EQUIVARIANT INFINITE LOOP SPACE THEORY, I. THE SPACE LEVEL STORY

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Abstract. We rework and generalize equivariant infinite loop space theory, which shows how to construct G-spectra from G-spaces with suitable structure. There is a naive version which gives naive G-spectra for any topological group G, but our focus is on the construction of genuine G-spectra when G is finite.

We give new information about the Segal and operadic equivariant infinite loop space machines, supplying many details that are missing from the literature, and we prove by direct comparison that the two machines give equivalent output when fed equivalent input. The proof of the corresponding nonequivariant uniqueness theorem, due to May and Thomason, works for naive G-spectra for general G but fails hopelessly for genuine G-spectra when G is finite. Even in the nonequivariant case, our comparison theorem is considerably more precise, giving a direct point-set level comparison.

We have taken the opportunity to update this general area, equivariant and nonequivariant, giving many new proofs, filling in some gaps, and giving some corrections to results in the literature.

Contents

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1. Introduction and preliminaries

Equivariant homotopy theory is much richer than nonequivariant homotopy theory. Equivariant generalizations of nonequivariant theory are often non-trivial and often admit several variants. Nonequivariantly, symmetric monoidal (or equivalently permutative) categories and E_∞ spaces give rise to spectra. There are several "machines" that take such categorical or space level input and deliver spectra as output, and there are comparison theorems showing that all such machines are equivalent [29, 30, 38, 51].

Equivariantly, there are different choices of G-spectra that can be taken as output of infinite loop space machines. One choice is naive G-spectra, which are simply spectra with a G-action. They can be defined for *any* topological group G. The weak equivalences between them are the G-maps that induce nonequivariant weak equivalences on fixed point spectra. As we shall indicate, it is quite straightforward to generalize infinite loop space theory so as to accept naive input, such as G-spaces with actions by nonequivariant E_∞ operads, and deliver naive G-spectra. Moreover, the nonequivariant comparisons generalize effortlessly to this context. As we shall see, this much all works for any topological group G.

Naive G-spectra really are naive. For example, one cannot prove any version of Poincaré duality in the cohomology theories they represent. They can be indexed on the natural numbers, and n should be thought of geometrically as a stand-in for $\mathbb{R}^n$ or S^n, with trivial G-action. When G is a compact Lie group, we also have genuine G-spectra, which are indexed on representations or, more precisely, real vector spaces (better, inner product spaces) V with an action of G. Their one-point compactifications are the representation spheres, denoted S^V, and these spheres are inverted in the genuine G-stable homotopy category.

The weak equivalences between genuine G-spectra are again nonequivariant equivalences on fixed point spectra. However, since suspending and looping with respect to the spheres S^V does not commute with passage to H-fixed points when the action of G on V is nontrivial, the H-fixed spectra retain homotopical information about the representations of G. That information is encoded in the notion of equivalence of genuine G-spectra. One can also restrict attention to subclasses of representations and obtain a plethora of kinds of G-spectra intermediate between the naive ones (indexed only on trivial representations) and the genuine ones (indexed on all finite dimensional representations).

Our focus is on finite groups G and equivariant infinite loop space machines whose inputs are G-spaces X with extra structure and whose outputs are genuine Ω-G-spectra whose zeroth spaces are equivariant group completions of X. The two nonequivariant machines in most common use are those of Segal [51] and the senior author [27], and we call these the Segal and operadic machines. The Segal machine was generalized equivariantly in [54] and the operadic machine was generalized equivariantly in [10] and more recently in [12], which can be viewed in part as a prequel to this paper. We study these equivariant generalizations of the Segal and operadic infinite loop space machines and prove that when fed equivalent data they produce equivalent output.

Due to their very different constructions, the two machines have different advantages and disadvantages. Whereas the operadic machine is defined only for finite groups, the Segal machine can be used to construct genuine G-spectra for any compact Lie group G. These spectra are unfortunately not Ω-G-spectra unless G is finite, but their restrictions to finite subgroups H are Ω-H-spectra. The Segal machine also works simplicially and is likely to be the machine of choice in motivic contexts, if and when such a motivic theory is developed; it has yet to be developed even nonequivariantly. The operadic machine generalizes directly to give machines that manufacture intermediate types of G-spectra from intermediate types of input data, but we do not know a Segal type analogue. Due to its more topological flavor, the operadic machine was used to produce genuine G-spectra from categorical data in [12], where the machine was used to give categorical proofs of topological results.

We have several ways to generalize the Segal machine equivariantly, and we have comparisons among them. We develop the one closest to Segal's original version in §2, highlighting the role of its inductive simplicial definition in proving the group completion property. This version of the machine, which has not previously been developed equivariantly, starts from $\mathscr{F}$-G-spaces, namely functors from the category $\mathscr{F}$ of finite sets, the opposite of Segal's category Γ, to the category of based G-spaces. It produces naive G-spectra for any topological group G. We shall use it to prove the equivariant group completion property for our other versions of the Segal machine. Following Segal [51] nonequivariantly, we compare that machine to a conceptual version that is defined by categorical prolongation of functors defined on $\mathscr{F}$ to functors defined on the category $\mathscr{W}_G$ of G-CW complexes. However, the homotopical conditions needed to make the conceptual machine useful are seldom satisfied by the examples that arise in nature.

We recall the homotopically well-behaved machine in §3, which is based on the two-sided bar construction. This version of the Segal machine was first defined nonequivariantly by Woolfson [59], but the equivariant generalization of his definition starting from $\mathscr{F}$-G-spaces fails to be well-behaved homotopically. Following Shimakawa [54, 55, 56], we instead focus on an equivariant generalization that starts from the category of $\mathscr{F}_G$-G-spaces, which are functors from the category $\mathscr{F}_G$ of finite G-sets to the category of based G-spaces. As we reprove, these categories of input data are equivalent. The comparison of equivalences and the relevant "specialness" conditions shed considerable light on the underlying homotopy theory. To explain ideas without technical clutter, we defer the longer proofs about the Segal machine from §2 and §3 to §9.

To prove that the Segal and operadic machines are equivalent, we must first redevelop and generalize both so that they do in fact accept the same input. The generalizations follow the corresponding nonequivariant theory in May and Thomason [38]. Categories of operators were introduced there in order to simultaneously generalize $\mathscr{F}$-spaces and E_∞ spaces, and G-categories of operators serve the same purpose equivariantly. We have E_∞ G-categories of operators $\mathscr{D}$ over $\mathscr{F}$ and E_∞ G-categories $\mathscr{D}_G$ over $\mathscr{F}_G$, and we have algebras over each; $\mathscr{F} = \mathscr{D}$ and $\mathscr{F}_G = \mathscr{D}_G$ are special cases. Ignoring operads, we develop and compare Segal machines that produce genuine G-spectra from such algebras in §4.

Turning to operads, in the brief §5 we show how to construct a G-category of operators $\mathscr{D}(\mathscr{C}_G)$ over $\mathscr{F}$ and a G-category of operators $\mathscr{D}_G(\mathscr{C}_G)$ over $\mathscr{F}_G$ from an operad $\mathscr{C}_G$. Our focus in this paper is on E_∞ operads $\mathscr{C}_G$, and we prove that $\mathscr{D}(\mathscr{C}_G)$ and $\mathscr{D}_G(\mathscr{C}_G)$ are E_∞ G-categories of operators when $\mathscr{C}_G$ is an E_∞ operad.

Letting $\mathscr{C}_G$ be an E_∞ G-operad, we have algebras over $\mathscr{C}_G$, $\mathscr{D}(\mathscr{C}_G)$, and $\mathscr{D}_G(\mathscr{C}_G)$. In §6, we generalize the operadic machine to accept such generalized input. Starting from $\mathscr{C}_G$-algebras, the equivariant operadic machine was first developed by Costenoble and Waner [10][1] and is given a thorough modern redevelopment in [12]. Therefore we focus on the generalization and on comparisons of the machines starting from these three kinds of operadic input. This is conceptually the same as in May and Thomason [38], but the key proof equivariantly is considerably more intricate and is deferred to §8. A curious feature is that, in contrast to the Segal machine, there is no particular need to consider $\mathscr{F}_G$ rather than $\mathscr{F}$ when developing the operadic machine, although use of $\mathscr{F}_G$ is convenient for purposes of comparison.

The comparison of the Segal and operadic machines starting from the same input is given in §7. It seems quite amazing to us. Even nonequivariantly, it is far more precise than the comparison given in [38]. There is a family of operads, called the Steiner operads [58] (see also [12]). They are variants of the little cubes and little discs operads that share the good properties and lack the bad properties of each of those, as explained in [36, §3]. Nonequivariantly, they have played an important role in infinite loop space theory ever since their introduction in 1979. We see here

[1]In part they follow extensive unpublished work of Hauschild, May, and Waner.

that they mediate between the Segal and operadic infinite loop space machines just as if they had been invented for that purpose. That is wholly unexpected and truly uncanny.

Some background may help explain why we find such a precise point-set level comparison so surprising. In the Segal machine, higher homotopies are encoded in the specialness *property* of the structure map $\delta\colon X_n \longrightarrow X_1^n$ relating the nth space to the nth power of the first space of an $\mathscr{F}$-G-space. In the operadic machine, they are encoded in the *structure* given by the action maps $\mathscr{C}_G(n) \times X^n \longrightarrow X$ of a $\mathscr{C}_G$-algebra. Actions by the G-category of operators $\mathscr{D}_G(\mathscr{C}_G)$ encode both sources of higher homotopies in the same structure and yet, up to equivalence, carry no more information than either does alone.

The difference is perhaps illuminated by thinking about the commutativity operad $\mathscr{N}$ with $\mathscr{N}(j) = *$. The G-category $\mathscr{D}(\mathscr{N})$ over $\mathscr{F}$ is just $\mathscr{F}$ itself. An $\mathscr{N}$-G-space $X = X_1$ is the same thing as an $\mathscr{F}$-G-space with $X_n = X_1^n$. Both just give X the structure of a commutative monoid in $G\mathscr{T}$, and that is far too restricted to give the domain of an infinite loop space machine: the fixed point spaces of the infinite loop G-spaces resulting from such input are equivalent to products of Eilenberg-MacLane spaces. From the point of view of the Segal machine, we are replacing $\mathscr{N}$-G-spaces with homotopically well-behaved $\mathscr{F}$-G-spaces as input. From the point of view of the operadic machine, we are replacing $\mathscr{N}$-G-spaces by $\mathscr{C}_G$-spaces for any chosen E_∞ operad $\mathscr{C}_G$.

Consideration of categorical input is conspicuous by its absence in this paper and is crucial to the applications to equivariant algebraic K-theory that we have in mind. Both for application to the most natural input data and for the multiplicative theory, it is essential to work 2-categorically with lax functors, or at least pseudo-functors, rather than just with categories and functors, and it is desirable to work with symmetric monoidal G-categories rather than just the (genuine) permutative G-categories defined in [12]. That requires quite different categorical underpinnings than are discussed in this paper or in [12] and will be treated in detail in [15, 16]. We note that while permutative G-categories and their symmetric monoidal generalization are defined operadically in the cited papers, they are processed using the equivariant version of the Segal machine that we develop in this paper.

Model categorical interpretations may also be conspicuous by their paucity. All relevant model structures are developed in the papers [18, 24, 26, 44], and it is not hard to interpret some of our work, but not the essential parts, model theoretically. One point is that the group completion property, which is central to the theory and nearly all of its applications, is invisible to the relevant model categories. Another is that model categorical cofibrant and/or fibrant approximation might obscure what is intrinsically a quite intricate collection of very precisely interrelated notions. Perhaps unfashionably, we are interested in preserving as much point-set level structure as possible, which we find illuminating, and of course that is precisely what more abstract frameworks are designed to avoid.

We complete this section with some preliminaries that give common background for the various machines, fixing notations and definitions that are used throughout the paper. While our main interest is in finite groups, unless otherwise specified we let G be any topological group here and in §2 and §3. Subgroups of G are understood to be closed and homomorphisms are understood to be continuous. We restrict to finite groups starting in §4.

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1.1. **Preliminaries about G-spaces and Hopf G-spaces.** We let $\mathscr{U}$ be the category of compactly generated spaces weak Hausdorff spaces, let $\mathscr{U}_*$ be the category of based spaces, and let $\mathscr{T}$ be its subcategory of nondegenerately based spaces. We let $G\mathscr{U}$, $G\mathscr{U}_*$ and $G\mathscr{T}$ be the categories of G-spaces, based G-spaces, and nondegenerately based G-spaces, with left action by G; G acts trivially on basepoints. Maps in these categories are G-maps. We write $G\mathscr{T}(X,Y)$ for the based space of based G-maps $X \longrightarrow Y$, with basepoint the trivial map.

We let $\mathscr{T}_G$ be the category whose objects are the nondegenerately based G-spaces, but in contrast to $G\mathscr{T}$, whose morphisms are all based maps, not just the G-maps. Then G acts on maps by conjugation: for a map $f\colon X \longrightarrow Y$, $(gf)(x) = g\cdot f(g^{-1}\cdot x)$. We write $\mathscr{T}_G(X,Y)$ for the based G-space of based maps $X \longrightarrow Y$, with G acting by conjugation. Thus $\mathscr{T}_G(X,Y)$ is a G-space with fixed point space $G\mathscr{T}(X,Y)$. We can view $\mathscr{T}_G$ as a G-category such that G acts trivially on objects, and then $G\mathscr{T}$ can be viewed as the fixed point category of $\mathscr{T}_G$. In our treatment of the Segal machine, we will also need to consider the full subcategories $G\mathscr{W} \subset G\mathscr{T}$ and $\mathscr{W}_G \subset \mathscr{T}_G$ of based G-CW complexes.[2]

Properties of G-spaces are very often defined by passage to fixed point spaces. For example, a G-space X is said to be G-connected if X^H is (path) connected for all $H \subset G$.

Definition 1.1. Let $f\colon K \longrightarrow L$ be a map of G-spaces. We say that f is a *weak G-equivalence* if $f^H\colon K^H \longrightarrow L^H$ is a weak equivalence for all $H \subset G$. A family of subgroups of G is a set of subgroups closed under subconjugacy. For a family $\mathbb{F}$ of subgroups of G, we say that f is a *weak $\mathbb{F}$-equivalence* if f is a weak H-equivalence for all $H \in \mathbb{F}$. We often omit the word weak, taking it to be understood throughout.

The following families are central to equivariant bundle theory and to the analysis of equivariant infinite loop space machines. They will be used ubiquitously. Let Σ_n denote the nth symmetric group.

Definition 1.2. For a subgroup H of G and a homomorphism $\alpha\colon H \longrightarrow \Sigma_n$, let Λ_α be the subgroup $\{(h,\alpha(h)) \,|\, h \in H\}$ of $G \times \Sigma_n$. All subgroups Λ of $G\times\Sigma_n$ such that $\Lambda \cap \Sigma_n = \{e\}$ are of this form. Let $\mathbb{F}_n$ denote the family of all such subgroups. Taking α to be trivial, we see that $H \in \mathbb{F}_n$ for all n and all $H \subset G$.

[2] For our purposes, we need not restrict to essentially small full subcategories.

Remark 1.3. Since our homomorphisms are continuous, any $\alpha\colon G \longrightarrow \Sigma_n$ factors through a homomorphism $\pi_0(G) \longrightarrow \Sigma_n$. In particular, there are no non-trivial homomorphisms if G is connected. There are also infinite discrete groups G that admit no non-trivial homomorphisms to a finite group. Therefore, although the families $\mathbb{F}_n$ appear in general, they are only of real interest when G is finite.

When G is finite, we adopt the following conventions on finite G-sets.

Notation 1.4. Let $\mathbf{n}$ denote the based set $\{0, 1, \ldots, n\}$ with basepoint 0. For a finite group G, a homomorphism $\alpha\colon G \longrightarrow \Sigma_n$ determines the based G-set $(\mathbf{n}, \alpha)$ specified by letting G act on $\mathbf{n}$ by $g \cdot i = \alpha(g)(i)$ for $1 \leq i \leq n$. Conversely, a based G-action on $\mathbf{n}$ determines a G-homomorphism α by the same formula. Every based finite G-set with n non-basepoint elements is isomorphic to one of the form $(\mathbf{n}, \alpha)$ for some α. We understand based finite G-sets to be of this form throughout.

We also need some preliminaries about H-spaces, which we call Hopf spaces to avoid confusion with subgroups of G. Recall that a Hopf space is a based space X with a product such that the basepoint is a two-sided unit up to homotopy. For simplicity, we assume once and for all that our Hopf spaces are homotopy associative and homotopy commutative[3], since that holds in our examples. We say that a Hopf space is grouplike if, in addition, $\pi_0(X)$ is a group, necessarily abelian.

Definition 1.5. A Hopf map $f\colon X \longrightarrow Y$ is a *group completion* if Y is grouplike, $f_*\colon \pi_0(X) \longrightarrow \pi_0(Y)$ is the Grothendieck group of the commutative monoid $\pi_0(X)$, and for every every field of coefficients, $f_*\colon H_*(X) \longrightarrow H_*(Y)$ is the algebraic localization obtained by inverting the elements of the submonoid $\pi_0(X)$ of $H_*(X)$.[4]

Definition 1.6. A Hopf G-space is a based G-space X with a product G-map such that its basepoint e is a two-sided unit element, in the sense that left or right multiplication by e is a weak G-equivalence $X \longrightarrow X$. Then each X^H is a Hopf space, and we assume as before that each X^H is homotopy associative and commutative. A Hopf G-space X is grouplike if each X^H is grouplike. A Hopf G-map $f\colon X \longrightarrow Y$ is a group completion if Y is grouplike and the fixed point maps f^H are all nonequivariant group completions. Clearly a group completion of a G-connected Hopf G-space is a weak G-equivalence.

1.2. **Preliminaries about G-cofibrations and simplicial G-spaces.** Since G-cofibrations play an important role in our work, we insert some standard remarks about them.[5]

Remark 1.7. A map is a G-cofibration if it satisfies the G-homotopy extension property and a basepoint $* \in X$ is nondegenerate if the inclusion $* \longrightarrow X$ is a G-cofibration. Since we are working in $G\mathscr{U}$, a G-cofibration is an inclusion with closed image [33, Problem 5.1]. By [6, Proposition A.2.1] (or [33, p. 43]), if $i\colon A \longrightarrow B$ is a closed inclusion of G-spaces, then i is a G-cofibration if and only if (B, A) is a G-NDR pair. Using this criterion, we see that i is then also an H-cofibration for any (closed) subgroup H of G and that passage to orbits or to fixed points over

[3] It would suffice to assume that left and right translation by any element are homotopic.

[4] Segal [51, §4] describes the notion of group completion a bit differently, in a form less amenable to equivariant generalization, and he makes several reasonable restrictive hypotheses in his proof of the group completion property. In particular, he assumes that X is a topological monoid and that $\pi_0(X)$ contains a cofinal free abelian monoid.

[5] They are part of the h-model structure on $G\mathscr{U}$, as in [37] nonequivariantly.

H gives a cofibration. Moreover, just as nonequivariantly, a pushout of a map of G-spaces along a G-cofibration is a G-cofibration.

The Segal and operadic infinite loop space machines are both constructed using geometric realizations of simplicial G-spaces $X_\bullet$. Such realizations are only well-behaved when $X_\bullet$ is Reedy cofibrant.

Definition 1.8. Let $X_\bullet$ be a simplicial G-space with G-space of n-simplices X_n. The nth latching space of X is given by

$$L_nX = \bigcup_{i=0}^{n-1} s_i(X_{n-1}).$$

It is a G-space, and the inclusion $L_nX \longrightarrow X_n$ is a G-map. We say that $X_\bullet$ is Reedy cofibrant if this map is a G-cofibration for each n.

With different nomenclature, this concept was studied nonequivariantly in the early 1970's (e.g. [27, §11], [28, Appendix], [51, Appendix A]). We will use the following standard results.

Lemma 1.9. *A simplicial G-space $X_\bullet$ is Reedy cofibrant if all degeneracy operators s_i are G-cofibrations.*

Proof. The nonequivariant statement is proven by an inductive application of Lillig's union theorem stating that the union of cofibrations is a cofibration [23] (or [27, Lemma A.6]). The proof can be found in [51, proof of A.5] or [22, proof of 2.4.(b)]. The equivariant proof is the same, using the equivariant version of Lillig's theorem, which is a particular case of [7, Theorem A.2.7]. □

The converse to Lemma 1.9 is proved in [45, Proposition 4.11], but we shall not use it.

Theorem 1.10. *Let $f_\bullet\colon X_\bullet \longrightarrow Y_\bullet$ be a map of Reedy cofibrant simplicial G-spaces such that each f_n is a weak G-equivalence. Then the realization $|f_\bullet|\colon |X_\bullet| \longrightarrow |Y_\bullet|$ is a weak G-equivalence.*

Proof. Nonequivariantly this is in [28, Theorem A.4], and the equivariant version follows by application of the nonequivariant case to fixed point spaces, noting that geometric realization commutes with taking fixed points. □

The following result is well-known, but since we could not find a proof in the published literature, we provide one in §10.[6]

Theorem 1.11. *Let $f_\bullet\colon X_\bullet \longrightarrow Y_\bullet$ be a map of Reedy cofibrant simplicial G-spaces such that each f_n is a G-cofibration. Then the realization $|f_\bullet|\colon |X_\bullet| \longrightarrow |Y_\bullet|$ is a G-cofibration.*

Remark 1.12. Without exception, every simplicial G-space used in this paper is Reedy cofibrant. In each case, we can check from the definitions and the fact that we are working with nondegenerately based G-spaces that all s_i are G-cofibrations. For the examples appearing in the Segal machine, the verifications are straightforward. For the examples appearing in the operadic machine, the verifications follow those in [27, Proposition A.10] and elaborations of the arguments there.

[6]We were inspired by [45], an unpublished masters thesis, which gives a detailed exposition of simplicial spaces. However, its statement of Theorem 1.11 is missing a necessary hypothesis.

1.3. **Categorical preliminaries and basepoints.** Some familiarity with enriched category theory, especially in equivariant contexts, may be helpful. A more thorough treatment of the double enrichment present here is given in [17]. We need some general definitions that start with a closed symmetric monoidal category $\mathscr{V}$ with unit object U and product denoted by $\otimes$. Closed means that we have internal function objects $\underline{\mathscr{V}}(V,W)$ in $\mathscr{V}$ giving an adjunction

$$\underline{\mathscr{V}}(X\otimes Y,Z)\cong\underline{\mathscr{V}}(X,\underline{\mathscr{V}}(Y,Z))$$

in $\mathscr{V}$. We assume that $\mathscr{V}$ is complete and cocomplete.

A $\mathscr{V}$-category $\mathscr{E}$ is a category enriched in $\mathscr{V}$. This means that for each pair (m,n) of objects of $\mathscr{E}$ there is an object $\mathscr{E}(m,n)$ of $\mathscr{V}$ and there are unit and composition maps $I\colon U\longrightarrow\mathscr{E}(m,m)$ and $C\colon\mathscr{E}(n,p)\otimes\mathscr{E}(m,n)\longrightarrow\mathscr{E}(m,p)$ satisfying the identity and associativity axioms. It would be more categorically precise to write $\underline{\mathscr{E}}(m,n)$, saving $\mathscr{E}(m,n)$ for the underlying set of morphisms $m\longrightarrow n$. A $\mathscr{V}$-functor $F\colon\mathscr{E}\longrightarrow\mathscr{Q}$ between $\mathscr{V}$-categories is a functor enriched in $\mathscr{V}$. This means that for each pair (m,n), there is a map

$$F\colon\mathscr{E}(m,n)\longrightarrow\mathscr{Q}(F(m),F(n))$$

in $\mathscr{V}$, and these maps are compatible with the unit and composition of $\mathscr{E}$ and $\mathscr{Q}$. A $\mathscr{V}$-transformation $\eta\colon F\longrightarrow F'$ is given by maps $\eta_m\colon U\longrightarrow\mathscr{Q}(F(m),F'(m))$ in $\mathscr{V}$ such that the evident naturality diagram commutes in $\mathscr{V}$.

$$\begin{array}{ccc}
\mathscr{E}(m,n) & \xrightarrow{\;F\;} & \mathscr{Q}(F(m),F(n)) \\
{\scriptstyle F'}\big\downarrow & & \big\downarrow{\scriptstyle (\eta_n)_*} \\
\mathscr{Q}(F'(m),F'(n)) & \xrightarrow[(\eta_m)^*]{} & \mathscr{Q}(F(m),F'(n))
\end{array}$$

We are especially interested in the cases $\mathscr{V}=\mathscr{T}$ and $\mathscr{V}=G\mathscr{T}$. In this paper, topological G-categories are understood to mean categories enriched in $G\mathscr{T}$.[7] When we enrich in spaces, the hom set is obtained from the hom space just by forgetting the topology, and we omit the underline. Thus when thinking of enrichment in $\mathscr{T}$, we write $\mathscr{T}(X,Y)$ for the based space of based maps $X\longrightarrow Y$ when there is no G-action in sight and we write $G\mathscr{T}(X,Y)$ for the based space of based G-maps $X\longrightarrow Y$ when X and Y are G-spaces.

We previously defined $\mathscr{T}_G$ to be the category of based G-spaces and nonequivariant maps, with G acting by conjugation on the hom spaces $\mathscr{T}_G(X,Y)$. From the point of view of enriched category theory, $\mathscr{T}_G$ just gives another name for the hom objects that give the enrichment. That is,

$$\underline{G\mathscr{T}}(X,Y)=\mathscr{T}_G(X,Y),$$

and similarly for $\mathscr{U}$. We speak in general of $G\mathscr{T}$-categories, but we distinguish notationally by writing $G\mathscr{T}$-functors as $X\colon\mathscr{E}\longrightarrow G\mathscr{T}$ when G acts trivially on $\mathscr{E}$ and $X\colon\mathscr{E}\longrightarrow\mathscr{T}_G$ in general. In the first case, we are enriching just in $\mathscr{T}$, using spaces of G-maps, but this is a G-trivial special case of equivariant enrichment.

There is considerable confusion in the literature concerning the handling of basepoints. The category $\mathscr{T}$ has two symmetric monoidal products, $\wedge$ with unit S^0 and $\times$ with unit $*$. When enriching in $\mathscr{T}$, we must use $\wedge$ since we must use the closed structure given by the spaces $F(X,Y)$ of based maps, with basepoint given by

[7] In the sequels [15, 16], we work more generally with categories internal to $G\mathscr{U}$.

$X \to * \to Y$. However, we sometimes use the forgetful functors $\mathscr{T} \longrightarrow \mathscr{U}$ and $G\mathscr{T} \longrightarrow G\mathscr{U}$ to forget basepoints in our enrichments, and implicitly we are then thinking about $\times$.

In all variants and generalizations of the Segal machine, we start with a category $\mathscr{E}$ enriched over $\mathscr{T}$ or $G\mathscr{T}$. It has a zero object 0, so that there are unique maps $0 \to n$ and $n \to 0$ for every object $n \in \mathscr{E}$. Then $\mathscr{E}(n,0)$ and $\mathscr{E}(0,n)$ are each a point and the map $m \to 0 \to n$ is the basepoint of $\mathscr{E}(m,n)$, which must be nondegenerate to have the cited enrichment. We are concerned with $G\mathscr{T}$-functors defined on $\mathscr{E}$; by neglect of G-action, they are also $\mathscr{T}$-functors. The following trivial observation has been overlooked since the start of this subject.

Lemma 1.13. *For any $\mathscr{T}$-functor $X\colon \mathscr{E} \longrightarrow \mathscr{T}$, $X(0) = *$.*

Proof. The unique map $0 \to 0$ in $\mathscr{E}(0,0)$ is both the basepoint and the identity; $X\colon \mathscr{E}(0,0) \longrightarrow \mathscr{T}(X(0),X(0))$ must send it to both the trivial map $X(0) \longrightarrow X(0)$ that sends all points to the basepoint and the identity map. This can only happen if $X(0)$ is a point. □

Remark 1.14. For a $G\mathscr{U}$-category $\mathscr{J}$ and a $G\mathscr{T}$-category $\mathscr{Q}$, we can add disjoint basepoints to the hom objects of $\mathscr{J}$ to form a $G\mathscr{T}$-category $\mathscr{J}_+$ or we can forget basepoints to regard $\mathscr{Q}$ as a $G\mathscr{U}$-category $\mathbb{U}\mathscr{Q}$. Via the adjunction betwen $(-)_+$ and $\mathbb{U}$, $G\mathscr{U}$-functors $\mathscr{J} \longrightarrow \mathbb{U}\mathscr{Q}$ can be identified with $G\mathscr{T}$-functors $\mathscr{J}_+ \longrightarrow \mathscr{Q}$.

Remark 1.15. Let $\mathscr{E}$ be a $G\mathscr{T}$-category with a zero object. A $G\mathscr{U}$-functor $X\colon \mathbb{U}\mathscr{E} \longrightarrow \mathscr{U}_G$ is said to be *reduced* if $X(0)$ is a point. If X is reduced the map $X(0) \longrightarrow X(n)$ induced by $0 \longrightarrow n$ gives each $X(n)$ a basepoint, and composition preserves basepoints. Moreover, if these basepoints are nondegenerate, X will give a $\mathscr{T}$-functor $\mathscr{E} \longrightarrow \mathscr{T}_G$, as X then sends the zero map $m \to 0 \to n$ to the trivial map $X(m) \to * = X(0) \to X(n)$.

