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The image of J in the EHP sequence

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1. Introduction

The EHP sequence is based on a result of James [J] who showed that there is a map H such that $S^n \to \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$ is a fibration when localized at 2. The map $S^n \to \Omega S^{n+1}$ is usually labeled E. The boundary homomorphism in the homotopy sequence is usually labeled P. For our purposes it will be most convenient to combine all the EHP sequences into one system. This gives the following filtration of $\Omega^\infty \Sigma^\infty S^0 = Q(S^0)$:

$$\Omega S^1 \subset \Omega^2 S^2 \cdots \subset \Omega^n S^n \cdots \subset Q(S^0).$$

Associated to this filtration is a spectral sequence $\{E^r_{s,t}\}$ whose E^1 term is $E^1_{s,t} = \pi_t(\Omega^s S^s, \Omega^{s-1} S^{s-1})$ and $d^r\colon E^r_{s,t} \to E^r_{s-r,t-1}$. The E^∞-term is an associated graded group of $\pi_*(S^0)$. James' result referred to above gives $E^1_{s,t} = \pi_t(\Omega^s S^{2s-1})$ when localized at 2. Indeed, since this is the calculation of the homotopy groups of the base space in a fibration, if Y is any space then the homotopy theory $[\Sigma^j Y, \ ]$ can be used. The resulting spectral sequence $\{E^r_{s,t}(Y)\}$ has as E^1 term $E^1_{s,t} = [\Sigma^t Y, \Omega^s S^{2s-1}]$.

We will study this EHP spectral sequence. The primary tool will be a comparison of this homotopy functor spectral sequence with another spectral sequence which is a homology spectral sequence. This second spectral sequence is sometimes called the stable EHP sequence. We now describe it.

Snaith [S] has constructed maps $\Omega^n S^{n+1} \to Q(\Sigma P^n)$, where P^n is real projective space, which give maps $\Omega^{n+1} S^{n+1} \to Q(P^n)$. In [K] it is proved that skeleton filtration of P is comparable to the EHP sequence filtration. In particular for each n there is a map $(\Omega^n S^n, \Omega^{n-1} S^{n-1}) \to (QP^{n-1}, QP^{n-2})$. The skeleton filtration in P induces a spectral sequence $\{SE^r\}$ whose E^1 term is $SE^1_{s,t} = \pi_t(QS^{s-1})$ and Snaith and Kuhn's results together give a mapping of spectral sequences s: $\{E^r_{s,t}\} \to \{SE^r_{s,t}\}$. This map at E^1 is induced by $\Omega^s S^{2s-1} \subset QS^{s-1}$.

The Kahn-Priddy theorem [KP] shows that $\pi_*(Q(S^0)) \to \pi_*(QP)$ is monic in dimension > 0. It is in the nature of spectral sequences that this property should not hold at the E_∞ level and, indeed, the generator of the image of J in $\pi_{16}(QS^0)$ has bigrading (3, 16) in the *EHP* sequence and its image in $\pi_{16}(QP)$ has bigrading (2, 16).

The "image of J" for the purpose of this paper is the homotopy of the spectrum J which is the fiber of a map $\mathrm{bo} \xrightarrow{\psi} \Sigma^4\mathrm{bsp}$ where bo and bsp are the connected Ω-spectra whose E_0 term is $\mathbf{Z} \times \mathrm{BO}$ or $\mathbf{Z} \times \mathrm{Bsp}$ ($\mathbf{Z} \times$ the classifying space of the symplectic group) respectively. The map defining J was first given by Adams using the Adams operation $\psi^3 - 1$. Another such map is given in [M2]. The key result of [MM] is that for any map ψ: $\mathrm{bo} \to \Sigma^4\mathrm{bsp}$ such that $H^4(\psi)$ is a 2 primary isomorphism, there is a 2 primary isomorphism between $\pi_*(J)$ and $\pi_*(\psi)$. This paper studies the homotopy module $\pi_*(J)$ and the role this module plays in the *EHP* sequence.

First we can produce a stable *EHP* sequence with J_* as the functor. This gives a mapping of spectral sequences $\{E_{**}\} \xrightarrow{s(J)} \{SE^r_{**}(J)\}$ where

$$SE^1_{s,t}(J) = \pi_t\big(\Omega^\infty(S^{s-1} \wedge J)\big) = J_t(S^{s-1}).$$

We think of $SE^r_{**}(J)$ as the stable *EHP* sequence for the image of the J spectrum. We wish to compare this spectral sequence with $\{E^r_{**}\}$. The idea will be to find a homotopy theory which can be applied to the *EHP* sequence filtration (so that we can use James' result to calculate the E^1 term) and which stabilizes to a homology theory in some sense isomorphic to J_* (so that we can compare it with the stable *EHP* sequence for the J spectrum).

The homotopy theory we will use is based on $\mathbf{C}P^2 \wedge \mathbf{R}P^2$. Let $\{Y^n\}$ be the suspension spectrum based on this smash product indexed so that $Y^n = \Sigma^{n-6}\mathbf{C}P^2 \wedge \mathbf{R}P^2$. Let $\pi_n(X; Y) = [Y^n, X]$. For n large there is a self map $Y^{n+2} \xrightarrow{v_1} Y^n$ each of whose iterates is essential (Prop. 2.2). Thus $\pi_*(X; Y)$ is a $\mathbf{Z}/4[v_1]$ module. In Section 2 we show that $2v_1 = 0$ so $\pi_*(X, Y)$ is a $\mathbf{Z}/4[v_1]/2v_1$ module. Call this model R. The homotopy theory we wish to study is $\pi_*(\ ; Y) \oplus_R \mathbf{Z}/2[v_1, v_1^{-1}] = V^{-1}\pi_*(\ ; Y)$. (If M is any R module we will let $V^{-1}M = M \otimes_R \mathbf{Z}/2[v_i, v_1^{-1}]$.)

A corollary of the main result in this paper is the following.

THEOREM 1.0. *The Snaith map induces an isomorphism* $V^{-1}(\pi_*(QS^0); Y) \to V^{-1}(J \wedge Y)_*(P)$.

This result shows that $V^{-1}(\pi_*(\ ;Y))$ is the homotopy theory for which we are looking. For technical reasons it is easier to look at a modification of the *EHP* sequence. We wish to consider the following:

$$\begin{array}{ccccccccc}
\Omega S^1 & \subset \Omega^3 S^3 & \subset & \cdots & \subset \Omega^{2n+1}S^{2n+1} & \subset & \cdots & \subset & Q(S^0) \\
\downarrow & \downarrow & & & \downarrow & & & & \downarrow \\
QP^0 \to & QP^2 \to & & \cdots \to & QP^{2n} & \longrightarrow & \cdots & \to & Q(P).
\end{array}$$

Let $W(n)$ be the fiber of the map $S^{2n-1} \to \Omega^2 S^{2n-1}$. Then for this filtration we get an *EHP*-like sequence $\{\bar{E}^r_{s,t}\}$ with

$$\bar{E}^1_{s,t} = \pi_t(\Omega^{2s+1}S^{2s+1}, \Omega^{2s-1}S^{2s-1}) = \pi_t(\Omega^{2s-2}W(s)).$$

Let $\{S\bar{E}^r_{s,t}\}$ be the corresponding stable *EHP* sequence with $S\bar{E}^1_{s,t} = \pi_t(QP^{2s}_{2s-1})$. Using the homotopy theory $V^{-1}\pi_*(\ ;Y)$ and the homology theory based on $J \wedge Y$, we get a mapping of spectral sequences $\{\bar{E}^r_{*,*}(V^{-1}\pi_*(\ :Y)\} \to \{SE^r_{*,*}(V^{-1}(J \wedge Y_*(\ ))\}$. The E_1 terms give the following map:

$$V^{-1}\big(\pi_*(\Omega^{2s-2}W(s);Y)\big) \overset{s_*}{\to} V^{-1}(J \wedge Y)_*\big(P^{2s}_{2s-1}\big)$$

which is the map induced by the composite $\Omega^{2s-2}W(s) \to Q(P^{2n}_{2n-1}) \to \Omega^\infty(P^{2n}_{2n-1} \wedge J \wedge Y)$.

A principle result of this paper is the following:

THEOREM 1.1. *The map s_* is an isomorphism and induces an isomorphism of spectral sequences*

$$\big\{E^r_{*,*}\big(V^{-1}\pi_*(\ ;Y)\big)\big\} \to \big\{SE^r_{*,*}\big(V^{-1}(J \wedge Y)_*(\ )\big)\big\}.$$

Since we are really interested in $\pi_*(\Omega^{2n+1}S^{2n+1})$ and $\pi_*(QP^{2n})$, we need to get more information than just what Theorem 1.1 gives. To this end we calculate $V^{-1}\pi_*(\ ;Y)$ for the various spaces involved.

THEOREM 1.2. $V^{-1}(\pi_*(S^0;Y)) = E(a)$ *and* $V^{-1}(\pi_*(M_{2\iota};Y)) = E(a,b)$ *where the gradation of a and b is* 1. *(S^0 is the suspension spectrum of the sphere and $M_{2\iota}$ is the $\mathbf{Z}/2$ Moore spectrum.)*

THEOREM 1.3. *Let s_n be the Snaith map* [S] s_n: $\Omega^{2n}S^{2n+1} \to \Omega^\infty\Sigma^{\infty+1}P^{2n}$. *Then s_n induces an isomorphism*

$$V^{-1}\big(\pi_*(S^{2n+1};Y)\big) \to V^{-1}\big(\pi_*\big(\Omega^\infty\Sigma^\infty(\Sigma^{2n+1}P^{2n});Y\big)\big).$$

THEOREM 1.4. *Let g_n: $P^{2n} \to P^{2n} \wedge J$ be the Hurewicz map. Then g_n induces an isomorphism*

$$V^{-1}\big(\pi_*(P^{2n}; Y)\big) \to V^{-1}\big(\pi_*(P^{2n} \wedge J; Y)\big)$$

and $V^{-1}(\pi_*(P^{2n} \wedge J; Y)) = E(a, b_n)$. The gradations of a and b_n are both 1.

A comment about these results and K-theory is in order. Since K theory is a homology theory it is not very good on unstable objects and fibrations. The results of this paper suggest that the homotopy theory, $\pi_*(\ ; Y)$, is the theory of choice to use on such unstable objects to get the kind of results that K-theory gets on stable objects.

These results have another interpretation. Let $M^n = S^{n-1} \cup_{2\iota} e^n$ be the n^{th} space in the $\mathbf{Z}/2$ Moore space spectrum, $M_{2\iota}$. Adams [A] constructed a map A: $M^{8+n} \to M^n$ for n large with the property that all of the iterates of A are essential. Let α be a class in $[M^i, S^n]$. The question we ask is: Find all α such that αA^k is essential for all k. Adams in [A] found a collection of classes with this property and we proved in [M2] that his collection is everything which occurs stably. The above results describe everything which occurs unstably for odd spheres and which occurs stably for P^{2n}. From this description it is clear that we are really studying an unstable e-invariant and getting a complete description.

In [M1], results concerning the homotopy module $\pi_*(J)$ in the *EHP* sequence are described. Detailed proofs have not appeared. This paper describes those results and gives proofs of the main results in [M1]. The key results in [M1] follow directly from the following.

THEOREM 1.5. *The composite $\pi_j(\Omega^{2n+1}S^{2n+1}) \to \pi_j(QP^{2n}) \to J_j(P^{2n})$ is onto if $j \geq 2n+1$ and $j \not\equiv -2 \bmod 8$. If $j \geq 2n + 8 + 2i$ when $j \equiv (2^i - 2) \bmod 2^{i+1}$ then the composite is onto.*

The exceptional values of j correspond to the Kervaire invariant problem. This connection is discussed in Section 7. In particular see Theorem 7.11.

In proving Theorem 1.5 we will give an explicit construction of the homotopy classes involved. In particular we show that the classes in $\pi_*(\Omega^{2n+1}S^{2n+1})$ which map onto the classes in J_*P^{2n} are in the image of $\pi_*(\Omega^{2n+1}\Sigma^{2n+1}P^{2n+1})$ where $P^{2n} \to \Omega^{2n+1}S^{2n+1}$ factors through $\mathrm{SO}(2n+1)$. This shows that the homotopy classes are natural under suspension. One might ask whether the map $\Sigma^{2n+1}P^{2n} \to S^{2n+1}$ induces an epimorphism in homotopy in dimension greater than $2n+1$. This would be a finite Kahn-Priddy result.

Finally it is worthwhile mentioning at least one of the corollaries of 1.5.

THEOREM 1.6. *If $n \equiv 3, 5 \bmod 8$, $n > 3$, then $\pi_{j+n}(S^n)$ is nonzero for all $j \geq n$.*

This theorem follows from 1.5 and the calculations in Section 7 which show that in these cases, $J_i(P^{2n}) \neq 0$. In particular, compare the table given after 7.9.

Sections 5 and 6 of [M1] give some further applications. In particular the unstable composition properties of the classes described by 1.5 in terms of the form $8k-1$ are particularly interesting.

Section 2 describes the functor $\pi_*(\ ;Y)$ and redoes [M2] in this language. The whole theory of bo-resolutions takes on a particularly nice, simple form when studied this way. Theorem 1.2 is proved in this section. It is essentially a restatement of the main theorem of [M2].

In Section 2, the theory is studied effectively for the spectra S^0 and M_{2^ι}. In order to get at theorems like 1.1 and 1.3 we use Adams resolutions. By this we mean a resolution of a space by generalized Eilenberg-MacLane spaces to get unstable Adams spectral sequences. We generalize the notion slightly and study the situation where we have a map $f\colon X \to Z$, where X and Z are spaces with resolutions such that f extends to a map between the resolutions. We show in Section 3 how this situation leads to a resolution of the fiber of f. In this setting, if f induces an isomorphism in the resolution above a "1/5 line" then f induces an isomorphism in the v_1-periodic homotopy. Sections 3, 4 and 5 are devoted to describing this situation in general and to applying it in particular to the Snaith map $W(n) \to \Sigma^{4n-3}M_{2^\iota}$. In [M3], an algebraic map between resolutions of $W(n)$ and the stable resolution of M_{2^ι} is given. In Section 5, using a result of F. Cohen, we show how to get a geometric map between the two resolutions. Theorem 1.1 then follows quite easily.

Theorem 1.0 follows immediately from Theorem 1.1 and the map $\lambda\colon P \to S^0$ which is the stable version of $P \to \mathrm{SO} \to QS^0$. The composite $QP \xrightarrow{Q\lambda} QS^0 \to \Omega^\infty(P \wedge J)$ induces an isomorphism $V^{-1}\pi_*(P;Y) \to V^{-1}(J \wedge Y)_*(P)$ by 1.1 and this gives 1.0.

Theorems 1.3 and 1.4 are proved in Section 6.

The results of Section 2 on bo-resolutions give quickly a proof of 1.4. The proof of 1.5 occupies the bulk of Section 7.

Section 8 gives some results which connect the homotopy detected by $J_*(P^{2n})$ with the image of J. The connection of Theorem 1.5 with the EHP sequence is also discussed.

2. bo-resolutions

The key results on bo-resolutions which allow for a determination of the homotopy localized at v_1 are contained in [M2]. These results are summarized here from the point of view of the functor $\pi_*(\ ;Y)$.

First we will introduce $\pi_*(\ ;Y)$ formally and prove some of the elementary properties.

Definition 2.1. Let Y be the suspension spectrum of $\mathbf{CP}^2 \wedge \mathbf{RP}^2$ indexed so that $Y^n = \Sigma^{n-6}\mathbf{CP}^2 \wedge \mathbf{RP}^2$. Let $\pi_n(X;Y) = [Y^n, X]$. If X is a space, then $n \geq 6$. To get an Abelian group we need $n \geq 8$.

PROPOSITION 2.2. 1) *There is a self map* v_1: $Y^2 \to Y^0$ *such that the composite* v_1^k: $Y^{2k} \to Y^0$ *is essential for all* k. *The map* v_1 *can be chosen to be self dual.*

2) *The identity map of* Y *has order* 4. *The following composite is twice the identity map*: $Y^3 \xrightarrow[p]{} S^3 \xrightarrow[\nu]{} S^0 \xrightarrow[i]{} Y^3$ *where* p *pinches onto the top cell and* i *is the dual of* p.

Proof. Part 1 is quite easy and is covered in detail in [DM]. Part 2 can best be seen by looking at $M_{2\iota} \wedge Y$. The Cartan formula shows Sq^4 is non-zero and this shows that the three skeleton of $M_{2\iota} \wedge Y^3$ is $[Y \cup \Sigma^1 Y]^{(3)}$. Thus the map 2 has the same cofiber as the map given in 2.2.2.

Let $R = \mathbf{Z}/4\,[v_1]/2v_1 = 0$. Let v_1 have degree 2. Let $R' = \mathbf{Z}/2\,[v_1]$.

PROPOSITION 2.3. *The homotopy theory* $\pi_*(\ ;Y)$ *is an* R-*module.*

Proof. First note that because v_1 is chosen to be self dual $2v_1 = v_1 2$. Next we need to show $Y^5 \xrightarrow{2} Y^5 \xrightarrow{v_1} Y^3$ is null homotopic. But this is the same as $Y^5 \to S^5 \xrightarrow[\nu]{} S^2 \to Y^5 \to Y_3$. This composite has Adams filtration 2 and indeed the map $S^5 \to Y^3$ is also a map of Adams filtration 2. But $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(Y), \mathbf{Z}/2) = 0$ if $s \geq 2$ and $t - s = 5$. This calculation is easy.

PROPOSITION 2.4. $\pi_*(\mathrm{bo}; Y) = R'$.

Proof. By Spanier-Whitehead duality, $[Y^i, \mathrm{bo}] = \pi_i(\mathrm{bo} \wedge Y^3)$. But $\mathrm{bo} \wedge \Sigma^{-2}\mathbf{C}P^2 \simeq bu$. Thus $\pi_*(\mathrm{bo} \wedge Y^3) = \pi_*(bu; \mathbf{Z}/2) = \mathbf{Z}/2[v_1]$.

This result suggests that Y is the dual homotopy theory to the cohomology theory bo. The duality is strained but $K(\mathbf{Z})$ is the space whose homotopy is $\mathbf{Z}$ which is the universal ring for ordinary homotopy π_*. Now 2.4 says that bo is the "Eilenberg-MacLane" space for $\pi_*(\ ;Y)$ whose universal ring is R. (R and R' are different but the analogy is still amusing.) Because of this, bo-resolutions are fundamental for studying Y-homotopy.

Recall from [M2] the following results. By a bo-resolution we mean a tower of spectra

$$\begin{tikzcd} S^0 \arrow[d, "\iota"] & S_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & S_s \arrow[l] \arrow[d] & S_{s+1} \arrow[l] & \cdots \arrow[l] \\ \mathrm{bo} & S_1 \wedge \mathrm{bo} & & S_s \wedge \mathrm{bo} & & \end{tikzcd}$$

where $S_s \wedge \mathrm{bo} \leftarrow S_s \leftarrow S_{s+1}$ is a fibration and $\iota\colon S^0 \to \mathrm{bo}$ is the unit. If we apply π_*, then we get an exact couple with $E_1^{s,t} = \pi_{t-s}(S_s \wedge \mathrm{bo})$.

