, then one has the following basic result:

Theorem 5.2. *Suppose X is path-connected and locally contractible. Let Cov_X be the category of maps $Y \to X$ which are covering spaces. Then, we have an equivalence of* *categories* $\mathrm{Cov}_X \simeq \mathrm{Set}_{\pi_1(X,*)}$, *which sends a cover* $p\colon Y \to X$ *to the fiber* $p^{-1}(*)$ *with the monodromy action of* $\pi_1(X,*)$.

The fundamental group $\pi_1(X,*)$ can, in fact, be *recovered* from the structure of the category Cov_X. This result suggests that the theory of the fundamental group should be more primordial than its definition might suggest; at least, it might be expected to have avatars in other areas of mathematics in which the notion of a covering space makes sense.

Grothendieck realized, in [30], that there is a purely algebraic notion of a *finite cover* for a scheme (rather than a topological space): that is, given a scheme X, one can define a version of Cov_X that corresponds to the topological notion of a finite cover. When X is a variety over the complex numbers $\mathbb{C}$, the algebraic notion turns out to be equivalent to the topological notion of a finite cover of the complex points $X(\mathbb{C})$ with the analytic topology. As a result, in [30], it was possible to define a *profinite group* classifying these finite covers of schemes. Grothendieck had to prove a version of Theorem 5.2 without an a priori definition of the fundamental group, and did so by *axiomatizing* the properties that a category would have to satisfy in order to arise as the category of finite sets equipped with a continuous action of a profinite group. He could then *define* the group in terms of the category of finite covers. The main objective of this paper is to obtain similar categories from stable homotopy theories.

The categories that appear in this setting are called *Galois categories*, and the theory of Galois categories will be reviewed in this section. We will, in particular, describe a version of Grothendieck's Galois theory that does not require a fiber functor, relying primarily on versions of descent theory.

5.1. The fundamental group

To motivate the definitions, we begin by quickly reviewing how the classical étale fundamental group of [30] arises.

Definition 5.3. Let $f\colon Y \to X$ be a finitely presented map of schemes. We say that $f\colon Y \to X$ is **étale** if f is flat and the sheaf $\Omega_{Y/X}$ of relative Kähler differentials vanishes.

Étaleness is the algebro-geometric analog of being a "local homeomorphism" in the complex analytic topology. Given it, one can define the analog of a (finite) covering space.

Definition 5.4. A map $f\colon Y \to X$ is a **finite cover** (or finite covering space) if f is finite and étale. The collection of all finite covering spaces of X forms a category Cov_X, a full subcategory of the category of schemes over X.

The basic example of a finite étale cover is the map $\bigsqcup_S X \to X$. If X is connected, then a map $Y \to X$ is a finite cover if and only if it *locally* has this form with respect to the flat topology. In other words, a map $Y \to X$ is a finite cover if and only if there exists a finitely presented, faithfully flat map $X' \to X$ such that the pull-back

$$\begin{tikzcd} X' \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd},$$

is of the form $\bigsqcup_S X' \to X'$ where S is a finite set; if X is not connected, the number of sheets may vary over X. In other words, one has an analog of Definition 5.1, where "locally" is replaced by "locally in the flat topology." This strongly suggests that the algebro-geometric definition of a finite cover is a good analog of the conventional topological one.

Example 5.5. Suppose $X = \operatorname{Spec} k$ where k is an algebraically closed field. In this case, there is a canonical equivalence of categories

$$\mathrm{Cov}_X \simeq \mathrm{FinSet},$$

where FinSet is the category of finite sets, which sends an étale cover $Y \to X$ to its set of connected components.

Fix a geometric point $\overline{x} \to X$, and assume that X is a *connected* scheme. Grothendieck's idea is to extract the fundamental group $\pi_1(X, \overline{x})$ directly from the structure of the *category* Cov_X. In particular, as in Theorem 5.2, the category Cov_X will be equivalent to the category of representations (in finite sets) of a certain (profinite) group $\pi_1(X, \overline{x})$.

Definition 5.6. The **fundamental group** $\pi_1(X, \overline{x})$ of the pair $(X, \overline{x})$ is given by the automorphism group of the forgetful functor

$$\mathrm{Cov}_X \to \mathrm{FinSet},$$

which consists of the composite

$$\mathrm{Cov}_X \to \mathrm{Cov}_{\overline{x}} \simeq \mathrm{FinSet},$$

where the first functor is the pull-back and the second is the equivalence of Example 5.5.

The automorphism group of such a functor naturally acquires the structure of a *profinite* group, and the forgetful functor above naturally lifts to a functor $\mathrm{Cov}_X \to \mathrm{FinSet}_{\pi_1(X,\overline{x})}$, where $\mathrm{FinSet}_{\pi_1(X,\overline{x})}$ denotes the category of finite sets equipped with a continuous action of the profinite group $\pi_1(X, \overline{x})$.

Then, one has:

Theorem 5.7. *(See Grothendieck [30].) The above functor* $\mathrm{Cov}_X \to \mathrm{FinSet}_{\pi_1(X,\overline{x})}$ *is an equivalence of categories.*

Grothendieck proved Theorem 5.7 by *axiomatizing* the properties that a category would have to satisfy in order to be of the form FinSet_G for G a profinite group, and checking that any Cov_X is of this form. We review the axioms here.

Recall that, in a category $\mathcal{C}$, a map $X \to Y$ is a *strict epimorphism* if the natural diagram

$$X \times_Y X \rightrightarrows X \to Y,$$

is a coequalizer.

Definition 5.8. (See Grothendieck [30, Exp. V, sec. 4].) A **classical Galois category** is a category $\mathcal{C}$ equipped with a **fiber functor** $F\colon \mathcal{C} \to \mathrm{FinSet}$ satisfying the following axioms:

1. $\mathcal{C}$ admits finite limits and F commutes with finite limits.
2. $\mathcal{C}$ admits finite coproducts and F commutes with finite coproducts.
3. $\mathcal{C}$ admits quotients by finite group actions, and F commutes with those.
4. F is conservative and preserves strict epimorphisms.
5. Every map $X \to Y$ in $\mathcal{C}$ admits a factorization $X \to Y' \to Y$ where $X \to Y'$ is a strict epimorphism and where $Y' \to Y$ is a monomorphism, which is in addition an inclusion of a summand.

Let $\mathcal{C}$ be a classical Galois category with fiber functor $F\colon \mathcal{C} \to \mathrm{FinSet}$. Grothendieck's Galois theory shows that $\mathcal{C}$ can be recovered as the category of finite sets equipped with a continuous action of a certain profinite group.

Definition 5.9. The **fundamental (or Galois) group** $\pi_1(\mathcal{C})$ of a classical Galois category $(\mathcal{C}, F)$ in the sense of Grothendieck is the automorphism group of the functor $F\colon \mathcal{C} \to \mathrm{FinSet}$.

The fundamental group of $\mathcal{C}$ is naturally a profinite group, as a (non-filtered) inverse limit of finite groups. Note that if $\mathcal{C}$ is a classical Galois category with fiber functor F, if $\pi_1(\mathcal{C})$ is the Galois group, then the fiber functor $\mathcal{C} \to \mathrm{FinSet}$ naturally lifts to

$$\mathcal{C} \to \mathrm{FinSet}_{\pi_1(\mathcal{C})},$$

just as before.

Proposition 5.10. *(See Grothendieck [30, Exp. V, Theorem 4.1].) If* $(\mathcal{C}, F)$ *is a classical Galois category, then the functor* $\mathcal{C} \to \mathrm{FinSet}_{\pi_1(\mathcal{C})}$ *as above is an equivalence of categories.*

Given a connected scheme X with a geometric point $\overline{x} \to X$, then one can show that the category Cov_X equipped with the above fiber functor (of taking the preimage over $\overline{x}$ and taking connected components) is a classical Galois category. The resulting fundamental group is a very useful invariant of a scheme, and for varieties over an algebraically closed fields of characteristic zero can be computed by profinitely completing the topological fundamental group (i.e., that of the $\mathbb{C}$-points). In particular, Theorem 5.7 is a special case of Proposition 5.10.

5.2. Definitions

In this section, we will give an exposition of Galois theory appropriate to the non-connected setting. Namely, to a type of category which we will simply call a Galois category, we will attach a *profinite groupoid:* that is, a pro-object in the $(2,1)$-category of groupoids with finitely many objects and finite automorphism groups. The advantage of this approach, which relies heavily on descent theory, is that we will not start by assuming the existence of a fiber functor, since we might not have one a priori.

Axiomatic Galois theory in many forms has a voluminous literature. The original treatment, of course, is [30], reviewed in the previous subsection. A textbook reference for some of these ideas is [11]. In [29], an approach to Galois theory (in the connected case) for almost rings is given that does not assume a priori the existence of a fiber functor. The use of profinite groupoids in Galois theory is well-known (e.g., [19,51]), and the main result below (Theorem 5.36) is presumably familiar to experts; we have included a detailed exposition for lack of a precise reference and because our $(2,1)$-categorical approach may be of some interest. Certain types of infinite Galois theory have been developed in the work of Bhatt and Scholze [15] on the pro-étale topology; we will not touch on anything related to this here. Finally, we note that forthcoming work of Lurie will treat an embedding from the ∞-category of profinite spaces into that of ∞-topoi, which provides a vast generalization of these ideas.

We start by reviewing some category theory.

Definition 5.11. We say that an object $\emptyset$ in a category $\mathcal{C}$ is **empty** if any map $x \to \emptyset$ is an isomorphism, and if $\emptyset$ is initial.

For example, the empty set is an empty object in the category of sets. In the *opposite* to the category of commutative rings, the zero ring is empty.

Definition 5.12. Let $\mathcal{C}$ be a category admitting finite coproducts, such that the initial object (i.e., the empty coproduct) is empty. We shall say that $\mathcal{C}$ admits **disjoint coproducts**

if for any $x, y \in \mathcal{C}$, the natural square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & x \sqcup y \end{tikzcd},$$

is cartesian.

The category of sets (or more generally, any topos) admits disjoint coproducts. The *opposite* of the category of commutative rings also admits disjoint coproducts.

Definition 5.13. Let $\mathcal{C}$ be a category admitting finite coproducts and finite limits. We will say that **coproducts are distributive** if for every $y \to x$ in $\mathcal{C}$, the pullback functor $\mathcal{C}_{/x} \to \mathcal{C}_{/y}$ commutes with finite coproducts.

Similarly, the category of sets (or any topos) and the opposite to the category of commutative rings satisfy this property and are basic examples to keep in mind.

Suppose $\mathcal{C}$ admits disjoint and distributive coproducts. Then $\mathcal{C}$ acquires the following very useful feature (familiar from Proposition 2.39). Given an object $x \simeq x_1 \sqcup x_2$ in $\mathcal{C}$, then we have a natural equivalence of categories,

$$\mathcal{C}_{/x} \simeq \mathcal{C}_{/x_1} \times \mathcal{C}_{/x_2},$$

which sends an object $y \to x$ of $\mathcal{C}_{/x}$ to the pair $(y \times_x x_1 \to x_1, y \times_x x_2 \to x_2)$.

Definition 5.14. Let $\mathcal{C}$ be a category admitting finite limits. Given a map $y \to x$ in $\mathcal{C}$, we have an adjunction

$$\mathcal{C}_{/y} \rightleftarrows \mathcal{C}_{/x}, \tag{11}$$

where the left adjoint sends $y' \to y$ to the composite $y' \to y \to x$, and the right adjoint takes the pullback along $y \to x$. We will say that $y \to x$ is an **effective descent morphism** if the above adjunction is monadic.

By the Beck–Bénabou–Roubaud theorem that establishes the connection between monads and descent [12], we can reformulate the notion equivalently as follows. Form the bar construction in $\mathcal{C}$,

$$\cdots \substack{\to\\\to\\\to} y \times_x y \rightrightarrows y,$$

which is a simplicial object in $\mathcal{C}$ augmented over x. Applying the pullback functor everywhere, we get a cosimplicial category

$$\mathcal{C}_{/y} \rightrightarrows \mathcal{C}_{/y \times_x y} \substack{\to\\\to\\\to} \cdots,$$

receiving an augmentation from $\mathcal{C}_{/x}$. Then $y \to x$ is an effective descent morphism if the functor

$$\mathcal{C}_{/x} \to \mathrm{Tot}\left(\mathcal{C}_{/y} \rightrightarrows \mathcal{C}_{/y \times_x y} \substack{\to\\\to\\\to} \cdots\right),$$

is an equivalence of categories. If $\mathcal{C}$ is an ∞-category, we can make the same definition.

We note that whether or not a map $y \to x$ is an effective descent morphism can be checked using the Barr–Beck theorem applied to the adjunction (11). Namely, the pullback along $y \to x$ needs to preserve reflexive coequalizers which are split under pullback, and it needs to be conservative.

Finally, we are ready to define a Galois category.

Definition 5.15. A **Galois category** is a category $\mathcal{C}$ such that:

1. $\mathcal{C}$ admits finite limits and coproducts, and the initial object $\emptyset$ is empty.
2. Coproducts are disjoint and distributive in $\mathcal{C}$.
3. Given an object x in $\mathcal{C}$, there is an effective descent morphism $x' \twoheadrightarrow *$ (where $*$ is the terminal object) and a decomposition $x' = x'_1 \sqcup \cdots \sqcup x'_n$ such that each map $x \times x'_i \to x'_i$ decomposes as the fold map $x \times x'_i \simeq \bigsqcup_{S_i} x'_i \to x'_i$ for a finite set S_i.

The collection of all Galois categories and functors between them (which are required to preserve coproducts, effective descent morphisms, and finite limits) can be organized into a $(2,1)$-category GalCat. Given $\mathcal{C}, \mathcal{D} \in$ GalCat, we will let $\mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}, \mathcal{D})$ denote the groupoid of functors $\mathcal{C} \to \mathcal{D}$ in GalCat.

In other words, we might say that an object $x \in \mathcal{C}$ is in *elementary form* if $x \simeq \bigsqcup_S *$ for some finite set S. More generally, if there exists a decomposition $* \simeq *_1 \sqcup \cdots \sqcup *_n$, such that, as an object of $\mathcal{C} \simeq \prod_i \mathcal{C}_{/*_i}$, each $y \times_* *_i \to *_i$ is in elementary form, we say that y is in *mixed elementary form.* Then the *defining* feature of a Galois category is that, locally, every object is in mixed elementary form.

Our first goal is to develop some of the basic properties of Galois categories. First, we need a relative version of the previous paragraph.

Definition 5.16. Let $\mathcal{C}$ be a category satisfying the first two conditions of Definition 5.15 (which we note are preserved by passage to $\mathcal{C}_{/x}$ for any $x \in \mathcal{C}$). We say that a map $f : x \to y$ is *setlike* if there are finite sets S, T such that $x \simeq \bigsqcup_S *$, $y \simeq \bigsqcup_T *$ and the map $x \to y$ comes from a map of finite sets $S \to T$. This implies that $x \in \mathcal{C}_{/y}$ is in mixed elementary form.

For example, if $y = *$, then $x \to y$ is setlike if and only if x is in elementary form. Suppose x, y are in elementary form, so that $x \simeq \bigsqcup_S *$ and $y \simeq \bigsqcup_T *$. Then a map $x \to y$ is not necessarily setlike. However, by the disjointness of coproducts, it follows that the map $\bigsqcup_S * \to \bigsqcup_T *$ gives, for each $s \in S$, a decomposition of the terminal object $*$ as a disjoint union of objects $*_t^{(s)}$ for each $t \in T$. It follows that, refining these decompositions, there exists a decomposition $* \simeq *_1 \sqcup \cdots \sqcup *_n$ such that the map $x \to y$ becomes setlike after pulling back along $*_i \to *$. In particular, $x \to y$ is locally setlike. The same argument works if x, y are disjoint unions of summands of the terminal object.

More generally, we have:

Proposition 5.17. *Let $f\colon x \to y$ be any map in the Galois category $\mathcal{C}$. Then there exist an effective descent morphism $t \twoheadrightarrow *$ and a decomposition $t \simeq \bigsqcup_{i=1}^n t_i$ such that the map $x \times t_i \to y \times t_i$ in $\mathcal{C}_{/t_i}$ is setlike. More generally, given any finite set of maps $f_j\colon x_j \to y_j$ we can find such a decomposition such that each $f_j \times t_i$ is setlike.*

Proof. We can choose t such that $(x \sqcup y) \times t$ is in mixed elementary form: in particular, we have a decomposition $t \simeq t_1 \sqcup \cdots \sqcup t_n$ such that $(x \sqcup y) \times t_i$ is a disjoint union of copies of t_i in $\mathcal{C}_{/t_i}$. It follows that $x \times t_i \to t_i$ and $y \times t_i \to t_i$ are objects in $\mathcal{C}_{/t_i}$ which are disjoint union of summands of copies of the terminal object $t_i \in \mathcal{C}_{/t_i}$. Using the previous discussion, it follows that we can refine the t_i further (by splitting into summands) to make $x \to y$ setlike on each summand. A similar argument would work for any finite set of morphisms in $\mathcal{C}$. $\square$

Corollary 5.18. *Let $\mathcal{C}$ be a Galois category and let $x \in \mathcal{C}$. Then $\mathcal{C}_{/x}$ is a Galois category.*

Proof. The first two axioms are evident. For the third, fix a map $y \to x$ in $\mathcal{C}$ (thus defining an object of $\mathcal{C}_{/x}$). By Proposition 5.17, we can find an object $x' \in \mathcal{C}$ together with an effective descent morphism $x' \twoheadrightarrow *$ such that $y \times x' \to x \times x'$ becomes, after decomposing x' into a disjoint union of summands, setlike in $\mathcal{C}_{/x'}$. It follows that $y \times x' \to x' \times x$ is in mixed elementary form as an object of $\mathcal{C}_{/x \times x'}$. $\square$

The notion of an effective descent morphism is a priori not so well-behaved, which might be a cause for worry. Our next goal is to show that this is not the case.

Proposition 5.19. *A Galois category $\mathcal{C}$ admits finite colimits, which are distributive over pullbacks.*

Proof. Let K be a finite category; choose a map $p\colon K \to \mathcal{C}_{/x}$ for some object $x \in \mathcal{C}$. Since $\mathcal{C}_{/x}$ is itself a Galois category, we can replace $\mathcal{C}_{/x}$ with $\mathcal{C}$ and show that if $y \in \mathcal{C}$ is arbitrary, then the natural map

$$\varinjlim_K (y \times p(k)) \to y \times \varinjlim_K p(k), \tag{12}$$

is an equivalence, and in particular the colimits in question exist.

There is one case in which the above would be automatic. Since $\mathcal{C}$ has finite coproducts, we can define the product of a finite set with any object in $\mathcal{C}$. Suppose there exists a diagram $\overline{p}\colon K \to \mathrm{FinSet}$ and an object $u \in \mathcal{C}$ such that $p = \overline{p} \times u$. For example, suppose that for every morphism in K, the image in $\mathcal{C}$ is setlike; then this would happen. In this case, both sides of (12) are defined and are given by $y \times u \times \varinjlim_K \overline{p}$, since finite coproducts distribute over pullbacks.

We will say that a diagram $p\colon K \to \mathcal{C}$ is *good* if it arises from a $\overline{p}\colon K \to \mathrm{FinSet}$ and an $u \in \mathcal{C}$; the good case is thus straightforward. If we have a finite decomposition of the terminal object $* = \bigsqcup_{i=1}^n *_i$ such that the restriction $p \times_* *_i$ is good, then we say that p is *weakly good*. In this case, using $\mathcal{C} \simeq \prod_{i=1}^n \mathcal{C}_{/*_i}$, we conclude that (12) is defined and holds.

We can reduce to the good (or weakly good) case via descent. There exists an effective descent morphism $x \to *$ such that $p \times x\colon K \to \mathcal{C}_{/x}$ is weakly good by Proposition 5.17. Using the expression $\mathcal{C} \simeq \mathrm{Tot}\left(\mathcal{C}_{/x\times\cdots\times x}\right)$, it follows that (12) must be true at each stage in the totalization, and the respective colimits are compatible with the coface and coboundary maps, so that it is (defined and) true in the totalization. □

Remark 5.20. In the above argument, we have tacitly used the following fact. Consider a category I and an I-indexed family of categories (or ∞-categories) $(\mathcal{C}_i)_{i\in I}$. Consider a functor $p\colon K \to \varprojlim_I \mathcal{C}_i$, where K is a fixed simplicial set. Suppose each composite $K \xrightarrow{p} \varprojlim_I \mathcal{C}_i \to \mathcal{C}_{i_1}$ (for each $i_1 \in I$) admits a colimit and suppose these colimits are preserved by the various maps in I. Then p admits a colimit compatible with the colimits in each $\mathcal{C}_i$.

Corollary 5.21. *The composite of two effective descent morphisms in a Galois category $\mathcal{C}$ is an effective descent morphism.*[4] *If $x \to y$ is any map in $\mathcal{C}$ and $y' \to y$ is an effective descent morphism, then $x \to y$ is an effective descent morphism if and only if $x \times_y y' \to y'$ is one.*

Proof. Since (Proposition 5.19) a Galois category has finite colimits, which distribute over pull-backs, it follows by the Barr–Beck theorem a map $x \to y$ is an effective descent morphism if and only if it is conservative. This is preserved under compositions. The second statement is proved similarly, since one only has to check conservativity locally. □

Proposition 5.22. *Given a map $f\colon x \to y$ in the Galois category $\mathcal{C}$, the following are equivalent:*

1. *f is an effective descent morphism.*
2. *f is a strict epimorphism.*
3. *For every $y' \to y$ with y' nonempty, the pullback $x \times_y y'$ is nonempty.*

[4] Results on this question in more general categories are contained in [75,72].

Proof. All three conditions can be checked locally. After base-change by an effective descent morphism $t \twoheadrightarrow *$ and a decomposition $t \simeq t_1 \sqcup \cdots \sqcup t_n$, we can assume that the map $x \to y$ is setlike, thanks to Proposition 5.17. In this case, the result is obvious. □

We now discuss a few facts about functors between Galois categories. These will be useful when we analyze GalCat as a 2-category in the next section.

Proposition 5.23. *Let $\mathcal{C}, \mathcal{D}$ be Galois categories. A functor $\mathcal{C} \to \mathcal{D}$ in* GalCat *preserves finite colimits.*

Proof. This is proved as in Proposition 5.19: any functor preserves colimits of *good* diagrams (in the terminology of the proof of Proposition 5.19), and after making a base change one may reduce to this case. □

Next, we include a result that shows that GalCat (or, rather, its opposite) behaves, to some extent, like a Galois category itself; at least, it satisfies a version of the first axiom of Definition 5.15.

Definition 5.24. A Galois category $\mathcal{C}$ is **connected** if there exists no decomposition $* \simeq *_1 \sqcup *_2$ with $*_1, *_2$ nonempty and if additionally $\emptyset \not\simeq *$.

Proposition 5.25. *Let $\mathcal{C}$ be a connected Galois category and let $\mathcal{C}_1, \mathcal{C}_2$ be Galois categories. Then $\mathcal{C}_1 \times \mathcal{C}_2 \in$ GalCat and we have an equivalence of groupoids*

$$\mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_1, \mathcal{C}) \sqcup \mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_2, \mathcal{C}).$$

The above equivalence of groupoids is as follows. Given a functor $\mathcal{C}_i \to \mathcal{C}$ for $i \in \{1, 2\}$, we obtain a functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ by composing with the appropriate projection.

Proof. The assertion that $\mathcal{C}_1 \times \mathcal{C}_2 \in$ GalCat is easy to check. Consider a functor $F \colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ in GalCat. Note that every object $(x, y) \in \mathcal{C}_1 \times \mathcal{C}_2$ decomposes as the disjoint union $(x, \emptyset) \sqcup (\emptyset, y)$. For example, in $\mathcal{C}_1 \times \mathcal{C}_2$, the terminal object $* = (*, *)$ decomposes as the union $*_1 \sqcup *_2$ where $*_1$ is terminal in $\mathcal{C}_1$ and empty in $\mathcal{C}_2$, and $*_2$ is terminal in $\mathcal{C}_2$ and empty in $\mathcal{C}_1$. It follows that $F(*_1) = \emptyset$ or $F(*_2) = \emptyset$ since $\mathcal{C}$ is connected. If $F(*_1) = \emptyset$ and therefore $F(*_2) = *$, then we have for $x \in \mathcal{C}_1, y \in \mathcal{C}_2$,

$$F((x, y)) \simeq F((x, y) \times *_2) \simeq F((\emptyset, y)),$$

so that F (canonically) factors through $\mathcal{C}_2$. The other case is analogous. □

More generally, let $\mathcal{C}$ be an arbitrary Galois category, and fix $\mathcal{C}_1, \mathcal{C}_2 \in$ GalCat. We find, by the same reasoning,

$$\mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{C}) \simeq \bigsqcup_{* = *_1 \sqcup *_2} \mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_1, \mathcal{C}_{/*_1}) \times \mathrm{Fun}^{\mathrm{Gal}}(\mathcal{C}_2, \mathcal{C}_{/*_2}), \tag{13}$$

where the disjoint union is taken over all decompositions of the terminal object in $\mathcal{C}$.

This concludes our preliminary discussion of the basic properties of Galois categories. In the next subsection, we will give another description of the $(2, 1)$-category of Galois categories. For now, though, we describe a basic method of extracting Galois categories from other sources.

Definition 5.26. A **Galois context** is an ∞-category $\mathcal{C}$ satisfying the first two axioms of Definition 5.15 together with a class $\mathcal{E} \subset \mathcal{C}$ of morphisms such that:

1. $\mathcal{E}$ is closed under composition and base change and contains every equivalence.
2. Every morphism in $\mathcal{E}$ is an effective descent morphism.
3. Given a cartesian diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[d] \\ y' \arrow[r] & y \end{tikzcd},$$

   where $y' \to y \in \mathcal{E}$, then $x \to y$ belongs to $\mathcal{E}$ if and only if $x' \to y'$ does.
4. A map $x \to y \simeq y_1 \sqcup y_2$ belongs to $\mathcal{E}$ if and only if $x \times_y y_1 \to y_1$ and $x \times_y y_2 \to y_2$ belong to $\mathcal{E}$.
5. For any object $x \in \mathcal{C}$ and any finite nonempty set S, the fold map $\bigsqcup_S x \to x$ belongs to $\mathcal{E}$.

Given Galois contexts $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{D}, \mathcal{E}')$, a **functor of Galois contexts** $F: (\mathcal{C}, \mathcal{E}) \to (\mathcal{D}, \mathcal{E}')$ will mean a functor of ∞-categories $\mathcal{C} \to \mathcal{D}$ which respects finite limits and coproducts and which carries morphisms in $\mathcal{E}$ to morphisms in $\mathcal{E}'$.

Definition 5.27. Given a Galois context $(\mathcal{C}, \mathcal{E})$, we say that an object $x \in \mathcal{C}$ is **Galoisable** (or $\mathcal{E}$-Galoisable) if there exists a map $y \to *$ in $\mathcal{E}$ such that the pullback $x \times y \to y$ is in mixed elementary form in $\mathcal{C}_{/y}$, as in the discussion after Definition 5.15. In other words, we require that there is a decomposition $y \simeq y_1 \sqcup \cdots \sqcup y_n$ such that each $x \times y_i \to y_i$ decomposes as a finite coproduct $\bigsqcup_{S_i} y_i \to y_i$.

Given a category satisfying the first two axioms of Definition 5.15, the following result enables us to extract a Galois category by considering the Galoisable objects.

Proposition 5.28. *Let $(\mathcal{C}, \mathcal{E})$ be a Galois context. Then the collection of Galoisable objects in $\mathcal{C}$ (considered as a full subcategory of $\mathcal{C}$) forms a Galois category.*

Proof. Note first that the collection of Galoisable objects actually forms a *category* rather than an ∞-category: that is, the mapping space between any two Galoisable objects is (homotopy) discrete. More precisely, if $x \in \mathcal{C}$ is Galoisable and $x' \in \mathcal{C}$ is arbitrary, then we claim that $\mathrm{Hom}_{\mathcal{C}}(x', x)$ is discrete. To see this, we choose an effective descent morphism $u_1 \sqcup \cdots \sqcup u_n \twoheadrightarrow *$ such that each map $u_i \times x \to x$ is in elementary form. Using the expression $\mathcal{C} \simeq \mathrm{Tot}(\mathcal{C}_{/u_1 \times \cdots \times u_n})$, one reduces to the case where x is a (disjoint) finite coproduct of copies of the terminal object $*$. In this case, $\mathrm{Hom}_{\mathcal{C}}(x', \bigsqcup_S *)$ is the *set* of all S-labeled decompositions of x' as direct sums of subobjects, using the expression $\mathcal{C}_{/\bigsqcup_S *} \simeq \prod_S \mathcal{C}_{/*} \simeq \prod_S \mathcal{C}$.

Suppose $y \in \mathcal{C}$ is a Galoisable object. We need to show that there is a Galoisable object t' and an $\mathcal{E}$-morphism $t' \twoheadrightarrow *$ such that the pullback $y \times t' \to t'$ is in mixed elementary form. By assumption, we know that we can do this with *some* object $t \in \mathcal{C}$ with an $\mathcal{E}$-morphism $t \twoheadrightarrow *$, but we do not have any control of t. We will find a *Galoisable* choice of t' by an inductive procedure.

Define the *rank* of a Galoisable object $y \in \mathcal{C}$ as follows. If y is mixed elementary, with respect to a decomposition $* \simeq \bigsqcup_{i=1}^n *_i$ (with the $*_i$ nonempty) and $y = \bigsqcup_{i=1}^n \bigsqcup_{S_i} *_i$ for finite sets S_i, we define the rank to be $\sup_i |S_i|$. In general, we make a base change in $\mathcal{C}$ along some $\mathcal{E}$-morphism $t \to *$ (by a not necessarily Galoisable object) to reduce to this case. In other words, to define the rank of y, we choose an $\mathcal{E}$-morphism $t \twoheadrightarrow *$ such that $y \times t \to t$ is in mixed elementary form in $\mathcal{C}_{/t}$, and then consider the rank of that.

If the rank is zero, then $y = \emptyset$. We now use induction on the rank of y. First, we claim that there is a decomposition $* \simeq *_1 \sqcup *_2$ such that $y \to *$ factors through an $\mathcal{E}$-morphism $y \to *_1$. (Meanwhile, $y \times_* *_2 = \emptyset$.) To see this decomposition and claim, we can work locally on $\mathcal{C} \simeq \mathrm{Tot}(\mathcal{C}_{/t \times \cdots \times t})$ to reduce to the case in which y is already in mixed elementary form, for which the assertion is evident. Thus we can reduce to the case where $y \to *$ is an $\mathcal{E}$-morphism.

Now consider the pullback $y \times y \to y$. This admits a section, so we have $y \times y \simeq y \sqcup c$ where c is another Galoisable object in $\mathcal{C}_{/y}$; to see that c exists, one works locally using t to reduce to the mixed elementary case. However, by working locally again, one sees that the rank of c is one less than the rank of y. We can reduce the rank one by one, splitting off pieces, to get down to the case where $y = \emptyset$. □

In fact, the above argument shows that if $x \in \mathcal{C}$ is Galoisable, there exists a Galoisable $y \in \mathcal{C}$ together with a morphism $y \twoheadrightarrow *$ which belongs to $\mathcal{E}$ such that $x \times y \to y$ is in mixed elementary form.

Corollary 5.29. *Let $(\mathcal{C}, \mathcal{E})$ be a Galois context. Then a map $x \to y$ between Galoisable objects in $\mathcal{C}$ is an effective descent morphism in the category of Galoisable objects if and only if it belongs to $\mathcal{E}$. Therefore, a functor of Galois contexts induces a functor of Galois categories.*

Proof. Working locally (because of the local nature of belonging to $\mathcal{E}$, and in view of the preceding remark), we may assume the map $x \to y$ is setlike, in which case it is evident. □

5.3. The Galois correspondence

The Galois correspondence for groupoids gives an alternate description of the $(2,1)$-category GalCat. To see this, we describe the building blocks in GalCat.

Example 5.30. Let G be a finite group. Then the category FinSet_G of finite G-sets is a Galois category. Only the last axiom requires verification. In fact, given any finite G-set T, we have an effective descent morphism $G \to *$ such that $T \times G$, as a G-set, is a disjoint union of copies of G (since it is free).

This Galois category enjoys a convenient universal property, following [19].

