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A NEW INFINITE FAMILY IN ${}_2\pi_*{}^S$†

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§ 1. INTRODUCTION

THE APPROACH toward homotopy theory which is the basis of this note is the Adams spectral sequence. Let A be the mod 2 Steenrod algebra. Then Adams has shown that there is a spectral sequence $\{E_r^{s,t}\}$ such that $E_\infty^{**} = E_0({}_2\pi_*{}^S)$ and $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_2, Z_2)$. (There is a similar spectral sequence for each prime.) A good reference is [1].

Let $h_i \in \mathrm{Ext}_A^{1,2^i}(Z_2, Z_2)$ be the family described by Adams in [1]. These classes are related to Sq^{2^i} which represent the generators over A of IA, the augmentation ideal of A. Adams also shows in [1] that $h_1h_i \neq 0$ in $\mathrm{Ext}_A^{2,2^i+2}(Z_2, Z_2)$ if $i \neq 2$. Let $\varphi_{1,i}$, $i \geq 3$ be the secondary operation based on the relation in A, $Sq^{2^i+1}Sq^1 + Sq^2Sq^{2^i} + Sq^4Sq^{2^i-2} + Sq^{2^i}Sq^2 = 0$. The main result of this paper is the following.

THEOREM 1. *For each $i \neq 2$ there is a class $\eta_i \in \pi^S_{2^i}$ such that in the mapping cone of η_i, $\varphi_{1,i}$ is not zero.*

In the Adams spectral sequence setting Theorem 1 is equivalent to

THEOREM 1′. *For each $i \neq 2$, the class h_1h_i is a permanent cycle.*

This result establishes an infinite family of elements at filtration 2 in the Adams Spectral Sequence. Adams in [1] showed that there were only a finite number of classes at level 1 and the results of [9] show that the image J includes only finitely many classes at each filtration level. In particular this family is not in the image of the J-homomorphism.

The technique to prove this result is much less technical than those in current vogue in stable homotopy theory. Using these more technical methods one can construct from each η_i a finite family of classes in ${}_2\pi_*{}^S$.

Previous results related to this problem are:

(1) If $\theta_j \in {}_2\pi^S_{2^{j+1}-2}$, represented by $\{h_j^2\}$ in the Adams spectral sequence, exists and $2\theta_j = 0$, then $\{h_1h_{j+1}\} \in \langle\theta_j, 2i, \eta\rangle$. Hence the theorem is known if $i \leq 6$.

(2) If there exists an element η_i detected by h_1h_i, $i \geq 3$, then $H(\eta_i) = \nu$ on S^{2^i-2} where $H(\eta_i)$ is the Hopf invariant and ν is the generator of $\pi_3{}^S$ [8]. This settles the open cases on $[\iota_n, \nu]$ which were left by [8].

(3) In [10] (and corrections in [2]) it is shown that h_ih_j does not project to a homotopy class except for h_i^2, h_1h_i and a finite number of exceptional cases.

The theorem follows directly from the following result.

THEOREM 2. *For each $i \geq 3$ there is a stable complex X_i of dimension $2^i - 1$ and a map f_i: $X_i \to S^0$ such that*

(a) $H^{2^i-1}(X_i; Z_2) = Z_2$ *and* $\tilde{H}^j(X_i; Z_2) = 0$ *for* $j < 2^i - 2^{i-3}$.

(b) *In* $S^0 \cup_{f_i} CX_i$, Sq^{2^i} *is non-zero.*

(c) *There is a map* $g_i: S^{2^i} \to X_i$ *such that the composite* $S^{2^i} \to X_i \to X_i/[X_i]^{2^i-2} \simeq S^{2^i-1}$ *represents* η, *the generator of* $\pi_1{}^S$.

Indeed, standard arguments show that the composite $f_i \cdot g_i$ represents h_1h_i, thus proving h_1h_i is a cycle for each r in the Adams spectral sequence. Since it cannot be a boundary, the proof of Theorem 1 is complete.

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§2 contains the proof of Theorem 2 with the major technical part delayed until a later section. §3 recalls some results on Ω^2S^{2l+1}. §4 contains some new results on Ω^2S^{2l+1} which may be of independent interest. In particular 2.6 and 4.1 may be of interest.

Theorem 1 seems to offer strong evidence for the existence of Kervaire manifolds of dimension 2^i-2. It does not seem to be useful in a proof of the Kervaire invariant conjecture.