There is a natural map $\varepsilon\colon \pi_*(\mathrm{bo}) \to \mathbf{Z}/2$ induced by $\mathrm{bo} \to K(\mathbf{Z}/2)$. We say a $\pi_*(\mathrm{bo})$ module M is a $\mathbf{Z}/2$-vector space if there is a commutative diagram

$$\begin{tikzcd} \pi_*(\mathrm{bo}) \otimes M \arrow[r] \arrow[dr, "\varepsilon \otimes \mathrm{id}"'] & M \\ & \mathbf{Z}/2 \otimes M \arrow[u, "\simeq"'] \end{tikzcd}$$

The main result of [M2] is the following.

THEOREM 2.5.

$$\text{a)}\quad E_\infty^{0,t} = \begin{cases} \mathbf{Z}, & t = 0 \\ \mathbf{Z}/2, & t \equiv 1, 2 \bmod 8 \\ 0 & \text{all other } t \end{cases}$$

$$\text{b)}\quad E_\infty^{1,t} = \begin{cases} \mathbf{Z}/2^{|8k|_2} & t = 4k \\ \mathbf{Z}/2 & t = 1, 2 \bmod 8 \\ 0 & \text{all other } t \end{cases}$$

where $|j|_2$ is the 2-adic norm of j (the power of 2 in the prime decomposition of j), and all classes are v_1-periodic.

c) *$E_\infty^{s,t} = 0$ for $6s > t + 12$ and is a $\mathbf{Z}/2$ vector space (as a $\pi_*\mathrm{bo}$ module) for all $s > 1$ and all t.*

This is proved in [M2] and another discussion of it is given in [DGM]

This theorem yields the following. Let $\{E_r(S^0; Y)\}$ be the exact couple which results from applying the functor $\pi_*(\ ; Y)$ to the bo resolution.

THEOREM 2.6. a) *$E_\infty^{0,*} = R'$ with the generator having dimension 0.*

b) *$E_\infty^{1,*} = \mathbf{Z}/2 + R'$ with the $\mathbf{Z}/2$ class having dimension 4 and the free class having dimension 6.*

c) *$E_\infty^{s,*}$ is a $\mathbf{Z}/2$ vector space as an R-module and $E_\infty^{s,t} = 0$ for $6s > t + 12$, $s > 1$. (There is a natural map of $R \to \mathbf{Z}/2$.)*

We will show below how 2.5 implies 2.6 and we will outline, using detailed results from [M2], a direct proof of 2.6. It is hoped that the second part will be useful in understanding [M2]. First we observe the following corollary.

COROLLARY 2.7. *$V^{-1}(\pi_*(S^0; Y)) = E(a)$ where $E(a)$ is the exterior algebra on one generator over the ring $v_1^{-1}R$.*

This is part 1 of 1.2. The second part of 1.2 follows immediately from the sequence $S^0 \to M_{2\iota} \to S^1$ and the fact that the homotopy sequence for $\pi_*(\ , Y)$ splits on the parts which are R' free.

As an illustration of $\pi_*(\ ,Y)$, we give the following low-dimensional calculations. For free R or R' modules only the generator is given.

	$j=0$	1	2	3	4	5	6,
$\pi_j(S^0;Y)=$	R'	0	0	$\mathbf{Z}/2$	0	R'	$\mathbf{Z}/2$,
$\pi_j(M_{2\iota};Y)=$	R'	R'		$\mathbf{Z}/2$	$\mathbf{Z}/2$	R'	R.

The only surprise is the R in $\pi_6(M_{2\iota},Y)$. These calculations are easily obtained from the known Ext calculations.

We will now give the proof of 2.6 from 2.5. The homotopy described in 2.5 for $s=0$ and 1 is the ker and coker respectively of the map τ: bo $\to \Sigma^4$bsp which is a homology isomorphism in dimension 4. Clearly $\pi_*(\text{bo};Y)=R'$ with generator in dimension 0 and $\pi_*(\Sigma^4\text{bsp};Y)=\mathbf{Z}/2\oplus R'$ with generators in dimension 4 and 6 respectively. The second statement may take a little work. Recall bsp $=B(1)\wedge$ bo and $B(1)$ is the cofiber of $\eta^\#\colon \Sigma M_{2\iota}\to S^0$. Consider the sequence

$$\cdots\to\pi_j(\Sigma M_{2\iota}\wedge\text{bo};Y)\underset{\eta^\#_*}{\to}\pi_j(S^0\wedge\text{bo};Y)\underset{i_*}{\to}\pi_j(\text{bsp};Y)\to\cdots.$$

The groups are:

$$\begin{aligned}
\pi_j(\Sigma M_{2\iota}\wedge\text{bo};Y)&=0 && j=0,3\\
&=R' && j=1,2 \text{ generated by } a_j.\\
\pi j(S^0\wedge\text{bo};Y)&=0 && j\neq 0\\
&=R' && j=0 \text{ generated by } 1.\\
\pi_j(\text{bsp};Y)&=\mathbf{Z}/2 && J=0\\
&=R' && j=2 \text{ generated by } b\\
&=0 && j=1,3.
\end{aligned}$$

It is easy to see that $\eta^\#_* a_2=v_1\iota$ so $j_*\colon (R'\iota)\to\mathbf{Z}/2$ is the augmentation. Then $\partial_* b=a_1$ is necessary for exactness.

This establishes parts a and b. To establish part c, let $Y^n\to S^0$ be a map which lifts to S_s, $s>1$. We have the following diagram:

$$\begin{tikzcd}
 & \Omega S_{s-1}\wedge\text{bo} \arrow[dr, "i_s"] & & \\
Y^{n+2} \arrow[ur, dashed, "\phi"] \arrow[r, "v_1"'] & Y^n \arrow[r, "f_s"] \arrow[dr, "f_1"'] & S_s \arrow[r, "g_s"] & S_s\wedge\text{bo}. \\
 & & S^0 &
\end{tikzcd}$$

The map v_1 has Adams filtration 1 so that the composite $g_s f_s v_1$ has Adams filtration ≥ 1. By 5.11 of [M2] (see also 3.6 of [DM2]), this implies there is a map ϕ so that $g_s i_s \phi = g_s f_s v_1$. Thus $(f_s v_1 - i_s \phi)$ is a lift of $f v_1$ which lifts to S_{s+1}. Thus if $f\colon Y^n \to S^0$ has filtration $s > 1$ in the bo resolution spectral sequence, then the composite $Y^{(n+2)} \xrightarrow{v_1} Y^n \xrightarrow{f} S^0$ has filtration $\geq s+1$ and this is part c.

Note that Corollary 2.7 actually asserts a little more. Indeed if $\alpha \in \pi_n(S^0; Y)$ has bo-resolution filtration > 1 then there is a k such that $\alpha v_1^k = 0$. The above argument shows that αv_1^k has filtration $> 1 + k$. The connectivity of S_s is $3s - 1$ while $\alpha v_1^k \in \pi_{n+2k}(S^0; Y)$. Thus there is a k such that $\pi_{n+2k}(S_{1+k}; Y) = 0$ and αv_1^k factors through this group.

Next we will recast the proof of 2.5 given in [M2] in terms of $\pi_*(\ ; Y)$. It has a particularly nice form when we invert v_1. These following paragraphs then contain a sketch of a direct proof of Corollary 2.7.

PROPOSITION 2.8. $\pi_*(\mathrm{bo} \wedge \mathrm{bo}; Y) = \bigoplus_{n=0}^{\infty} R'a_n \oplus W$ *where* a_n *is a generator in dimension* $8n - 2\alpha(n)$ *and* W *is some* $\mathbf{Z}/2$ *vector space* ($\alpha(n)$ *is the number of* 1's *in the dyadic expansion of* n; $\alpha(0) = 0$).

Proof. Theorem 2.4 of [M2] asserts that there are spectra $B(i)$ such that $\mathrm{bo} \wedge \mathrm{bo} = \vee \Sigma^{4i} B(i) \wedge \mathrm{bo}$. Theorem 3.6 of the same paper asserts that there are maps $f_i\colon B(2i) \wedge \mathrm{bo} \to \mathrm{bo}^{4i-\alpha(i)}$ and $g_i\colon B(2i+1) \wedge \mathrm{bo} \to \mathrm{bsp}^{4i-\alpha(i)}$ whose fibers are wedges of $K(\mathbf{Z}/2)$. (bo^k is the k-th level of an Adams resolution of bo. In particular $\mathrm{Ext}_A^{s,t}(H^*(\mathrm{bo}^k), \mathbf{Z}/2) = \mathrm{Ext}_{A_1}^{s+k,t+k}(\mathbf{Z}/2, \mathbf{Z}/2)$. bsp^k is similarly defined.) Since $\pi_*(\mathrm{bo}^{4i-\alpha(i)}; Y) = R' + W$ where the free R' generator is in dimension $8i - 2\alpha(i)$ and W is a $\mathbf{Z}/2$ vector space of some unspecified dimension, we have $\pi_*(\Sigma^{8i} B(2i) \wedge \mathrm{bo}; Y) = R'a_{2i} + W_{2i}$. Since $\pi_*(\mathrm{bsp}^{4i-\alpha(i)}; Y) = R' \oplus W$ where the free generator is in dimension $8i + 2\alpha(i) + 2$ and W is as before, we have $\pi_*(\Sigma^{8i+4} B(2i+1) \wedge \mathrm{bo}; Y) = R'a_{2i+1} + W_{2i+1}$. This completes the proof of the proposition.

Let $Z = \oplus_{n=0} R'a_n$ where the a_n are as in Proposition 2.8. This presentation of Z by just listing generators may seem strange. The next result justifies the notation. On the other hand there are several other ways to label generators. The notation, with $h_{i,j}$ as the class in the bar resolution for $|\xi_i^{2^j}|$, is frequently used. With this notation, $Z = E(h_{i,1}, i = 2, 3, \ldots)$. Since $h_{i,1}$ has homology dimension $2^{i+1} - 2$, the generator a_n corresponds to $\prod_{i \geq 2} h_{i,1}^{\varepsilon_i}$ where $n = \sum_{i=0} \varepsilon_i 2^i$ and $\varepsilon_i = 0$ or 1. Under this representation the next result is very complicated to state. We should also note that the formula of Ravenel [R] has some connection with the next proposition.

The bo resolution when $\pi_*(\ ;Y)$ is used as a functor gives rise to a chain complex where $Z = \oplus_{n\geq 0} R' a_n$:

$$R' \to Z \oplus W_1 \xrightarrow{d_1} Z \otimes Z \oplus W_2 \xrightarrow{d_2} \cdots \to Z^{\otimes s} \oplus W_s \to \cdots$$

where $Z = \oplus_{n\geq 0} R' a_n$ and W_s is a free $\mathbf{Z}/2$ module for each s. The next result asserts that d_1 at least on the R' free part behaves in a way analogous to a divided polynomial algebra.

PROPOSITION 2.9. *$d_1 a_n = \Sigma_{k+j=n} v_1^{\nu(k,j)} a_j \otimes a_k$ modulo classes in W_2, where $\nu(k,j)$ is the power of 2 present in the binomial coefficient $\binom{n}{j} = (j, k)$*

Proof. Theorem 5.8 [M2] implies that the composite 2.9.1,

2.9.1 $\quad \Sigma^{4n} B(n) \wedge \mathrm{bo} \to \mathrm{bo} \wedge \mathrm{bo} \to \mathrm{bo} \wedge \mathrm{bo} \wedge \mathrm{bo} \to \Sigma^{4n}(B(j) \wedge B(k) \wedge \mathrm{bo})$

for $j + k = n$ is a map of degree $\binom{n}{j}$ on the cell in dimension $4n$. This gives for n, j, and k even,

$$\mathrm{bo}^{2n-\alpha(n)} \to \mathrm{bo}^{2j+2k-\alpha(j)-\alpha(k)}$$

for filtration $\nu(j, k) = \alpha(j) + \alpha(k) - \alpha(n)$. But

$$\left(bo^{2-j+2k-\alpha(j)-\alpha(k)}\right)^{\alpha(j)+\alpha(k)-\alpha(n)} = bo^{2n-\alpha(n)}$$

and so the induced map gives rise to the map generated by

$$a_n \to v_1^{\alpha(j)+\alpha(k)-\alpha(n)} a_j \otimes a_k.$$

The proposition is the summing of all these parts. Note that if n, j or k is odd the argument works exactly the same way.

To complete the analysis of the bo resolution we note that 5.8 of [M2] implies that d_s for every s is the sum of maps like d_1. Thus the chain complex is that of a "divided polynomial algebra". Precisely we have

THEOREM 2.10. *In $V^{-1}(\mathbf{Z})$ let $\bar{a}_n = v_1^{-\alpha(n)} a_n$. Then*

$$d_1 \colon V^{-1}(Z) \to V^{-1}Z \otimes V^{-1}(Z) \text{ is } d_1 \bar{a}_n = \sum_{i+j=n} \bar{a}_i \otimes \bar{a}_j.$$

This is immediate from 2.9. Thus the bo resolution and the functor $V^{-1}(\pi_*(\ ;Y))$ give rise to the standard complex for computing the homology of a polynomial algebra on one generator. The answer is an exterior algebra on one generator. This is 2.6 and more specifically 2.7.

We close this section with some simple calculations.

2.11. Let $M_\eta^n = \Sigma^{n-4} \mathbf{C}P^2$ be considered as a suspension spectrum. There is a natural isomorphism of $[M_\eta^n; M_{2\iota}]$ and $\pi_n(S^0; Y)$.

This follows directly from Spanier-Whitehead duality.

2.12. $V^{-1}(\pi_*(P_{2n+1}^{2n+2k}; Y)) \simeq V^{-1}(\pi_*(S^0, Y))(b_n, c_{n,k})$. ($M(a, b)$ is a free M module on generators a and b.)

We prove this by induction. Suppose we have the result for fixed n and k. Then we have

$$\begin{tikzcd}[column sep=small]
V^{-1}\left(\pi_*\left(P_{2n+1}^{2n+2k}; Y\right)\right) \arrow[rr] & & V^{-1}\left(\pi_*\left(P_{2n+1}^{2n+2k+2}; Y\right)\right) \arrow[r] \arrow[dl, "i_*"] & {} \\
& V^{-1}\left(\rho_*\left(P_{2n+2k+1}^{2n+2k+2}; Y\right)\right) \arrow[ul, "\partial_*"] & &
\end{tikzcd}$$

The proposition would be established if we proved $\partial_*(1) = c_{n,k}$ and $i_*c_{n,k+1} = b$ where $V^{-1}(\pi_*(M_{2\iota}; Y)) = V^{-1}(\pi_*(S^0; Y)(1 + b))$. To prove the claim it is sufficient to consider

$$P_{2n+1}^{2n+2k} \wedge \mathrm{bo} \to P_{2n+1}^{2n+2k+2} \wedge \mathrm{bo} \to P_{2n+2k+1}^{2n+2k+2} \wedge \mathrm{bo}.$$

Applying $\pi_*(\ ; Y)$ to this sequence, we have, with W', W and W'' arbitrary, $\mathbf{Z}/2$ modules,

$$\begin{aligned} R'\left(c_{2n+2}, c_{2n+2k}\right) \oplus W' \to R'\left(c_{2n+2}, c_{2n+2k+2}\right) \oplus W \\ \to R'\left(c_{2n+2k+1}, c_{2n+2k+2}\right) \oplus W'' \xrightarrow{\bar{\partial}_*} \end{aligned}$$

and $\partial_*(c_{2n+2k+1}) = c_{2n+2k}$. The calculations now follow easily.

3. Unstable resolutions

The last paragraph of Section 2 describes the general strategy to be used to prove 1.1. We first introduce some definitions. The basic tool which we will use is a resolution of a space by Eilenberg-MacLane spaces. Such resolutions are the heart of the Adams spectral sequence approach but we would like to consider slightly more general resolutions.

Definition 3.1. A resolution of a space X, $\mathfrak{X}$, is a quintuple $(\{X_i\}, \{F_i\}, \{p_i\}, \{f_i\}, \{g_i\})$ giving a diagram

$$\begin{tikzcd}
& F_1 \arrow[d] & F_2 \arrow[d] & & F_s \arrow[d] & \\
X_0 & X_1 \arrow[l, "p_0"'] & X_2 \arrow[l, "p_1"'] & \cdots \arrow[l] & X_s \arrow[l] & \cdots \arrow[l] \\
& & & X \arrow[ulll, "f_0"] \arrow[ul, "f_2"] \arrow[ur, "f_s"'] & &
\end{tikzcd}$$

where $X_s \leftarrow X_{s+1} \leftarrow F_{s+1}$ is a fibration classified by $X_s \xrightarrow{g_s} BF_{s+1}$ and F_s is a product of Eilenberg-MacLane spaces. We say that the resolution is proper if

$\ker p_s^* = \ker f_s^*$. To get an Adams resolution we would require in addition that f_s^* be onto for each s. We will not usually require this. For our purposes here we also require that F_s be a product of $K(\mathbf{Z}/2, n)$'s.

Clearly the most simply described resolution for a locally finite space is one where $X_0 = K(\tilde{H}_*(X))$ and inductively $BF_s = K(\ker f_s^*)$ and the maps are defined in the obvious way. This gives an Adams resolution and the problem addressed by Massey and Peterson [MP2], Wellington [W] and others is to get some manageable description of this resolution and in particular its E_2-term. We will usually use resolutions with nice properties. The general question will not concern us.

Associated to any resolution $\mathfrak{X}$ there is an Adams type spectral sequence. Let $E_r^{s,t}(\mathfrak{X})$ be the homotopy exact couple of the resolution. We will usually index the spectral sequence by $E_1^{s,t}(\mathfrak{X}) = \pi_{t-s}(F_s)$. The spectral sequence converges to $E_0\pi_*(X_\infty)$ and if f_∞ is a homotopy equivalence, this is $E_0\pi_*(X)$.

Also associated to any resolution $\mathfrak{X}$ is a filtration of $H^*(X)$ given by $F_s(H^*(X)) = \operatorname{im} f_s^*$. For an Adams resolution this filtration is trivial. Note also that if $\ker p_s^* \neq \ker f_s^*$ then we would get a spectral sequence, which converges to $H^*(X)$, from the Serre spectral sequences of the fibrations.

If we need to talk about two spaces X and Z and resolutions between them, we will use the functorial notation $\mathfrak{X}(X)$, $\mathfrak{X}(Z)$. The resolutions we use are usually not functorial but the notation is very useful.

Proposition 3.2. *Let h: $X \to Z$ be a map. If $\mathfrak{X}(X)$ is an Adams resolution and $\mathfrak{X}(Z)$ is any resolution then the map h induces $h_\#$: $\mathfrak{X}(X) \to \mathfrak{X}(Z)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f_0(X)"'] & Z \arrow[d, "f_0(Z)"] \\
X_0(X) & X_0(Z)
\end{tikzcd}$$

Since $f_0^*(X)$ is onto and $X_0(Z)$ is a product of Eilenberg-MacLane spaces, we can find h_0: $X_0(X) \to X_0(Z)$, making the resulting diagram commute. Now suppose we have the following:

$$\begin{tikzcd}
X \arrow[rrr, "h"] \arrow[dr, "f_{s+1}"] \arrow[ddr, "f_s"'] & & & Z \arrow[dl, "f'_{s+1}"'] \arrow[dd, "f'_s"] \\
& X_{s+1}(X) \arrow[d, "p_s"] & X_{s+1}(Z) \arrow[dr, "p'_s"] & \\
& X_s(X) \arrow[rr, "h_s"] & & X_s(Z) \arrow[d, "g'_s"] \\
& & & BF_{s+1}(Z)
\end{tikzcd}$$

with $h_s f_s = f_s' h$. We need to get h_{s+1} such that $f_{s+1} h_{s+1} = f_{s+1}' h$. To construct an h_{s+1} we need $g_s' h_s p_s \sim 0$. But this is so if and only if $p_s^* h_s^* g_s'^* = 0$. Now $\ker p_s^* = \ker f_s^*$ so h_{s+1} exists if $f_s^* h_s^* g_s'^* = 0$. But this is $h^* f_s'^* g_s'^*$ which is clearly zero. Thus h_{s+1} exists. The difference of $f_{s+1}' h$ and $h_{s+1} f_{s+1}$ is a map λ: $X \to F_{s+1}(Z)$. This difference can be eliminated if and only if λ factors through $X \xrightarrow{f_{s+1}} X_{s+1}(X)$. Since f_{s+1}^* is onto this always happens.