Definition 5.31. Let $\mathcal{C}$ be a Galois category and let G be a finite group. A ***G*-torsor** in $\mathcal{C}$ consists of an object $x \in \mathcal{C}$ with a G-action such that there exists an effective descent morphism $y \twoheadrightarrow *$ such that $y \times x \in \mathcal{C}_{/y}$, as an object with a G-action, is given by

$$y \times x \simeq \bigsqcup_G y,$$

where G acts on the latter by permuting the summands. For instance, x could be $\bigsqcup_G *$. The collection of G-torsors forms a full subcategory $\mathrm{Tors}_G(\mathcal{C}) \subset \mathrm{Fun}(BG, \mathcal{C})$.

The Galois category FinSet_G has a natural example of a G-torsor: namely, G itself. The next result states that it is *universal* with respect to that property.

Proposition 5.32. *If $\mathcal{C}$ is a Galois category, there is a natural equivalence between* $\mathrm{Fun}^{\mathrm{Gal}}(\mathrm{FinSet}_G, \mathcal{C})$ *and the category* $\mathrm{Tors}_G(\mathcal{C})$ *of G-torsors in $\mathcal{C}$.*

Proof. Any functor of Galois categories preserves torsors for any finite group. In particular, given a functor $F\colon \mathrm{FinSet}_G \to \mathcal{C}$ in GalCat, one gets a natural choice of G-torsor in $\mathcal{C}$ by considering $F(G)$. Since everything in FinSet_G is a colimit of copies of G, the choice of $F(G)$ determines everything else. Together with the Yoneda lemma, this implies that the functor from $\mathrm{Fun}^{\mathrm{Gal}}(\mathrm{FinSet}_G, \mathcal{C})$ to G-torsors is fully faithful.

It remains to argue that, given a G-torsor in $\mathcal{C}$, one can construct a corresponding functor $\mathrm{FinSet}_G \to \mathcal{C}$ in GalCat. In other words, we want to show that the fully faithful functor

$$\mathrm{Fun}^{\mathrm{Gal}}(\mathrm{FinSet}_G, \mathcal{C}) \to \mathrm{Tors}_G(\mathcal{C}),$$

is essentially surjective. However, writing $\mathcal{C}$ as a totalization of $\mathcal{C}_{/x \times \cdots \times x}$, one may assume the G-torsor is trivial, in which case the claim is evident. $\square$

More generally, we can build Galois categories from finite groupoids. This will be very important from a 2-categorical point of view.

Definition 5.33. We say that a groupoid $\mathscr{G}$ is **finite** if $\mathscr{G}$ has finitely many isomorphism classes of objects and, for each object $x \in \mathscr{G}$, the automorphism group $\mathrm{Aut}_{\mathscr{G}}(x)$ is finite. The collection of all finite groupoids, functors, and natural transformations is naturally organized into a $(2,1)$-category $\mathrm{Gpd}_{\mathrm{fin}}$.

In other words, a finite groupoid is a 1-truncated homotopy type such that π_0 is finite, as is π_1 with any choice of basepoint.

Given a finite groupoid $\mathscr{G}$, the category $\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})$ of functors from $\mathscr{G}$ into the category of finite sets forms a Galois category. This is a generalization of Example 5.30 and follows from it since the categories $\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})$ are finite products of the Galois categories of finite G-sets as G varies over the automorphism groups. If we interpret $\mathscr{G}$ as a 1-truncated homotopy type, then this is precisely the category of finite *covering spaces* of $\mathscr{G}$, or of local systems of finite sets on $\mathscr{G}$.

It follows that we get a functor of $(2,1)$-categories

$$\mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}} \to \mathrm{GalCat},$$

sending a finite groupoid $\mathscr{G}$ to the associated functor category $\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})$. Note that, for example, a natural transformation between functors of finite groupoids gives a natural transformation at the level of Galois categories.

In order to proceed further, we need a basic formal property of GalCat:

Proposition 5.34. *The $(2,1)$-category* GalCat *admits filtered colimits, which are computed at the level of the underlying categories: the colimit of a diagram of Galois categories and functors between them (which respect coproducts, finite limits, and effective descent morphisms) in the $(2,1)$-category of categories is again a Galois category.*

Proof. Let $F\colon I \to \mathrm{GalCat}$ be a filtered diagram of Galois categories. Our claim is that the colimit $\varinjlim_I F$ is a Galois category and the natural functors $F(j) \to \varinjlim_I F$ respect the relevant structure. We first observe that $\varinjlim_I F$ has all finite limits and colimits, and the functors $F(j) \to \varinjlim_I F$ respect those. This holds for any filtered diagram of ∞-categories and functors preserving finite limits (resp. colimits) as a formal consequence of the commutation of finite limits and filtered colimits in the ∞-category of spaces. For example, every finite diagram in $\varinjlim_I F$ factors through a finite stage. From this, the first two axioms of Definition 5.15 follow.

Next, we want to claim that the functors $F(j) \to \varinjlim_I F$ respect effective descent morphisms. Once we have shown this, the last axiom of Definition 5.15 will follow, since we know it at each stage $F(j)$. In fact, let $x \to y$ be an effective descent morphism in $F(j)$. Then, we need to check that pull-back along $x \to y$ is conservative and respects finite colimits in $\varinjlim_I F$; however, this follows since it holds in each $F(j')$, since finite colimits and pullbacks are preserved under $F(j') \to \varinjlim_I F$.

Finally, it follows from the previous paragraph that since every object in each $F(j)$ is locally in mixed elementary form, with respect to effective descent morphisms in $F(j)$, the same is true in $\varinjlim_I F$, since every object in the colimit comes from a finite stage. $\square$

It follows that we get a natural functor

$$\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \simeq \mathrm{Ind}(\mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}}) \to \mathrm{GalCat},$$

i.e., a *contravariant* functor from the $(2,1)$-category $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}}$ into the $(2,1)$-category of Galois categories. We give this a name.

Definition 5.35. A **profinite groupoid** is an object of $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}}^{\mathrm{op}})$.

We will describe some features of the $(2,1)$-category of profinite groupoids in the next subsection. In the meantime, the main result can now be stated as follows.

Theorem 5.36 *(The Galois correspondence). The functor* $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{GalCat}$ *is an equivalence of 2-categories.*

Proof. We first check that the functor is fully faithful. To do this, first fix *finite* groupoids $\mathscr{G}, \mathscr{G}'$. We want to compare the categories of functors $\mathrm{Fun}(\mathscr{G}, \mathscr{G}')$ and $\mathrm{Fun}^{\mathrm{Gal}}(\mathrm{Fun}(\mathscr{G}', \mathrm{FinSet}), \mathrm{Fun}(\mathscr{G}, \mathrm{FinSet}))$. In particular, we want to show that

$$\mathrm{Fun}(\mathscr{G}, \mathscr{G}') \to \mathrm{Fun}^{\mathrm{Gal}}(\mathrm{Fun}(\mathscr{G}', \mathrm{FinSet}), \mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})), \tag{14}$$

is an equivalence of groupoids. We can reduce to the case where $\mathscr{G}$ has one isomorphism class of objects, since both sides of (14) send coproducts in $\mathscr{G}$ to products of groupoids. We can also reduce to the case where $\mathscr{G}'$ has a single connected component, since if $\mathscr{G}$ is connected, then both sides of (14) take coproducts in $\mathscr{G}'$ to coproducts. This is clear for the left-hand-side. For the right-hand-side, note that coproducts in $\mathscr{G}'$ go over to *products* in GalCat for $\mathrm{Fun}(\mathscr{G}', \mathrm{FinSet})$. Now use Proposition 5.25 to describe the corepresented functor for a product in GalCat. In order to show that (14) is an equivalence when $\mathscr{G}, \mathscr{G}'$ are finite groupoids, it thus suffices to work with *groups*. We can do this extremely explicitly.

In the case of *finite groups*, given any two such G, G', the groupoid of maps between the associated groupoids has connected components given by the conjugacy classes of homomorphisms $G \to G'$. Given any $f\colon G \to G'$, the automorphism group of f is the centralizer of the image $f(G)$. To understand $\mathrm{Fun}^{\mathrm{Gal}}(\mathrm{FinSet}_{G'}, \mathrm{FinSet}_G)$, we can use Proposition 5.32. We need to describe the category of G'-torsors in FinSet_G. Any such gives a G'-torsor in FinSet by forgetting, so a G'-torsor in FinSet_G yields in particular a copy of G' with G acting G'-equivariantly (i.e., G acts by right multiplication by various elements of G'). It follows that any torsor arises by considering a homomorphism $\phi\colon G \to G'$ and using that to equip the G-torsor $G' \in \mathrm{FinSet}_{G'}$ with the structure of a G-set. A natural transformation of functors, or a morphism of torsors, is given by a conjugacy in G' between two homomorphisms $G \to G'$: an automorphism of the torsor comes from right multiplication by an element of G' which centralizes the image of $G \to G'$. This verifies full faithfulness for finite groupoids, i.e., that (14) is an equivalence if $\mathscr{G}, \mathscr{G}'$ are finite.

Finally, we need to check that the full faithfulness holds for all *profinite* groupoids. That is a formal consequence of the fact that $\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})$ is a *compact* object in GalCat for $\mathscr{G}$ a finite groupoid. If $\mathscr{G}$ is connected, this is a consequence of the universal property, Proposition 5.32, since a torsor involves a finite amount of data. In general, the observation follows from the connected case together with Proposition 5.25 (and the remarks immediately following, in particular (13)).

To complete the proof, we need to show that the functor is essentially surjective: that is, every Galois category arises from a profinite groupoid. For this, we need another lemma on the formal structure of GalCat.

Lemma 5.37. GalCat *admits finite limits, which are preserved under* GalCat $\to \mathrm{Cat}_\infty$.

Proof. Since GalCat has a terminal object (the terminal category), it suffices to show that given a diagram

$$\begin{tikzcd} & \mathcal{C}' \arrow[d] \\ \mathcal{C}'' \arrow[r] & \mathcal{C} \end{tikzcd},$$

in GalCat, the category-theoretic fiber product is still a Galois category. Of the axioms in Definition 5.15, only the third needs checking. Note first that a map $x \to y$ in $\mathcal{C}' \times_{\mathcal{C}} \mathcal{C}''$ is an effective descent morphism if it is one in $\mathcal{C}'$ and $\mathcal{C}''$. This follows from the fact that the formation of overcategories and totalizations are compatible with fiber products of categories.

Let x be an object of the fiber product. We want to show that x is locally in mixed elementary form. As before, we can perform induction on the *rank* of x (defined as the maximum of the ranks of the images in $\mathcal{C}', \mathcal{C}''$). The natural map $x \to *$ has the property that $* \simeq *_1 \sqcup *_2$ where $x \to *$ factors through an effective descent morphism $x \twoheadrightarrow *_1$. In fact, we can construct these on $\mathcal{C}', \mathcal{C}''$ and they have to match up on $\mathcal{C}$. So, we can assume that $x \to *$ is an effective descent morphism. Now after base-change along $x \to *$, we can find a section of $x \times x \to x$ and thus obtain a splitting of $x \times x$ (since we can in $\mathcal{C}', \mathcal{C}''$). Using induction on the rank, we can conclude as before. $\square$

Remark 5.38. The same logic shows that GalCat admits arbitrary limits, although they are no longer preserved under the forgetful functor GalCat $\to \mathrm{Cat}_\infty$; one has to take the subcategory of the categorical limit consisting of objects whose rank is bounded.

Let $\mathcal{C}$ be any Galois category, which we want to show lies in the image of the fully faithful functor $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{GalCat}$. In order to do this, we will write $\mathcal{C}$ as a filtered colimit of subcategories which do belong to the image.

Let $\mathcal{C}$ be a Galois category. Then, if $\mathcal{C}$ is not the terminal category (i.e., if the map $\emptyset \to *$ in $\mathcal{C}$ is not an isomorphism), there is a faithful functor $\mathrm{FinSet} \to \mathcal{C}$ which sends a finite set S to $\bigsqcup_S *$. This is a functor in GalCat and defines, for every nonempty Galois category $\mathcal{C}$, a (non-full) Galois subcategory $\mathcal{C}_{\mathrm{triv}}$. In other words, we take the objects which are in elementary form and the setlike maps between them. More generally, if $*$ decomposes as $* = *_1 \sqcup \cdots \sqcup *_n$, we can define a subcategory $\mathcal{C}^{\mathrm{loc}}_{\mathrm{triv}} \subset \mathcal{C}$ by writing $\mathcal{C} \simeq \prod_{i=1}^n \mathcal{C}_{/*_i}$ and taking the subcategory $\mathcal{C}^{\mathrm{loc}}_{\mathrm{triv}} = \prod_{i=1}^n (\mathcal{C}_{/*_i})_{\mathrm{triv}}$.

Let $y \twoheadrightarrow *$ be an effective descent morphism and let $y \simeq y_1 \sqcup \cdots \sqcup y_n$ be a decomposition of y. We define a map $f \colon x \to x'$ in $\mathcal{C}$ to be *split* with respect to y and the above decomposition if $f \times y_i \in \mathcal{C}_{/y_i}$ is setlike for each $i = 1, 2, \dots, n$. Via descent theory, we can write this subcategory as

$$\mathcal{C}' = \mathrm{Tot}\left(\prod_{i=1}^n \mathcal{C}^{\mathrm{triv}}_{/y_i} \rightrightarrows \prod_{i,j=1}^n \mathcal{C}^{\mathrm{triv}}_{y_i \times y_j} \substack{\to\\\to\\\to} \cdots \right).$$

In other words, this subcategory of $\mathcal{C}$ arises as an inverse limit (indexed by a cosimplicial diagram) of products of copies of FinSet. Any such is the category of finite covers of a finite CW complex (presented by 3-skeleton of the dual simplicial set[5]) and is thus in the image of $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}}$. However, $\mathcal{C}$ is the filtered union over all such subcategories as we consider effective descent morphisms $y_1 \sqcup \cdots y_n \twoheadrightarrow *$ with the $\{y_i\}$ varying. It follows that $\mathcal{C}$ is the filtered colimit in GalCat of objects which belong to the image of $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{GalCat}$, and is therefore in the image of $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}}$ itself. □

Theorem 5.36 enables us to make the following fundamental definition.

Definition 5.39. Given a Galois category $\mathcal{C}$, we define the **fundamental groupoid** or **Galois groupoid** $\pi_{\leq 1}\mathcal{C}$ of $\mathcal{C}$ as the associated profinite groupoid under the correspondence of Theorem 5.36.

We next use the Galois correspondence to obtain a few technical results on torsors.

Corollary 5.40. *The Galois categories* FinSet_G *jointly detect equivalences: given a functor in* GalCat, $F \colon \mathcal{C} \to \mathcal{D}$, *if* F *induces an equivalence on the categories of* G*-torsors for each finite group* G*, then* F *is an equivalence. In other words, if the map*

$$\mathrm{Tors}_G(\mathcal{C}) \to \mathrm{Tors}_G(\mathcal{D}) \tag{15}$$

is an equivalence of groupoids for each G*, then* F *is an equivalence.*

[5] We recall that the 3-truncation of the above totalization is sufficient to compute the totalization.

Proof. By (13), it follows that if (15) is always an equivalence, then the map

$$\mathrm{Hom}_{\mathrm{GalCat}}(\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet}), \mathcal{C}) \to \mathrm{Hom}_{\mathrm{GalCat}}(\mathrm{Fun}(\mathscr{G}, \mathrm{FinSet}), \mathcal{D}),$$

is an equivalence for each finite *groupoid* $\mathscr{G}$. Dualizing, and using the Galois correspondence, we find that the map $\pi_{\leq 1}\mathcal{D} \to \pi_{\leq 1}\mathcal{C}$ of profinite groupoids has the property that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(\pi_{\leq 1}\mathcal{C}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(\pi_{\leq 1}\mathcal{D}, \mathscr{G})$$

is always an equivalence, for every finite groupoid $\mathscr{G}$. However, we know that finite groupoids generate $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ under cofiltered limits, so we are done. $\square$

Corollary 5.41. *Let $\mathcal{C}$ be a Galois category and $x \in \mathcal{C}$ be an object. Then there exists a G-torsor y in $\mathcal{C}$ for some finite group G such that $x \times y \to y$ is in mixed elementary form in $\mathcal{C}_{/y}$.*

Proof. We can reduce to the case where $\mathcal{C} = \mathrm{Fun}(\mathscr{G}, \mathrm{FinSet})$ for $\mathscr{G}$ a finite groupoid, since $\mathcal{C}$ is a filtered colimit of such. Let $\mathscr{G}$ have objects $x_1, \dots, x_n$ up to isomorphism with automorphism groups $G_1, \dots, G_n$. Then, there is a natural $G_1 \times \cdots \times G_n$-torsor y on $\mathscr{G} \simeq \bigsqcup_{i=1}^n BG_i$ (which on the ith summand is the universal cover times the trivial $\prod_{j \neq i} G_j$-torsor) such that any object x in $\mathcal{C}$ has the property that $y \times x$ is in mixed elementary form. $\square$

5.4. Profinite groupoids

Given Theorem 5.36, it behooves us to discuss the 2-category $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ of profinite groupoids in more detail. We begin by studying connected components.

We have a natural functor $\pi_0 \colon \mathrm{Gpd}_{\mathrm{fin}} \to \mathrm{FinSet}$ sending a groupoid to its set of isomorphism classes of objects. Therefore, we get a functor $\pi_0 \colon \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}}) \to \mathrm{Pro}(\mathrm{FinSet})$ which is uniquely determined by the properties that it recovers the old π_0 for finite groupoids and that it commutes with cofiltered limits. Recall that the category $\mathrm{Pro}(\mathrm{FinSet})$ is the category of compact, Hausdorff, and totally disconnected topological spaces, under the realization functor which sends a profinite set to its inverse limit (in the category of sets) with the inverse limit topology. It follows that the collection of "connected components" of a profinite groupoid is one of these.

Remark 5.42. Note that $\pi_0 \colon \mathrm{Gpd}_{\mathrm{fin}} \to \mathrm{FinSet}$ does not commute with finite inverse limits, so that its right Kan extension to $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ does not. While the reader might object that there should be a $\lim^1$ obstruction to the commutation of π_0 and cofiltered limits (of towers, say), we remark that $\lim^1$-terms always vanish for towers of finite groups.

In practice, we will mostly be concerned with the case where the (profinite) set π_0 of connected components is a singleton.

Definition 5.43. We say that a profinite groupoid is **connected** if its π_0 is a singleton. The collection of connected profinite groupoids spans a full subcategory $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0} \subset \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$.

In general, it will thus be helpful to have an explicit description of this profinite set. Recall that there is an algebraic description of Pro(FinSet) given by *Stone duality.* Given a Boolean algebra B, the spectrum $\mathrm{Spec}\, B$ of prime ideals (with its Zariski topology) is an example of a profinite set, i.e., it is compact, Hausdorff, and totally disconnected. Recall now:

Theorem 5.44 *(Stone duality). The functor $B \mapsto \mathrm{Spec}\, B$ establishes an anti-equivalence $\mathrm{Bool}^{\mathrm{op}} \simeq \mathrm{Pro}(\mathrm{FinSet})$.*

For a textbook reference on Stone duality, see [42]. The Galois correspondence in the form of Theorem 5.36 can be thought of as a mildly categorified version of Stone duality. In particular, we can use Stone duality to describe π_0 of a profinite groupoid.

Proposition 5.45. *Let $\mathcal{C}$ be a Galois category. Then $\pi_0(\pi_{\leq 1}\mathcal{C})$ corresponds, under Stone duality, to the Boolean algebra of subobjects $x \subset *$.*

Let $\mathcal{C}$ be a Galois category. Given two subobjects $x, y \subset *$ of the terminal object, we define their product to be the categorical product $x \times y$. Their sum is the minimal subobject of $*$ containing both x, y: in other words, the image of $x \sqcup y \to *$. By working locally, it follows that this actually defines a Boolean algebra.

Proof. In fact, if $\mathcal{C}$ is a Galois category corresponding to a *finite* groupoid, the result is evident. Since the construction above sends filtered colimits of Galois categories to filtered colimits of Boolean algebras, we can deduce it for any Galois category in view of Theorem 5.36. □

In practice, the Galois categories that we will be considering will be connected (in the sense of Definition 5.24). By Proposition 5.45, it follows that a Galois category $\mathcal{C}$ is connected if and only if $\pi_{\leq 1}\mathcal{C}$ is connected as a profinite groupoid. In our setting, this will amount to the condition that certain commutative rings are free from idempotents. With this in mind, we turn our attention to the *connected* case. Here we will be able to obtain a very strong connection with the (somewhat more concrete) theory of *profinite groups.*

The 2-category $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ has a terminal object $*$, the contractible profinite groupoid. Under the Galois correspondence, this corresponds to the category FinSet of finite sets.

Definition 5.46. A **pointed profinite groupoid** is a profinite groupoid $\mathscr{G}$ together with a map $* \to \mathscr{G}$ in $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$. The collection of pointed profinite groupoids forms a $(2,1)$-category, the undercategory $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$.

For example, let G be a profinite group, so that G is canonically a pro-object in finite groups. Applying the classifying space functor to this system, we obtain a *pointed* profinite groupoid $BG \in \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ as the formal inverse limit of the finite groupoids $B(G/U)$ as $U \subset G$ ranges over the open normal subgroups, since each $B(G/U)$ is pointed. By construction, the associated Galois category is $\varinjlim_{U \subset G} \mathrm{FinSet}_{G/U}$, or equivalently, the category of finite sets equipped with a *continuous* G-action (i.e., an action which factors through G/U for U an open normal subgroup). We thus obtain a functor

$$B\colon \mathrm{Pro}(\mathrm{FinGp}) \to \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/},$$

where FinGp is the category of finite groups and Pro(FinGp) is the category of profinite groups. Observe that this functor is *fully faithful*, since the analogous functor $B\colon \mathrm{FinGp} \to (\mathrm{Gpd}_{\mathrm{fin}})_{*/}$ is fully faithful, and each BG for G finite defines a cocompact object of $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$.

There is a rough inverse to this construction, given by taking the "fundamental group." In general, if $\mathcal{C}$ is an ∞-category with finite limits, and $C \in \mathcal{C}$ is an object, then the natural functor

$$\mathrm{Pro}(\mathcal{C}_{C/}) \to \mathrm{Pro}(\mathcal{C})_{C/}$$

is an equivalence of ∞-categories. In the case of $\mathcal{C} = \mathrm{Gpd}_{\mathrm{fin}}$, we know that there is a functor

$$\pi_1\colon (\mathrm{Gpd}_{\mathrm{fin}})_{*/} \to \mathrm{FinGp}, \tag{16}$$

to the category FinGp of finite groups, given by the usual fundamental group of a pointed space, or more categorically as the automorphism group of the distinguished point. As above, let Pro(FinGp) be the category of profinite groups and continuous homomorphisms.

Definition 5.47. We define a functor $\pi_1\colon \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/} \to \mathrm{Pro}(\mathrm{FinGp})$ from the 2-category of pointed profinite groupoids to the category of profinite groups given by right Kan extension of (16), so that π_1 agrees with the old π_1 on pointed finite groupoids and commutes with filtered limits.

Given a pointed finite groupoid $\mathscr{G}$, we have a natural map

$$B\pi_1(\mathscr{G}) \to \mathscr{G}, \tag{17}$$

and by general formalism, we have a natural transformation of the form (17) on $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$.

Proposition 5.48. *Given an object $\mathscr{G} \in \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$, the following are equivalent:*

1. *$\mathscr{G}$ is connected, i.e., $\pi_0\mathscr{G}$ is a singleton.*
2. *The map $B\pi_1\mathscr{G} \to \mathscr{G}$ is an equivalence in $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$.*

In particular, the functor $B\colon \mathrm{Pro}(\mathrm{FinGp}) \to \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})_{*/}$ is fully faithful with image consisting of the pointed connected profinite groupoids.

Proof. The second statement clearly implies the first: any BG for G a profinite group is connected, as the inverse limit of connected profinite groupoids. We have also seen that the functor B is fully faithful, since it is fully faithful on finite groups. It remains to show that if $\mathscr{G}$ is a pointed, connected profinite groupoid, then the map $B\pi_1\mathscr{G} \to \mathscr{G}$ is an equivalence.

For this, we write $\mathscr{G}$ as a cofiltered limit $\varprojlim_I \mathscr{G}_i$, where I is a filtered partially ordered set indexing the $\mathscr{G}_i$ and each $\mathscr{G}_i$ is a pointed finite groupoid. We know that $\mathscr{G}$ is connected, though each $\mathscr{G}_i$ need not be. However, we obtain a new inverse system $\{B\pi_1\mathscr{G}_i\}$ equipped with a map to the inverse system $\{\mathscr{G}_i\}$ and we want to show that the two inverse systems are pro-isomorphic. We need to produce an inverse map of pro-systems $\{\mathscr{G}_i\} \to \{B\pi_1\mathscr{G}_i\}$. For this, we need to produce for each $i \in I$ an element $j \geq i$ and a map

$$\mathscr{G}_j \to B\pi_1\mathscr{G}_i.$$

These should define an element of $\varprojlim_i \varinjlim_j \mathrm{Hom}(\mathscr{G}_j, B\pi_1\mathscr{G}_i)$. In order to do this, we simply note that there exists $j \geq i$ such that the map $\mathscr{G}_j \to \mathscr{G}_i$ lands inside the connected component $B\pi_1\mathscr{G}_i$ of $\mathscr{G}_i$ at the basepoint, because otherwise the pro-system would not be connected as a filtered inverse limit of nonempty finite sets is nonempty. One checks easily that the two maps of pro-systems define an isomorphism between $\{\mathscr{G}_i\}$ and $\{B\pi_1\mathscr{G}_i\}$. $\square$

Let $\mathscr{G} \in \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ be a *connected* profinite groupoid. This means that the space of maps $* \to \mathscr{G}$ in $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ is connected, i.e., there is only one such map up to homotopy. (This is not entirely immediate, but will be a special case of Proposition 5.49 below.) Once we choose a map, we point $\mathscr{G}$ and then the data is essentially equivalent to that of a profinite group in view of Proposition 5.48. If we do not point $\mathscr{G}$, then what we have is essentially a profinite group "up to conjugacy."

Proposition 5.49. *Let G, G' be profinite groups. Then the space $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(BG, BG')$ is given as follows:*

1. *The connected components are in one-to-one correspondence with conjugacy classes of continuous homomorphisms $f\colon G \to G'$.*

2. *The group of automorphisms of a given continuous homomorphism $f\colon G \to G'$ is given by the centralizer in G' of the image of f.*

In other words, if we restrict our attention to the subcategory $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0} \subset \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ consisting of *connected* profinite groupoids, then it has a simple explicit description as a 2-category where the objects are the profinite groups, maps are continuous homomorphisms, and 2-morphisms are conjugations.

Proof. This assertion is well-known when G, G' are finite groups: maps between BG and BG' in $\mathrm{Gpd}_{\mathrm{fin}}$ are as above. The general case follows by passage to cofiltered limits. Let $G = \varprojlim_U G/U, G' = \varprojlim_V G'/V$ where U (resp. V) ranges over the open normal subgroups of G (resp. G'). In this case, we have

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(BG, BG') \simeq \varprojlim_V \varinjlim_U \mathrm{Hom}_{\mathrm{Gpd}_{\mathrm{fin}}}(B(G/U), B(G/V)),$$

and passing to the limit, we can conclude the result for G, G' profinite, if we observe that the set of conjugacy classes of continuous homomorphisms $G \to G'$ is the inverse limit of the sets of conjugacy classes of continuous homomorphisms $G \to G'/V$ as $V \subset G$ ranges over open normal subgroups. The assertion about automorphisms, or conjugacies, is easier.

To see this in turn, suppose given continuous homomorphisms $\phi_1, \phi_2\colon G \to G'$ such that, for every continuous map $\psi\colon G' \to G''$ where G'' is finite, the composites $\psi \circ \phi_1, \psi \circ \phi_2$ are conjugate. We claim that ϕ_1, ϕ_2 are conjugate. The collection of all surjections $\psi\colon G' \to G''$ with G'' finite forms a filtered system, and for each ψ, we consider the (finite) set $F_\psi \subset G''$ of $x \in G''$ such that $\psi \circ \phi_2 = x(\psi \circ \phi_1)x^{-1}$. Since by hypothesis each F_ψ is nonempty, it follows that the inverse limit is nonempty, so that ϕ_1, ϕ_2 are actually conjugate as homomorphisms $G \to G'$. Conversely, suppose given for each $\psi\colon G' \to G''$ with G'' finite a *conjugacy class* of continuous maps $\phi_\psi\colon G \to G''$, and suppose these are compatible with one another; we want to claim that there exists a conjugacy class of continuous homomorphisms $\phi\colon G \to G'$ that lifts all the ϕ_ψ. For this, we again consider the *finite* nonempty sets G_ψ of all continuous homomorphisms $G \to G''$ in the conjugacy class of ϕ_ψ, and observe the inverse limit of these is nonempty. Any point in the inverse limit gives a continuous homomorphism $G \to G''$ with the desired property. □

6. The Galois group and first computations

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. In this section, we will make the main definition of this paper, and describe two candidates for the *Galois group* (or, in general, groupoid) of $\mathcal{C}$. Using the descent theory described in Section 3, we will define a category of *finite covers* in the ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebra objects in $\mathcal{C}$. Finite covers will be those commutative algebra objects which "locally" look like direct factors of products of copies of the unit. There are two possible definitions of "locally," which lead to slightly different Galois groups. We will show that these ∞-categories of finite covers are actually Galois categories in the sense of Definition 5.15. Applying the Galois correspondence, we will obtain a profinite groupoid.

The rest of this paper will be devoted to describing the Galois group in certain special instances. In this section, we will begin that process by showing that the Galois group is entirely algebraic in two particular instances: connective $\mathbf{E}_\infty$-rings and even periodic $\mathbf{E}_\infty$-rings with regular π_0. In either of these cases, one has various algebraic tricks to study modules via their homotopy groups. The associated ∞-categories of modules turn out to be extremely useful building blocks for a much wider range of stable homotopy theories.

6.1. Two definitions of the Galois group

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory, as before. We will describe two possible analogs of "finite étaleness" appropriate to the categorical setting.

Definition 6.1. An object $A \in \mathrm{CAlg}(\mathcal{C})$ is a **finite cover** if there exists an $A' \in \mathrm{CAlg}(\mathcal{C})$ such that:

1. A' admits descent, in the sense of Definition 3.18.
2. $A \otimes A' \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(A'))$ is of the form $\prod_{i=1}^n A'[e_i^{-1}]$, where for each i, e_i is an idempotent in A'.

The finite covers span a subcategory $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C}) \subset \mathrm{CAlg}(\mathcal{C})$.

Definition 6.2. An object $A \in \mathrm{CAlg}(\mathcal{C})$ is a **weak finite cover** if there exists an $A' \in \mathrm{CAlg}(\mathcal{C})$ such that:

1. The functor $\otimes A' \colon \mathcal{C} \to \mathcal{C}$ commutes with all homotopy limits.
2. The functor $\otimes A'$ is conservative.
3. $A \otimes A' \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{C}}(A'))$ is of the form $\prod_{i=1}^n A'[e_i^{-1}]$, where for each i, e_i is an idempotent in A'.

The weak finite covers span a subcategory $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C}) \subset \mathrm{CAlg}(\mathcal{C})$.

Our goal is to show that both of these definitions give rise to Galois categories in the sense of the previous section, which we will do using the general machine of Proposition 5.28. Observe first that $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ satisfies the first two conditions of Definition 5.15.

Lemma 6.3. *Given $\mathcal{C}$ as above, consider the ∞-category $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ and the collection of morphisms $\mathcal{E}$ given by the maps $A \to B$ which admit descent. Then $(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}, \mathcal{E})$ is a Galois context in the sense of Definition 5.26.*

Proof. The composite of two descendable morphisms is descendable by Proposition 3.24, descendable morphisms are effective descent morphisms by Proposition 3.22, and the locality of descendability (i.e., the third condition of Definition 5.26) follows from the second part of Proposition 3.24. The remaining conditions are straightforward. □

Lemma 6.4. *Given $\mathcal{C}$ as above, consider the ∞-category $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ and the collection of morphisms $\mathcal{E}$ given by the maps $A \to B$ such that the functor $\otimes_A B\colon \mathrm{Mod}_{\mathcal{C}}(A) \to \mathrm{Mod}_{\mathcal{C}}(B)$ commutes with limits and is conservative. Then $(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}, \mathcal{E})$ is a Galois context in the sense of Definition 5.26.*

Proof. It is easy to see that $\mathcal{E}$ satisfies the first axiom of Definition 5.26, and we can apply Barr–Beck–Lurie to see comonadicity of $\otimes_A B$ (i.e., the second axiom). The fourth and fifth axioms are straightforward.