This paper is concerned only with the prime 2. All coefficient groups for cohomology and homology are assumed to be Z_2. Where appropriate, spaces should be localized at the prime 2.

The author would also like to thank D. Kahn and Ed Brown for several profitable conversations on the material of this note and Frank Adams who pointed out a gap in the original version of this paper.

§2. THE PROOF OF THEOREM 2.

Let $S^9 \underset{g}{\to} B^2O$ represent a generator of $\pi_9(B^2O)$. Let $\Omega g\colon \Omega S^9 \to BO$ be the loop map of g.

PROPOSITION 2.1. *$(\Omega g)^* w_{8k} \neq 0$ for all k.*

Proof. Since $(S^8\times\cdots\times S^8)$ maps to ΩS^9 by multiplication and since Ωg is an H map, the Stiefel Whitney classes of the composite is $\prod_{i=1,k}(1+\kappa_i)$ where $\{\kappa_i\}$ are the standard generators of $H^8(S^8\times\cdots\times S^8)$.

Let $\tilde\Omega^2 g\colon \Sigma\Omega^2S^9\to BO$ be the adjoint of $\Omega^2 g$.

PROPOSITION 2.2. *$\tilde\Omega^2g^*(w_{2^i})\neq 0$, $i\geq 3$.*

Proof. $\tilde\Omega^2g$ factors through Ωg and if $f\colon \Sigma\Omega^2S^9\to\Omega S^9$ is the adjoint of the identity then f^* is a monomorphism in dimension 2^i, $i\geq 3$.

PROPOSITION 2.3. *Let J be the inclusion $SO\subset\Omega^\infty S^\infty$. Then the Thom complex of $\tilde\Omega^2 g$ is $S^0\cup_{J\Omega^2 g} C\Omega^2S^9$.*

This is a standard result on Thom complexes of bundles over a space which is a suspension.

By Proposition 2.2 we see that in the Thom complex of $\tilde\Omega^2g$, $Sq^{2^i}U\neq 0$ for all $i\geq 3$ where U is the Thom class. The spaces and maps that are needed for Theorem 2 will turn out to be subcomplexes and attaching maps of this Thom complex.

Milgram[12] gives a definition of Ω^2S^{2l+1} which exhibits the space as a filtered CW complex. This filtration can also be described in the following way. Let $C_n(2)$ be the space of "little cubes" in R^2, i.e., $C_n(2)$ is the configuration space of n disjoint embedded rectangles in R^2. Then Ω^2S^{2l+1} is an identification space of $\bigcup_{q<\infty} C_q(2)X_{\Sigma_q}(S^{2l-1})^q$ where Σ_q is the symmetric group on q-letters. Then $F_n(2l+1)\subset\Omega^2S^{2l+1}$ is the image of $\bigcup_{q\leq n} C_q(2)X_{\Sigma_q}(S^{2l-1})^q$ under this identification. Details can be found in [11] and also [14].

The following theorem of Snaith[14] is crucial.

THEOREM 2.4. *As a stable complex, $\Omega^2S^{2l+1} = \vee F_n(2l+1)/F_{n-1}(2l+1)$.*

Thus as a stable map $J\Omega^2g\colon \Omega^2S^9\to S^0$ is a sum of maps $\Sigma_n\bar f_n\colon \vee F_n(9)/F_{n-1}(9)\to S^0$. We will show that the choice of $(X_i, f_i) = (F_{2^{i-3}}(9)/F_{2^{i-3}-1}(9), \bar f_{2^{i-3}})$ satisfies the conclusion of Theorem 2. Property (a) follows immediately from the definition of the filtration, [11]. Property (b) follows from 2.2. Property (c) remains to be proved.

The proof of part (c) will involve a detailed analysis of the spaces $F_n(9)/F_{n-1}(9)$ at the prime 2. The rest of the paper is devoted to this. We will need the following.

THEOREM 2.5. *For all n, k and l positive integers there is a homotopy equivalence*

$$\Sigma^{n(2k-1)}F_n(2l+1)/F_{n-1}(2l+1)\simeq\Sigma^{n(2l-1)}F_n(2k+1)/F_n(2k+1).$$

This is proved in [5].

In [4] Brown and Gitler discussed the module $M(k)$ over the Steenrod algebra, A, defined by $M(k) = A/A\{\chi Sq^i | i > k\}$.

THEOREM 2.6. *As modules over the Steenrod algebra*

$$\tilde{H}^*(F_n(2l+1)/F_{n-1}(2l+1)) \simeq M\left(\left[\frac{n}{2}\right]\right)$$

with an appropriate dimension shift.