The next result is crucial for our application but it is a very easy result.

Proposition 3.3. *Suppose X and Z are spaces with resolutions and h: $X \to Z$ is given with a map $h_\#$: $\mathfrak{X}(X) \to \mathfrak{X}(Z)$. Then there is a resolution of the fiber of h, $\mathfrak{X}(h)$, and we have a long exact sequence*

$$\cdots \to E_2^{s,t}(\mathfrak{X}(h)) \to E_2^{s,t}(\mathfrak{X}(X)) \to E_2^{s,t}(\mathfrak{X}(Z)) \to E_2^{s+1,t}(\mathfrak{X}(h)) \to \cdots .$$

Proof. Let $X_s(h)$ be the fiber of the map $X_s(X) \xrightarrow{p_1 h_2} X_{s-1}(Z)$. It is easy to see that $X_{s+1}(h) \to X_s(h)$ is classified by $BF_s(X) \times F_{s-1}(Z)$. Indeed, the diagram

$$\begin{tikzcd}
F_{s+1}(h) \arrow[r] \arrow[d] & F_{s+1}(X) \arrow[r] \arrow[d] & F_s(Z) \arrow[d] \\
X_{s+1}(h) \arrow[r] \arrow[d] & X_{s+1}(X) \arrow[r] \arrow[d] & X_s(Z) \arrow[d] \\
X_s(h) \arrow[r] & X_s(X) \arrow[r] & X_{s-1}(Z)
\end{tikzcd}$$

exhibits $F_{s+1}(h)$ as a fibration over $F_{s+1}(X)$ with $\Omega F_s(X)$ as fiber.

Let G be the fiber of $X_s(X) \to X_s(Z)$. Then we have

$$\begin{tikzcd}
F_{s+1}(X) \arrow[d] \arrow[dr] & & \\
X_{s+1}(h) \arrow[r] \arrow[d] & X_{s+1}(X) \arrow[r] \arrow[d] & X_s(Z) \\
G \arrow[r] & X_s(X) \arrow[ur] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega F_s(Z) \arrow[d] & & F_s(Z) \arrow[d] \\
G \arrow[r] \arrow[d] & X_s(X) \arrow[r] \arrow[dr] & X_s(Z) \arrow[d] \\
X_s(h) \arrow[ur] & & X_{s-1}(Z)
\end{tikzcd}$$

Thus

$$\begin{tikzcd}
X_{s+1}(h) \arrow[r] & G \arrow[r] & X_s(h) \\
F_{s+1}(X) \arrow[u] & \Omega F_s(Z) \arrow[u] &
\end{tikzcd} ,$$

which is a tower of fibrations, represents $F_{s+1}(h)$ as a fibration over $\Omega F_s(Z)$. This implies $F_{s+1}(h) = F_{s+1}(X) \times \Omega F_s(Z)$ as claimed. Thus for the E_1 terms we have a short exact sequence:

$$0 \to E_1^{s-1,t}(\mathfrak{X}(Z)) \to E_1^{s,t}(\mathfrak{X}(h)) \to E_1^{s,t}(\mathfrak{X}(X)) \to 0.$$

Taking homology gives the proposition.

We are now in a position to outline the proof of 1.1. Let $W(n)$ be the fiber of $S^{2n-1} \xrightarrow{i} \Omega^2 S^{2n+1}$. If we use an Adams resolution for S^{2n-1}, $\mathfrak{X}(S^{2n-1})$ and $\Omega^2\mathfrak{X}(S^{2n+1})$ for S^{2n+1}, where this means the double loop of an Adams resolution of S^{2n+1}, then we have a mapping $i\colon \mathfrak{X}(S^{2n-1}) \to \Omega^2\mathfrak{X}(S^{2n+1})$ by 3.2. By 3.3 we have a resolution $\mathfrak{X}(i)$ which is a resolution of $W(n)$. The E_2 term of this resolution was analyzed in [M3]. The principal result of that paper established an isomorphism between $E_2^{s,t}(\mathfrak{X}(i))$ and $\operatorname{Ext}_A^{s',t'}(H(M_{2\iota}), \mathbf{Z}/2)$ for s and s', t and t' satisfying a "$1/5$ line" requirement. Precise statements are given in 4.4. We need now to construct a map from $W(n)$ to $Q(M_{2\iota})$, that is, from an unstable object to a stable object, which can be extended to resolutions. ($QM_{2\iota}$ is the zero-th space in the Ω spectrum constructed from the suspension spectrum $S^0 \cup_{2\iota} e^1$.) Snaith [S] has constructed the maps. The map between the resolutions does not immediately follow. We will get it by constructing a map $\sigma_n\colon W(n) \to \Omega^4 W(n+1)$ and showing that this map extends to a map of the resolution for $W(n)$ and $W(n+1)$. Then using 3.3 we get a resolution of the fiber of σ_n and by [M3] this resolution has a "$1/5$ edge". This will be enough to show $V^{-1}(\pi_*(\sigma_n; Y)) = 0$. This then will give a proof of 1.1.

4. The fiber of the double suspension map

In this section we will investigate $W(n)$, the fiber of $S^{2n-1} \xrightarrow{i} \Omega^2 S^{2n+1}$. Our first goal is to describe a particular resolution for $W(n)$. The easiest way to do this is via the Λ-algebra of [6A]. We recall these results.

Let Λ be a differential graded algebra over $\mathbf{Z}/2$ generated by symbols λ_i of degree $i+1$. The relations are given by

$$\sum_{k=0}^{j} \binom{j}{k} \lambda_{m+j-k}\lambda_{2m+1+k} = 0$$

for all $m \geq 0$ and $j \geq 0$. The differential is given by

$$\lambda_j \to \sum_{k=1}^{j} \binom{j+1}{k} \lambda_{j-k}\lambda_{k-1}.$$

These are not necessarily the easiest formulae to work with but they are the easiest to remember. In this paper we do not make any explicit calculations. The main result of [6A] is:

THEOREM 4.1. $H_{**}(\Lambda, d) = \operatorname{Ext}_A^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$.

By use of the relations, it is possible to get a $\mathbf{Z}/2$ basis of Λ given by λ_I where $I = (i_1, \ldots, i_l)$ and $2i_j > i_{j+1} - 1$. Let Λ^n be the subspace of Λ spanned by λ_I with $i_1 < n$.

THEOREM 4.2. [R]. *The subspace of* Λ, Λ^n, *is a differential sub-complex of* Λ *and* $H_{**}(\Lambda^n, d) = E_2^{*,*}(S^n)$ *where* S^n *is the* E_2*-term of the spectral sequence* $\{E_r(\mathfrak{X}(S^n))\}$ *where* $\mathfrak{X}(S^n)$ *is an Adams resolution of* S^n.

This result was first proved by Rector. We will give a proof in Section 5 because we need the following result which follows easily from a particular proof of 4.2.

The following result has been known for many years and I am not sure who first noticed it. The resolution $\Omega^k\mathfrak{X}$, for any resolution $\mathfrak{X}$, is $(\{\Omega^k X_i\}, \{\Omega^k F_i\}, \{\Omega^k P_i\}, \{\Omega^k f_i\}, \{\Omega^k g_i\})$. It is a resolution of $\Omega^k X$ if $\mathfrak{X}$ is a resolution of X. It may not be a proper resolution.

THEOREM 4.3. *Let* $k < j$ *and let* $\mathfrak{X}$ *be an Adams resolution of* S^j. *Then* $(\Omega^k f_s)^*\colon H^*(X_s(\Omega^k\mathfrak{X})) \to H^*(\Omega^k S^j)$ *is onto in dimensions* $< 2^{s+1}(j-k)$.

This result is proved in Section 5.

We now apply this theory to $W(n)$. First note that $\Lambda^{2n-1} \subset \Lambda^{2n+1}$ as differential graded objects. In our notation Λ^{2n-1} is the E_1 term of an unstable Adams spectral sequence of S^{2n-1}. $(\Lambda^{2n-1})_{s,t} = \pi_{t-s-2n-1}\ (F_s(S^{2n-1})$ for a particular resolution. We will use $\mathfrak{X}(S^{2n-1})$ to refer to this particular resolution.) The double suspension map gives a map $S^{2n-1} \xrightarrow{i} \Omega^2 S^{2n+1}$. By 3.2 we have a map $\mathfrak{X}(S^{2n-1}) \to \Omega^2\mathfrak{X}(S^{2n+1})$ and this map is just $\Lambda^{2n-1} \subset \Lambda^{2n+1}$. By 3.3 we have a resolution of $W(n)$. By considering the complex $\Lambda^{2n+1}/\Lambda^{2n-1}$ we have a candidate for a resolution of $W(n)$. The five lemma shows that the homologies of $\Lambda^{2n+1}/\Lambda^{2n-1}$ and $E_2(\mathfrak{X}(i))$ are isomorphic.

We find that a shift in indexing these groups is convenient. Also when we mean the appropriate Λ-algebra resolution we will drop the $\mathfrak{X}$ and just write $E_2^{s,t}(S^n)$, for example. The normalization of the groups $E_2^{s,t}(S^n)$ is such that $t-s$ refers to the stem. Thus $E_2^{s,t}(S^n) = \mathbf{Z}/2$ if $s = t = 0$ and is 0 for $t < 0$. The indexing we use for $W(n)$ gives $E_2^{1,2}(W(n)) = \mathbf{Z}/2$ and $E_2^{s,t}(W(n)) = 0$ if $s = 0$ or $t < 2$. The long exact sequence becomes

$$\cdots \to E_2^{s,t}(S^{2n-1}) \to E_2^{s,t}(S^{2n+1}) \to E_2^{s,t-2n+2}(W(n)) \to E_2^{s+1,t}(S^{2n-1}) \to .$$

A key step in proving Theorem 1.1 is the following:

PROPOSITION 4.4. *There is a map* $\sigma_n\colon W(n) \to \Omega^4 W(n+1)$ *of degree* 1 *in homology in dimension* $4n-3$. *This map induces a map* $\sigma_{n\#}\colon \mathfrak{X}(W(n)) \to \Omega^4\mathfrak{X}(W(n+1))$ *and* $\sigma_{n\#}\colon E_2^{s,t}(W(n)) \to E_2^{s,t}(W(n+1))$ *is an isomorphism for* $6s > t + 20 - 4n$.

We will first deduce 1.1 from 4.4. The proof of 4.4 follows. Let X be any space and let $\mathfrak{X}$ be any resolution of X. A map $k\colon Y \to X$ has filtration i with respect to $\mathfrak{X}$ if there is a map k_i: $Y \to (f_i)$ where (f_i) is the fiber of the map f_i: $X \to X_i$ such that the composite $Y \to (f_i) \to X$ is k.

LEMMA 4.5. *If k: $Y \to X$ has filtration i then kv_1 has filtration $i + 1$.*

Proof. The fibration $(f_{i+1}) \to (f_i) \to F_i$ is clear. The map $k_i v_1$ is zero in cohomology so it lifts to (f_{i+1}).

Now let (h_n) be the fiber of h_n and let $\mathfrak{X}(h_n)$ be the resolution given from 3.3 and 4.4. Let k: $Y^m \to (h_n)$ be any map. Then kv_1^a has filtration a in the resolution $\mathfrak{X}(h_n)$. The isomorphism part of 4.4 implies that the fiber of f_s: $(h_n) \to X_s(H_n)$ is $5s - 20 + 8n - 3$ connected. Thus for a large enough, we see that kv_1^a: $Y^{m+2a} \to (h_n)$ factors through a point and so is inessential. Thus $V^{-1}(\pi_*((h_n); Y)) = 0$ and this implies $V^{-1}(\pi_*(W(n)) \simeq V^{-1}(\pi_*(W(n+1))$ for every n. This then implies 1.1.

The rest of this section is devoted to a proof of 4.4. The isomorphism part of 4.4 is the main theorem of [M3]. What we need to do here is to construct the map σ_n and show that σ_n extends to a map between the resolutions. Cohen, May and Taylor first constructed maps like this [CMT]. The best which follows from their work is a map $W(n) \to \Omega^8 W(n + 2)$. The following proposition is interesting because it is not done via Snaith type maps. I am indebted to F. Cohen for helpful conversation about this map. In particular a key step in the proof which follows is due to him.

PROPOSITION 4.6 (F. Cohen [Co]). *There is a map σ_n: $W(n) \to \Omega^4 W(n + 1)$ which is an isomorphism in $H_{4n-3}(\ ;\mathbf{Z})$.*

Proof. We will state three lemmas whose proofs we will give later. The proposition will follow from them. If H is an H-space, let $H\{2\}$ be the fiber of the H-space squaring map $H \xrightarrow{\Delta} H \times H \xrightarrow{\mu} H$. If X is a suspension let $X\{2\}$ be the fiber of the degree 2 map $X \xrightarrow{p} X \vee X \xrightarrow{f} X$ where p is the pinch map and f is the folding map. Typically $(\Omega S^n)\{2\}$ and $\Omega(S^n\{2\})$ are different spaces.

LEMMA 4.7. *There is a map h: $\Omega^2 S^{2n+1} \to (\Omega^2 S^{4n+1})\{2\}$ which is an isomorphism in $H_{4n-2}(\ ;\mathbf{Z})$.*

LEMMA 4.8. *There is a map i_n: $(\Omega^2 S^{4n+1})\{2\} \to \Omega^3(S^{4n+2}\{2\})$ which is an isomorphism in $H_{4n-2}(\ ;\mathbf{Z})$.*

LEMMA 4.9. *There is a map γ_n: $S^{4n+2}\{2\} \to W(n + 1)$ which is an isomorphism in $H_{4n+1}(\ ;\mathbf{Z})$.*

Now the proposition follows from the fiber diagram

$$\begin{tikzcd} W_n \arrow[d, "\sigma_n"] & S^{2n-1} \arrow[r] \arrow[d] & \Omega^2 S^{2n+1} \arrow[d, "g"] \\ \Omega^4 W_n \arrow[r] & * \arrow[r] & \Omega^3 W_n \end{tikzcd}$$

where g is the composite $\Omega^3\gamma_n \circ i_n \circ h$ and σ_n is some lifting of the fiber.

Proof of 4.7. This proof is an adaptation of an argument due to James [J] and Moore [Mo]. Let ϕ be the composite

$$S^{2n+1} \xrightarrow{p} S^{2n+1} \vee S^{2n+1} \xrightarrow{(-1,1)} S^{2n+1} \vee S^{2n+1} \xrightarrow{f} S^{2n+1}.$$

Clearly ϕ is null homotopic. The Hilton-Milnor theorem ([Wh], pages 511–540) gives a decomposition of $\Omega(S^{2n+1} \vee S^{2n+1})$ which allows us to study $\Omega\phi$. We get a factorization of $\Omega\phi$ as follows:

$$\Omega S^{2n+1} \xrightarrow{\Delta} (\Omega S^{2n+1})^3 \xrightarrow{\Omega(-1)\times\Omega(1)\times h_2} (\Omega S^{2n+1})^2$$

$$\times\ \Omega S^{4n+1} \xrightarrow{(\mathrm{id})^2\times\Omega w} (\Omega S^{2n+1})^2 \xrightarrow{m} \Omega S^{2n+1}$$

where w is the Whitehead product $S^{4n+1} \to S^{2n+1}$ and $m(x, y, z) = (xy)z$. Thus $0 = \Omega\phi = \Omega(-1) + \Omega(1) + (\Omega w) \circ h_2$, and $h_2 \circ (\Omega(-1) + \Omega(1) + \Omega w \circ h_2) = 0$. Since w is a suspension class which is annihilated by a suspension, $h_2 \Omega w = 0$. Thus $\Omega h_2 \cdot (\Omega^2(-1) + \Omega^2(1)) = 0$. Also note that $h_2(\Omega(-1)) = h_2$ and so we have $\Omega h_2 \cdot (2) = 0$. This gives h in the diagram

$$\begin{tikzcd} (\Omega^2 S^{4n+1})\{2\} \arrow[rr] & & \Omega^2 S^{4n+1} \arrow[r, "2"] & \Omega^2 S^{4n+1} \\ & \Omega^2 S^{2n+1} \arrow[ul, "h"] \arrow[ur, "h_2"'] & & \end{tikzcd}$$

Proof of 4.8. The argument is very similar. Consider

$$S^{4n+1} \xrightarrow{p} S^{2n+1} \wedge S^{2n+1} \xrightarrow{f} S^{2n+1},$$

which represents 2, and look at $\Omega(2)$. As above, using the Hilton-Milnor theorem, we have

$$\Omega S^{4n+1} \xrightarrow{\Delta} (\Omega S^{4n+1})^3 \xrightarrow{(\mathrm{id})^2\times h_2} (\Omega S^{4n+1})^2 \times \Omega S^{8n+1} \xrightarrow{\mathrm{id}\times\Omega w} (\Omega S^{4n+1})^3 \xrightarrow{m} \Omega S^{4n+1}.$$

Thus $\Omega 2 = 2 + \Omega w \circ h_2$. Since w maps to zero in S^{4n+2} we have a commutative diagram

$$\begin{tikzcd}
\Omega S^{4n+1} \arrow[r, "2"] \arrow[d] & \Omega S^{4n+1} \arrow[d] \\
\Omega^2 S^{4n+2} \arrow[r, "\Omega^2(2)"] & \Omega^2 S^{4n+2}
\end{tikzcd}$$

This implies 4.8.

Proof of 4.9. Splicing two *EHP* sequences together we see that $W(n+1)$ is the fiber of a map $\Omega^3 S^{4n+5} \to \Omega S^{4n+3}$. This map has degree 2 on the cell in dimension $4n+2$. This gives the desired map.

The final step in the proof of 4.4 is the following.

PROPOSITION 4.10. *The map* σ_n *of* 4.6 *extends to a map between the resolutions of* $W(n)$ *and* $W(n+1)$.