Let $\mathscr{E}$ be a $G\mathscr{T}$-category with a zero object 0, and let X and Y be respectively covariant and contravariant $G\mathscr{T}$-functors $\mathscr{E} \longrightarrow \mathscr{T}_G$. Then the tensor product of functors $Y \otimes_{\mathscr{E}} X$ is defined as the coequalizer of the diagram

$$\bigvee_{m,n} Y_n \wedge \mathscr{E}(m,n) \wedge X_m \rightrightarrows \bigvee_n Y_n \wedge X_n,$$

where the arrows are given by the action of $\mathscr{E}$ on X and on Y, respectively.

We obtain $G\mathscr{U}$-functors $\mathbb{U}X$, $\mathbb{U}Y\colon \mathbb{U}\mathscr{E} \longrightarrow \mathscr{U}_G$ by forgetting basepoints. The tensor product of functors $\mathbb{U}Y \otimes_{\mathbb{U}\mathscr{E}} \mathbb{U}X$ is the coequalizer of the diagram

$$\coprod_{m,n} Y_n \times \mathscr{E}(m,n) \times X_m \rightrightarrows \coprod_n Y_n \times X_n.$$

The following result shows that the difference between these two constructions is only apparent. Later on, we shall sometimes use wedges and smash products and sometimes instead use disjoint unions and products, whichever seems convenient.

Lemma 1.16. *With $\mathscr{E}$, X, and Y as above, there is a natural isomorphism*

$$Y \otimes_{\mathscr{E}} X \cong \mathbb{U}Y \otimes_{\mathbb{U}\mathscr{E}} \mathbb{U}X.$$

Proof. We will show that the quotient of $\coprod_n Y_n \times X_n$ given by the coequalizer encodes the required basepoint identifications. Let $(y, *_n) \in Y_n \times X_n$. As noted above, the basepoint in X_n is given by $*_n = (0_{0,n})_*(*_0)$, where $*_0$ is the unique point in X_0 and $0_{0,n}$ is the unique map $0 \longrightarrow n$ in $\mathscr{E}$. Then

$$(y, *_n) = (y, (0_{0,n})_*(*_0)) \sim (0^*_{0,n}(y), *_0) = (*_0, *_0),$$

the last equation following from the fact that Y_0 is a singleton. A similar argument shows that $(*_n, x) \sim (*_0, *_0)$. □

1.4. **Spectrum level preliminaries.**

Definition 1.17. A naive G-spectrum, which we prefer to call a G-prespectrum, is a sequence of based G-spaces $\{T_n\}_{n\geq 0}$ and based G-maps $\sigma_n \colon \Sigma T_n \longrightarrow T_{n+1}$. It is a naive Ω-G-spectrum if the adjoint maps $\tilde{\sigma}_n \colon T_n \longrightarrow \Omega T_{n+1}$ are weak G-equivalences. It is a positive Ω-G-spectrum if the $\tilde{\sigma}_n$ are weak G-equivalences for $n \geq 1$. We let $G\mathscr{P}$ denote the category of G-prespectra. We call zeroth spaces T_0 of naive Ω-G-spectra naive infinite loop G-spaces.

When we restrict to compact Lie groups, our preferred category of (genuine) G-spectra will be the category $G\mathscr{S}$ of orthogonal G-spectra. Orthogonal G-spectra and their model structures are studied in [24], to which we refer the reader for details and discussion of the following definition. We use Remark 1.14.

Definition 1.18. Let G be a compact Lie group and let $\mathscr{I}_G$ be the $G\mathscr{U}$-category of finite dimensional real G-inner product spaces and linear isometric isomorphisms, with G acting on morphism spaces by conjugation. Note that $\mathscr{I}_G$ is symmetric monoidal under $\oplus$. An $\mathscr{I}_G$-G-space is a $G\mathscr{U}$-functor $\mathscr{I}_G \longrightarrow \mathscr{T}_G$ or, equivalently by Remark 1.14, a $G\mathscr{T}$-functor $\mathscr{I}_{G+} \longrightarrow \mathscr{T}_G$. The sphere $\mathscr{I}_G$-G-space S is given by $S(V) = S^V$. The external smash product

$$X \barwedge Y \colon \mathscr{I}_G \times \mathscr{I}_G \longrightarrow \mathscr{T}_G$$

of $\mathscr{I}_G$-G-spaces X and Y is the $G\mathscr{U}$-functor given by

$$(X \barwedge Y)(V, W) = X(V) \wedge Y(W).$$

A (genuine orthogonal) G-spectrum is an $\mathscr{I}_G$-G-space $E \colon \mathscr{I}_G \longrightarrow \mathscr{T}_G$ together with a $G\mathscr{U}$-transformation $E \barwedge S \longrightarrow E \circ \oplus$ between $G\mathscr{U}$-functors $\mathscr{I}_G \times \mathscr{I}_G \longrightarrow \mathscr{T}_G$. Thus we have G-spaces $E(V)$, morphism G-maps

$$\mathscr{I}_G(V, V') \longrightarrow \mathscr{U}_G(E(V), E(V'))$$

and structure G-maps

$$\sigma \colon E(V) \wedge S^W \longrightarrow E(V \oplus W).$$

natural in V and W. Note in particular that $\mathscr{I}_G(V, V)$ is the orthogonal group $O(V)$, with G acting by conjugation, so that $E(V)$ is both a G-space and an $O(V)$-space and σ is a map of both G-spaces and $O(V) \times O(W)$-spaces. A G-spectrum E is an Ω-G-spectrum if the adjoint maps

$$\tilde{\sigma} \colon E(V) \longrightarrow \Omega^W E(V \oplus W)$$

are weak G-equivalences. It is a positive Ω-G-spectrum if these maps are weak G-equivalences when $V^G \neq 0$. We let $G\mathscr{S}$ denote the category of G-spectra. We call zeroth spaces $E(0)$ of Ω-G-spectra genuine infinite loop G-spaces, or simply infinite loop G-spaces.

The Ω-G-spectra are the fibrant objects in the stable model structure on $G\mathscr{S}$ and the positive Ω-G spectra are the fibrant objects in the positive stable model structure. The identity functor is a left Quillen equivalence from the stable model structure to the positive stable model structure. We have the following change of universe functor, which is a right Quillen adjoint.

Definition 1.19. The forgetful functor $i^*\colon G\mathscr{S} \longrightarrow G\mathscr{P}$ sends a G-spectrum X to the (naive) G-prespectrum with nth space $X_n = X(\mathbb{R}^n)$.

Remark 1.20. If $V^G \neq 0$, we can write $V = \mathbb{R} \oplus W$ and thus $S^V = S^1 \wedge S^W$ and $\Omega^V = \Omega\Omega^W$. Then $\tilde{\sigma}\colon X_0 \longrightarrow \Omega^V X(V)$ factors as the composite

$$X_0 \xrightarrow{\tilde{\sigma}} \Omega X_1 \xrightarrow{\Omega\tilde{\sigma}} \Omega\Omega^W X(\mathbb{R} \oplus W) = \Omega^V X(V).$$

If X is a positive Ω-G-spectrum, then the second arrow is a weak G-equivalence. Therefore, if $X_0 \longrightarrow \Omega X_1$ is a group completion, then so is $X_0 \longrightarrow \Omega^V X(V)$ for all V such that $V^G \neq 0$.

1.5. **Equivariant infinite loop space machines.** In this section, we give a very quick overview of equivariant infinite loop space machines. The details are worked out in the next few sections.

Nonequivariantly, there are several recognition principles that one can apply to spaces to determine whether they become infinite loop spaces after group completion. One is the operadic approach developed by the first author in [27] and another is the approach using Γ-spaces developed by Segal in [51]. The opposite category of Segal's Γ is the category $\mathscr{F}$ of finite sets, and we shall call Γ-spaces $\mathscr{F}$-spaces. Infinite loop space machines take some appropriate input Y, part of which is an underlying Hopf space X, and construct from it an Ω-spectrum $\mathbb{E}Y$ together with a group completion $X \longrightarrow \mathbb{E}_0 Y$. This form was taken as the definition of an infinite loop space machine in [38]. If G acts on the input data Y through maps that are compatible with the structure, then both the operadic machine and the Segal machine generalize immediately to give infinite loop space machines landing in naive Ω-G-spectra.

The *genuine* equivariant theory is much harder since genuine infinite loop G-spaces have deloopings not only with respect to all spheres S^n, but also with respect to all representation spheres S^V for all finite dimensional G-representations V. For finite G, the genuine equivariant generalization of the operadic approach was first worked out in [10] and is worked out more fully in [12]. The genuine equivariant version of Segal's approach was first worked out in [54] and is worked out more fully here. Both machines are generalized here to forms which accept the same input, and then they are proven to give equivalent output when fed the same input.

When G is a finite group, genuine equivariant infinite loop space machines $\mathbb{E}_G$ take appropriate input Y with underlying Hopf G-spaces X to genuine Ω-G-spectra $\mathbb{E}Y$. They restrict to give underlying naive G-spectra, and the group completion $X \longrightarrow \mathbb{E}_0 Y$ is seen by the underlying naive G-spectrum. As we shall see, the machines as they appear most naturally do not take precisely this form, and we then have to tweak them into the form just specified. There are general features common to any equivariant infinite loop space machine, and we refer the reader to [12, §2.3] for a discussion. For example, the group completion property directly implies that any machine commutes with products and with passage to fixed points.

2. The simplicial and conceptual versions of the Segal machine

There are several variants of the Segal infinite loop space machine, as originally developed by Segal [51] and Woolfson [59]. Later sources include Bousfield and Friedlander [8], working simplicially, and, much later, Mandell, May, Schwede, and Shipley [25]. Equivariant versions appear in Shimada and Shimakawa [53, 54, 55] and, later, Blumberg [5].

We here give a simplicial variant[8] and two equivalent conceptual variants, one starting from finite sets and the other starting from finite G-sets. Of course, the use of finite G-sets is mainly of interest when G is finite, but it applies in general. We defer consideration of our preferred homotopical variant to the next section.

The simplicial variant is the equivariant version of Segal's original definition [51]. As far as we know, his paper is the only source in the literature that actually proves the crucial group completion property, and his proof makes essential use of his original simplicial definition.[9] This version does not directly generalize to give genuine G-spectra when G is a compact Lie group or even a finite group, and it does not appear in the equivariant literature. Therefore, even at this late date, there is no published account of the equivariant Segal machine that proves the group completion property. Just as nonequivariantly, this property is central to the applications, especially to algebraic K-theory.

In fact, we do not know a direct proof of the group completion property starting from the conceptual or homotopical variants treated in [8, 25, 53, 59] and, equivariantly, [5, 54]. Rather, we derive it for the conceptual variants from their equivalence with the simplicial variant. To give the group completion property equivariantly, to differentiate the theory for varying types of groups, and to prepare for a comparison with the operadic machine, we give a fully detailed exposition of the Segal machine in all its forms. This may also be helpful to the modern reader since even nonequivariantly the original sources make for hard reading and are sketchy in some essential respects.

2.1. Definitions: the input of the Segal machine.

Definition 2.1. Let $\mathscr{F}$ be the opposite of Segal's category Γ.[10] It is the category of finite based sets $\mathbf{n} = \{0, 1, \dots, n\}$ with 0 as basepoint. The morphisms are the based maps, and the unique morphism that factors through $\mathbf{0}$ is a nondegenerate basepoint for $\mathscr{F}(\mathbf{m}, \mathbf{n})$. Let $\Pi \subset \mathscr{F}$ be the subcategory with the same objects and those morphisms $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $\phi^{-1}(j)$ has at most one element for $1 \leq j \leq n$; these are composites of projections, injections, and permutations. Let $\Sigma \subset \Pi$ be the subgroupoid with the same objects and the elements of the symmetric groups Σ_n, regarded as based isomorphisms $\mathbf{n} \longrightarrow \mathbf{n}$, as morphisms.

Since the composition in $\mathscr{F}$ factors through the smash product, we can view $\mathscr{F}$ as a category enriched in $\mathscr{T}$, with the discrete topology on the based hom sets.

Definition 2.2. An *$\mathscr{F}$-space* is a $\mathscr{T}$-functor $X\colon \mathscr{F} \longrightarrow \mathscr{T}$, written $\mathbf{n} \mapsto X_n$; Π-spaces are defined similarly. A map of $\mathscr{F}$-spaces or of Π-spaces is a $\mathscr{T}$-natural transformation.

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[8] We are referring to simplicial spaces, not simplicial sets, here.

[9] His proof imposes some unnecessary restrictive hypotheses that generally hold in practice.

[10] As in [25] and elsewhere, we use the notation $\mathscr{F}$ to avoid confusion between Γ and $\Gamma^{op} = \mathscr{F}$.

Definition 2.3. The *Segal maps* $\delta_i\colon \mathbf{n} \longrightarrow \mathbf{1}$ in $\mathscr{F}$ send i to 1 and j to 0 for $j \neq i$. These maps are all in Π. For an $\mathscr{F}$-space X, the *Segal map* $\delta\colon X_n \longrightarrow X_1^n$ has coordinates induced by the δ_i. If $n = 0$, we interpret δ as the terminal map $X_0 \longrightarrow *$. The "multiplication map" $\phi_n\colon \mathbf{n} \longrightarrow \mathbf{1}$, which is not in Π, sends j to 1 for $1 \leq j \leq n$. It induces an "n-fold multiplication" $X_n \longrightarrow X_1$ on an $\mathscr{F}$-space X.

Definition 2.4. An *$\mathscr{F}$-G-space* is a $\mathscr{T}$-functor $X\colon \mathscr{F} \longrightarrow G\mathscr{T}$; Π-G-spaces are defined similarly. A map of $\mathscr{F}$-G-spaces or Π-G-spaces is a $G\mathscr{T}$-natural transformation. Since $\Sigma \subset \Pi \subset \mathscr{F}$, X_n and X_1^n are $(G \times \Sigma_n)$-spaces and $\delta\colon X_n \longrightarrow X_1^n$ is a map of $G \times \Sigma_n$-spaces.

Remark 2.5. It is usual, starting in [51, Definition 1.2], to define an $\mathscr{F}$-space to be a (non-enriched) functor $\mathscr{F} \longrightarrow \mathscr{T}$, requiring X_0 to be contractible, and to say that X is reduced if X_0 is a point. This led to mistakes and confusion, as explained in [35]. As we observed in Lemma 1.13, our requirement that X be a $\mathscr{T}$-functor forces X_0 to be a point for trivial reasons; compare Remark 1.15.

Recall from Definition 1.2 that for a homomorphism $\alpha\colon G \longrightarrow \Sigma_n$, Λ_α is the subgroup $\{(g, \alpha(g))|g \in G\}$ of $G \times \Sigma_n$.

Definition 2.6. Let X, Y be Π-G-spaces, and $f\colon X \longrightarrow Y$ be a map of Π-G-spaces.

(i) The Π-space X is $\mathbb{F}_\bullet$-*special* if $\delta\colon X_n \longrightarrow X_1^n$ is a weak Λ_α-equivalence for all $n \geq 0$ and all homomorphisms $\alpha\colon G \longrightarrow \Sigma_n$ (where $\Sigma_0 = \{e\} = \Sigma_1$).
(ii) The Π-space X is *special* if each $\delta\colon X_n \longrightarrow X_1^n$ is a weak G-equivalence.
(iii) The map f is an $\mathbb{F}_\bullet$-*level equivalence* if each $f_n\colon X_n \longrightarrow Y_n$ is a weak Λ_α-equivalence for all homomorphisms $\alpha\colon G \longrightarrow \Sigma_n$.
(iv) The map f is a *level G-equivalence* if each $f_n\colon X_n \longrightarrow Y_n$ is a weak G-equivalence.

An $\mathscr{F}$-G-space X is $\mathbb{F}_\bullet$-special or special if its underlying Π-G-space is so. A special $\mathscr{F}$-G-space X is *grouplike* or, synonymously, *very special* if $\pi_0(X_1^H)$ is a group (necessarily abelian) under the induced product for each $H \subset G$. A map $f\colon X \longrightarrow Y$ of $\mathscr{F}$-G-spaces is an $\mathbb{F}_\bullet$-level equivalence or level equivalence if it is so as a map of Π-G-spaces.

When there are no non-trivial homomorphisms $G \longrightarrow \Sigma_n$, for example when G is connected, a Π-G-space is $\mathbb{F}_\bullet$-special if and only if it is special. In fact, the notion of an $\mathbb{F}_\bullet$-special Π-G-space is only of substantial interest when G is finite. However, §2.4 will give motivation that applies in general.

We need several technical results about these notions, the first of which is the key to the following two.

Lemma 2.7. *Let X be a G-space, and let $\Lambda = \{(h, \alpha(h)) \mid h \in H\} \subset G \times \Sigma_n$, where $H \subset G$ and $\alpha\colon H \longrightarrow \Sigma_n$ is a homomorphism. Then there is a natural homeomorphism*

$$(X^n)^\Lambda \cong \prod X^{K_i},$$

where the product is taken over the orbits of the H-set $(\mathbf{n}, \alpha)$ and the $K_i \subset H$ are the stabilizers of chosen elements in the corresponding orbit.

Proof. The Λ-action on X^n is given by

$$(h, \alpha(h))(x_1, \ldots, x_n) = (hx_{\alpha(h^{-1})(1)}, \ldots, hx_{\alpha(h^{-1})(n)}).$$

The partition of $\mathbf{n}$ into H-orbits decomposes $\mathbf{n}$ as the wedge of finite subsets, each with a transitive set of shuffled indices, so it is enough to consider each H-orbit separately. Thus we may as well assume that the H-action on $(\mathbf{n}, \alpha)$ is transitive. Note that this reduction is natural in X.

Let $K \subset H$ be the stabilizer of $1 \in \mathbf{n}$. We claim that projection onto the first coordinate induces the required natural homeomorphism $\pi\colon (X^n)^\Lambda \longrightarrow X^K$. If $(x_1, \dots, x_n) \in X^n$ is a Λ-fixed point, then x_1 is a K-fixed point since

$$kx_1 = kx_{\alpha(k^{-1})(1)} = x_1$$

for $k \in K$, the second equality holding because $(x_1, \dots, x_n)$ is fixed by Λ.

To construct $\pi^{-1}\colon X^K \longrightarrow (X^n)^\Lambda$, for $1 \le j \le n$ choose $h_j \in H$ such that $\alpha(h_j)(1) = j$. This choice amounts to choosing a system of coset representatives for H/K, and the map $j \mapsto [h_j]$ gives a bijection of H-sets between $(\mathbf{n}, \alpha)$ and H/K. We claim that the map $: X \longrightarrow X^n$ that sends x to the n-tuple $(h_1x, \dots, h_nx)$ restricts to the required inverse π^{-1}. This map is clearly continuous. We first show that if $x \in X^K$, then $(h_1x, \dots, h_nx)$ is fixed by Λ. Let $h \in H$ and note that

$$\alpha(hh_{\alpha(h^{-1})(j)})(1) = \alpha(h)(\alpha(h_{\alpha(h^{-1})(j)})(1)) = \alpha(h)(\alpha(h^{-1})(j)) = j.$$

In view of our bijection between $(\mathbf{n}, \alpha)$ and H/K, there exists $k \in K$ such that $hh_{\alpha(h^{-1})(j)} = h_jk$. The jth coordinate of $(h, \alpha(h)) \cdot (h_1x, \dots, h_nx)$ is given by

$$hh_{\alpha(h^{-1})(j)}x = h_jkx = h_jx,$$

the second equality holding because $x \in X^K$. Thus $(h_1x, \dots, h_nx)$ is fixed by Λ.

Since K is the stabilizer of 1, $h_1 \in K$. Thus the first coordinate of $(h_1x, \dots, h_nx)$ is x itself, and $\pi \circ \pi^{-1} = \mathrm{id}$. If $(x_1, \dots, x_n)$ is fixed by Λ, then $x_j = h_jx_1$ for all j. By the definition of h_j, the jth coordinate of $(h_j, \alpha(h_j)) \cdot (x_1, \dots, x_n)$ is $h_jx_{\alpha(h_j^{-1})(j)} = h_jx_1$. Since$(x_1, \dots, x_n)$ is a Λ-fixed point, this shows that $x_j = h_jx_1$, hence $\pi^{-1} \circ \pi = \mathrm{id}$. □

Definition 2.8. For a based G-space X, let $\mathbb{R}X$ denote the Π-G-space with nth G-space X^n. The Π-space structure is given by basepoint inclusions, projections, and permutations.

Lemma 2.9. *If $f\colon X \longrightarrow Y$ is a weak equivalence of based G-spaces, then the induced map $\mathbb{R}f\colon \mathbb{R}X \longrightarrow \mathbb{R}Y$ is an $\mathbb{F}_\bullet$-level equivalence of Π-G-spaces.*

Proof. This is immediate from Lemma 2.7. □

Lemma 2.10. *Let $f\colon X \longrightarrow Y$ be an $\mathbb{F}_\bullet$-level equivalence of Π-G-spaces. Then X is $\mathbb{F}_\bullet$-special if and only if Y is $\mathbb{F}_\bullet$-special. Similarly, if f is a level G-equivalence, then X is special if and only if Y is special.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ X_1^n & \xrightarrow[f_1^n]{} & Y_1^n. \end{array}$$

For the first statement, the horizontal arrows are weak Λ_α-equivalences by assumption and Lemma 2.9, so one of the vertical arrows is a weak Λ_α-equivalence if and only if the other one is. The proof of the second statement is similar but simpler. □

We defined $\mathbb{F}_\bullet$-special Π-G-spaces in terms of those subgroups Λ in the family $\mathbb{F}_n$ (see Definition 1.2) which are defined by homomorphisms $\alpha\colon G \longrightarrow \Sigma_n$, ignoring those which are defined by homomorphisms $\beta\colon H \longrightarrow \Sigma_n$ for proper subgroups H of G. The following result shows that when G is finite we obtain the same notion if we instead use all of the groups in $\mathbb{F}_n$. The result implicitly relates $\mathbb{F}_\bullet$-special Π-G-spaces to equivariant covering space theory and relates $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-spaces to the operadic approach to equivariant infinite loop space theory. It therefore explains and justifies the terms $\mathbb{F}_\bullet$-special and $\mathbb{F}_\bullet$-level equivalence.

Lemma 2.11. *Assume that G is finite. Then a Π-G-space X is $\mathbb{F}_\bullet$-special if and only if the Segal maps $\delta\colon X_n \longrightarrow X_1^n$ are weak $\mathbb{F}_n$-equivalences for all $n \geq 0$. Similarly, a map $f\colon X \longrightarrow Y$ of Π-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only if f_n is a weak $\mathbb{F}_n$-equivalence for all $n \geq 0$.*

Proof. If $\delta\colon X_n \longrightarrow X_1^n$ is a weak $\mathbb{F}_n$-equivalence, then it is a weak Λ_β-equivalence for all $H \subset G$ and all homomorphisms $\beta\colon H \longrightarrow \Sigma_n$. Restricting to those homomorphisms with domain G, this condition for all n implies that X is $\mathbb{F}_\bullet$-special. Conversely, assume that X is $\mathbb{F}_\bullet$-special. We must prove that each δ is a weak $\mathbb{F}_n$-equivalence.

Thus consider a subgroup $\Lambda_\beta \subset G \times \Sigma_n$, $\beta\colon H \longrightarrow \Sigma_n$. We will show that $\delta\colon X_n \longrightarrow X_1^n$ is a weak Λ_β-equivalence by displaying it as a retract of a suitable weak equivalence. The homomorphism β gives rise to an H-set $B = (\mathbf{n}, \beta)$. Embed B as a subset of the G-set $A = G_+ \wedge_H B$ and observe that, as an H-set, A splits as $B \vee C$, where $C = (A \setminus B)_+$. Let $p = |A \setminus B|$ and $q = n + p$. Use the given ordering of B and an ordering of C to identify A with $(\mathbf{q}, \alpha)$. Here $\mathbf{q} = \mathbf{n} \vee \mathbf{p}$ and α is a homomorphism $G \longrightarrow \Sigma_q$ which when restricted to H is of the form $\beta \vee \gamma$. That is, $(\mathbf{q}, \alpha|_H) = (\mathbf{n}, \beta) \vee (\mathbf{p}, \gamma|_H)$. Let

$$\iota\colon (\mathbf{n}, \beta) \longrightarrow (\mathbf{q}, \alpha|_H) \quad \text{and} \quad \pi\colon (\mathbf{q}, \alpha|_H) \longrightarrow (\mathbf{n}, \beta)$$

be the inclusion that sends i to i for $0 \leq i \leq n$ and the projection that sends i to i for $0 \leq i \leq n$ and i to 0 for $i > n$. Then the following diagram displays a retraction. Its bottom arrows are the evident inclusion and projection.

$$\begin{array}{ccccc} X_n & \xrightarrow{\iota_*} & X_q & \xrightarrow{\pi_*} & X_n \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ X_1^n & \longrightarrow & X_1^q & \longrightarrow & X_1^n \end{array}$$

Since X is $\mathbb{F}_\bullet$-special, the middle vertical arrow δ is a weak Λ_α-equivalence and thus a weak $\Lambda_{\alpha|H}$-equivalence. Therefore the left arrow δ is a weak Λ_β-equivalence.

Similarly, if $f\colon X \longrightarrow Y$ is an $\mathbb{F}_\bullet$-level equivalence, we have a retract diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\iota_*} & X_q & \xrightarrow{\pi_*} & X_n \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_n} \\ Y_n & \xrightarrow[\iota_*]{} & Y_q & \xrightarrow[\pi_*]{} & Y_n \end{array}$$

in which f_q is a weak Λ_α-equivalence and therefore f_n is a weak Λ_β-equivalence. $\square$

2.2. **The simplicial version of the Segal machine.** Let Δ be the usual simplicial category. A simplicial object is a contravariant functor defined on Δ, and a cosimplicial object is a covariant functor. Regarding $\mathscr{F}$ as a full subcategory of the category of based sets, we may regard the simplicial circle $S^1_s = \Delta[1]/\partial\Delta[1]$ as a contravariant functor $F\colon \Delta^{op} \longrightarrow \mathscr{F}$. By pullback along F, an $\mathscr{F}$-G-space X can be viewed as a simplicial G-space, and it has a geometric realization $|X| = |X \circ F|$; we use the standard realization, taking degeneracies into account. The evident G-map $X_1 \times I \longrightarrow |X|$ factors through a natural G-map $\Sigma X_1 \longrightarrow |X|$ with adjoint $\eta\colon X_1 \longrightarrow \Omega|X_1|$. We shall give the proof of the following result in §9.2. It is implicit in many early sources; we will follow [40, §15].

Proposition 2.12. *If X is a special $\mathscr{F}$-G-space, then the G-map $\eta\colon X_1 \longrightarrow \Omega|X|$ is a group completion of Hopf G-spaces.*

From here, the Segal machine in its first avatar is constructed as follows [51, §1]. Working equivariantly, we use a slight reformulation that is given in [38].[11]

Remark 2.13. We have the smash product $\wedge\colon \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}$. It sends $(\mathbf{m}, \mathbf{n})$ to $\mathbf{mn}$ and is strictly associative and unital using lexicographic ordering. The unit is $\mathbf{1}$. We also have the wedge sum $\vee\colon \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}$ which sends $(\mathbf{m}, \mathbf{n})$ to $\mathbf{m} + \mathbf{n}$. It is also strictly associative and unital, with unit $\mathbf{0}$, and $\mathscr{F}$ is bipermutative under this sum and product.

Definition 2.14. Let X be a special $\mathscr{F}$-G-space. We have the functor

$$X \circ \wedge\colon \mathscr{F} \times \mathscr{F} \longrightarrow G\mathscr{T}.$$

For each $\mathbf{q}$, let $X[q]$ be the $\mathscr{F}$-G-space that sends $\mathbf{p}$ to $X(\mathbf{p} \wedge \mathbf{q})$; thus $X[0] = *$ and $X[1] = X$. Following Segal, define the *classifying $\mathscr{F}$-G-space* $\mathbb{B}X$ to be the $\mathscr{F}$-G-space whose qth G-space is the realization $|X[q]|$. Iterating, with $\mathbb{B}^0X = X$, define $\mathbb{B}^{n+1}X = \mathbb{B}(\mathbb{B}^nX)$ for $n \geq 0$. The $\mathscr{F}$-G-spaces $\mathbb{B}^nX$ for $n \geq 1$ are again special; since $(\mathbb{B}^nX)_1$ is G-connected, they are also grouplike.

Notation 2.15. Let $\mathbb{S}^N_G X$ denote the resulting naive G-prespectrum with nth space $(\mathbb{S}^N_G X)_n = (\mathbb{B}^nX)_1$ for $n \geq 0$. Thus its 0th G-space is X_1 and, by Proposition 2.12, its structure map $X_1 \longrightarrow \Omega(\mathbb{S}^N_G X)_1$ is a group completion and the structure maps $(\mathbb{S}^N_G X)_n \longrightarrow \Omega(\mathbb{S}^N_G X)_{n+1}$ for $n \geq 1$ are weak G-equivalences.

With this definition, $\mathbb{S}^N_G X$ is a positive Ω-G-spectrum. Varying the definition by taking the 0th G-space to be $\Omega(\mathbb{B}X)_1$, the nonequivariant Segal machine plays a special role. As proven in [38], any infinite loop space machine that takes $\mathscr{F}$-spaces, or appropriate more general input, to Ω-spectra and has a natural group completion map from X_1 to its zeroth space is equivalent to the Segal machine. The proof makes essential use of the fact that the Segal machine produces $\mathscr{F}\mathscr{F}$-spaces, namely functors $\mathscr{F} \longrightarrow \mathscr{F}\mathscr{T}$.