Finally, suppose $A \to B$ is a morphism in $\mathrm{CAlg}(\mathcal{C})$ and $A \to A'$ belongs to $\mathcal{E}$, i.e., tensoring $\otimes_A A'$ commutes with limits and is conservative. Suppose $A' \to B' \stackrel{\mathrm{def}}{=} A' \otimes_A B$ has the same property. Then we want to claim that $A \to B$ belongs to $\mathcal{E}$.

First, observe that $\otimes_A B$ is conservative. If $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ is such that $M \otimes_A B \simeq 0$, then $(M \otimes_A A') \otimes_{A'} B'$ is zero, so that $M \otimes_A A'$ is zero as $A' \to B'$ belongs to $\mathcal{E}$, and thus $M = 0$. Finally, we need to check the claim about $\otimes_A B$ commuting with limits. In other words, given $\{M_i\} \in \mathrm{Mod}_{\mathcal{C}}(A)$, we need to show that the natural map

$$B \otimes_A \prod M_i \to \prod (M_i \otimes_A B)$$

is an equivalence. We can do this after tensoring with A', so we need to see that

$$A' \otimes_A B \otimes_A \prod M_i \to A' \otimes_A \prod (M_i \otimes_A B)$$

is an equivalence. However, since tensoring with A' commutes with limits, this map is

$$B' \otimes_{A'} \prod (M_i \otimes_A A') \to \prod (M_i \otimes_A A') \otimes_{A'} B',$$

which is an equivalence since $\otimes_{A'} B'$ commutes with limits by assumption. □

The basic result of this section is the following.

Theorem 6.5. *Given $\mathcal{C}$, $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})^{\mathrm{op}}$ and $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})^{\mathrm{op}}$ are Galois categories, with $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C}) \subset \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$. If $\mathbf{1} \in \mathcal{C}$ is compact, then the two are the same.*

Proof. This follows from Proposition 5.28 if we take $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ as our input ∞-category. As we checked above, we have two candidates for $\mathcal{E}$, both of which yield Galois contexts. The Galoisable objects yield either the finite covers or the weak finite covers.

Next, we need to note that a finite cover is actually a weak finite cover. Note first that either a finite cover or a weak finite cover is dualizable, since dualizability can be checked locally in a limit diagram of symmetric monoidal ∞-categories. However, the argument of Proposition 5.28 (or the following corollary) shows that, given a finite cover $A \in \mathrm{CAlg}(\mathcal{C})$, we can choose the descendable $A' \in \mathrm{CAlg}(\mathcal{C})$ such that $A \otimes A'$ is in mixed elementary form so that A' itself is a finite cover: in particular, so that A' is dualizable. Therefore, we can choose A' so that $\otimes A'$ commutes with arbitrary homotopy limits.

Finally, we need to see that the two notions are equivalent in the case where $\mathbf{1}$ is compact. For this, we use the reasoning of the previous paragraph to argue that if $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$, then there exists an object $A' \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ such that the dual to $\mathbf{1} \to A'$ is a distinguished effective descent morphism (i.e., tensoring with A' is conservative and commutes with homotopy limits) and such that $A' \to A \otimes A'$ is in mixed elementary form. However, in this case, A' is dualizable, as an element of $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$, so it admits descent in view of Theorem 3.38. Therefore, A is actually a finite cover. $\square$

Proposition 6.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism of stable homotopy theories, so that F induces a functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$. Then F carries $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$ into $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{D})$ and $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ into $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{D})$.*

Proof. Let $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$. Then there exists $A' \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$, which is a G-torsor for some finite group G, such that $A \otimes A'$ is a finite product of localizations of A' at idempotent elements, in view of Corollary 5.41. Therefore, $F(A) \otimes F(A')$ is a finite product of localizations of $F(A')$ at idempotent elements.

Now $F(A') \in \mathrm{CAlg}(\mathcal{D})$ is dualizable since A' is, so tensoring with $F(A')$ commutes with limits in $\mathcal{D}$. If we can show that tensoring with $F(A')$ is *conservative* in $\mathcal{D}$, then it will follow that $F(A)$ satisfies the conditions of Definition 6.2. In fact, we will show that the smallest *ideal* of $\mathcal{D}$ closed under arbitrary colimits and containing $F(A')$ is all of $\mathcal{D}$. This implies that any object $Y \in \mathcal{D}$ with $Y \otimes F(A') \simeq 0$ must actually be contractible.

To see this, recall that A' has a G-action. We have a *norm map* (cf. [49, sec. 2.1] for a general reference in this context)

$$A'_{hG} \to A'^{\,hG} \simeq \mathbf{1},$$

which we claim is an equivalence (Lemma 6.7 below). After applying F, we find that $F(A')_{hG} \simeq \mathbf{1}$, which proves the claim and thus shows that tensoring with $F(A')$ is faithful.

If $A \in \mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$, then we could choose the torsor A' so that it actually belonged to $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$ as well. The image $F(A')$ thus is a descendable commutative algebra object in $\mathcal{D}$ since descendability is a "finitary" condition that does not pose any convergence issues with infinite limits. So, by similar (but easier) logic, we find that $F(A) \in \mathrm{CAlg}^{\mathrm{cov}}(\mathcal{D})$. $\square$

Lemma 6.7. *Let $\mathcal{C}$ be a stable homotopy theory and let $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})^{\mathrm{op}}$ be a G-torsor, where G is a finite group. Then the norm map $A_{hG} \to A^{hG} \simeq \mathbf{1}$ is an equivalence.*

Proof. It suffices to prove this after tensoring with A; note that tensoring with A is conservative and commutes with all homotopy limits. However, after tensoring with A, the G-action on A becomes induced, so the norm map is an equivalence. □

Finally, we can make the main definition of this paper.

Definition 6.8. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. The **Galois groupoid** $\pi_{\leq 1}(\mathcal{C})$ of $\mathcal{C}$ is the Galois groupoid of the Galois category $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})^{\mathrm{op}}$. The **weak Galois groupoid** $\pi_{\leq 1}^{\mathrm{weak}}(\mathcal{C})$ is the Galois groupoid of $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})^{\mathrm{op}}$. When $\mathbf{1}$ has no nontrivial idempotents, we will write $\pi_1(\mathcal{C}), \pi_1^{\mathrm{weak}}(\mathcal{C})$ for the **Galois group** (resp. **weak Galois group**) of $\mathcal{C}$ with the understanding that these groups are defined "up to conjugacy."

As above, we have an inclusion $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C}) \subset \mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$ of Galois categories. In particular, we obtain a morphism of profinite groupoids

$$\pi_{\leq 1}^{\mathrm{weak}}(\mathcal{C}) \to \pi_{\leq 1}(\mathcal{C}). \tag{18}$$

The dual map on Galois categories is fully faithful. In particular, if $\mathcal{C}$ is *connected*, so that $\pi_1, \pi_1^{\mathrm{weak}}$ can be represented by profinite groups, the map (18) is *surjective.* Moreover, by Theorem 6.5, if $\mathbf{1}$ is compact, (18) is an equivalence.

In the following, we will mostly be concerned with the Galois groupoid, which is more useful for computational applications because of the rapidity of the descent. The weak Galois groupoid is better behaved as a functor out of the ∞-category of stable homotopy theories. We will discuss some of the differences further below. The weak Galois groupoid seems in particular useful for potential applications in $K(n)$-local homotopy theory where $\mathbf{1}$ is not compact. Note, however, that the Galois groupoid depends only on the 2-ring of *dualizable objects* in a given stable homotopy theory, because the property of admitting descent (for a commutative algebra object which is dualizable) is a finitary one. So, the Galois groupoid can be viewed as a functor 2-Ring $\to \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\mathrm{op}}$.

Definition 6.9. We will define the **Galois group(oid)** of an $\mathbf{E}_\infty$-ring R to be that of $\mathrm{Mod}(R)$. Note that the weak Galois group(oid) and the Galois group(oid) of $\mathrm{Mod}(R)$ are canonically isomorphic, by Theorem 6.5.

In any event, both the profinite groupoids of (18) map to something purely algebraic. Given a finite étale cover of the ordinary commutative ring $R_0 = \pi_0 \mathrm{End}_{\mathcal{C}}(\mathbf{1})$, we get a commutative algebra object in $\mathcal{C}$.

Proposition 6.10. *Let R_0' be a finite étale R_0-algebra. The induced classically étale object of* $\mathrm{CAlg}(\mathcal{C})$ *is a finite cover, and we have a fully faithful embedding*

$$\mathrm{Cov}_{\mathrm{Spec}\, R_0} \subset \mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})^{\mathrm{op}},$$

from the category $\mathrm{Cov}_{\mathrm{Spec}\, R_0}$ *of schemes finite étale over* $\mathrm{Spec}\, R_0$ *into the opposite to the category* $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$*.*

This was essentially first observed in [13].

Proof. We can assume that $\mathcal{C} = \mathrm{Mod}(R)$ for R an $\mathbf{E}_\infty$-ring, because if $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$, we always have an embedding $\mathrm{Mod}^\omega(R) \subset \mathcal{C}$ and everything here happens inside $\mathrm{Mod}^\omega(R)$ anyway. It follows from Theorem 2.32 that we have a fully faithful embedding $\mathrm{Cov}_{\mathrm{Spec}\, R_0} \subset \mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$, so it remains only to show that any classically étale algebra object coming from a finite étale R_0-algebra R_0' is in fact a finite cover. However, we know that there exists a finite étale R_0-algebra R_0'' such that:

1. R_0'' is faithfully flat over R_0.
2. $R_0' \otimes_{R_0} R_0''$ is the localization of $\prod_S R_0''$ at an idempotent element, for some finite set S.

We can realize R_0', R_0'' topologically by $\mathbf{E}_\infty$-rings R', R'' under R. Now R'' admits descent over R', as a finite faithfully flat R-module, and $R' \otimes_R R''$ is the localization of $\prod_S R''$ at an idempotent element, so that $R' \in \mathrm{CAlg}^{\mathrm{cov}}(\mathrm{Mod}(R))$. $\square$

The classically étale algebras associated to finite étale R_0-algebras give the "algebraic" part of the Galois group and fit into a sequence

$$\pi_1^{\mathrm{weak}}(\mathcal{C}) \twoheadrightarrow \pi_1(\mathcal{C}) \twoheadrightarrow \pi_1^{\mathrm{et}}\, \mathrm{Spec}\, R_0. \tag{19}$$

Definition 6.11. We will say that the Galois theory of $\mathcal{C}$ is **algebraic** if these maps are isomorphisms.

It is an insight of [71] that the second map in (19) is generally not an isomorphism: that is, there are examples of finite covers that are genuinely topological and do not appear so at the level of homotopy groups. We will review the connection between our definitions and Rognes's work in the next section.

6.2. Rognes's Galois theory

In [71], Rognes introduced the definition of a *G-Galois extension* of an $\mathbf{E}_\infty$-ring R for G a finite group. (Rognes also considered the case of a *stably dualizable group*, which will be discussed only incidentally in this paper.) Rognes worked in the setting of E-local spectra for E a fixed spectrum. The same definition would work in a general stable homotopy theory. In this subsection, we will connect Rognes's definition with ours.

Definition 6.12 *(Rognes).* Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory. An object $A \in \mathrm{CAlg}(\mathcal{C})$ with the action of a finite group G (in $\mathrm{CAlg}(\mathcal{C})$) is a G**-Galois extension** if:

1. The map $\mathbf{1} \to A^{hG}$ is an equivalence.
2. The map $A \otimes A \to \prod_G A$ (given informally by $(a_1, a_2) \mapsto \{a_1 g(a_2)\}_{g \in G}$) is an equivalence.

We will say that A is a **faithful G-Galois extension** if further tensoring with A is conservative.

General G-Galois extensions in this sense are outside the scope of this paper. In general, there is no reason for a G-Galois extension to be well-behaved at all with respect to descent theory. By an example of Wieland (see [70]), the map $C^*(B\mathbb{Z}/p; \mathbb{F}_p) \to \mathbb{F}_p$ given by evaluating on a point is a $\mathbb{Z}/p$-Galois extension, but one cannot expect to carry out descent along it in any manner. However, one has:

Proposition 6.13. *A faithful G-Galois extension in $\mathcal{C}$ is equivalent to a G-torsor in the Galois category* $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$.

This in turn relies on:

Proposition 6.14. *(See [71, Proposition 6.2.1].) Any G-Galois extension A of the unit is dualizable.*

The proof in [71] is stated for the E-localization of $\mathrm{Mod}(A)$ for A an $\mathbf{E}_\infty$-ring, but it is valid in any such setting.

Proof of Proposition 6.13. A G-torsor in $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ is, by definition, a commutative algebra object A with an action of G such that there exists an $A' \in \mathrm{CAlg}(\mathcal{C})$ such that $\otimes A'$ is conservative and commutes with limits, with $A' \otimes A \simeq \prod_G A'$ as an A'-algebra and compatibly with the G-action. This together with descent along $\mathbf{1} \to A'$ implies that the map $\mathbf{1} \to A^{hG}$ is an equivalence. Similarly, the map $A \otimes A \to \prod_G A$ is well-defined in $\mathcal{C}$ and becomes an equivalence after base-change to A' (by checking for the trivial torsor), so that it must have been an equivalence to begin with.

Finally, if $\mathbf{1} \to A$ is a faithful G-Galois extension in the sense of Definition 6.12, then A is dualizable by Proposition 6.14, so that $\otimes A$ commutes with limits. Moreover, $\otimes A$ is faithful by assumption. Since $A \otimes A$ is in elementary form, it follows that $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ and is in fact a G-torsor. $\square$

The use of G-torsors will be very helpful in making arguments. For example, given a connected Galois category, any nonempty object is a quotient of a G-torsor for some finite group G; in fact, understanding the Galois theory is equivalent to understanding torsors for finite groups.

Corollary 6.15. *A G-torsor in the Galois category* $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$ *is equivalent to a G-Galois extension $A \in \mathrm{CAlg}(\mathcal{C})$ such that A admits descent.*

Proof. Given a G-torsor in $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$, it follows easily that it generates all of $\mathcal{C}$ as a thick $\otimes$-ideal, since descendability can be checked locally and since a trivial torsor is descendable. Conversely, if A is a G-Galois extension with this property, then A is a finite cover of the unit: we can take as our descendable commutative algebra object (required by Definition 6.1) A itself. $\square$

Corollary 6.16. *If $|G|$ is invertible in $\pi_0\mathrm{End}(\mathbf{1})$, then a G-torsor in $\mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ actually belongs to $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$. In particular, if $\mathbb{Q} \subset \pi_0\mathrm{End}(\mathbf{1})$, then the two fundamental groups are the same: (19) is an isomorphism.*

Proof. In any stable ∞-category $\mathcal{D}$ where $|G|$ is invertible (i.e., multiplication by $|G|$ is an isomorphism on each object), then for any object $X \in \mathrm{Fun}(BG, \mathcal{C})$, X^{hG} is a retract of X. In fact, the composite

$$X^{hG} \to X \to X_{hG} \xrightarrow{N} X^{hG},$$

is an equivalence, where N is the norm map.

In particular, given a G-torsor $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$, we have $\mathbf{1} \simeq A^{hG}$, so that $\mathbf{1}$ is a retract of A: in particular, the thick $\otimes$-ideal A generates contains all of $\mathcal{C}$, so that (by Corollary 6.15) it belongs to $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C})$. This proves the first claim of the corollary.

Finally, if $\mathbb{Q} \subset \pi_0\mathrm{End}(\mathbf{1})$, then fix a weak finite cover $B \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$. There is a G-torsor $A \in \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C})$ for some finite group G such that $A \otimes B$ is a localization of a product of copies of A at idempotent elements. Since the thick $\otimes$-ideal that A generates contains all of $\mathcal{C}$ by the above, it follows that B is actually a finite cover. $\square$

6.3. The connective case

The rest of this paper will be devoted to computations of Galois groups. These computations are usually based on descent theory together with results stating that we can identify the Galois theory in certain settings as entirely algebraic. Our first result along these lines shows in particular that we can recover the classical étale fundamental group of a commutative ring. More generally, we can describe the Galois group of a connective $\mathbf{E}_\infty$-ring purely algebraically.

Theorem 6.17. *Let A be a connective $\mathbf{E}_\infty$-ring. Then the map $\pi_1(\mathrm{Mod}(A)) \to \pi_1^{\mathrm{et}} \operatorname{Spec} \pi_0 A$ is an equivalence; that is, all finite covers or weak finite covers are classically étale.*

Remark 6.18. This result, while not stated explicitly in [71], seems to be folklore. One has the following intuition: a connective $\mathbf{E}_\infty$-ring consists of its π_0 (which is a discrete commutative ring) together with higher homotopy groups $\pi_i, i > 0$ which can be thought of as "fuzz," a generalized sort of nilthickening. Since nilpotents should not affect the étale site, we would expect the Galois theory to be invariant under the map $A \to \tau_{\leq 0} A$ in this case.

Proof. Let A be a connective $\mathbf{E}_\infty$-ring. The argument was explained for $\pi_0 A$ noetherian in [57, Example 5.5], and the general case can be reduced to this using the commutation of Galois theory and filtered colimits (Theorem 6.20 below). In fact, the ∞-category of connective $\mathbf{E}_\infty$-rings is compactly generated and any compact object has noetherian π_0. Therefore, the result assuming $\pi_0 A$ noetherian implies it in general since any connective $\mathbf{E}_\infty$-ring is a filtered colimit of compact objects. □

The above argument illustrates a basic technique one has: one tries, whenever possible, to reduce to the case of $\mathbf{E}_\infty$-rings which satisfy *Künneth isomorphisms*. In this case, one can attempt to study G-Galois extensions using algebra.

Example 6.19. (Cf. [71, Theorem 10.3.3].) The Galois group of Sp is trivial, since Sp is the ∞-category of modules over the sphere S^0, and the étale fundamental group of $\pi_0(S^0) \simeq \mathbb{Z}$ is trivial by Minkowski's theorem that the discriminant of a number field is always > 1 in absolute value.

6.4. Galois theory and filtered colimits

In this subsection, we will prove that Galois theory behaves well with respect to filtered colimits.

Theorem 6.20. *The functor* $A \mapsto \mathrm{CAlg}^{\mathrm{cov}}(\mathrm{Mod}(A)), \mathrm{CAlg} \to \mathrm{Cat}_\infty$ *commutes with filtered colimits. In particular, given a filtered diagram* $I \to \mathrm{CAlg}$*, the map*

$$\pi_{\leq 1} \mathrm{Mod}(\varinjlim_I A_i) \to \varprojlim_I \pi_{\leq 1} \mathrm{Mod}(A_i),$$

is an equivalence of profinite groupoids.

Theorem 6.20 will be a consequence of some categorical technology together and is a form of "noetherian descent." To prove it, we can work with G-torsors in view of Corollary 5.40. Given an $\mathbf{E}_\infty$-ring $A \in \mathrm{CAlg}$, we let $\mathrm{Gal}_G(A)$ be the category of faithful G-Galois extensions of A: that is, the category of G-torsors in $\mathrm{CAlg}^{\mathrm{cov}}(A)$. We need to show that given a filtered diagram $\{A_i\}$ of $\mathbf{E}_\infty$-rings, the functor

$$\varinjlim \mathrm{Gal}_G(A_i) \to \mathrm{Gal}_G(\varinjlim A_i),$$

is an equivalence of categories: i.e., that it is fully faithful and essentially surjective. We start by showing that faithful Galois extensions are compact $\mathbf{E}_\infty$-algebras.

Lemma 6.21. *Let $A \to B$ be a faithful G-Galois extension. Then B is a compact object in the ∞-category $\mathrm{CAlg}_{A/}$ of $\mathbf{E}_\infty$-algebras over A. Moreover, $\mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, \cdot)$ takes values in homotopy discrete spaces.*

Proof. First, recall that if $A \to B$ is a *classically étale* extension, then the result is true. In fact, if $A \to B$ is classically étale, then for any $\mathbf{E}_\infty$-A-algebra A', the natural map

$$\mathrm{Hom}_{\mathrm{CAlg}_{A/}}(B, A') \to \mathrm{Hom}_{\mathrm{Ring}_{\pi_0 A/}}(\pi_0 B, \pi_0 A'),$$

is an equivalence. Moreover, $\pi_0 B$, as an étale $\pi_0 A$-algebra, is finitely presented or equivalently compact in $\mathrm{Ring}_{\pi_0 A/}$. The result follows for an étale extension.

Now, a Galois extension need not be classically étale, but it becomes étale after an appropriate base change, so we can use descent theory. Recall that we have an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Mod}(A) \simeq \mathrm{Tot}\left(\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots\right).$$

Upon taking commutative algebra objects, we get an equivalence of ∞-categories

$$\mathrm{CAlg}_{A/} \simeq \mathrm{Tot}\left(\mathrm{CAlg}_{B/} \rightrightarrows \mathrm{CAlg}_{B \otimes_A B/} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots\right).$$

The object $B \in \mathrm{CAlg}_{A/}$ becomes classically étale, thus compact, after base-change along $A \to B$. We may now apply the next sublemma to conclude. $\square$

Sublemma. *Let $\mathcal{C}^{-1} \in \mathrm{Pr}^L$ be a presentable ∞-category and $\mathcal{C}^\bullet$ a cosimplicial object in Pr^L with an equivalence of ∞-categories*

$$\mathcal{C}^{-1} \simeq \mathrm{Tot}(\mathcal{C}^\bullet).$$

Suppose that $x \in \mathcal{C}^{-1}$ is an object such that:

- *The image x^i of x in $\mathcal{C}^i, i \geq 0$ is compact for each i.*
- *There exists n such that the image x^i of x in each $\mathcal{C}^i$ is n-cotruncated in the sense that*

$$\mathrm{Hom}_{\mathcal{C}^i}(x^i, \cdot) \colon \mathcal{C}^i \to \mathcal{S}$$

  takes values in the subcategory $\tau_{\leq n}\mathcal{S} \subset \mathcal{S}$ of n-truncated spaces. (This follows once x^0 is n-cotruncated.)

Then x is compact (and n-cotruncated) in $\mathcal{C}^{-1}$).

Proof. Given objects $w, z \in \mathcal{C}^{-1}$, the natural map

$$\mathrm{Hom}_{\mathcal{C}}(w, z) \to \mathrm{Tot}\, \mathrm{Hom}_{\mathcal{C}^\bullet}(w^\bullet, z^\bullet)$$

is an equivalence, where for each $i \geq 0$, w^i, z^i are the objects in $\mathcal{C}^i$ that are the images of w, z.

Therefore, it follows that $\mathrm{Hom}_{\mathcal{C}^{-1}}(x, \cdot): \mathcal{C}^{-1} \to \mathcal{S}$ is the totalization of a cosimplicial functor $\mathcal{C}^{-1} \to \mathcal{S}$ given by $\mathrm{Hom}_{\mathcal{C}^\bullet}(x^\bullet, \cdot^\bullet)$. Each of the terms in this cosimplicial functor, by assumption, commutes with filtered colimits and takes values in n-truncated spaces. The sublemma thus follows because the totalization functor

$$\mathrm{Tot}\colon \mathrm{Fun}(\Delta, \tau_{\leq n}\mathcal{S}) \to \mathcal{S},$$

lands in $\tau_{\leq n}\mathcal{S}$, and commutes with filtered colimits: a totalization of n-truncated spaces can be computed by a partial totalization, and finite limits and filtered colimits of spaces commute with one another. $\square$

Next, we prove a couple of general categorical lemmas about compact objects in undercategories and filtered colimits.

Lemma 6.22. *Let $\mathcal{C}$ be a compactly generated, presentable ∞-category and let $\mathcal{C}^\omega$ denote the collection of compact objects. Then, for each $x \in \mathcal{C}$, the undercategory $\mathcal{C}_{x/}$ is compactly generated. Moreover, the subcategory $(\mathcal{C}_{x/})^\omega$ is generated under finite colimits and retracts by the morphisms of the form $x \to x \sqcup y$ for $y \in \mathcal{C}^\omega$.*

Proof. To prove this, recall that if $\mathcal{D}$ is any presentable ∞-category and $\mathcal{E} \subset \mathcal{D}$ is a (small) subcategory of compact objects, closed under finite colimits, then there is induced a map in Pr^L

$$\mathrm{Ind}(\mathcal{E}) \to \mathcal{D},$$

which is an equivalence of ∞-categories precisely when $\mathcal{E}$ *detects equivalences*: that is, when a map $x \to y$ in $\mathcal{D}$ is an equivalence when $\mathrm{Hom}_{\mathcal{D}}(e, x) \to \mathrm{Hom}_{\mathcal{D}}(e, y)$ is a homotopy equivalence for all $e \in \mathcal{E}$. Indeed, in this case, it follows that $\mathrm{Ind}(\mathcal{E}) \to \mathcal{D}$ is a fully faithful functor, which embeds $\mathrm{Ind}(\mathcal{E})$ as a full subcategory of $\mathcal{D}$ closed under colimits. But any fully faithful left adjoint whose right adjoint is conservative is an equivalence of ∞-categories. This argument is a very slight variant of Proposition 5.3.5.11 of [44].

Now, we apply this to $\mathcal{C}_{x/}$. Clearly, the objects $x \to x \sqcup y$ in $\mathcal{C}_{x/}$, for $y \in \mathcal{C}^\omega$, are compact. Since

$$\mathrm{Hom}_{\mathcal{C}_{x/}}(x \sqcup y, z) = \mathrm{Hom}_{\mathcal{C}}(y, z),$$

it follows from the above paragraph if $\mathcal{C}$ is compactly generated, then the $x \to x \sqcup y$ in $\mathcal{C}_{x/}$ *detect equivalences* and thus generate $\mathcal{C}_{x/}$ under colimits. More precisely, if $\mathcal{E} \subset \mathcal{C}_{x/}$ is the subcategory generated under finite colimits by the $x \to x \sqcup y, y \in \mathcal{C}^\omega$, then the natural functor $\mathrm{Ind}(\mathcal{E}) \to \mathcal{C}_{x/}$ is an equivalence. Since $(\mathrm{Ind}(\mathcal{E}))^\omega$ is the idempotent completion of $\mathcal{E}$ (Lemma 5.4.2.4 of [44]), the lemma follows. $\square$

Let $\mathcal{C}$ be a compactly generated, presentable ∞-category. We observe that the association $x \in \mathcal{C} \mapsto (\mathcal{C}_{x/})^{\omega}$ is actually functorial in x. Given a morphism $x \to y$, we get a functor

$$\mathcal{C}_{x/} \to \mathcal{C}_{y/}$$

given by pushout along $x \to y$. Since the right adjoint (sending a map $y \to z$ to the composite $x \to y \to z$) commutes with filtered colimits, it follows that $\mathcal{C}_{x/} \to \mathcal{C}_{y/}$ restricts to a functor on the compact objects. We get a functor

$$\Phi \colon \mathcal{C} \to \mathrm{Cat}_{\infty}, \quad x \mapsto (\mathcal{C}_{x/})^{\omega}.$$

Our next goal is to show that Φ commutes with filtered colimits.

Lemma 6.23. *The functor Φ has the property that for any filtered diagram $x \colon I \to \mathcal{C}$, the natural functor*

$$\varinjlim_{I} \Phi(x_i) \to \Phi(\varinjlim_{I} x_i), \tag{20}$$

is an equivalence of ∞-categories.

Proof. Full faithfulness of Φ is a formal consequence of the definition of a compact object. In fact, an element of $\varinjlim_I \Phi(x_i)$ is represented by an object $i \in I$ and a map $x_i \to y_i$ that belongs to $(\mathcal{C}_{x_i/})^{\omega}$. We will denote this object by (i, y_i). This object is the same as that represented by $x_j \to y_i \sqcup_{x_i} x_j$ for any map $i \to j$ in I.

Given two such objects in $\varinjlim_I \Phi(x_i)$, we can represent them both by objects $x_i \to y_i, x_i \to z_i$ for some index i. Then

$$\mathrm{Hom}_{\varinjlim_I}((i, y_i), (i, z_i)) = \varinjlim_{j \in I_{i/}} \mathrm{Hom}_{\mathcal{C}_{x_j/}}(y_j, z_j),$$

where y_j, z_j denotes the pushforwards of y_i, z_i along $x_j \to z_j$.

Let $x = \varinjlim_I x_i$, and let y, z denote the pushforwards of y_i, z_i all the way along $x_i \to x$. Then our claim is that the map

$$\varinjlim_{j \in I_{i/}} \mathrm{Hom}_{\mathcal{C}_{x_j/}}(y_j, z_j) \to \mathrm{Hom}_{\mathcal{C}_{x/}}(y, z)$$

is an equivalence. Now, we write

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{x/}}(y, z) &\simeq \mathrm{Hom}_{\mathcal{C}_{x_i/}}(y_i, z) \\ &\simeq \mathrm{Hom}_{\mathcal{C}_{x_i/}}(y_i, \varinjlim_{j \in I_{i/}} z_j)\end{aligned}$$

$$\begin{aligned}
&\simeq \varinjlim_{j \in I_{i/}} \mathrm{Hom}_{\mathcal{C}_{x_i/}}(y_i, z_j) \\
&\simeq \varinjlim_{j \in I_{i/}} \mathrm{Hom}_{\mathcal{C}_{x_j/}}(y_j, z_j),
\end{aligned}$$

and we get the equivalence as desired.

Finally, to see that (20) establishes the right hand side as the idempotent completion of the first, we use the description of compact objects in $\mathcal{C}_{x/}$. To complete the proof, note now that a filtered colimit of idempotent complete ∞-categories is itself idempotent complete [50, Lemma 7.3.5.16]. $\square$

Corollary 6.24. *Hypotheses as above, the functor $\Psi : x \mapsto (\mathcal{C}_{x/})^{\omega, \leq 0}$ sending x to the category of 0-cotruncated, compact objects in $\mathcal{C}_{x/}$ has the property that the natural functor $\varinjlim_I \Psi(x_i) \to \Psi(\varinjlim x_i)$ is an equivalence.*

This follows from the previous lemma, because 0-cotruncatedness of an object y is equivalent to the claim that the map $S^1 \otimes y \to y$ is an equivalence.

Proof of Theorem 6.20. For A an $\mathbf{E}_\infty$-ring, let $(\mathrm{CAlg}_{A/})^{\omega, \leq 0}$ be the (ordinary) category of 0-cotruncated, compact $\mathbf{E}_\infty$-A-algebras; this includes any finite cover of A, for example, since finite covers of A are locally étale. Then we have a fully faithful inclusion of ∞-categories

$$\mathrm{Gal}_G(A) \subset \mathrm{Fun}(BG, (\mathrm{CAlg}_{A/})^{\omega, \leq 0}).$$

Although BG is not compact in the ∞-category of ∞-categories, the truncation to n-categories for any n is: BG can be represented as a simplicial set with finitely many simplices in each dimension. Therefore, the right-hand-side has the property that it commutes with filtered colimits in A by Corollary 6.24. Thus, for any filtered diagram $A \colon I \to \mathrm{CAlg}$, the functor

$$\varinjlim_{i \in I} \mathrm{Gal}_G(A_i) \to \mathrm{Gal}_G(\varinjlim_{i \in I} A_i),$$

is fully faithful.

Moreover, given a G-Galois extension B of $A = \varinjlim_I A_i$, there exists $i \in I$ and a compact, 0-cotruncated A_i-algebra B_i with a G-action, such that $A \to B$ is obtained by base change from $A_i \to B_i$. It now suffices to show that $A_i \to B_i$ becomes G-Galois after some base change $A_i \to A_j$. For any $j \in I$ receiving a map from I, we let $B_j = A_j \otimes_{A_i} B_i$. We are given that $\varinjlim_{j \in I_{i/}} B_j$ is a faithful G-Galois extension of $\varinjlim_{j \in I_{i/}} A_j$ and we want to claim that there exists j such that B_j is a faithful G-Galois extension of A_j.

Now, the condition for $A_j \to B_j$ to be faithfully G-Galois has two parts:

1. $B_j \otimes_{A_j} B_j \to \prod_G B_j$ should be an equivalence.
2. $A_j \to B_j$ should be descendable.

The first condition is detected at a "finite stage" since both the source and target are compact objects of $\mathrm{CAlg}_{A_j/}$.

Unfortunately, we do not know how to use this line of argument alone to argue that the $A_j \to B_j$'s are faithful G-Galois for some j, although we suspect that it is possible.