This is proved in §4.

In their paper Brown and Gitler proved

THEOREM [4]. *There are spectra $\bar{B}(k)$ satisfying*

(i) $H^*(\bar{B}(k)) \simeq M(k)$ *as A-modules.*

(ii) *Let H be the Eilenberg–MacLane spectrum $K(Z_2)$. If α: $\bar{B}(k) \to H$ is the map representing $1 \in M(k)$ then α_*: $\bar{B}(k)_q(X) \to H_q(X)$ is an epimorphism for any CW Complex X if $q < 2k+2$.*

The spaces F_n/F_{n-1} satisfy (i) but so far have not been shown to satisfy (ii). The main theorem could be proved more easily if (ii) were valid for F_n/F_{n-1}.

CONJECTURE. *The spaces $\{F_{2n}(2l+1)/F_{2n-1}(2l+1)\}$ realize the Brown–Gitler spectrum $\{\bar{B}(n)\}$.*

Instead of proving the conjecture we calculate enough of the homotopy of $F_{2n}(9)/F_{2n-1}(9)$ to complete the proof. This will be done in §5.

Also for the remainder of the paper we will let $B(n)$ be suspension spectra such that $B(n)_{2n(2l-1)} = F_{2n}(2l+1)/F_{2n-1}(2l+1)$.

§3. THE SPACE $\Omega^2 S^{2l+1}$

In this section results on the filtration of $\Omega^2 S^{2l+1}$ are stated.

PROPOSITION 3.1. $H_*(\Omega^2 S^{2l+1}) = P(x_i; i = 1, 2, \ldots)$ *where dimension of x_i is $2^{i-1}(2l) - 1$.*

Let x_i have filtration 2^{i-1} and assign to each monomial $x_1^{j_1} \cdot x_2^{j_2} \cdots x_n^{j_n}$ the filtration $\Sigma j_i 2^{i-1}$. (The $x_i = Q_1 \cdots Q_1 x_1$ in the usual Dyer Lashof notation.)

PROPOSITION 3.2. $H_*(F_n(2l+1)) = A_n$ *where A_n is the subset of $P(x_i)$ generated additively by monomials of filtration $\leq n$.*

This follows immediately from the definition [11].

Another property of the filtration which is needed is:

PROPOSITION 3.3. *There are maps $\mu_{n,m}$: $F_n(2l+1) \times F_m(2l+1) \to F_{n+m}(2l+1)$ such that $\mu_{n,m*}$ is an epimorphism if $n + m = N$, $n = 2^i$, $m = N - 2^i$ and $m < n$. If $n = 2^{i+1}$ then x_{i+1} is the only class not in the image of $\mu_{m,m*}$.*

This is clear from 3.2. The maps $\mu_{n,m}$ are induced by the multiplication in $\Omega^2(S^{2l+1})$.

§4. BROWN–GITLER TYPE SPECTRUM

In this section some properties of the module $M(k)$ are derived. Some connections with double loop spaces are also discussed.

Let h: $S^3 \to B^3O$ represent a generator of $\pi_3(B^3O)$. Let $\Omega^2 h$: $\Omega^2 S^3 \to BO$ be the appropriate loop map.

THEOREM 4.1. *Let h_n be the composite $F_n(3) \subset \Omega^2 S^3 \to BO$. Then for the Thom complex of h_n, $T(h_n)$, we have $\tilde{H}^*(T(h_n)) \simeq M(n)$.*

Proof. The proof will proceed by induction. $T(h_1)$ is the Z_2-Moore space whose cohomology is also $M(1)$. Suppose we have shown $\tilde{H}^*(T(h_k)) \simeq M(k)$ for all $k < n$. To complete the proof we need to show

(1) that $\chi Sq^k U_n = 0$ for $k > n$ if U_n is the Thom class of h_n;

(2) that the map $\tilde{H}^*(T(h_n)) \to M(n)$ is onto;

(3) that $H^j(T(h_n)) \simeq M_n^j$ for each j as Z_2 vector spaces.