We emphasize that this construction of the Segal machine works for any topological group G. Moreover, the uniqueness proof for infinite loop space machines in [38] works verbatim to compare any other infinite loop space machine landing in naive G-spectra to the Segal machine. However, even when G is finite, this construction does *not* work to construct *genuine* G-spectra from $\mathscr{F}$-G-spaces: there is no evident way to build in deloopings by non-trivial representations of G.

[11] Perversely, [38] takes Δ to be the opposite of the category every other reference calls Δ.

2.3. **The conceptual version of the Segal machine.** Returning to $\mathscr{F}$-G-spaces, the more conceptual variants of the nonequivariant Segal machine generalize to give genuine G-spectra when G is compact Lie. These variants do *not* make use of the functor $F\colon \Delta^{op} \longrightarrow \mathscr{F}$. That is, underlying simplicial G-spaces play no role in their construction. We follow the nonequivariant exposition of [25]; [5] gives some relevant equivariant details. While [5] dealt with compact Lie groups, much of it applies equally well to general topological groups G. We introduce notation for the categories of enriched functors that we shall be using. Recall §1.3.

Notation 2.16. For a (small) $G\mathscr{T}$-category $\mathscr{D}$, let $\mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$ denote the category of $G\mathscr{T}$-functors $\mathscr{D}\longrightarrow\mathscr{T}_G$ and $G\mathscr{T}$-natural transformations between them. When G acts trivially on $\mathscr{D}$, as is the case of $\mathscr{F}$, a $G\mathscr{T}$-functor defined on $\mathscr{D}$ take values on morphisms in the fixed point spaces $\mathscr{T}_G(X,Y)^G = G\mathscr{T}(X,Y)$ of based G-maps $X\longrightarrow Y$. We therefore use the alternative notation $\mathrm{Fun}(\mathscr{D},G\mathscr{T})$ in that case.

Definition 2.17. A *$\mathscr{W}_G$-G-space* Y is a $G\mathscr{T}$-functor $\mathscr{W}_G\longrightarrow\mathscr{T}_G$. A map of $\mathscr{W}_G$-G-spaces is a $G\mathscr{T}$-natural transformation between them. Regarding $\mathscr{F}$ as a G-trivial G-category, it is both a full subcategory of $G\mathscr{W}$ and a G-trivial full G-subcategory of $\mathscr{W}_G$. We have the functor categories

$$\mathrm{Fun}(\mathscr{F},G\mathscr{T}) = \mathrm{Fun}(\mathscr{F},\mathscr{T}_G)$$

of $\mathscr{F}$-G-spaces and $\mathrm{Fun}(\mathscr{W}_G,\mathscr{T}_G)$ of $\mathscr{W}_G$-G-spaces. The inclusion $\mathscr{F}\subset\mathscr{W}_G$ induces a forgetful functor

$$\mathbb{U}\colon\ \mathrm{Fun}(\mathscr{W}_G,\mathscr{T}_G)\longrightarrow\mathrm{Fun}(\mathscr{F},\mathscr{T}_G).$$

We say that a $\mathscr{W}_G$-G-space Y is $\mathbb{F}_\bullet$-special, special, or grouplike if the $\mathscr{F}$-G-space $\mathbb{U}Y$ is so.

As a matter of elementary category theory (see e.g. [25]), the functor $\mathbb{U}$ has a left adjoint prolongation functor

$$\mathbb{P}\colon\ \mathrm{Fun}(\mathscr{F},\mathscr{T}_G)\longrightarrow\mathrm{Fun}(\mathscr{W}_G,\mathscr{T}_G).$$

The study of model structures on $\mathscr{W}_G$-G-spaces given in [5] when G is a compact Lie group applies verbatim when G is any topological group.

Remark 2.18. Let Y be any $\mathscr{W}_G$-G-space, such as $Y=\mathbb{P}X$ for an $\mathscr{F}$-G-space X. For G-spaces $A,B\in G\mathscr{W}$, the adjoint $B\longrightarrow\mathscr{W}_G(A,A\wedge B)$ of the identity map on $A\wedge B$ can be composed with Y to obtain a G-map

$$B\longrightarrow\mathscr{T}_G(Y(A),Y(A\wedge B)).$$

Its adjoint is a G-map

$$Y(A)\wedge B\longrightarrow Y(A\wedge B). \tag{2.19}$$

Letting $A=S^n$ and $B=S^1$ with trivial G-action, these maps give the structure maps

$$\Sigma Y(S^n)\longrightarrow Y(S^{n+1})$$

of a naive G-prespectrum $\mathbb{U}_{G\mathscr{P}}Y$.

When G is a compact Lie group we can define an orthogonal G-spectrum given at level V by $Y(S^V)$. The composites

$$\mathscr{I}_G(V,V')\longrightarrow\mathscr{W}_G(S^V,S^{V'})\xrightarrow{\ Y\ }\mathscr{T}_G(Y(S^V),Y(S^{V'}))$$

of Y and the map induced by one-point compactification of maps $V \longrightarrow V$ give a $G\mathscr{U}$-functor $\mathscr{I}_G \longrightarrow \mathscr{T}_G$ or, equivalently and more sensibly here, a $G\mathscr{T}$-functor $\mathscr{I}_{G+} \longrightarrow \mathscr{T}_G$. Just as in the nonequivariant case [25], letting $A = S^V$ and $B = S^W$ in (2.19) for representations V and W, we obtain the structure G-maps

$$\Sigma^W Y(S^V) \longrightarrow Y(S^{V\oplus W})$$

of an orthogonal G-spectrum $\mathbb{U}_{G\mathscr{S}}Y$ such that $i^*\mathbb{U}_{G\mathscr{S}}Y = \mathbb{U}_{G\mathscr{P}}Y$.

More generally, as in [25], we have forgetful functors

$$\mathbb{U}_{\mathscr{C}}\colon \operatorname{Fun}(\mathscr{W}_G, \mathscr{T}_G) \longrightarrow \mathscr{C},$$

where $\mathscr{C}$ can be the category of G-prespectra, symmetric G-spectra, or orthogonal G-spectra. Of course, nonequivariantly, Segal took $\mathscr{C}$ to be prespectra. We choose naive G-prespectra, $G\mathscr{P}$, for general topological groups G and genuine orthogonal G-spectra, $G\mathscr{S}$, for compact Lie groups G.

Definition 2.20. For a general topological group G, the conceptual Segal machine on $\mathscr{F}$-G-spaces is the composite

$$\mathbb{U}_{G\mathscr{P}} \circ \mathbb{P}\colon \operatorname{Fun}(\mathscr{F}, G\mathscr{T}) = \operatorname{Fun}(\mathscr{F}, \mathscr{T}_G) \longrightarrow G\mathscr{P}.$$

For compact Lie groups G, the conceptual Segal machine on $\mathscr{F}$-G-spaces is the analogous composite

$$\mathbb{U}_{G\mathscr{S}} \circ \mathbb{P}\colon \operatorname{Fun}(\mathscr{F}, G\mathscr{T}) \longrightarrow G\mathscr{S}.$$

Its composite with $i^*\colon G\mathscr{S} \longrightarrow G\mathscr{P}$ is $\mathbb{U}_{G\mathscr{P}} \circ \mathbb{P}$.

The functor $\mathbb{P}$ is a left Kan extension that is best viewed as a tensor product of functors. For $A \in G\mathscr{W}$, we have the contravariant $G\mathscr{T}$-functor $A^\bullet\colon \mathscr{F} \longrightarrow G\mathscr{T}$. Conceptually, it is the represented functor that sends $\mathbf{n}$ to the function space $\mathscr{W}_G(\mathbf{n}, A) \cong A^n$ with its induced action by G. By definition,

$$(\mathbb{P}X)(A) = A^\bullet \otimes_{\mathscr{F}} X. \tag{2.21}$$

Taking $A = \mathbf{n}$, the unit $\eta\colon X \longrightarrow \mathbb{U}\mathbb{P}X$ of the adjunction sends $x \in X_n$ to $(\mathrm{id}_{\mathbf{n}}, x)$; by Yoneda, η is a natural isomorphism. For a $\mathscr{W}_G$-G-space $Y\colon \mathscr{W}_G \longrightarrow \mathscr{T}_G$, the counit $\varepsilon\colon \mathbb{P}\mathbb{U}Y \longrightarrow Y$ is given on $A \in G\mathscr{W}$ by the composites

$$\mathscr{W}_G(\mathbf{n}, A) \wedge Y(\mathbf{n}) \xrightarrow{Y\wedge \mathrm{id}} \mathscr{T}_G(Y(\mathbf{n}), Y(A)) \wedge Y(\mathbf{n}) \xrightarrow{eval} Y(A).$$

For an $\mathscr{F}$-G-space $X\colon \mathscr{F} \longrightarrow G\mathscr{T}$, we write X_n for $X(\mathbf{n})$ as before, but we follow the usual convention of abbreviating notation by writing $(\mathbb{P}X)(A) = X(A)$ for general $A \in G\mathscr{W}$. The following result is a variant of Segal's [51, Proposition 3.2 and Lemma 3.7].

Proposition 2.22. *The naive G-prespectrum $\mathbb{U}_{G\mathscr{P}}\mathbb{P}X$ is naturally isomorphic to the G-prespectrum $\mathbb{S}_G^N X$ (of Notation 2.15). Thus, if X is special, then $\mathbb{U}_{G\mathscr{P}}\mathbb{P}X$ is a positive Ω-G-prespectrum with bottom structural map a group completion of X_1.*

Segal's proof is very briefly sketched in [51, §3], in different language. For the reader's convenience, we give a more complete argument in §9.1. The following result is the key observation, and it is the crucial point for us. Via Proposition 2.12, it makes the group completion property for $\mathbb{U}_{G\mathscr{P}}\mathbb{P}X$ transparent.

Proposition 2.23. *For $\mathscr{F}$-G-spaces X, there is a natural G-homeomorphism*

$$|X| \longrightarrow (S^1)^\bullet \otimes_{\mathscr{F}} X = (\mathbb{P}X)(S^1).$$

2.4. **A factorization of the conceptual Segal machine.** The previous sections apply to general topological groups. We continue in that generality. The results of this section are illuminating in general, but they are only really useful when G is finite. Here we consider the G-category $\mathscr{F}_G$ of finite based G-sets rather than just the category $\mathscr{F}$ of finite based sets. Use of $\mathscr{F}_G$ in tandem with $\mathscr{F}$ is essential to our work. In practice, input arises most often as $\mathscr{F}$-G-spaces but, by a result of Shimakawa [55] that we shall reprove with different details,[12] these are interchangeable with $\mathscr{F}_G$-G-spaces.

Definition 2.24. Let $\mathscr{F}_G$ be the G-category of finite based G-sets and all based functions, with G acting by conjugation on function sets. For convenience and precision, we restrict the objects of $\mathscr{F}_G$ to be the finite G-sets $A = (\mathbf{n}, \alpha)$, as in Notation 1.4. Let Π_G be the G-subcategory with the same objects and those morphisms $\phi\colon (\mathbf{m}, \alpha) \longrightarrow (\mathbf{n}, \beta)$ such that $\phi^{-1}(j)$ has at most one element for $1 \le j \le n$. We obtain inclusions $\mathscr{F} \subset \mathscr{F}_G$ and $\Pi \subset \Pi_G$ by restricting to the trivial homomorphisms $\varepsilon_n\colon G \longrightarrow \Sigma_n$.

As with $\mathscr{F}$, we view $\mathscr{F}_G$ as a category enriched in $G\mathscr{T}$, with the discrete topology on the based hom sets of maps, on which G acts by conjugation. The basepoint of $\mathscr{F}_G((\mathbf{m}, \alpha), (\mathbf{n}, \beta))$ is the unique map that factors through $\mathbf{0}$. When G is finite, a finite G-set is evidently a G-CW complex and for a general G we can enlarge $\mathscr{W}_G$ if necessary to obtain an inclusion $\mathscr{F}_G \subset \mathscr{W}_G$.

Definition 2.25. An *$\mathscr{F}_G$-G-space* Y is a $G\mathscr{T}$-functor $Y\colon \mathscr{F}_G \longrightarrow \mathscr{T}_G$; a Π_G-G-space Y is a $G\mathscr{T}$-functor $Y\colon \Pi_G \longrightarrow \mathscr{T}_G$. Morphisms are $G\mathscr{T}$-natural transformations. We write $Y(A)$ for the value of Y on $A = (\mathbf{n}, \alpha)$, writing Y_n for the value of Y on $(\mathbf{n}, \varepsilon_n)$. We let $(Y_1^n)^\alpha$ denote Y_1^n with the G-action

$$g(y_1, \ldots, y_n) = (gy_{\alpha(g^{-1})(1)}, \ldots, gy_{\alpha(g^{-1})(n)}).$$

It can be identified with the G-space $\mathscr{T}_G(A, Y_1)$.

Definition 2.26. For $A = (\mathbf{n}, \alpha)$ and a Π_G-G-space Y, define a based G-map $\varepsilon\colon A \wedge A \longrightarrow \mathbf{1} = S^0$ by the Kronecker δ function: $(i, j) \mapsto 1$ if $i = j$ and to 0 if $i \neq j$. Its adjoint is a G-map $A \longrightarrow \mathscr{F}_G(A, \mathbf{1})$. Composing with

$$Y\colon \mathscr{F}_G(A, \mathbf{1}) \longrightarrow \mathscr{T}_G(Y(A), Y_1)$$

and adjointing, we obtain a G-map $\partial_A\colon A \wedge Y(A) \longrightarrow Y_1$. Thus $\partial_A(j, y) = (\delta_j)_*(y)$ for $1 \le j \le n$, where δ_j is induced by the jth projection $(\mathbf{n}, \alpha) \longrightarrow (\mathbf{1}, \varepsilon_1)$. The *Segal map*

$$\delta_A\colon Y(A) \longrightarrow \mathscr{T}_G(A, Y_1) \cong (Y_1^n)^\alpha$$

is the adjoint of ∂_A; we usually abbreviate δ_A to δ. Note that δ is a G-map, although it components δ_j are usually not.

Definition 2.27. A Π_G-G-space Y is *special* if the δ_A are weak G-equivalences for all $A = (\mathbf{n}, \alpha)$. A map $f\colon Y \longrightarrow Z$ of Π_G-G-spaces is a *level G-equivalence* if each $f\colon Y(A) \longrightarrow Z(A)$ is a weak G-equivalence. We say that an $\mathscr{F}_G$-G-space is special if its underlying Π_G-G-space is so and that a map of $\mathscr{F}_G$-G-spaces is a level G-equivalence if its underlying map of Π_G-G-spaces is so.

[12]The starting point of [55] came from conversations during a long and mutually profitable visit Shimakawa made to the first author.

The inclusion $\mathscr{F} \hookrightarrow \mathscr{F}_G$ induces a restriction functor

$$\mathbb{U}\colon \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{F}, G\mathscr{T})$$

which has a left adjoint prolongation functor

$$\mathbb{P}\colon \mathrm{Fun}(\mathscr{F}, G\mathscr{T}) \longrightarrow \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G).$$

The adjunction $(\mathbb{P}, \mathbb{U})$ of Definition 2.17 factors as the composite of the analogous adjunctions given by the functors

$$\mathrm{Fun}(\mathscr{F}, G\mathscr{T}) \xrightarrow{\mathbb{P}} \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \xrightarrow{\mathbb{P}} \mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G)$$

$$\mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G) \xrightarrow{\mathbb{U}} \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \xrightarrow{\mathbb{U}} \mathrm{Fun}(\mathscr{F}, G\mathscr{T}).$$

The units of these adjunctions are isomorphisms since the forgetful functors $\mathbb{U}$ are induced by the full and faithful inclusions $\mathscr{F} \hookrightarrow \mathscr{F}_G$ and $\mathscr{F}_G \hookrightarrow \mathscr{W}_G$.

Notation 2.28. For $A \in \mathscr{W}_G$, we now write $A^\bullet$ ambiguously for both the restriction to $\mathscr{F}_G \subset \mathscr{T}_G$ and the restriction to $\mathscr{F} \subset \mathscr{F}_G \subset \mathscr{T}_G$ of the represented functor $\mathscr{T}_G(-, A)\colon \mathscr{T}_G^{op} \longrightarrow \mathscr{T}_G$.

Then the factorization of $\mathbb{P}$ as $\mathbb{P}\mathbb{P}$ takes the explicit form

$$A^\bullet \otimes_{\mathscr{F}} X \cong A^\bullet \otimes_{\mathscr{F}_G} (\mathscr{F}_G \otimes_{\mathscr{F}} X) = A^\bullet \otimes_{\mathscr{F}_G} \mathbb{P}X. \tag{2.29}$$

While our main interest is in $\mathscr{F}_G$-G-spaces and $\mathscr{F}$-G-spaces, we will also use the analogous forgetful and prolongation functors relating Π_G-G-spaces and Π-G-spaces.

Observe that we have no analogue for Π_G-G-spaces (or for $\mathscr{F}_G$-G-spaces) of the dichotomies between $\mathbb{F}_\bullet$-special and special and between $\mathbb{F}_\bullet$-level equivalences and level G-equivalences that we had for Π-G-spaces (and thus for $\mathscr{F}$-G-spaces). The following result shows that the notions defined in Definition 2.27 for Π_G-G-spaces correspond to the $\mathbb{F}_\bullet$-notions for Π-G-spaces. That should help motivate the latter, which may at first sight have seemed unnatural.

Theorem 2.30. *The adjoint pairs of functors*

$$\mathrm{Fun}(\Pi, G\mathscr{T}) \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathrm{Fun}(\Pi_G, \mathscr{T}_G)$$

and

$$\mathrm{Fun}(\mathscr{F}, G\mathscr{T}) \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$$

specify equivalences of categories. Moreover, the following statements hold.

(i) A Π_G-G-space Y is special if and only if the Π-G-space $\mathbb{U}Y$ is $\mathbb{F}_\bullet$-special.

(ii) A map $f\colon Y \longrightarrow Z$ of Π_G-G-spaces is a level G-equivalence if and only if the map $\mathbb{U}f\colon \mathbb{U}Y \longrightarrow \mathbb{U}Z$ of Π-G-spaces is an $\mathbb{F}_\bullet$-level equivalence.

(iii) A Π-G-space X is $\mathbb{F}_\bullet$-special if and only if the Π_G-G-space $\mathbb{P}X$ is special.

(iv) A map f of Π-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only if the map $\mathbb{P}f$ of Π_G-G-spaces is a level G-equivalence.

All of these statements remain true with with Π and Π_G replaced by $\mathscr{F}$ and $\mathscr{F}_G$.

Proof. For a Π-G-space X and a finite G-set $A = (\mathbf{n}, \alpha)$,

$$(\mathbb{P}X)(A) = A^\bullet \otimes_\Pi X,$$

where $A^\bullet \colon \Pi \longrightarrow G\mathscr{T}$ is the functor that sends $\mathbf{m}$ to $\Pi_G(\mathbf{m}, A)$. Recall that the underlying set of $\Pi_G(\mathbf{m}, A)$ is just $\Pi(\mathbf{m}, \mathbf{n})$ with G-action induced by the action of G on $\mathbf{n}$ given by α. The action of G on $A^\bullet \otimes_\Pi X$ is induced by the diagonal action. For $n \geq 0$, the unit

$$\eta\colon X_n \longrightarrow \Pi(-, \mathbf{n}) \otimes_\Pi X$$

is the G-map given by $\eta(x) = (\mathrm{id}_n, x)$. It is a G-homeomorphism with inverse given by $\eta^{-1}(\mu, x) = \mu_*(x)$ for $\mu\colon \mathbf{m} \longrightarrow \mathbf{n}$ in Π and $x \in X_m$. Clearly η^{-1} is well-defined, $\eta^{-1}\eta = \mathrm{id}$, and $\eta\eta^{-1} = \mathrm{id}$ since $(\mu, x) \sim (\mathrm{id}_n, \mu_* x)$. Since η^{-1} is inverse to a G-map, it is a G-map.

We must show that the counit $\varepsilon\colon \mathbb{P}\mathbb{U}Y \longrightarrow Y$ is an isomorphism for a Π_G-G-space Y. Again let $A = (\mathbf{n}, \alpha) \in \Pi_G$. Then

$$\varepsilon\colon (\mathbb{P}\mathbb{U}Y)(A) = A^\bullet \otimes_\Pi (\mathbb{U}Y) \longrightarrow Y(A)$$

is the G-map given by $\varepsilon(\mu, y) = \mu_* y$ for $\mu\colon \mathbf{m} \longrightarrow A$ and $y \in Y_m$, where $\mu_*\colon Y_m \longrightarrow Y(A)$. It is a G-homeomorphism with inverse given by $\varepsilon^{-1}(y) = (\iota^{-1}, \iota_* y)$ for $y \in Y(A)$, where $\iota \in \Pi_G(A, \mathbf{n})$ is the morphism whose underlying function on $\mathbf{n}$ is the identity. Clearly $\varepsilon\varepsilon^{-1} = \mathrm{id}$, and $\varepsilon^{-1}\varepsilon = \mathrm{id}$ since $(\mu, y) \sim (\iota^{-1}, \iota_*\mu_* y)$. The identification uses the morphism $\iota \circ \mu$ in Π. Again, since ε^{-1} is inverse to a G-map, it is a G-map. The proof with Π and Π_G replaced by $\mathscr{F}$ and $\mathscr{F}_G$ is the same.

To prove (i) and (ii), we describe more explicitly how a Π_G-G-space Y is reconstructed from its underlying Π-G-space. For a finite G-set $A = (\mathbf{n}, \alpha)$, let Y_n^α denote Y_n with a new action $\cdot_\alpha$ of G specified in terms of α and the original action of G by $g \cdot_\alpha y = \alpha(g)_*(g \cdot y)$. In effect, ε^{-1} identifies $Y(A)$ with the G-space Y_n^α.

Consider $\Lambda_\alpha = \{(g, \alpha(g))\}$. Projection onto the first coordinate gives an isomorphism $\Lambda_\alpha \longrightarrow G$. The Λ_α-action on Y_n obtained by restriction of the action of $G \times \Sigma_n$ is given by

$$(g, \alpha(g)) \cdot y = \alpha(g)_*(g \cdot y).$$

Thus it coincides with the G-action that we used to define Y_n^α. This immediately implies (ii). Similarly, the Λ_α-action on Y_1^n obtained by restriction of the action of $G \times \Sigma_n$ given by the diagonal action of G and the permutation action of Σ_n is

$$g(y_1, \ldots, y_n) = (gy_{\alpha(g^{-1})(1)}, \ldots, gy_{\alpha(g^{-1})(n)}).$$

Thus it coincides with the G-action that we used to define $(Y_1^n)^\alpha$. Therefore ε^{-1} identifies the Segal G-map $\delta\colon Y(\mathbf{n}, \alpha) \longrightarrow (Y_1^n)^\alpha$ with the Λ_α-map $\delta\colon Y_n \longrightarrow Y_1^n$. This immediately implies (i), and (iii) and (iv) follow formally from (i) and (ii) since $\eta\colon \mathrm{id} \longrightarrow \mathbb{U}\mathbb{P}$ is an isomorphism.

Since statements (i)-(iv) for $\mathscr{F}$-G-spaces and $\mathscr{F}_G$-G-spaces depend only on their underlying Π-G-spaces and Π_G-G-spaces, they follow immediately. □

We record analogues for $\mathscr{F}_G$-G-spaces of Lemmas 2.9 and 2.10 for $\mathscr{F}$-G-spaces. While they could be proven directly, we just observe that the first follows immmediately from Theorem 2.30(iii), and the second follows as in the proof of Lemma 2.10.

Definition 2.31. For a based G-space X, let $\mathbb{R}_G X$ denote the Π_G-G-space with $(\mathbf{n}, \alpha)$th G-space $(X^n)^\alpha$. Conceptually, it is obtained by prolonging the Π-G-space $\mathbb{R}X$ from Definition 2.8 to a Π_G-G-space.

Lemma 2.32. *If $f\colon X \longrightarrow Y$ is a weak equivalence of based G-spaces, then the induced map $\mathbb{R}_G f\colon \mathbb{R}_G X \longrightarrow \mathbb{R}_G Y$ is a level G-equivalence of Π_G-G-spaces.*

Lemma 2.33. *If $f\colon X \longrightarrow Y$ is a level G-equivalence of Π_G-G-spaces, then X is special if and only if Y is special.*

The factorization of the Segal machine holds for any topological group G. Since actions of G on finite G-sets factor through actions of $\pi_0(G)$, it is clear that $\mathscr{W}_G$-G-spaces generally incorporate much more information than $\mathscr{F}$-G-spaces do.

3. The homotopical version of the Segal machine

For the moment, we continue to work with a general topological group G. However, our interest is to understand the Segal machine homotopically when G is finite or compact Lie. While $\mathbb{U}_{G,\mathscr{S}}\mathbb{P}(X)$ gives the most conceptually natural equivariant version of the Segal machine on $\mathscr{F}$-G-spaces X, it is by itself of negligible use since the functor $\mathbb{P}$ does not enjoy good homotopical properties before some kind of homotopical approximation of X. We define a naive homotopical Segal machine in §3.2 and show its defects. Restricting to compact Lie groups, we define a genuine homotopical Segal machine and summarize its homotopical properties in §3.3, deferring the longer proofs to §9. In §3.4, we discuss the case of finite groups model theoretically and contrast the case of compact Lie groups.

The homotopical version of the Segal machine is defined in terms of an appropriate enriched version of the two-sided categorical bar construction. We start in §3.1 with a general discussion of this construction.

3.1. **The categorical bar construction.** We here define the variant of the bar construction used in the homotopical Segal machine. We begin with some general definitions that start with a closed symmetric monoidal category $\mathscr{V}$ and a $\mathscr{V}$-category $\mathscr{E}$, as in §1.3. Let Y be a contravariant and X be a covariant $\mathscr{V}$-functor $\mathscr{E} \longrightarrow \mathscr{V}$. They are given by objects Y_n and X_n in $\mathscr{V}$ and maps in $\mathscr{V}$

$$Y\colon \mathscr{E}(m,n) \longrightarrow \underline{\mathscr{V}}(Y_n, Y_m) \quad \text{and} \quad X\colon \mathscr{E}(m,n) \longrightarrow \underline{\mathscr{V}}(X_m, X_n)$$

that are compatible with composition and identity. These have adjoint evaluation maps

$$E_Y\colon \mathscr{E}(m,n) \otimes Y_n \longrightarrow Y_m \quad \text{and} \quad E_X\colon \mathscr{E}(m,n) \otimes X_m \longrightarrow X_n.$$

Associated to the triple $(Y, \mathscr{E}, X)$ we have a categorical two-sided bar construction $B_\bullet(Y, \mathscr{E}, X)$. It is a simplicial object in $\mathscr{V}$. Its q-simplex object in $\mathscr{V}$ is the coproduct

$$(3.1) \qquad B_q(Y, \mathscr{E}, X) = \coprod_{(n_0,\dots,n_q)} Y_{n_q} \otimes \mathscr{E}(n_{q-1}, n_q) \otimes \cdots \otimes \mathscr{E}(n_0, n_1) \otimes X_{n_0},$$

where $(n_0, \dots, n_q)$ runs over the $(q+1)$-tuples of objects of $\mathscr{E}$. Its faces d_i for $0 \le i \le q$ are induced by the evaluation maps of Y and X and by composition in $\mathscr{E}$, and its degeneracies s_i for $0 \le i \le q$ are induced by the unit maps of $\mathscr{E}$. In more detail, the d_i are induced by

$$\mathscr{E}(n_0, n_1) \otimes X_{n_0} \xrightarrow{E_X} X_{n_1} \quad \text{if } i = 0,$$

$$\mathscr{E}(n_i, n_{i+1}) \otimes \mathscr{E}(n_{i-1}, n_i) \xrightarrow{C} \mathscr{E}(n_{i-1}, n_{i+1}) \quad \text{if } 0 < i < q,$$

$$Y_{n_q} \otimes \mathscr{E}(n_{q-1}, n_q) \xrightarrow{E_Y} Y_{n_{q-1}} \quad \text{if } i = q.$$

The s_i are induced by

$$U \xrightarrow{I} \mathscr{E}(n_i, n_i) \ \text{ if } 0 \le i \le q.$$

When $\mathscr{V}$ is cartesian closed, so that $\otimes = \times$, $B_\bullet(Y, \mathscr{E}, X)$ is the nerve of an internal "Grothendieck category of elements" $\mathscr{C}(Y, \mathscr{E}, X)$. The objects and morphisms of this category are both objects of $\mathscr{V}$. The object of objects is the coproduct

$$\mathscr{C}_0 = \coprod_n Y_n \times X_n,$$

where n runs over the objects of $\mathscr{E}$. The object of morphisms is the coproduct

$$\mathscr{C}_1 = \coprod_{m,n} Y_n \times \mathscr{E}(m,n) \times X_m,$$

where (m, n) runs over the pairs of objects of $\mathscr{E}$. We have source, target, and identity maps S, T, and I given as follows: S and T are given on components by the evaluation maps of Y and X.

$$S = E_Y \times \mathrm{id} \colon Y_n \times \mathscr{E}(m,n) \times X_m \longrightarrow Y_m \times X_m$$

$$T = \mathrm{id} \times E_X \colon Y_n \times \mathscr{E}(m,n) \times X_m \longrightarrow Y_n \times X_n;$$

I is induced by the identity maps $U \longrightarrow \mathscr{E}(n,n)$ of $\mathscr{E}$.