Instead, we use some obstruction theory. The map $A \to B$ exhibits B as a perfect A-module. For any $\mathbf{E}_1$-ring R, let $\mathrm{Mod}^\omega(R)$ be the stable ∞-category of perfect R-modules. Then the natural functor

$$\varinjlim_I \mathrm{Mod}^\omega(A_i) \to \mathrm{Mod}^\omega(A),$$

is an equivalence of ∞-categories.[6] It follows that we can "descend" the perfect A-module B to a perfect A_j-module B'_j for some j (asymptotically unique), and we can descend the multiplication map $B \otimes_A B \to B$ (resp. the unit map $A \to B$) to $B'_j \otimes_{A_j} B'_j \to B'_j$ (resp. $A_j \to B'_j$). We can also assume that homotopy associativity holds for j "large." The G-action on B in the *homotopy category* of perfect A-modules descends to an action on B'_j in the *homotopy category* of perfect A_j-modules, and the equivalence $B \otimes_A B \simeq \prod_G B$ descends to an equivalence $B'_j \otimes_{A_j} B'_j \simeq \prod_G B'_j$. Finally, the fact that the thick subcategory that B generates contains A can also be tested at a finite stage.

The upshot is that, for j large, we can "descend" the G-Galois extension $A \to B$ to a perfect A_j-module B'_j with the portion of the structure of a G-Galois extension that one could see *solely from the homotopy category.* However, using obstruction theory one can promote this to a genuine Galois extension. In Theorem 6.25 below, we show that B'_j can be promoted to an $\mathbf{E}_\infty$-algebra (in A_j-modules) for $j \gg 0$ with a G-action, which is a faithful G-Galois extension.

It follows that the B'_j lift B to A_j for $j \gg 0$, and even with the G-action (which is unique in a faithful Galois extension; see Theorem 11.1.1 of [71]). $\square$

Theorem 6.25. *Let A' be an $\mathbf{E}_\infty$-ring, and let B' be a perfect A'-module such that the thick subcategory generated by B' contains A'. Suppose given:*

1. *A homotopy commutative, associative and unital multiplication $B' \otimes_{A'} B' \to B'$.*
2. *A G-action on B' in the homotopy category, commuting with the multiplication and unit maps, such that the map $B' \otimes_{A'} B' \to \prod_G B'$ is an equivalence of A-modules.*

Then B' has a unique $\mathbf{E}_\infty$-multiplication extending the given homotopy commutative one, and $A \to B$ is faithful G-Galois (in particular, the G-action in the homotopy category extends to a strict one of $\mathbf{E}_\infty$-maps on B).

[6] One does not even need to worry about idempotent completeness here because we are in a stable setting, and any self-map $e \colon A \to A$ with $e^2 \simeq e$ can be extended to an idempotent.

Here we use an argument, originally due to Hopkins in a different setting, that will be elaborated upon further in joint work with Heuts; as such, we give a sketch of the proof.

Proof. We use the obstruction theory of [69] (see also [2, Sec. 3]) to produce a unique $\mathbf{E}_1$-structure. Since $B' \otimes_{A'} B'$ is a finite product of copies of B', it follows that B' satisfies a perfect universal coefficient formula in the sense of that paper. The obstruction theory developed there states that the obstructions to producing an $\mathbf{E}_1$-structure lie in $\mathrm{Ext}^{n,3-n}_{\pi_*(B'\otimes_{A'}B')}(B'_*, B'_*)$ for $n \geq 4$, and the obstructions to uniqueness in the groups $\mathrm{Ext}^{n,2-n}_{\pi_*(B'\otimes_{A'}B')}(B'_*, B'_*)$ for $n \geq 3$. The hypotheses of the lemma imply that B'_* is a projective $\pi_*(B' \otimes_{A'} B')$-module, though, so that all the obstructions (both to uniqueness and existence) vanish.

Our next goal is to promote this to an $\mathbf{E}_\infty$-multiplication extending the given $\mathbf{E}_1$-structure. We claim that the space of $\mathbf{E}_1$-maps between any tensor power $B'^{\otimes m}$ and any other tensor power $B'^{\otimes n}$ of B' is homotopy discrete and equivalent to the collection of maps of A-*ring spectra*: that is, homotopy classes of maps $B'^{\otimes m} \to B'^{\otimes n}$ (in A-modules) that commute with the multiplication laws *up to homotopy*. This is a consequence of the analysis in [68] (in particular, Theorem 14.5 there), and the fact that the $B'^{\otimes n}$-homology of $B'^{\otimes m}$ is étale, so that the obstructions of [68] all vanish.

It follows that if $\mathcal{C}$ is the smallest symmetric monoidal ∞-category of $\mathrm{Alg}(\mathrm{Mod}(A'))$ (i.e., $\mathbf{E}_1$-algebras in $\mathrm{Mod}(A')$) containing B', then $\mathcal{C}$ is equivalent to an ordinary symmetric monoidal category, which is equivalent to a full subcategory of the category of A-ring spectra. Since B' is a commutative algebra object in that latter category, it follows that it is a commutative algebra object of $\mathrm{Alg}(\mathrm{Mod}(A'))$, and thus gives an $\mathbf{E}_\infty$-algebra. The G-action, since it was by maps of A-ring spectra, also comes along. □

6.5. The even periodic and regular case

Our first calculation of a Galois group was in Theorem 6.17, where we showed that the Galois group of a connective $\mathbf{E}_\infty$-ring was entirely algebraic. In this section, we will show (Theorem 6.29) that the analogous statement holds for an even periodic $\mathbf{E}_\infty$-ring with regular (noetherian) π_0. As in the proof of Theorem 6.17, the strategy is to reduce to considering ring spectra with Künneth isomorphisms. Unfortunately, in the nonconnective setting, the "residue field" ring spectra one wants can be constructed only as $\mathbf{E}_1$-algebras (rather than as $\mathbf{E}_\infty$-algebras), so one has to work somewhat harder.

Definition 6.26. An $\mathbf{E}_\infty$-ring A is **even periodic** if:

1. $\pi_i A = 0$ if i is odd.
2. The multiplication map $\pi_2 A \otimes_{\pi_0 A} \pi_{-2} A \to \pi_0 A$ is an isomorphism.

In particular, $\pi_2 A$ is an invertible $\pi_0 A$-module; if it is free of rank one, then $\pi_*(A) \simeq \pi_0(A)[t_2^{\pm 1}]$ where $|t_2| = 2$.

Even periodic $\mathbf{E}_\infty$-rings (such as complex K-theory KU) play a central role in chromatic homotopy theory because of the connection, beginning with Quillen, with the theory of *formal groups*. We will also encounter even periodic $\mathbf{E}_\infty$-rings in studying stable module ∞-categories for finite groups below. The ∞-categories of modules over them turn out to be fundamental building blocks for many other stable homotopy theories, so an understanding of their Galois theory will be critical for us.

We begin with the simplest case.

Proposition 6.27. *Suppose A is an even periodic $\mathbf{E}_\infty$-ring with $\pi_0 A \simeq k[t^{\pm 1}]$ where $|t| = 2$ and k a field. Then the Galois theory of A is algebraic: $\pi_1 \operatorname{Mod}(A) \simeq \operatorname{Gal}(k^{\mathrm{sep}}/k)$.*

Proof. We want to show that any finite cover of A is étale at the level of homotopy groups; flat would suffice. Let B be a G-Galois extension of A. Then $B \otimes_A B \simeq \prod_G B$. Since $\pi_*(A)$ is a graded field, it follows that

$$\pi_*(B) \otimes_{\pi_*(A)} \pi_*(B) \simeq \prod_G \pi_*(B).$$

Moreover, since B is a perfect A-module, it follows that $\pi_*(B)$ is a finite-dimensional $\pi_*(A)$-module.

Making a base-change $t \mapsto 1$, we can work in $\mathbb{Z}/2$-graded k-vector spaces rather than graded $k[t^{\pm 1}]$-modules. So we get a $\mathbb{Z}/2$-graded commutative (in the graded sense) k-algebra $B'_* = B'_0 \oplus B'_1$ with the property that we have an equivalence of $\mathbb{Z}/2$-graded B'_*-algebras

$$B'_* \otimes_k B'_* \simeq \prod_G B'_*. \tag{21}$$

Observe that this tensor product is the *graded* tensor product.

From this, we want to show that $B'_1 = 0$, which will automatically force B'_0 to be étale over k. Suppose first that the characteristic of k is not 2. By Lemma 6.28 below, there exists a map of graded k-algebras $B'_* \to \overline{k}$. We can thus compose with the map $k \to B'_* \to \overline{k}$ and use (21) to conclude that $B'_* \otimes_k \overline{k} \simeq \prod_G \overline{k}$ as a graded k-algebra. This in particular implies that $B'_1 = 0$ and that B'_0 is a finite separable extension of k, which proves Proposition 6.27 away from the prime 2.

Finally, at the prime 2, we need to show that (21) still implies that $B'_1 = 0$. In this case, $B'_0 \oplus B'_1$ is a *commutative* k-algebra and (21) implies that it must be étale. After extending scalars to $\overline{k}$, $B'_0 \oplus B'_1$ must, as a commutative ring, be isomorphic to $\prod_G \overline{k}$. However, any idempotents in $B'_0 \oplus B'_1$ are clearly concentrated in degree zero. So, we can make the same conclusion at the prime 2. □

Lemma 6.28. *Let k be an algebraically closed field with $2 \neq 0$, and A'_* a nonzero finite-dimensional $\mathbb{Z}/2$-graded commutative k-algebra. Then there exists a map of graded k-algebras $A'_* \to k$.*

Proof. Induction on $\dim A_1'$. If $A_1' = 0$, we can use the ordinary theory of artinian rings over algebraically closed fields. If there exists a nonzero $x \in A_1'$, we can form the two-sided ideal (x): this is equivalently the left or right ideal generated by x. In particular, anything in (x) has square zero. It follows that $1 \notin (x)$ and we get a map of k-algebras

$$A_*' \to A_*'/(x),$$

where $A_*'/(x)$ is a *nontrivial* finite-dimensional $\mathbb{Z}/2$-graded commutative ring of smaller dimension in degree one. We can thus continue the process. $\square$

We can now prove our main result.

Theorem 6.29. *Let A be an even periodic $\mathbf{E}_\infty$-ring with $\pi_0 A$ regular noetherian. Then the Galois theory of A is algebraic.*

Most of this result appears in [14], where the Galois group of E_n is identified at an odd prime (as the Galois group of its π_0). Our methods contain the modifications needed to handle the prime 2 as well.

Remark 6.30. This will also show that all Galois extensions of A in the sense of [71] are faithful.

Proof of Theorem 6.29. Fix a finite group G and let B be a G-Galois extension of A, so that

$$A \simeq B^{hG}, \quad B \otimes_A B \simeq \prod_G B.$$

By Proposition 6.14, B is a perfect A-module; in particular, the homotopy groups of B are finitely generated $\pi_0 A$-modules.

Our goal is to show that B is even periodic and that $\pi_0 B$ is étale over $\pi_0 A$. To do this, we may reduce to the case of $\pi_0 A$ regular *local*, by checking at each localization. We are now in the following situation. The $\mathbf{E}_\infty$-ring A is even periodic, with $\pi_0 A$ local with its maximal ideal generated by a regular sequence $x_1, \dots, x_n \in \pi_0 A$ for $n = \dim \pi_0 A$. Let k be the residue field of $\pi_0 A$. In this case, then one can define a *multiplicative homology theory* P_* on the category of A-modules via

$$P_*(M) \stackrel{\text{def}}{=} \pi_*(M/(x_1, \dots, x_n)M) \simeq \pi_*(M \otimes_A A/(x_1, \dots, x_n)),$$

where $A/(x_1, \dots, x_n) \simeq A/x_1 \otimes_A \cdots \otimes_A A/x_n$. More precisely, it is a consequence of the work of Angeltveit [2, Sec. 3] that $A/(x_1, \dots, x_n)$ can be made (noncanonically) an $\mathbf{E}_1$-algebra in $\mathrm{Mod}(A)$. In particular, $A/(x_1, \dots, x_n)$ is, at the very least, a ring object in the homotopy category of A-modules; this weaker assertion, which is all that we need, is proved directly in [23, Theorem 2.6]. The fact that each A/x_i acquires the structure of a ring object in the homotopy category of A-modules already means that for any A-module M, the homotopy groups of $M/x_iM \simeq M \otimes_A A/x_i$ are actually $\pi_0(A)/(x_i)$-modules.

In any event, $M \mapsto P_*(M)$ is a multiplicative homology theory taking values in $k[t^{\pm 1}]$-modules. It satisfies a Künneth isomorphism,

$$P_*(M) \otimes_{k[t^{\pm 1}]} P(N) \simeq P_*(M \otimes_A N),$$

by a standard argument: with N fixed, both sides define homology theories on A-modules; there is a natural map between the two; moreover, this map is an isomorphism for $M = A$. This implies that the natural map is an isomorphism by a five-lemma argument. Note that the $\mathbf{E}_1$-ring $A/(x_1, \ldots, x_n)$ is usually not homotopy commutative if $p = 2$.

For convenience, rather than working in the category of graded $k[t^{\pm 1}]$-modules, we will work in the (equivalent) category of $\mathbb{Z}/2$-graded k-vector spaces, and denote the modified functor by Q_* (instead of P_*). Since $A \to B$ is G-Galois, it follows from $B \otimes_A B \simeq \prod_G B$ that there is an isomorphism of $\mathbb{Z}/2$-graded $k[G]$-modules,

$$Q_*(B) \otimes_k Q_*(B) \simeq \prod_G Q_*(B).$$

In particular, it follows that

$$\dim Q_0(B) + \dim Q_1(B) = |G|. \tag{22}$$

We now use a Bockstein spectral sequence argument to bound the rank of $\pi_0 B$ and $\pi_1 B$.

Lemma 6.31. *Let M be a perfect A-module. Suppose that $\dim_k Q_0(M) = a$. Then the rank of $\pi_0 M$ as a $\pi_0 A$-module (that is, the dimension after tensoring with the fraction field) is at most a.*

Proof. Choose a system of parameters $x_1, \ldots, x_n$ for the maximal ideal of $\pi_0 A$. If M is as in the statement of the lemma, then we are given that

$$\dim \pi_0(M/(x_1, \ldots, x_n)M) \leq a.$$

We consider the sequence of A-modules

$$M_i = M/(x_1, \ldots, x_i)M = M \otimes_A A/x_1 \otimes_A \cdots \otimes_A A/x_i;$$

here $\pi_0(M_i)$ is a finitely generated module over the regular local ring $\pi_0(A)/(x_1, \ldots, x_i)$. For instance, $\pi_0(M_n)$ is a module over the residue field k, and our assumption is that its rank is at most a.

We make the following inductive step.

Inductive step. If $\pi_0(M_{i+1})$ has rank $\leq a$ as a module over the regular local ring $\pi_0(A)/(x_1, \dots, x_{i+1})$, then $\pi_0(M_i)$ has rank $\leq a$ as a module over the regular local ring $\pi_0(A)/(x_1, \dots, x_i)$.

To see this, consider the cofiber sequence

$$M_i \xrightarrow{x_i} M_i \to M_{i+1},$$

and the induced injection in homotopy groups

$$0 \to \pi_0(M_i)/x_i\pi_0 M_i \to \pi_0(M_{i+1}).$$

We now apply the following sublemma. By descending induction on i, this will imply the desired claim.

Sublemma. *Let $(R, \mathfrak{m})$ be a regular local ring, $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Consider a finitely generated R-module N. Given an injection*

$$0 \to N/xN \to N',$$

where N' is a finitely generated $R/(x)$-module, we have

$$\operatorname{rank}_R N \leq \operatorname{rank}_{R/(x)} N'.$$

Proof. When R is a discrete valuation ring (so that $R/(x)$ is a field), this follows from the structure theory of finitely generated modules over a PID.

To see this in general, we may localize at the prime ideal $(x) \subset R$ (and thus replace the pair $(R, R/(x))$ with $R_{(x)}, R_{(x)}/(x)R_{(x)})$, which does not affect the rank of either side, and which reduces us to the DVR case. $\square$

With the sublemma, we can conclude that $\operatorname{rank}_{\pi_0(A)/(x_1,\dots,x_i)} \pi_0(M_i) \leq a$ for all i by descending induction on i, which completes the proof of Lemma 6.31. $\square$

By Lemma 6.31, it now follows that $\pi_0 B$, as a $\pi_0 A$-module, has rank at most $a = \dim_k Q_0(B)$, where $a \leq |G|$. However, when we invert everything in $\pi_0 A$ (i.e., form the fraction field $k(\pi_0 A)$), then ordinary Galois theory goes into effect (Proposition 6.27) and $\pi_0 B \otimes_{\pi_0 A} k(\pi_0 A)$ is a finite étale $\pi_0 A$-algebra with Galois group G. In particular, it follows that $a = |G|$.

As a result, by (22), $Q_1(B) = 0$. It follows, again by the Bockstein spectral sequence, in the form of Lemma 6.32 below, that B is evenly graded and $\pi_* B$ is free as an A-module. In particular, $\pi_0(B \otimes_A B) \simeq \pi_0 B \otimes_{\pi_0 A} \pi_0 B$, which means that we get an isomorphism

$$\pi_0 B \otimes_{\pi_0 A} \pi_0 B \simeq \prod_G \pi_0 B,$$

so that $\pi_0 B$ is étale over $\pi_0 A$ (more precisely, $\operatorname{Spec} \pi_0 B \to \operatorname{Spec} \pi_0 A$ is a G-torsor), as desired. This completes the proof of Theorem 6.29. $\square$

Lemma 6.32. *Let A be an even periodic $\mathbf{E}_\infty$-ring such that $\pi_0 A$ is regular local and n-dimensional, with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Let M be a perfect A-module such that the A-module $M/(x_1, \dots, x_n)M$ satisfies $\pi_1(M/(x_1, \dots, x_n)M) = 0$. Then $\pi_1(M) = 0$ and $\pi_0(M)$ is a free $\pi_0(A)$-module.*

Proof. Lemma 6.32 follows from a form of the Bockstein spectral sequence: the evenness implies that there is no room for differentials; Proposition 2.5 of [40] treats the case of $A = E_n$. We can give a direct argument as follows.

Namely, we show that $\pi_1(M/(x_1, \dots, x_i)M) = 0$ for $i = 0, 1, \dots, n$, by descending induction on i. By assumption, it holds for $i = n$. The inductive step is proved as in the proof of Lemma 6.31, except that Nakayama's lemma is used to replace the sublemma. This shows that $\pi_1(M) = 0$.

Now, inducting in the other direction (i.e., in ascending order in i), we find that $x_1, \dots, x_n$ defines a regular sequence on $\pi_0(M)$ and the natural map

$$\pi_0(M)/(x_1, \dots, x_i) \to \pi_0(M/(x_1, \dots, x_i)),$$

is an isomorphism. This implies that the depth of $\pi_0(M)$ as a $\pi_0(A)$-module is equal to n, so that $\pi_0(M)$ is a free $\pi_0(A)$-module. $\square$

7. Local systems, cochain algebras, and stacks

The rest of this paper will be focused on the calculations of Galois groups in certain examples of stable homotopy theories, primarily those arising from chromatic homotopy theory and modular representation theory. The basic ingredient, throughout, is to write a given stable homotopy theory as an *inverse limit* of simpler stable homotopy theories to which one can apply known algebraic techniques such as Theorem 6.29 or Theorem 6.17. Then, one puts together the various Galois groupoids that one has via techniques from descent theory.

In the present section, we will introduce these techniques in slightly more elementary settings.

7.1. Inverse limits and Galois theory

Our approach can be thought of as an elaborate version of van Kampen's theorem. To begin, let us recall the setup of this. Let X be a topological space, and let $U, V \subset X$ be open subsets which cover X. In this case, the diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd},$$

is a homotopy pushout. In order to give a covering space $Y \to X$, it suffices to give a covering space $Y_U \to U$, a covering space $Y_V \to V$, and an isomorphism $Y_U|_{U\cap V} \simeq Y_V|_{U\cap V}$ of covers of $U \cap V$. In other words, the diagram of categories

$$\begin{tikzcd} \mathrm{Cov}_X \arrow[r] \arrow[d] & \mathrm{Cov}_U \arrow[d] \\ \mathrm{Cov}_V \arrow[r] & \mathrm{Cov}_{U\cap V} \end{tikzcd} \tag{23}$$

is cartesian, where for a space Z, Cov_Z denotes the category of topological covering spaces of Z. It follows that the dual diagram on fundamental *groupoids*

$$\begin{tikzcd} \pi_{\leq 1}(U \cap V) \arrow[r] \arrow[d] & \pi_{\leq 1}(V) \arrow[d] \\ \pi_{\leq 1}(V) \arrow[r] & \pi_{\leq 1}(X) \end{tikzcd}$$

is, dually, *cocartesian.* In particular, van Kampen's theorem is a formal consequence of descent theory for covers.

As a result, one can hope to find analogs of van Kampen's theorem in other setting. For instance, if X is a *scheme* and $U, V \subset X$ are open subschemes, then descent theory implies that the diagram (23) (where Cov now refers to *finite* étale covers) is cartesian, so the dual diagram on étale fundamental groupoids is cocartesian.

Our general approach comes essentially from the next result:

Proposition 7.1. *Let K be a simplicial set and let $p\colon K \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ be a functor to the ∞-category $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ of stable homotopy theories. Then we have a natural equivalence in* GalCat*,*

$$\mathrm{CAlg}^{\mathrm{w.cov}}\left(\varprojlim_K p\right) \simeq \varprojlim_{k\in K} \mathrm{CAlg}^{\mathrm{w.cov}}(p(k)). \tag{24}$$

Proof. The statement that (24) is an equivalence equates to the statement that for any finite group G, to give a G-torsor in the stable homotopy theory $\varprojlim_K p$ is equivalent to giving a compatible family of G-torsors in $p(k), k \in K$. (Recall, however, from Remark 5.38 that infinite limits in GalCat exist, but they do not commute with the restriction $\mathrm{GalCat} \to \mathrm{Cat}_\infty$ in general.) We observe that we have a natural functor from the left-hand-side of (24) to the right-hand-side which is fully faithful (as both are subcategories of the ∞-category of commutative algebra objects in $\varprojlim_K p$), so that the functor

$$\mathrm{Tors}_G\left(\mathrm{CAlg}^{\mathrm{w.cov}}\left(\varprojlim_K p\right)\right) \to \varprojlim_{k\in K} \mathrm{Tors}_G(\mathrm{CAlg}^{\mathrm{w.cov}}(p(k)))$$

is fully faithful.

We need to show that if $A \in \mathrm{Fun}(BG, \mathrm{CAlg}^{\mathrm{w.cov}}(\varprojlim_K p))$ has the property that its image in $\mathrm{Fun}(BG, \mathrm{CAlg}^{\mathrm{w.cov}}(p(k)))$ for each $k \in K$ is a G-torsor, then it is a G-torsor to begin with. However, A is dualizable, since it is dualizable locally (cf. [50, Prop. 4.6.1.11]), and it is faithful, since it is faithful locally, i.e., at each $k \in K$. The map $A \otimes A \to \prod_G A$ is an equivalence since it is an equivalence locally, and putting these together, A is a G-torsor. $\square$

In the case where we work with finite covers, rather than weak finite covers, additional finiteness hypotheses are necessary.

Proposition 7.2. *Let K be a simplicial set and let $p\colon K \to$ 2-Ring be a functor. Then we have a natural fully faithful inclusion*

$$\mathrm{CAlg}^{\mathrm{cov}}(\varprojlim_K p(k)) \to \varprojlim_K \mathrm{CAlg}^{\mathrm{cov}}(p(k)), \tag{25}$$

which is an equivalence if K is finite.

Proof. Since both sides are subcategories of $\mathrm{CAlg}(\varprojlim_K p(k)) = \varprojlim_K \mathrm{CAlg}(p(k))$, the fully faithful inclusion is evident. The main content of the result is that if K is finite, then the inclusion is an equivalence. In other words, we want to show that given a commutative algebra object in $\varprojlim_K p(k)$ which becomes a finite cover upon restriction to each $p(k)$, then it is a finite cover in the inverse limit. Since both sides of (25) are Galois categories (thanks to Lemma 5.37), it suffices to show that G-torsors on either side are equivalent. In other words, given a compatible diagram of G-torsors in the $\mathrm{CAlg}^{\mathrm{cov}}(p(k))$, we want the induced diagram in $\mathrm{CAlg}(\varprojlim_K p(k))$ to be a finite cover.

So let $A \in \mathrm{Fun}(BG, \mathrm{CAlg}(\varprojlim_K p))$ be such that its evaluation at each vertex $k \in K$ defines a G-torsor in $\mathrm{CAlg}^{\mathrm{cov}}(p(k))$. We need to show that $A \in \mathrm{CAlg}^{\mathrm{cov}}(\varprojlim_K p)$. For this, in view of Corollary 6.15, it suffices to show that A admits descent. But this follows in view of Proposition 3.25 and the fact that the image of A in each $k \in K$ admits descent in the stable homotopy theory $p(k)$. $\square$

Using the Galois correspondence, one finds:

Corollary 7.3. *In the situation of Proposition 7.2 or Proposition 7.1, we have an equivalence in* $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$*:*

$$\varinjlim_K \pi_{\leq 1}^{\mathrm{weak}} p(k) \simeq \pi_{\leq 1}^{\mathrm{weak}}(\varprojlim_K p(k)), \quad \varinjlim_K \pi_{\leq 1} p(k) \simeq \pi_{\leq 1}(\varprojlim_K p(k)). \tag{26}$$

For example, let $U, V \subset X$ be open subsets of a scheme X. Then we have an equivalence

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(U) \times_{\mathrm{QCoh}(U \cap V)} \mathrm{QCoh}(V),$$

by descent theory. The resulting homotopy pushout diagram that one obtains on fundamental groupoids (by (26)) is the van Kampen theorem for open immersions of schemes.

Using this, one can also obtain a van Kampen theorem for gluing *closed* immersions of schemes. For simplicity, we state the result for commutative rings. Let A', A, A'' be (discrete) commutative rings and consider *surjections* $A' \twoheadrightarrow A, A'' \twoheadrightarrow A$. In this case, one has a pull-back square (as we recalled in Example 2.23)

$$\begin{tikzcd} \mathrm{Mod}^{\omega}(A' \times_A A'') \arrow[r] \arrow[d] & \mathrm{Mod}^{\omega}(A') \arrow[d] \\ \mathrm{Mod}^{\omega}(A'') \arrow[r] & \mathrm{Mod}^{\omega}(A) \end{tikzcd}.$$

Note that the analog without the compactness, or more generally connectivity, hypothesis would fail. Using (26), and the observation that the Galois groupoid depends only on the dualizable objects, we obtain the following well-known corollary:

Corollary 7.4. *We have a pushout of profinite groupoids*

$$\pi_{\leq 1}^{\mathrm{et}}(\mathrm{Spec}(A' \times_A A'')) \simeq \pi_{\leq 1}^{\mathrm{et}}(\mathrm{Spec}\, A') \sqcup_{\pi_{\leq 1}^{\mathrm{et}}(\mathrm{Spec}\, A)} \pi_{\leq 1}^{\mathrm{et}}(\mathrm{Spec}\, A'').$$

This result is one expression of the intuition that $\mathrm{Spec}(A' \times_A A'')$ is obtained by "gluing together" the schemes $\mathrm{Spec}\, A', \mathrm{Spec}\, A''$ along the closed subscheme $\mathrm{Spec}\, A$. This idea in derived algebraic geometry has been studied extensively in [45].

These ideas are often useful even in cases when one can only *approximately* resolve a stable homotopy theory as an inverse limit of simpler ones; one can then obtain *upper bounds* for Galois groups. For example, let K be a simplicial set, and consider a diagram $f\colon K \to \mathrm{CAlg}$. Let $A = \varprojlim_K f(k)$. In this case, one always has a functor

$$\mathrm{Mod}(A) \to \varprojlim_K \mathrm{Mod}(f(k)),$$

which is *fully faithful* on the perfect A-modules since the right adjoint preserves the unit. If K is finite, it is fully faithful on all of $\mathrm{Mod}(A)$. It follows that, *regardless* of any finiteness hypotheses on K, there are fully faithful inclusions

$$\mathrm{CAlg}^{\mathrm{cov}}(\mathrm{Mod}(A)) \subset \mathrm{CAlg}^{\mathrm{cov}}(\varprojlim_K \mathrm{Mod}(f(k))) \subset \varprojlim_K \mathrm{CAlg}^{\mathrm{cov}}(\mathrm{Mod}(f(k))). \tag{27}$$

We will explore the interplay between these different Galois categories in the next section. They can be used to give upper bounds on the Galois group of A since fully faithful inclusions of connected Galois categories are dual to *surjections* of profinite groups.

7.2. ∞-categories of local systems

In this subsection, we will introduce the first example of the general van Kampen approach (Proposition 7.2), for the case of a *constant* functor.

Let X be a connected space, which we consider as an ∞-groupoid. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a stable homotopy theory, which we will assume connected for simplicity.

Definition 7.5. The functor category $\mathrm{Fun}(X, \mathcal{C})$ acquires the structure of a symmetric monoidal ∞-category via the "pointwise" tensor product. We will call this the ∞-category of **$\mathcal{C}$-valued local systems** on X and denote it by $\mathrm{Loc}_X(\mathcal{C})$.

This is a special case of the van Kampen setup of the previous section, when we are considering a functor from X to 2-Ring or $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ which is constant with value $\mathcal{C}$. This means that, with no conditions whatsoever, we have

$$\pi_1^{\mathrm{weak}}(\mathrm{Loc}_X(\mathcal{C})) \simeq \widehat{\pi_1 X} \times \pi_1^{\mathrm{weak}}(\mathcal{C}),$$

in view of Proposition 7.1, where $\widehat{\pi_1 X}$ denotes the profinite completion of the fundamental group $\pi_1 X$. Explicitly, given a functor $f\colon X \to \mathrm{FinSet}$, we obtain (by mapping into $\mathbf{1}$) a local system in $\mathrm{CAlg}(\mathcal{C})$ parametrized by X. These are always weak finite covers in $\mathrm{Loc}_X(\mathcal{C})$, and these come from finite covers of X or local systems of finite sets on X. Given weak finite covers in $\mathcal{C}$ itself, we can take the constant local systems at those objects to obtain weak finite covers in $\mathrm{Loc}_X(\mathcal{C})$.

If, further, X is a finite CW complex, it follows that

$$\pi_1(\mathrm{Loc}_X(\mathcal{C})) \simeq \widehat{\pi_1 X} \times \pi_1(\mathcal{C}),$$

in view of Proposition 7.2. We will use this to begin describing the Galois theory of a basic class of nonconnective $\mathbf{E}_\infty$-rings, the cochain algebras on connective ones.

In particular, let $\mathcal{C} = \mathrm{Mod}(E)$ for an $\mathbf{E}_\infty$-algebra E, so that we can regard $\mathrm{Loc}_X(\mathrm{Mod}(E)) = \mathrm{Fun}(X, \mathrm{Mod}(E))$ as parametrizing "local systems of E-modules on X." The unit object in $\mathrm{Loc}_X(\mathrm{Mod}(E))$ has endomorphism $\mathbf{E}_\infty$-ring given by the cochain algebra $C^*(X; E)$. Therefore, we have an adjunction of stable homotopy theories

$$\mathrm{Mod}(C^*(X; E)) \rightleftarrows \mathrm{Loc}_X(\mathrm{Mod}(E)),$$

between modules over the E-valued cochain algebra $C^*(X; E)$ and $\mathrm{Loc}_X(\mathrm{Mod}(E))$, where the right adjoint Γ takes the global sections (i.e., inverse limit) over X. The left adjoint is fully faithful when restricted to the perfect $C^*(X;E)$-modules and in general if $\mathbf{1}$ is compact in $\mathrm{Loc}_X(\mathrm{Mod}(E))$. Therefore, we get surjections of fundamental groups

$$\begin{aligned}&\widehat{\pi_1 X} \times \pi_1(\mathrm{Mod}(E))\\ &\quad \simeq \pi_1^{\mathrm{weak}}(\mathrm{Loc}_X(\mathrm{Mod}(E))) \twoheadrightarrow \pi_1(\mathrm{Loc}_X(\mathrm{Mod}(E))) \twoheadrightarrow \pi_1(\mathrm{Mod}(C^*(X;E))).\end{aligned} \tag{28}$$

In this subsection and the next, we will describe the objects and maps in (28) in some specific instances.