To prove 1, note that $T(h_1)$ is the Z_2-Moore space and $B(1)$ is also the Z_2-Moore space. Suppose for each $p < n$, $\chi Sq^k U_p = 0$ for $k > p$. Then by 3.3 we have a map $T(h_p) \wedge T(h_q) \to T(h_n)$ if $p + q = n$ and, for suitable p and q, the map is a monomorphism in cohomology except if $n = 2^i$ and then fails only in dimension $2^{i+1} - 1$. By the Cartan formula we are thus finished except for $\chi Sq^{2^{i+1}-1} U_{2^i}$. Davis [6] has shown that $\chi Sq^{2^{i+1}-1} = Sq^{2^{i-1}} \cdots Sq^2 Sq^1$ and $\chi Sq^{2^{i+1}-2} = Sq^{2^i} \cdots Sq^2$. Thus $\chi Sq^{2^{i+1}-1} U_{2^i} = Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^2 [U \cdot x_1]$. Since as a space $\Omega^2 S^3 = S^1 \times W$ where W is 1-connected, all $Sq^I x_1 = 0$ and so the formula is $[Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^2] U \cdot x_1 = (\chi Sq^{2^i - 2} U) x_1 = 0$ by the preceding argument. This shows that $H^*(T(f_n))$ satisfies Property 1.

The proof of Parts 2 and 3 requires a little more information about M_k. Some of this is in [4] or is easily implied by results there.

PROPOSITION 4.2. *The kernel of the natural map $M_k \to M_{k-1}$ is $M\left[\frac{k}{2}\right] \chi Sq^k$.*

Proof. A Z_2 basis for M_k is given by $\{\chi Sq^I;\ I \text{ is admissible in the usual sense and } i_1 \leq k\}$, (1.3 [3]). Thus a Z_2-basis for $M_k/M_{k-1} = \{\chi Sq^I;\ I \text{ admissible and } i_1 = k\}$. Hence $I = \left\{k, i_2, i_3, \ldots, i_j \,\middle|\, i_2 \leq \left[\frac{k}{2}\right]\right\}$. The proposition is now clear.

PROPOSITION 4.3. *Let $n = 2^i + l$ where $l \leq 2^i$. Then the natural map $M(n) \to M(2^i) \oplus M(l)$ is a monomorphism except if $l = 2^i$ and then fails only in dimension $2^{i+2} - 1$.*

Proof. By direct calculation the result is true for $n = 2$ and 3. Suppose the result is true for all $n < N$. Suppose $N \neq 2^i$ or $2^i + 1$. Consider the diagram

$$\begin{array}{ccc}
\Sigma^N M\left(\left[\frac{N}{2}\right]\right) & \xrightarrow{\mu_1} & \Sigma^N\left(M(2^{i-1}) \otimes M\left(\left[\frac{l}{2}\right]\right)\right) \\
\downarrow i_1 & & \downarrow i_2 \\
M(N) & \xrightarrow{\mu} & M(2^i) \otimes M(l) \\
\downarrow a & & \downarrow b \\
M(N-1) & \xrightarrow{\mu_2} & M(2^i) \otimes M(l-1).
\end{array}$$

The left column is exact; the map i_1 is defined by $i_1(\chi Sq^I) = \chi Sq^N Sq^I$; i_2 is defined by $i_2(\chi Sq^I \otimes \chi Sq^J) = \chi(Sq^{2^i} Sq^I) \otimes \chi(Sq^l Sq^J)$. The maps a and b are the natural projections. On the set generated by $\{\chi Sq^I | i_1 < N\}$ the map a is a monomorphism and, by induction μ_2 is also. By direct calculation b is a monomorphism on $\operatorname{Im} \mu$. Thus μ is a monomorphism on this set. The image of i_1 together with this set generates $M(N)$ and $i_2\mu_1$ is clearly a monomorphism. Thus μ is a monomorphism. The minor modifications necessary if $N = 2^i$ or $2^i + 1$ are clear.

It is worth noting that the class $Sq^1 Sq^2 \cdots Sq^{2^i} \in M(2^i)$ is the only class which μ maps to zero.

We return to the proof of 4.1. Suppose by induction we have shown $H^*(T(h_n)) \simeq M(n)$ for all $n < N$. Then we have

$$\begin{array}{ccc}
H^*(T(h_N)) & \xrightarrow{\bar{\mu}} & H^*(T(h_{2^i})) \otimes H^*(T(h_l)) \\
\downarrow & & \downarrow \\
M_N & \xrightarrow[\mu]{} & M_{2^i} \otimes M_l.
\end{array}$$

Since μ is a monomorphism except as noted, the classes $p(\chi Sq^I)$ form a Z_2 basis. Thus $\chi Sq^I U_N$ are linearly independent as Z_2 classes. Finally, we have to check $Sq^1 Sq^2 \cdots Sq^{2^i} U_{2^i}$. But $Sq^2 \cdots Sq^{2^i} U_{2^i} \neq 0$ and $Sq^1(Sq^2 \cdots Sq^{2^i} U_{2^i}) = Sq^1 U(x_{i-1}^2) = U \cdot x_i + U \cdot x_1 x_{x-1}^2$. The second term is zero in $T(h_n)$. This completes the proof of part 2.