The composition

$$C \colon \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \longrightarrow \mathscr{C}_1$$

is induced by the composition in $\mathscr{E}$. The point is that when $\otimes = \times$ we have the identification

$$\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \cong \coprod_{(m,n,p)} Y_p \times \mathscr{E}(n,p) \times \mathscr{E}(m,n) \times X_m.$$

where (m, n, p) runs over the triples of objects of $\mathscr{E}$.

When $\mathscr{V}$ has a suitable covariant simplex functor $\Delta \longrightarrow \mathscr{V}$ so that we have a "geometric realization" functor from simplicial objects in $\mathscr{V}$ to $\mathscr{V}$, we define $B(Y, \mathscr{E}, X)$ to be the realization of $B_\bullet(Y, \mathscr{E}, X)$.

Remark 3.2. The two-sided bar construction goes back to [40, §12]. It is described in the form just given in Shulman [57, Definition 12.1], where more details can be found. The construction in this generality and in this form is central to the study of weighted colimits in enriched category theory.

Now let us return to our space level context. The discussion just given applies with $\mathscr{V} = G\mathscr{U}$ for any G, where we take $\otimes = \times$. It also applies with $\mathscr{V} = G\mathscr{T}$, where we take $\otimes = \wedge$. Given a based triple $(Y, \mathscr{E}, X)$, so taking $\mathscr{V} = G\mathscr{T}$ and using $\otimes = \wedge$ everywhere, specialization of our general construction gives a bar construction $B^\wedge(Y, \mathscr{E}, X)$. Alternatively, forgetting about basepoints, we can take $\mathscr{V} = G\mathscr{U}$ and use $\otimes = \times$ everywhere to get a bar construction $B^\times(Y, \mathscr{E}, X)$. Neither is right for our purposes. With $B^\wedge$, a key later proof, that of the wedge axiom in §9.4, would fail. With $B^\times$, we could not enrich over $G\mathscr{T}$, as we now explain.

As in §1.3, we assume that $\mathscr{E}$ has a zero object 0 such that each $\mathscr{E}(0, n)$ and $\mathscr{E}(n, 0)$ is a point. Then each $\mathscr{E}(m, n)$ has the basepoint $m \to 0 \to n$. For each object n of $\mathscr{E}$, we have the represented functor $\mathscr{E}_n = \mathscr{E}(-, n)$. In particular, $\mathscr{E}_0$ is the constant functor $*$ at a point. As n varies, the bar constructions $B^\times(\mathscr{E}_n, \mathscr{E}, X)$ give a covariant functor

$$B^\times(\mathscr{E}, \mathscr{E}, X) \colon \mathscr{E} \longrightarrow G\mathscr{T}.$$

It is a $G\mathscr{U}$-enriched functor but it is not $G\mathscr{T}$-enriched because the map

$$\mathscr{E}(m,n) \longrightarrow \mathscr{T}_G\big(B^\times(\mathscr{E}_m, \mathscr{E}, X), B^\times(\mathscr{E}_n, \mathscr{E}, X)\big)$$

does not send the basepoint of the source to the basepoint (zero map) of the target.

Let $\varepsilon \colon B^\times(\mathscr{E}, \mathscr{E}, X) \longrightarrow X$ be the canonical map of (non-enriched) functors $\mathscr{E} \longrightarrow G\mathscr{T}$, constructed at level n by passing to realization from the map of simplicial G-spaces

$$\varepsilon_n \colon B_\bullet^\times(\mathscr{E}_n, \mathscr{E}, X) \longrightarrow (X_n)_\bullet$$

that is given by composition in $\mathscr{E}$ and the action of $\mathscr{E}$ on X. Here $(X_n)_\bullet$ denotes the constant simplicial G-space at X_n. Each ε_n is a G-homotopy equivalence with homotopy inverse

$$\eta_n \colon X_n \longrightarrow B^\times(\mathscr{E}_n, \mathscr{E}, X)$$

given by sending $x \in X_n$ to the zero simplex $(\mathrm{id}_n, x) \in \mathscr{E}(n,n) \times X_n$. Then $B^\times(*, \mathscr{E}, X)$ is contractible (since $X_0 = *$ and $\mathscr{E}_0 = *$).

We have an inclusion $* \longrightarrow Y$ given by the basepoints of the Y_n and we define

$$B(Y, \mathscr{E}, X) = B^\times(Y, \mathscr{E}, X)/B^\times(*, \mathscr{E}, X). \tag{3.3}$$

The inclusions of basepoints $* \longrightarrow Y_n$ are G-cofibrations. Since our bar constructions are Reedy cofibrant simplicial G-spaces, so that the geometric realization of a level G-cofibration is also a G-cofibration (see Theorem 1.11), these inclusions induce G-cofibrations $B^\times(*, \mathscr{E}, X) \longrightarrow B^\times(Y, \mathscr{E}, X)$. Therefore the quotient map

$$B^\times(Y, \mathscr{E}, X) \longrightarrow B(Y, \mathscr{E}, X) \tag{3.4}$$

is a G-homotopy equivalence. With $Y = \mathscr{E}_n$, this gives the following result.

Proposition 3.5. *$B(\mathscr{E}, \mathscr{E}, X)$ is a $G\mathscr{T}$-functor $\mathscr{E} \longrightarrow \mathscr{T}_G$, and ε induces a levelwise G-homotopy equivalence $B(\mathscr{E}, \mathscr{E}, X) \longrightarrow X$ of such functors with level inverses induced by the η_n.*

Remark 3.6. Again using Reedy cofibrancy, we see that $B(Y, \mathscr{E}, X)$ is the geometric realization of the simplicial based G-space whose space of q-simplices is

$$B_q(Y, \mathscr{E}, X)/B_q(*, \mathscr{E}, X).$$

This can be rewritten as the wedge of half-smash products

$$\bigvee_n Y_n \wedge B_{q-1}^\times(\mathscr{E}_n, \mathscr{E}, X)_+.$$

As explained in complete categorical generality in [57, Lemma 19.7], we can commute realization and $\otimes_{\mathscr{E}}$ to obtain the isomorphism

$$B(Y, \mathscr{E}, X) \cong Y \otimes_{\mathscr{E}} B(\mathscr{E}, \mathscr{E}, X).$$

One proof uses a direct comparison of definitions on the level of q-simplices for each q, but the result is also an application of the (enriched) categorical Fubini theorem.

More generally, if $\mathscr{D}$ and $\mathscr{E}$ are both as above, $\nu \colon \mathscr{D} \longrightarrow \mathscr{E}$ is a $G\mathscr{T}$-functor, Y is a contravariant $G\mathscr{T}$-functor $\mathscr{E} \longrightarrow \mathscr{T}_G$, and X is a covariant $G\mathscr{T}$-functor $\mathscr{D} \longrightarrow \mathscr{T}_G$, there is an isomorphism

$$B(\nu^* Y, \mathscr{D}, X) \cong Y \otimes_{\mathscr{E}} B(\mathscr{E}, \mathscr{D}, X),$$

where $\nu^*Y = Y \circ \nu$ and, similarly, each $\mathscr{E}_n$ is viewed as a contravariant $G\mathscr{T}$-functor $\mathscr{D} \longrightarrow \mathscr{T}_G$ by precomposition with ν.

Remark 3.7. The homotopical Segal machine is obtained by using examples of two-sided bar constructions to construct $\mathscr{W}_G$-G-spaces. The structure maps of the G-spectra they give by restricting to spheres must come from comparison maps of the form

$$B(Y, \mathscr{E}, X) \wedge C \longrightarrow B(Y \wedge C, \mathscr{E}, X). \tag{3.8}$$

We have such maps with our definition of the bar construction, but we would not have them if we tried to use $B^\times$. The point is that (2.19) in Remark 2.18 does not apply if we use $B^\times$ due to the difference between unbased and based enrichment. To make this more precise, consider the relationship between smash products and products. For based spaces A, B, and C, there is no natural map

$$(A \times B) \wedge C \longrightarrow (A \wedge C) \times B.$$

Under the natural isomorphism $(A\times B)\times C \cong (A\times C)\times B$, we collapse out different subspaces to construct the source and target, and neither is contained in the other. Explicitly, writing a, b, c for the basepoints of A, B, C and x, y, z for general points of A, B, C, we identify all points (x, y, c) and (a, b, z) with (a, b, c) in the source, but we identify all points (a, z, y) and (x, c, y) with the point (a, c, y) in the target.

Our interest is in the case where $\mathscr{E}$ is $\mathscr{F}$ or $\mathscr{F}_G$ or the more general categories of operators $\mathscr{D}$ and $\mathscr{D}_G$ to be introduced later. Note that although $\mathscr{F}$ is topologically discrete and G-trivial, we still view it as a category enriched in $\mathscr{T}_G$.

3.2. The naive homotopical Segal machine.

Specializing from the previous section, let $Y \colon \mathscr{F} \longrightarrow G\mathscr{T}$ be a contravariant $G\mathscr{T}$-functor and $X \colon \mathscr{F} \longrightarrow G\mathscr{T}$ be a covariant $G\mathscr{T}$-functor. We then have the bar construction $B(Y, \mathscr{F}, X)$. The action of G on it is induced diagonally by the actions on the Y_n and X_n.

For $G\mathscr{T}$-functors $Y \colon \mathscr{F}_G^{op} \longrightarrow \mathscr{T}_G$ and $X \colon \mathscr{F}_G \longrightarrow \mathscr{T}_G$, we have the resulting two-sided bar construction $B(Y, \mathscr{F}_G, X)$. For its construction, we must remember the action of G on the finite G-sets $\mathscr{F}_G((\mathbf{m}, \alpha), (\mathbf{n}, \beta))$. While we are interested in general X, in both cases we are only interested in particular Y, namely those of the form $Y = A^\bullet$, as in Notation 2.28.

Nonequivariantly, Woolfson [59] constructed a homotopical Segal machine by restricting $B(A^\bullet, \mathscr{F}, X)$ to spheres $A = S^n$.[13] Equivariantly, we can apply the same construction, taking G to be any topological group, X to be an $\mathscr{F}$-G-space, and A to be in $G\mathscr{W}$. For reasons we now explain, this construction fails to lead to genuine Ω-G-spectra when G is finite, even when X is $\mathbb{F}_\bullet$-special.

When $A = \mathbf{n}$, $A^\bullet$ is the represented functor $\mathscr{F}_n = \mathscr{F}(-, \mathbf{n})$, and as n varies we obtain the $\mathscr{F}$-G-space $B(\mathscr{F}, \mathscr{F}, X)$ whose nth G-space is $B(\mathscr{F}_n, \mathscr{F}, X)$. We have an implicit and important action of Σ_n on source and target; $\Sigma_n \subset \mathscr{F}(\mathbf{n}, \mathbf{n})$ acts from the left on $\mathscr{F}_n$ by postcomposition in $\mathscr{F}$, and that induces the action on $B(\mathscr{F}_n, \mathscr{F}, X)$. Observe that $B_\bullet(\mathscr{F}_n, \mathscr{F}, X)$ is a simplicial $(G \times \Sigma_n)$-space and ε_n is the geometric realization of a map $B_\bullet(\mathscr{F}_n, \mathscr{F}, X) \longrightarrow (X_n)_\bullet$ of simplicial $(G \times \Sigma_n)$-spaces. Proposition 3.5 specializes to give the following result.

[13] This is revisionist. He was writing before the two-sided bar construction was formally defined.

Proposition 3.9. *Let X be an $\mathscr{F}$-G-space X. Then the map*

$$\varepsilon\colon B(\mathscr{F},\mathscr{F},X) \longrightarrow X$$

of $\mathscr{F}$-G-spaces is a level G-equivalence, hence X is special if and only if $B(\mathscr{F},\mathscr{F},X)$ is special.

Warning 3.10. While the map ε_n is a map of $(G\times\Sigma_n)$-spaces, the map

$$\eta_n\colon X_n \longrightarrow B(\mathscr{F}_n,\mathscr{F},X)$$

is *not* Σ_n-equivariant since the action of Σ_n occurs on $\mathscr{F}(-,\mathbf{n})$ in the target and on X_n in the source, so that $\eta(\sigma x) = (\mathrm{id}_n, \sigma x)$ while $\sigma\eta(x) = (\sigma, x)$. Thus there is no reason to expect ε to be a level $\mathbb{F}_\bullet$-equivalence and no reason to expect $B(\mathscr{F},\mathscr{F},X)$ to be $\mathbb{F}_\bullet$-special even if X is so.

By Propositions 2.22 and 3.9, $B(\mathscr{F},\mathscr{F},X)$ and X can be used interchangeably when passing to naive G-prespectra. Recall that we also have the two-sided bar construction with $\mathscr{F}$ replaced by $\mathscr{F}_G$. We elaborate our comparison to include $\mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X)$. Here $B(\mathscr{F}_G,\mathscr{F}_G,Y)$ is defined at level $(\mathbf{n},\alpha)$ by replacing the left variable $\mathscr{F}_G$ by the functor $\mathscr{F}_G(-,(\mathbf{n},\alpha))\colon \mathscr{F}_G^{op} \longrightarrow \mathscr{T}_G$ represented by $(\mathbf{n},\alpha)$. This comparison will also pave the way towards the construction of genuine G-spectra when G is a compact Lie group.

Still letting G be any topological group and using Lemma 2.33, we have the following analogue of Proposition 3.9.

Proposition 3.11. *Let Y be an $\mathscr{F}_G$-G-space. Then $\varepsilon\colon B(\mathscr{F}_G,\mathscr{F}_G,Y) \longrightarrow Y$ is a level G-equivalence, hence Y is special if and only if $B(\mathscr{F}_G,\mathscr{F}_G,Y)$ is special.*

We view $\mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X)$ as a genuine homotopical approximation to X in view of the following corollary, which is immediate from Theorem 2.30. Note that we can identify X with $\mathbb{U}\mathbb{P}X$ via the unit isomorphism.

Corollary 3.12. *For any $\mathscr{F}$-G-space X, the map $\mathbb{U}\varepsilon\colon \mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X) \longrightarrow X$ is an $\mathbb{F}_\bullet$-level equivalence, hence X is $\mathbb{F}_\bullet$-special if and only if $\mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X)$ is $\mathbb{F}_\bullet$-special.*

Remark 3.13. We compare bar constructions along the adjoint equivalence $(\mathbb{P},\mathbb{U})$ between $\mathscr{F}$-G-spaces and $\mathscr{F}_G$-G-spaces. For $\mathscr{F}$-G-spaces X, we have

$$\mathbb{P}B(\mathscr{F},\mathscr{F},X) = \mathscr{F}_G \otimes_{\mathscr{F}} B(\mathscr{F},\mathscr{F},X) \cong B(\mathscr{F}_G,\mathscr{F},X).$$

The inclusion $\iota\colon \mathscr{F} \longrightarrow \mathscr{F}_G$ induces a natural map of $\mathscr{F}_G$-G-spaces

$$\iota_*\colon \mathbb{P}B(\mathscr{F},\mathscr{F},X) \longrightarrow B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X)$$

such that the following diagrams commute; the second is obtained from the first by applying $\mathbb{U}$.

$$\begin{array}{ccc} \mathbb{P}B(\mathscr{F},\mathscr{F},X) & \xrightarrow{\mathbb{P}\varepsilon} & \mathbb{P}X \\ {\scriptstyle\iota}\downarrow & \nearrow_{\varepsilon} & \\ B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X) & & \end{array} \qquad \begin{array}{ccc} B(\mathscr{F},\mathscr{F},X) & \xrightarrow{\varepsilon} & X \\ {\scriptstyle\mathbb{U}\iota}\downarrow & \nearrow_{\mathbb{U}\varepsilon} & \\ \mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X) & & \end{array}$$

In the first, the diagonal arrow ε is a level G-equivalence, but we cannot expect ι and $\mathbb{P}\varepsilon$ to be level G-equivalences since that would imply that all three arrows in the second diagram are $\mathbb{F}_\bullet$-level equivalences, contradicting Warning 3.10. In the second diagram, $\mathbb{U}\varepsilon$ is an $\mathbb{F}_\bullet$-level equivalence and the other two arrows are level G-equivalences.

Using Proposition 2.22, we see that this comparison implies the following comparison of naive G-prespectra.

Proposition 3.14. *Let X be a special $\mathscr{F}$-G-space. Then the positive naive Ω-G-prespectra obtained by prolonging X, $B(\mathscr{F},\mathscr{F},X)$, and $\mathbb{U}B(\mathscr{F}_G,\mathscr{F}_G,\mathbb{P}X)$ to $\mathscr{W}_G$-G-spaces and then restricting to spheres S^n are level G-equivalent. Their bottom structural maps are compatible group completions of G-spaces equivalent to X_1.*

Proof. For any A, functoriality of prolongation applied to the second diagram of Remark 3.13 gives a commutative diagram

$$\begin{array}{ccc} B(A^\bullet,\mathscr{F},X) & \longrightarrow & X(A). \\ \big\downarrow & \nearrow & \\ B(A^\bullet,\mathscr{F}_G,\mathbb{P}X) & & \end{array}$$

Here we have used the factorization of prolongation from $\mathscr{F}$-G-spaces to $\mathscr{W}_G$-G-spaces through $\mathscr{F}_G$-G-spaces and the isomorphism $\mathbb{P}\mathbb{U}\cong \mathrm{Id}$, where $\mathbb{P}$ is prolongation from $\mathscr{F}$-G-spaces to $\mathscr{F}_G$-G-spaces. By Lemma 2.10 and Remark 3.13, all three of our $\mathscr{F}$-G-spaces are special. Restricting to spheres S^n, we can apply Proposition 2.22 to each of them. Taking $A=S^0$, we see compatible G-equivalences with X_1, and taking $A=S^1$, we see that the bottom structure maps are compatible group completions. That implies that we have weak G-equivalences at level 1. In turn, since we are comparing Ω-G-prespectra, that implies that we have weak G-equivalences at all levels n. □

3.3. **The genuine homotopical Segal machine.** Let G be a compact Lie group. We cannot expect to construct genuine Ω-G-spectra from special $\mathscr{F}$-G-spaces, but we show here how to construct a genuine G-spectrum $\mathbb{S}_G X$ from an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space X. Equivalently, we construct a genuine G-spectrum $\mathbb{S}_G Y$ from a special $\mathscr{F}_G$-G-space Y. We think of $Y=\mathbb{P}X$ or, equivalently, $X=\mathbb{U}Y$. When G is finite, $\mathbb{S}_G X$ is a (genuine) positive Ω-G-spectrum whose bottom structural G-map is a group completion of X_1. Prolongation of $B(\mathscr{F},\mathscr{F},X)$ to a $\mathscr{W}_G$-G-space does not give a positive Ω-G-spectum; prolongation of $B(\mathscr{F}_G,\mathscr{F}_G,Y)$ to a $\mathscr{W}_G$-G-space does.

The following definition gives a modernized version of Shimakawa's equivariant Segal machine [54].[14] The strange looking notation id_* anticipates a later generalization. Recall Notation 2.28.

Definition 3.15. Write $\mathrm{id}_* Y = B(\mathscr{F}_G,\mathscr{F}_G,Y)$ for an $\mathscr{F}_G$-G-space Y; thus id_* is a functor $\mathrm{Fun}(\mathscr{F}_G,\mathscr{T}_G)\longrightarrow \mathrm{Fun}(\mathscr{F}_G,\mathscr{T}_G)$. For a compact Lie group G, the Segal machine $\mathbb{S}_G$ on $\mathscr{F}_G$-G-spaces is the composite

$$\mathrm{Fun}(\mathscr{F}_G,\mathscr{T}_G)\xrightarrow{\mathrm{id}_*}\mathrm{Fun}(\mathscr{F}_G,\mathscr{T}_G)\xrightarrow{\mathbb{P}}\mathrm{Fun}(\mathscr{W}_G,\mathscr{T}_G)\xrightarrow{\mathbb{U}_{G\mathscr{S}}}G\mathscr{S}.$$

More explicitly, taking $A=S^V$,

$$\mathbb{S}_G(Y)(V)=B((S^V)^\bullet,\mathscr{F}_G,Y)=(S^V)^\bullet\otimes_{\mathscr{F}_G}\mathrm{id}_* Y.$$

[14] Orthogonal G-spectra had not been developed when [54] was written; he worked with Lewis-May G-spectra.

The Segal machine $\mathbb{S}_G$ on $\mathscr{F}$-G-spaces X is defined by

$$\mathbb{S}_G X = \mathbb{S}_G \mathbb{P} X.$$

The definition makes sense for any X. When X is special, Proposition 3.14 shows that the underlying naive G-prespectrum of $\mathbb{S}_G X$ is equivalent to $\mathbb{S}_G^N X$ hence is a positive Ω-G-prespectrum with bottom structural map a group completion of X_1.

Remark 3.16. The group completion property is not easy to see directly from the definition of $\mathbb{S}_G$. Shimakawa's strategy [56, p 357], not fully detailed, was to show that for $H \subset G$, Woolfson's version of the nonequivariant Segal machine $\mathbb{S}(Y^H)$ is equivalent to $(\mathbb{S}_G Y)^H$, where Y^H is the composite of restriction to $\mathscr{F}$ and the H-fixed point functor, so that $(Y^H)_n = Y(\mathbf{n})^H$, and then to quote the equivalence of Woolfson's version with Segal's original version. With our proof, the equivalence on fixed points follows formally from the group completion property, as is shown quite generally in [12, Theorem 2.20].

We are primarily interested in understanding $\mathbb{S}_G X$ when G is finite and X is $\mathbb{F}_\bullet$-special. However, the following variant of the standard notion of a linear functor (compare [5, 25]) makes sense for any topological group G. Recall that $G\mathscr{W}$ and $\mathscr{W}_G$ are the categories of based G-CW complexes whose respective morphisms are based G-maps and all based maps, with G acting by conjugation.

Definition 3.17. A $\mathscr{W}_G$-G-space Z is *positive linear* if for any G-connected A and any G-map $f\colon A \longrightarrow B$ in $G\mathscr{W}$,

$$Z(A) \xrightarrow{f_*} Z(B) \xrightarrow{i_*} Z(Cf)$$

is a fibration sequence of based G-spaces, where $i\colon B \longrightarrow Cf$ is the cofiber of f. That is, the induced map from $Z(A)$ to the homotopy fiber of i_* is a weak G-equivalence.

The "positive" refers to the assumption that A is G-connected.

In §9.3, we adapt and extend nonequivariant arguments of Segal and Woolfson [51, 59] to prove the following result, which applies to a general topological group G and is of independent interest. It is perhaps surprising that we only need X to be special, not $\mathbb{F}_\bullet$-special, for the first statement and that we do not know how to derive either statement from the other. However, we will only make use of the second statement in this paper.

Theorem 3.18. *Let G be a topological group. If X is a special $\mathscr{F}$-G-space, then the $\mathscr{W}_G$-G-space that sends A to $B(A^\bullet, \mathscr{F}, X)$ is positive linear. If Y is a special $\mathscr{F}_G$-G-space, such as $\mathbb{P}X$ for an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space X, then $B(A^\bullet, \mathscr{F}_G, Y)$ is positive linear.*

Now let G be finite. We want to understand the structure maps of the genuine G-spectrum $\mathbb{S}_G X$. The following result is closely related to nonequivariant results in [8, 25, 46, 51, 59] and equivariant results of Segal [52] and Shimakawa [54]. Our formulation is a slight variant of the specialization to finite groups G of a result of Blumberg [5, Theorem 1.2] about $\mathscr{W}_G$-G-spaces for compact Lie groups G.

Theorem 3.19. *Let G be finite and let Z be a positive linear $\mathscr{W}_G$-G-space such that the restriction of Z to $\mathscr{F}_G$ is a special $\mathscr{F}_G$-G-space. Then $\mathbb{U}_{G\mathscr{S}} Z$ is a positive Ω-G-spectrum.*

Remark 3.20. This result depends on the Wirthmüller isomorphism, and the proof of its specialization to finite groups can be simplified quite a bit by use of the simplified proof of that result in [34]. In turn, that depends implicitly on Atiyah duality for finite based G-sets and, as explained in [13, §3.2], the map ε of Definition 2.26 plays a central role in that. This ties the proof of Theorem 3.19 to the Segal map $Y(G/H) \longrightarrow \mathscr{T}_G(G/H, Y_1)$; compare [5, Remark 3.18].

Remark 3.21. Clearly Theorem 3.19 applies to $Z = \mathbb{P}X$ when X is $\mathbb{F}_\bullet$-special. Despite Theorem 3.18, we have no such conclusion when X is only special.

Here now is the fundamental theorem about the Segal machine for finite groups.

Theorem 3.22. *Let G be finite and let X be an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space. Then $\mathbb{S}_G X$ is a positive Ω-G-spectrum. Moreover, if $V^G \neq 0$, then the composite*

$$X_1 \longrightarrow B(\mathscr{F}_G, \mathscr{F}_G, \mathbb{P}X)_1 = (\mathbb{S}_G X)(S^0) \longrightarrow \Omega^V(\mathbb{S}_G X)(S^V)$$

of η_1 and the structure G-map is a group completion.

Proof. Let Z be the $\mathscr{W}_G$-G-space $B((-)^\bullet, \mathscr{F}_G, \mathbb{P}X)$. Then Z is positive linear by Theorem 3.18 and its restriction to $\mathscr{F}$ is an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space by Corollary 3.12. Therefore Theorem 3.19 implies the first statement, and the second follows from Proposition 3.14 and Remark 1.20. □

3.4. **Change of groups and compact Lie groups.** We summarize our conclusions. For any topological group G, we have a functor $\mathbb{S}_G^N$ that takes $\mathscr{F}$-G-spaces to G-prespectra. It has four variants. The first uses Segal's original simplicially defined inductive machine. The second is the conceptual machine and the third and fourth are composites that first take $\mathscr{F}$-G-spaces to $\mathscr{W}_G$-G-spaces by one of two choices of a bar construction and then take $\mathscr{W}_G$-G-spaces to G-prespectra.

We restrict attention to special $\mathscr{F}$-G-spaces X, reduced as always, for clarity in this digressive section. Then the four choices are equivalent and the functor $\mathbb{S}_G^N$ assigns a positive naive Ω-G-spectrum to X, together with a group completion $\eta\colon X_1 \simeq (\mathbb{S}_G^N X)_0 \longrightarrow \Omega(\mathbb{S}_G^N X)_1$.

If H is a subgroup of G and we write $\iota\colon H \longrightarrow G$ for the inclusion, then we have various functors ι^* that restrict given G actions to H actions. By inspection, these functors commute with all constructions in sight. Therefore $\iota^*\mathbb{S}_G^N X \cong \mathbb{S}_H^N \iota^* X$. In fact, these functors ι^* are Quillen right adjoints with respect to the various model structures in [5, 24, 25] on our categories. Moreover, they also commute with the group completion maps η, so that $\eta\iota^* = \iota^*\eta\colon \iota^* X_1 \longrightarrow (\mathbb{S}_H^N \iota^* X)_1$.

For finite groups G, if X is $\mathbb{F}_\bullet$-special rather than just special, we have a positive genuine Ω-G-spectrum $\mathbb{S}_G X$ with underlying naive Ω-G-spectrum $\mathbb{S}_G^N X$. Formally we have a forgetful functor from genuine orthogonal G-spectra indexed on a complete G-universe to naive G-spectra indexed on the trivial universe, and this functor is a Quillen right adjoint with respect to to the various model structures in [5, 24, 25]. Restriction to subgroups works the same way on the level of genuine G-spectra as it does on the level of naive G-spectra. Note that if H is a finite subgroup of a topological group G and X is an $\mathscr{F}$-G-space which is $\mathbb{F}_\bullet$-special as an $\mathscr{F}$-H-space, then $\iota^*\mathbb{S}_G^N X \cong \mathbb{S}_H^N \iota^* X$ is the underlying naive Ω-H-spectrum of a genuine Ω-H-spectrum.

Now let G be a compact Lie group. Just assuming that X is special, our construction still gives a genuine G-spectrum $\mathbb{S}_G X$ whose underlying naive G-prespectrum is the Ω-G-spectrum $\mathbb{S}_G^N X$, and we still have the group completion

$\eta\colon X_1 \longrightarrow \Omega(\mathbb{S}_G X)_1$. However, in contrast to the case of finite groups, even if X is $\mathbb{F}_\bullet$-special, $\mathbb{S}_G X$ need not be a positive genuine Ω-G-spectrum. Let $\mathbb{R}$ denote fibrant approximation in the positive stable model structure on G-spectra. We agree to replace $\mathbb{S}_G X$ by its fibrant approximation $\mathbb{R}\mathbb{S}_G X$, and we regard the new $\mathbb{S}_G X$ as the genuine G-spectrum constructed from the $\mathscr{F}$-G-space X. Its underlying naive G-spectrum is positive fibrant, and its structure maps $(\mathbb{S}_G X)_0 \longrightarrow \Omega^V(\mathbb{S}_G X)(V)$ are group completions when $V^G \neq 0$. However, they are not group completions of the originally given X_1, as we now explain in a model theoretic framework. Nevertheless, even though we know little about how its homotopy type relates to X_1, we view the new $\mathbb{S}_G$ as the best equivariant infinite loop space machine we can hope for when G is a compact Lie group.