Example 7.6. If X is simply connected, then this map is an isomorphism, given the natural section $\mathrm{Mod}(E) \to \mathrm{Loc}_X(\mathrm{Mod}(E))$ which sends an E-module to the constant local system with that value, so E and $C^*(X;E)$ have the same fundamental group.

Suppose X has the homotopy type of a *finite* CW complex, so that the functor Γ is obtained via a finite homotopy limit and in particular commutes with all homotopy colimits. In this case, as we mentioned earlier, the unit object in $\mathrm{Loc}_X(\mathrm{Mod}(E))$ is compact, so that the map $\pi_1^{\mathrm{weak}}(\mathrm{Loc}_X(\mathrm{Mod}(E))) \to \pi_1(\mathrm{Loc}_X(\mathrm{Mod}(E)))$ is an isomorphism. In this case, the entire problem boils down to understanding the image of the fully faithful, colimit-preserving functor $\mathrm{Mod}(C^*(X;E)) \to \mathrm{Loc}_X(\mathrm{Mod}(E))$.

By definition, $\mathrm{Mod}(C^*(X;E))$ is generated by the unit object, so its image in $\mathrm{Loc}_X(\mathrm{Mod}(E))$ consists of the full subcategory of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ generated by the unit object, which is the *trivial* constant local system. In particular, we should think of $\mathrm{Mod}(C^*(X;E)) \subset \mathrm{Loc}_X(\mathrm{Mod}(E))$ as the "ind-unipotent" local systems of E-modules parametrized by X. We can see some of that algebraically.

Definition 7.7. Let A be a module over a commutative ring R and let G be a group acting on A by R-endomorphisms. We say that the action is **unipotent** if there exists a finite filtration of R-modules

$$0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = A,$$

which is preserved by the action of G, such that the G-action on each A_i/A_{i-1} is trivial. We say that the G-action is **ind-unipotent** if A is a filtered union of G-stable submodules $A_\alpha \subset A$ such that the action of G on each A_α is unipotent.

Proposition 7.8. *Let X be a connected space. Consider an object M of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ and let M_x be the underlying E-module for some $x \in X$. Suppose M belongs to the localizing subcategory of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ generated by the unit. Then, the action of $\pi_1(X,x)$ on each $\pi_0 E$-module $\pi_k(M_x)$ is ind-unipotent.*

Conversely, suppose E is connective. Given $M \in \mathrm{Loc}_X(\mathrm{Mod}(E))$ such that the monodromy action of $\pi_1(X,x)$ on each $\pi_k(M_x)$ is ind-unipotent, then if M is additionally n-coconnective for some n and if X is a finite CW complex, we have $M \in \mathrm{Mod}(C^*(X;E)) \subset \mathrm{Loc}_X(\mathrm{Mod}(E))$.

Proof. Clearly the unit object of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ has unipotent action of $\pi_1(X,x)$ on its homotopy groups: the monodromy action by $\pi_1(X,x)$ is trivial. The collection of objects of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ with ind-unipotent action of $\pi_1(X,x)$ is seen to be a localizing subcategory using long exact sequences. The first assertion follows.

For the final assertion, since X is a finite CW complex, the functor $\mathrm{Mod}(C^*(X;E)) \to \mathrm{Loc}_X(\mathrm{Mod}(E))$ is fully faithful and commutes with colimits. We can write M as a colimit of the local systems of E-modules

$$0 \simeq \tau_{\geq n} M \to \tau_{\geq n-1} M \to \tau_{\geq n-2} M \to \cdots,$$

where each term in the local system has only finitely many homotopy groups. It suffices to show that each $\tau_{\geq k} M$ belongs to $\mathrm{Mod}(C^*(X;E)) \subset \mathrm{Loc}_X(\mathrm{Mod}(E))$. Working inductively, one reduces to the case where M itself has a single nonvanishing homotopy group (say, a π_0) with ind-unipotent action of $\pi_1(X,x)$. Since the subcategory of $\mathrm{Loc}_X(\mathrm{Mod}(E))$ consisting of local systems M with $\pi_*(M_x) = 0$ for $* \neq 0$ is an ordinary category, equivalent to the category of local systems of $\pi_0 E$-modules on X, our task is one of algebra. One reduces (from the algebraic definition of ind-unipotence) to showing that if M_0 is a $\pi_0 E$-module, then the induced object in $\mathrm{Loc}_X(\mathrm{Mod}(E))$ with trivial $\pi_1(X,x)$-action belongs to $\mathrm{Mod}(C^*(X;E))$. However, this object comes from the $C^*(X;E)$-module $C^*(X;\tau_{\leq 0}E) \otimes_{\pi_0 E} M_0$. $\square$

Remark 7.9. Suppose X is *one-dimensional*, so that X is a wedge of finitely many circles. Then, for any E, any $M \in \mathrm{Loc}_X(\mathrm{Mod}(E))$ such that the action of $\pi_1(X,x)$ is ind-unipotent on $\pi_*(M_x)$ belongs to the image of $\mathrm{Mod}(C^*(X;E)) \to \mathrm{Loc}_X(\mathrm{Mod}(E))$. In other words, one needs no further hypotheses on E or M_x.

To see this, we need to show (by Theorem 2.29) that the inverse limit functor

$$\Gamma = \varprojlim_X \colon \mathrm{Loc}_X(\mathrm{Mod}(E)) \to \mathrm{Mod}(C^*(X;E)),$$

is *conservative* when restricted to those local systems with the above ind-unipotence property on homotopy groups. Recall that one has a spectral sequence

$$E_2^{s,t} = H^s(X; \pi_t M_x) \implies \pi_{t-s}\Gamma(X, M),$$

for computing the homotopy groups of the inverse limit. The $s = 0$ line of the E_2-page is *never* zero if the action is ind-unipotent unless $M = 0$: there are always fixed points for the action of $\pi_1(X,x)$ on $\pi_*(M_x)$. If X is one-dimensional, the spectral sequence degenerates at E_2 for dimensional reasons; this forces the inverse limit $\varprojlim_X M$ to be nonzero unless $M = 0$.

As we saw earlier in this subsection, in order to construct finite covers of the unit object in $\mathrm{Loc}_X(\mathrm{Mod}(E))$, we can consider a local system of finite sets $\{Y_x\}_{x \in X}$ on X

(i.e., a finite cover of X), and consider the local system $\{C^*(Y_x; E)\}_{x \in X}$ of $\mathbf{E}_\infty$-algebras under E. The induced object in $\mathrm{Loc}_X(\mathrm{Mod}(E))$ will generally not be unipotent in this sense. In fact, unless there is considerable torsion, this will almost never be the case.

For example, suppose G is a finite group, and let R be a commutative ring. Consider the G-action on $\prod_G R$. The group action is ind-unipotent if G is a p-group (for some prime number p) where p is nilpotent in R.

Proof. Suppose $q \mid |G|$ and q is not nilpotent in R, but the G-action on $\prod_G R$ is ind-unipotent. It follows that we can invert q and, after some base extension, assume that R is a *field* with $q \neq 0$. We can even assume $\zeta_q \in R$. We need to show that the standard representation is not ind-unipotent when $q \mid |G|$; this follows from restricting G to $\mathbb{Z}/q \subset G$, and observing that various nontrivial one-dimensional characters occur and these must map trivially into any unipotent representation.

Conversely, if G is a p-group and p is nilpotent in R, then by filtering R, we can assume $p = 0$ in R. Now in fact *any* $R[G]$-module is ind-unipotent, because the augmentation ideal of $R[G]$ is nilpotent. □

Corollary 7.10. *Suppose p is not nilpotent in the $\mathbf{E}_\infty$-ring R. Then the surjection $\widehat{\pi_1 X} \times \pi_1 \mathrm{Mod}(E) \twoheadrightarrow \pi_1 \mathrm{Mod}(C^*(X; E))$ factors through $\widehat{\pi_1 X}_{p^{-1}}$ where $\widehat{\pi_1 X}_{p^{-1}}$ denotes the profinite completion away from p.*

Corollary 7.11. *If R is a $\mathbf{E}_\infty$-ring such that $\mathbb{Z} \subset \pi_0 R$, then the map $\pi_1 \mathrm{Mod}(R) \to \pi_1 \mathrm{Mod}(C^*(X; R))$ is an isomorphism of profinite groups.*

Remark 7.12. In $K(n)$-local stable homotopy theory, the comparison question between modules over the cochain $\mathbf{E}_\infty$-ring and local systems has been studied in [33, sec. 5.4].

Putting these various ideas together, it is not too hard to prove the following result, whose essential ideas are contained in [71, Proposition 5.6.3]. Here $\widehat{\pi_1 X}_p$ denotes the pro-p-completion of $\pi_1 X$.

Theorem 7.13. *Let X be a finite CW complex. Then if R is an $\mathbf{E}_\infty$-ring with p nilpotent and such that $\pi_i R = 0$ for $i \gg 0$ (e.g., a field of characteristic p), then the natural map*

$$\widehat{\pi_1 X}_p \times \pi_1 \mathrm{Mod}(R) \to \pi_1 \mathrm{Mod}(C^*(X; R)) \tag{29}$$

is an isomorphism.

Proof. By Corollary 7.10, the natural map $\widehat{\pi_1 X} \times \pi_1 \mathrm{Mod}(R) \twoheadrightarrow \pi_1 \mathrm{Mod}(C^*(X; R))$ does in fact factor through the quotient of the source where $\widehat{\pi_1 X}$ is replaced by its pro-p-completion. It suffices to show that the induced map (29) is an isomorphism. Equivalently, we need to show that if $Y \to X$ is a finite G-torsor for G a p-group, then $C^*(X; R) \to C^*(Y; R)$ is a faithful G-Galois extension. Equivalently, we need to show that if $\{Y_x\}_{x\in X}$ is the local system of finite sets defined by the finite cover $Y \to X$, then the local system of R-modules $\{C^*(Y_x; R)\}_{x\in X}$ (which gives a G-Galois cover of the unit in $\mathrm{Loc}_X(\mathrm{Mod}(R))$) actually belongs to the image of $\mathrm{Mod}(C^*(X; R))$. However, this is a consequence of Proposition 7.8 because the monodromy action is by elements of the p-group G. Any G-module over a ring with p nilpotent is ind-unipotent. $\square$

Remark 7.14. Let $Y \to X$ be a map of spaces, and let R be as above. Then there are two natural local systems of R-module spectra on X that one can construct:

1. The object of $\mathrm{Loc}_X(\mathrm{Mod}(R))$ obtained from the $C^*(X; R)$-module $C^*(Y; R)$, i.e., the local system $C^*(Y; R) \otimes_{C^*(X;R)} C^*(*; R)$ which is a local system as $*$ ranges over X.
2. Consider the fibration $Y \to X$ as a local system of spaces $\{Y_x\}$ on X, $x \in X$, and apply $C^*(\cdot; R)$ everywhere.

In general, these local systems are not the same: they are the same only if the R-valued *Eilenberg–Moore spectral sequence* for the square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & Y \arrow[d] \\ \{x\} \arrow[r] & X \end{tikzcd},$$

converges, for every choice of basepoint $x \in X$. This question can be quite subtle, in general. Theorem 7.13 is essentially equivalent to the convergence of the R-valued Eilenberg–Moore spectral sequence when $Y \to X$ is a G-torsor for G a p-group. This is the approach taken by Rognes in [71].

Finally, we close with an example suggesting further questions.

Example 7.15. The topological part of the Galois group of $C^*(S^1; \mathbb{F}_p)$ is precisely $\widehat{\mathbb{Z}}_p$. The Galois covers come from the maps

$$C^*(S^1; \mathbb{F}_p) \to C^*(S^1; \mathbb{F}_p),$$

dual to the degree p^n maps $S^1 \to S^1$. This would not work over the sphere S^0 replacing $\mathbb{F}_p$, in view of Corollary 7.10. However, this *does* work in p-adically completed homotopy theory.

Let Sp_p be the ∞-category of p-complete (i.e., S^0/p-local) spectra, and let $\widehat{S}_p$ be the p-adic sphere, which is the unit of Sp_p. The map $C^*(S^1; \widehat{S}_p) \to C^*(S^1; \widehat{S}_p)$ which is dual to the degree p map $S^1 \to S^1$ is a $\mathbb{Z}/p$-weak Galois extension in Sp_p. In particular, it will follow that the weak Galois group of Sp_p is the product of $\widehat{\mathbb{Z}_p}$ with that of Sp_p itself.

To see this, note that we have a fully faithful embedding

$$L_{S^0/p}\operatorname{Mod}(C^*(S^1;\widehat{S}_p)) \simeq \operatorname{Mod}_{\mathrm{Sp}_p}(C^*(S^1;\widehat{S}_p)) \subset \operatorname{Loc}_{S^1}(\mathrm{Sp}_p).$$

In $\operatorname{Loc}_{S^1}(\mathrm{Sp}_p)$, we need to show that the local system of p-complete spectra obtained from the cover $S^1 \xrightarrow{p} S^1$ actually belongs to the subcategory of $\operatorname{Loc}_{S^1}(\mathrm{Sp}_p)$ generated under colimits by the unit (equivalently, by the constant local systems).

In order to prove this claim, it suffices to prove the analog after quotienting by p^n for each p, since for any p-complete spectrum X, we have

$$X \simeq \Sigma^{-1} L_{S^0/p}(\varinjlim_n (X \otimes S^0/p^n)),$$

as the colimit $\varinjlim_n S^0/p^n$ (where the successive maps are multiplication by p) has p-adic completion given by the suspension of the p-adic sphere. But on the other hand, we can apply Remark 7.9 to the cofiber of p^n on our local system, since an order p automorphism on a p-torsion abelian group is always ind-unipotent.

By contrast, the analogous assertion would fail if we worked in the setting of *all* $C^*(S^1;\widehat{S}_p)$-modules (not p-complete ones): the (weakly) Galois covers constructed are only Galois after p-completion. This follows because $C^*(S^1;\widehat{S}_p)$ has coconnective rationalization, and all the Galois covers of it are étale (as we will show in Theorem 8.17).

7.3. Stacks and finite groups

To start with, let k be a separably closed field of characteristic p and let G be a finite group. Consider the stable homotopy theory $\operatorname{Mod}_G(k)$ of k-module spectra equipped with an action of G, or equivalently the ∞-category $\operatorname{Loc}_{BG}(\operatorname{Mod}(k))$ of local systems of k-module spectra on BG. We will explore the Galois theory of $\operatorname{Mod}_G(k)$ and the various inclusions (27).

Theorem 7.16. *Let k be separably closed of characteristic p. $\pi_1^{\mathrm{weak}}(\operatorname{Mod}_G(k)) \simeq G$ but $\pi_1(\operatorname{Mod}_G(k))$ is the quotient of G by the normal subgroup generated by the order p elements.*

Proof. The assertion of $\pi_1^{\mathrm{weak}}(\operatorname{Mod}_G(k))$ is immediate: the weak "Galois closure" (i.e., maximal connected object in the Galois category) of the unit in $\operatorname{Mod}_G(k)$ is $\prod_G k$, thanks to Proposition 7.1. The more difficult part of the result concerns the (non-weak) Galois group.

Any finite cover $A \in \operatorname{CAlg}(\operatorname{Mod}_G(k))$ must be given by an action of G on an underlying $\mathbf{E}_\infty$-k-algebra which must be $\prod_S k$ for S a finite set; S gets a natural G-action, which determines everything. In particular, we get that A must be a product of copies of $\prod_{G/H} k$. We need to determine which of these are actually finite covers. We can always reduce to the Galois case, so given a surjection $G \twoheadrightarrow G'$, we need a criterion for when $\prod_{G'} k \in \operatorname{CAlg}(\operatorname{Mod}_G(k))$ is a finite cover.

Fix an order p element $g \in G$. We claim that if $\prod_{G'} k \in \mathrm{CAlg}(\mathrm{Mod}_G(k))$ is a finite cover, then g must map to the identity in G'. In fact, otherwise, we could restrict to $\mathbb{Z}/p \subset G$ to find (after inverting an idempotent of the restriction) that $\prod_{\mathbb{Z}/p} k$ would be a finite cover in $\mathrm{Mod}_{\mathbb{Z}/p}(k)$. This is impossible since $(\prod_{\mathbb{Z}/p} k)^{h\mathbb{Z}/p} \simeq k$ while $k^{h\mathbb{Z}/p}$ has infinitely many homotopy groups; thus the unit cannot be in the thick $\otimes$-ideal generated by $\prod_{\mathbb{Z}/p}(k)$. It follows from this that if $\prod_{G'} k$ is a finite cover in $\mathrm{Mod}_G(k)$, then every order p element must map to the identity in G'.

Conversely, suppose $G \twoheadrightarrow G'$ is a surjection annihilating every order p element. We claim that $\prod_{G'} k$ is a finite cover in $\mathrm{Mod}_G(k)$. Since it is a G'-Galois extension of the unit, it suffices to show that it is descendable by Corollary 6.15. For this, by the Quillen stratification theory (in particular, Theorem 4.8), one can check this after restricting to an elementary abelian p-subgroup. But after such a restriction, our commutative algebra object becomes a finite product of copies of the unit. $\square$

Corollary 7.17. *Let k be a separably closed field of characteristic $p > 0$. The Galois group $C^*(BG; k) \simeq k^{hG}$ is given by the quotient of the pro-p-completion of G by the order p elements in G.*

By the pro-p-completion of G, we mean the maximal quotient of G which is a p-group. In other words, we take the smallest normal subgroup $N \subset G$ such that $|G|/|N|$ is a power of p, and then take the normal subgroup N' generated by N and the order p elements in G. The Galois group of $C^*(BG; k)$ is the quotient G/N'.

Proof. Observe that the ∞-category of perfect $C^*(BG; k)$-modules is a full subcategory of the ∞-category $\mathrm{Loc}_{BG}(\mathrm{Mod}(k)) \simeq \mathrm{Mod}_G(k)$ of k-module spectra equipped with a G-action. We just showed in Theorem 7.16 that the Galois group of the latter was the quotient of G by the normal subgroup generated by the order p elements. In other words, the descendable connected Galois extensions of the unit in $\mathrm{Mod}_G(k)$ were the products $\prod_{G'} k$ where $G \twoheadrightarrow G'$ is a surjection of groups annihilating the order p elements.

It remains to determine which of these Galois covers actually belong to the thick subcategory generated by the unit $\mathbf{1} \in \mathrm{Mod}_G(k)$. As we have seen, that implies that the monodromy action of $\pi_1(BG) \simeq G$ on homotopy groups is ind-unipotent; this can only happen (for a permutation module) if G' is a p-group. If G' is a p-group, though, then the unipotence assumption holds and $\prod_{G'} k$ does belong to the thick subcategory generated by the unit, so these do come from $\mathrm{Mod}(C^*(BG; k))$. $\square$

Remark 7.18. Even if we were interested only in $\mathbf{E}_\infty$-rings and their modules, for which the Galois group and weak Galois group coincide, the proof of Corollary 7.17 makes clear the importance of the distinction (and the theory of descent via thick subcategories) in general stable homotopy theories. We needed thick subcategories and Quillen stratification theory to run the argument.

Example 7.19. We can thus obtain a weak *invariance result* for Galois groups (which we will use later). Let R be an $\mathbf{E}_\infty$-ring under $\mathbb{F}_p$, given trivial $\mathbb{Z}/p$-action. Then the Galois theories of R and $R^{h\mathbb{Z}/p}$ are the same, i.e., $R \to R^{h\mathbb{Z}/p}$ induces an equivalence on Galois groupoids. In fact, we know from $\mathrm{Mod}^\omega(R^{h\mathbb{Z}/p}) \subset \mathrm{Fun}(B\mathbb{Z}/p, \mathrm{Mod}^\omega(R))$ that Galois extensions of $R^{h\mathbb{Z}/p}$ come either from those of R or from the $\mathbb{Z}/p$-action. However, $\prod_{\mathbb{Z}/p} R$ is not a $\mathbb{Z}/p$-torsor because the thick $\otimes$-ideal it generates in $\mathrm{Fun}(B\mathbb{Z}/p, \mathrm{Mod}^\omega(R))$ cannot contain the unit: in fact, the Tate construction on R with $\mathbb{Z}/p$ acting trivially is nonzero, while the Tate construction on anything in the thick $\otimes$-ideal generated by $\prod_{\mathbb{Z}/p} R$ is trivial.

Consider now, instead of a finite group, an algebraic stack $\mathfrak{X}$. As discussed in Example 2.22, one has a natural stable homotopy theory $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent complexes on $\mathfrak{X}$, obtained via

$$\mathrm{QCoh}(\mathfrak{X}) = \varprojlim_{\mathrm{Spec}\, A \to \mathfrak{X}} D(\mathrm{Mod}(A)),$$

where we take the inverse limit over all maps $\mathrm{Spec}\, A \to X$; we could restrict to smooth maps. It follows from Theorem 6.17 that a *weak finite cover* in $\mathrm{QCoh}(\mathfrak{X})$ is the compatible assignment of a finite étale A-algebra for each map $\mathrm{Spec}\, A \to \mathfrak{X}$. In other words, the weak Galois group of $\mathrm{QCoh}(\mathfrak{X})$ is the étale fundamental group of the stack $\mathfrak{X}$.

If the unit object in $\mathrm{QCoh}(\mathfrak{X})$ is compact, the weak Galois group and the Galois group of $\mathrm{QCoh}(\mathfrak{X})$ are the same. One can make this conclusion if $\mathfrak{X}$ is *tame*, which roughly means that (if $\mathfrak{X}$ is Deligne–Mumford) the orders of the stabilizers are invertible (cf. [38, Theorems B and C]). If this fails, then the weak Galois group and the Galois group need not be the same, and one gets a *canonical quotient* of the étale fundamental group of an algebraic stack, the Galois group of $\mathrm{QCoh}(\mathfrak{X})$.

Example 7.20. Let G be a finite group, and let $\mathfrak{X} = BG$ over a separably closed field of characteristic p. Then $\mathrm{QCoh}(\mathfrak{X})$ is precisely the ∞-category $\mathrm{Mod}_G(k)$ considered in the previous section. The fundamental group of $\mathfrak{X}$ is G, and the main result of the previous subsection (Theorem 7.16) implies that the difference between the Galois group of $\mathrm{QCoh}(\mathfrak{X})$ and the étale fundamental group of $\mathfrak{X}$ is precisely the order p elements in the latter.

Thus, we know that for any map of stacks $B\mathbb{Z}/p \to \mathfrak{X}$ where p is not invertible on $\mathfrak{X}$, the $\mathbb{Z}/p$ must vanish in the fundamental group of $\mathrm{QCoh}(\mathfrak{X})$ (but not necessarily in the fundamental group of $\mathfrak{X}$). When $\mathfrak{X} = BG$ for some finite group, this is the *only* source of the difference between two groups. We do not know what the difference looks like in general.

Next, as an application of these ideas, we include an example that shows that the Galois group is a sensitive invariant of an $\mathbf{E}_\infty$-ring: that is, it can vary as the $\mathbf{E}_\infty$-structure varies within a fixed $\mathbf{E}_1$-structure.

Example 7.21. Let k be a separably closed field of characteristic $p > 0$. Let α_{p^2} be the usual rank p^2 group scheme over k and let $(\alpha_{p^2})^\vee$ be its Cartier dual, which is another infinitesimal commutative group scheme. Let $\mathbb{Z}/p^2$ be the usual constant group scheme. Consider the associated classifying stacks $B\mathbb{Z}/p^2$ and $B(\alpha_{p^2})^\vee$, and the associated cochain $\mathbf{E}_\infty$-rings $C^*(B\mathbb{Z}/p^2; k)$ and $C^*(B(\alpha_{p^2})^\vee; k)$ defined as endomorphisms of the unit of quasi-coherent sheaves.

Since $\alpha_{p^2}^\vee$ is infinitesimal, it follows that the fundamental group of the stack $B(\alpha_{p^2})^\vee$ is trivial and in particular that $\pi_1 \operatorname{Mod}(C^*(B(\alpha_{p^2})^\vee; k))$ is trivial. In other words, we are using the geometry of the stack to *bound above* the possible Galois group for the $\mathbf{E}_\infty$-ring of cochains with values in the structure sheaf. However, by Corollary 7.17, we have $\pi_1 \operatorname{Mod}(C^*(B\mathbb{Z}/p^2; k)) \simeq \mathbb{Z}/p$.

Finally, we note that there is a canonical equivalence of $\mathbf{E}_1$-rings between the two cochain algebras. In fact, the k-linear *abelian* category of (discrete) quasi-coherent sheaves on $B\mathbb{Z}/p^2$ can be identified with the category of modules over the group ring $k[\mathbb{Z}/p^2]$, which is noncanonically isomorphic to the algebra $k[x]/(x^{p^2})$. The k-linear *abelian* category of discrete quasi-coherent sheaves on $B(\alpha_{p^2})^\vee$ is identified with the category of modules over the ring of functions on α_{p^2}, which is $\mathbb{F}_p[x]/x^{p^2}$. In particular, we get a k-linear *equivalence* between either the abelian or derived categories of sheaves in either case. Since the cochain $\mathbf{E}_\infty$-rings we considered are (as $\mathbf{E}_1$-algebras) the endomorphism rings of the object k (which is the same representation either way), we find that they are equivalent as $\mathbf{E}_1$-algebras.

8. Invariance properties

Let R be a (discrete) commutative ring and let $I \subset R$ be a nilpotent ideal. Then it is a classical result in commutative algebra, the "topological invariance of the étale site," [30, Theorem 8.3, Exp. I], that the étale site of $\operatorname{Spec} R$ and the closed subscheme $\operatorname{Spec} R/I$ are equivalent. In particular, given an étale R/I-algebra $\overline{R}'$, it can be lifted *uniquely* to an étale R-algebra R' such that $R' \otimes_R R/I \simeq \overline{R}'$.

In this section, we will consider analogs of this result for $\mathbf{E}_\infty$-rings. For example, we will prove:

Theorem 8.1. *Let R be an $\mathbf{E}_\infty$-algebra under $\mathbb{Z}$ with p nilpotent in $\pi_0 R$. Then the map*

$$R \to R \otimes_{\mathbb{Z}} \mathbb{Z}/p,$$

induces an isomorphism on fundamental groups.

Results such as Theorem 8.1 will be extremely useful for us. For example, it will be integral to our computation of the Galois groups of stable module ∞-categories over finite groups. Theorem 8.1, which is immediate in the case of R *connective* (thanks to Theorem 6.17 together with the classical topological invariance result), seems to be very non-formal in the general case.

Throughout this section, we assume that our stable homotopy theories are *connected.*

8.1. Surjectivity properties

We begin with some generalities from [30]. We have the following easy lemma.

Lemma 8.2. *Let $G \to H$ be a morphism of profinite groups. Then the following are equivalent:*

1. *$G \to H$ is surjective.*
2. *For every finite (continuous) H-set S, S is connected if and only if the G-set obtained from S by restriction is connected.*

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a (connected) stable homotopy theory. Given a commutative algebra object $A \in \mathcal{C}$, we have functors $\mathrm{CAlg}^{\mathrm{cov}}(\mathcal{C}) \to \mathrm{CAlg}^{\mathrm{cov}}(\mathrm{Mod}_{\mathcal{C}}(A)), \mathrm{CAlg}^{\mathrm{w.cov}}(\mathcal{C}) \to \mathrm{CAlg}^{\mathrm{w.cov}}(\mathrm{Mod}_{\mathcal{C}}(A))$ given by tensoring with A. Using the Galois correspondence, this comes from the map of profinite groups $\pi_1(\mathrm{Mod}_{\mathcal{C}}(A)) \to \pi_1(\mathcal{C})$ by restricting continuous representations in finite sets. The following is a consequence of Lemma 8.2.

Proposition 8.3. *Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object with the following property: given any $A' \in \mathrm{CAlg}(\mathcal{C})$ which is a weak finite cover, the map*

$$\mathrm{Idem}(A') \to \mathrm{Idem}(A \otimes A') \tag{30}$$

is an isomorphism. Then the induced maps

$$\pi_1(\mathrm{Mod}_{\mathcal{C}}(A)) \to \pi_1(\mathcal{C}), \quad \pi_1^{\mathrm{weak}}(\mathrm{Mod}_{\mathcal{C}}(A)) \to \pi_1^{\mathrm{weak}}(\mathcal{C}),$$

are surjections of profinite groups.

Thus, it will be helpful to have some criteria for when maps of the form (30) are isomorphisms.

Definition 8.4. Given $A \in \mathrm{CAlg}(\mathcal{C})$, we will say that A is **universally connected** if for every $A' \in \mathrm{CAlg}(\mathcal{C})$, the map $\mathrm{Idem}(A') \to \mathrm{Idem}(A' \otimes A)$ in (30) is an isomorphism.

It follows by Proposition 8.3 that if A is universally connected, then $\pi_1^{\mathrm{weak}}(\mathrm{Mod}_{\mathcal{C}}(A)) \to \pi_1^{\mathrm{weak}}(\mathcal{C})$ and $\pi_1(\mathrm{Mod}_{\mathcal{C}}(A)) \to \pi_1(\mathcal{C})$ are surjections; moreover, this holds after any base change in $\mathrm{CAlg}(\mathcal{C})$. That is, if $A' \in \mathrm{CAlg}(\mathcal{C})$, then the map $\pi_1(\mathrm{Mod}_{\mathcal{C}}(A \otimes A')) \to \pi_1(\mathrm{Mod}_{\mathcal{C}}(A'))$ is a surjection, and similarly for the weak Galois group.

Note first that if A *admits descent*, then (30) is always an injection, since for any A', we can recover A' as the totalization of the cobar construction on A tensored with A' and since Idem commutes with limits (Corollary 2.40). In fact, it thus follows that if A admits descent, then $\mathrm{Idem}(A')$ is the equalizer of the two maps $\mathrm{Idem}(A\otimes A') \rightrightarrows \mathrm{Idem}(A\otimes A\otimes A')$. More generally, one can obtain a weaker conclusion under weaker hypotheses:

Proposition 8.5. *If $A \in \mathrm{CAlg}(\mathcal{C})$ is faithful (i.e., tensoring with A is a conservative functor $\mathcal{C} \to \mathcal{C}$), then the map (30) is always an injection, for any $A' \in \mathrm{CAlg}(\mathcal{C})$.*

Proof. It suffices to show that if $e \in \mathrm{Idem}(A')$ is an idempotent which maps to zero in $\mathrm{Idem}(A \otimes A')$, then e was zero to begin with. The hypothesis is that $A'[e^{-1}]$ becomes contractible after tensoring with A, and since A is faithful, it was contractible to begin with; that is, e is zero. □

We thus obtain the following criterion for universal connectedness.

Proposition 8.6. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a connected stable homotopy theory. Suppose $A \in \mathrm{CAlg}(\mathcal{C})$ is an object with the properties:*

1. *A is descendable.*
2. *The multiplication map $A \otimes A \to A$ is faithful.*

Then A is universally connected.

Proof. We will show that if $B \in \mathrm{CAlg}(\mathcal{C})$ is arbitrary, then the map $\mathrm{Idem}(B) \to \mathrm{Idem}(A \otimes B)$ is an *isomorphism*. Since A is descendable, we know that there is an equalizer diagram

$$\mathrm{Idem}(B) \to \mathrm{Idem}(A \otimes B) \rightrightarrows \mathrm{Idem}(A \otimes A \otimes B).$$

To prove the proposition, it suffices to show that the two maps $\mathrm{Idem}(A\otimes B) \rightrightarrows \mathrm{Idem}(A\otimes A \otimes B)$ are equal.