PROPOSITION 4.4. *$H_j(F_n(2l+1)/F_{n-1}(2l+1)) = H_j(\Sigma^{n \cdot 2l-1} T(h(h_{[n/2]}))$ as Z_2-modules.*

Proof. Let $\{x_i\}$ be the generators of $H_*(\Omega^2 S^3)$ and let $\{y_i\}$ be the generators of $H_*(\Omega^2 S^{2l+1})$. Recall that dim $x_i = 2^i - 1$ and dim $y_i = 2^i \cdot l - 1$. The 1–1 correspondence is given by $x^I = y^{J(I)}$ where $J(I) = (j_1, j_2, \ldots, j_{k+1})$ satisfies

$$j_1 = n - 2i_1 - 4i_2 - 8i_3 - \cdots - 2^k i_k; \qquad j_p = i_{p-1} \qquad p > 1.$$

It is clear that this map is 1–1 and if the filtration of I is less than or equal to $\left[\frac{n}{2}\right]$ then the filtration of J is n.

To complete the proof of Theorem 4.1 we note that as Z_2 vector spaces $M_n{}^i = \bigotimes M^{i-k}$. Also as Z_2 vector spaces $H^i(T(h_n)) = \bigotimes_{k \leq n} H^{i-k}(T(h_{[k/2]}))$ and so, if by induction, $H^*(T(h_p)) \simeq M_p$ for $p < n$, then $H^*(T(h_n)) = M_n$ as Z_2 vector spaces. Again note that $H^*(T(h_1)) = M_1$ and so theorem 4.1 is established.

COROLLARY 4.5. *The Thom complex of $\Omega^2 h$ is the Eilenberg–MacLane spectrum.*

This result was first observed several years ago. The proof was computational (related to the above). In [7] a very elegant proof was given. (That proof has some serious misprints and at least one error but the corrections are easily seen and a valid proof follows the lines suggested.)

From the above it follows easily that

THEOREM 2.6. $\tilde{H}^*(F_n(2l+1)/F_{n-1}(2l+1)) \cong M\left(\left[\frac{n}{2}\right]\right)$, *as modules over the Steenrod algebra.*

Proof. Proposition 4.4 implies that the homology modules are isomorphic as graded Z_2 vector spaces.

After 4.4 we need to show that in $\tilde{H}^*(F_n(2l+1)/F_{n-1}(2l+1)) Sq^I x_1{}^{n\dagger} \neq 0$ if $Sq^I 1 \neq 0$ in $M\left(\left[\frac{n}{2}\right]\right)$ and $\chi Sq^k x_1{}^n = 0$ if $k > n$.

First note that F_2/F_1 is a Z_2 Moore space. Suppose 2.6 is true for $p < n$. Then

$$\begin{array}{ccc} \tilde{H}^*(F_n/F_{n-1}) & \to & \tilde{H}^*(F_p/F_{p-1} \wedge F_{n-p}/F_{n-p-1}) \\ & & \wr\wr \\ M\left(\left[\frac{n}{2}\right]\right) & \xrightarrow{\mu} & M\left(\left[\frac{p}{2}\right]\right) \otimes M\left(\left[\frac{n-p}{2}\right]\right). \end{array}$$

The map $\bar{\mu}$ comes from 4.3. If $Sq^I 1 \neq 0$ in $M\left(\left[\frac{n}{2}\right]\right)$ then $\bar{\mu} Sq^I 1 \neq 0$ for suitably chosen p unless $n = 2^i$ and $Sq^I = Sq^{2^i - 1}$. Thus except for the case just noted $Sq^I(x_1{}^n) \neq 0$. Finally $Sq^{2^i-1} x_1{}^{2^i} = x_i$ and this handles the one exception. Thus $H^*(F_n/F_{n-1}) \simeq M\left(\left[\frac{n}{2}\right]\right)$ and this establishes 2.6.

§5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

To complete the proof of Theorem 2 we need to verify.