The adjoint pair relating $\mathscr{W}_G$-G-spaces to orthogonal G-spectra is a Quillen equivalence even when G is a compact Lie group, as noted in [5, Theorem A.13]. However, consider the prolongation functor $\mathbb{P}$ from $\mathscr{F}$-G-spaces to $\mathscr{W}_G$-G-spaces and its right adjoint $\mathbb{U}$. We have noted that the unit of the adjunction is the identity. When G is finite, we have a complementary result about the counit. It is best expressed model theoretically, and we digress to summarize relevant background.

The absolute stable model structure on the category $\mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G)$ is defined in [5, §A.4]. It starts with the absolute level model structure in which a map $Y \longrightarrow Z$ is a fibration or weak equivalence if each $Y(A) \longrightarrow Z(A)$ is a Serre G-fibration or a weak G-equivalence. The absolute stable model structure has the same cofibrations, but a map is a weak equivalence if it is a stable equivalence in the sense that the underlying map of G-prespectra is a π_*-isomorphism. The fibrant $\mathscr{W}_G$-G-spaces Z are those for which the maps $Z(A) \longrightarrow \Omega^V Z(A \wedge S^V)$ of (I.3.4) are weak G-equivalences for all $A \in \mathscr{W}_G$ and all G-representations V. There is a Quillen equivalent stable model structure with the same weak equivalences, and it is Quillen equivalent to the category of orthogonal G-spectra.

It has long been understood[15] that the categories $\mathrm{Fun}(\mathscr{F}, \mathscr{T}_G)$ and $\mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$ admit stable model structures such that the three pairs $(\mathbb{P}, \mathbb{U})$ in sight are Quillen adjunctions. The proofs are similar to those of [25, §18] and [50]. More recent relevant expositions are in [44, 48, 49]. We start with the level model structures, which are defined in the same way as for $\mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G)$. The stable equivalences are the maps f such that $\mathbb{P}f$ is a stable equivalence of $\mathscr{W}_G$-G-spaces. The fibrant objects are the grouplike $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-spaces or the grouplike special $\mathscr{F}_G$-G-spaces.[16]

Proposition 3.23. *Let G be finite. Let Y be a positive linear $\mathscr{W}_G$-G-space whose underlying $\mathscr{F}_G$-G-space is special. Let $\lambda\colon X \longrightarrow \mathbb{U}Y$ be a bifibrant approximation of the underlying $\mathscr{F}$-G-space $\mathbb{U}Y$ in the stable model structure on $\mathscr{F}$-G-spaces. Then the composite of $\mathbb{P}\lambda$ and the counit ε is a stable equivalence $\mathbb{P}X \longrightarrow \mathbb{P}\mathbb{U}Y \longrightarrow Y$.*

Proof. This is proven nonequivariantly in [25, Lemma 18.10], and we can mimic the argument indicated there. As in [25, Proposition 18.8], $\mathbb{P}X$ is positive linear. Therefore, since G is finite, Theorem 3.19 implies that, after applying $\mathbb{U}_{G\mathscr{F}}$ to the composite, we obtain a map of connective orthogonal Ω-G-spectra which is a weak G-equivalence on 0th spaces and is therefore a stable equivalence. □

As in [25, Theorem 0.10], this implies that $(\mathbb{P}, \mathbb{U})$ is a connective Quillen equivalence between $\mathscr{F}$-G-spaces and $\mathscr{W}_G$-G-spaces, that is, a Quillen adjoint pair that

[15] The paper [5] deferred exposition, which would have been digressive there.

[16] Note that the group completion property is invisible from the model theoretic perspective.

induces an equivalence between the respective homotopy categories of connective objects. These conclusions do not generalize to compact Lie groups G. As was noted by Blumberg [5, Appendix C], following Segal [52], they already fail when $G = S^1$. In fact, taking $G = S^1$, Blumberg concludes that no reasonable condition on an $\mathscr{F}$-G-space can imply that $\mathbb{U}_{G\mathscr{S}}\mathbb{P}X$ is a positive Ω-G-spectrum. However, there was no reason to expect any such result. From the point of view of constructing genuine G-spectra with good properties from naturally occurring space level data, our construction works as well as can be expected. However, unlike the case of finite groups, there is no reasonable specification of space level data on $\mathscr{F}$-G-spaces sufficient to construct all connective G-spectra. We cannot even expect to construct suspension G-spectra since we do not have an analog of the Barratt-Priddy-Quillen theorem for compact Lie groups.

4. The generalized Segal machine

Unless otherwise specified, we take our group G to be **finite** from here on out.

The input of the Segal infinite loop space machine looks nothing like the input of the operadic machine. To compare them, we must generalize the natural input of both to obtain common input to which generalizations of both machines apply. We explain the generalized Segal machine in this section, postponing consideration of operads to the next. We define two equivariant versions of the categories of operators introduced in [38]. One version has finite sets as objects, the other finite G-sets, generalizing $\mathscr{F}$ and $\mathscr{F}_G$ respectively. As in §2.4, we show how to construct examples of the second kind from examples of the first kind.

After defining what it means for a G-category of operators to be an E_∞ G-category of operators, we generalize the homotopical version of the Segal machine by generalizing its input from $\mathscr{F}$-G-spaces to $\mathscr{D}$-G-spaces, where $\mathscr{D}$ is any E_∞ G-category of operators over $\mathscr{F}$. We compare the $\mathscr{D}$ and $\mathscr{D}_G$ machines to the $\mathscr{F}$ and $\mathscr{F}_G$ machines by proving that they have equivalent inputs and that they produce equivalent output when fed equivalent input. Thus the increased generality is more apparent than real. The point of the generalization is that operadic data feed naturally into the $\mathscr{D}$-G-space rather than the $\mathscr{F}$-G-space machine. We reiterate that the categorical input data of the sequels [15, 16] is intrinsically operadic.

4.1. G-categories of operators $\mathscr{D}$ over $\mathscr{F}$ and $\mathscr{D}$-G-spaces.

Definition 4.1. A G-category of operators $\mathscr{D}$ over $\mathscr{F}$, abbreviated *G-CO* over $\mathscr{F}$, is a category enriched in $G\mathscr{T}$ whose objects are the based sets $\mathbf{n}$ for $n \geq 0$ together with G-functors

$$\Pi \xrightarrow{\iota} \mathscr{D} \xrightarrow{\xi} \mathscr{F}$$

such that ι and ξ are the identity on objects and $\xi \circ \iota$ is the inclusion. Here G acts trivially on Π and $\mathscr{F}$. We say that $\mathscr{D}$ is reduced if $\mathscr{D}(\mathbf{m}, \mathbf{n})$ is a point if either $m = 0$ or $n = 0$, and we restrict attention to reduced *G-COs* over $\mathscr{F}$ henceforward. A morphism $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of *G-CO*s over $\mathscr{F}$ is a $G\mathscr{T}$-functor over $\mathscr{F}$ and under Π.

Remark 4.2. We have omitted cofibration conditions that will be added later, since what we need is a bit different for the Segal and the operadic machines. See Remark 4.16 for the Segal and Remark 6.15 for the operadic machine. The purpose of these conditions is to ensure that all bar constructions in sight are realizations of Reedy cofibrant simplicial G-spaces, as claimed in Remark 1.12.

Since we have morphism G-spaces $\mathscr{D}(\mathbf{m},\mathbf{n})$ such that composition is given by equivariant maps, we have G-fixed identity elements, and we have maps

$$\Pi(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{F}(\mathbf{m},\mathbf{n})$$

whose composite is the inclusion. Note that $\Pi(\mathbf{m},\mathbf{n})$ is contained in $\mathscr{D}(\mathbf{m},\mathbf{n})^G$. Of course, $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ is a map of *G-COs* over $\mathscr{F}$ for any *G-CO* $\mathscr{D}$ over $\mathscr{F}$.

Recall that $\mathscr{T}_G$ denotes the $G\mathscr{T}$-category of nondegenerately based G-spaces and all based G-maps, with G-acting by conjugation.

Definition 4.3. A $\mathscr{D}$-G-space X is a $G\mathscr{T}$-functor $X\colon \mathscr{D} \longrightarrow \mathscr{T}_G$. A map of $\mathscr{D}$-G-spaces is a $G\mathscr{T}$-natural transformation. Composing X with $\iota\colon \Pi \longrightarrow \mathscr{D}$ gives X an underlying Π-G-space. We say that X is $\mathbb{F}_\bullet$-special if its underlying Π-G-space is $\mathbb{F}_\bullet$-special. We say that a map of $\mathscr{D}$-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if its underlying map of Π-G-spaces is an $\mathbb{F}_\bullet$-level equivalence. Let $\mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$ denote the category of $\mathscr{D}$-G-spaces.

4.2. ***G*-categories of operators $\mathscr{D}_G$ over $\mathscr{F}_G$ and $\mathscr{D}_G$-*G*-spaces.** We can further generalize the input, following §2.4.

Definition 4.4. A G-category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$, abbreviated *G-CO* over $\mathscr{F}_G$, is a category enriched in $G\mathscr{T}$ with objects the based G-sets $(\mathbf{n},\alpha)$ for $n \geq 0$ and $\alpha\colon G \longrightarrow \Sigma_n$, together with G-functors

$$\Pi_G \xrightarrow{\iota_G} \mathscr{D}_G \xrightarrow{\xi_G} \mathscr{F}_G$$

such that ι_G and ξ_G are the identity on objects and $\xi_G \circ \iota_G$ is the inclusion. We say that $\mathscr{D}_G$ is reduced if $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$ is a point if either $m = 0$ or $n = 0$, and we restrict attention to reduced *G-COs* over $\mathscr{F}_G$ henceforward. A morphism $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$ of *G-CO*s over $\mathscr{F}_G$ is a $G\mathscr{T}$-functor over $\mathscr{F}_G$ and under Π_G.

Remark 4.5. As in Remark 4.2, we have omitted the cofibration condition specified in Remark 4.16 and the evident analogue of the cofibration condition specified in Remark 6.15.

Of course, G acts non-trivially on the morphism sets of Π_G and $\mathscr{F}_G$. We have morphism G-spaces $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$ such that composition is given by G-maps, we have G-fixed identity elements, and we have G-maps

$$\Pi_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \longrightarrow \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \longrightarrow \mathscr{F}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$$

whose composite is the inclusion. Again, $\xi_G\colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$ is a map of *G-COs* over $\mathscr{F}_G$ for any *G-CO* $\mathscr{D}_G$ over $\mathscr{F}_G$.

Regarding sets $\mathbf{n}$ as G-trivial G-sets, we have the following observation.

Lemma 4.6. *The full subcategory $\mathscr{D}$ with objects $\mathbf{n}$ of a G-category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$ is a G-category of operators over $\mathscr{F}$.*

Conversely, just as we constructed Π_G and $\mathscr{F}_G$ from Π and $\mathscr{F}$, we can construct a *G-CO* $\mathscr{D}_G$ over $\mathscr{F}_G$ from any *G-CO* $\mathscr{D}$ over $\mathscr{F}$. We shall only be interested in those $\mathscr{D}_G$ that are constructed in this fashion.

Construction 4.7. Let $\mathscr{D}$ be a G-category of operators over $\mathscr{F}$. We define a G-category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$ whose full subcategory of objects $\mathbf{n}$ is $\mathscr{D}$. The morphism G-space $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$ is the space $\mathscr{D}(\mathbf{m},\mathbf{n})$, with G-action induced by conjugation and the original G-action on $\mathscr{D}(\mathbf{m},\mathbf{n})$. Explicitly, for $f \in \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$,

$$g \cdot f = \beta(g) \circ (gf) \circ \alpha(g^{-1});$$

We check that $g\cdot(h\cdot f) = (gh)\cdot f$ using that G acts trivially on permutations since they are in the image of Π. Composition and identity maps are inherited from $\mathscr{D}$ and are appropriately equivariant.

A routine verification shows the following.

Lemma 4.8. *The inclusion $\mathscr{D} \hookrightarrow \mathscr{D}_G$ makes the following diagram of $G\mathscr{T}$-categories commute.*

$$\begin{array}{ccccc} \Pi & \longrightarrow & \mathscr{D} & \longrightarrow & \mathscr{F} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Pi_G & \longrightarrow & \mathscr{D}_G & \longrightarrow & \mathscr{F}_G. \end{array} \tag{4.9}$$

Moreover, a map $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of G-COs over $\mathscr{F}$ induces a map

$$\nu_G \colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$$

of G-COs over $\mathscr{F}_G$, which is compatible with the inclusions.

Definition 4.10. A $\mathscr{D}_G$-G-space Y is a $G\mathscr{T}$-functor $Y\colon \mathscr{D}_G \longrightarrow \mathscr{T}_G$. A map of $\mathscr{D}_G$-G-spaces is a $G\mathscr{T}$-natural transformation. Composing Y with $\iota_G\colon \Pi_G \longrightarrow \mathscr{D}_G$ gives Y an underlying Π_G-G-space. We say that Y is special if its underlying Π_G-G-space is special. We say that a map of $\mathscr{D}_G$-G-spaces is a level G-equivalence if its underlying map of Π_G-G-spaces is a level G-equivalence. Let $\mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G)$ denote the category of $\mathscr{D}_G$-G-spaces.

4.3. **The equivalence between** $\mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$ **and** $\mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G)$**.** We can now generalize §2.4 to a comparison between $\mathscr{D}$-G-spaces and $\mathscr{D}_G$-G-spaces. The forgetful functor

$$\mathbb{U}\colon \mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$$

has a left adjoint prolongation functor

$$\mathbb{P}\colon \mathrm{Fun}(\mathscr{D},\mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G).$$

Explicitly,

$$(\mathbb{P}X)(\mathbf{n},\alpha) = \mathscr{D}_G(-,(\mathbf{n},\alpha)) \otimes_{\mathscr{D}} X = \bigvee_m \mathscr{D}_G(\mathbf{m},(\mathbf{n},\alpha)) \wedge X_m/\sim,$$

where $(f,\phi_* x) \sim (f\phi, x)$ for a map $\phi\colon \mathbf{k} \longrightarrow \mathbf{m}$ in $\mathscr{D}$, an element $x \in X_k$, and a map $f\colon \mathbf{m} \longrightarrow (\mathbf{n},\alpha)$ in $\mathscr{D}_G(\mathbf{m},(\mathbf{n},\alpha))$. (We have written out this coequalizer of G-spaces explicitly to facilitate checks of details). The following result generalizes Theorem 2.30 from $\mathscr{F}$ to an arbitrary G-CO over $\mathscr{F}$.

Theorem 4.11. *The adjoint pair of functors*

$$\mathrm{Fun}(\mathscr{D},\mathscr{T}_G) \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G)$$

specifies an equivalence of categories.

The proof is very similar to that of the special case $\mathscr{D} = \mathscr{F}$ dealt with in Theorem 2.30. Only points of equivariance require comment, and the following lemma is the key to understanding the relevant G-actions. It identifies the G-space $(\mathbb{P}X)(\mathbf{n},\alpha)$ with the space X_n with a new G-action induced by α and the original G-action on X_n. We denote this G-space X_n^α.

Lemma 4.12. *For a $\mathscr{D}$-G-space X, the G-space $(\mathbb{P}X)(\mathbf{n},\alpha)$ is G-homeomorphic to the G-space X_n^α, namely X_n with the G-action $\cdot_\alpha$ specified by $g\cdot_\alpha x = \alpha(g)_*(gx)$. Via this homeomorphism, the evaluation maps*

$$\mathscr{D}_G((\mathbf{n},\alpha),(\mathbf{p},\beta)) \wedge (\mathbb{P}X)(\mathbf{n},\alpha) \longrightarrow (\mathbb{P}X)(\mathbf{p},\beta)$$

are given on the underlying spaces by the corresponding maps for X,

$$\mathscr{D}(\mathbf{n},\mathbf{p}) \wedge X_n \longrightarrow X_p.$$

Proof. Modulo equivariance, this is an application of the Yoneda lemma. Write $\mathrm{id}_\alpha\colon \mathbf{n} \longrightarrow (\mathbf{n},\alpha)$ for $\mathrm{id}\colon \mathbf{n}\longrightarrow\mathbf{n}$ regarded as an element of $\mathscr{D}_G(\mathbf{n},(\mathbf{n},\alpha))$. Define

$$F\colon X_n^\alpha \longrightarrow \mathscr{D}_G(-,(\mathbf{n},\alpha))\otimes_{\mathscr{D}} X$$

by sending $x\in X_n$ to the equivalence class of (id_α, x). Then F is a G-map since

$$F(g\cdot_\alpha x) = (\mathrm{id}_\alpha, g\cdot_\alpha x) = (\mathrm{id}_\alpha, \alpha(g)_*(gx)) \sim (\mathrm{id}_\alpha\circ\alpha(g), gx) = (g\,\mathrm{id}_\alpha, gx).$$

Define an inverse map

$$F^{-1}\colon \mathscr{D}_G(-,(\mathbf{n},\alpha))\otimes_{\mathscr{D}} X \longrightarrow X_n^\alpha$$

by sending the equivalence class of $(f,x)\in\mathscr{D}_G(\mathbf{m},(\mathbf{n},\alpha))\times X_m$ to $f_*(x)$, where we think of f as a map $\mathbf{m}\longrightarrow\mathbf{n}$ in $\mathscr{D}$ and interpret $f_*(x)$ to mean $X(f)(x)\in X_n$. Note that F^{-1} is well defined. We have

$$F^{-1}F(x) = F^{-1}(\mathrm{id}_\alpha, x) = \mathrm{id}_* x = x$$

and

$$FF^{-1}(f,x) = F(f_*x) = (\mathrm{id}_\alpha, f_*x) \sim (f,x),$$

hence F and F^{-1} are inverse homeomorphisms. Note that F^{-1} is automatically a G-map since it is inverse to the G-map F. The compatibility with the $\mathscr{D}_G$-G-space structure is clear. □

Using this, we mimic the proof of Theorem 2.30 to prove the equivalence of the categories of $\mathscr{D}$-G-spaces and $\mathscr{D}_G$-G-spaces.

Proof of Theorem 4.11. Clearly, since the inclusion $\mathscr{D}\longrightarrow\mathscr{D}_G$ is full and faithful, the unit $X\longrightarrow\mathbb{U}\mathbb{P}X$ of the adjunction is an isomorphism for any $\mathscr{D}$-G-space X. Let Y be a $\mathscr{D}_G$-G-space. We must show that the counit $\mathbb{P}\mathbb{U}Y\longrightarrow Y$ of the adjunction is an isomorphism. A check of definitions shows that the counit G-map $(\mathbb{P}\mathbb{U}Y)(\mathbf{n},\alpha)\longrightarrow Y(\mathbf{n},\alpha)$ agrees under the isomorphism of Lemma 4.12 with the map, necessarily a G-map,

$$\mathrm{id}_{\alpha *}\colon Y_n^\alpha \longrightarrow Y(\mathbf{n},\alpha)$$

induced by the morphism $\mathrm{id}_\alpha\in\mathscr{D}_G(\mathbf{n},(\mathbf{n},\alpha))$. Writing ${}_\alpha\mathrm{id}\colon(\mathbf{n},\alpha)\longrightarrow\mathbf{n}$ for $\mathrm{id}\colon\mathbf{n}\longrightarrow\mathbf{n}$ regarded as an element of $\mathscr{D}_G((\mathbf{n},\alpha),\mathbf{n})$, we see that ${}_\alpha\mathrm{id}$ induces the inverse homeomorphism

$${}_\alpha\mathrm{id}_*\colon Y(\mathbf{n},\alpha)\longrightarrow Y_n^\alpha$$

to $\mathrm{id}_{\alpha *}$. Again, ${}_\alpha\mathrm{id}_*$ is automatically a G-map since it is inverse to a G-map. □

Just as for $\mathscr{F}$-G-spaces in §2.4, a $\mathscr{D}$-G-space X has two Π_G-G-spaces associated to it. We can either apply $\mathbb{P}$ to its underlying Π-G-space or we can apply $\mathbb{P}$ to X and take its underlying Π_G-G-space. The proof of Theorem 4.11 implies that these two Π_G-G-spaces coincide. Therefore the four statements about Π and Π_G that are listed in Theorem 2.30 also hold for $\mathscr{D}$ and $\mathscr{D}_G$. We record them in the following two corollaries.

Corollary 4.13. *A $\mathscr{D}$-G-space X is $\mathbb{F}_\bullet$-special if and only if the $\mathscr{D}_G$-G-space $\mathbb{P}X$ is special. A $\mathscr{D}_G$-G-space Y is special if and only if the $\mathscr{D}$-G-space $\mathbb{U}Y$ is $\mathbb{F}_\bullet$-special.*

Corollary 4.14. *A map f of $\mathscr{D}$-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only $\mathbb{P}f$ is a level G-equivalence of $\mathscr{D}_G$-G-spaces. A map f of $\mathscr{D}_G$-G-spaces is a level G-equivalence if and only if $\mathbb{U}f$ is an $\mathbb{F}_\bullet$-level equivalence of $\mathscr{D}$-G-spaces.*

4.4. **Comparisons of $\mathscr{D}$-G-spaces and $\mathscr{E}$-G-spaces for $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$.** Let X be an $\mathscr{F}$-G-space. Then X and the $\mathscr{D}$-G-space $\xi^*X = X\circ\xi\colon \mathscr{D} \longrightarrow \mathscr{T}_G$ have the same underlying Π-G-space, hence one is $\mathbb{F}_\bullet$-special or special if and only if the other is so. The left adjoint of $\xi^*\colon \mathrm{Fun}(\mathscr{F},\mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$ is given by the evident left Kan extension along $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$. Following [38, Theorem 1.8] nonequivariantly, we expect the bar construction to give a homotopically well-behaved variant. With $\mathscr{F}$ replaced by $\mathscr{D}$, the analogue of Proposition 3.9 holds and admits the same proof.

To implement this strategy, we start with an $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-space Y and construct from it an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space ξ_*Y together with a zigzag of $\mathbb{F}_\bullet$-equivalences between Y and $\xi^*\xi_*Y$. We shall use this to construct a Segal machine whose input is an $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-space Y and whose output is equivalent to $\mathbb{S}_G\xi_*Y$.

As in [38], we work more generally here, starting from a map $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of G-COs over $\mathscr{F}$ and comparing $\mathscr{D}$-G-spaces and $\mathscr{E}$-G-spaces. We are mainly interested in the case $\nu = \xi$. We write $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$ for the induced map of G-COs over $\mathscr{F}_G$. Focus on ν_G rather than ν allows us to focus on G-equivalence rather than $\mathbb{F}_\bullet$-equivalence. For clarity, we sometimes write $\mathbb{U}_\mathscr{D}$ and $\mathbb{P}_\mathscr{D}$ instead of $\mathbb{U}$ and $\mathbb{P}$ for the adjunction between $\mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$ and $\mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G)$, and similarly for $\mathscr{E}$.

Definition 4.15. For $Z\in \mathrm{Fun}(\mathscr{D}_G,\mathscr{T}_G)$, define $\nu_{G*}Z \in \mathrm{Fun}(\mathscr{E}_G,\mathscr{T}_G)$ by

$$\nu_{G*}Z = B(\mathscr{E}_G,\mathscr{D}_G,Z).$$

Here the target is defined levelwise by replacing $\mathscr{E}_G$ by the composite

$$\mathscr{E}_G(-,(\mathbf{n},\alpha))\circ\nu_G\colon \mathscr{D}^{op} \longrightarrow \mathscr{T}_G,$$

of ν_G and the $G\mathscr{T}$-functor $\mathscr{E}_G^{op} \longrightarrow \mathscr{T}_G$ represented by $(\mathbf{n},\alpha)$. For $Y\in\mathrm{Fun}(\mathscr{D},\mathscr{T}_G)$, define $\nu_*Y\in \mathrm{Fun}(\mathscr{E},\mathscr{T}_G)$ by

$$\nu_*Y = \mathbb{U}_\mathscr{E}\nu_{G*}\mathbb{P}_\mathscr{D}Y = \mathbb{U}_\mathscr{E}B(\mathscr{E}_G,\mathscr{D}_G,\mathbb{P}_\mathscr{D}Y).$$

Remark 4.16. To ensure that the bar constructions we use are geometric realizations of Reedy cofibrant simplicial G-spaces, we require the unit maps

$$* \longrightarrow \mathscr{D}_G((\mathbf{n},\alpha),(\mathbf{n},\alpha))$$

of G-COs over $\mathscr{F}_G$ to be G-cofibrations, and similarly for G-COs over $\mathscr{F}$. This holds when $\mathscr{D}_G$ is constructed from a G-operad $\mathscr{C}$ such that $\mathrm{id}\colon * \longrightarrow \mathscr{C}(1)$ is a G-cofibration, as is true in our examples.

Definition 4.17. A map $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$ of G-COs over $\mathscr{F}_G$ is a G-equivalence if each map

$$\nu_G\colon \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \longrightarrow \mathscr{E}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$$

is a weak G-equivalence. A map $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ of G-COs over $\mathscr{F}$ is an $\mathbb{F}_\bullet$-equivalence if the associated map $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$ of G-COs over $\mathscr{F}_G$ is a G-equivalence.

Recall the notion of an $\mathbb{F}_\bullet$-level equivalence of Π-G-spaces from Definition 2.6 and Lemma 2.11. Recall too that a map of Π-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only if its associated map of Π_G-G-spaces is a level G-equivalence; see Definition 2.27 and Theorem 2.30(iv). These definitions and results are inherited by $\mathscr{D}$ and $\mathscr{D}_G$ (as in Corollary 4.14). The following definition recalls notation from §3.1.

Definition 4.18. For a G-CO $\mathscr{D}$ over $\mathscr{F}$ and a fixed n, let $\mathscr{D}_n$ be the corepresented $\mathscr{D}$-G-space specified by $\mathscr{D}_n(\mathbf{p}) = \mathscr{D}(\mathbf{n},\mathbf{p})$, with the action of $\mathscr{D}$ given by composition. Analogously, we have corepresented $\mathscr{D}_G$-G-spaces $\mathscr{D}_{G(\mathbf{n},\alpha)}$.

Lemma 4.19. *If $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ is an $\mathbb{F}_\bullet$-equivalence of G-COs over $\mathscr{F}$, then for each n, ν restricts to an $\mathbb{F}_\bullet$-level equivalence of $\mathscr{D}$-G-spaces $\mathscr{D}_n \longrightarrow \nu^*\mathscr{E}_n$.*

Proof. One can check that the cited restriction is a map of $\mathscr{D}$-G-spaces. Moreover, an easy comparison of definitions shows that $\mathbb{P}\mathscr{D}_n$ can be identified with $\mathscr{D}_{G\mathbf{n}}$. Therefore the conclusion follows from Corollary 4.14 and our definition of an $\mathbb{F}_\bullet$-equivalence of G-COs over $\mathscr{F}$, which of course was chosen in order to make this and cognate results true. □

Theorem 4.20. *Let $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{E}_G$ be a G-equivalence of G-COs over $\mathscr{F}_G$.*

(i) Let X be an $\mathscr{E}_G$-G-space and Y a $\mathscr{D}_G$-G-space. Then there are natural zigzags of level G-equivalences between $\nu_{G*}\nu_G^*X$ and X and between $\nu_G{}^*\nu_{G*}Y$ and Y.
(ii) Y is a special $\mathscr{D}_G$-G-space if and only if $\nu_{G*}Y$ is a special $\mathscr{E}_G$-G-space.
(iii) A map f of $\mathscr{D}_G$-G-spaces is a level G-equivalence if and only if $\nu_{G*}f$ is a level G-equivalence of $\mathscr{E}_G$-G-spaces.

Proof. Abbreviate notation here by writing α for a finite G-set $(\mathbf{n},\alpha)$ and X_α for $X(\mathbf{n},\alpha)$ when X is an $\mathscr{E}_G$-space. Starting with X, we have the natural maps

$$(4.21)\qquad (\nu_{G*}\nu_G{}^*X)_\alpha = B(\mathscr{E}_G,\mathscr{D}_G,\nu_G{}^*X)_\alpha \xrightarrow{B(\mathrm{id},\nu_G,\mathrm{id})} B(\mathscr{E}_G,\mathscr{E}_G,X)_\alpha \xrightarrow{\varepsilon} X_\alpha.$$

Starting with Y we have the natural maps

$$(4.22)\qquad (\nu_G{}^*\nu_{G*}Y)_\alpha = \nu_G{}^*B(\mathscr{E}_G,\mathscr{D}_G,Y)_\alpha \xleftarrow{B(\nu_G,\mathrm{id},\mathrm{id})} B(\mathscr{D}_G,\mathscr{D}_G,Y)_\alpha \xrightarrow{\varepsilon} Y_\alpha.$$

The maps ε with targets X_α and Y_α are G-equivalences, with the usual inverse equivalences η, as in the proof of Proposition 3.9(ii). At each level α, the other two maps are realizations of levelwise simplicial G-equivalences, and the Reedy cofibrancy of the simplicial bar construction ensures that these realizations are themselves G-equivalences, by Theorem 1.10. Note that we do not need X or Y to be special to prove (i).