Proof. The resolution for $W(n)$ is the resolution of the fiber of $S^{2n-1} \to \Omega^2 S^{2n+1}$ where the resolution of S^{2n-1} is $\mathfrak{X}(S^{2n-1})$ and the resolution of $\Omega^2 S^{2n+1}$ is $\Omega^2 \mathfrak{X}(S^{2n+1})$. Clearly there is no trouble getting the map between $X_0(W(n))$ and $\Omega^4 X_0(W(n+1))$. Suppose we have the following commutative diagram (without $(\sigma_n)_{i+1}$):

$$\begin{tikzcd}
W(n) \arrow[rrr, "\sigma_n"] \arrow[dr, "f_{i+1}"] \arrow[ddr, "F_i"'] & & & \Omega^4 W_{n+1} \arrow[dl, "f'_{i+1}"'] \arrow[ddl, "f'_i"] \\
& X_{i+1}(W(n)) \arrow[r, "(\sigma_n)_{i+1}"] & \Omega^4 X_{i+1}(W(n+1)) \arrow[d] & \\
& X_i(W(n)) \arrow[r, "(\sigma_n)_i"] & \Omega^4 X_i(W(n+1)) \arrow[d, "g_i"] & \\
& & \Omega^4 BF_{i+1}(W(n+1)). &
\end{tikzcd}$$

We need to construct $(\sigma_n)_{i+1}$ so the top square commutes. First we construct a lift of $(\sigma_n)_i p_i$. To do this we need the fact that $p_i^*(\sigma_n)_i^* g_i^* = 0$. The $\ker p_i^* = \ker f_i^*$. Thus we need that $f_i^*(\sigma_n)_i^* g_i^* = 0$. This is the same as $\sigma_n^* f_i'^* g_i^*$, since f_i' lifts $f_i'^* g_i^* = 0$. Thus $(\sigma_n)_{i+1}$ exists. The difference of $(\sigma_n)_{i+1} f_{i+1}$ and $f'_{i+1}\sigma_n$ is represented by a map $\lambda\colon W(n) \to \Omega^4 F_i(W(n+1))$. If λ factors through f_{i+1}, then we can alter the lift $(\sigma_n)_{i+1}$ to get rid of this difference. By 4.3 we see that f_{i+1}^* is onto through dimension $2^{i+2}(4n-3)-1$. (In homology we have $H_* W(n+1)) = H_*(\Omega^2 S^{4n-1}) \otimes H_*(\Omega^3 S^{4n+1})$ and the first class not in f_{i+1}^* is the dual of $(a_{4n-3})^{2^{i+2}}$ where a_{4n-3} generates $H_{4n-3}(\Omega^2 S^{4n-1})$.) On the other hand, $\Omega^4 F_i(W(n+1))$ is a product of Eilenberg-MacLane spaces such that

$\Pi_*(\Omega^4 F_i(W(n+1)) \cong \Sigma\Lambda_{i+1}(W(n+1))$ where $\Lambda_{i+1}(W(n+1))$ is the $\mathbf{Z}/2$ vector space spanned by λ^I, I admissible, length of $(I) = i+2$ and if $I = (i_1,\ldots,i_l)$ then $i_1 = 2n+1$ or $2n+2$. The dimension of λ^I is $\Sigma i_j + 2n - 4$. The largest one is $I = (2n+2, 4n+4,\ldots,2^{i+1}(2n+2))$. Thus its dimension is $(2^{i+2}-1)(2n+2) + 2n - 4$. If $2^{i+2}(4n-3) - 1 > (2^{i+2}-1)(2n+2) + 2n - 4$ then λ factors through f_{i+1} and $(\sigma_n)_{i+1}$ can be chosen to give a commutative diagram. Clearly the inequality is satisfied if $n \geq 3$. Thus we have two special cases to consider.

Case 1. $W(2) \to \Omega^4 W(3)$. To handle this case we note that the map h of 4.7 gives a map $\bar{h}$: $W(2) \to (\Omega^3 S^9)\{2\}$. The maps i_2 and γ_2 of 4.8 and 4.9 give a map $\bar{\bar{h}}$: $\Omega S^9\{2\} \to \Omega^2 W(3)$. We will show that each of these extends to a cover. The argument is the same as above except that the maximum dimension in $F_i(\Omega^3 S^9\{2\})$ corresponds to a λ_I with leading $i = 8$ and length i. In particular the maximum dimension is $(2^{i+1}-1)\cdot 8 + 6$ which is $< 2^{i+2}\cdot 5$. Thus $\bar{h}$ lifts to a map of resolutions. To see that $\bar{\bar{h}}$ lifts to a map of resolutions, note that $f_{i+1}^* H^*(X_{i+1}((\Omega S^9)\{2\}) \to H^*(\Omega S^9)\{2\}$ is onto in dimension less than $2^{i+2}\cdot 7$ while, as above, the maximum dimension of nonzero homotopy in $F_i(\Omega^2 W(3))$ is $(2^{i+2}-1)\cdot 6 + 1$. Again it is easily checked that $2^{i+2}\cdot 7 > (2^{i+2}-1)\cdot 6 + 1$. Thus $\bar{\bar{h}}$ lifts to a map of resolutions. The desired map between resolutions covering σ_2 is the composite $\Omega^2\bar{h}\bar{\bar{h}}$.

Case 2. $W(1) \to \Omega^4 W(2)$. The argument is similar to the above except we need to observe that there is a map $\bar{h}$: $BW(1) \to (\Omega^2 S^5)\{2\}$. This map lifts to a map of resolutions by similar dimensional analysis. The map i_1: $(\Omega S^5)\{2\} \to \Omega^2(S^6\{2\})$ lifts to a map of resolutions since f_{i+1}^*: $H^*(X_{i+1}((\Omega S^5)\{2\}))$ is onto in dimensions less than $2^{i+2}\cdot 3$ while $\pi_j(F_i(\Omega^2(S^6\{2\}))$ is zero if $j > (2^{i+1}-1)\cdot 5 + 3$. Finally it is easy to verify that γ_2: $S^6\{2\} \to W(2)$ lifts to a map of resolutions.

This completes the proof of 4.4 and thus also 1.1.

5. Proof of 4.3

This section contains a proof of 4.3. In doing this we give a development of the Λ-algebra and the resolution it gives. We first summarize the work of Massey and Peterson [MP1] and [MP2].

Let M be a graded module over A. For each $m \in M$, let $|m|$ be the dimension of m. The module M is an *unstable module* if for each $m \in M$, $\mathrm{Sq}^i m = 0$ for all $i > |m|$.

Let M be an unstable module. Let s: $M \to M$ be defined by $s(m) = \mathrm{Sq}^{|m|} m$ for all $m \in M$. (s is called λ in [MP]. In order to avoid confusion with the

Λ-algebra we use s.) Thus M can be considered as a $\mathbf{Z}/2\ [s]$-module with $s^i(m) = s(s^{i-1}(m))$. M with this module structure is called an *s-module.* More generally if N is a graded module over $\mathbf{Z}/2$ and s is a $\mathbf{Z}/2$ vector space homomorphism from N to N with $s(N)^i \subset N^{2i}$, then N inherits a $\mathbf{Z}/2\ [s]$-module structure. As usual, an s-module is free if it has a basis.

Let M be an s-module. Then $U(M)$ is the free symmetric algebra on M, modulo the ideal generated by $m^2 - s(m)$.

PROPOSITION 5.1 (10.4 of [MP1]). *Let M be an s-module which is locally finite. Then $U(M)$ is a polynomial algebra if and only if M is a free s-module.*

Let M be any graded module over $\mathbf{Z}/2$. Let $\boldsymbol{\sigma} M$ be the free s-module generated by $\overline{M}$ where $(\overline{M})^i = (M)^{i-1}$.

Let M be a graded module over $\mathbf{Z}/2$ and let N be an s-module. A *boundary type* map f: $M \to N$ is any map which can be factored as $M \xrightarrow{i} \boldsymbol{\sigma} M \xrightarrow{\tilde{f}} N$ where $\tilde{f}$ is an s-module map and i is the obvious degree 1 inclusion.

An amusing exercise in these definitions is the following version of the Cartan basis theorem.

PROPOSITION 5.2. *The $\mathbf{Z}/2$ cohomology of $K(\mathbf{Z}/2, n)$ is $U(\boldsymbol{\sigma}^n(\mathbf{Z}/2)$.*

The proof is the same as given in [MT].

A *chain complex* of free s-modules is a collection of free s-modules C_i and boundary type maps d_i: $C_i \to C_{i-1}$ so that $d_{i-1}d_i = 0$. $H_n(C_i, d_i) = \ker d_i/\operatorname{im} \tilde{d}_{i+1}$ where $\tilde{d}_{i+1}$ is the s-module map determined by the boundary type map d_{i+1}.

A key result of [MP1] is the following.

THEOREM 5.3 (7.4 OF [MP1]). *If C_1 and C_0 are s-free unstable A-modules and $H^*(X_i) = U(C_i)$, $i = 0, 1$, and $X_1 \to E \to X_0$ is a fiber space with $C_1 \subset H^*(X_1)$ transgressive, then $\tau(C_1) \subset C_0$ and $H^*(E) \simeq U(\ker \tau)/\operatorname{im} p^*$ and* $\ker p^* = U(\operatorname{im} \bar{\tau})$.

We can construct a particular resolution of S^n which gives the unstable Λ-algebra at least formally. There is some connection between this treatment and the work of Priddy [P]. I would also like to thank E. Ossa who pointed out a difficulty with an earlier treatment of this material.

Let $\overline{A}$ be the algebra generated by $\overline{\mathrm{Sq}^i}$, $i > 0$, subject to the relations $\{\overline{\mathrm{Sq}}^a\overline{\mathrm{Sq}}^b = 0 \text{ if } a < 2b\}$. As $\mathbf{Z}/2$ graded modules $\overline{A}$ and A are isomorphic under the map ϕ: $\mathrm{Sq}^J \to \overline{\mathrm{Sq}}^J$ where $J = (j_1, \ldots, j_s)$ and $j_i \geq 2j_{i+1}$. Let $L_n \subset A$ (or $\overline{L}_n \subset \overline{A}$) be the vector space spanned by Sq^J, J admissible, $J = (j_1, \ldots, j_l)$ and

$j_l = n$. Let $L_n(j) \subset L_n$ be the subspace spanned by Sq^J with excess $e(J) \leq j$ ($e(J) = 2j_1 + \dim J$). Change the gradation of $L_n(j)$ so that $(L_n(j))^t = \{\mathrm{Sq}^J \mid \dim J + j = t\}$. Then $\bar{L}_n(j)$ is an unstable $\bar{A}$ module and is a free s-module. Also $\boldsymbol{\sigma}\bar{L}_n(j) = \bar{L}_n(j+1)$. Then 5.2 can be interpreted to be

$$\tilde{H}^*(K(\mathbf{Z}/2, j)) = U\Big(\bigoplus_{n \leq j} L_n(j)\Big)$$

but not as A modules.

Now we can write down an explicit resolution of $\mathbf{Z}/2$ by unstable $\bar{A}$ modules for each j.

$$\Sigma^j\mathbf{Z}/2 \leftarrow \bigoplus_{n \leq j} \bar{L}_n(j) \leftarrow \bigoplus_{i_1 < j}\Big(\bigoplus_{n < j + i_1} \bar{L}_n(j + i_1 - 1)\Big)\sigma_{i_1} \leftarrow \cdots \leftarrow$$

$$\leftarrow \bigoplus_{I \in \mathcal{I}_s(j)}\Big(\bigoplus_{n \leq j + \Sigma i_k - s} \bar{L}(j + \Sigma i_k - s)\Big)\sigma_I \leftarrow \cdots$$

where $I = (i_s, i_{s-1}, \cdots, i_1)$ and $i_j < 2i_{j-1}$ and $\mathcal{I}_s(j) = \{I \mid i_j < 2i_{j-1} \text{ and } i_1 \leq j\}$.

The maps $\bar{d}_s$ are boundary type maps defined by

$$\bar{d}_s\sigma_I = \bar{\mathrm{S}}\mathrm{q}^{i_s}\sigma_{I'} \text{ where } I = (i_s, \ldots, i_1) \text{ and } I' = (i_{s-1}, \ldots, i_1).$$

The observation that $\bar{L}_n(j) = \bigoplus_{k \geq 2n} \bar{L}_k(n + j)\bar{\mathrm{S}}\mathrm{q}^n$ allows one to show easily that the homology of this complex is just $\mathbf{Z}/2$ in dimension j and degree 0. The maps $\bar{d}_s$ are boundary type maps and this accounts for the "$-s$" which occurs in the formulae.

From this resolution of unstable $\bar{A}$ modules we wish to produce a resolution of unstable A modules,

$$\text{5.4} \quad \boldsymbol{\sigma}^j\mathbf{Z}/2 \xleftarrow{d_1} \bigoplus_{i_1 \leq j}(\boldsymbol{\sigma}^{j+i_1-1}\mathbf{Z}/2)\sigma_{i_1} \leftarrow \cdots \xleftarrow{d_2} \bigoplus_{I \in \mathcal{I}_s(j)}(\boldsymbol{\sigma}^{j+\Sigma i_k - s}\mathbf{Z}/2)\sigma_I \leftarrow \cdots,$$

with d_s being a boundary type map equal to $\bar{d}_s$ modulo a relation. The relation we use is a modification of the notion of "moment" as discussed in [SE], Chapter 1.

By the moment of $\mathrm{Sq}^J\sigma_I$, $m(Sq^J\sigma_I)$, we mean $\Sigma_{s=1}^l s j_s + \Sigma_{s=1}^k (l + s) I_{k-s+1}$ where $J = (j_1, \ldots, j_l)$ and $I = (i_k, \ldots, i_1)$ and $i_j < 2i_{j-1}$. Let $\|\mathrm{Sq}^J\sigma_I\| = \min\{m(\mathrm{Sq}^{J'}\sigma_I) \mid \mathrm{Sq}^J = \Sigma\mathrm{Sq}^{J'}$ and J' is admissible in the Steenrod algebra sense, $(j_i \geq 2j_{i-1})\}$. Call $\|\mathrm{Sq}^J\sigma_I\|$ the norm of $\mathrm{Sq}^J\sigma_I$.

Proposition 5.5. *There is a resolution* 5.4 *of unstable A modules with* d_s *being a boundary type map such that* $\phi(d_s(\mathrm{Sq}^J\sigma_I)) = \bar{d}_s\bar{\mathrm{S}}\mathrm{q}^J\sigma_I$ *modulo classes of norm* $< \|\mathrm{Sq}^J\sigma_I\|$.

Proof. We begin the construction with d_1 as follows:

$$d_1\sigma_{i_1} = \mathrm{Sq}^{i_1} \in \boldsymbol{\sigma}^j\mathbf{Z}/2.$$

The map d_1 considered as a boundary type map extends to

$$\boldsymbol{\sigma}^j\mathbf{Z}/2 \leftarrow \bigoplus_{i_1 \le j} \left(\boldsymbol{\sigma}^{j+i_1-1}\mathbf{Z}/2\right)\sigma_{i_1}$$

and has coker $= \mathbf{Z}/2$ in degree j. The kernel of d_1 is spanned by classes

$$\mathrm{Sq}^{J'}\left(\mathrm{Sq}^{j_l}\sigma_{i_1} + \sum_c \binom{i_1 - c - 1}{j_l - 2c}\mathrm{Sq}^{j_l+i_1-c}\sigma_c\right)$$

where (J', j_l) is admissible and $j_l < 2i_1$. Indeed the argument in [SE, page 8] shows that if J is admissible then $\|d_1\,\mathrm{Sq}^J\sigma_{i_1}\| \le \|\mathrm{Sq}^J\sigma_{i_1}\|$ and equality holds if and only if $j_l \ge 2i_1$.

Now suppose by induction we have constructed the complex 5.4 through dimension $s - 1$ with all the required properties. Suppose further that the kernel of d_{s-1} is spanned by classes of the form $\mathrm{Sq}^{J'}(\mathrm{Sq}^{j_l}\sigma_I + a(j_l, I))$ where (J', j_l) is admissible, $I = (i_{s-1}, \dots, i_1)$ and $j_l < 2i_{s-1}$, and $a(j_l, I) = \Sigma\mathrm{Sq}^{J_i}\sigma_{I_i}$ where J_i are admissible and $\|\mathrm{Sq}^{J_i}\sigma_{I_i}\| < \|\mathrm{Sq}^{j_l}\sigma_I\|$. This induction hypothesis is what we have shown for d_1 and is the consequence of d_{s-1} being equal to $\bar{d}_{s-1}$ modulo the norm relation. Then

$$d_s\colon \bigoplus_{I \in \mathcal{I}_s(j)} \left(\boldsymbol{\sigma}^{(j+\Sigma i_i - s)}\mathbf{Z}/2\right)\sigma_I \to \bigoplus_{I \in \mathcal{I}_{s-1}(j)} \left(\boldsymbol{\sigma}^{(j+\Sigma i_i - s+1)}\mathbf{Z}/2\right)\sigma_I$$

is defined by $d_s\sigma_I = \mathrm{Sq}^{i_s}\sigma_{I'} + a(i_s, I')$ where $(i_s, I') = I$. The norm property again shows that $\|d_s\mathrm{Sq}^J\sigma_I\| \le \|\mathrm{Sq}^J\sigma_I\|$ and equality holds if and only if $j_l \ge 2i_s$. Thus d_s and $\bar{d}_s$ agree on the norm preserving part, d_s maps onto the $\ker d_{s-1}$; thus, as graded vector spaces, $\ker d_s$ and $\ker \bar{d}_s$ are isomorphic. Then for each $j < 2i_s$, $\|d_s\mathrm{Sq}^j\sigma_I\| < \|\mathrm{Sq}^j\sigma_I\|$ and $d_s\mathrm{Sq}^j\sigma_I \in \ker d_{s-1}$. Now $d_s\mathrm{Sq}^j\sigma_I = d_s a(j, I)$ where a is a sum of classes $\mathrm{Sq}^{J_i}\sigma_{I_i}$, Sq^{J_i} admissible, and $\|\mathrm{Sq}^{J_i}\sigma_{I_i}\| < \|\mathrm{Sq}^j\sigma_I\|$. Thus $\mathrm{Sq}^j\sigma_I + a(j, I) \in \ker d_s$ and the classes $\mathrm{Sq}^{J'}(\mathrm{Sq}^j\sigma_I + a(j, I))$, as (J', j) ranges over admissible sequences in $(\boldsymbol{\sigma}^{j+\Sigma i_k - s}\mathbf{Z}/2)$, are linearly independent and span $\ker d_s$ since they are in 1–1 correspondence with a set spanning $\ker \bar{d}_s$. This completes the induction and proves 5.5.

The Steenrod algebra includes Sq^0 and the relation $\mathrm{Sq}^0 = 1$. Following Priddy [P], we note that the resolution we have is one constructed from an associated graded algebra to the Steenrod algebra. Getting the differential thus follows by standard arguments.

From 5.5 we can now prove the following result where

$$C_s = \bigoplus_{I \in d_s} \left(\sigma^{j+\Sigma i_k - s}\mathbf{Z}/2\right)\sigma_I.$$

THEOREM 5.6. *There is a sequence of spaces X_i and maps p_i and f_i and h_i such that*

$$\begin{tikzcd}
 & & & S^j \arrow[llld, "f_0"'] \arrow[lld, "f_1"] \arrow[rrd, "f_s"] & & & \\
X_0 & & X_1 \arrow[ll, "p_1"'] & \cdots \arrow[l, "p_2"'] & & X_s \arrow[ll, "p_s"'] & \cdots \arrow[l, "p_{s+1}"'] \\
K(\mathbf{Z}/2, j) \arrow[u] & & K(V_1) \arrow[u, "h_1"] & & & K(V_s) \arrow[u, "h_s"] &
\end{tikzcd}$$

1) *p_i is a fibration with $K(V_i)$ as fiber and* $\ker p_i^* = \ker f_{i-1}^*$.

2) *f_i^* is an epimorphism.*

3) *V_s is a graded $\mathbf{Z}/2$ vector space generated by σ_I where $I = (i_s, \ldots, i_1)$ $i_k < 2i_{k-1}$. The dimension of σ_I is $\Sigma_{k=1}^s (i_k - 1) + j$.*

4) *Let $M(V_s)$ be the s-free unstable A module such that $U(M(V_s)) = H^*(K(V_s))$. Then $M(V_s) = C_s$ and the composite $C_s \xrightarrow{d_s} C_{s-1} \to U(C_{s-1})$ is the transgression homomorphism of the fibration $X_s \to X_{s-1}$.*

Proof. Let $X_0 = K(\mathbf{Z}/2, j)$ and let X_1 be the fiber of the map g_1: $X_0 \to \Pi_{i=1}^j K(\mathbf{Z}/2, j+1)$ defined by $(\mathrm{Sq}^1, \ldots, \mathrm{Sq}^j)$. Let V_1 be generated by $\{\sigma_{i_1} \mid 0 < i_1 \leq j\}$ and the dimension of σ_{i_1} be $j + i_1 - 1$. Then $X_0 \leftarrow X_1 \leftarrow K(V_1)$ is a fibration. Theorem 5.3 applies and gives $H^*(X_1) \simeq \Sigma^j M/U(\ker \tau)$. But τ is just d_1 followed by a monomorphism and so $H^*(X_1) \simeq \Sigma^j M/U(\ker d_1)$. The map f_1 is the lift of the nontrivial map f_0: $S^j \to X_0$.