PROPERTY C OF THEOREM 2. *There is a map $g_i: S^{2^i} \to X_i$ such that the composition $S^{2^i} \to X_i \to X_i/[X_i]^{2^i-1} \simeq S^{2^i-1}$ represents η, the generator of $\pi_1{}^S$.*

†Here and for the rest of this argument a basis for the cohomology group is represented by the same symbol as the homology basis. The reader should not assume that an indicated product, which is a homology product, is also a cohomology product.

Recall that $X_i = F_{2^{i-3}}(9)/F_{2^{i-3}-1}(9)$. Let $B(n)$ be the stable complex $F_{2n}(2l+1)/F_{2n-1}(2l+1)$ which by 2.5 does not depend on l. Normalize $B(n)$ so the first essential cell is in dimension zero. Since $X_i = \Sigma^{(2^{i-3})(7)}B(2^{i-4})$, we need to prove

PROPOSITION 5.1. *There is a map* g_i': $S^{2^{i+1}} \to B(2^i)$ *such that the composite*

$$S^{2^{i+1}} \to B(2^i) \to B(2^i)/[B(2^i)]^{2^{i+1}-2} \simeq S^{2^{i+1}-1}$$

is essential.

We will use the Adams spectral sequence approach. Let Λ be the Λ algebra of Kan *et al.*[3]. Recall that Λ is generated by symbols $\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_l}$ with $2i_j \geq i_{j+1}$. We abbreviate such a symbol by λ_I where $I = \{i_1, i_2, \dots, i_l\}$. Let $\Lambda_n \subset \Lambda$ be the subset of Λ generated by $\{\lambda_I | I$ admissible, $i_l < n\}$. In [4] the following is proved

PROPOSITION 5.2[4]. *The subset* Λ_n *is a sub-chain complex and if* $\mathscr{A}(n) = \Lambda/\Lambda_n$ *then* $H_*(\mathscr{A}(n), d) \simeq \mathrm{Ext}_A{}^*(M(n), Z_2)$.

Note that a basis for $\mathscr{A}(n)$ can be taken to be $\{\lambda_I | I$ admissible, $i_l \geq n\}$.

We are indebted to Ed. Brown for the following lemma.

LEMMA 5.3. *If* $\lambda_I \in \mathscr{A}(n)_j$ *for* $j \leq 2n+1$, *then* $d\lambda_I = 0$.

Proof. Let $I = (i_1, \dots, i_l)$ be admissible; let $L(I) = i_l$ and $l(I) = l$. The relations in Λ are given by

$$\lambda_i\lambda_j = \Sigma \binom{s-1}{2s-(j-2i)} \lambda_{i+s}\lambda_{j-s}.$$

The differential on λ_j is given by the same formula for $i = -1$. To prove the lemma we will use induction on $l(I)$. The lemma is true for l_i by an easy calculation. Now suppose it is true for all λ_I such that $l(I) < l$. Since $\lambda_{i_1}\lambda_{-1}\lambda_{i_2} = \lambda_{i_1}(\Sigma\lambda_r\lambda_s)$ when arranged in admissible form, we need only verify the lemma for $l_{-1}\lambda_I$. Suppose $\lambda_{-1}\lambda_{i_1}\cdots\lambda_{i_{l-1}} = \Sigma\lambda_{I_t}$. Let $s_t = L(I_t)$. By induction we have

$$\sum_{j=1}^{l-1} i_j \geq 2s_t + 2$$

for each t. Hence $\lambda_{I_t}\lambda_{i_l} = \lambda_{I_t'}\lambda_{s_t}\lambda_{i_l}$

$$= \sum_r \lambda_{I_t'}\lambda_{u_r^t}\lambda_{v_r^t}$$

where $2v_r^t \leq i_l + 2s_t$. This last conclusion uses the formula for relations. Therefore $\lambda_{I_t}\lambda_{i_l} = \sum_{r,p} \lambda_{I_{t,r,p}}$, $L(I_{t,r,p}) \leq v_r^t$ and $I_{t,r,p}$ admissible. Hence $2L(I_{t,r,p}) \leq 2v_r^t \leq i_l + 2s_t \leq i_l + \sum_{j=1}^{l-1} i_j \leq |I| - 2$. Thus if $L(I_{t,r,p}) \geq n$ for some t, r, p, then $2 + 2n \leq |I|$. This implies 5.3.

COROLLARY 5.4. *In the Adams spectral sequence for* $B(2^i)$, *the class* $\lambda_{2^{i+1}} \in E_2^{1,2^{i+1}+1}$ *is not zero.*

To prove 5.1 we will show that $\lambda_{2^{i+1}}$ projects to a non-zero homotopy class which has the desired property. To prove that it is a cycle for each r we need another lemma.