By Lemma 2.33, (i) implies (ii). Indeed, Y is special if and only if $\nu_G{}^*\nu_{G*}Y$ is special and, since $\nu_G{}^*\nu_{G*}Y$ and $\nu_{G*}Y$ have the same underlying Π-G-space, one is special if and only if the other is so. Part (iii) follows from the naturality of the G-equivalences in (4.22). □

As in Lemma 4.8, the following diagram of $G\mathscr{T}$-categories commutes.

$$\begin{array}{ccccccc}
\Pi & \longrightarrow & \mathscr{D} & \xrightarrow{\nu} & \mathscr{E} & \xrightarrow{\xi} & \mathscr{F} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Pi_G & \longrightarrow & \mathscr{D}_G & \xrightarrow[\nu_G]{} & \mathscr{E}_G & \xrightarrow[\xi_G]{} & \mathscr{F}_G.
\end{array}$$

Therefore $\nu^*\mathbb{U}_{\mathscr{E}} = \mathbb{U}_{\mathscr{D}}\nu_G^*$, as we shall use in the proof of the following analogue of Theorem 4.20 for G-COs over $\mathscr{F}$.

Theorem 4.23. *Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be an $\mathbb{F}_\bullet$-equivalence of G-COs over $\mathscr{F}$.*

(i) Let X be an $\mathscr{E}$-G-space and Y be a $\mathscr{D}$-G-space. Then there are natural zigzags of $\mathbb{F}_\bullet$-equivalences between $\nu_*\nu^*X$ and X and between $\nu^*\nu_*Y$ and Y.

(ii) Y is an $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-space if and only if ν_*Y is an $\mathbb{F}_\bullet$-special $\mathscr{E}$-G-space.

(iii) A map f of $\mathscr{D}$-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only if ν_*f is an $\mathbb{F}_\bullet$-level equivalence of $\mathscr{E}$-G-spaces.

Proof. Recall from Theorem 4.11 that $(\mathbb{P}_{\mathscr{D}}, \mathbb{U}_{\mathscr{D}})$ is an adjoint equivalence of categories, and similarly for $\mathscr{E}$. Note that we have the following sequence of natural isomorphisms of $\mathscr{D}_G$-G-spaces

$$\mathbb{P}_{\mathscr{D}}\nu^*X \cong \mathbb{P}_{\mathscr{D}}\nu^*\mathbb{U}_{\mathscr{E}}\mathbb{P}_{\mathscr{E}}X = \mathbb{P}_{\mathscr{D}}\mathbb{U}_{\mathscr{D}}\nu_G^*\mathbb{P}_{\mathscr{E}}X \cong \nu_G^*\mathbb{P}_{\mathscr{E}}X.$$

To prove (i), write $\simeq$ to indicate a zigzag of $\mathbb{F}_\bullet$-level equivalences. Recall from Corollary 4.14 that $\mathbb{P}$ takes $\mathbb{F}_\bullet$-level equivalences to level G-equivalences and $\mathbb{U}$ takes level G-equivalences to $\mathbb{F}_\bullet$-level equivalences, while ν_{G*} preserves level G-equivalences by Theorem 4.20(iii). Therefore, by Theorem 4.20(i), we have the zigzags

$$\nu_*\nu^*X = \mathbb{U}_{\mathscr{E}}\nu_{G*}\mathbb{P}_{\mathscr{D}}\nu^*X \cong \mathbb{U}_{\mathscr{E}}\nu_{G*}\nu_G^*\mathbb{P}_{\mathscr{E}}X \simeq \mathbb{U}_{\mathscr{E}}\mathbb{P}_{\mathscr{E}}X \cong X$$

and

$$\nu^*\nu_*Y = \nu^*\mathbb{U}_{\mathscr{E}}\nu_{G*}\mathbb{P}_{\mathscr{D}}Y = \mathbb{U}_{\mathscr{D}}\nu_G^*\nu_{G*}\mathbb{P}_{\mathscr{D}}Y \simeq \mathbb{U}_{\mathscr{D}}\mathbb{P}_{\mathscr{D}}Y \cong Y$$

of level G-equivalences. Using Corollary 4.14, (ii) and (iii) follow as in the proof of Theorem 4.20. □

4.5. Comparisons of inputs and outputs of the generalized Segal machine.

Definition 4.24. We say that a G-CO $\mathscr{D}_G$ over $\mathscr{F}_G$ is an E_∞ G-CO if the map $\xi_G\colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$ is a G-equivalence. We say that a G-CO $\mathscr{D}$ over $\mathscr{F}$ is an E_∞ G-CO if its associated $\mathscr{D}_G$ is an E_∞ G-CO over $\mathscr{F}_G$.

The term "E_∞" is convenient, but it is a little misleading, as will become clear when we turn to operads.

We assume throughout this section that $\mathscr{D}$ is an E_∞ G-CO over $\mathscr{F}$, and we specialize the results of the previous section to $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ and $\varepsilon_G\colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$. The following results are just specializations of Theorems 4.20 and 4.23.

Theorem 4.25. *The following conclusions hold.*

(i) Let X be an $\mathscr{F}_G$-G-space and Y a $\mathscr{D}_G$-G-space. Then there are natural zigzags of level G-equivalences between $\xi_{G*}\xi_G^*X$ and X and between $\xi_G^*\xi_{G*}Y$ and Y.

(ii) Y is a special $\mathscr{D}_G$-G-space if and only if $\xi_{G*}Y$ is a special $\mathscr{F}_G$-G-space.

(iii) A map f of $\mathscr{D}_G$-G-spaces is a level G-equivalence if and only if $\xi_{G*}f$ is a level G-equivalence of $\mathscr{F}_G$-G-spaces.

Theorem 4.26. *The following conclusions hold.*

(i) Let X be an $\mathscr{F}$-G-space and Y a $\mathscr{D}$-G-space. Then there are natural zigzags of $\mathbb{F}_\bullet$-equivalences between $\xi_*\xi^*X$ and X and between $\xi^*\xi_*Y$ and Y.

(ii) Y is an $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-space if and only if ξ_*Y is an $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-space.

(iii) A map f of $\mathscr{D}$-G-spaces is an $\mathbb{F}_\bullet$-level equivalence if and only if ξ_*f is an $\mathbb{F}_\bullet$-level equivalence of $\mathscr{F}$-G-spaces.

Therefore the pair of functors

$$\xi_G^*\colon \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) \quad \text{and} \quad \xi_{G*}\colon \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$$

induce inverse equivalences between the homotopy categories obtained by inverting the respective level G-equivalences, and this remains true if we restrict to special objects. Similarly, the pair of functors

$$\xi^*\colon \mathrm{Fun}(\mathscr{F}, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) \quad \text{and} \quad \xi_*\colon \mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{F}, \mathscr{T}_G)$$

induce inverse equivalences between the homotopy categories obtained by inverting the respective $\mathbb{F}_\bullet$-level equivalences, and this remains true if we restrict to $\mathbb{F}_\bullet$-special objects. We conclude that the four input categories for Segal machines displayed in the square of the following diagram are essentially equivalent.

(4.27)
$$\begin{array}{ccccccc}
\mathrm{Fun}(\mathscr{F}, \mathscr{T}_G) & \xleftarrow{\mathbb{U}} & \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) & \xleftarrow{\mathbb{U}} & \mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G) & \xrightarrow{\mathbb{U}_{G\mathscr{S}}} & G\mathscr{S} \\
\Big\downarrow{\scriptstyle \xi^*} & & \Big\downarrow{\scriptstyle \xi_G^*} & & & & \\
\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) & \xleftarrow[\mathbb{U}]{} & \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) & & & &
\end{array}$$

Here $\xi^*\mathbb{U} = \mathbb{U}\xi_G^*$. We regard the four categories in the square as possible domain categories for generalized Segal infinite loop space machines.

The functors in the square are right adjoints. By Theorems 2.30 and 4.11, the inclusions of $\mathscr{F}$ in $\mathscr{F}_G$ and $\mathscr{D}$ in $\mathscr{D}_G$ induce equivalences of categories, denoted $\mathbb{U}$ in the diagram. Their left adjoints $\mathbb{P}$ give inverses, and the functors $\mathbb{U}$ and $\mathbb{P}$ preserve the relevant special objects and levelwise equivalences. Theorems 4.25 and 4.26 show that the vertical arrows ξ^* and ξ_G^* become equivalences of homotopy categories with inverses ξ_* and ξ_{G*} (not the left adjoints) after inverting the relevant equivalences. Consider the following diagram of functors.

(4.28)
$$\begin{array}{ccccccc}
\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) & \xrightarrow{\mathbb{P}} & \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) & & & & \\
\Big\downarrow{\scriptstyle \xi_*} & & \Big\downarrow{\scriptstyle \xi_{G*}} & & & & \\
\mathrm{Fun}(\mathscr{F}, \mathscr{T}_G) & \xrightarrow[\mathbb{P}]{} & \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) & \xrightarrow[\mathbb{P}]{} & \mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G) & \xrightarrow[\mathbb{U}_{G\mathscr{S}}]{} & G\mathscr{S}
\end{array}$$

The isomorphism $\mathbb{P}\mathbb{U} \cong \mathrm{id}$ on $\mathscr{F}_G$-G-spaces and the definitions of ξ_* and ξ_{G*} imply that the square commutes up to natural isomorphism. The composite in the bottom row is our original conceptual Segal machine. We can specialize by letting $\xi = \mathrm{id}\colon \mathscr{F} \longrightarrow \mathscr{F}$ and $\xi_G = \mathrm{id}\colon \mathscr{F}_G \longrightarrow \mathscr{F}_G$. According to Definition 3.15, the composite $\mathbb{U}_{G\mathscr{S}}\mathbb{P}\,\mathrm{id}_*$ starting at $\mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G)$ is the Segal machine $\mathbb{S}_G$ on $\mathscr{F}_G$-G-spaces Y and the composite $\mathbb{U}_{G\mathscr{S}}\mathbb{P}\,\mathrm{id}_*\,\mathbb{P}$ is the Segal machine on $\mathscr{F}$-G-spaces X. That is

(4.29)
$$\mathbb{S}_G Y = \mathbb{U}_{G\mathscr{S}}\mathbb{P}\,\mathrm{id}_*\, Y \quad \text{and} \quad \mathbb{S}_G X = \mathbb{U}_{G\mathscr{S}}\mathbb{P}\,\mathrm{id}_*\,\mathbb{P}X \cong \mathbb{U}_{G\mathscr{S}}\mathbb{P}\mathbb{P}\,\mathrm{id}_*\, X.$$

We regard the composites obtained by replacing the functors id_* with ξ_* and ξ_{G*} as generalized Segal machines $\mathbb{S}_G$ defined on $\mathscr{D}$-G-spaces X and $\mathscr{D}_G$-G-spaces Y. Explicitly, we define the corresponding orthogonal G-spectra to be

$$\mathbb{S}_G Y = \mathbb{U}_{G\mathscr{S}}\mathbb{P}\xi_{G*}Y \quad \text{and} \quad \mathbb{S}_G X = \mathbb{U}_{G\mathscr{S}}\mathbb{P}\xi_{G*}\mathbb{P}X \cong \mathbb{U}_{G\mathscr{S}}\mathbb{P}\mathbb{P}\xi_* X. \tag{4.30}$$

Clearly the machines starting with $\mathscr{F}$-G-spaces or $\mathscr{F}_G$-G-spaces are equivalent and similarly for $\mathscr{D}$ and $\mathscr{D}_G$. Theorem 4.32 below will show that the machines starting with $\mathscr{F}$-G-spaces or $\mathscr{D}$-G-spaces and the machines starting with $\mathscr{F}_G$-G-spaces or $\mathscr{D}_G$-G-spaces are equivalent. Thus the four machines in sight are equivalent under our equivalences of input data. That is, we obtain equivalent output by starting at any of the four vertices of the square, converting input data from the other three vertices to that one, and taking the relevant machine $\mathbb{S}_G$. We use the following invariance principle in the proof of Theorem 4.32.

Proposition 4.31. *The following conclusions about homotopy invariance hold.*

(i) If $f\colon X \longrightarrow Y$ is a level G-equivalence of $\mathscr{D}$-G-spaces, then

$$f\colon B(A^\bullet, \mathscr{D}, X) \longrightarrow B(A^\bullet, \mathscr{D}, Y)$$

is a weak G-equivalence for all $A \in G\mathscr{W}$.

(ii) If $f\colon X \longrightarrow Y$ is a level G-equivalence of $\mathscr{D}_G$-G-spaces, then the induced map

$$f\colon B(A^\bullet, \mathscr{D}_G, X) \longrightarrow B(A^\bullet, \mathscr{D}_G, Y)$$

is a weak G-equivalence for all $A \in G\mathscr{W}$.

(iii) If $f\colon X \longrightarrow Y$ is an $\mathbb{F}_\bullet$-level equivalence of $\mathscr{D}$-G-spaces, then the induced map

$$f\colon B(A^\bullet, \mathscr{D}_G, \mathbb{P}X) \longrightarrow B(A^\bullet, \mathscr{D}_G, \mathbb{P}Y)$$

is a weak G-equivalence for all $A \in G\mathscr{W}$.

Proof. By Remark 1.12 (see also Remark 4.16), our bar constructions are all geometric realizations of Reedy cofibrant simplicial G-spaces, hence Theorem 1.10 gives the first conclusion. The second statement follows similarly. The third follows from the second using that $\mathbb{P}f$ is a level G-equivalence of $\mathscr{D}_G$-G-spaces by Corollary 4.14. □

The limitations of the first part and need for the second are clear from the fact that $B(\mathscr{F}, \mathscr{D}, X)$ is only level G-equivalent, not $\mathbb{F}_\bullet$-level equivalent, to X.

Theorem 4.32. *The following four equivalences of outputs hold.*

(i) If X is a $\mathscr{D}$-G-space, then the G-spectra $\mathbb{S}_G X$ and $\mathbb{S}_G \xi_* X$ are equivalent.
(ii) If Y is a $\mathscr{D}_G$-G-space, then the G-spectra $\mathbb{S}_G Y$ and $\mathbb{S}_G \xi_{G*} Y$ are equivalent.
(iii) If X is an $\mathscr{F}$-G-space, then the G-spectra $\mathbb{S}_G X$ and $\mathbb{S}_G \xi^* X$ are equivalent.
(iv) If Y is an $\mathscr{F}_G$-G-space, then the G-spectra $\mathbb{S}_G Y$ and $\mathbb{S}_G {\xi_G}^* Y$ are equivalent.

Proof. We first prove (ii), which is the hardest part, and then show the rest. Thus let Y be a $\mathscr{D}_G$-G-space. We claim that the $\mathscr{W}_G$-G-spaces $\mathbb{P}\xi_{G*}Y$ and $\mathbb{P}\,\mathrm{id}_*\,\xi_{G*}Y$ are level equivalent. Applying $\mathbb{U}_{G\mathscr{S}}$, this will give (ii). Thus let $A \in \mathscr{W}_G$. Then

$$(\mathbb{P}\xi_{G*}Y)(A) = A^\bullet \otimes_{\mathscr{F}_G} B(\mathscr{F}_G, \mathscr{D}_G, Y) \cong B(A^\bullet, \mathscr{D}_G, Y)$$

and

$$(\mathbb{P}\,\mathrm{id}_*\,\xi_{G*}Y)(A) = A^\bullet \otimes_{\mathscr{F}_G} B(\mathscr{F}_G, \mathscr{F}_G, \xi_{G*}Y) \cong B(A^\bullet, \mathscr{F}_G, \xi_{G*}Y).$$

In both cases, the isomorphism comes from Remark 3.6. By Theorem 4.25(i) there is a natural zigzag of level G-equivalences between $\xi_G^*\xi_{G*}Y$ and Y. By Proposition 4.31(ii), this gives a zigzag of level G-equivalences of $\mathscr{D}_G$-G-spaces

$$B(A^\bullet, \mathscr{D}_G, Y) \simeq B(A^\bullet, \mathscr{D}_G, \xi_G^*\xi_{G*}Y).$$

Since $\xi_G \colon \mathscr{D}_G \longrightarrow \mathscr{F}_G$ is a G-equivalence of G-categories of operators, $B(\mathrm{id}, \xi_G, \mathrm{id})$ induces an equivalence at the level of q-simplices of the bar constructions

$$B(A^\bullet, \mathscr{D}_G, \xi_G^*\xi_{G*}Y) \longrightarrow B(A^\bullet, \mathscr{F}_G, \xi_{G*}Y).$$

Again, since these bar constructions are geometric realizations of Reedy cofibrant simplicial G-spaces, we get a weak G-equivalence on geometric realizations by Theorem 1.10. This proves the claim and thus proves (ii).

To prove (i), let X be a $\mathscr{D}$-G-space. Applying (ii) to $Y = \mathbb{P}X$ and using (4.29) and (4.30), we see that

$$\mathbb{S}_G X \cong \mathbb{S}_G \mathbb{P}X \simeq \mathbb{S}_G \xi_{G*}\mathbb{P}X \cong \mathbb{S}_G \mathbb{P}\xi_* X \cong \mathbb{S}_G \xi_* X.$$

To prove (iv), let Y be an $\mathscr{F}_G$-G-space. We claim that the $\mathscr{W}_G$-G-spaces $\mathbb{P}\,\mathrm{id}_* Y$ and $\mathbb{P}\xi_{G*}\xi_G^* Y$ are level equivalent. Here

$$(\mathbb{P}\,\mathrm{id}_* Y)(A) = A^\bullet \otimes_{\mathscr{F}_G} B(\mathscr{F}_G, \mathscr{F}_G, Y) \cong B(A^\bullet, \mathscr{F}_G, Y)$$

and

$$(\mathbb{P}\xi_{G*}\xi_G^* Y)(A) = A^\bullet \otimes_{\mathscr{F}_G} B(\mathscr{F}_G, \mathscr{D}_G, \xi_G^* Y) \cong B(A^\bullet, \mathscr{D}_G, \xi_G^* Y).$$

The isomorphisms again come from Remark 3.6. Just as before, $B(\mathrm{id}, \xi_G, \mathrm{id})$ induces a weak G-equivalence

$$B(A^\bullet, \mathscr{D}_G, \xi_G^* Y) \longrightarrow B(A^\bullet, \mathscr{F}_G, Y).$$

Finally, (iii) follows by application of (iv) to $Y = \mathbb{P}X$. □

Use of the generalized homotopical Segal machine will be convenient when we compare the Segal and operadic machines, but it is logically unnecessary. We could just as well replace $\mathscr{D}$-G-spaces Y by the $\mathscr{F}$-G-spaces $\xi_* Y$ and apply the Segal machine $\mathbb{S}_G$ on them. We have just shown that we obtain equivalent outputs from these two homotopical variants of the Segal machine. We conclude that all homotopical Segal machines in sight produce equivalent output when fed equivalent input. The resulting G-spectra are equivalent via compatible natural zigzags. We conclude that all of our machines are essentially equivalent, and they are all equivalent to our preferred machine $\mathbb{S}_G$ on $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-spaces.

5. From G-operads to G-categories of operators

We show here how operadic input, like Segalic input, can be generalized to G-categories of operators. This section, like the previous one, is based on the nonequivariant theory developed in [38], but considerations of equivariance require a little more work. We show how to construct G-categories of operators from G-operads in §5.1. We show that the construction takes E_∞ G-operads to E_∞ G-categories of operators, which is not obvious equivariantly, in §5.2.

5.1. **G-categories of operators associated to a G-operad $\mathscr{C}_G$.** We assume that the reader is familiar with operads, as originally defined in [27]. More recent brief expositions can be found in [31, 32]. Operads make sense in any symmetric monoidal category. Ours will be in the cartesian monoidal category $G\mathscr{U}$. We assume once and for all that our operads $\mathscr{C}$ are reduced, meaning that $\mathscr{C}(0)$ is a point. We have a slight clash of notation since we follow [12] in writing $\mathscr{C}$ for an operad in $\mathscr{U}$, regarded as a G-trivial G-operad, whereas we write $\mathscr{C}_G$ for a general G-operad. This clashes with the dichotomy between $\mathscr{D}$ and $\mathscr{D}_G$.[17]

Definition 5.1. Let $\mathscr{C}_G$ be an operad of G-spaces. We construct a G-CO over $\mathscr{F}$, which we denote by $\mathscr{D}(\mathscr{C}_G)$, abbreviated $\mathscr{D}$ when there is no risk of confusion. Similarly, we write $\mathscr{D}_G = \mathscr{D}_G(\mathscr{C}_G)$ for the associated G-CO over $\mathscr{F}_G$. The morphism G-spaces of $\mathscr{D}$ are

$$\mathscr{D}(\mathbf{m},\mathbf{n}) = \coprod_{\phi\in\mathscr{F}(\mathbf{m},\mathbf{n})} \prod_{1\le j\le n} \mathscr{C}_G(|\phi^{-1}(j)|)$$

with G-action induced by the G-actions on the $\mathscr{C}_G(n)$. Write elements in the form (ϕ, c), where $c = (c_1, \dots, c_n)$. For $(\phi, c)\colon \mathbf{m} \longrightarrow \mathbf{n}$ and $(\psi, d)\colon \mathbf{k} \longrightarrow \mathbf{m}$, define

$$(\phi, c) \circ (\psi, d) = (\phi\circ\psi, \prod_{1\le j\le n} \gamma(c_j; \prod_{\phi(i)=j} d_i)\sigma_j).$$

Here γ denotes the structural maps of the operad. The d_i with $\phi(i) = j$ are ordered by the natural order on their indices i and σ_j is that permutation of $|(\phi\circ\psi)^{-1}(j)|$ letters which converts the natural ordering of $(\phi\circ\psi)^{-1}(j)$ as a subset of $\{1, \dots, k\}$ to its ordering obtained by regarding it as $\coprod_{\phi(i)=j} \psi^{-1}(i)$, so ordered that elements of $\psi^{-1}(i)$ precede elements of $\psi^{-1}(i')$ if $i < i'$ and each $\psi^{-1}(i)$ has its natural ordering as a subset of $\{1, \dots, k\}$.

The identity element in $\mathscr{D}(\mathbf{n},\mathbf{n})$ is (id, id^n), where id on the right is the unit element in $\mathscr{C}_G(1)$. The map $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ sends (ϕ, c) to ϕ. The inclusion $\iota\colon \Pi \longrightarrow \mathscr{D}$ sends $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ to (ϕ, c), where $c_i = \mathrm{id} \in \mathscr{C}_G(1)$ if $\phi(i) = 1$ and $c_i = * \in \mathscr{C}_G(0)$ if $\phi(i) = 0$. This makes sense since Π is the subcategory of $\mathscr{F}$ with morphisms ϕ such that $|\phi^{-1}(j)| \le 1$ for $1 \le j \le n$.

Observe that $\mathscr{D}$ is reduced as a G-CO over $\mathscr{F}$ since $\mathscr{C}_G$ is reduced as an operad.

Observation 5.2. Since we will need it later and it illustrates the definition, we describe explicitly how composition behaves when the point c or d in one of the maps is of the form $(\mathrm{id}, \dots, \mathrm{id}) \in \mathscr{C}(1) \times \cdots \times \mathscr{C}(1)$.

For $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ and a permutation $\tau\colon \mathbf{m} \longrightarrow \mathbf{m}$,

$$\begin{aligned}(\phi, c_1, \dots, c_n) \circ (\tau, \mathrm{id}, \dots \mathrm{id}) &= (\phi\circ\tau, \prod_j (\gamma(c_j, \mathrm{id}, \dots, \mathrm{id})\sigma_j) \\ &= (\phi\circ\tau, c_1\sigma_1, \dots, c_n\sigma_n),\end{aligned}$$

where $c_j \in \mathscr{C}(|\phi^{-1}(j)|)$, and $\sigma_j \in \Sigma_{(|(\phi\circ\tau)^{-1}(j)|} = \Sigma_{|\phi^{-1}(j)|}$. Note that σ_j depends only on ϕ and τ and not on c.

For $\psi\colon \mathbf{m} \longrightarrow \mathbf{n}$ and a permutation $\tau\colon \mathbf{n} \longrightarrow \mathbf{n}$,

$$\begin{aligned}(\tau, \mathrm{id}, \dots, \mathrm{id}) \circ (\psi, d_1, \dots, d_n) &= (\tau\circ\psi, \prod_j (\gamma(\mathrm{id}, d_{\tau^{-1}(j)})\sigma_j)) \\ &= (\tau\circ\psi, d_{\tau^{-1}(1)}, \dots, d_{\tau^{-1}(n)})\end{aligned}$$

[17] For this and related reasons, we do not adopt the original notation $\hat{\mathscr{C}}$ from [38].

since each σ_j is the identity (because there is only one i such that $\tau(i) = j$).

Example 5.3. The commutativity operad $\mathscr{N}$ has nth space a point for all n. We think of it as a G-trivial G-operad. Then $\mathscr{F} = \mathscr{D}(\mathscr{N})$, again regarded as G-trivial.

For any operad $\mathscr{C}_G$, an $\mathscr{F}$-G-space Y can be viewed as the $\mathscr{D}(\mathscr{C}_G)$-$G$-space ξ^*Y. Thus $\mathscr{D}(\mathscr{C}_G)$-$G$-spaces give a generalized choice of input to the Segal machine. As we shall discuss in §6.4, they also give generalized input to the operadic machine. Indeed, an action of the operad $\mathscr{C}_G$ on a G-space X gives rise to an action of $\mathscr{D}(\mathscr{C}_G)$ on the Π-G-space $\mathbb{R}X$ with $(\mathbb{R}X)_n = X^n$.

5.2. **E_∞ G-operads and E_∞ G-categories of operators.** Let $\mathscr{C}_G$ be a (reduced) operad of G-spaces, or G-operad for short. We say that $\mathscr{C}_G$ is an E_∞ G-operad if $\mathscr{C}_G(n)$ is a universal principal (G, Σ_n)-bundle for each n. This means that $\mathscr{C}_G(n)$ is a Σ_n-free $(G \times \Sigma_n)$-space such that

$$\mathscr{C}_G(n)^\Lambda \simeq * \quad \text{if } \Lambda \subset G \times \Sigma_n \text{ and } \Lambda \cap \Sigma_n = e \text{ (that is, if } \Lambda \in \mathbb{F}_n).$$

Since $\mathscr{C}_G(n)$ is Σ_n-free, $\mathscr{C}_G(n)^\Lambda = \emptyset$ if $\Lambda \notin \mathbb{F}_n$. We call G-spaces with an action by an E_∞ G-operad E_∞ G-spaces. As we recall from [10, 12] in the next section, they provide input for an infinite loop space machine that sends an E_∞ G-space to a genuine Ω-G-spectrum whose zeroth space is a group completion of X. We prove the following theorem.

Theorem 5.4. *If $\mathscr{C}_G$ is an E_∞ G-operad, then $\mathscr{D} = \mathscr{D}(\mathscr{C}_G)$ is an E_∞ G-CO over $\mathscr{F}$ or, equivalently, $\mathscr{D}_G$ is an E_∞ G-CO over $\mathscr{F}_G$.*

Consider the trivial map of G-operads $\xi\colon \mathscr{C}_G \longrightarrow \mathscr{N}$ that sends each $\mathscr{C}_G(n)$ to the point $\mathscr{N}(n)$. Of course, $\mathscr{N}$ is not an E_∞ G-operad, but it is clear from the definitions that $\mathscr{F} = \mathscr{D}(\mathscr{N})$ is an E_∞ G-CO over $\mathscr{F}$. The map $\xi\colon \mathscr{D} \longrightarrow \mathscr{F}$ is $\mathscr{D}(\xi)$. The following result with $\mathscr{C}' = \mathscr{N}$ has Theorem 5.4 as an immediate corollary. We give pedantic details of equivariance since this is the crux of the comparison of inputs of the Segal and operadic machines.

Theorem 5.5. *Let $\nu\colon \mathscr{C}_G \longrightarrow \mathscr{C}'_G$ be a map of G-operads such that the fixed point map $\nu^\Lambda\colon \mathscr{C}_G(n)^\Lambda \longrightarrow \mathscr{C}'_G(n)^\Lambda$ is a weak equivalence for all n and all $\Lambda \in \mathbb{F}_n$. Then the induced map $\nu_G\colon \mathscr{D}_G \longrightarrow \mathscr{D}'_G$ of G-COs over $\mathscr{F}_G$ is a G-equivalence.*

Proof. We must prove that the map

$$\nu_G\colon \mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta)) \longrightarrow \mathscr{D}'_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$$

is a G-equivalence for all $(\mathbf{m},\alpha)$ and $(\mathbf{n},\beta)$. Recall that $\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))$ is just $\mathscr{D}(\mathbf{m},\mathbf{n})$ with the G-action given by $g \cdot f = \beta(g) \circ (gf) \circ \alpha(g^{-1})$.

Let H be a subgroup of G. We claim that there is a homeomorphism

$$\big[\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))\big]^H \cong \coprod_\phi \prod_i \mathscr{C}(q_i)^{\Lambda_i},$$

where ϕ runs over the H-equivariant maps $(\mathbf{m},\alpha) \longrightarrow (\mathbf{n},\beta)$, i runs over the H-orbits of $(\mathbf{n},\beta)$, and q_i and Λ_i depend only on ϕ and H, with Λ_i in $\mathbb{F}_{q_i}$. This homeomorphism is moreover compatible with the map ν_G. It follows from the assumption that the map

$$(\nu_G)^H\colon \big[\mathscr{D}_G((\mathbf{m},\alpha),(\mathbf{n},\beta))\big]^H \longrightarrow \big[\mathscr{D}'_G((\mathbf{m},\alpha),(\mathbf{n},\beta))\big]^H$$

is a weak equivalence for all subgroups H, as wanted.