However, these maps become equal after composing with the map $\mathrm{Idem}(A\otimes A\otimes B) \to \mathrm{Idem}(A \otimes B)$ induced by the multiplication $A \otimes A \to A$. Since $A \otimes A \to A$ is faithful, the map $\mathrm{Idem}(A \otimes A \otimes B) \to \mathrm{Idem}(A \otimes B)$ is injective by Proposition 8.5, which thus proves the result. □

Proposition 8.6 is thus almost a tautology, although the basic idea will be quite useful for us. Unfortunately, the hypotheses are rather restrictive. If A is a local artinian ring and k the residue field, then the map $A \to k$ admits descent. However, the multiplication map $k \otimes_A k \to k$ need not be faithful: $k \otimes_A k$ has always infinitely many homotopy groups (unless $A = k$ itself). Nonetheless, we can prove:

Proposition 8.7. *Let k be a field. Let A be a connective $\mathbf{E}_\infty$-ring with a map $A \to k$ inducing a surjection on π_0. Suppose $A \to k$ admits descent. Then $A \to k$ is universally connected.*

Proof. Once again, we show that for any $A' \in \mathrm{CAlg}_{A/}$, the map $A' \to A' \otimes_A k$ induces an isomorphism on idempotents. Since $A \to k$ is descendable, it suffices to show that the two maps

$$\mathrm{Idem}(A' \otimes_A k) \rightrightarrows \mathrm{Idem}(A' \otimes_A k \otimes_A k)$$

are the same. For this, we know that the two maps become the same after composition with the multiplication map $A' \otimes_A (k \otimes_A k) \to A' \otimes_A k$. To show that the two maps are the same, it will suffice to show that they are *isomorphisms.* In other words, since we have a commutative diagram

$$\mathrm{Idem}(A' \otimes_A k) \rightrightarrows \mathrm{Idem}(A' \otimes_A k \otimes_A k) \to \mathrm{Idem}(A' \otimes_A k), \tag{31}$$

where the composite arrow is the identity, it suffices to show that *either one* of the two maps $\mathrm{Idem}(A' \otimes_A k) \rightrightarrows \mathrm{Idem}(A' \otimes_A k \otimes_A k)$ is an isomorphism.

More generally, we claim that for any k-algebra R, the map

$$R \to R \otimes_k (k \otimes_A k),$$

induced by the map of k-algebras $k \to k \otimes_A k$, induces an *isomorphism* on idempotents. (In (31), this is the map that we get from free, without using the fact that $A' \otimes_A k$ was the base-change of an A-algebra.) Since we have a Künneth isomorphism, this follows from the following purely algebraic lemma.

Lemma 8.8. *Let R_* be a graded-commutative k-algebra and let R'_* be a graded-commutative* connected *k-algebra: $R'_0 \simeq k$ and $R'_i = 0$ for $i < 0$. Then the natural map from idempotents in R_* to idempotents in the graded tensor product $R_* \otimes_k R'_*$ is an isomorphism.*

Proof. We have a map

$$\mathrm{Idem}(R_*) \to \mathrm{Idem}(R_* \otimes_k R'_*),$$

which is injective, since the map $k \to R'_*$ admits a section in the category of graded-commutative k-algebras. But the "reduction" map $\mathrm{Idem}(R_* \otimes_k R'_*) \to \mathrm{Idem}(R_*)$ is also injective. In fact, since idempotents form a Boolean algebra, it suffices to show that an idempotent in $R_* \otimes_k R'_*$ that maps to zero in R_* must have been zero to begin with. However, such an idempotent would belong to the ideal $R_* \otimes_k R'_{>0}$, which easily forces it to be zero. □

Example 8.9. Proposition 8.7 applies in the setting of an artinian ring mapping to its residue field. However, we also know that the map $A \to A/\mathfrak{m}$ for A artinian and $\mathfrak{m}$ a maximal ideal can be obtained as a finite composition of square-zero extensions, so we could also appeal to Corollary 8.12 below.

8.2. Square-zero extensions

Given the classical topological invariance of the étale site, the following is not so surprising.

Proposition 8.10. *If A is an $\mathbf{E}_\infty$-ring and M an A-module, then the natural map $A \to A \oplus M$ (where $A \oplus M$ denotes the trivial "square zero" extension of A by M), induces an isomorphism on fundamental groups.*

This will follow from the following more general statement.

Proposition 8.11. *Let R be an $\mathbf{E}_\infty$-ring with no nontrivial idempotents. Let X be a two-fold loop object in the ∞-category $\mathrm{CAlg}_{R//R}$ of $\mathbf{E}_\infty$-R-algebras over R. Then the map $R \to X$ induces an isomorphism on fundamental groups.*

Note that a one-fold delooping is insufficient, because of the example of cochains on S^1 (cf. Theorem 7.13).

Proof. In view of Corollary 2.40, we see that X has no nontrivial idempotents. Next, observe that we have maps $R \to X \to R$ by assumption, so that, at the level of fundamental groups, we get a section of the map $\pi_1(\mathrm{Mod}(X)) \to \pi_1(\mathrm{Mod}(R))$. In particular, the map $\pi_1(\mathrm{Mod}(X)) \to \pi_1(\mathrm{Mod}(R))$ is surjective. We thus need to show that the map $\pi_1(\mathrm{Mod}(R)) \to \pi_1(\mathrm{Mod}(X))$ (coming from $X \to R$) is also surjective, which we can do via Proposition 8.3.

To see that, suppose $X \simeq \Omega^2 Y$ where Y is an object in $\mathrm{CAlg}_{R//R}$. We want to show that the fundamental group of $\mathrm{Mod}(X)$ is surjected onto by that of $\mathrm{Mod}(R)$. Consider the pull-back diagram of $\mathbf{E}_\infty$-algebras,

$$\begin{tikzcd} \Omega^2 Y \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & \Omega Y \end{tikzcd}.$$

Using Corollary 2.40 again, we find that ΩY has no nontrivial idempotents. Therefore, we have maps

$$\pi_1(\mathrm{Mod}(R)) \to \pi_1(\mathrm{Mod}(R) \times_{\mathrm{Mod}(\Omega Y)} \mathrm{Mod}(R))) \twoheadrightarrow \pi_1(\mathrm{Mod}(\Omega^2 Y)).$$

The second map is a surjection since it comes from a fully faithful inclusion of stable homotopy theories $\mathrm{Mod}(\Omega^2 Y) \subset \mathrm{Mod}(R) \times_{\mathrm{Mod}(\Omega Y)} \mathrm{Mod}(R)$. Since ΩY has no nontrivial idempotents, $\pi_1 \mathrm{Mod}(\Omega Y)$ receives a map from $\pi_1 \mathrm{Mod}(R)$ and we have $\pi_1(\mathrm{Mod}(R) \times_{\mathrm{Mod}(\Omega Y)} \mathrm{Mod}(R)) \simeq \pi_1(\mathrm{Mod}(R)) \sqcup_{\pi_1(\mathrm{Mod}(\Omega Y))} \pi_1(\mathrm{Mod}(R))$. This implies that the first map is a surjection too, as desired. $\square$

We can also consider the behavior of the Galois group under (not necessarily trivial) square-zero extensions. Recall (see [50, sec. 7.4.1]) that these are obtained as follows. Given an $\mathbf{E}_\infty$-ring A and an A-module M, for every map $\phi\colon A \to A \oplus M$ in $\mathrm{CAlg}_{/A}$, we can form the pull-back

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A \arrow[r, "\phi"] & A \oplus M \end{tikzcd},$$

where $0\colon A \to A \oplus M$ is the standard map (informally, $a \mapsto (a, 0)$). The resulting map $A' \to A$ is referred to as a **square-zero extension** of A, by ΩM.

Corollary 8.12. *Notation as above, the map $\pi_1 \operatorname{Mod}(A') \to \pi_1 \operatorname{Mod}(A)$ is a surjection. In fact, $A' \to A$ is universally connected.*

Proof. It suffices to show that $A' \to A$ is universally connected. This follows from the fact that Idem commutes with inverse limits, since one checks that the two maps $A \rightrightarrows A \oplus M$ are universally connected. □

The Galois group is not invariant under arbitrary square-zero extensions. Let $A = \mathbb{C}[x^{\pm 1}]$ where $|x| = 0$ be the free $\mathbf{E}_\infty$-algebra under $\mathbb{C}$ on an invertible degree zero generator (so that A is discrete). Consider the $\mathbb{C}$-derivation $A \to A$ sending a Laurent polynomial $f(x)$ to its derivative. Then, when we form the pull-back

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "0"] \\ A \arrow[r, "f \mapsto (f, f')"] & A \oplus A \end{tikzcd},$$

the pull-back is given by an $\mathbf{E}_\infty$-algebra A' with $\pi_0 A' \simeq \mathbb{C}, \pi_{-1} A' \simeq \mathbb{C}$, and $\pi_i A' = 0$ otherwise. The Galois theory of this $\mathbf{E}_\infty$-ring is algebraic because this $\mathbf{E}_\infty$-ring is necessarily the free $\mathbf{E}_\infty$-ring on a degree -1 generator, or equivalently the trivial square-zero extension $\mathbb{C} \oplus \Omega\mathbb{C}$. So its Galois group is trivial, by Proposition 8.10. However, the map $\mathbb{C} \oplus \Omega\mathbb{C} \to \mathbb{C}[x^{\pm 1}]$ does not induce an isomorphism on Galois groups: that of the former is trivial, while that of the latter is $\widehat{\mathbb{Z}}$.

8.3. Stronger invariance results

We will now prove the main invariance results of the present section.

Theorem 8.13. *Let A be a regular local ring with residue field k and maximal ideal $\mathfrak{m} \subset A$. Let R be an $\mathbf{E}_\infty$-ring under A such that $\mathfrak{m}$ is nilpotent in $\pi_0 R$. Then the natural map*

$$R \to R \otimes_A k,$$

induces an isomorphism on fundamental groups.

Proof. We start by showing that $\pi_1(\mathrm{Mod}(R \otimes_A k)) \to \pi_1(\mathrm{Mod}(R))$ is always a surjection; in other words, we must show that for any $\mathbf{E}_\infty$-algebra R' under R, the natural map

$$\mathrm{Idem}(R') \to \mathrm{Idem}(R' \otimes_R (R \otimes_A k)) \simeq \mathrm{Idem}(R' \otimes_A k) \tag{32}$$

is an isomorphism.

Since k is a perfect A-module, it follows that $R \otimes_A k$ is a perfect R-module. Moreover, $R \otimes_A k$ is faithful as an R-module because tensoring over A with k is faithful on the subcategory of $\mathrm{Mod}(A)$ consisting of A-modules whose homotopy groups are $\mathfrak{m}$-power torsion. It follows that $R \to R \otimes_A k$ is descendable in view of Theorem 3.38. Therefore, the map (32) is an injection. Since the map

$$k \otimes_A k \to k,$$

is descendable, as $k \otimes_A k$ is connective with bounded homotopy groups and π_0 given by k, it follows from Proposition 8.6 (by tensoring this with R) that $\pi_1(\mathrm{Mod}(R \otimes_A k)) \to \pi_1(\mathrm{Mod}(R))$ is a surjection.

Consider the cobar construction

$$R \to R \otimes_A k \rightrightarrows R \otimes_A k \otimes_A k \overset{\to}{\underset{\to}{\to}} \cdots, \tag{33}$$

where all $\mathbf{E}_\infty$-rings in question have no nontrivial idempotents. We will use this and descent theory to complete the proof.

Note that we can make the two maps $\pi_{\leq 1}(\mathrm{Mod}(R \otimes_A k \otimes_A k)) \rightrightarrows \pi_{\leq 1}(\mathrm{Mod}(R \otimes_A k))$ into pointed maps by choosing a basepoint of $\pi_{\leq 1}\mathrm{Mod}(R \otimes_A k)$ and using the multiplication map $R \otimes_A (k \otimes_A k) \to R \otimes_A k$. We conclude (by descent theory and (33)) that $\pi_1(\mathrm{Mod}(R))$ is the coequalizer of the two maps

$$\pi_1(\mathrm{Mod}(R \otimes_A k \otimes_A k)) \rightrightarrows \pi_1(\mathrm{Mod}(R \otimes_A k)),$$

choosing basepoints as above.

We claim here that the multiplication map $R \otimes_A (k \otimes_A k) \to R \otimes_A k$ induces a *surjection* on fundamental groups. Given this, we can construct a diagram

$$\pi_1(\mathrm{Mod}(R \otimes_A k)) \twoheadrightarrow \pi_1(\mathrm{Mod}(R \otimes_A k \otimes_A k)) \rightrightarrows \pi_1(\mathrm{Mod}(R \otimes_A k)),$$

where the two composites are equal. This completes the proof that $\pi_1(\mathrm{Mod}(R)) \simeq \pi_1(\mathrm{Mod}(R \otimes_A k))$, subject to the proof of surjectivity.

To prove surjectivity, we observe that $R \otimes_A k \otimes_A k \to R \otimes_A k$ induces a surjection on fundamental groups, in view of Proposition 8.7, since $k \otimes_A k \to k$ satisfies the conditions of that result; since A is regular, $k \otimes_A k$ is connective and has only finitely many nonzero homotopy groups, so $k \otimes_A k \to k$ admits descent. $\square$

It seems likely that Theorem 8.13 can be strengthened considerably, although we have not succeeded in doing so. For example, one would like to believe that if R is a discrete commutative ring and $I \subset R$ is an ideal of square zero, then given an $\mathbf{E}_\infty$-R-algebra R', the map $R' \to R' \otimes_R R/I$ would induce an isomorphism on fundamental groups. We do *not* know whether this is true in general. By Corollary 8.12, it does induce a surjection at least. The worry is that one does not have good control on the homotopy groups of a relative tensor product of $\mathbf{E}_\infty$-ring spectra; there is a spectral sequence, but the filtration is in the opposite direction than what one wants.

However, in the case when the $\mathbf{E}_\infty$-rings satisfy mild connectivity hypotheses, one can prove the following much stronger result.

Theorem 8.14. *Suppose R is a connective $\mathbf{E}_\infty$-ring with finitely many homotopy groups and $I \subset \pi_0 R$ a nilpotent ideal. Let R' be an $\mathbf{E}_\infty$-R-algebra which is $(-n)$-connective for $n \gg 0$. Then the map $R' \to R' \otimes_R \pi_0(R)/I$ induces an isomorphism on fundamental groups.*

For example, one could take $I = 0$, and the statement is already nontrivial. We need first two lemmas:

Lemma 8.15. *Let A be a connective $\mathbf{E}_\infty$-ring and let A' be an $\mathbf{E}_\infty$-A-algebra which is $(-n)$-connective for $n \gg 0$. Then the natural map*

$$\mathrm{Idem}(A') \to \mathrm{Idem}(A' \otimes_A \pi_0 A) \tag{34}$$

is an isomorphism. In particular, it follows that $\pi_1 \mathrm{Mod}(A' \otimes_A \pi_0 A) \to \pi_1 \mathrm{Mod}(A')$ is a surjection.

Proof. In fact, by a connectivity argument (taking an inverse limit over Postnikov systems), the Adams spectral sequence based on the map $A \to \pi_0 A$ converges for any A-module which is $(-n)$-connective for $n \gg 0$. In other words, we have that

$$A' = \mathrm{Tot}\left(A' \otimes_A \pi_0 A \rightrightarrows A' \otimes_A \pi_0 A \otimes_A \pi_0 A \substack{\to\\ \to\\ \to} \cdots\right),$$

so that, since Idem commutes with limits, we find that $\mathrm{Idem}(A')$ is the equalizer of the two maps $\mathrm{Idem}(A' \otimes_A \pi_0 A) \rightrightarrows \mathrm{Idem}(A' \otimes_A \pi_0 A \otimes_A \pi_0 A)$. In particular, (34) is always injective. Moreover, by the same reasoning, the multiplication map $\pi_0 A \otimes_A \pi_0 A \to \pi_0 A$ (which is also a map from a connective $\mathbf{E}_\infty$-ring to its zeroth Postnikov section) induces an injection

$$\mathrm{Idem}(A' \otimes_A \pi_0 A \otimes_A \pi_0 A) \hookrightarrow \mathrm{Idem}(A' \otimes_A \pi_0 A),$$

which equalizes the two maps $\mathrm{Idem}(A' \otimes_A \pi_0 A) \rightrightarrows \mathrm{Idem}(A' \otimes_A \pi_0 A \otimes_A \pi_0 A)$. It follows that the two maps were equal to begin with, which proves that (34) is an isomorphism. □

Lemma 8.16. *Let A be a discrete $\mathbf{E}_\infty$-ring and $J \subset A$ a square-zero ideal. Then, given any $\mathbf{E}_\infty$-A-algebra A', the natural map $A' \to A' \otimes_A A/J$ induces an isomorphism on idempotents.*

Proof. This is a consequence of Corollary 8.12. □

Proof of Theorem 8.14. Let R_0 be the $\mathbf{E}_\infty$-R-algebra given by $\pi_0(R)$ and consider R_0/I as well. Then we have maps $R \to R_0 \to R_0/I$ and we want to show that, after base-changing to R', the Galois groups are invariant. We will do this in a couple of stages following the proof of Theorem 8.13.

First, suppose $I = 0$. Using descent along $R \to R_0$, one concludes that $\pi_1(\mathrm{Mod}(R'))$ is the coequalizer of the two maps $\pi_1(\mathrm{Mod}(R' \otimes_R R_0 \otimes_R R_0)) \rightrightarrows \pi_1(\mathrm{Mod}(R' \otimes_R R_0))$. We wish to claim that the two maps are equal. Now the multiplication map $R_0 \otimes_R R_0 \to R_0$ satisfies the conditions of Lemma 8.15, so one concludes that the map $\pi_1(\mathrm{Mod}(R' \otimes_R R_0)) \to \pi_1(\mathrm{Mod}(R' \otimes_R R_0 \otimes_R R_0))$ is a surjection, which coequalizes the two maps considered above. Therefore, the two maps are equal.

Next, we need to allow $I \neq 0$. By composition $R \to \tau_{\leq 0} R \to R_0/I$, we may assume that R itself is discrete. We may also assume that I is square-zero. In this case, the map $R \to R_0/I$ satisfies descent and is universally connected by Lemma 8.16. Therefore, we can apply the same argument as above, to write $\pi_1(\mathrm{Mod}(R'))$ as the coequalizer of the two maps $\pi_1(\mathrm{Mod}(R' \otimes_{R_0} R_0/I \otimes_{R_0} R_0/I)) \rightrightarrows \pi_1(\mathrm{Mod}(R' \otimes_{R_0} R_0/I))$. Moreover, these two maps are the same using the surjection $\pi_1(\mathrm{Mod}(R' \otimes_{R_0} R_0/I)) \twoheadrightarrow \pi_1(\mathrm{Mod}(R' \otimes_{R_0} R_0/I \otimes_{R_0} R_0/I))$ given to us by Lemma 8.15 as above. □

8.4. Coconnective rational $\mathbf{E}_\infty$-algebras

Let k be a field of characteristic zero, and let A be an $\mathbf{E}_\infty$-k-algebra such that:

1. $\pi_i A = 0$ for $i > 0$.
2. The map $k \to \pi_0 A$ is an isomorphism.

Following [47], we will call such $\mathbf{E}_\infty$-k-algebras **coconnective;** these are the $\mathbf{E}_\infty$-rings which enter, for instance, in rational homotopy theory. In the following, we will prove:

Theorem 8.17. *If A is a coconnective $\mathbf{E}_\infty$-k-algebra, then every finite cover of A is étale. In particular,*

$$\pi_1 \mathrm{Mod}(A) \simeq \mathrm{Gal}(\overline{k}/k).$$

Proof. We will prove Theorem 8.17 using tools from [47]. Namely, it is a consequence of [47, Proposition 4.3.13] that every coconnective $\mathbf{E}_\infty$-k-algebra A can be obtained as a totalization of a cosimplicial $\mathbf{E}_\infty$-k-algebra $A^\bullet$ where A^i, for each $i \geq 0$, is in the form $k \oplus V[-1]$ where V is a vector space over k, and this is considered as a trivial "square zero" extension. In rational homotopy theory, this assertion is dual to the statement that a connected space can be built as a geometric realization of copies of wedges of S^1.

Now we know from Proposition 8.10 that the Galois groupoid is invariant under trivial square-zero extensions, so it follows that $\pi_1 \operatorname{Mod}(A^i) \simeq \operatorname{Gal}(\overline{k}/k)$, with the finite covers arising only from the étale extensions (or equivalently, finite étale extensions of k itself). It follows easily from this that the finite covers in the ∞-category $\operatorname{Tot}\operatorname{Mod}(A^\bullet)$ are in natural equivalence with the finite étale extensions of k, and this completes the proof, since the ∞-category of perfect A-modules embeds fully faithfully into this totalization. $\square$

Note that the strategy of this proof is to give an *upper bound* for the Galois theory of the $\mathbf{E}_\infty$-ring A by writing it as an inverse limit of square-zero $\mathbf{E}_\infty$-rings. One might, conversely, hope to use Galois groups to prove that $\mathbf{E}_\infty$-rings *cannot* be built as inverse limits of certain simpler ones. For example, in characteristic p, the example of cochain algebras shows that the analog of Theorem 8.17 is false; in particular, one cannot write a given coconnective $\mathbf{E}_\infty$-ring in characteristic p as a totalization of square-zero extensions.

9. Stable module ∞-categories

Let G be a finite group and let k be a perfect field of characteristic $p > 0$, where p divides the order of G. The theory of G-representations in k-vector spaces is significantly more complicated than it would be in characteristic zero because the group ring $k[G]$ is not semisimple: for example, the group G has k-valued cohomology. If one wishes to focus primarily on, for example, the cohomological information specific to characteristic p, then projective $k[G]$-modules are essentially irrelevant and, factoring them out, one has the theory of *stable module categories* reviewed earlier in Example 2.26. One obtains a compactly generated, symmetric monoidal stable ∞-category $\operatorname{St}_G(k)$ obtained as the Ind-completion of the Verdier quotient of $\operatorname{Fun}(BG, \operatorname{Mod}^\omega(k))$ by the thick $\otimes$-ideal of perfect $k[G]$-module spectra.

Our goal in this section is to describe the Galois group of a stable module ∞-category for a finite group. Since any element in the stable module ∞-category can be viewed as an ordinary linear representation of G (for compact objects, finite-dimensional representations) modulo a certain equivalence relation, these results ultimately come down to concrete statements about the tensor structure on linear representations of G modulo projectives.

Our basic result (Theorem 9.9) is that the Galois theory of a stable module category for an *elementary abelian* p-group is entirely algebraic. We will use this, together with the Quillen stratification theory, to obtain a formula for the Galois group of a general stable module ∞-category, and calculate this in special cases.

9.1. The case of $\mathbb{Z}/p$

Our first goal is to determine the Galois group of $\mathrm{St}_V(k)$ when V is *elementary abelian*, i.e. of the form $(\mathbb{Z}/p)^n$. In this case, recall (Theorem 2.30) that $\mathrm{St}_V(k)$ is symmetric monoidally equivalent to the ∞-category of modules over the Tate construction k^{tV}. We will start by considering the case $V = \mathbb{Z}/p$.

Proposition 9.1. *Let k be a field of characteristic $p > 0$. The Galois theory of the Tate construction $k^{t\mathbb{Z}/p}$ is algebraic.*

Proof. Without loss of generality, we can assume k perfect. In the case $p = 2$, $k^{t\mathbb{Z}/2}$ has homotopy groups given by

$$k^{t\mathbb{Z}/2} \simeq k[t^{\pm 1}],$$

where $|t| = -1$. A (simpler) version of Proposition 6.27 shows that any Galois extension of $k^{t\mathbb{Z}/2}$ is étale, since π_0 satisfies a perfect Künneth isomorphism for $k^{t\mathbb{Z}/2}$-modules and every module over $k^{t\mathbb{Z}/2}$ is algebraically flat. It follows that if $k^{t\mathbb{Z}/2} \to R$ is G-Galois, for G a finite group, then $\pi_0 R$ is a finite G-Galois extension of k.

The case of an odd prime is slightly more subtle. In this case, we have

$$k^{t\mathbb{Z}/p} \simeq k[t^{\pm 1}] \otimes_k E(u), \quad |t| = -2, |u| = -1,$$

so that we get a tensor product of a Laurent series ring and an exterior algebra. Since the homotopy ring is no longer regular, we will have to show that any G-Galois extension of $k^{t\mathbb{Z}/p}$ is flat at the level of homotopy groups. We can do this by comparing with the Tate construction $W(k)^{t\mathbb{Z}/p}$, where $W(k)$ is the ring of Witt vectors on k and $\mathbb{Z}/p$ acts trivially on $W(k)$. The $\mathbf{E}_\infty$-ring $W(k)^{t\mathbb{Z}/p}$ has homotopy groups given by

$$\pi_* W(k)^{t\mathbb{Z}/p} \simeq k[t^{\pm 1}], \quad |t| = 2,$$

and the $\mathbf{E}_\infty$-ring that we are interested in is given by

$$k^{t\mathbb{Z}/p} \simeq W(k)^{t\mathbb{Z}/p} \otimes_{W(k)} k.$$

Now Proposition 6.27 tells us that the Galois theory of $W(k)^{t\mathbb{Z}/p}$ is algebraic, and the invariance result Theorem 8.13 enables us to conclude the same for $k^{t\mathbb{Z}/p}$. $\square$

9.2. Tate spectra for elementary abelian subgroups

Let k be a field of characteristic p. We know that $k^{t\mathbb{Z}/p}$ has homotopy groups given by a tensor product of an exterior and Laurent series algebra on generators in degrees $-1, -2$, respectively. For an elementary abelian p-group of higher rank, the picture is somewhat more complicated: the homotopy ring behaves irregularly (with entirely square-zero material in positive homotopy groups), but the Tate construction is still built up from a diagram of $\mathbf{E}_\infty$-rings whose homotopy rings come from tensor products of polynomial (or Laurent series) rings and exterior algebras. This diagram roughly lives over $\mathbb{P}^{n-1}_k$ where n is the rank of the given elementary abelian p-group, and the stable module ∞-category $\mathrm{St}_{(\mathbb{Z}/p)^n}(k)$ can be described as quasi-coherent sheaves on a derived version of projective space (Theorem 9.2). In this subsection, we will review this picture, which will be useful when we describe the Galois groups in the next section.

We consider the case of $p > 2$, and leave the minor modifications for $p = 2$ to the reader. Fix an elementary abelian p-group $V = (\mathbb{Z}/p)^n$, and let $V_k = V \otimes_{\mathbb{F}_p} k$. Consider first the homotopy fixed points k^{hV}, whose homotopy ring is given by

$$\pi_*(k^{hV}) \simeq E(V_k^\vee) \otimes \mathrm{Sym}^*(V_k^\vee),$$

where the exterior copy of $V_k^\vee$ is concentrated in degree -1, and the polynomial copy of $V_k^\vee$ is concentrated in degree -2. For each nonzero homogeneous polynomial $f \in \mathrm{Sym}^*(V_k^\vee)$, we can form the localization $k^{hV}[f^{-1}]$, whose degree zero part *modulo nilpotents* is given by the localization $\mathrm{Sym}^*(V_k^\vee)_{(f)}$ (i.e., the degree zero part of the localization $\mathrm{Sym}^*(V_k^\vee)[f^{-1}]$). There is also a small nilpotent part that comes from the evenly graded portion of the exterior algebra. In particular, we find, using natural maps between localizations:

1. For every Zariski open *affine* subset $U \subset \mathbb{P}(V_k^\vee)$, we obtain a (canonically associated) $\mathbf{E}_\infty$-ring $\mathcal{O}^{\mathrm{top}}(U)$ by localizing k^{hV} at an appropriate homogeneous form. Precisely, U is given as the complement to the zero locus of a homogeneous form $f \in \mathrm{Sym}^*(V_k^\vee)$, and we invert f in k^{hV}: $\mathcal{O}^{\mathrm{top}}(U) = k^{hV}[f^{-1}]$.
2. For every inclusion $U \subset U'$ of Zariski open affines, we obtain a map of $\mathbf{E}_\infty$-algebras (under k^{hV}) $\mathcal{O}^{\mathrm{top}}(U') \to \mathcal{O}^{\mathrm{top}}(U)$. These maps are canonical; $\mathcal{O}^{\mathrm{top}}(U'), \mathcal{O}^{\mathrm{top}}(U)$ are localizations of k^{hV} and $\mathcal{O}^{\mathrm{top}}(U)$ has more elements inverted.
3. For each $U \subset \mathbb{P}(V_k^\vee)$, the $\mathbf{E}_\infty$-ring $\mathcal{O}^{\mathrm{top}}(U)$ has a unit in degree two. The ring $\pi_0(\mathcal{O}^{\mathrm{top}}(U))$ is canonically an algebra over the (algebraic) ring of functions $\mathcal{O}_{\mathrm{alg}}(U)$ on $U \subset \mathbb{P}(V_k^\vee)$, and is a tensor product of $\mathcal{O}_{\mathrm{alg}}(U)$ with the even components of an exterior algebra over k.
4. We have natural isomorphisms of sheaves of graded $\mathcal{O}_{\mathrm{alg}}$-modules

$$\pi_*(\mathcal{O}^{\mathrm{top}}) \simeq E(V_k^\vee) \otimes_{\mathcal{O}_{\mathrm{alg}}} \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(r),$$

where $\mathcal{O}(1)$ is the usual hyperplane bundle on $\mathbb{P}(V_k^\vee)$ and $\mathcal{O}(r) \simeq \mathcal{O}(1)^{\otimes r}$ is concentrated in degree $-2r$. The generators of the exterior algebra $E(V_k^\vee)$ are in degree -1.

It follows that the homotopy groups $\pi_*(\mathcal{O}^{\mathrm{top}}(U))$ for $U \subset \mathbb{P}(V_k^\vee)$ fit together into *quasi-coherent sheaves* on the site of affine Zariski opens $U \subset \mathbb{P}(V_k^\vee)$ and inclusions between them. In particular, we can view the association $U \mapsto \mathcal{O}^{\mathrm{top}}(U)$ as defining a *sheaf* of $\mathbf{E}_\infty$-ring spectra (under k, or even under k^{hV}) over the Zariski site of $\mathbb{P}(V_k^\vee)$, whose sections over an affine open $U \subset \mathbb{P}(V_k^\vee)$ are given by $\mathcal{O}^{\mathrm{top}}(U)$.

We will now describe our basic comparison result. Since $\mathcal{O}^{\mathrm{top}}$ is a sheaf of $\mathbf{E}_\infty$-algebras under k^{hV}, we obtain a symmetric monoidal, colimit-preserving functor

$$\mathrm{Mod}(k^{hV}) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}),$$

into the ∞-category $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ of *quasi-coherent* $\mathcal{O}^{\mathrm{top}}$*-modules*, defined as the homotopy limit

$$\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}) = \varprojlim_{U \subset \mathbb{P}(V_k^\vee)} \mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(U)),$$

where the homotopy limit is taken over all open affine subsets of $\mathbb{P}(V_k^\vee)$. Restricting to $\mathrm{Mod}^\omega(k^{hV}) \simeq \mathrm{Fun}(BV, \mathrm{Mod}^\omega(k))$, we obtain a symmetric monoidal exact functor

$$\mathrm{Fun}(BV, \mathrm{Mod}^\omega(k)) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}).$$

We observe that the standard representation of V, as an object of the former, is sent to zero in $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$. In fact, the standard representation of V corresponds to a k^{hV}-module with only one nonvanishing homotopy group, and it therefore vanishes under the types of *periodic* localization that one takes in order to form $\mathcal{O}^{\mathrm{top}}(U)$ for $U \subset \mathbb{P}(V_k^\vee)$ an open affine. Using the universal property of the stable module ∞-category, we obtain a factorization

$$\mathrm{Fun}(BV, \mathrm{Mod}^\omega(k)) \to \mathrm{St}_V(k) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}),$$

where the functor $\mathrm{St}_V(k) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ is symmetric monoidal and colimit-preserving.

Theorem 9.2. *The functor* $\mathrm{Mod}(k^{tV}) \simeq \mathrm{St}_V(k) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ *is an equivalence of symmetric monoidal* ∞*-categories.*

Proof. We start by observing that, by construction of the Verdier quotient (Definition 2.10), the stable module ∞-category $\mathrm{St}_V(k)$ is obtained as a *localization* of $\mathrm{Mod}(k^{hV}) \simeq \mathrm{Ind}(\mathrm{Fun}(BV, \mathrm{Mod}^\omega(k)))$, and in particular k^{tV} is a localization of the $\mathbf{E}_\infty$-ring k^{hV}.

By construction, k^{tV} is the localization of k^{hV} at the map of k^{hV}-modules $M \to 0$, where M is the k^{hV}-module corresponding to the standard representation of V. So, in particular, the localization functor

$$\mathrm{Mod}(k^{hV}) \to \mathrm{Mod}(k^{tV}),$$

given by tensoring up, has a fully faithful right adjoint which embeds $\mathrm{Mod}(k^{tV})$ as the subcategory of all k^{hV}-modules N such that $\mathrm{Hom}_{\mathrm{Mod}(k^{hV})}(M, N)$ is contractible. If we write $e_1, \dots, e_n \in \pi_{-2}(k^{hV})$ for polynomial generators of k^{hV}, then $k^{hV}/(e_1, \dots, e_n) \in \mathrm{Mod}^\omega(k^{hV})$ generates the same thick subcategory as M, as we observed in the discussion immediately preceding Definition 4.13. So, the k^{tV}-modules are precisely the k^{hV}-modules N such that

$$N/(e_1, \dots, e_n)N \simeq 0 \in \mathrm{Mod}(k^{hV}),$$

using self-duality of $k^{hV}/(e_1, \dots, e_n)$.