LEMMA 5.5. *There is a cofiber sequence* $B(n-1) \to B(n) \to \sum^{2^n} B\left[\frac{n}{2}\right]$.

Proof. Let $(\Omega^2 S^{2^i+1})^k$ be the k skeleton. Then $F_n(2^i+1) = [\Omega^2 S^{2^i+1}]^{(n+1)(2^i-1)-1}$ if $n+1 < 2^i$. Let

$\Omega^2S^{2^i+1} \to \Omega^2S^{2^{i+1}+1}$ be the James map H. Then H_*, the induced map in homology, is an epimorphism. This shows that there is a map $F_{2j}(2^i+1) \to F_j(2^{i+1}+1)$ for all $j < 2^{i-1}$. Finally note that at the prime 2 we have a principal fibration

$$\Omega S^{2^i} \to \Omega^2 S^{2^i+1} \to \Omega^2 S^{2^{i+1}+1}.$$

Hence $S^{2^i} \times S^{2^i-1} \times F_{2j}(2^i+1) \to F_{2j+2}(2^i+1) \to F_{j+1}(2^{i+1}+1)$ factors through $F_j(2^{i+1}+1) \to F_{j+1}(2^{i+1}+1)$. Thus for each n and i such that $n+1 < 2^{i-1}$, the composite

$$\frac{S^{2^i-1} \times S^{2^i-1} \times F_{2(n-1)}(2^i+1)}{S^{2^i-1} \times S^{2^i-1} \times F_{2(n-1)-1}(2^i+1)} \xrightarrow{j} \frac{F_{2n}(2^i+1)}{F_{2n-1}(2^i+1)} \xrightarrow{p} \frac{F_n(2^{i+1}+1)}{F_{n-1}(2^i+1)}$$

is trivial but in homology j_* is a monomorphism and p_* is an epimorphism so the sequence is a cofiber sequence. This establishes the lemma.

(This argument is due to R. J. Milgram and replaces the more complicated original proof.)

We also need

LEMMA 5.6. *The cofiber sequence of* 5.5 *induces a long exact sequence*

$$\cdots \to E_2^{s,t}(B(2^i-1)) \xrightarrow{i_*} E_2^{s,t}(B(2^i)) \xrightarrow{p_*} E_2^{s,t-2^i}(B(2^{i-1})) \xrightarrow{\delta} E_2^{s+1,t}(B(2^i-1)) \to \cdots$$ *and* p_*: $E_2^{s,2^{i+1}-1+s}(B(2^i)) \to E_2^{s,2^i+s-1}(B(2^{i-1}))$ *is zero for each s.*

Proof. The long exact sequence follows from the fact that the cohomology modules form a short exact sequence[1]. The homomorphism property is obtained by combining 5.5 and 5.3. At the chain level we have

$$\mathcal{A}(2^{i-1}) \xrightarrow{q} \mathcal{A}(2^i-1) \xrightarrow{i} \mathcal{A}(2^i) \text{ where } q\lambda_I = \lambda_I\lambda_{2^i-1}.$$

Recall that if $\lambda_I \in \mathcal{A}(2^{i-1})$ then $\lambda_I = \lambda_{I'}\lambda_{i_l}$ with $i_l \geq 2^{i-1}$ and so $g\lambda_I$ is already admissible if λ_I is. Lemma 5.3 implies that if $|I| < 2^{i+1}$ then $d\lambda_I = 0$ in $\mathcal{A}(2^i-1)$. Thus q induces a monomorphism on classes λ_J with $|J| < 2^i$. But q induces δ in the exact sequence and this implies the claim on p.

Now we prove

LEMMA 5.7 *The class* $\lambda_{2^{i+1}} \in E_2^{1,2^{i+1}+1}(B(2^i))$ *is a cycle for each r in the Adams spectral sequence.*

Proof. Clearly $\lambda_{2^{i+1}}$ is never a boundary and so if it is an r cycle it is non-zero in E_{r+1}. Define for each i an integer or ∞, r_i, by the following rule: if $\delta_{r_i}\lambda_{2^{i+1}} \neq 0$ in $E_{r_i}(B(2^i))$ then this defines r_i. If $\lambda_{2^{i+1}}$ is a cycle for all r, then let $r_i = \infty$.