To prove the claim, recall that

$$\mathscr{D}(\mathbf{m},\mathbf{n}) = \coprod_{\phi\colon \mathbf{m}\longrightarrow\mathbf{n}} \prod_{1\le j\le n} \mathscr{C}(|\phi^{-1}(j)|).$$

Using Definition 5.1 and in particular Observation 5.2, the new action on $\mathscr{D}(\mathbf{m},\mathbf{n})$ is given by

$$g\cdot(\phi;x_1,\dots,x_n) = (\beta(g)\phi\alpha(g^{-1}); gx_{\beta(g^{-1})(1)}\sigma_{\beta(g^{-1})(1)}(g^{-1}),\dots,gx_{\beta(g^{-1})(n)}\sigma_{\beta(g^{-1})(n)}(g^{-1})),$$

where $\sigma_j(g^{-1})\in\Sigma_{|(\phi\circ\alpha(g^{-1}))^{-1}(j)|}=\Sigma_{|\phi^{-1}(j)|}$ is that permutation of $|(\phi\circ\alpha(g^{-1}))^{-1}(j)|$ letters which converts the natural ordering of $(\phi\circ\alpha(g^{-1}))^{-1}(j)$ as a subset of $\{1,\dots,m\}$ to its ordering obtained by regarding it as $\coprod_{\phi(i)=j}\alpha(g)(i)$, so ordered so that $\alpha(g)(i)$ precedes $\alpha(g)(i')$ if $i<i'$.

Note that the component corresponding to ϕ is nonempty in the H-fixed points if and only if ϕ is H-equivariant. In what follows we fix such a ϕ. Careful analysis of the definition of $\sigma_j(h)$ shows that for $j\in\{1,\dots,n\}$, and $h,k\in H$, we have

$$\sigma_j(hk)=\sigma_j(h)\sigma_{\beta(h^{-1})(j)}(k). \tag{5.6}$$

The H-action shuffles the indices within each H-orbit of $(\mathbf{n},\beta|_H)$, so it is enough to consider each H-orbit separately. We can assume then that the H-action on $(\mathbf{n},\beta|_H)$ is transitive. The rest of the proof is analogous to the proof of Lemma 2.7.

Since ϕ is H-equivariant and the action is transitive, all the sets $\phi^{-1}(j)$ have the same cardinality, say q. Let K be the stabilizer of $1\in\mathbf{n}$ under the action of H. By (5.6), σ_1 restricted to K is homomorphism, and thus

$$\Lambda=\{(k,\sigma_1(k)\mid k\in K\}\subseteq G\times\Sigma_q$$

is a subgroup that belongs to $\mathbb{F}_q$. To complete the proof of the claim, we note that the projection to the first coordinate induces a homeomorphism

$$\big(\mathscr{C}(q)\times\cdots\times\mathscr{C}(q)\big)^H \longrightarrow \mathscr{C}(q)^\Lambda.$$

One can easily check that if $(x_1,\dots,x_n)\in\mathscr{C}(q)^n$ is an H-fixed point, then x_1 is a Λ-fixed point, since for all $k\in K$ we have that

$$(k,\sigma_1(k))\cdot x_1 = kx_1\sigma_1(k^{-1}) = kx_{\beta(k^{-1})(1)}\sigma_{\beta(k^{-1})(1)}(k^{-1}) = x_1,$$

the last equality being true by the assumption that $(x_1,\dots,x_n)$ was fixed by H. To construct an inverse, for every j, choose $h_j\in H$ such that $\beta(h_j)(1)=j$. Note that choosing these amounts to choosing a system of coset representatives for H/K. Consider the map $\mathscr{C}(q)\longrightarrow\mathscr{C}(q)^n$ that sends x to the n-tuple with jth coordinate

$$x_j = h_jx\sigma_1(h_j^{-1}).$$

We leave it to the reader to check that this map restricts to the fixed points and is inverse to the projection. □

6. The generalized operadic machine

Having redeveloped the Segal infinite loop space machine equivariantly, we now review and generalize the equivariant operadic infinite loop space machine. Since the prequel [12] of Guillou and the first author also reviews that machine, in part following the earlier treatment of Costenoble and Waner [10], and since the equivariant generalization of the basic definitions of [27] is entirely straightforward, we shall be brief, focusing on the material that is needed here and is not treated in [12]. In particular, again following the nonequivariant work of Thomason and the first author [38], we develop a generalization of the equivariant operadic machine analogous to our generalization of the equivariant Segal machine. We compare the inputs and outputs of the classical and generalized machines and show that they are equivalent. Starting from an E_∞ G-operad, the generalized input is the same as the generalized input to the Segal machine that we saw in §5.2.

6.1. **The Steiner operads.** The advantages of the Steiner operads over the little cubes or little discs operads are explained in detail in [36, §3]. The little cubes operads $\mathscr{C}_n$ work well nonequivariantly, but are too square for multiplicative and equivariant purposes. The little discs operads are too round to allow maps of operads $\mathscr{D}_n \longrightarrow \mathscr{D}_{n+1}$ that are compatible with the natural map $\Omega^n X \longrightarrow \Omega^{n+1}\Sigma X$. The Steiner operads are more complicated to define, but they enjoy all of the good properties of both the little cubes and little discs operads. Some such family of operads must play a central role in any version of the operadic machine, but the special features of the Steiner operads will play an entirely new and unexpected role in our comparison of the operadic and Segal machines.

We review the definition and salient features of the equivariant Steiner operads from [12, §1.1] and [58], referring to those sources for more detailed treatments. Let V be a finite dimensional real inner product space and let G act on V through linear isometries. We are only interested in the group action when G is finite; for more general groups G, we only use these operads to construct naive G-spectra, taking the action to be trivial. Define a *Steiner path* to be a continuous map h from I to the space of distance-reducing embeddings $V \longrightarrow V$ such that $h(1)$ is the identity map; thus $|h(t)(v) - h(t)(w)| \leq |v - w|$ for all $v, w \in V$ and $t \in I$. Define $\pi(h)\colon V \longrightarrow V$ by $\pi(h) = h(0)$ and define the "center point" of h to be the value of $0 \in V$ under the embedding $h(0)$, that is $c(h) = \pi(h)(0) \in V$. Crossing embeddings $V \longrightarrow V$ with id_W sends Steiner paths in V to Steiner paths in $V \oplus W$.

For $j \geq 0$, define $\mathscr{K}_V(j)$ to be the G-space of j-tuples $(h_1, \ldots, h_j)$ of Steiner paths such that the embeddings $\pi(h_r) = h_r(0)$ for $1 \leq r \leq j$ have disjoint images; G acts by conjugation on embeddings and thus on Steiner paths and on j-tuples thereof. Pictorially (albeit imprecisely), one can think of a point in the Steiner operad as a continuous deformation of V into a point in the little disks operad. The symmetric group Σ_j permutes j-tuples. We take $\mathscr{K}_V(0) = *$ and let $\mathrm{id} \in \mathscr{K}_V(1)$ be the constant path at the identity $V \longrightarrow V$. Compose Steiner paths pointwise, $(\ell \circ h)(t) = \ell(t) \circ h(t)\colon V \longrightarrow V$. Define the structure maps

$$\gamma\colon \mathscr{K}_V(k) \times \mathscr{K}_V(j_1) \times \cdots \times \mathscr{K}_V(j_k) \longrightarrow \mathscr{K}_V(j_1 + \cdots + j_k)$$

by sending

$$(\langle \ell_1, \ldots, \ell_k\rangle; \langle h_{1,1}, \ldots, h_{1,j_1}\rangle, \ldots, \langle h_{k,1}, \ldots, h_{k,j_k}\rangle)$$

to

$$\langle \ell_1 \circ h_{1,1}, \ldots, \ell_1 \circ h_{1,j_1}, \ldots, \ell_k \circ h_{k,1}, \ldots, \ell_k \circ h_{k,j_k}\rangle.$$

Note that $\mathscr{K}_0$ is the trivial operad, $\mathscr{K}_0(0) = *$, $\mathscr{K}_0(1) = \{\mathrm{id}\}$ and $\mathscr{K}_0(j) = \emptyset$ for $j \geq 2$. Via π, the operad $\mathscr{K}_V$ acts on $\Omega^V X$ for any G-space X in the same way that the little cubes operad acts on n-fold loop spaces or the little discs operad acts on V-fold loop spaces.

Define $\zeta\colon \mathscr{K}_V(j) \longrightarrow \mathbf{Conf}(V, j)$, where $\mathbf{Conf}(V, j)$ is the configuration G-space of ordered j-tuples of distinct points of V, by sending $\langle h_1, \ldots, h_j\rangle$ to $(c(h_1), \ldots, c(h_j))$. The original argument of Steiner [58] generalizes without change equivariantly to prove that ζ is a $(G \times \Sigma_j)$-deformation retraction. With G finite, we may take the colimit over V in a complete G-universe U to obtain an E_∞ G-operad $\mathscr{K}_U$.

6.2. **The classical operadic machine.** Recall the definition of an E_∞ G-operad from §5.2 and let $\mathscr{C}_G$ be a fixed chosen E_∞ G-operad throughout this section.

A $\mathscr{C}_G$-algebra (X, θ) is a G-space X together with $(G \times \Sigma_j)$-maps

$$\theta_j \colon \mathscr{C}_G(j) \times X^j \longrightarrow X$$

for $j \geq 0$ such that the diagrams specified in [27, §1] or [31, 32] commute. We call an algebra over $\mathscr{C}_G$ a $\mathscr{C}_G$-space. Since $\mathscr{C}_G(0) = \{*\}$, the action determines a basepoint in X, and we assume that it is nondegenerate. Several examples of E_∞ operads $\mathscr{C}_G$ and $\mathscr{C}_G$-spaces appear in [12, 14]. As explained in [36, §8], the operadic infinite loop space machine is a homotopical adaptation of Beck's categorical monadicity theorem. If G is finite, this machine is a functor $\mathbb{E}_G$ from $\mathscr{C}_G$-spaces to (genuine) orthogonal G-spectra. We summarize the construction of $\mathbb{E}_G$, following [12].

An operad $\mathscr{C}_G$ determines a monad $\mathbb{C}_G$ on based G-spaces such that the category $\mathscr{C}_G[G\mathscr{T}]$ of $\mathscr{C}_G$-algebras is isomorphic to the category $\mathbb{C}_G[G\mathscr{T}]$ of $\mathbb{C}_G$-algebras. Nonequivariantly, this motivated the definition of operads [27, 31, 32], and it is proven equivariantly in [10, 12]. Intuitively, $\mathbb{C}_G X$ is constructed as the quotient of $\coprod_{j \geq 0} \mathscr{C}_G(j) \times_{\Sigma_j} X^j$ by basepoint identifications. Formally, it is the categorical tensor product of functors $\mathscr{C}_G \otimes_{\mathcal{I}} X^\bullet$, where $\mathcal{I}$ is the category[18] of finite sets $\underline{j} = \{1, \ldots, j\}$ and injections. To be explicit, for $1 \leq i \leq j$, let $\sigma_i \colon \underline{j-1} \longrightarrow \underline{j}$ be the ordered injection that skips i in its image. Note that the morphisms in $\mathcal{I}$ are generated by the maps σ_i and the permutations. Then $\mathscr{C}_G$ is regarded as a functor $\mathcal{I}^{op} \longrightarrow G\mathscr{U}$ via the right action of Σ_j and the "degeneracy maps" $\sigma_i \colon \mathscr{C}_G(j) \longrightarrow \mathscr{C}_G(j-1)$ specified in terms of the structure map γ of $\mathscr{C}_G$ by

$$\sigma_i(c) = \gamma(c; \mathrm{id}^{i-1} \times 0 \times \mathrm{id}^{j-i}) \tag{6.1}$$

where $0 \in \mathscr{C}_G(0)$ and $\mathrm{id} \in \mathscr{C}_G(1)$; $X^\bullet$ is the covariant functor $\mathcal{I} \longrightarrow G\mathscr{U}$ that sends $\underline{j}$ to X^j and uses the left action of Σ_j and the injections $\sigma_i \colon X^{j-1} \longrightarrow X^j$ given by insertion of the basepoint in the ith position.

There are several choices that can be made in the construction of the machine $\mathbb{E}_G$, as discussed in [12]. We use the construction landing in orthogonal G-spectra. It is more natural topologically to land in Lewis-May or EKMM G-spectra [11, 21, 24] since all such G-spectra are fibrant and the relationship of $\mathbb{E}_G$ to the equivariant Barratt-Priddy-Quillen theorem is best explained using them. That variant of the operadic machine is discussed and applied in [12], and we shall say nothing more about it here.

Regardless of such choices, the construction of $\mathbb{E}_G$ is based on the two-sided monadic bar construction defined in [27, §9]. In our context, that specializes to give a based G-space $B(F, \mathbb{C}_G, X)$ for a monad $\mathbb{C}_G$ in $G\mathscr{T}$, a $\mathbb{C}_G$-algebra X, and a $\mathbb{C}_G$-functor $F = (F, \lambda)$. Here $F \colon G\mathscr{T} \longrightarrow G\mathscr{T}$ is a functor and $\lambda \colon F\mathbb{C}_G \longrightarrow F$ is a natural transformation such that $\lambda \circ F\eta = \mathrm{Id} \colon F \longrightarrow F$ and

$$\lambda \circ F\mu = \lambda \circ \lambda\mathbb{C}_G \colon F\mathbb{C}_G\mathbb{C}_G \longrightarrow F.$$

There results a simplicial G-space $B_\bullet(F, \mathbb{C}_G, X)$ with q-simplices $F\mathbb{C}_G^q X$. Its geometric realization is $B(F, \mathbb{C}_G, X)$. We emphasize that while the bar construction can be specified in sufficiently all-embracing generality that both the categorical

[18] $\mathcal{I}$ was often denoted Λ in the 1970's and is nowadays often denoted FI.

version used in the Segal machine and the monadic version used in the operadic machine are special cases [41, 42, 43, 57], these constructions look very different.

To incorporate the relationship to loop spaces encoded in the Steiner operads, we define $\mathscr{C}_V = \mathscr{C}_G \times \mathscr{K}_V$. We view $\mathscr{C}_G$-spaces as $\mathscr{C}_V$-spaces via the projection to $\mathscr{C}_G$, and we view G-spaces $\Omega^V X$ as $\mathscr{C}_V$-spaces via the projection to $\mathscr{K}_V$. We write $\mathbb{C}_V$ for the monad on $G\mathscr{T}$ associated to $\mathscr{C}_V$.

Theorem 6.2. *The composite of the map $\mathbb{C}_V X \longrightarrow \mathbb{C}_V\Omega^V\Sigma^V X$ induced by the unit of the adjunction (Σ^V, Ω^V) and the action map $\mathbb{C}_V\Omega^V\Sigma^V X \longrightarrow \Omega^V\Sigma^V X$ specifies a natural map $\alpha\colon \mathbb{C}_V X \longrightarrow \Omega^V\Sigma^V X$ which is a group completion if V contains $\mathbb{R}^2$. These maps specify a map of monads $\mathbb{C}_V \longrightarrow \Omega^V\Sigma^V$.*

The second statement is proven by the same formal argument as in [27, Theorem 5.2]. The first statement is discussed and sharpened to the case $V \supset \mathbb{R}$ in [12]. The adjoint $\tilde{\alpha}\colon \Sigma^V\mathbb{C}_V \longrightarrow \Sigma^V$ of α is an action of the monad $\mathbb{C}_V$ on the functor Σ^V and we have the monadic bar construction

$$(\mathbb{E}_G X)(V) = B(\Sigma^V, \mathbb{C}_V, X)$$

for $\mathbb{C}_G$-spaces X. An isometric isomorphism $V \longrightarrow V'$ in $\mathscr{I}_G$ induces natural transformations $\Sigma^V \longrightarrow \Sigma^{V'}$ and $\mathbb{C}_V \longrightarrow \mathbb{C}_{V'}$, which in turn induce a map

$$B(\Sigma^V, \mathbb{C}_V, X) \longrightarrow B(\Sigma^{V'}, \mathbb{C}_{V'}, X).$$

These maps assemble to make $\mathbb{E}_G X$ into an $\mathscr{I}_G$-G-space. Smashing with G-spaces commutes with realization of based simplicial G-spaces, by the same proof as in the nonequivariant case [27, Proposition 12.1], and inclusions $V \longrightarrow W$ induce maps of monads $\mathbb{C}_V \longrightarrow \mathbb{C}_W$. This gives the structural maps

$$\sigma\colon \Sigma^{W-V}\mathbb{E}_G X(V) \cong B(\Sigma^W, \mathbb{C}_V, X) \longrightarrow B(\Sigma^W, \mathbb{C}_W, X) = \mathbb{E}_G Y(W)$$

which are compatible with the $\mathscr{I}_G$-G-space structure, so $\mathbb{E}_G X$ is an orthogonal spectrum. The structure maps of $\mathbb{E}_G X$ and their adjoints are closed inclusions. As explained in [12], and as goes back to [27] nonequivariantly and to Costenoble and Waner equivariantly [10], we have the following theorem, which gives the basic homotopical property of the infinite loop space machine $\mathbb{E}_G$.

Theorem 6.3. *There are natural maps*

$$X \xleftarrow{\ \varepsilon\ } B(\mathbb{C}_V, \mathbb{C}_V, X) \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} B(\Omega^V\Sigma^V, \mathbb{C}_V, X) \xrightarrow{\ \zeta\ } \Omega^V B(\Sigma^V, \mathbb{C}_V, X)$$

of $\mathscr{C}_V$-spaces. The map ε is a G-homotopy equivalence with a natural G-homotopy inverse ν (which is not a $\mathscr{C}_V$-map), the map $B(\alpha, \mathrm{id}, \mathrm{id})$ is a group completion when V contains $\mathbb{R}^2$, and the map ζ is a weak G-equivalence.

Proof. The first statement is a standard property of the bar construction that works just as well equivariantly as nonequivariantly [27, Proposition 9.8] or [57, Lemma 9.9]. The second statement is deduced from Theorem 6.2 by passage to fixed point spaces and use of the same argument as in the nonequivariant case [28, Theorem 2.3(ii)]. The last statement is an equivariant generalization of [27, Theorem 12.7] that is proven carefully in [10, Lemmas 5.4, 5.5]. See [12] for further discussion and variants of the construction. □

Define

$$\xi = \zeta \circ B(\alpha, \mathrm{id}, \mathrm{id}) \circ \nu\colon X \longrightarrow \Omega^V\mathbb{E}_G X(V).$$

Then ξ is a natural group completion when $V \supset \mathbb{R}^2$ and is thus a weak G-equivalence when X is grouplike. The following diagram commutes, where $\tilde{\sigma}$ is adjoint to σ.

$$\begin{array}{ccc} & X & \\ {\scriptstyle \xi}\swarrow & & \searrow{\scriptstyle \xi} \\ \Omega^V \mathbb{E}_G X(V) & \xrightarrow[\Omega^V \tilde{\sigma}]{} & \Omega^{V\oplus W}\mathbb{E}_G X(V \oplus W). \end{array}$$

Therefore $\Omega^V\tilde{\sigma}$ is a weak equivalence if V contains $\mathbb{R}^2$. For general topological groups, everything works exactly the same way provided that we restrict to those V with trivial G-action. However, even if $G = S^1$, the group completion property of α fails if V is a non-trivial representation of G, as was first noticed by Segal [52]. A proof can be found in [5, Appendix B]. We restrict G to be finite from now on.

Remark 6.4. With G finite, it is harmless to think of $\mathbb{E}_G X(V)$ as an Ω-G-spectrum. If we set $\mathbb{E}'_G X(V) = \Omega^2 \mathbb{E}_G X(V \oplus \mathbb{R}^2)$, then the maps $\tilde{\sigma}\colon \mathbb{E}_G X(V) \longrightarrow \mathbb{E}'_G X(V)$ specify an equivalence from $\mathbb{E}_G X$ to an Ω-G-spectrum, giving a simple and explicit fibrant approximation whose zeroth space is a group completion of X.

6.3. **The monads $\mathbb{D}$ and $\mathbb{D}_G$ associated to the G-categories $\mathscr{D}$ and $\mathscr{D}_G$.** Recall that the category $\mathscr{C}_G[G\mathscr{T}]$ of algebras over an operad $\mathscr{C}_G$ is isomorphic to the category $\mathbb{C}_G[G\mathscr{T}]$ of algebras over the associated monad $\mathbb{C}_G$. Let $\mathscr{D} = \mathscr{D}(\mathscr{C}_G)$ be the G-CO over $\mathscr{F}$ associated to $\mathscr{C}_G$ and let $\mathscr{D}_G$ be the associated G-CO over $\mathscr{F}_G$. As worked out nonequivariantly in [38, §5], we define monads $\mathbb{D}$ on the category of Π-G-spaces and $\mathbb{D}_G$ on the category of Π_G-G-spaces. The commutative diagram of inclusions of categories

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi_G \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_G} \\ \mathscr{D} & \longrightarrow & \mathscr{D}_G \end{array}$$

gives rise to a commutative diagram of forgetful functors

$$\begin{array}{ccc} \mathrm{Fun}(\Pi, \mathscr{T}_G) & \xleftarrow{\ \mathbb{U}\ } & \mathrm{Fun}(\Pi_G, \mathscr{T}_G) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i_G^*} \\ \mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) & \xleftarrow[U]{} & \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G). \end{array}$$

Categorical tensor products then give left adjoints making the following diagram commute up to natural isomorphism.

$$\begin{array}{ccc} \mathrm{Fun}(\Pi, \mathscr{T}_G) & \xrightarrow{\ \mathbb{P}\ } & \mathrm{Fun}(\Pi_G, \mathscr{T}_G) \\ {\scriptstyle \mathbb{D}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}_G} \\ \mathrm{Fun}(\mathscr{D}, \mathscr{T}_G) & \xrightarrow[\mathbb{P}]{} & \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) \end{array}$$

Here $\mathbb{D}$ and $\mathbb{D}_G$ are the left adjoints of i^* and i_G^*, respectively. By a standard abuse of notation, we write $\mathbb{D}$ and $\mathbb{D}_G$ for the resulting endofunctors $i^*\mathbb{D}$ on $\mathrm{Fun}(\Pi_G, \mathscr{T}_G)$ and $i_G^*\mathbb{D}_G$ on $\mathrm{Fun}(\Pi_G, \mathscr{T}_G)$. Explicitly, the monads $\mathbb{D}$ and $\mathbb{D}_G$ are defined as

$$(\mathbb{D}X)(\mathbf{n}) = \mathscr{D}(-, \mathbf{n}) \otimes_\Pi X \tag{6.5}$$

for a Π-G-space X, where $\mathscr{D}(-,\mathbf{n})$ is the represented functor induced by i, and

$$(\mathbb{D}_G Y)(\mathbf{n},\alpha) = \mathscr{D}_G(-,(\mathbf{n},\alpha)) \otimes_{\Pi_G} Y \tag{6.6}$$

for a Π_G-G-space Y, where $\mathscr{D}_G(-,(\mathbf{n},\alpha))$ is the represented functor induced by i_G.

The units η of the adjunctions $(\mathbb{D}, i^*)$ and $(\mathbb{D}_G, i_G^*)$ give the unit maps of the monads, and the action maps of the $\mathscr{D}$-G-spaces $\mathbb{D}X$ and $\mathscr{D}_G$-G-spaces $\mathbb{D}_G Y$ give the products μ. More concretely, $\mu\colon \mathbb{D}\mathbb{D} \longrightarrow \mathbb{D}$ and $\mu\colon \mathbb{D}_G\mathbb{D}_G \longrightarrow \mathbb{D}_G$ are derived from the compositions in $\mathscr{D}$ and $\mathscr{D}_G$, respectively, and thus from the structure maps γ of $\mathscr{C}_G$. The unit maps η are derived from the identity morphisms in these categories and thus from the unit element $\mathrm{id} \in \mathscr{C}_G(1)$.

As shown nonequivariantly in [38, §5], there are isomorphisms of categories from the category $\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G)$ of $\mathscr{D}$-G-spaces to the category $\mathbb{D}[\mathrm{Fun}(\Pi, G\mathscr{T})]$ of algebras over the monad $\mathbb{D}$ and from the category $\mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G)$ of $\mathscr{D}_G$-G-spaces to the category $\mathbb{D}_G[\mathrm{Fun}(\Pi_G, G\mathscr{T})]$ of algebras over the monad $\mathbb{D}_G$.

Proposition 6.7. *The categories* $\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G)$ *and* $\mathbb{D}[\mathrm{Fun}(\Pi, \mathscr{T}_G)]$ *are isomorphic. The categories* $\mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G)$ *and* $\mathbb{D}_G[\mathrm{Fun}(\Pi_G, \mathscr{T}_G)]$ *are isomorphic. Therefore the categories* $\mathbb{D}[\mathrm{Fun}(\Pi, \mathscr{T}_G)]$ *and* $\mathbb{D}_G[\mathrm{Fun}(\Pi_G, \mathscr{T}_G)]$ *are equivalent.*

Proof. The proof for $\mathbb{D}$ is a comparison of action maps $\mathscr{D}(\mathbf{m},\mathbf{n}) \wedge X(\mathbf{m}) \longrightarrow X(\mathbf{n})$ and $(\mathbb{D}X)(\mathbf{n}) \longrightarrow X(\mathbf{n})$; it is entirely analogous to the original argument for algebras over operads in [27, Proposition 2.8]. A similar proof works for $\mathbb{D}_G$. The result there can also be derived from the result for $\mathbb{D}$, using $\mathbb{D}_G\mathbb{P} \cong \mathbb{P}\mathbb{D}$ and Theorem 4.11, and that result also implies the last statement. □

Since the categories of $\mathbb{D}$-algebras and of $\mathbb{D}_G$-algebras are equivalent, the monads $\mathbb{D}$ and $\mathbb{D}_G$ can be used interchangeably. This contrasts markedly with the Segal machine, where considerations of specialness led us to focus on $\mathscr{D}_G$ rather than $\mathscr{D}$.

We need some homotopical and some formal properties of the monads $\mathbb{D}$ and $\mathbb{D}_G$, following [38]. We first establish the formal properties, whose proofs are identical to those in [38]. We first write the following results in terms of $\mathscr{D}$ and $\mathbb{D}$ for simplicity. With attention to enrichment, the parallel results for $\mathscr{D}_G$ work in exactly the same way, and they can also be derived from the results for $\mathbb{D}$ by use of the isomorphism $\mathbb{P}\mathbb{D} \cong \mathbb{D}_G\mathbb{P}$ and Proposition 6.7.

Recall that we have the functor $\mathbb{R}\colon G\mathscr{T} \longrightarrow \mathrm{Fun}(\Pi, G\mathscr{T})$ given by $(\mathbb{R}X)_n = X^n$. Define $\mathbb{L}\colon \mathrm{Fun}(\Pi, G\mathscr{T}) \longrightarrow G\mathscr{T}$ by $\mathbb{L}Y = Y_1$. Then $(\mathbb{L},\mathbb{R})$ is an adjoint pair such that $\mathbb{L}\mathbb{R} = \mathrm{Id}$. On a Π-G-space Y, the unit $\delta\colon \mathrm{Id} \longrightarrow \mathbb{R}\mathbb{L}$ of the adjunction is given by the Segal maps. Letting $\mathscr{C}_G[G\mathscr{T}]$ denote the category of $\mathscr{C}_G$-spaces, we show that $(\mathbb{L},\mathbb{R})$ induces an adjunction between that category and $\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G)$.

Proposition 6.8. *The adjunction* $(\mathbb{L},\mathbb{R})$ *between* $\mathrm{Fun}(\Pi, G\mathscr{T})$ *and* $G\mathscr{T}$ *induces an adjunction between* $\mathrm{Fun}(\mathscr{D}, \mathscr{T}_G)$ *and* $\mathscr{C}_G[G\mathscr{T}]$ *such that* $\mathbb{L}\mathbb{R} = \mathrm{Id}$ *and the unit* $\delta\colon \mathrm{Id} \longrightarrow \mathbb{R}\mathbb{L}$ *is given by the Segal maps. A* $\mathscr{D}$*-*G*-space with underlying* Π*-*G*-space* $\mathbb{R}X$ *determines and is determined by a* $\mathscr{C}_G$*-space structure on* $\mathbb{L}\mathbb{R}X = X$*.*

Proof. The nonequivariant proof of [38, Lemma 4.2] applies verbatim. □

We require an analyis of the behavior of the monad $\mathbb{D}$ with respect to the adjunction $(\mathbb{L},\mathbb{R})$.