Now suppose we have defined X_s, and all associated maps and have shown that $H^*(X_s) \simeq \Sigma^j M/U(\ker d_s)$. Then we define X_{s+1} as the fiber of the map $X_s \xrightarrow{g_s} BK(V_{s+1})$ where g_s is induced by $\tilde{d}_{s+1}$: $C_{s+1} \to C_s$. This is well defined since $\mathrm{im}(\tilde{d}_{s+1})$ is $\simeq \ker d_s$. Also $U(C(s+1)) \simeq H^*(BK(V_{s+1}))$. This yields the fibration

$$K(V_{s+1}) \xrightarrow{h_{s+1}} X_{s+1} \xrightarrow{P_{s+1}} X_s.$$

Thus $\ker p_{s+1}$ is generated by $\mathrm{im}\, d_{s+1}$ which clearly equals $\ker f_{s+1}^*$ by 5.3. Again by 5.3 we have $H^*(X_{s+1}) \simeq \Sigma^j M/U(\ker d_{s+1})$ and the induction is complete.

The homotopy exact couple induced by this resolution has an E_1 term isomorphic to the Λ-algebra Adams spectral sequence for S^j at least as graded $\mathbf{Z}/2$ vector spaces. To get the Λ-algebra of [6A] explicitly, map $\sigma_i \to \lambda_{i-1}$ and send $\sigma_i \sigma_j \to \lambda_{j-1}\lambda_{i-1}$. This proves 4.2.

What we are explicitly interested in is what happens to the resolution when each space is replaced by Ω^k of the space. Consider

$$\Omega^k X_0 \leftarrow \Omega^k X_1 \leftarrow \cdots \leftarrow \Omega^k X_s \leftarrow \cdots .$$

This resolution is related to the complex of unstable modules:

5.7 $$\boldsymbol{\sigma}^{j-k}\mathbf{Z}/2 \leftarrow \bigoplus_{i_1 \le j} \left(\boldsymbol{\sigma}^{j+i_1-1-k}\mathbf{Z}/2\right)\sigma_{i_1} \leftarrow \cdots \leftarrow \bigoplus_{I \in \mathcal{I}_s(j)} \left(\boldsymbol{\sigma}^{j+\Sigma i_k - s - k}\right)\sigma_I \leftarrow \cdots .$$

Let $\{\overline{C}_s\}$ be the complex of 5.7.

THEOREM 5.8. *The homology of the complex* $\{\overline{C}_s\}$ *is the* $\mathbf{Z}/2$ *vector space generated by* σ_I *where* $\Sigma_{k=2}^{s}(2i_{k-1} - i_k) \ge k - 2 + s$, $I \in \mathcal{I}_s(j)$.

Proof. Associated to the resolution 5.7 is the corresponding $\overline{A}$ resolution. The theorem is obvious in that context; hence it is true.

The proof of 4.3 follows easily from 5.8. The homology $H_*(\overline{C})$ gives a filtration of $H_*(\Omega^k S^j)$. The correspondence given by $Q_i \to \sigma_{j-k+i-1}$ allows one to compare the two groups as $\mathbf{Z}/2$ vector spaces. The conclusion of 4.3 is implied by $H_{s,l}(\overline{C}) = 0$ if $l < 2^s(j-k)$.

6. The proof of Theorems 1.3 and 1.4

The key tool in proving 1.3 will be to use the Snaith maps s_n. We need the following result. It seems to have been first proved in [CT] and it is also in [K]. A modification of [S] can also be used to prove it. One might note that the Kahn-Priddy theorem is an easy corollary.

THEOREM 6.1. *The following diagram commutes for each* n:

$$\begin{tikzcd} Q(P_{2n-1}^{2n}) \arrow[r] & Q(\Sigma P^{2n-2}) \arrow[r] & Q(\Sigma P^{2n}) \\ \Omega^{2n-2}(W(n)) \arrow[r] \arrow[u, "s_n"] & \Omega^{2n-2}S^{2n-1} \arrow[r] \arrow[u, "s_{2n-1}"] & \Omega^{2n}S^{2n+1} \arrow[u, "s_{2n+1}"] \end{tikzcd}$$

where the s_i are the Snaith maps [S].

From 6.1 we can construct an induction argument to complete the proof of 1.3. Assume that s_{2n-1} induces an isomorphism

$$V^{-1}\left(\pi_*(\Omega^{2n-2}S^{2n-1}; Y)\right) \to V^{-1}\left(\pi_*(Q\Sigma P^{2n-2}; Y)\right).$$

We have seen that the map construction, via 4.4., of $W(n) \to Q(\Sigma^{2n-3}P_{2n-1}^{2n})$ introduces an isomorphism

* $$V^{-1}\pi_*(W(n); Y) \to V^{-1}\pi_*\left(Q\Sigma^{2n-3}P_{2n-1}^{2n}; Y\right).$$

It is easily seen that if $\Sigma^{2n-3}P^{2n}_{2n-1} \to Q\Sigma^{2n-3}P^{2n}_{2n-1}$ is the stabilization map, then

$$V^{-1}\pi_*\big(\Sigma^{2n-3}P^{2n}_{2n-1}; Y\big) \to V^{-1}\pi_*Q\Sigma^{2n-3}P^{2n}_{2n-1}; Y\big)$$

is an epimorphism and so $*$ will be an isomorphism for any map which is correct in dimension $2n-3$, and therefore, in particular, for the Snaith map. Now the five lemma completes the proof.

Instead of proving 1.4, we will prove the following more general result.

THEOREM 6.2. *Let X be a suspension spectrum of a finite dimensional locally finite* CW *complex. Then* $V^{-1}(\pi_*(X;Y)) \simeq V^{-1}(\pi_*(X \wedge J; Y))$.

Proof. Let

$$X^1 \to X^2 \to \cdots \to X^n \to \cdots \to X$$

be the skeleton decomposition of X. The cofibers at each stage are wedges of spheres. In the stable category, these cofibrations are fibrations and we have the homotopy exact couple, $E_1(X) = \pi_*(X^n, X^{n-1}; Y)$ and $E_\infty = E_0\pi(X;Y)$. If we consider the tower

$$X^1 \wedge J \to X^2 \wedge J \to \cdots \to X^n \wedge J \to \cdots,$$

we again have a homotopy exact couple with $E_1(X \wedge J) = \pi_*(X^n \wedge J, X^{n-1} \wedge J; Y)$. The Hurewicz map gives a map $E_1(X) \to E_1(X \wedge J)$ which is a map of spectral sequences. Theorem 1.2 asserts $V^{-1}(E_1(X)) \simeq V^{-1}(E_1(X \wedge J))$. Thus these spectral sequences are isomorphic and so $E_\infty(V^{-1}(E_1(X)) \simeq E_\infty(V^{-1}(E_1(X \wedge J))$. This implies the theorem.

In general when one studies homotopy with coefficients, one has the problem of recovering the actual homotopy. A Bockstein spectral sequence is a frequently used device to accomplish this. Results such as 6.2 make this task much easier in the case of $\pi_*(X;Y)$ if it is possible to calculate $\pi_*(X \wedge J)$. In the cases that we are interested in, S^{2n+1} and P^{2n}, we have the following which follows directly from 6.2:

PROPOSITION 6.3. *Let F_n be the fiber of the Snaith map followed by the Hurewicz image $\Omega^{2n}S^{2n+1} \to Q(\Sigma P^{2n} \wedge J)$. Then $\pi_j(F_n; Y)$ is annihilated by v_1^k for some k which may depend on j.*

Remark: This result is discussed again in Section 9, problem 1.

7. Proof of 1.5

In this section we will prove 1.5. The proof is not really hard but is long. We need to calculate $J_*(P^{2n})$. To do this we need to calculate $J_*(P)$ and $J_*(P_{2n+1})$.

Then we will calculate the images of $\pi_*(P_k^l) \to J_*(P_k^l)$ for $k = 1$, $l = 2n$ or ∞ and $k = 2n + 1$, $l = \infty$. Using these results we will be able to prove 1.5.

The chapter is long and so a guide as to what results are located where might be useful. Results 7.1 though 7.6 calculate the relevant Ext groups. The answers are given as charts following 7.6. These are charts for the resolution constructed for the fiber of the map $P \wedge \mathrm{bo} \to P \wedge \Sigma^4 \mathrm{bsp}$ following 3.3. The differentials are calculated in Theorem 7.7. This result contains a new proof, then, of the vector field problem for spheres and, in particular, calculates $(\psi^3 - 1)$ action in stunted projective spaces. Theorem 7.9 specializes the result to P^{2n} and tables are given to list the groups $J_*(P^{2n})$. The rest of the section is devoted to proving 1.5. The idea is to use the composite

$$\Omega^{2n+1}\Sigma^{2n+1}P^{2n} \to \Omega^{2n+1}S^{2n+1} \to QP^{2n}$$

where the first map is obtained from $P^{2n} \subset \mathrm{SO}(2n+1) \subset \Omega^{2n+1}S^{2n+1}$. A periodicity-type map is stably constructed giving stable maps

$$P_{8k+1}^{2n+8k} \to P^{2n}$$

of Adams filtration $4k$ which induces isomorphism in $V^{-1}\pi_*(\ ;Y)$ homotopy. These maps give a means to construct elements in $\pi_*(S^{2n+1})$ which map onto generators in $J_*(P^{2n})$. The details are given in results 7.10 through 7.19.

We begin with the J-groups. Recall that J is the fiber of a map $\mathrm{bo} \xrightarrow{\phi} \Sigma^4 \mathrm{bsp}$. We get a resolution of $P^{2n} \wedge J$ from a stable Adams resolution of $P^{2n} \wedge \mathrm{bo}$ and $P^{2n} \wedge \Sigma^4 \mathrm{bsp}$ by a stable version of 3.3. Call this resolution $\mathfrak{X}(P^{2n})$. There are analogously described resolutions for P and P_{2n+1} which are labeled $\mathfrak{X}(P)$ and $\mathfrak{X}(P_{2n+1})$ respectively.

Let X be one of the spaces P^{2n}, P or P_{2n+1}; $B(1)$ is defined in Section 2.

LEMMA 7.1. *There are resolutions of* $X \wedge \mathrm{bo}$ *and* $X \wedge \Sigma^4 \mathrm{bsp}$ *so that*

$$E_1^{s,t}(\mathfrak{X}(X)) = \mathrm{Ext}_{A_1}^{s,t}\big(\tilde{H}^*(X), \mathbf{Z}/2\big) \oplus \mathrm{Ext}_{A_1}^{s-1,t-4}\big(\tilde{H}^*(X \wedge B(1)), \mathbf{Z}/2\big).$$

Proof. The resolutions of $X \wedge \mathrm{bo}$ and $X \wedge \mathrm{bsp}$ we wish to use are minimal ones used to construct $\mathrm{Ext}_A^{s,t}(H^*(X \wedge \mathrm{bo}), \mathbf{Z}/2)$ and $\mathrm{Ext}_A^{s,t}(H^*(X \wedge \mathrm{bsp}), \mathbf{Z}/2)$. Since $H^*\mathrm{bo} = A \otimes_{A_1} \mathbf{Z}/2$, we use a change of rings to get

$$\mathrm{Ext}_A^{s,t}(H^*(X \wedge \mathrm{bo}), \mathbf{Z}/2) \simeq \mathrm{Ext}_{A_1}^{s,t}\big(\tilde{H}^*(X), \mathbf{Z}/2\big).$$

Note that $\mathrm{bsp} = B(1) \wedge \mathrm{bo}$. The rest of the argument is similar.

The groups $\mathrm{Ext}_{A_1}(\tilde{H}^*(X), \mathbf{Z}/2)$ have been calculated in many places. The following is possibly novel and the ideas will be used later. Let $P \xrightarrow{\lambda} S^0$ be the standard map. (λ is the adjoint of the composite $P \xrightarrow{W} \mathrm{SO} \to \Omega^\infty S^\infty$ where W is the map used to define the homology of SO.) Let R be the fiber.

LEMMA 7.2. *The spectrum $R \wedge \mathrm{bo}$, when localized at 2, is homotopy equivalent to* $\bigvee_{n\geq 0} K(\mathbf{Z}_{(2)}, 4n)$.

Proof. We need only look at the A_1 module structure of R. Filter H^*R by letting $F_i(H^*(R)) = \mathrm{im}\{A_1 \otimes (\oplus_{j\leq i} H^j(R))\} \to H^*(R))$. Then $F_i = F_{i+1}$ unless $i + 1 = 4k$. It is easily seen that $F_{4k}/F_{4k-1} = \Sigma^{4k} A_1 \otimes_{A_0} \mathbf{Z}/2$. Thus $\mathrm{Ext}_{A_1}^{s,t}(\oplus_{k\geq 0} F_{4k}/F_{4k-1}, \mathbf{Z}/2) = \oplus_{k\geq 0} \mathrm{Ext}_{A_0}(\Sigma^{4k}\mathbf{Z}/2, \mathbf{Z}/2)$. This is the E_1 term of a spectral sequence, associated to the filtration, converging to $E^0 \mathrm{Ext}_{A_1}(H^*(R), \mathbf{Z}/2)$. There can be no differentials for dimensional reasons. In $H^*(R \wedge \mathrm{bo})$ there are infinite cyclic classes in dimension $4k$. Let $R \wedge \mathrm{bo} \to \bigvee_{k\geq 0} K(\mathbf{Z}, 4k)$ be defined by these. This map induces an isomorphism in Ext groups and so is a homotopy equivalence when localized at 2.

Since $0 \to H^j(\Sigma P \wedge \mathrm{bo}) \to H^j(R \wedge \mathrm{bo}) \to H^j(\mathrm{bo}) \to 0$ is exact for each j, we have long exact sequences for each t

$$\cdots \to \mathrm{Ext}_{A_1}^{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) \xrightarrow{p^\#} \mathrm{Ext}_{A_1}^{s,t}(H^*(R), \mathbf{Z}/2)$$

$$\to \mathrm{Ext}_{A_1}^{s,t-1}\big(\tilde{H}^*(P), \mathbf{Z}/2\big) \to \mathrm{Ext}_{A_1}^{s+1,t}(\mathbf{Z}/2, \mathbf{Z}/2).$$

If $t - s = 4k$ the map $p^\#$ is a monomorphism. This proves the following.

LEMMA 7.3.

$$\mathrm{Ext}_{A_1}^{s,t}\big(\tilde{H}^*P, \mathbf{Z}/2\big) = \begin{cases} \mathbf{Z}/2 & (s,t) = (4k, 1+12k), (1+4k, 3+12k) \\ \mathbf{Z}/2 & \textit{if } t - s = 4k - 1 \textit{ and } s < 2k \textit{ if } k \equiv 0 \bmod 2, \\ & \qquad\qquad s \leq 2k \textit{ if } k \equiv 1 \bmod 2 \\ 0 & \textit{otherwise}. \end{cases}$$

$$\pi_j(P \wedge \mathrm{bo}) = \begin{cases} \mathbf{Z}/2 & j \equiv 1, 2 \bmod 8 \\ \mathbf{Z}/2^{2k} & j = 4k-1,\ k \equiv 0 \bmod 2 \\ \mathbf{Z}/2^{2k+1} & j = 4k-1,\ k \equiv 1 \bmod 2, \\ 0 & \textit{all other } j. \end{cases}$$

To calculate $\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P \wedge B(1)), \mathbf{Z}/2)$ we have the following sequence of A_1 modules:

$$0 \leftarrow H^*(B(1)) \leftarrow A_1 \otimes_{A_0} \mathbf{Z}/2 \leftarrow \Sigma^5 \mathbf{Z}/2 \leftarrow 0.$$

Hence we have long exact sequences for each t:

$$\mathrm{Ext}_{A_1}^{s-1,t-5}\big(\tilde{H}^*(P), Z/2\big) \to \mathrm{Ext}_{A_1}^{s,t}\big(\tilde{H}^*(P \wedge B(1)), \mathbf{Z}/2\big)$$

$$\to \mathrm{Ext}_{A_0}^{s,t}\big(\tilde{H}^*(P), \mathbf{Z}/2\big)$$

$$\xrightarrow{p} \mathrm{Ext}_{A_1}^{s,t-5}\big(\tilde{H}^*(P), \mathbf{Z}/2\big) \to \cdots$$

where

$$\mathrm{Ext}_{A_0}^{s,t}\left(\tilde{H}^*(P), \mathbf{Z}/2\right) = \begin{cases} 0, & s > 0, \\ \mathbf{Z}/2, & s = 0, t = 2l - 1, \end{cases}$$

and the map p is zero. This proves the following:

LEMMA 7.4.

$$\mathrm{Ext}_{A_1}^{s,t}\left(\tilde{H}^*(P \wedge B(1)), \mathbf{Z}/2\right) \simeq \mathrm{Ext}_{A_1}^{s-1,t-5}\left(\tilde{H}^*(P), \mathbf{Z}/2\right) \oplus \mathrm{Ext}_{A_0}^{s,t}\left(\tilde{H}^*(P), \mathbf{Z}/2\right).$$

$$\pi_j P \wedge \mathrm{bsp} = \begin{cases} \mathbf{Z}/2 & j = 1 \text{ or } 6 \bmod 8, \\ \mathbf{Z}/2 + \mathbf{Z}/2 & j \equiv 5 \bmod 8, \\ \mathbf{Z}/2^{2k-1} & j = 4k-1, \quad k \equiv 1 \bmod 2, \\ \mathbf{Z}/2^{2k} & j = 4k-1, \quad k \equiv 0 \bmod 2, \\ 0 & \textit{all other } j. \end{cases}$$

LEMMA 7.5.

$$\mathrm{Ext}_{A_1}^{s,t}\left(\tilde{H}^*(P_{4n+1}), \mathbf{Z}/2\right) \simeq \mathrm{Ext}_{A_1}^{s,t-4n}\left(\tilde{H}^*(P), \mathbf{Z}/2\right).$$

$$\begin{aligned} \mathrm{Ext}_{A_1}^{s,t}\left(\tilde{H}^*(P_{4n-1}), \mathbf{Z}/2\right) &\simeq \mathrm{Ext}_{A_1}^{s-1,t-4n}\left(\tilde{H}^*(P), \mathbf{Z}/2\right), \quad s > 0 \\ &= \mathbf{Z}/2, \qquad s = 0, \quad t = 4k-1, \quad k \geq n. \end{aligned}$$

Proof. By [MR], $\Sigma^{4n} P \wedge \mathrm{bo} \simeq P_{4n+1} \wedge \mathrm{bo}$ and this gives the first statement. The map $P_{4n-1} \xrightarrow{\mathrm{id} \wedge 2} P_{4n-1}$ factors through $P_{4n-1} \xrightarrow{p} P_{4n+1} \xrightarrow{f} P_{4n-1}$ where p is the pinch map and f is a map of filtration 1. It is easy to verify that $P_{4n-1} \cup_f CP_{4n+1}$ is A_1 free; thus f induces an isomorphism for $s \geq 0$, $\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P_{4n+1}), \mathbf{Z}/2) \to \mathrm{Ext}_{A}^{s+1,t+1}(\tilde{H}^*(P_{4n-1}), \mathbf{Z}/2)$. The rest of the calculation is now simple.