Now fix i so that r_i is the smallest integer in the set $\{r_i\}$. We need to show $r_i = \infty$. Consider the cofiber sequence

$$B(2^{i+1}-1) \to B(2^{i+1}) \to \Sigma^{2^{i+1}}B(2^i).$$

By hypothesis on $\lambda_{2^{i+2}} \in E^{1,2^{i+2}+1}(B(2^{i+1}))$ δ_{r_i} is defined. By direct calculation we have $p_*\lambda_{2^{i+2}} = \lambda_{2^{i+1}}$, thus $p_*\delta_{r_i}\lambda_{2^{i+2}} = \delta_{r_i}\lambda_{2^{i+1}} \neq 0$ but in this gradation p_* is zero by Lemma 5.6. This implies Lemma 5.7.

To complete the proof of 5.1 we must show that if g_i' represents $\lambda_{2^{i+1}}$ it has the additional property required. It will be sufficient to show that the map

$$E_2^{s,t}(B(2^i)) \xrightarrow{h_*} E_2^{s,t}(S^{2^{i+1}-1})\ h_*\lambda_{2^{i+1}} = h_1.$$

satisfies

LEMMA 5.8. *If N is any power of 2 larger than 2^i, there is a map of A-modules*

$$\tilde{H}^*(\Sigma^{-N+2^{i+1}+1}P^{N-2}_{N-2^{i+1}-1}) \underset{k}{\leftarrow} M(2^i) \quad \text{with} \quad k(1) \neq 0.$$

Proof. The easiest way to see this is to observe that if $\bar{B}(2^i)$ is the actual Brown–Gitler spectrum then according to [4] there is a map of $\mathcal{D}(P^{2^{i+1}}) \to \bar{B}(2^i)$ whose induced map in cohomology is the one we want.

Note that k is an isomorphism in dimension $2^{i+1}-1$ since the single generator in each module is $Sq^{2^{i+1}-1}$ on the zero dimensional cell.

LEMMA 5.9 *For each i and for N = order $\tilde{K}(RP^{2^i})$ there is a map $\bar{g}$: $S^{N-1} \to P^{N-2}_{N-2^i-1}$ which is coextension by $\eta \epsilon \pi_1{}^S$ on the top cell.*

Proof. Recall that the Radon Hurwitz theorem on vector fields on spheres implies that there is a map $S^{N-1} \underset{p}{\to} P^{N-1}_{N-2^i-1}$ of degree 1. Consider the composite $S^{N-1} \to P^{N-1}_{N-2^i-1} \subset P^{\infty}_{N-2^i-1}$ which we will call $\bar{p}$. Then $2\bar{p}$ factors through the $N-2$ skeleton but not through the $N-3$ skeleton. To see this consider

$$\begin{array}{ccccc} P_{N-2^i-1} & \xrightarrow{\rho_1} & P_{N-2} & \xrightarrow{\rho_2} & P_{N-1} \\ \uparrow{\scriptstyle \bar{p}} & & & & \\ S^{N-1} & & & & \end{array}$$

Since the first two cells of the right hand space form a Z_2 Moore space, $\rho_2\rho_1(2\bar{p}) \simeq 0$. Thus $\bar{p}$ factors through the $N-2$ skeleton. Let $\bar{g}$: $S^{N-1} \to P^{N-2}_{N-2^i-1}$ be such a factorization. Since $\pi_{N-1}(P_{N-2}) = Z_4$ generated by $\rho_1\bar{p}$ we see that $\rho_1(2\bar{p}) \neq 0$[13]. Hence $2\bar{p}$ does not factor through $N-3$ skeleton. This implies $\bar{g}$ is an essential coextension of the $N-2$ cell and this implies 5.9.

We now can complete the proof of 5.1. By construction the map $\bar{g}$ has filtration 1 and so we have

$$\begin{array}{ccc} & E_2(S^{2^{i+1}-1}) & \\ {\scriptstyle \bar{h}_*} \nearrow & & \nwarrow {\scriptstyle h_*} \\ E_2(\Sigma^{-N+2^{i+1}+1}P^{N-2}_{N-2_{i+1}-1}) & \xrightarrow[k_*]{} & E_2(B(2^i)). \end{array}$$

Let $[g]$ be the class representing $\bar{g}$ then $\bar{h}_*[g] = h_1$ hence $h_*k_*[g] = h_1$. Thus $k_*[g] \neq 0$. Thus $k_*[g] = \lambda_{2^{i+1}}$ and $h_*\lambda_{2^{i+1}} = h_1$ and this completes the proof of property (c) and hence Theorem 2.

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