Now, we have a morphism of $\mathbf{E}_\infty$-rings

$$k^{hV} \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}}), \tag{35}$$

and our first task is to show that this morphism induces an equivalence $k^{tV} \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})$. Observe first that, after inverting any of $e_1, \dots, e_n \in \pi_{-2}(k^{hV})$, (35) becomes an equivalence since we already know what $\mathcal{O}^{\mathrm{top}}$ looks like on the basic open affines; we also know that taking global sections over $\mathbb{P}(V_k^\vee)$ is a finite homotopy limit and thus commutes with arbitrary homotopy colimits. However, we also know that $k^{hV}/(e_1, \dots, e_n)$ maps to the zero $\mathcal{O}^{\mathrm{top}}$-module since, on every basic open affine of $\mathbb{P}(V_k^\vee)$, one of the $\{e_i\}$ is always invertible. Thus we get a map $k^{tV} \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})$ of k^{hV}-modules with the dual properties:

1. Both modules smash to zero with $k^{hV}/(e_1, \dots, e_n)$.
2. The map induces an equivalence after inverting each e_i, $1 \leq i \leq n$.

By a formal argument, it now follows that $k^{tV} \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})$ is an equivalence to begin with. In fact, we show that, for each i, the map

$$k^{tV}/(e_1, \dots, e_i) \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})/(e_1, \dots, e_i) \tag{36}$$

is an equivalence by *descending* induction on i. For $i = n$, both sides are contractible. If we are given that (36) is an equivalence, then the map $k^{tV}/(e_1, \dots, e_{i-1}) \to \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})/(e_1, \dots, e_{i-1})$ has the property that it becomes an equivalence after either inverting e_i (by the second property above) or by smashing with $k^{hV}/(e_i)$ (by the inductive hypothesis); it thus has to be an equivalence in turn. This completes the inductive step and the proof that $k^{tV} \simeq \Gamma(\mathbb{P}(V_k^\vee), \mathcal{O}^{\mathrm{top}})$.

All in all, we have shown that the functor

$$\mathrm{Mod}(k^{tV}) \simeq \mathrm{St}_V(k) \to \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$$

is *fully faithful*. To complete the proof of Theorem 9.2, we need to show that the global sections functor is conservative on $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$. However, if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ has the property that $\Gamma(\mathbb{P}(V_k^\vee), \mathcal{F})$ is contractible, then the same holds for $\mathcal{F}[e_i^{-1}]$. By analyzing the descent spectral sequence, it follows that the global sections of $\mathcal{F}[e_i^{-1}]$ are precisely the sections of $\mathcal{F}$ over the ith basic open affine chart of $\mathbb{P}(V_k^\vee)$. Thus, if $\Gamma(\mathbb{P}(V_k^\vee), \mathcal{F})$ is contractible, then $\mathcal{F}$ has contractible sections over each of the basic open affines, and is thus contractible to begin with. (This argument is essentially the ampleness of $\mathcal{O}(1)$.) $\square$

9.3. G-Galois extensions for topological groups

Our next goal is to calculate the Galois group for k^{tV} for any elementary abelian p-group V. In the case of rank one, we had a trick for approaching the Galois group. Although k^{tV} was not even periodic, there was a good integral model (namely, $W(k)^{tV}$) which was related to k^{tV} by reducing mod p, so that we could use an invariance property to reduce to the (much easier) $\mathbf{E}_\infty$-ring $W(k)^{tV}$.

When the rank of V is greater than one, both these tricks break down. There is no longer a comparable integral model of an $\mathbf{E}_\infty$-ring such as $k^{h\mathbb{Z}/p} \otimes k^{t\mathbb{Z}/p}$, as far as we know. Our strategy is based instead on a comparison with the Tate spectra for *tori*, which are much more accessible. To interpolate between the Tate spectra for tori and the Tate spectra for elementary abelian p-groups, we will need a bit of the theory of Galois extensions for topological groups, which was considered in [71]. We will describe the associated theory of descent in this section. We refer to [53] for further applications of these ideas to the Picard group and the classification of localizing subcategories of the stable module category (recovering older results), as well as a discussion of how this formulation of G-Galois extensions relates to that of Rognes [71] (who uses a definition similar to Definition 6.12).

Definition 9.3. Fix a topological group G which has the homotopy type of a finite CW complex (e.g., a compact Lie group). Let R be an $\mathbf{E}_\infty$-ring and let R' be an $\mathbf{E}_\infty$-R-algebra with an action of G (in the ∞-category of $\mathbf{E}_\infty$-R-algebras).

We will say that R' is a faithful G**-Galois extension** of R if there exists a descendable $\mathbf{E}_\infty$-R-algebra R'' such that we have an equivalence of $\mathbf{E}_\infty$-R''-algebras

$$R' \otimes_R R'' \simeq C^*(G; R''),$$

which is compatible with the G-action.

Note that the cochain $\mathbf{E}_\infty$-ring $C^*(G; R'')$ is the "coinduced" G-action on an R''-module. It follows in particular that the natural map $R \to R'^{hG}$ is an equivalence, and is so universally; for any $\widetilde{R} \in \mathrm{CAlg}_{R/}$, the natural map $\widetilde{R} \to (R' \otimes_R \widetilde{R})^{hG}$ is an equivalence. Moreover, R' is perfect as an R-module, since this can be checked locally (after base-change to R'') and G has the homotopy type of a finite CW complex. It follows from general properties of descendable morphisms that faithful G-Galois extensions are preserved under base-change.

We will need the following version of classical Galois descent, which has been independently considered in various forms by several authors, for instance [32,28,55,6].

Theorem 9.4. *Let G be a topological group of the homotopy type of a finite CW complex, and let $R \to R'$ be a faithful G-Galois extension of $\mathbf{E}_\infty$-rings. The natural functor*

$$\mathrm{Mod}(R) \to \mathrm{Mod}(R')^{hG}, \tag{37}$$

is an equivalence of ∞-categories.

The "natural functor" comes from the expression $R \simeq R'^{hG}$; the G-action on R' induces one on the symmetric monoidal ∞-category $\mathrm{Mod}(R')$. In particular, we get a *fully faithful* embedding $\mathrm{Mod}^\omega(R) \to \mathrm{Mod}(R')^{hG}$ for free.

Proof. Suppose first that $R' \simeq C^*(G; R)$ with the G-action coming from the translation action of G on itself. Then, we have a fully faithful, colimit-preserving embedding

$$\mathrm{Mod}(R') \subset \mathrm{Loc}_G(\mathrm{Mod}(R)),$$

as we saw in Section 7.2. The G-action here on $\mathrm{Loc}_G(\mathrm{Mod}(R))$ comes from the translation action again. Taking homotopy fixed points, we get

$$\mathrm{Mod}(R')^{hG} \subset \mathrm{Loc}_{G_{hG}}(\mathrm{Mod}(R)) \simeq \mathrm{Loc}_*(\mathrm{Mod}(R)) \simeq \mathrm{Mod}(R), \tag{38}$$

because the construction $X \mapsto \mathrm{Loc}_X(\mathrm{Mod}(R))$ sends homotopy colimits in X to homotopy limits of stable ∞-categories. The natural functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R')^{hG}$ now composes all the way over in (38) to the identity, so that it must have been an equivalence to begin with since all the maps in (38) are fully faithful.

Now suppose $R \to R'$ is a general G-Galois extension, so that there exists a descendable $\mathbf{E}_\infty$-R-algebra T such that $R \to R'$ becomes a trivial Galois extension after base-change along $R \to T$. The functor (37) is a functor of R-linear ∞-categories so, to show that it is an equivalence, it suffices to show that (37) induces an equivalence after applying the construction $\otimes_{\mathrm{Mod}(R)} \mathrm{Mod}(T)$: that is, after considering T-module objects in each ∞-category (cf. Proposition 3.45). In other words, to show that (37) is an equivalence, it suffices to tensor up and show that

$$\begin{aligned}\mathrm{Mod}(T) &\to (\mathrm{Mod}(R'))^{hG} \otimes_{\mathrm{Mod}(R)} \mathrm{Mod}(T)\\ &\simeq \left(\mathrm{Mod}(R') \otimes_{\mathrm{Mod}(R)} \mathrm{Mod}(T)\right)^{hG} \simeq \mathrm{Mod}(C^*(G;T))^{hG},\end{aligned}$$

is an equivalence of ∞-categories, which we just proved. □

It follows in particular that whenever we have a G-Galois extension in the above sense, for G a *topological* group then we can relate the fundamental groups of R and R'. In fact, we have, in view of Theorem 9.4,

$$\mathrm{CAlg}^{\mathrm{cov}}(R) \simeq \mathrm{CAlg}^{\mathrm{cov}}(R')^{hG}.$$

Using the Galois correspondence, it follows that there is a G-action on the object $\pi_{\leq 1}\operatorname{Mod}(R') \in \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$, and the homotopy *quotient* in $\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ by this G-action is precisely the fundamental groupoid of $\operatorname{Mod}(R)$, i.e.,

$$\pi_{\leq 1}\operatorname{Mod}(R) \simeq (\pi_{\leq 1}\operatorname{Mod}(R'))_{hG} \in \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}}).$$

We now describe homotopy orbits in $\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ in the case that will be of interest. Let $X \in \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ be a *connected* profinite groupoid and consider an action of a *connected* topological group G on X.

Proposition 9.5. *To give an action of G on $X \in \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}$ is equivalent to giving a homomorphism of groups $\pi_1(G) \to \pi_1(X)$ whose image is contained in the center of $\pi_1(X)$. In other words, the 2-category $\operatorname{Fun}(BG, \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0})$ can be described as follows:*

1. *Objects are profinite groups F together with maps $\phi\colon \pi_1(G) \to F$ whose image is contained in the center of F.*
2. *1-morphisms between pairs (F, ϕ) and (F', ϕ') are continuous homomorphisms $\psi\colon F \to F'$ such that the diagram*

$$\begin{tikzcd} \pi_1(G) \arrow[d, "\phi"'] \arrow[dr, "\phi'"] & \\ F \arrow[r, "\psi"] & F' \end{tikzcd} ,$$

   commutes.
3. *2-morphisms are given by conjugacies between homomorphisms.*

In particular, the forgetful functor $\operatorname{Fun}(BG, \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}) \to \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}$ induces fully faithful maps on the hom-spaces.

Proof. In order to give an action of $X \in \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}$, we need to construct a map of $\mathbf{E}_1$-spaces $G \to \operatorname{Aut}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}}(X)$, where $\operatorname{Aut}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}}(X)$ is the automorphism $\mathbf{E}_1$-algebra of X. Since, however, G is connected, it is equivalent to specifying a map of $\mathbf{E}_1$-algebras (or loop spaces) into $\tau_{\geq 1}\operatorname{Aut}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}}(X)$. However, we know from Proposition 5.49 that $\tau_{\geq 1}\operatorname{Aut}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}}(X)$ is precisely a $K(Z(\pi_1(X)), 1)$, so the space of $\mathbf{E}_1$-maps as above is simply the *set* of homomorphisms $\pi_1(G) \to Z(\pi_1(X))$.

Finally, we need to understand the mapping spaces in $\operatorname{Fun}(BG, \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0})$. Consider two connected profinite groupoids X, Y with G-actions. The space of maps $X \to Y$ in $\operatorname{Fun}(BG, \operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}}))$ is equivalent to the homotopy fixed points $\operatorname{Hom}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(X, Y)^{hG}$, where $\operatorname{Hom}_{\operatorname{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(X, Y)$ is a groupoid as discussed earlier. In general, given any groupoid $\mathscr{G}$ with an action of G, the functor $\mathscr{G}^{hG} \to \mathscr{G}$ is fully faithful. The action of G means that every element in $\pi_1(G)$ determines a natural transformation from the identity to itself on $\mathscr{G}$, and the homotopy fixed points pick out the full subcategory of $\mathscr{G}$ spanned by elements on which that natural transformation is the identity (for any $\gamma \in \pi_1(G)$).

In the case of $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(X, Y)$, the objects are continuous homomorphisms $\psi\colon \pi_1 X \to \pi_1 Y$, and the morphisms between objects are conjugacies. For $\gamma \in \pi_1(G)$, we obtain elements $\gamma_x \in \pi_1(X)$ and $\gamma_y \in \pi_1(Y)$ (in view of the G-action on X, Y), and the action of γ on $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})}(X, Y)$ at the homomorphism ψ is given by the element $\psi(\gamma_x)\psi(\gamma_y)^{-1}$, which determines a self-conjugacy from ψ to itself. To say that this self-conjugacy is the identity for any γ, i.e., that the map is G-equivariant (which here is a *condition* instead of extra data), is precisely the second description of the 1-morphisms. $\square$

Remark 9.6. The above argument would have worked in any $(2,1)$-category where we could write down the π_1 of the automorphism $\mathbf{E}_1$-algebra easily.

In particular, if G acts trivially on $Y \in \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})^{\geq 0}$, then to give a map $X \to Y$ is equivalent to giving a map in $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ which annihilates the image of $\pi_1(G) \to \pi_1(X)$. It follows that the *homotopy quotients* X_{hG} in $\mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$ can be described by taking the quotient of $\pi_1 X$ by the closure of the image of $\pi_1(G)$: this is the universal profinite groupoid with a trivial G-action to which X maps.

Putting all of this together, we find:

Corollary 9.7. *Let G be a connected topological group of the homotopy type of a finite CW complex, and let $R \to R'$ be a faithful G-Galois extension. Then we have an exact sequence of profinite groups*

$$\widehat{\pi_1 G} \to \pi_1 \mathrm{Mod}(R') \to \pi_1 \mathrm{Mod}(R) \to 1. \tag{39}$$

Remark 9.8. Throughout this section, we shall be somewhat cavalier about the use of basepoints, since we will be working with connected profinite groupoids.

9.4. The general elementary abelian case

Let V be an elementary abelian p-group and let k be a field of characteristic p. In this section, we will prove our main result that the Galois theory of k^{tV} is algebraic. In order to do this, we will use the presentation in Theorem 9.2 of $\mathrm{Mod}(k^{tV})$ via quasi-coherent sheaves on a "derived" version of $\mathbb{P}(V_k^\vee)$. Any G-Galois extension of k^{tV} clearly gives a G-Galois extension of $\mathcal{O}^{\mathrm{top}}(U)$ for any $U \subset \mathbb{P}(V_k^\vee)$ by base-change. Conversely, the affineness result Theorem 9.2 implies that to give a G-Galois extension of k^{tV} is equivalent to giving G-Galois extensions of $\mathcal{O}^{\mathrm{top}}(U)$ for $U \subset \mathbb{P}(V_k^\vee)$ affine together with the requisite compatibilities. This would be doable if $\mathcal{O}^{\mathrm{top}}(U)$ was even periodic with regular π_0, although the exterior generators present an obstacle. Nonetheless, by a careful comparison with the analog for *tori*, we will prove:

Theorem 9.9. *Let V be an elementary abelian p-group. If k is a field of characteristic p, all finite coverings of k^{tV} are étale, so $\pi_1(\mathrm{Mod}(k^{tV})) \simeq \mathrm{Gal}(k^{\mathrm{sep}}/k)$.*

Proof. Since projective space is (geometrically) simply connected, it suffices to show that the Galois theory of

$$k^{t\mathbb{Z}/p} \otimes_k k^{h(\mathbb{Z}/p)^n} \simeq k^{t\mathbb{Z}/p} \otimes_k C^*(B(\mathbb{Z}/p)^n; k),$$

for $n > 0$, is algebraic, and thus given by the (algebraic) étale fundamental group of the corresponding affine open cell in $\mathbb{P}(V_k^\vee)$. These $\mathbf{E}_\infty$-rings are the $\mathcal{O}^{\mathrm{top}}(U)$ for $U \subset \mathbb{P}(V_k^\vee)$ the basic open affines of projective space. It will follow that a faithful Galois extension of k^{tV} is locally algebraically étale over $\mathbb{P}(V_k^\vee)$.

For this, we will use the fibration sequence

$$S^1 \to B\mathbb{Z}/p \to BS^1,$$

induced by the inclusion $\mathbb{Z}/p \subset S^1$ with quotient S^1. This is a principal S^1-bundle and we find in particular an S^1-action on $C^*(B\mathbb{Z}/p; k)$ such that

$$C^*(BS^1; k) \simeq C^*(B\mathbb{Z}/p; k)^{hS^1} \tag{40}$$

In fact, the map $C^*(BS^1; k) \to C^*(B\mathbb{Z}/p; k)$ is a faithful S^1-Galois extension (in the sense of Definition 9.3): by the Eilenberg–Moore spectral sequence, and the fiber square

$$\begin{tikzcd} B\mathbb{Z}/p \times S^1 \arrow[r] \arrow[d] & B\mathbb{Z}/p \arrow[d] \\ B\mathbb{Z}/p \arrow[r] & BS^1 \end{tikzcd},$$

expressing the earlier claim that $B\mathbb{Z}/p \to BS^1$ is an S^1-torsor, it follows that

$$C^*(B\mathbb{Z}/p; k) \otimes_{C^*(BS^1;k)} C^*(B(\mathbb{Z}/p); k) \simeq C^*(S^1; k) \otimes_k C^*(B\mathbb{Z}/p; k),$$

with the "coinduced" S^1-action on the right. Moreover, $C^*(BS^1; k) \to C^*(B\mathbb{Z}/p; k)$ is descendable: in fact, a look at homotopy groups shows that the latter is a wedge of the former and its shift.

Let $\mathbb{T}^n \simeq (S^1)^n$ be the n-torus, which contains $(\mathbb{Z}/p)^n$ as a subgroup. Similarly, we find that there is a $\mathbb{T}^n$-action on $C^*(B(\mathbb{Z}/p)^n; k)$ in the ∞-category of $C^*(B\mathbb{T}^n; k)$-algebras which exhibits $C^*(B(\mathbb{Z}/p)^n; k)$ as a faithful $\mathbb{T}^n$-Galois extension of $C^*(B(\mathbb{Z}/p)^n; k)$. We can now apply a bit of descent theory. Fix any $C^*(B\mathbb{T}^n; k)$-algebra R, and let $R' \simeq$

$R \otimes_{C^*(B\mathbb{T}^n;k)} C^*(B(\mathbb{Z}/p)^n;k)$. Since R' is a faithful $\mathbb{T}^n$-Galois extension of R, we have a (natural) exact sequence given by Corollary 9.7:

$$\widehat{\mathbb{Z}}^n \to \pi_1(\mathrm{Mod}(R')) \to \pi_1(\mathrm{Mod}(R)) \to 1. \tag{41}$$

Finally, we may attack the problem of determining the Galois theory of $k^{t\mathbb{Z}/p} \otimes_k k^{h(\mathbb{Z}/p)^n}$ where $n > 0$. We have

$$\pi_* C^*(B(\mathbb{Z}/p)^{n+1};k) \simeq k[e_0, e_1, \dots, e_n] \otimes E(\epsilon_0, \dots, \epsilon_n), \quad |e_i| = -2,\ |\epsilon_i| = -1.$$

Our goal is to determine the Galois theory of the localization $k^{t\mathbb{Z}/p} \otimes_k k^{h(\mathbb{Z}/p)^n} \simeq C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]$. Now, we also have

$$\pi_* C^*(B\mathbb{T}^{n+1};k) \simeq k[e_0, \dots, e_n], \quad |e_i| = -2,$$

and the map $C^*(B\mathbb{T}^{n+1};k) \to C^*(B(\mathbb{Z}/p)^{n+1};k)$ sends the $\{e_i\}$ to the $\{e_i\}$. This map is a faithful $\mathbb{T}^{n+1}$-Galois extension. As we did for $C^*(B(\mathbb{Z}/p)^{n+1};k)$, consider the localization $C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}]$, whose homotopy groups are given by

$$\pi_* C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}] \simeq k[e_0^{\pm 1}, f_1, \dots, f_n], \quad |f_i| = 0, \tag{42}$$

where for $i \geq 1$, $f_i = e_i/e_0$. In particular, the Galois theory of $C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}]$ is algebraic thanks to Theorem 6.29, and by (41), we have an exact sequence

$$\widehat{\mathbb{Z}}^{n+1} \to \pi_1 \mathrm{Mod}(C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]) \to \pi_1 \mathrm{Mod}(C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}]) \to 1 \tag{43}$$

Our argument will be that the first map is necessarily *zero*, which will show that the Galois theory of $C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]$ is algebraic as desired. In order to do this, we will use a naturality argument.

We can form the completion

$$A = C^*(\widehat{B\mathbb{T}^{n+1};k})[e_0^{-1}]_{(f_1,\dots,f_n)},$$

at the ideal $(f_1, \dots, f_n)$, whose homotopy groups now become the tensor product of the Laurent series ring $k[e_0^{\pm 1}]$ together with a *power series* ring $k[[f_1, \dots, f_n]]$. We will prove:

Lemma 9.10. *The Galois theory of* $A' \stackrel{\text{def}}{=} A \otimes_{C^*(B\mathbb{T}^{n+1};k)} C^*(B(\mathbb{Z}/p)^{n+1};k)$ *is entirely algebraic (and, in particular, that of* A*).*

Proof. The $\mathbf{E}_\infty$-ring $A' = A \otimes_{C^*(B\mathbb{T}^{n+1};k)} C^*(B(\mathbb{Z}/p)^{n+1};k)$, which by definition is the $\mathbf{E}_\infty$-ring obtained from $C^*(B(\mathbb{Z}/p)^{n+1};k)$ obtained by inverting the generator e_0 and completing with respect to the ideal $(f_1, \dots, f_n)$, admits another description: it is the homotopy fixed points $(k^{t\mathbb{Z}/p})^{h(\mathbb{Z}/p)^n}$ where $(\mathbb{Z}/p)^n$ acts trivially.[7] Since we have computed the Galois theory of $k^{t\mathbb{Z}/p}$ and found it to be algebraic in Proposition 9.1, this, together with Example 7.19, implies the claim. □

Finally, consider the diagram obtained from the faithful $\mathbb{T}^{n+1}$-Galois extensions $C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}] \to C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]$ and $A \to A'$,

$$\begin{tikzcd}
\widehat{\mathbb{Z}}^{n+1} \arrow[r] \arrow[d] & \pi_1(\mathrm{Mod}(A')) \arrow[r] \arrow[d] & \pi_1(\mathrm{Mod}(A)) \arrow[r] \arrow[d] & 1 \\
\widehat{\mathbb{Z}}^{n+1} \arrow[r] & \pi_1 \mathrm{Mod}(C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]) \arrow[r] & \pi_1 \mathrm{Mod}(C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}]) \arrow[r] & 1
\end{tikzcd}.$$

In the top row, in view of Lemma 9.10, the map out of $\widehat{\mathbb{Z}}^{n+1}$ must be zero. It follows that the same must hold in the bottom row. In other words, the Galois theory of $C^*(B(\mathbb{Z}/p)^{n+1};k)[e_0^{-1}]$ is equivalent to the (algebraic) Galois theory of $C^*(B\mathbb{T}^{n+1};k)[e_0^{-1}]$. As we saw at the beginning, this is precisely the step we needed to see that the Galois theory of the Tate construction k^{tV} is "locally" algebraic over $\mathbb{P}(V_k^\vee)$, and this completes the proof of Theorem 9.9. □

Remark 9.11. This argument leaves open a natural question: is the Galois theory of a general localization $C^*(B(\mathbb{Z}/p)^{n+1};k)[f^{-1}]$ algebraic?

9.5. General finite groups

Let G be any finite group. In this section, we will put together the various pieces (in particular, Theorem 9.9 and Quillen stratification theory) to give a description of the Galois group of the stable module ∞-category $\mathrm{St}_G(k)$ over a field k of characteristic $p > 0$.

For each subgroup $H \subset G$, recall the commutative algebra object $A_H = \prod_{G/H} k \in \mathrm{CAlg}(\mathrm{Mod}_G(k))$. A_H has the property that $\mathrm{Mod}_{\mathrm{Mod}_G(k)}(A_H) \simeq \mathrm{Mod}_H(k)$, and the adjunction $\mathrm{Mod}_G(k) \rightleftarrows \mathrm{Mod}_{\mathrm{Mod}_G(k)}(A_H)$ whose left adjoint tensors with A_H can be identified with *restriction* to the subgroup H. We will need an analog of this at the level of stable module categories. We refer to [59, sec. 5.3] for a discussion of these types of equivalences and for a proof of a general result including this in the ∞-categorical setting.

Proposition 9.12. *(See Balmer [5].) Let $\mathscr{A}_H \in \mathrm{CAlg}(\mathrm{St}_G(k))$ be the image of A_H in the stable module ∞-category. Then we can identify $\mathrm{Mod}_{\mathscr{A}_H}(\mathrm{St}_G(k)) \simeq \mathrm{St}_H(k)$ and we can identify the adjunction $\mathrm{St}_G(k) \rightleftarrows \mathrm{Mod}_{\mathscr{A}_H}(\mathrm{St}_G(k))$ with the restriction-coinduction adjunction $\mathrm{St}_G(k) \rightleftarrows \mathrm{St}_H(k)$.*

[7] In general, the formation of homotopy fixed points do not commute with localization from $k^{h\mathbb{Z}/p}$ to $k^{t\mathbb{Z}/p}$: the failure is precisely measured by the need to take the completion.

Proposition 9.12 suggests that we can perform a type of descent in stable module ∞-categories by restricting to appropriate subgroups. In particular, we can hope to reduce the calculation of certain invariants in $\mathrm{St}_G(k)$ to those of $\mathrm{St}_H(k)$ where $H \subset G$ are certain subgroups, by performing descent along commutative algebra objects of the form $\mathscr{A}_H$. We shall carry this out for the Galois group.

Let G be any finite group, and let $\mathcal{A}$ be a collection of subgroups of G such that any elementary abelian p-subgroup of G is contained in a conjugate of an element of $\mathcal{A}$. For each $H \in \mathcal{A}$, we consider the object $\prod_{G/H} k \in \mathrm{CAlg}(\mathrm{Mod}_G(k))$.

Proposition 9.13. *The commutative algebra object*

$$A = \prod_{H \in \mathcal{A}} \left(\prod_{G/H} k \right) \in \mathrm{CAlg}(\mathrm{Mod}_G(k))$$

admits descent.

Proof. In order to prove this, by Theorem 4.8, it suffices to prove that the above commutative algebra admits descent after restriction from G to each elementary abelian p-subgroup. However, when we restrict from G to each elementary abelian p-subgroup, the above commutative algebra object contains a copy of the unit object as a direct factor (as commutative algebras), so that it clearly admits descent. $\square$

In particular, it follows that the image $\mathscr{A} \in \mathrm{CAlg}(\mathrm{St}_G(k))$ of the above commutative algebra object $A = \prod_{H \in \mathcal{A}} \left(\prod_{G/H} k \right) \in \mathrm{Mod}_G(k)$ in the stable module ∞-category also admits descent. It follows that we have an equivalence of symmetric monoidal ∞-categories

$$\mathrm{St}_G(k) \simeq \mathrm{Tot} \left(\mathrm{Mod}_{\mathrm{St}_G(k)}(\mathscr{A}) \rightrightarrows \mathrm{Mod}_{\mathrm{St}_G(k)}(\mathscr{A} \otimes \mathscr{A}) \substack{\to\\ \to\\ \to} \cdots \right). \tag{44}$$

There is a classical cofinality argument that enables us to rewrite this inverse limit in a different fashion. Recall:

Definition 9.14. The **orbit category** $\mathcal{O}(G)$ is the category of all finite G-sets of the form G/H for $H \subset G$ a subgroup.

We have a functor

$$\mathcal{O}(G) \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}), \quad G/H \mapsto \mathrm{St}_H(k) = \mathrm{Mod}_{\mathrm{St}_G(k)} \left(\prod_{G/H} k \right).$$

Note that given any finite G-set S, we can form a commutative algebra object in $\mathrm{St}_G(k)$ given by $\prod_S k = k^S$. This construction takes coproducts of G-sets to products.

Suppose $\mathcal{A}$ is a collection of subgroups of G which is closed under finite intersections and conjugation by elements of G. We will use the following notation:

Definition 9.15. We let $\mathcal{O}_{\mathcal{A}}(G) \subset \mathcal{O}(G)$ be the full subcategory spanned by the G-sets G/H for $H \in \mathcal{A}$. We let $\mathcal{O}'_{\mathcal{A}}(G) \subset \mathcal{O}_{\mathcal{A}}(G)$ be the full subcategory including only the $\{G/H\}$ for $H \in \mathcal{A}$ and $H \neq 1$.

Using standard cofinality arguments (cf. [59, sec. 6.5]), we obtain from the descent statement (44):

Corollary 9.16. *Let $\mathcal{A}$ be a collection of subgroups of G. Suppose that $\mathcal{A}$ is closed under conjugation and finite intersections. Suppose every elementary abelian p-subgroup of G is contained in a subgroup belonging to $\mathcal{A}$. Then we have a decomposition*

$$\mathrm{St}_G(k) \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{St}_H(k). \tag{45}$$

These types of descent statements at the level of homotopy categories have been developed in [5]. We also have an analogous (but easier) decomposition

$$\mathrm{Mod}_G(k) \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{Mod}_H(k).$$

Using Theorem 9.9, we get:

Theorem 9.17. *Let $\mathcal{A}$ be the collection of elementary abelian p-subgroups of G. If k is a separably closed field of characteristic p, then the Galois group of $\mathrm{St}_G(k)$ is the profinite completion of the fundamental group of the nerve of the category $\mathcal{O}'_{\mathcal{A}}(G)$.*

Proof. The decomposition (45) implies that there is a decomposition

$$\pi_{\leq 1}(\mathrm{St}_G(k)) = \varinjlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)} \pi_{\leq 1}(\mathrm{St}_H(k)).$$

Now by Theorem 9.9, when H is nontrivial we have $\pi_{\leq 1}(\mathrm{St}_H(k)) = *$. When $H = 1$, then $\mathrm{St}_H(k) = 0$ so that the Galois groupoid is empty. It follows that the functor $\mathcal{O}_{\mathcal{A}}(G) \to \mathrm{Pro}(\mathrm{Gpd}_{\mathrm{fin}})$, $G/H \mapsto \pi_{\leq 1}(\mathrm{St}_H(k))$ is the left Kan extension of the constant functor $*$ on $\mathcal{O}'_{\mathcal{A}}(G) \subset \mathcal{O}_{\mathcal{A}}(G)$. This implies the result. $\square$

Unfortunately, we do not know in general an explicit description of the above fundamental group. We will give a couple of simple examples below.

Theorem 9.18.

1. *Let G be a finite group whose center contains an order p element (e.g., a p-group). Then the Galois group of* $\mathrm{St}_G(k)$ *is the quotient of G by the normal subgroup generated by the order p elements: the functor* $\mathrm{Mod}_G(k) \to \mathrm{St}_G(k)$ *induces an isomorphism on fundamental groups.*
2. *Suppose G is a finite group such that the intersection of any three p-Sylow subgroups of G is nontrivial. Then* $\mathrm{Mod}_G(k) \to \mathrm{St}_G(k)$ *induces an isomorphism on fundamental groups.*

Proof. Consider the first case. Choose an order p subgroup C contained in the center of G, and consider the collection $\mathcal{A}$ of all nontrivial elementary abelian p-subgroups of G which contain C. Note that $\mathcal{A}$ does not contain the trivial subgroup. Then we get decompositions $\mathrm{St}_G(k) \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{St}_H(k)$ and similarly for $\mathrm{Mod}_G(k)$. In both cases, the Galois groupoid of each term in the inverse limit is a point. It follows that

$$\pi_1(\mathrm{St}_G(k)) \simeq \pi_1(\mathrm{Mod}_G(k)) \simeq \pi_1 N(\mathcal{O}_{\mathcal{A}}(G)),$$

and since we have already computed the Galois group of $\mathrm{Mod}_G(k)$ (Theorem 7.16), we are done.