Let a_s be the function

$$a_s = \begin{cases} 8k \text{ if } s = 4k \text{ or } 4k+1 \\ 8k+1 \text{ if } s = 4k+2 \\ 8k+3 \text{ if } s = 4k+3 \ . \end{cases}$$

LEMMA 7.6.

$$\mathrm{Ext}_{A_1}^{s,t}\left(\tilde{H}^*(P^{4n}), \mathbf{Z}/2\right)$$

$$= \begin{cases} \mathrm{Ext}_{A_1}^{s,t}\left(\tilde{H}^*(P), \mathbf{Z}/2\right), & s > t - 4n - a_s \\ \mathbf{Z}/2, & (s,t) = (1+4k, 4n+1+12k) \\ & \qquad = (2+4k, 4n+3+12k) \\ 0 & \textit{otherwise}. \end{cases}$$

$$\operatorname{Ext}_{A_1}^{s,t}\big(\tilde{H}^*(P^{4n-2}),\mathbf{Z}/2\big)$$

$$=\begin{cases}\operatorname{Ext}_{A_1}^{s,t}\big(\tilde{H}^*(P),\mathbf{Z}/2\big), & s>t-4n-a_{s-1}\\ \mathbf{Z}/2, & (s,t)=(2+4k,4n+2+12k)\\ & \qquad=(3+4k,4n+5+12k)\\ 0 & \textit{otherwise}.\end{cases}$$

Proof. We use the short exact sequence $0\to H^*(P_{2n+1})\to H^*(P)\to H^*(P^{2n})\to 0$ to get a long exact sequence in the Ext groups from which the theorem follows immediately from the previous calculation.

The following charts summarize all the above calculations. Periodicity gives the complete picture from these charts since

$$\operatorname{Ext}_{A_1}^{s,t}\big(H^*(P^{2n}),\mathbf{Z}/2\big)\simeq\operatorname{Ext}_{A_1}^{s-4,t+12}\big(H^*(P^{2n}),\mathbf{Z}/2\big)$$

for all s and t. In these charts, vertical lines indicate classes connected by h_0 multiplications and slanting lines indicate classes connected by h_1 multiplications.

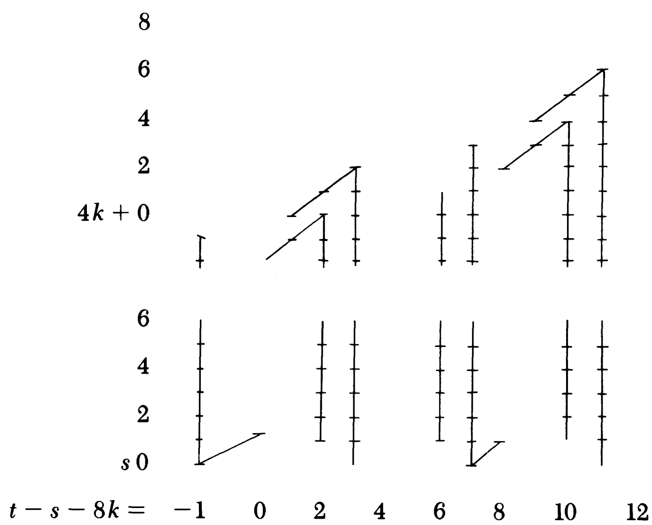


CHART 1

$E_1(\mathfrak{X}(P))$ for large values of $t-s$

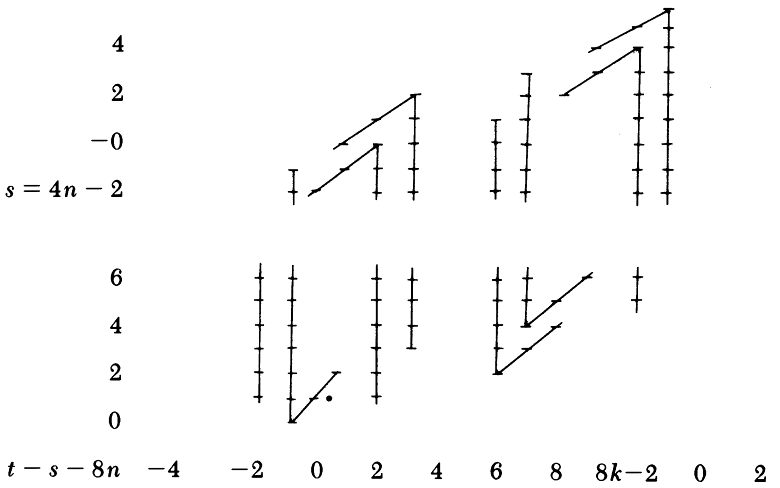


CHART 2

$E_1(\mathfrak{X}(P^{8n}))$

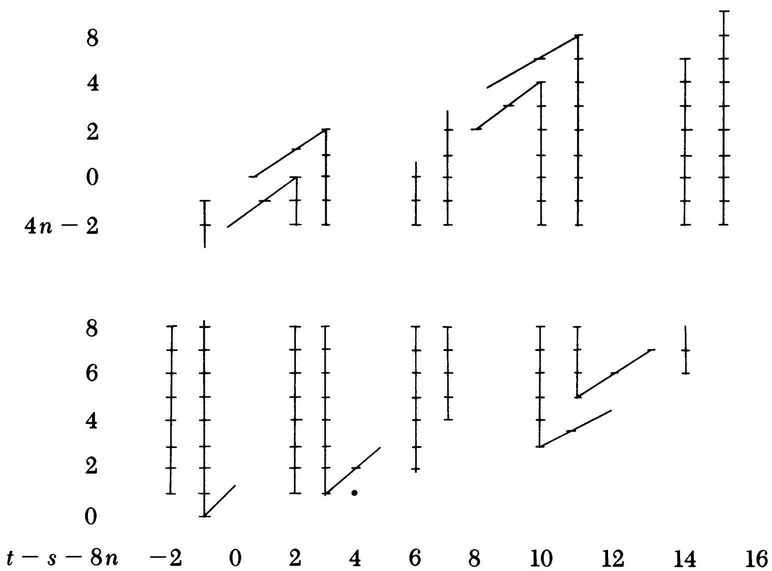


CHART 3

$E_1(\mathfrak{X}(P^{8n+2}))$

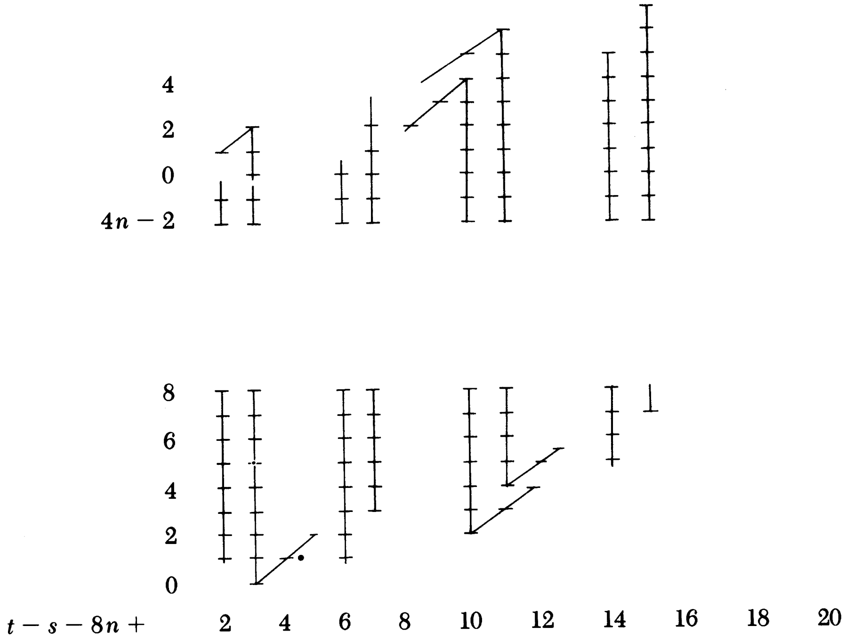


CHART 4

$E_1(\mathfrak{X}(P^{8n+4}))$

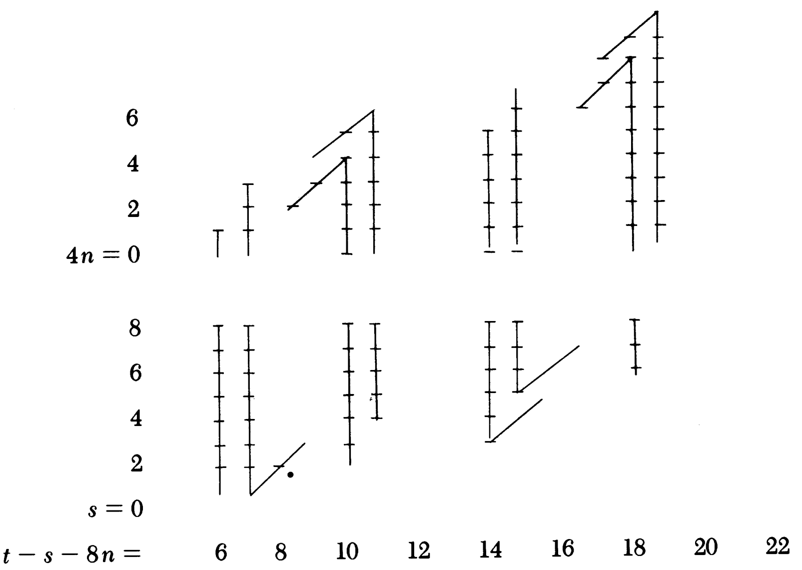


CHART 5

$E_1(\mathfrak{X}(P^{8n+6}))$

Next we need to calculate the resulting spectral sequence. We will first calculate the differentials in $E_*(\mathfrak{X}(P))$. It is clear from chart 1 that the only differentials which are possible go from $t-s=4j-1$ to $t-s=4j-2$. Let $|j|_2$ be the 2-adic norm of j, i.e., the coefficient of 2 in the prime factorization of j.

THEOREM 7.7. *In the spectral sequence* $\{E_r(\mathfrak{X}(P))\}$, *the class* 1_{4j-1} *in* $E^{0,4j-1}$ *is a cycle in* $E_{|j|_2}$ *and* $\delta_{|j|_2}1_{4j-1} \neq 0$ *if* $E_1^{|j|_2,4j-2+|j|_2} \neq 0$. *This formula implies all differentials.*

Proof. Using 7.2, we have a commutative diagram

$$\begin{tikzcd}
 & D \arrow[r, "f"] \arrow[d] & \Sigma P \wedge J \arrow[d] \\
\mathrm{bo} \arrow[r] \arrow[d, "\phi"] & \vee_{n\geq 0} K(\mathbf{Z},4n) \arrow[r] \arrow[d, "p"] & \Sigma P \wedge \mathrm{bo} \arrow[d] \\
\Sigma^4\mathrm{bsp} \arrow[r] & \vee_{n>0} K(\mathbf{Z},4n) \times K(\mathbf{Z}/2,4n+2) \arrow[r] & \Sigma^5 P \wedge \mathrm{bsp}
\end{tikzcd}$$

where D is the fiber of p and p is defined to make the square with ϕ commute. Clearly $E_1^{s,t}(\mathfrak{X}(P))$ maps under f onto $E_1^{s,t}(\mathfrak{X}(P))$ if $t-s=4j-1$ or $4j-2$. Thus the spectral sequence for $P \wedge J$ is determined by that of D. The spectral sequence of D follows easily from what the map p does in homotopy. The map p in homotopy follows from what ϕ does in homology. We have the following lemma.

LEMMA 7.8. $\phi_*\colon H_{4k}(\mathrm{bo}) \to H_{4k}(\Sigma^4 bsp)$ *is multiplication by* k *on a choice of the free Abelian summand.*

We delay the proof of the lemma.

$$p_*\colon \pi_{4j}(\vee_n K(\mathbf{Z},4n) \to \pi_{4j}(\wedge_{n>0}(K(\mathbf{Z},4n) \vee K(\mathbf{Z}/2,4n+2)))$$

is multiplication by $2^{|j|_2} \times l$ where l is odd. This translates exactly into the conclusion of the theorem applied to $\{E_r(\mathfrak{X}((p)))\}$ and by naturality to $\{E_r(\mathfrak{X}(P))\}$ which gives the theorem.

Proof of 7.8. Let $S^5 \xrightarrow{g} B^2O$ be a generator. Then $\Omega g\colon \Omega S^5 \to BO$ is well defined. Let X_5 be the Thom complex. Then X_5 is a ring spectrum and $X_5 \wedge X_5 \overset{f}{\cong} \Omega S^5_+ \wedge X_5 = \vee_{i\geq 0}\Sigma^{4i}X_5$ [M8]. Let $\bar{f}$ be the composite $X_5 \xrightarrow{\mathrm{id}\wedge\iota_0}$

$X_5 \wedge X_5 \to \Omega S^5_+ \wedge X_5 \to \Sigma^4 X_5$. In [M8] it is shown that $\bar{f}_*$: $H_{4k}(X_5; \mathbf{Z}) \to H_{4k}(\Sigma^4 X_5, \mathbf{Z})$ is just multiplication by k. By using 5.10 of [M2] we have a commutative diagram of 2-adic completed spaces (bspin in [M2] is bsp here).

$$\begin{tikzcd} \mathrm{bo} \arrow[r, "\phi"] & \Sigma^4\mathrm{bsp} \\ X_5 \arrow[u, "g"] \arrow[r, "\bar{f}"] & \Sigma^4 X_5 \arrow[u, "g'"'] \end{tikzcd}$$

where g and g′ induce isomorphisms

$$g_*: H_*(X_5; \mathbf{Z}) \to H_*(\mathrm{bo}, \mathbf{Z})/\mathrm{Torsion}$$

$$g'_*: H_*\big(\Sigma^4 X_5; \mathbf{Z}\big) \to H_*\big(\Sigma^4\mathrm{bsp}; \mathbf{Z}\big)/\mathrm{Torsion}.$$

This completes the proof of 7.8.

The differentials in $\{E_r(\mathfrak{X}(P^{2n}))\}$ are all implied by the following:

THEOREM 7.9. *Each class a in $E_1^{s,t}(\mathfrak{X}(P^{2n}))$ for $t - s = 4j - 1$ projects a cycle in $E_{|j|_2}$ and $\delta_{|j|_2} a \neq 0$ if $E_1^{|j|_2, 4j-2+|j|_2} \neq 0$. This formula implies all the differentials.*

The proof is immediate from 7.7 and the inclusion $P^{2n} \to P$. Using the charts we have the following:

1) In $E_*(\mathfrak{X}(P^{2n}))$ there are no differentials if $n \leq 3$. Thus

$$J_*(P^{2n}) = \mathrm{bo}_*(P^{2n}) \oplus \mathrm{bsp}_*(\Sigma^4 P^{2n}) \text{ if } n \leq 3.$$

2) $E_2(\mathfrak{X}(P^8)) = E_\infty(\mathfrak{X}(P^8))$ and the only non-zero δ_1 is

$$\delta_1: E_1^{3+4s,14+12s} \to E_1^{4+4s,14+12s}, \ s \geq 0.$$

Thus we have

$$J_j(P^8) = \begin{cases} \mathbf{Z}/2 & j = 1, 2, 4, \text{ or } 6, \text{ and } \equiv 0(8) \text{ with } j \geq 16 \\ \mathbf{Z}/2 + \mathbf{Z}/2 & j \equiv 1(8), j \geq 9 \\ \mathbf{Z}/8 & j \equiv 3(8) \\ \mathbf{Z}/16 & j \equiv 6 \text{ or } 7(8), j \geq 7 \\ \mathbf{Z}/2 + \mathbf{Z}/2 + \mathbf{Z}/2 & j = 8 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/8 & j \equiv 2(8), j \geq 10 \\ 0 & j \equiv 4 \text{ or } 5(8), j \geq 5. \end{cases}$$

3) For P we have the following where $A_k = \mathbf{Z}/2^{|16k|_2}$:

$$J_j(P) = \begin{cases} J_j(P^8) & j \leq 6 \text{ otherwise} \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & j \equiv 0,1(8) \\ \mathbf{Z}/2 & j \equiv 2,9(8) \\ \mathbf{Z}/8 & j \equiv 3(8) \\ 0 & j \equiv 5(8) \\ \mathbf{Z}/\left(2^{|k|_2+1}\right) & j = 8k-2 \\ A_k & j = 8k-1. \end{cases}$$

To complete the calculation we give the following tables. Let

$$b_{k,j} = \min\left\{|\,16k\,|_2,\ |\,\text{order of the line bundle over } P^j\,|_2\right\}.$$

Let $\bar{A}_k(j) = \mathbf{Z}/2^{b_{k,j}}$.

$$J_j(P^{8n}) = \begin{cases} J_j(P) & j < 8n \text{ otherwise} \\ (\mathbf{Z}/2)^3 & j \equiv 0,1(8) \\ (\mathbf{Z}/2) + \mathbf{Z}/8 & j \equiv 2(8) \\ \mathbf{Z}/8 & j \equiv 3(8) \\ 0 & j \equiv 4,5(8) \\ \bar{A}_k(8n) & j = 8k-2 \\ \bar{A}_k(8n) + \mathbf{Z}/2 & j = 8k-1. \end{cases}$$

$$J_j(P^{8n+2}) = \begin{cases} J_j(P) & j < 8n+2 \text{ otherwise} \\ \mathbf{Z}/2 + \mathbf{Z}/2 & j = 8n+2 \\ \mathbf{Z}/8 & j = 8n+3 \\ \mathbf{Z}/2 & j \equiv 0,5(8) \\ \mathbf{Z}/2 + \mathbf{Z}/2 & j \equiv 1,4(8) \\ \mathbf{Z}/8 + \mathbf{Z}/2 & j \equiv 2,3(8) \\ \bar{A}_k(8n+2) & j = 8k-i,\ i = 2 \text{ or } 1. \end{cases}$$

$$J_j(P^{8n+4}) = \begin{cases} J_j(P) & j < 8n+4 \\ J_j(P^{8n+2}) & j \geq 8n+4,\ j \neq 8k+i,\ i = 2 \text{ or } 1 \\ \bar{A}_k(8n+4) & j = 8k+i,\ i = 2 \text{ or } 1. \end{cases}$$

$$J_j(P^{8n+6}) = \begin{cases} J_j(P) & j < 8n+6 \\ \mathbf{Z}/2^{|4n|_2} & j = 8n+6 \text{ otherwise} \\ (\mathbf{Z}/2)^3 & j \equiv 0,1(8) \\ \mathbf{Z}/8 + \mathbf{Z}/2 & j \equiv 2(8) \\ \mathbf{Z}/8 & j \equiv 3(8) \\ 0 & j \equiv 4,5(8) \\ \bar{A}_k(8n+6) & j = 8k-2 \\ \bar{A}_k(8n+6) + \mathbf{Z}/2 & j = 8k-1. \end{cases}$$

As a first step toward proving 1.5 we prove the following.

THEOREM 7.10. *Let s be the map $\Omega^\infty S^\infty S^1 \to Q\Sigma P$. Then $s_*\colon \pi_j(S^0) \to J_j(P)$ has the following cokernel*:

$$\begin{array}{lll} \mathbf{Z}/2 & j \equiv 4 \bmod 8 & \\ \mathbf{Z}/2 & j \equiv 0 \bmod 8, & j \neq 2^i \\ J_j(P) & j \equiv 6 \bmod 8 & j \neq 2^i - 2 \\ J_j(P)/\mathbf{Z}/2 & j = 2^i - 2 & 6 \leq j \leq 62 \\ 0 & \textit{all other } j \textit{ except if } j = 2^i - 2. & \end{array}$$

Remark: It has been conjectured that if the Kervaire invariant manifold exists in dimension $2^i - 2$ then $\mathrm{Coker}(s_*)_{2^i-2} = J_{2^i-2}(P)/\mathbf{Z}/2$. This is equivalent to the conjecture in [M6], page 4. Partial results are known but apparently the strongest form of the conjecture has not been proved.

Proof. We first consider $j \equiv 0 \bmod 4$. Let $p\colon P \to P_{4n-3}$ be the pinch map. Then the coset in $\pi_{4n}(P)$ corresponding to the class in $E_1^{1,4n+1}$ projects under p to a class in the image of $\pi_{4n}(\Sigma^{4n-3}J) \to \pi_{4n}(P_{4n-3} \wedge J)$. The class on the left represents ν, the generator of the three stem. Thus if there is a map $f\colon S^{4n} \to P$ such that the composite $S^{4n} \to P \to P \wedge J$ is in this coset, then f fits in the diagram

$$\begin{tikzcd} P \arrow[r, "p"] & P_{4n-3} \\ S^{4n} \arrow[u, "f"] \arrow[r, "\nu"'] & S^{4n-3} \arrow[u, "i"'] \end{tikzcd}.$$

Compare this with the following diagram where the top row is the *EHP* sequence, the middle row is the fibration induced by $S^{4n-3} \subset Q(S^{4n-3})$ and the lower right hand square is obtained from the map $P^{n-1} \to \Omega^n S^\Omega$ which factors through $\mathrm{SO}(n)$.

$$\begin{tikzcd}
\Omega^2 S^{8n-5} \arrow[d] \arrow[dr] & & & \Omega S^{4n-2} \arrow[dll] \arrow[d] \\
(\Sigma) \arrow[r] & S^{4n-3} \arrow[r, "\Sigma"] & Q & (S^{4n-3}) \\
(\Sigma') \arrow[u] \arrow[r] & \Sigma^{4n-3} P^{4n-2} \arrow[u] \arrow[r, "\Sigma'"] & Q & (\Sigma^{4n-3} P) \arrow[u]
\end{tikzcd}$$

We see that the existence of such an f implies there is a class in $\pi_{8n-2}(S^{4n-2})$ with Hopf invariant ν. Results of [M5] and [M9] show that this happens only if $n = 2^i$. The resulting homotopy classes are called η_i in [M5].

Next we consider the case $j \equiv 6 \bmod 8$. What we can prove here is related to the Kervaire invariant as remarked above so we isolate the statement as a theorem.

THEOREM 7.11. *If the image of* $\pi_{8n-2}(S) \to J_{8n-2}(P)$ *is* $\mathbf{Z}/2$ *then* $n = 2^i$ *and there is a manifold with Kervaire invariant* 1 *in dimension* $2^{i+3} - 2$.

Proof. The argument uses the following diagram. The maps P and E are the usual ones in the EHP sequence. The maps Σ and Σ' are suspension type maps. As usual (f) is the fiber of a map f.

$$\begin{tikzcd}
S^{16-n-3} \arrow[r, "j"] & \Sigma^{8n-2} P_{8n-1} \arrow[d, "k"] & & \\
& (\Sigma') \arrow[r, "i_1"] \arrow[d, "k_1"] & \Sigma^{8n-1} P^{8n-2} \arrow[r, "\Sigma'"] \arrow[d, "k_2"] & Q(\Sigma^{8n-1} P) \arrow[d] \\
& (\Sigma) \arrow[r, "i_2"] & S^{8n-1} \arrow[r, "\Sigma"] \arrow[dr, "E"] & Q(S^{8n-1}) \\
& \Omega^2 S^{16n-1} \arrow[u] \arrow[ur, "P"] & & \Omega S^{8n} \arrow[u]
\end{tikzcd}$$

Now $P(\iota) = [\iota_{8n-1}, \iota_{8n-1}]$. But $P(\iota) = (i_2 k_1 k j)_*(\iota)$ where j is the degree one map. Thus $[\iota_{8n-1}, \iota_{8n-1}]$ is in the image of $(k_2)_*$. Suppose

$$f\colon S^{16n-3} \to Q(\Sigma^{8n-1} P)$$

projects to the class of order 2 in $J_{8n-2}(P)$. This class factors through $\Sigma^{8n-1} P^{8n-2}$ where it has order at least 4 since it has order 4 in $J_{8n-2}(P^{8n-2})$. Let $\alpha \in \pi_{8n-2}(P^{8n-2})$ be such a class. The image of 2α in $J_{8n-2}(P^{8n-2})$ is the same as the image of $S^{8n-2} \to P^{8n-2}$ which defines P^ι. There

$(k_2)_*(2\alpha + \beta) = [\iota_{8n-1}, \iota_{8n-1}]$ for some β which maps to zero in $J_{8n-2}(P^{8n-2})$. Since β is a stable homotopy class, the Adams filtration of β is ≥ 3 except if $n - 2^i$. The Adams filtration of α must also be ≥ 3 except if $n = 2^i$. But the Adams filtration of $[\iota_{8n-1}, \iota_{8n-1}]$ is 2 unless $n = 2^i$ when it is equal to 3.

Using the results of [BP], we see that $\alpha \in \{h_{i+2}^2\}$ and this completes the proof of 7.11 (and the case $j \equiv 6 \bmod 8$ of 7.10).

Remark: This argument shows clearly the connection between the class of order 2 in $J_{8n-2}(P)$ and the Kervaire invariant. Also if there is a class $\theta_i \in \{h_i^2\}$ and if $2\theta_i = 0$ then we see immediately that $s_*(\theta_i) \in J_{2^{i+1}-2}(P)$ is non-zero. This result should be true if θ_i exists but is not of order 2. I do not know of a proof of this.

To complete the proof of 7.10 we must show that the rest of the classes in $J_*(P)$ are in the Hurewicz image. Consider the following diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[dr] & P \wedge J \arrow[r, "\lambda\wedge\mathrm{id}"] & J \\
 & P \wedge \Sigma^3\mathrm{bsp} \arrow[u, "i_1"'] \arrow[r, "\lambda\wedge\mathrm{id}"] & \Sigma^3\mathrm{bsp} \arrow[u, "i_2"'] \\
M^7 \arrow[uu, "\bar{f}"] \arrow[ur, "f"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

where $\lambda\colon P \to S^0$ is the stable map obtained from the composite $P \to SO \subset Q(S^0)$. In the following we write $\lambda \wedge \mathrm{id}$ as λ. All the other maps except f, g and $\bar{f}$ have been defined. By 7.2 we see that every map $g\colon M^7 \to \Sigma^3\mathrm{bsp}$ of Adams filtration 1 factors through λ. Let $f\colon M^7 \to P \wedge \Sigma^3\mathrm{bsp}$ be a factorization of the composite $M^7 \xrightarrow{P} S^7 \xrightarrow{\bar{g}} \Sigma^3\mathrm{bsp}$ where P is the pinch map and $\bar{g}$ is a generator. Since P and $P \wedge J$ agree through dimension 6 we can construct $\bar{f}$ to give a commutative diagram. Let $A\colon M^{15} \to M^7$ be the Adams map [A]; then $i_2 g A^k$ is essential for all k. Thus fA^k is essential for all k. We need to show that $S^{8k+6} \to M^{8k+7} \xrightarrow{A^k} M^7 \xrightarrow{\bar{f}} P$ is null homotopic. Since $\bar{f}$ has filtration 1 and A^k has filtration $4k$, the composite has Adams filtration $4k + 1$. We will prove the following later.

LEMMA 7.12. *There is no non-zero homotopy class in* $\pi_{8k+6}(P)$, $k \geq 1$, *of Adams filtration* $4k + 1$.

This lemma shows that $M^{8k+7} \to P$ factors through a map $S^{8k+7} \xrightarrow{\bar{\bar{f}}} P$ and since the composite $M^{8k+7} \to P \to P \wedge J$ is essential, $\bar{\bar{f}}$ is essential and projects to a generator of $J_{8k+7}(P)$. Let $\bar{\eta}$ generate the $\mathbf{Z}/4$ in $\pi_2(M^1)$. Then

$S^{8k+8} \xrightarrow{\bar{\eta}} M^{8k+7} \xrightarrow{A^k} M^7 \to P$ generates the $\mathbf{Z}/2$ in $J_{8k+8}(P)$ of filtration $4k+6$. All of the other classes are constructed in a similar fashion.

Proof of 7.12. Consider the composite $SO \xrightarrow{J} \Omega^\infty S^\infty \xrightarrow{s} QP$ where s is the loop of the Barratt Eccles map [BE]. Let $E_2(SO)$ and $E_2(\Omega^\infty S^\infty)$ be E_2 terms for unstable Adams spectral sequences for SO and $\Omega^\infty S^\infty$ respectively. Let $E_2(\mathfrak{X}(P))$ be the looping of a stable Adams spectral sequence for P. Then we have maps $E_2(SO) \to E_2(\Omega^\infty S^\infty) \to E_2(\mathfrak{X}(P))$. Let f: $M^{11} \to SO$ be the composite $M^{11} \xrightarrow{p} S^{11} \xrightarrow{\text{gen}} SO$. By Curtis [C, page 191] f has filtration 3. Thus $M^{3+8k} \xrightarrow{A^{k-1}} M^{11} \to SO$ has filtration $3+4k$ and the composite $M^{3+8k} \to QP$ has filtration $\geq -1+4k$. Since the class of Adams filtration $1+4k$ in stem $2+8k$ cannot be in the image of J, the composite $S^{12+8k} \to M^{11+8k} \to QP$ is zero. Thus the class of Adams filtration $4k$ in stem $3+8k$ represents the generator of the image of the J homomorphism. Since $\pi_{11+8k}(SO) \circ \nu = 0$, the class in Adams filtration $1+4k$ and stem $6+8k$, which is h_2 composed with the class of Adams filtration $4k$ in stem $3+8k$, cannot represent a nonzero homotopy class.

Next we study $\{E_r(\mathfrak{X}(P_{2n+1}))\}$ and $J_*(P_{2n+1})$. Lemma 7.5 calculates the E_1-term and Theorem 7.6 implies all the differentials. The result of this calculation is the following.

THEOREM 7.13. *The groups $J_*(P_{2n+1})$ are given by the following table where $a_n = 0$, $n \equiv 0(2)$ and $a_n = 2$ if $n \equiv 1(2)$.*

$$J_j(P_{2n+1}) = \begin{cases} \mathbf{Z}/2 & j = 2n+1 \\ \mathbf{Z}/2 & j = 2n+2,\ n \equiv 0(2) \\ 0 & j = 2n+2,\ n \equiv 1(2) \\ \mathbf{Z}/8 & j = 2n+3,\ n \equiv 0(2) \\ \mathbf{Z}/2 & j = 2n+3,\ n \equiv 1(2) \\ (\mathbf{Z}/2)^2 & j \equiv (2n+a_n)(8),\ \equiv (2n+1+a_n)(8) \\ \mathbf{Z}/2 & \quad \equiv (2n+a_n+4)(8) \\ A(j,n) & j-2n-a_n \equiv 2(8) \\ B(j,n) & j-2n-a_n \equiv 3(8) \\ C(j,n) & j-2n-a_n \equiv 6(8) \\ D(j,n) & j-2n-a_n \equiv 7(8) \end{cases}$$

The groups A, B, C, D fit into the following exact sequences.

$$0 \to B(j,n) \to \mathrm{bo}_j(P_{2n+1}) \xrightarrow{(j+1)} \mathrm{bsp}_{j-4}(P_{2n+1}) \to A(j-1,n) \to 0,$$
$$0 \to D(j,n) \to \mathrm{bo}_j(P_{2n+1}) \xrightarrow{(j+1)} \mathrm{bsp}_{j-4}(P_{2n+1}) \to C(j-1,n) \to 0.$$

Remarks. The map $(j+1)$ means multiplication by the integer $j+1$ which, since the groups on both sides are 2-groups, is multiplication by $2^{|j+1|_2}$. For values of j large compared to $2n+1$ we have

$$\begin{aligned} &j \equiv 2(8), && A(j,n) = 0 && = C(j,n), && B(j+1,n) = \mathbf{Z}/8 && = D(j+1,n) \\ &j = 8k-2, && A(j,n) = \mathbf{Z}/2^{|k|_2} && = C(j,n), && B(j+1,n) = \mathbf{Z}/2^{|16k|_2} && = D(j+1,n). \end{aligned}$$

(Note that for a given $j \equiv 2(8)$ and a given n only one of $A(j,n)$ or $C(j,n)$ is defined, and similarly for B and D. For example $A(26,5)$ is not defined since $26 - 10 - 2 \equiv 6 \bmod 8$ but $C(26,5)$ is defined (and is zero).)

THEOREM 7.14. *Let σ_j be the cokernel of the Hurewicz map $\pi_j(P_{2n+1}) \to J_j(P_{2n+1})$. Suppose $j > 2n+10$. Then $\sigma_j = Z/2$ if $j \equiv 4(8)$, $= 0$ if $j \equiv -1, 2, 3, 5 \bmod 8$ and $= Z/2$ or 0 if $j \equiv 0 \bmod 8$.*

Remarks on unsettled cases. If $j \equiv 6 \bmod 8$ then knowledge of the Kervaire invariant is necessary to determine the cokernel in all cases. Partial results are known. If $j \equiv 0 \bmod 8$ then the results on η_j [M5] are relevant. The results in [M6] settle the story if $j \equiv 8 \bmod 16$ completely. Also all small values of $j - 2n$ are settled in [M6]. The dependence on n and j is very complicated in these cases.

Proof. The case $j \equiv 4(8)$ is effectively handled in [M6]. The balance of the theorem essentially asserts that the "v_1 periodic" homotopy is mapped isomorphically. We know this for Y homotopy. If we consider the sequence

$$(P_{2n+1} \wedge J) \wedge M_{2\iota} \to (P_{2n+1} \wedge \mathrm{bo}) \wedge M_{2\iota} \to \left(P_{2n+1} \wedge \Sigma^4 \mathrm{bsp}\right) \wedge M_{2\iota},$$

we see that $\pi_j(P_{2n+1} \wedge J \wedge M_{2\iota}) = \pi_j(\Sigma^k \mathrm{bo} \wedge M_{2\iota}) \oplus \pi_j(\Sigma^{k+1}\mathrm{bo} \wedge M_{2\iota}) + W_j$ where W is a $\mathbf{Z}/2$ vector space (as a $\pi_*(\mathrm{bo})$ module) if $j \geq k$ and $k \geq 2n+8$ and $k \equiv -1 (\bmod 8)$. From this it follows immediately that the $(\mathbf{Z}/2)$'s claimed in the image for $j \equiv 0, 1, 2 \bmod (8)$ are in the image. All that remains is to verify the result for $j \equiv -1(4)$. We will give an explicit construction for the homotopy class. The following result is necessary.

For any suspension spectrum X and any $\alpha \in \pi_k(S^0)$ we have a map α: $\Sigma^k X \to X$ which is $\alpha \wedge \mathrm{id}$. The map 2: $P_{4n-1} \to P_{4n-1}$ factors through P_{4n+1}. Let f_1: $P_{4n+1} \to P_{4n-1}$ be such a factorization. The map 2: $P_{4n+1} \to P_{4n+1}$ factors through P_{4n+2}. Let f_2: $P_{4n+2} \to P_{4n+1}$ be such a factorization. Let $\bar{q}$: $P_{4n+2} \to S^{4n+2}$ be a degree 2 map. Then $(2 - \iota_{4n+2}\bar{q})$: $P_{4n+2} \to P_{4n+2}$ factors through f_3: $P_{4n+3} \to P_{4n+2}$, where ι_{4n+2}: $S^{4n+2} \to P_{4n+2}$ is a generator.

LEMMA 7.15. *There is a map q: $\Sigma^{4n+2}B(1) \to P_{4n+2}$ with cofiber $C_q(4n+3)$ and a map p: $\Sigma^{4n+5}M_{2\iota} \to P_{4n+3}$ with cofiber $C_p(4n+3)$ such that* *the following diagram commutes*:

$$
\begin{tikzcd}
P_{4n+5} \arrow[r, "f_1"] \arrow[dd, "i_3"] & P_{4n+3} \arrow[r, "f_3"] \arrow[dd, "i_2"] & P_{4n+2} \arrow[dr, "f_2"] \arrow[dd, "i_1"] & & \\
& & & P_{4n+1} \arrow[r, "f_1"] & P_{4n-1}. \\
P_{4n+7} \arrow[r, "j_1"'] & C_p(4n+3) \arrow[r, "j_3"'] & CC_q(4n+3) \arrow[ur, "j_2"'] & &
\end{tikzcd}
$$

In addition each map f_i and each map j_i has Adams filtration 1 *and each j_i and f_1 induce an isomorphism in* Ext_A *for* $6s > t + 14$. ($B(1)$ *is defined in* § 2.)

Proof. The construction of the maps q and p need the homotopy calculations of [M6]. The map q is constructed by observing that the composite $\Sigma^{4n+3}M_{2\iota} \underset{\eta}{\to} S^{4n+2} \underset{2\iota_{4n+2}}{\to} P_{2n+2}$ is null homotopic. It clearly has Adams filtration two and [M6, table 8.3] shows that there is nothing of Adams filtration two in these dimensions. Thus $S^{4n+2} \cup_\eta C\Sigma^{4n+3}M_{2\iota} \xrightarrow{q} P_{4n+2}$ and this defines C_q. Choose q to be a map of Adams filtration 1. Then f_2q has filtration 2 and again by inspection of [M6; table 8.4] we see that the composite is trivial. Hence j_2 exists. The map p: $\Sigma^{4n+5}M_{2\iota} \to P_{4n+3}$ is an extension of the non-trivial map of $S^{4n+5} \to P_{4n+3}$. The composite f_3p is zero and so $j_3C_p \to C_q$ exists. The map f_3 is the standard map. The composite $P^{4n+6}_{4n+5} \to P_{4n+5} \xrightarrow{f_2} P_{4n+3} \xrightarrow{i_2} C_p$ is inessential since f_1 has Adams filtration 1. Thus $P_{4n+5} \cup CP^{4n+6}_{4n+5} = P_{4n+7} \to C_n$. This constructs the maps. The statements about Adams filtration follow from the constructions. The fact, that j_i and f_1 induce isomorphism in Ext groups, asserted follows from the easily verified fact that $H^*C_{f_1}$ and $H^*C_{j_i}$, $i = 1, 2$, are all A_1 free modules. (C_g is the mapping cone of g.) This completes the proof.

We return to the proof of 7.14. The argument is easier for $2n + 1 = 8N + 3$ or $8N + 5$, and the two cases are similar. We will do $2n + 1 = 8N + 3$.

First recall that if $|8j|_2 = 4a + b$, $0 \le b \le 3$, then the solution to the vector field on spheres problem gives us a map g: $S^{8j-1} \to P_{8j-(8a+2^b)}$ which projects to a generator of $\mathrm{bo}_{8k-1}(P_{8j-(8a+2^b)})$. Lemma 7.15 gives a map g_1: $P_{8j-(8a+2^b)} \to P_{8N+3}$ which induces multiplication by 2 to the power $4(j - a - N) + b$ in bo_{8j-1}. Thus the composite

$$S^{8j-1} \to P_{8j-(8a+2^b)} \to P_{8N+3} \to P_{8N+3} \wedge J \to P_{8N+3} \wedge \mathrm{bo}$$

has the same image in homotopy in the $8j - 1$ stem as g_3. Hence g_3 induces a surjection in homotopy in stem $8j - 1$. To help check this we recall the calculation of 7.13. For this it is easier to just write down the powers of 2 involved. The generator of $J_{8j-1}(P_{8N+3})$ maps to $2(4j - 4N - 1 - 2 - 4a - b)$.