For the second case, let G be a finite group such that the intersection of any three p-Sylows in G is nontrivial. Here we will argue slightly differently. We fix a p-Sylow $P \subset G$ and consider the commutative algebra object $B = \prod_{G/P} k \in \mathrm{CAlg}(\mathrm{Mod}_G(k))$ and its image $\mathscr{B} \in \mathrm{CAlg}(\mathrm{St}_G(k))$. We observe that $B, B \otimes B, B \otimes B \otimes B$ have the same fundamental groupoids as $\mathscr{B}, \mathscr{B}\otimes\mathscr{B}, \mathscr{B}\otimes\mathscr{B}\otimes\mathscr{B}$, respectively: in fact, this follows from the previous item (that the Galois groups for $\mathrm{Mod}_H(k)$ and $\mathrm{St}_H(k)$ where H is a *nontrivial* p-group are isomorphic), since the hypotheses imply that the G-set $G/P \times G/P \times G/P$ has no free component to it. Therefore, by descent theory, the Galois groups of $\mathrm{Mod}_G(k)$ and $\mathrm{St}_G(k)$ must be isomorphic; note that the Galois group only depends on the 3-truncation of the descent diagram. □

On the other hand, there are cases in which there are finite covers in the stable module ∞-category that do not come from the representation category.

Corollary 9.19. *Let k be a separably closed field of characteristic p. Let G be a finite group such that the maximal elementary abelian p-subgroup of G has rank one (i.e., there is no embedding $\mathbb{Z}/p \times \mathbb{Z}/p \subset G$) and any two such are conjugate. In this case, the Galois group of* $\mathrm{St}_G(k)$ *is the Weyl group of a subgroup $\mathbb{Z}/p \subset G$.*

Proof. This is an immediate consequence of Theorem 9.17. □

For example, we find that the Galois group of the stable module ∞-category of Σ_p is precisely a $(\mathbb{Z}/p)^\times$, which is the Weyl group of $\mathbb{Z}/p \subset \Sigma_p$. We can see this very explicitly. The Tate construction $k^{t\Sigma_p}$ has homotopy groups given by

$$\pi_*(k^{t\Sigma_p}) \simeq E(\alpha_{2p-1}) \otimes P(\beta_{2p-2}^{\pm 1}),$$

whereas we have $k^{t\mathbb{Z}/p} \simeq E(\alpha_{-1}) \otimes P(\beta_2^{\pm 1})$. The extension $k^{t\Sigma_p} \to k^{t\mathbb{Z}/p}$ is Galois, and is obtained roughly by adjoining a $(p-1)$st root of the invertible element β_{2p-2}.

10. Chromatic homotopy theory

In this section, we begin exploring the Galois group in chromatic stable homotopy theory; this was the original motivating example for this project. In particular, we consider Galois groups over certain E_n-local $\mathbf{E}_\infty$-rings such as TMF and $L_n S^0$, and over the ∞-category $L_{K(n)}\,\mathrm{Sp}$ of $K(n)$-local spectra.

10.1. Affineness and TMF

Consider the $\mathbf{E}_\infty$-ring TMF of (periodic) topological modular forms. We refer to [20] for a detailed treatment. Our goal in this section is to describe its Galois theory. The homotopy groups of TMF are very far from regular; there is considerable torsion and nilpotence in $\pi_*(\mathrm{TMF})$ at the primes 2 and 3, coming from the stable stems. This presents a significant difficulty in the computation of arithmetic invariants of TMF and Mod(TMF).

Nonetheless, TMF itself is built up as an inverse limit of much simpler (at least, simpler at the level of homotopy groups) $\mathbf{E}_\infty$-ring spectra. Recall the construction of Goerss–Hopkins–Miller–Lurie, which builds TMF as the global sections of a sheaf of $\mathbf{E}_\infty$-ring spectra on the étale site of the moduli stack of elliptic curves M_{ell}. Given a commutative ring R, and an elliptic curve $C \to \operatorname{Spec} R$ such that the classifying map $\operatorname{Spec} R \to M_{ell}$ is étale, the construction assigns an $\mathbf{E}_\infty$-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ with the basic properties:

1. $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ is even periodic.
2. We have a canonical identification $\pi_0 \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \simeq R$ and a canonical identification of the formal group of $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ and the formal completion $\widehat{C}$.

The construction makes the assignment $(\operatorname{Spec} R \to M_{ell}) \mapsto \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ into a *functor* from the affine étale site of M_{ell} to the ∞-category of $\mathbf{E}_\infty$-rings, and one defines

$$\mathrm{TMF} = \Gamma(M_{ell}, \mathcal{O}^{\mathrm{top}}) \stackrel{\mathrm{def}}{=} \varprojlim_{\operatorname{Spec} R \to M_{ell}} \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R). \tag{46}$$

The moduli stack of elliptic curves is *regular*: any étale map $\mathrm{Spec}\, R \to M_{ell}$ has the property that R is a regular, two-dimensional domain. The Galois theory of each $\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R)$ is thus purely algebraic in view of Theorem 6.29. It follows that from the expression (46) that we have a fully faithful embedding

$$\mathrm{Mod}^{\omega}(\mathrm{TMF}) \subset \varprojlim_{\mathrm{Spec}\, R \to M_{ell}} \mathrm{Mod}^{\omega}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R)), \tag{47}$$

which proves that an *upper bound* for the Galois group of TMF is given by the Galois group of the moduli stack of elliptic curves. It is a folklore result that the moduli stack of elliptic curves, over $\mathbb{Z}$, is simply connected; see for instance [41]. Therefore, one has:

Theorem 10.1. TMF *is separably closed, i.e., has trivial Galois group.*

Using more sophisticated arguments, one can calculate the Galois groups not only of TMF, but also of various localizations (where the algebraic stack is no longer simply connected). This proceeds by a strengthening of (47).

Definition 10.2. The ∞-category $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ of **quasi-coherent $\mathcal{O}^{\mathrm{top}}$-modules** is the inverse limit $\varprojlim_{\mathrm{Spec}\, R \to M_{ell}} \mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R))$.

As usual, we have an adjunction

$$\mathrm{Mod}(\mathrm{TMF}) \rightleftarrows \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}),$$

since TMF is the $\mathbf{E}_\infty$-ring of endomorphisms of the unit in $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$. At least away from the prime 2 (this restriction is removed in [57]), it is a result of Meier, proved in [55], that the adjunction is an equivalence: TMF-modules are equivalent to quasi-coherent $\mathcal{O}^{\mathrm{top}}$-modules. In particular, the unit object in $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ is compact, which would not have been obvious a priori. It follows that we can make a stronger version of the argument in Theorem 10.1. We will do this below in more generality.

In [57], L. Meier and the author formulated a more general context for "affineness" results such as this. We review the results. Let M_{FG} be the moduli stack of formal groups. Let X be a Deligne–Mumford stack and let $X \to M_{FG}$ be a flat map. It follows that for every étale map $\mathrm{Spec}\, R \to X$, the composite $\mathrm{Spec}\, R \to X \to M_{FG}$ is flat and there is a canonically associated even periodic, *Landweber-exact* multiplicative homology theory associated to it. An *even periodic refinement* of this data is a lift of the diagram of homology theories to $\mathbf{E}_\infty$-rings. In other words, it is a sheaf $\mathcal{O}^{\mathrm{top}}$ of even periodic $\mathbf{E}_\infty$-rings on the affine étale site of X with formal groups given by the map $X \to M_{FG}$. This enables in particular the construction of an $\mathbf{E}_\infty$-ring $\Gamma(X, \mathcal{O}^{\mathrm{top}})$ of *global sections*, obtained as a homotopy limit in a similar manner as (46), and a stable homotopy theory $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ of quasi-coherent modules.

Now, one has:

Theorem 10.3. *(See [57, Theorem 4.1].) Suppose $X \to M_{FG}$ is a flat, quasi-affine map and let the sheaf $\mathcal{O}^{\mathrm{top}}$ of $\mathbf{E}_\infty$-rings on the étale site of X define an even periodic refinement of X. Then the natural adjunction*

$$\mathrm{Mod}(\Gamma(X, \mathcal{O}^{\mathrm{top}})) \rightleftarrows \mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}),$$

is an equivalence of ∞-categories.

In particular, in [57, Theorem 5.6], L. Meier and the author showed that, given $X \to M_{FG}$ *quasi-affine*, then one source of Galois extensions of $\Gamma(X, \mathcal{O}^{\mathrm{top}})$ was the Galois theory of the *algebraic* stack. If X is regular, we can give the following refinement.

Theorem 10.4. *Let X be a regular Deligne–Mumford stack. Let $X \to M_{FG}$ be a flat, quasi-affine map and fix an even periodic sheaf $\mathcal{O}^{\mathrm{top}}$ as above. Then we have a canonical identification*

$$\pi_1(\mathrm{Mod}(\Gamma(X, \mathcal{O}^{\mathrm{top}}))) \simeq \pi_1^{\mathrm{et}} X.$$

Proof. This is now a quick corollary of the machinery developed so far. By Theorem 10.3, we can identify modules over $\Gamma(X, \mathcal{O}^{\mathrm{top}})$ with quasi-coherent sheaves of $\mathcal{O}^{\mathrm{top}}$-modules. In particular, we can equivalently compute the Galois group, which is necessarily the same as the *weak* Galois group, of $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$. Using

$$\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}}) = \varprojlim_{\mathrm{Spec}\, R \to X} \mathrm{Mod}(\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R)),$$

where the inverse limit ranges over all étale maps $\mathrm{Spec}\, R \to X$, we find that the weak Galois groupoid of $\mathrm{QCoh}(\mathcal{O}^{\mathrm{top}})$ is the colimit of the weak Galois groupoids of the various $\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R)$. Since we know that these are algebraic (Theorem 6.29), we conclude that we arrive precisely at the colimit of Galois groupoids that computes the Galois groupoid of X. $\square$

In addition to the case of TMF, we find:

Corollary 10.5.

1. *The Galois group of $\mathrm{Tmf}_{(p)}$ (for any prime p) is equal to the étale fundamental group of $\mathbb{Z}_{(p)}$.*
2. *The Galois group of KO is $\mathbb{Z}/2$: the map $KO \to KU$ exhibits KU as the Galois closure of KO.*

Here Tmf is the non-connective, non-periodic flavor of topological modular forms associated to the compactified moduli stack of elliptic curves.

Proof. The first claim follows because the compactified moduli stack of elliptic curves is geometrically simply connected; this follows via the expression as a weighted projective stack $\mathbb{P}(4,6)$ when 6 is inverted. The second assertion follows from Theorem 6.29, which shows that KU is simply connected, since $\operatorname{Spec}\mathbb{Z}$ is. $\square$

10.2. $K(n)$-local homotopy theory

Let $K(n)$ be a Morava K-theory at height n. The ∞-category $L_{K(n)}\,\mathrm{Sp}$ of $K(n)$-local spectra, which plays a central role in modern chromatic homotopy theory, has been studied extensively in the monograph [40]. $L_{K(n)}\,\mathrm{Sp}$ is a basic example of a stable homotopy theory where the unit object is *not* compact, although $L_{K(n)}\,\mathrm{Sp}$ is compactly generated (by the localization of a finite type n complex, for instance). We describe the Galois theory of $L_{K(n)}\,\mathrm{Sp}$ here, following ideas of [21,14,71], and many other authors.

According to the "chromatic" picture, phenomena in stable homotopy theory are approximated by the geometry of the moduli stack M_{FG} of formal groups. When localized at a prime p, there is a basic open substack $M_{FG}^{\leq n}$ of M_{FG} parametrizing formal groups whose *height* (after specialization to any field of characteristic p) is $\leq n$. There is a closed substack $M_{FG}^{n} \subset M_{FG}^{\leq n}$ parametrizing formal groups of height *exactly* n over $\mathbb{F}_p$-algebras. The operation of $K(n)$-localization corresponds roughly to formally completing along this closed substack (after first restricting to the open substack $M_{FG}^{\leq n}$, which is E_n-localization). In particular, the Galois theory of $L_{K(n)}\,\mathrm{Sp}$ should be related to that of this closed substack.

It turns out that M_{FG}^{n} has an extremely special geometry. The substack M_{FG}^{n} is essentially the "classifying stack" of a large profinite group (with a slight Galois twist) known as the *Morava stabilizer group.*

Definition 10.6. Let $k = \overline{\mathbb{F}_p}$ and consider a height n formal group $\mathfrak{X}$ over k. We define the n**th Morava stabilizer group** $\mathbb{G}_n$ to be the automorphism group of $\mathfrak{X}$ (in the category of formal groups).

Any two height n formal groups over k are isomorphic, so it does not matter which one we use.

Definition 10.7. We define the n**th extended Morava stabilizer group** $\mathbb{G}_n^{\mathrm{ext}}$ to be the group of pairs (σ, ϕ) where $\sigma \in \operatorname{Aut}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ and $\phi\colon \mathfrak{X} \to \sigma^*\mathfrak{X}$ is an isomorphism of formal groups.

In fact, $\mathfrak{X}$ can be defined over the prime field $\mathbb{F}_p$ itself, so that $\sigma^*\mathfrak{X}$ is canonically identified with $\mathfrak{X}$, and in this case, every automorphism of $\mathfrak{X}$ is defined over $\mathbb{F}_{p^n}$. This gives $\mathbb{G}_n$ a natural profinite structure (by looking explicitly at coefficients of power series), and $\mathbb{G}_n^{\mathrm{ext}} \simeq \mathbb{G}_n \rtimes \operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.

The picture is that the stack M_{FG}^{n} is the classifying stack of the group *scheme* of automorphisms of a height n formal group over $\mathbb{F}_p$. This itself is a pro-étale group scheme which becomes constant after extension of scalars to $\mathbb{F}_{p^n}$. This picture is justified by the result that any two n formal group are étale locally isomorphic, and the scheme of automorphisms is in fact as claimed.

This picture has been reproduced closely in chromatic homotopy theory. Some of the most important objects in $L_{K(n)} \mathrm{Sp}$ are the *Morava E-theories* E_n. Let κ be a perfect field of characteristic p and let $\mathfrak{X}$ be a formal group of height n over κ, defining a map $\operatorname{Spec} \kappa \to M_{FG}^n$. The "formal completion" of M_{FG} along this map can be described by *Lubin–Tate theory*; in other words, the universal deformation $\mathfrak{X}_{\mathrm{univ}}$ of the formal group $\mathfrak{X}$ lives over the ring $W(\kappa)[[v_1, \dots, v_{n-1}]]$ for $W(\kappa)$ the ring of Witt vectors on κ. The association $(\kappa, \mathfrak{X}) \mapsto (W(\kappa)[[v_1, \dots, v_{n-1}]], \mathfrak{X}_{\mathrm{univ}})$ defines a functor from pairs $(\kappa, \mathfrak{X})$ to pairs of complete local rings and formal groups over them.

The result of Goerss, Hopkins and Miller [27,68] is that the above functor can be lifted to topology. Each pair $(W(\kappa)[[v_1, \dots, v_{n-1}]], \mathfrak{X}_{\mathrm{univ}})$ can be realized by a homotopy commutative ring spectrum $E_n = E_n(\kappa; \mathfrak{X})$ in view of the Landweber exact functor theorem. However, in fact one can construct a functor (essentially uniquely)

$$(\kappa, \mathfrak{X}) \mapsto E_n(\kappa; \mathfrak{X})$$

to the ∞-category of $\mathbf{E}_\infty$-rings, lifting this diagram of formal groups: for each $(\kappa, \mathfrak{X})$, $E_n(\kappa; \mathfrak{X})$ is even periodic with formal group identified with the universal deformation $\mathfrak{X}_{\mathrm{univ}}$ over $W(\kappa)[[v_1, \dots, v_{n-1}]]$.

We formally now state a definition that we have used before.

Definition 10.8. Any $E_n(\kappa; \mathfrak{X})$ will be referred to as a **Morava *E*-theory** and will be sometimes simply written as E_n.

Since M_{FG}^n is the classifying stack of a pro-étale group scheme, we should expect, if we take $\kappa = \overline{\mathbb{F}_p}$, an appropriate action of the extended Morava stabilizer group on $E_n(\kappa; \mathfrak{X})$. An action of the group $\mathbb{G}_n^{\mathrm{ext}}$ is given to us on $E_n(\kappa; \mathfrak{X})$ by the Goerss–Hopkins–Miller theorem. However, we should expect a "continuous" action of $\mathbb{G}_n^{\mathrm{ext}}$ on $E_n(\kappa; \mathfrak{X})$ on $\mathrm{Mod}(E_n(\kappa; \mathfrak{X}))$ whose homotopy fixed points are $L_{K(n)} \mathrm{Sp}$.

Although this does not seem to have been fully made precise, given an open subgroup $U \subset \mathbb{G}_n^{\mathrm{ext}}$, Devinatz–Hopkins [21] construct homotopy fixed points $E_n(\kappa; \mathfrak{X})^{hU}$ which have the desired properties (for example, if $U \subset \mathbb{G}_n^{\mathrm{ext}}$, one obtains $L_{K(n)} S^0$). It was observed in [71] that for $U \subset \mathbb{G}_n^{\mathrm{ext}}$ open normal, the maps

$$L_{K(n)} S^0 \to E_n(\kappa; \mathfrak{X})^{hU}$$

are $\mathbb{G}_n^{\mathrm{ext}}/U$-Galois in $L_{K(n)} \mathrm{Sp}$; they become étale after base-change to $E_n(\kappa; \mathfrak{X})$. The main result of this section is that this gives precisely the Galois group of $K(n)$-local homotopy theory.

Theorem 10.9. *The Galois group of $L_{K(n)}\,\mathrm{Sp}$ (which is also the weak Galois group) is the extended Morava stabilizer group $\mathbb{G}_n^{\mathrm{ext}}$.*

Away from the prime 2, this result is essentially due to Baker and Richter [14]. We will give a direct proof using descent theory. Let E_n be a Morava E-theory. Using descent for linear ∞-categories along $L_n S^0 \to E_n$ (Corollary 3.42 and Theorem 4.18), we find:

Proposition 10.10. *$E_n \in \mathrm{CAlg}(L_{K(n)}\,\mathrm{Sp})$ satisfies descent. In particular, we have an equivalence*

$$L_{K(n)}\,\mathrm{Sp} \simeq \mathrm{Tot}\left(L_{K(n)}\,\mathrm{Mod}(E_n) \rightrightarrows L_{K(n)}\,\mathrm{Mod}(L_{K(n)}(E_n \otimes E_n)) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots\right).$$

Proof. This follows directly from the fact that since the cobar construction $L_n S^0 \to E_n$ defines a constant pro-object in Sp (with limit $L_n S^0$), it defines a constant pro-object (with limit $L_{K(n)} S^0$) in $L_{K(n)}\,\mathrm{Sp}$ after $K(n)$-localizing everywhere. □

Therefore, we need to understand the Galois groups of stable homotopy theories such as $L_{K(n)}\,\mathrm{Mod}(E_n)$. We did most of the work in Theorem 6.29, although the extra localization adds a small twist that we should check first.

Let A be an even periodic $\mathbf{E}_\infty$-ring with $\pi_0 A$ a complete regular local ring with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ is a system of parameters for $\mathfrak{m}$. Let $\kappa(A) = A/(x_1, \ldots, x_n)$ be the topological “residue field” of A, as considered earlier.

Proposition 10.11. *Given a $\kappa(A)$-local A-module M, the following are equivalent:*

1. *M is dualizable in $L_{\kappa(A)}\,\mathrm{Mod}(A)$.*
2. *M is a perfect A-module.*

Proof. Only the claim that the first assertion implies the second needs to be shown. If M is dualizable in $L_{\kappa(A)}\,\mathrm{Mod}(A)$, then it follows that, since the *homology theory* $\kappa(A)_*$ is a monoidal functor, $\kappa(A)_*(M)$ must be dualizable in the category of graded $\kappa(A)_*$-modules. In particular, $\kappa(A)_0(M)$ and $\kappa(A)_1(M)$ are finite-dimensional vector spaces. From this, it follows that $\pi_*(M)$ itself must be a finitely generated $\pi_*(A)$-module, since $\pi_*(M)$ is (algebraically) complete. For example, given any i, we show that the $\pi_0(A)$-module $\pi_0(M/(x_1, \ldots, x_i)M)$ is finitely generated by descending induction on i; when $i = 0$ it is the assertion we want. When $i = n$, the finite generation follows from our earlier remarks. If we know finite generation at i, then we use the cofiber sequence

$$M/(x_1, \ldots, x_{i-1}) \xrightarrow{x_i} M/(x_1, \ldots, x_{i-1}) \to M/(x_1, \ldots, x_i),$$

to find that

$$\pi_0(M/(x_1,\dots,x_{i-1}))\otimes_{\pi_0(A)}\pi_0(A)/(x_i)\subset\pi_0(M/(x_1,\dots,x_i)),$$

is therefore finitely generated. However, by the x_i-adic completeness of $\pi_0(M/(x_1,\dots,x_{i-1}))$, this implies that $\pi_0(M/(x_1,\dots,x_{i-1}))$ is finitely generated.

Finally, since $\pi_*(A)$ has finite global dimension, this is enough to imply that M is perfect as an A-module. □

Proof of Theorem 10.9. We thus get an equivalence

$$\begin{aligned}&\mathrm{CAlg}^{\mathrm{w.cov}}(L_{K(n)}\,\mathrm{Sp})\\ &\quad\simeq \mathrm{Tot}\left(\mathrm{CAlg}^{\mathrm{w.cov}}(L_{K(n)}\,\mathrm{Mod}(E_n))\rightrightarrows \mathrm{CAlg}^{\mathrm{w.cov}}(L_{K(n)}\,\mathrm{Mod}(E_n\otimes E_n))\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\cdots\right).\end{aligned}$$

However, we have shown, as a consequence of Proposition 10.11 and Theorem 6.29, that $\mathrm{CAlg}^{\mathrm{w.cov}}(L_{K(n)}\,\mathrm{Mod}(E_n))$ is actually equivalent to the full subcategory spanned by the *finite étale* commutative algebra objects. Since finite étale algebra objects are preserved under base change, we can replace the above totalization via

$$\begin{aligned}&\mathrm{CAlg}^{\mathrm{w.cov}}(L_{K(n)}\,\mathrm{Sp})\\ &\quad\simeq \mathrm{Tot}\left(\mathrm{CAlg}^{\mathrm{w.cov}}_{\mathrm{alg}}(L_{K(n)}\,\mathrm{Mod}(E_n))\rightrightarrows \mathrm{CAlg}^{\mathrm{w.cov}}_{\mathrm{alg}}(L_{K(n)}\,\mathrm{Mod}(E_n\otimes E_n))\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\cdots\right),\end{aligned}$$

where the subscript alg means that we are only looking at the classical finite covers, i.e., the category is equivalent to the category of finite étale covers of π_0. In other words, we obtain a cosimplicial commutative ring, and we need to take the geometric realization of the étale fundamental groupoids to obtain the fundamental group of $L_{K(n)}\,\mathrm{Sp}$.

Observe that each commutative ring $\pi_0 L_{K(n)}(E_n^{\otimes m})$ is complete with respect to the ideal $(p, v_1, \dots, v_{n-1})$, in view of the $K(n)$-localization. The algebraic fundamental group is thus invariant under quotienting by this ideal. After we do this, we obtain precisely a presentation for the moduli stack M^n_{FG}, so the Galois group of $L_{K(n)}\,\mathrm{Sp}$ is that of this stack. As we observed earlier, this is precisely the extended Morava stabilizer group. □

10.3. Purity

We next describe a "purity" phenomenon in the Galois groups of $\mathbf{E}_\infty$-rings in chromatic homotopy theory: they appear to depend only on their L_1-localization. We conjecture below that this is true in general, and verify it in a few special (but important) cases.

We return to the setup of Section 10.1. Let R be an $\mathbf{E}_\infty$-ring that arises as the global sections of the structure sheaf ("functions") on a derived stack $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ which is a refinement of a flat map $X \to M_{FG}$. Suppose further that $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ is *0-affine*, i.e., the natural functor $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \to \mathrm{QCoh}(\mathfrak{X})$ is an equivalence, and that X is *regular*.

In this case, we have:

Theorem 10.12 *(KU-purity). The map $R \to L_{KU}R$ induces an isomorphism on Galois groups.*

In order to prove this result, we recall the *Zariski–Nagata purity theorem*, for which a useful reference is Exposé X of [31].

Theorem 10.13 *(Zariski–Nagata). Let X be a regular noetherian scheme and $U \subset X$ an open subset such that $X \setminus U$ has codimension ≥ 2 in X. Then the restriction functor establishes an equivalence of categories between finite étale covers of X and finite étale covers of U.*

If X is instead a regular Deligne–Mumford stack, and $U \subset X$ is an open substack whose complement has codimension ≥ 2 (a condition that makes sense étale locally, and hence for X), then it follows from the above and descent theory that finite étale covers of X and U are still equivalent.

Proof of Theorem 10.12. First we work localized at a prime p, so that $L_{KU} \simeq L_1$. In this case, the result is a now a direct consequence of various results in the preceding sections together with Theorem 10.13.

Choose a derived stack $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ whose global sections give R; suppose $\mathfrak{X}$ is an even periodic refinement of an ordinary Deligne–Mumford stack X, with a flat, affine map $X \to M_{FG}$. Then $L_1 R$ can be recovered as the $\mathbf{E}_\infty$-ring of functions on the open substack of $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ corresponding to the open substack U of X complementary to closed substack cut out by the ideal (p, v_1). The derived version of U is also 0-affine, as observed in [57, Proposition 3.27].

Now, in view of Theorem 10.4, the Galois group of $L_1 R$ is that of the open substack U, and the Galois group of R is that of X. However, the Zariski–Nagata theorem implies that the inclusion $U \subset X$ induces an isomorphism on étale fundamental groups. Indeed, the complement of $U \subset X$ has codimension ≥ 2 as (p, v_1) is a regular sequence on X by flatness and thus cuts out a codimension two substack of X.

To prove this integrally, we need to piece together the different primes involved. Given any $\mathbf{E}_\infty$-ring A, it follows from descent theory that there is a sheaf Gal_G of (ordinary) categories on the Zariski site of $\operatorname{Spec} \pi_0 A$, such that on a basic open affine $U_f = \operatorname{Spec} \pi_0 A[f^{-1}] \subset \operatorname{Spec} \pi_0 A$, $\mathrm{Gal}_G(U_f)$ is the groupoid of G-Galois extensions of the localization $A[f^{-1}]$. Thus we can prove:

Lemma 10.14. *Fix a finite group G. Let $R \to R'$ be a morphism of $\mathbf{E}_\infty$-rings with the following properties:*

1. *$R \to R'$ induces an equivalence of categories $\mathrm{Gal}_G(R_{(p)}) \to \mathrm{Gal}_G(R'_{(p)})$ for each p.*
2. *$R_{\mathbb{Q}} \to R'_{\mathbb{Q}}$ induces an equivalence of categories $\mathrm{Gal}_G(R_{\mathbb{Q}}) \to \mathrm{Gal}_G(R'_{\mathbb{Q}})$.*

Then the natural functor $\mathrm{Gal}_G(R) \to \mathrm{Gal}_G(R')$ is an equivalence of categories.

Proof. By the above, there is a sheaf $\mathrm{Gal}(G; R)$ (resp. $\mathrm{Gal}(G; R'))$ of categories on $\operatorname{Spec}\mathbb{Z}$, whose value over an open affine $\operatorname{Spec}\mathbb{Z}[N^{-1}]$ is the category of G-Galois extensions of $R[N^{-1}]$ (resp. of $R'[N^{-1}]$). These are the pushforwards of the sheaves Gal_G on $\operatorname{Spec}\pi_0 R, \operatorname{Spec}\pi_0 R'$ discussed above. Now Theorem 6.20, together with the hypotheses of the lemma, imply that the map of sheaves $\mathrm{Gal}(G; R) \to \mathrm{Gal}(G'; R)$ induces an *equivalence* of categories on each stalk over every point of $\operatorname{Spec}\mathbb{Z}$. It follows that the map induces an equivalence upon taking global sections, which is the conclusion we desired. □

This lemma let us conclude the proof of Theorem 10.12. Namely, the map $R \to L_K R$ satisfies the two hypotheses of the lemma above, since in fact $R_{\mathbb{Q}} \simeq (L_K R)_{\mathbb{Q}}$, and we have already checked the p-local case above. □

Using similar techniques, we can prove a purity result for the Galois groups of the E_n-local spheres.

Theorem 10.15. *The Galois theory of $L_n S^0$ is algebraic and is given by that of $\mathbb{Z}_{(p)}$.*

Proof. We can prove this using descent along the map $L_n S^0 \to E_n$. Since this map admits descent, we find that

$$\mathrm{CAlg}^{\mathrm{cov}}(L_n S^0) \simeq \mathrm{Tot}\left(\mathrm{CAlg}^{\mathrm{cov}}(E_n) \rightrightarrows \mathrm{CAlg}^{\mathrm{cov}}(E_n \otimes E_n) \overset{\to}{\underset{\to}{\to}} \cdots\right).$$

Now, $E_n \otimes E_n$ does not have a regular noetherian π_0. However, $\mathrm{CAlg}^{\mathrm{cov}}(E_n)$ is simply the ordinary category of finite étale covers of $\pi_0 E_n$, in view of Theorem 6.29. Therefore, we can replace the above totalization by the analogous totalization where we only consider the *algebraic* finite covers at each stage (since the two are the same at the first stage). In particular, since the cosimplicial (discrete) commutative ring

$$\pi_0(E_n) \rightrightarrows \pi_0(E_n \otimes E_n) \overset{\to}{\underset{\to}{\to}} \cdots,$$

is a presentation for the algebraic stack $M_{FG}^{\leq n}$ of formal groups (over $\mathbb{Z}_{(p)}$-algebras) of height $\leq n$, we find that the Galois theory of $L_n S^0$ is the Galois theory of this stack. The next lemma thus completes the proof. □

Lemma 10.16. *For $n \geq 1$, the maps of stacks $M_{FG}^{\leq n} \to M_{FG} \to \operatorname{Spec}\mathbb{Z}_{(p)}$ induce isomorphisms on fundamental groups.*

Proof. The moduli stack of formal groups M_{FG} has a presentation in terms of the map $\operatorname{Spec} L \to M_{FG}$, where L is the *Lazard ring* (localized at p). L is a polynomial ring on a countable number of generators over $\mathbb{Z}_{(p)}$. Similarly, $\operatorname{Spec} L \times_{M_{FG}} \operatorname{Spec} L$ is a polynomial ring on a countable number of generators over $\operatorname{Spec}\mathbb{Z}_{(p)}$. The étale fundamental group of $\mathbb{Z}_{(p)}[x_1, \dots, x_n]$ is that of $\mathbb{Z}_{(p)}$,[8] and by taking filtered colimits, the same follows for a polynomial ring over $\mathbb{Z}_{(p)}$ over a countable number of variables. Thus, the étale fundamental group M_{FG} is that of $\operatorname{Spec} \mathbb{Z}_{(p)}$. The last assertion follows because, again, the deletion of the closed subscheme cut out by (p, v_1) does not affect the étale fundamental group in view of the Zariski–Nagata theorem (applied to the infinite-dimensional rings by the filtered colimit argument). $\square$

The above results suggest the following purity conjecture.

Conjecture 10.17. *Let R be* any *L_n-local $\mathbf{E}_\infty$-ring. The map $R \to L_1 R$ induces an isomorphism on fundamental groups.*

Conjecture 10.17 is supported by the observation that, although not every L_n-local $\mathbf{E}_\infty$-ring has a regular π_0 (or anywhere close), L_n-local $\mathbf{E}_\infty$-rings seem to built from such at least to some extent. For example, the free $K(1)$-local $\mathbf{E}_\infty$-ring on a generator is known to have an infinite-dimensional regular π_0.

Remark 10.18. Conjecture 10.17 cannot be valid for general $L_n S^0$-linear stable homotopy theories: it is specific to $\mathbf{E}_\infty$-rings. For example, it fails for $L_{K(n)} \operatorname{Sp}$.

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[8] Let R be a regular, torsion-free ring. Then we claim that the fundamental group of $\operatorname{Spec} R[x]$ and $\operatorname{Spec} R$ are isomorphic under the natural map. In fact, this is evident (e.g., by topology) if R is a field of characteristic zero. Now, if K is the fraction field of R, then to give an étale cover of $\operatorname{Spec} R[x]$ is equivalent to giving an étale cover of $\operatorname{Spec} K[x]$ (i.e., an étale $K[x]$-algebra R'_K) such that the normalization of $R[x]$ in R'_K is étale over $R[x]$, since étale extensions preserve normality. The étale $K[x]$-algebra R'_K is necessarily of the form $L[x]$ where L is a finite separable extension of K (if it is connected, at least). In order for this normalization to be étale over $R[x]$, the normalization of R in L must be étale over R.

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