($2(a) = 2^a$.) The map g_1 from 7.15 is obtained essentially by multiplying 2 to a power which is $4j - 4a - b - 4N - 3$. These two exponents are the same. (These numbers are just the filtration of the classes in $E_1(\mathfrak{X}(P_{2n+1}))$.)

Minor modifications are needed to handle $8N + 5$. More serious modifications are needed to handle $8N + 1$ and $+7$. Again they are similar and if $b \neq 1, 2$ the proof is as before. If $b = 1$, or 2 then as before we get a map

$$g_1\colon P_{8j-(8a+2^b)-4} \to P_{8N+3}$$

but the vector field solution no longer applies. We will prove the following.

LEMMA 7.16. *If* $|4j|_2 = 4a + b$ *and* $b = 2$ *then there is a map* $S^{8j-1} \to C_q(8j - 8a - 2^b - 2)$ *of degree* 1.

Proof. Let ϕ be large compared to $|8j|_2$. Then the Spanier Whitehead dual of

$$P^{8j-1}_{8j-8a-6} \quad \text{is} \quad P^{2^\phi-1-8j+8a+6}_{2^\phi-1-8j+1}.$$

This is the Thom complex of $2^\phi - 8j$ times the line bundle over P^{8a+5}. This bundle is classified by the composite $P^{8a+5} \to P^{8a+5}_{8a+4} \xrightarrow{g} \mathrm{BO}$ where g extends a generator in dimension $8a + 4$. Now C_q is defined by $\Sigma^{8j-8a-6}B(1) \to P_{8j-8a-6} \to C_q$. Consider the finite skeleton

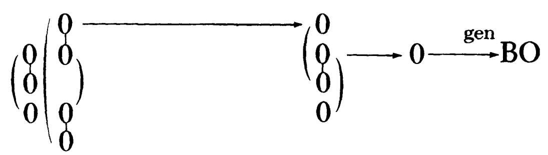
$$\Sigma^{8j-8a-6}B(1) \to P^{8j-8a-6}_{8j-8a-6} \to \left[C_q(8j - 8a - 6)\right]^{8j-8a-2}.$$

The Spanier-Whitehead dual of this gives the map h of the composite k

$$\mathfrak{D}\left(C_q(8j - 8a - 6)\right)^{8j-8a-2} \xrightarrow{h} P^{8a+5}_{8a+2} \to P^{8a+5}_{8a+4} \xrightarrow{g} \mathrm{BO}.$$

It is an easy calculation to verify that the composite k is inessential.

The following cell diagram may help. Vertical lines mean that 2ι is the attaching map and curved lines mean η is the attaching map.



$D\left(C_q(8j - 8a - 6)\right)$ $\quad P^{8a+5}_{8a+2} \quad S^{8a+4}$

This calculation induces maps among the Thom complexes:

$$\begin{array}{ccc} S^{2^\phi-1-8j-1} \vee \mathfrak{D}(C_q(8j - 8a - 6) & {}^{8j-8a-2} \to & \mathfrak{T}(g) \\ \uparrow & & \uparrow \\ [\mathfrak{D}(C_q(8j - 8a - 6)]^{2^\phi-1-8j+8a+6}_{2^\phi-8j} & \to & P^{2^\phi-1-8j+8a+6}_{2^\phi-8j} \end{array}$$

with degree 1 on the cell in dim $2^{\phi} - 8j$. Dualizing back gives

$$S^{8j-1} \vee \left[C_q(8j - 8a - 6)\right]^{8j-8a-2} \to C_q(8j - 8a - 6)$$

and the composite

$$S^{8j-1} \to S^{8j-1} \vee \left[C_q(8j - 8a - 6)\right]^{8j-8a-2} \to C_q(8j - 8a - 6)$$

has degree 1. By an almost identical argument we show

LEMMA 7.17. *For a and b as functions of j as before, if $b = 1$, then there is a map of S^{8j-1} into $C_p(8j - 8a - 2^b - 4)$ of degree* 1.

Using these two lemmas, instead of the vector field solution in the argument for 7.14, and $2n + 1 = 8N + 3$, allows one to complete the proof for $2n + 1 = 8N + 1$ or $8N + 7$. Indeed, as before we have

$$P_{8j-8a-1} \to C_p(8j - 8a - 5) \to C_q(8j - 8a - 6) \to P_{8j-8a-7} \to P_{8N+1}.$$

If $b = 3$ then there is a map $S^{8j-1} \to P_{8j-8a-7}$ by the vector field theorem. The case of dimension $8j + 3$ is handled in a very similar fashion.

This completes the proof of 7.14. From 7.14 we prove easily

THEOREM 7.18. *If $j \geq 2n + 8$ or if $j = 2^i - 2 \bmod 2^{i+1}$, then $j \geq 2n + 8 + 2i$; then the Hurewicz map $\pi_j(P^{2n}) \to J_j(P^{2n})$ is surjective.*

Proof. Again because the result is true for coefficients in Y and in M_{2^ι}, it is only necessary to verify the theorem if $j = 8p - 2$ and $8p - 1$. Consider the diagram

$$\begin{tikzcd}
\arrow[r] & \pi_{8p-1}(P^{2n}) \arrow[r] \arrow[d, "\phi'"] & \pi_{8p-1}(P) \arrow[r, "i_1"] \arrow[d] & \pi_{8p-1}(P_{2n+1}) \arrow[r] \arrow[d, "\phi''"] & \pi_{8p-2}(P^{2n}) \arrow[r] \arrow[d, "\phi'"] & {} \\
\arrow[r] & J_{8p-1}(P^{2n}) \arrow[r] & J_{8p-1}(P) \arrow[r, "i_2"] \arrow[d] & J_{8p-1}(P_{2n+1}) \arrow[r] \arrow[d] & J_{8p-2}(P^{2n}) \arrow[r] & {} \\
& & \mathrm{bo}_{8p-1}(P) \arrow[r, "i_3"] & \mathrm{bo}_{8p-1}(P_{2n+1}) & &
\end{tikzcd}$$

By the proof of 7.14, there is a map $f\colon S^{8p-1} \to P$ of Adams filtration $4p - |8p|_2$ such that $\phi(f)$ is a generator. The Adams vanishing line for P_{2n+1} in stem $8p - 1$ is $4p - n + a_n$ where $a_n = -1, 0, 0, -1$ for $n \equiv 0, 1, 2, 3 \bmod 4$ respectively. If $4p - |8p|_2 > 4p - n + a_n$, then f factors through P^{2n} and $\phi'(f)$ is a generator. If $4p - |8p|_2 \leq 4p - n + a_n$ then, if we let $n - a_n - |8p|_2 + 1 = k$, $2^k f$ has filtration $4p - n + a_n + 1$ and thus factors through P^{2n}. On the other hand, $i_3 \psi \phi 2^{k-1} f \neq 0$ so if $f'\colon S^{8p-1} \to P^{2n}$ is such a factorization, $\phi' f'$ is a generator. This handles the case $j = 8p - 1$.

Let f'': $S^{8p-1} \to P_{2n+1}$ represent a generator. Because of the filtration argument just presented, $f'\phi''$ cannot be in the image of j_2. (It is in the image of j_3.) Thus $\partial_2\phi''f'$ is non-zero and $\partial_1 f'$ maps onto the image. The exceptional case handles the situation where ϕ' is not onto.

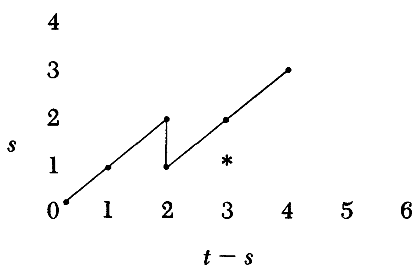
The case $j = 8p + 1$ is handled in an analogous fashion. The argument is quite simple and is left to the reader. This argument can be helped by considering the charts given earlier.

The proof of 1.5 is now easily finished. First suppose $n \geq 2$. Recall that there is a map $\Sigma^{2n+1}P^{2n} \to S^{2n+1}$. Thus from an easy modification of 7.15 we have for each k, $\Sigma^{2n+1}P_{8k+1}^{8k+2n} \to \Sigma^{2n+1}P^{2n} \to S^{2n+1}$. If we loop all of this $2n + 1$ times we have

$$P_{8k+1}^{8k+2n} \to \Omega^{2n+1}\left(\Sigma^{2n+1}P_{8k+1}^{8k+2n}\right) \to \Omega^{2n+1}\Sigma^{2n+1}P^{2n}$$
$$\to \Omega^{2n+1}S^{2n+1} \to Q\left(P^{2n} \wedge J\right).$$

In the stable range of the complex on the left, 7.18 applies. Thus the Hurewicz image $\pi_{j+2n+1}(S^{2n+1}) \to J_j(P^{2n})$ is onto for $8k + 2n < j < 16k + 1$. For any j there is a suitable k.

To handle $n = 1$ and 2 we use the discussion following 7.9. For S^3 we have a map $M^{11} \xrightarrow{\varepsilon^{\#}} S^3$ which is an extension of ε. It is easily verified that $M^9 \to \Omega^2S^3 \to Q(\Sigma P^2 \wedge J)$ is essential. The beginning homotopy of M^n satisfies: $\pi_n(M) = \pi_{n+1}(M) = \pi_{n+4}(M) = \mathbf{Z}/2$, $\pi_{n+2}(M) = \mathbf{Z}/4$ and $\pi_{n+3}(M) = (\mathbf{Z}/2)^2$. The Ext groups form a pattern as given in the following chart.



$\mathrm{Ext}_A(A_0, \mathbf{Z}/2)$ for t and s small.

These groups except * clearly map essentially in $Q(\Sigma P^2 \wedge J)$. Similarly there is a map $M^{12} \xrightarrow{\mu^{\#}} S^3$ extending μ. These maps with the Adams map complete the picture.

Very similar arguments apply when $n = 2$ (and 3). The result summarizes nicely as follows.

PROPOSITION 7.19. *The coker of* $\pi_j(\Omega^{2n}S^{2n+1}) \to J_j(\Sigma P^{2n})$ *when* $n \le 3$ *is* $\mathbf{Z}/2$ *if* $j = 5$ *and equals zero in all other dimensions.*

8. The image of J and v_1-periodic homotopy

Stable homotopy theorists prefer to discuss the spectrum J and the map $S^0 \to J$. In some sense J "detects" the image of J. We will make this precise. To talk about the image of J we look at $\mathrm{SO} \subset \Omega^\infty S^\infty$ and consider the induced map in homotopy. If we look at an Adams unstable spectral sequence for SO with E_2 represented as $E_2(\mathrm{SO})$ we have by 3.2 a map

$$J_\#: E_2^{s,t}(\mathrm{SO}) \to E_2^{2+1,t+1}(S^0) \simeq \mathrm{Ext}_A^{s+1,t+1}(\mathbf{Z}/2, \mathbf{Z}/2).$$

LEMMA 8.1. *In* $\{E_r(\mathrm{SO})\}$ *the generators of* $\pi_j(\mathrm{SO})$, $j = 8k,\ 8k+1$, *have filtration* $\ge 4k-3,\ 4k-2$ *respectively. The map* $M^{8k+3} \to S^{8k+3} \xrightarrow{g} \mathrm{SO}$ *has filtration* $4k+3$.

Proof. Let $S^7 \to \mathrm{SO}$ represent a generator. Then there is a map $M^{15} \xrightarrow{\bar{A}} S^7$ which extends 4τ where τ generates the $\mathbf{Z}/8$ summand in $\pi_{14}(S^7)$. (Compare 7.19.) Now consider

$$\begin{tikzcd} M^{15+8k} \arrow[r, "A^k"] & M^{15} \arrow[r] \arrow[rr, bend left, "j"] & S^7 \arrow[r] & \mathrm{SO}. \end{tikzcd}$$

It is easily verified that this map factors as $M^{15+8k} \to S^{15+8k} \xrightarrow{g} \mathrm{SO}$ where g is a generator. Let $\bar{\eta}_{15}$: $S^{16} \to M^{15}$ generate the $\mathbf{Z}/4$ summand. The composite $A_k\bar{\eta}_{15+8k}\eta$ generates $\pi_{17+8k}(\mathrm{SO})$ and has filtration $\ge 4k+2$. The composite $M^{19} \xrightarrow{(\eta^2\bar{\eta})^\#} M^{15} \to \mathrm{SO}$ factors as $M^{19} \to S^{19} \xrightarrow{g} \mathrm{SO}$ where g is a generator. Thus the map $M^{19+8k} \to \mathrm{SO} \to \Omega^\infty S^\infty$ represents the generator and has filtration $\ge 4k+3$.

THEOREM 8.2. *If* $k > 1$, *the generator of the image of* J *in* $8k$, $8k+1$ *and* $8k+3$ *has Adams filtration* $4k-1$, $4k$ *and* $4k+1$.

Proof. The calculations of [MT] show that in $t - s = 8k$, $k > 1$, the only class of Adams filtration $\ge 4k-2$ is $P^{(k-1)}c_0$. Thus the generator in $8k$ is in the coset $P^{k-1}c_0$ which is a unique class; $8k+1$ is similar. For the generator $8k+3$ we have $M^{8k+3} \to S^0$ of filtration $\ge 4k$ such that $S^{8k+2} \to M^{8k+3} \to S^0$ is null. Again, inspection of the tables in [MT] implies that the coset defined by $P^k h_2$ represents the generator in the image of J in the $8k+3$ stem.

Note that we say nothing about the filtration of the generator of the image of J in $8k-1$. There is a class in $\pi_{8k-1}(S^0)$ of filtration $4k - |8k|_2$ which projects to a generator of $J_{8k-1}(S^0)$.

THEOREM 8.3. *The composite* $M^{15+8k} \xrightarrow{A^k} M^{15} \xrightarrow{j} \mathrm{SO} \to Q(S^0) \to Q(P \wedge J)$ *is essential for all* k.

This is immediate and combined with the above shows that the homotopy we have been dealing with is the image of J. Another discussion of this is given in [DM].

A question one might ask is what is the sphere of origin of the image of J. By this we mean the following. What is the smallest k such that there is a class $\alpha \in \pi_j(\Omega^k S^k)$ such that the composite Σ: $\pi_j(\Omega^k S^k) \to \pi_j(QS^0)$ has the property that $\Sigma\alpha$ is a generator of the image of the J-homomorphism? Call this k, $\mathcal{O}(j)$.

THEOREM 8.4. *If* $j \geq 15$, *the origin of the image of* J *is*:

$$(j) = \begin{cases} j \equiv 0(8) \\ 2 \equiv 1(8) \\ 5 \equiv 3(8). \end{cases}$$

If $j = -1(8)$ *then an element of order* 2^a *has sphere of origin* $\bar{\mathcal{O}}_a$ *given by*

$$\begin{aligned} \bar{\mathcal{O}}_a &= 4 + i + 8l, \qquad & a &= 4l + i, \qquad & i &= i, 2, 3 \\ &= l + 8l, & a &= 4l. \end{aligned}$$

If $j = 8k - 1, j \geq 15$, *then the element in the image of* J *of order* 2^a *has sphere of origin*

$$\begin{aligned} k &= 5 + 8l, \qquad & a &= 4l + 1 \\ &= 6 + 8l & &= 4l + 2 \\ &= 7 + 8l & &= 4l + 3 \\ &= 1 + 8l & &= 4l. \end{aligned}$$

This, too, follows directly from the calculations, in particular, the tables given after 7.9. The $\mathrm{SO}(n)$ of origin for the stable classes of SO is, of course, different. These results are not part of this discussion but $\pi_{4n-1}(\mathrm{SO})$ is in the image of $\pi_{4n-1}(\mathrm{SO}(2n+1))$ if $n > 4$. $\pi_{8n+i}(\mathrm{SO})$ is in the image of $\pi_{8n+i}(\mathrm{SO}(6))$ for $i = 0$ and 1. (The map in question is the one induced by $\mathrm{SO}(n) \subset \mathrm{SO}$.)

9. Some problems

These results suggest some exponent problems. We first have the following which is proved in [M7].

THEOREM 9.1. *For each* k *there is a class* $\alpha \in \pi(S^0\colon Y)$ *such that* $\alpha v_1^k \neq 0$ *but* $\alpha v_1^{k+1} = 0$.

The classes which work are obtained from η_j's [M5].

The importance of Theorem 9.1 is analogous to the following. Let $S_1 \to S^0 \to K(\mathbf{Z})$ be a fibration. Then $\pi_*(S_1)$ is a torsion group but has torsion of arbitrary order. The fiber S_{v_1} of $S^0 \to J$ has a similar property with respect to v_1.

Problem 1. Let $S^n\langle n\rangle$ be the n-connected cover of S^n. Then, as is well known, the torsion in $\pi_*(S^n\langle n\rangle)$ is bounded. If $n = 2l+1$ then $\to S^{2l+1} \to K(\mathbf{Z}, 2l+1)$ is a rational equivalence. The fiber contains all the torsion. A useful notation would be $S^{2l+1}\{h_0\}$. Then $\Omega^{2l}S^{2l+1}\{h_0\} \to Q(\Sigma P^{2l} \wedge J)$ is a $V^{-1}R$ equivalence. Let $S^{2l+1}\{h_0, v_1\}$ be the fiber.

Conjecture 1. It seems likely that $\pi_*(S^{2l+1}\{h_0, v_1\}; Y)$ is annihilated by $v_1^{k(l)}$ for some $k(l)$. The function $k(l)$ should be $\leq [l/2]+3$. No upper bound is known even for S^3.

Problem 2. The maximum torsion in S^{2l+1} seems to be detected by $J_*(P^{2l})$.

Conjecture 2. The 2 primary torsion of $\pi_*(S^{2l+1}\{h_0, v_1\})$ should be about half that of $\pi_*(S^{2l+1})$. It may be the same function $k(l)$ in 1.

Problem 3. Let F_n be the fiber of $W(n) \to \Omega^4 W(n+1)$. The results of this paper and [M3] suggest that F_n has many of the properties of the space $\bar{A}_1$, a space whose cohomology realizes the subalgebra of A generated by Sq^1 and Sq^2.

Conjecture 3. It seems likely that $v_1^2\pi_*(F_n; Y) = 0$. A result like this should help in problems 1 and 2.

Problem 4. In [DM] the starting point for a v_2-theory is given. Can one find a spectrum J_2 so that $S^0 \to J_2$ induces a v_2-isomorphism and J_2 contains only torsion-free, v_1-free and v_2-free homotopy? Preliminary results give a v_2-periodic family into which the η_j's fit and these correspond to the "β"'s at odd prime. They include ν, ν^2, ν^3, $\eta^2 h_4$, $\nu\bar{\kappa}$, $\varepsilon\bar{\kappa}$, $\{hq_1\}$, $\mu\eta\eta_5, \ldots$ and a second family beginning with $\{c_1\}, \nu\{c_1\}, \{n\}, \{t\}, \{N\}, \ldots$. There should be several more "independent families". The detailed calculations available are not adequate to shed much light on this. This problem is really a version of a conjecture, due to Ravenel, which suggests that all the homotopy of spheres fits into v_i periodic families.

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