Proposition 6.9. *Let* X *be a* G*-space and* Y *be a* Π*-*G*-space.*

(i) *The* G*-space* $\mathbb{L}\mathbb{D}\mathbb{R}X = (\mathbb{D}\mathbb{R}X)_1$ *is naturally* G*-homeomorphic to* $\mathbb{C}_G X$*.*

(ii) *The* Π*-*G*-space* $\mathbb{D}\mathbb{R}X$ *is naturally isomorphic to the* Π*-*G*-space* $\mathbb{R}\mathbb{C}_G X$*.*

(iii) *The following diagram is commutative for each* n*.*

$$\begin{array}{ccc} (\mathbb{D}Y)_n & \xrightarrow{(\mathbb{D}\delta)_n} & (\mathbb{D}\mathbb{R}\mathbb{L}Y)_n \cong (\mathbb{C}_G\mathbb{L}Y)^n \\ \delta\downarrow & & \delta\downarrow\cong \\ (\mathbb{D}Y)_1^n & \xrightarrow[(\mathbb{D}\delta)_1^n]{} & (\mathbb{D}\mathbb{R}\mathbb{L}Y)_1^n \cong (\mathbb{C}_G\mathbb{L}Y)^n \end{array}$$

(iv) *The functor* $\mathbb{R}\mathbb{C}_G\mathbb{L}$ *on* Π*-*G*-spaces is a monad with product and unit induced from those of* $\mathbb{C}_G$ *via the composites*

$$\mathbb{R}\mathbb{C}_G\mathbb{L}\mathbb{R}\mathbb{C}_G\mathbb{L} = \mathbb{R}\mathbb{C}_G\mathbb{C}_G\mathbb{L} \xrightarrow{\mathbb{R}\mu\mathbb{L}} \mathbb{R}\mathbb{C}_G\mathbb{L} \quad \textit{and} \quad \mathrm{Id} \xrightarrow{\delta} \mathbb{R}\mathbb{L} \xrightarrow{\mathbb{R}\eta\mathbb{L}} \mathbb{R}\mathbb{C}_G\mathbb{L}.$$

(v) *The natural transformation* $\mathbb{D}\delta\colon \mathbb{D} \longrightarrow \mathbb{D}\mathbb{R}\mathbb{L} \cong \mathbb{R}\mathbb{C}_G\mathbb{L}$ *is a morphism of monads in the category* $\mathrm{Fun}(\Pi, G\mathscr{T})$ *of* Π*-*G*-spaces.*

(vi) *If* (F,λ) *is a* $\mathbb{C}_G$*-functor in* $\mathscr{V}$*, then* $F\mathbb{L}\colon \mathrm{Fun}(\Pi, G\mathscr{T}) \longrightarrow \mathscr{V}$ *is an* $\mathbb{R}\mathbb{C}_G\mathbb{L}$*-functor in* $\mathscr{V}$ *with action* $\lambda\mathbb{L}\colon F\mathbb{L}\mathbb{R}\mathbb{C}_G\mathbb{L} = F\mathbb{C}_G\mathbb{L} \longrightarrow F\mathbb{L}$*. Therefore, by pullback,* $F\mathbb{L}$ *is a* $\mathbb{D}$*-functor in* $\mathscr{V}$ *with action the composite*

$$F\mathbb{L}\mathbb{D} \xrightarrow{F\mathbb{L}\mathbb{D}\delta} F\mathbb{L}\mathbb{D}\mathbb{R}\mathbb{L} \cong F\mathbb{L}\mathbb{R}\mathbb{C}_G\mathbb{L} = F\mathbb{C}_G\mathbb{L} \xrightarrow{\lambda\mathbb{L}} F\mathbb{L}.$$

Proof. Nonequivariantly, these results are given in [38, §6] and the equivariance adds no complications. The proofs are inspections of definitions and straightforward diagram chases. □

Using the adjunction $(\mathbb{P}, \mathbb{U})$ and the isomorphism $\mathbb{D}_G\mathbb{P} \cong \mathbb{P}\mathbb{D}$, we derive the analogue for $\mathbb{D}_G$. As in Definition 2.31, we write

$$\mathbb{R}_G = \mathbb{P}\mathbb{R}\colon \mathscr{C}_G[G\mathscr{T}] \longrightarrow \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G)$$

and

$$\mathbb{L}_G = \mathbb{L}\mathbb{U}\colon \mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G) \longrightarrow \mathscr{C}_G[G\mathscr{T}].$$

With $\mathbb{D}$, $\mathbb{R}$, and $\mathbb{L}$ replaced by $\mathbb{D}_G$, $\mathbb{R}_G$, and $\mathbb{L}_G$, we then have the following analogue of Proposition 6.9.

Proposition 6.10. *Let* X *be a* G*-space and* Y *be a* Π_G*-*G*-space.*

(i) *The* G*-space* $\mathbb{L}_G\mathbb{D}_G\mathbb{R}_G X$ *is naturally* G*-homeomorphic to* $\mathbb{C}_G X$*.*

(ii) *The* Π_G*-*G*-space* $\mathbb{D}_G\mathbb{R}_G X$ *is naturally isomorphic to* $\mathbb{R}_G\mathbb{C}_G X$*.*

(iii) *The following diagram is commutative for each* $(\mathbf{n},\alpha)$*.*

$$\begin{array}{ccc} (\mathbb{D}_G Y)(\mathbf{n},\alpha) & \xrightarrow{\mathbb{D}_G\delta} & (\mathbb{D}_G\mathbb{R}_G\mathbb{L}_G Y)(\mathbf{n},\alpha) \cong (\mathbb{C}_G\mathbb{L}_G Y)^{(\mathbf{n},\alpha)} \\ \delta\downarrow & & \delta\downarrow\cong \\ (\mathbb{D}_G Y)_1^{(\mathbf{n},\alpha)} & \xrightarrow[(\mathbb{D}_G\delta)_1^{(\mathbf{n},\alpha)}]{} & (\mathbb{D}_G\mathbb{R}_G\mathbb{L}_G Y)_1^{(\mathbf{n},\alpha)} \cong (\mathbb{C}_G\mathbb{L}_G Y)^{(\mathbf{n},\alpha)} \end{array}$$

(iv) *The functor* $\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G$ *on* Π_G*-*G*-spaces is a monad with product and unit induced from those of* $\mathbb{C}_G$ *via the composites*

$$\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G = \mathbb{R}_G\mathbb{C}_G\mathbb{C}_G\mathbb{L}_G \xrightarrow{\mathbb{R}_G\mu\mathbb{L}_G} \mathbb{R}_G\mathbb{C}_G\mathbb{L}_G$$

and

$$\mathrm{Id} \xrightarrow{\delta} \mathbb{R}_G\mathbb{L}_G \xrightarrow{\mathbb{R}_G\eta\mathbb{L}_G} \mathbb{R}_G\mathbb{C}_G\mathbb{L}_G.$$

(v) *The natural transformation* $\mathbb{D}_G\delta\colon \mathbb{D}_G \longrightarrow \mathbb{D}_G\mathbb{R}_G\mathbb{L}_G \cong \mathbb{R}_G\mathbb{C}_G\mathbb{L}_G$ *is a morphism of monads in the category* $\mathrm{Fun}(\Pi_G, \mathscr{T}_G)$ *of* Π_G*-*G*-spaces.*

(vi) *If* (F,λ) *is a* $\mathbb{C}_G$*-functor in* $\mathscr{V}$*, then* $F\mathbb{L}_G\colon \mathrm{Fun}(\Pi_G,\mathscr{T}_G) \longrightarrow \mathscr{V}$ *is an* $\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G$*-functor in* $\mathscr{V}$ *with action* $\lambda\mathbb{L}_G\colon F\mathbb{L}_G\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G = F\mathbb{C}_G\mathbb{L}_G \longrightarrow F\mathbb{L}_G$*. Therefore, by pullback,* $F\mathbb{L}_G$ *is a* $\mathbb{D}_G$*-functor in* $\mathscr{V}$ *with action the composite*

$$F\mathbb{L}_G\mathbb{D}_G \xrightarrow{F\mathbb{L}_G\mathbb{D}_G\delta} F\mathbb{L}_G\mathbb{D}_G\mathbb{R}_G\mathbb{L}_G \cong F\mathbb{L}_G\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G = F\mathbb{C}_G\mathbb{L}_G \xrightarrow{\lambda\mathbb{L}_G} F\mathbb{L}_G.$$

We now turn to the homotopical properties of the monads $\mathbb{D}$ and $\mathbb{D}_G$ and their algebras. The proofs of the homotopical properties are similar to those in [38], but considerably more difficult, so some will be deferred to §8.3. In contrast with the Segal machine, we start with $\mathbb{D}$ rather than $\mathbb{D}_G$. Our interest is in E_∞ operads, but we allow more general operads until otherwise indicated.

Reedy cofibrancy of Π and $\mathscr{F}$-spaces can be defined as in [4], but we shall be informal about the former and not use the latter (see Remark 8.5). We give a quick definition, which mimics Definition 1.8.

Definition 6.11. For a Π-G-space X, the ordered injections $\sigma_i\colon \mathbf{n-1} \longrightarrow \mathbf{n}$ induce maps $\sigma_i\colon X_{n-1} \longrightarrow X_n$. A point of X_n in the image of some σ_i is said to be degenerate. The nth latching space of X is the set of degenerate points in X_n:

$$L_nX = \bigcup_{i=1}^{n} \sigma_i(X_{n-1}).$$

It is a $(G\times\Sigma_n)$-space, and the inclusion $L_nX \longrightarrow X_n$ is a $(G\times\Sigma_n)$-map. We say that X is *Reedy cofibrant* if this map is a $(G\times\Sigma_n)$-cofibration for each n.

Observe that $L_n\mathbb{R}X$ is the subspace of X^n consisting of those points at least one coordinate of which is the basepoint. By our standing assumption that basepoints are nondegenerate, $\mathbb{R}X$ is Reedy cofibrant.

Remark 6.12. Just as nonequivariantly [38, Definition 1.2], we can impose a cofibration condition on general Π-G-spaces X which ensures that they are Reedy cofibrant. Given an injection $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ in Π, we let Σ_ϕ be the subgroup of Σ_n consisting of those permutations τ such that $\tau(\operatorname{im}\phi) = \operatorname{im}\phi$. Then Σ_ϕ acts on the set $\mathbf{m}$ and ϕ is a Σ_ϕ-map. If the map $\phi_*\colon X_m \longrightarrow X_n$ is a $(G\times\Sigma_\phi)$-cofibration for all injections ϕ, then a direct application of [7, Theorem A.2.7] shows that X is Reedy cofibrant.

Following [27] nonequivariantly, we say that a G-operad $\mathscr{C}_G$ is Σ-free if the action of Σ_j on $\mathscr{C}_G(j)$ is free for each j. Surprisingly, we only need that much structure to prove the following result. It is the equivariant generalization of [38, Lemma 5.6], which implicitly used Reedy cofibrancy of Π-spaces via the remark above.

Theorem 6.13. *Let* $\mathscr{C}_G$ *be a* Σ*-free* G*-operad.*

(i) *If* $f\colon X \longrightarrow Y$ *is an* $\mathbb{F}_\bullet$*-level equivalence of Reedy cofibrant* Π*-*G*-spaces, then* $\mathbb{D}f\colon \mathbb{D}X \longrightarrow \mathbb{D}Y$ *is an* $\mathbb{F}_\bullet$*-level equivalence.*

(ii) *If* X *is an* $\mathbb{F}_\bullet$*-special Reedy cofibrant* Π*-*G*-space, then* $\mathbb{D}X$ *is* $\mathbb{F}_\bullet$*-special.*

(iii) *For any Reedy cofibrant* Π*-*G*-space* X*,* $\mathbb{D}X$ *is a Reedy cofibrant* Π*-*G*-space.*

Proof. The proof of (i) requires quite lengthy combinatorics about the structure of $\mathbb{D}X$ and its fixed point subspaces, so we defer it to §8.3. By (i) applied to the Segal map $\delta\colon X \longrightarrow \mathbb{R}\mathbb{L}X$, the map $\mathbb{D}\delta$ is an $\mathbb{F}_\bullet$-level equivalence. Since its target

$\mathbb{D}\mathbb{R}\mathbb{L}X \cong \mathbb{R}\mathbb{C}_G\mathbb{L}X$ is $\mathbb{F}_\bullet$-special, Lemma 2.10 implies that $\mathbb{D}X$ is also $\mathbb{F}_\bullet$-special. As we explain in §8.2, (iii) follows from Remark 6.12 and a more explicit description of $\mathbb{D}X$. □

By Theorem 4.11 and Corollaries 4.13 and 4.14, the isomorphism $\mathbb{P}\mathbb{D} \cong \mathbb{D}_G\mathbb{P}$ and Theorem 6.13 imply the following analogue of that result.

Proposition 6.14. *Assume that each $\mathscr{C}_G(j)$ is Σ_j-free.*

(i) If $f\colon X \longrightarrow Y$ is an $\mathbb{F}_\bullet$-level equivalence of Reedy cofibrant Π-G-spaces, then $\mathbb{D}_G\mathbb{P}f\colon \mathbb{D}_G\mathbb{P}X \longrightarrow \mathbb{D}_G\mathbb{P}Y$, is a level G-equivalence.

(ii) If X is a $\mathbb{F}_\bullet$-special Reedy cofibrant Π-G-space, then $\mathbb{D}_G\mathbb{P}X$ is a special Π_G-G-space.

6.4. **Comparisons of inputs and outputs of the operadic machine.** We have three equivalent ways to construct G-spectra from $\mathscr{D}_G$-G-spaces. We can convert $\mathscr{D}_G$-G-spaces to $\mathscr{D}$-G-spaces via Proposition 6.7, and we can convert those to $\mathscr{C}_G$-spaces by Proposition 6.18 below. We can then apply the original machine, or we can generalize the machine to both $\mathscr{D}$-G-spaces and $\mathscr{D}_G$-G-spaces. All three make use of the two-sided monadic bar construction of [27], starting from the formalities of Propositions 6.9 and 6.10.

Remark 6.15. Just as nonequivariantly [38, Addendum 1.7], we impose an additional cofibration condition on $\mathscr{D}$ to ensure that our bar constructions are given by Reedy cofibrant simplicial G-spaces. As in Remark 6.12, for an injection $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$, let $\Sigma_\phi \subset \Sigma_n$ be the subgroup of permutations τ such that $\tau(\operatorname{im}\phi) = \operatorname{im}\phi$. Then the map $\mathscr{D}(\mathbf{q},\mathbf{m}) \longrightarrow \mathscr{D}(\mathbf{q},\mathbf{n})$ induced by ϕ is a $(G\times\Sigma_\phi)$-map, and we require it to be a $(G\times\Sigma_\phi)$-cofibration. This holds for our categories of operators $\mathscr{D} = \mathscr{D}(\mathscr{C})$ since we are assuming that the inclusion of the identity $* \longrightarrow \mathscr{C}_G(1)$ is a G-cofibration.

In fact, we really only have two machines in view of the following result. We emphasize how different this is from the Segal machine, where the $\mathscr{D}$ and $\mathscr{D}_G$ bar constructions are not even equivalent, let alone isomorphic.

Proposition 6.16. *Let $\mathscr{C}_G$ be a G-operad in $G\mathscr{T}$ with category of operators $\mathscr{D}$. For any $\mathscr{D}$-space X and any $\mathbb{C}_G$-functor $F\colon G\mathscr{T} \longrightarrow G\mathscr{T}$, there is a natural isomorphism of G-spaces*

$$B(F\mathbb{L},\mathbb{D},X) \cong B(F\mathbb{L}_G,\mathbb{D}_G,\mathbb{P}X).$$

Proof. Since $\mathrm{Id} \cong \mathbb{U}\mathbb{P}$, $\mathbb{P}\mathbb{D} \cong \mathbb{D}_G\mathbb{P}$ and $\mathbb{L}_G = \mathbb{L}\mathbb{U}$, we have isomorphisms of q-simplices

$$F\mathbb{L}\mathbb{D}^q X \cong F\mathbb{L}_G\mathbb{D}_G^q\mathbb{P}X.$$

Formal checks show that these isomorphisms commute with the face and degeneracy operators. The conclusion follows on passage to geometric realization. □

For variety, and because that is what we shall use in the next section, we focus on $\mathscr{D}_G$ rather than $\mathscr{D}$ in this section. The following result compares the inputs to machines given by $\mathscr{D}_G$-G-spaces and $\mathscr{C}_G$-spaces.

Definition 6.17. For a $\mathscr{D}_G$-G-space Y, define a $\mathscr{C}_G$-space $\mathbb{X}(Y)$ by

$$\mathbb{X}(Y) = B(\mathbb{C}_G\mathbb{L}_G,\mathbb{D}_G,Y).$$

Here $\mathbb{C}_G$ is regarded as a functor $G\mathscr{T} \longrightarrow \mathscr{C}_G[G\mathscr{T}]$, and the construction makes sense since the realization of a simplicial $\mathscr{C}_G$-space is a $\mathscr{C}_G$-space, exactly as nonequivariantly [27, Theorem 12.2].

Proposition 6.18. *For special $\mathscr{D}_G$-G-spaces Y whose underlying Π-spaces are Reedy cofibrant, there is a zigzag of natural level G-equivalences of $\mathscr{D}_G$-G-spaces between Y and $\mathbb{R}_G\mathbb{X}(Y)$. For $\mathscr{C}_G$-spaces X, there is a natural G-equivalence of $\mathscr{C}_G$-spaces from $\mathbb{X}(\mathbb{R}_G X)$ to X.*

Proof. Proposition 6.14 implies that $\mathbb{D}_G\delta\colon \mathbb{D}_G Y \longrightarrow \mathbb{D}_G\mathbb{R}_G\mathbb{L}_G Y \cong \mathbb{R}_G\mathbb{C}_G\mathbb{L}_G Y$ is a level G-equivalence of $\mathscr{D}_G$-G-spaces. The realization of simplicial Π_G-G-spaces is defined levelwise, and since realization commutes with products of G-spaces, we have the right-hand isomorphism in the diagram

$$Y \xleftarrow{\ \varepsilon\ } B(\mathbb{D}_G,\mathbb{D}_G,Y) \xrightarrow{B(\mathbb{D}_G\delta,\mathrm{id},\mathrm{id})} B(\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G,\mathbb{D}_G,Y) \cong \mathbb{R}_G B(\mathbb{C}_G\mathbb{L}_G,\mathbb{D}_G,Y) = \mathbb{R}_G\mathbb{X}(Y).$$

By standard properties of the bar construction, as in [27] nonequivariantly, ε is a level G-equivalence of $\mathscr{D}_G$-G-spaces. Since the bar constructions are geometric realizations of Reedy cofibrant simplicial G-spaces (see Remark 1.12), it follows from Theorem 1.10 that $B(\mathbb{D}_G\delta,\mathrm{id},\mathrm{id})$ is a level G-equivalence of $\mathscr{D}_G$-G-spaces. For the second statement, we apply $\mathbb{L}_G$ to the level G-equivalence of $\mathscr{D}_G$-G-spaces

$$\mathbb{R}_G\mathbb{X}(\mathbb{R}_G X) \cong B(\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G,\mathbb{D}_G,\mathbb{R}_G X) \cong B(\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G,\mathbb{R}_G\mathbb{C}_G\mathbb{L}_G,\mathbb{R}_G X) \xrightarrow{\ \varepsilon\ } \mathbb{R}_G X,$$

where the second isomorphism follows from Proposition 6.10 and inspection. □

Therefore, after inverting the respective equivalences, the functors $\mathbb{R}_G$ and $\mathbb{X}$ induce an equivalence of categories between $\mathscr{C}_G$-spaces and special $\mathscr{D}_G$-G-spaces whose underlying Π-G-spaces are Reedy cofibrant. We conclude that, for an E_∞-operad $\mathscr{C}_G$, the input categories for operadic machines given by $\mathscr{C}_G$-spaces and by $\mathscr{D}_G$-G-spaces are essentially equivalent.

To generalize the machine from $\mathscr{C}_G$-spaces to $\mathscr{D}_G$-G-spaces, we again use the product operads $\mathscr{C}_V = \mathscr{C}_G \times \mathscr{K}_V$, where $\mathscr{K}_V$ is the Vth Steiner operad. We write $\mathbb{D}_{G,V}$ for the monad associated to the resulting category of operators $\mathscr{D}_{G,V}$ over $\mathscr{F}_G$. Then a $\mathscr{D}_G$-space is a $\mathscr{D}_{G,V}$-space for any representation V by pullback along the projection $\mathscr{D}_{G,V} \longrightarrow \mathscr{D}_G$.

Definition 6.19. For a $\mathscr{D}_G$-G-space Y, define the Vth space of the orthogonal G-spectrum $\mathbb{E}_G Y$ to be the monadic two-sided bar construction

$$\mathbb{E}_G(Y)(V) = B(\Sigma^V\mathbb{L}_G,\mathbb{D}_{G,V},Y). \tag{6.20}$$

The right action of $\mathbb{D}_{G,V}$ on $\Sigma^V\mathbb{L}_G$ is obtained from the projection $\mathbb{D}_{G,V} \longrightarrow \mathbb{K}_V$ and the action of $\mathbb{K}_V$ on Σ^V, via Proposition 6.9(vi). The $\mathscr{I}_G$-G-space structure is given as follows. For an isometric isomorphism $V \longrightarrow V'$ in $\mathscr{I}_G$, the map

$$B(\Sigma^V\mathbb{L}_G,\mathbb{D}_{G,V},Y) \longrightarrow B(\Sigma^{V'}\mathbb{L}_G,\mathbb{D}_{G,V'},Y)$$

is the geometric realization of maps induced at all simplicial levels by $S^V \longrightarrow S^{V'}$ and $\mathscr{K}_V \longrightarrow \mathscr{K}_{V'}$. Similarly, since smashing commutes with geometric realization, the structure maps

$$B(\Sigma^V\mathbb{L}_G,\mathbb{D}_{G,V},Y) \wedge S^W \longrightarrow B(\Sigma^{V\oplus W}\mathbb{L}_G,\mathbb{D}_{G,V\oplus W},Y)$$

are induced from the maps of monads $\mathbb{D}_{G,V} \longrightarrow \mathbb{D}_{G,V\oplus W}$.

Just as we used $\mathbb{S}_G$ for all variants of the Segal machine, we are using $\mathbb{E}_G$ for all variants of the operadic machine. To justify this, we must show that the machine $\mathbb{E}_G$ on $\mathscr{D}_G$-G-spaces does indeed generalize the machine $\mathbb{E}_G$ on $\mathscr{C}_G$-spaces X. To see that, observe that Proposition 6.10(ii) implies that we have a natural isomorphism $\mathbb{D}_{G,V}\mathbb{R}_G X \cong \mathbb{R}_G \mathbb{C}_V X$. Since $\mathbb{L}_G\mathbb{R}_G = \mathrm{Id}$, that gives us a natural isomorphism

$$B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, \mathbb{R}_G X) \cong B(\Sigma^V, \mathbb{C}_V, X), \tag{6.21}$$

where we regard $\mathbb{R}_G X$ as a $\mathscr{D}_G$-G-space via Proposition 6.8. Together with Proposition 6.18, this gives the following comparison of outputs of our machines.

Corollary 6.22. *For $\mathscr{C}_G$-spaces X, $\mathbb{E}_G X$ is naturally isomorphic to $\mathbb{E}_G\mathbb{R}_G X$. For special $\mathscr{D}_G$-G-spaces Y whose underlying Π-G-space is Reedy cofibrant, there is a zigzag of natural equivalences connecting $\mathbb{E}_G Y$ to $\mathbb{E}_G\mathbb{R}_G\mathbb{X}(Y) \cong \mathbb{E}_G\mathbb{X}(Y)$.*

Thus the machines $\mathbb{E}_G$ on $\mathscr{C}_G$-spaces and on special $\mathscr{D}_G$-G-spaces are essentially equivalent. Properties of the machine on special $\mathscr{D}_G$-G-spaces are essentially the same as properties of the machine on $\mathscr{C}_G$-spaces, as can either be proven directly or read off from the equivalence of machines.

7. The equivalence between the Segal and operadic machines

We give an explicit comparison between the generalized Segal and generalized operadic infinite loop space machines. The comparison is needed for consistency and because each machine has significant advantages over the other. That was already clear nonequivariantly, and it seems even more true equivariantly. As in [12, 36], in the previous section we used the Steiner operads rather than the little cubes operads that were used in [27, 38]. That change made equivariant generalization easy, and [36] gave other good reasons for the change. However, nothing like the present comparison was envisioned in earlier work. As we have recalled, the Steiner operad is built from paths of embeddings. We shall see that these paths give rise to a homotopy that at one end relates to the generalized Segal machine and at the other end relates to the generalized operadic machine. That truly seems uncanny.

7.1. **The statement of the comparison theorem.** To set the stage, we recapitulate some of what we have done. We fix an E_∞ operad $\mathscr{C}_G$ of G-spaces. We then have an E_∞ G-CO $\mathscr{D} = \mathscr{D}(\mathscr{C}_G)$ over $\mathscr{F}$ and an E_∞ G-CO $\mathscr{D}_G$ over $\mathscr{F}_G$. Our primary interest here is in infinite loop space machines defined either on special $\mathscr{F}_G$-G-spaces or on $\mathscr{C}_G$-spaces. The Segal machine is defined on the former and the operadic machine is defined on the latter. We have generalized both machines so that they accept special $\mathscr{D}_G$-G-spaces as input. Moreover, we have compared inputs and shown that both special $\mathscr{F}_G$-G-spaces and $\mathscr{C}_G$-spaces are equivalent to special $\mathscr{D}_G$-G-spaces and therefore to each other. Further, we have compared outputs. We have shown that application of the generalized Segal machine to $\mathscr{D}_G$-G-spaces is equivalent to application of the original homotopical Segal machine to $\mathscr{F}_G$-G-spaces, and that application of the generalized operadic machine to $\mathscr{D}_G$-G-spaces is equivalent to application of the original operadic machine to $\mathscr{C}_G$-spaces.

In more detail, $\mathbb{F}_\bullet$-special $\mathscr{F}$-G-spaces, $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-spaces, special $\mathscr{F}_G$-G-spaces, and special $\mathscr{D}_G$-G-spaces are all equivalent by Theorems 4.25 and 4.26, and the Segal machines on all four equivalent inputs give equivalent output by Theorem 4.32. We may therefore focus on the Segal machine $\mathbb{S}_G$ defined on special

$\mathscr{D}_G$-G-spaces Y. Similarly, $\mathscr{C}_G$-spaces, $\mathbb{F}_\bullet$-special $\mathscr{D}$-G-spaces, and special $\mathscr{D}_G$-G-spaces are equivalent by Propositions 6.8 and 6.18, and the operadic machine on $\mathscr{C}_G$-spaces is a special case of the operadic machine on $\mathscr{D}_G$-G-spaces by Corollary 6.22. Thus we may again focus on the operadic machine $\mathbb{E}_G$ defined on special $\mathscr{D}_G$-G-spaces Y.

Thus, fixing an E_∞ operad $\mathscr{C}_G$ with associated category of operators $\mathscr{D}_G$ over $\mathscr{F}_G$, we consider special $\mathscr{D}_G$-G-spaces Y. The Vth G-space of $\mathbb{S}_G Y$ is

$$(\mathbb{S}_G Y)(V) = B((S^V)^\bullet, \mathscr{D}_G, Y),$$

where, as before, $(S^V)^\bullet$ denotes the composite $G\mathscr{T}$-functor

$$\mathscr{D}_G^{op} \xrightarrow{\ \xi\ } \mathscr{F}_G^{op} \longrightarrow \mathscr{T}_G$$

that sends the object $(\mathbf{n}, \alpha)$ to the cartesian power $(S^V)^{(n,\alpha)} = \mathscr{T}_G((\mathbf{n}, \alpha), S^V)$. With $\star$ thought of as a place holder for the representation V in the Vth level of the spectrum, we adopt the notation

$$\mathbb{S}_G Y = B((S^\star)^\bullet, \mathscr{D}_G, Y).$$

Let $\mathscr{C}_V$ be the product operad $\mathscr{C}_G \times \mathscr{K}_V$ and let $\mathscr{D}_{G,V} = \mathscr{D}_G(\mathscr{C}_V)$ with associated monad $\mathbb{D}_{G,V}$ on the category of Π_G-G-spaces. The Vth G-space of $\mathbb{E}_G Y$ is

$$(\mathbb{E}_G Y)(V) = B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y).$$

Again using $\star$ as a place holder for the representation V of the Vth level of the spectrum, we adopt the notation

$$\mathbb{E}_G Y = B(\Sigma^\star \mathbb{L}_G, \mathbb{D}_{G,\star}, Y).$$

Note the different uses of the $\star$ notation. In both machines, it is a placeholder for representations V. However, in the Segal machine, we are using cartesian powers of G-spheres S^V to obtain functors $(S^V)^\bullet \colon \mathscr{F}_G^{op} \longrightarrow \mathscr{T}_G$, whereas in the operadic machine we are using the suspension functor Σ^V associated to S^V together with the Steiner operad $\mathscr{K}_V$. While a two-sided bar construction is used in both machines, the similarity of notation hides how different these bar constructions really are: the use of categories and contravariant and covariant functors in one is quite different from the use of monads, (right) actions on functors, and (left) actions on objects in the other. Nevertheless, our goal is to give a constructive proof of the following comparison theorem.

Theorem 7.1. *For special $\mathscr{D}_G$-G-spaces Y, there is a natural zigzag of equivalences of orthogonal G-spectra between $\mathbb{S}_G Y$ and $\mathbb{E}_G Y$.*

7.2. **The proof of the comparison theorem.** We display the zigzag and then fill in the required constructions and proofs in subsequent sections. In addition to using $\star$ as a placeholder for representations V, we use $\bullet$ as a placeholder for finite G-sets $(\mathbf{n}, \alpha)$.

$$
\begin{array}{ccc}
\mathbb{S}_G Y & = & B((S^\star)^\bullet, \mathscr{D}_G, Y) \\
 & & \uparrow \pi \\
 & & B((S^\star)^\bullet, \mathscr{D}_{G,\star}, Y) \\
 & & \downarrow i_1 \\
 & & B(I_+ \wedge (S^\star)^\bullet, \mathscr{D}_{G,\star}, Y) \\
 & & \uparrow i_0 \\
 & & B((S_0^\star)^\bullet, \mathscr{D}_{G,\star}, Y) \\
 & & \uparrow \iota \\
 & & B({}^\bullet(S^\star), \mathscr{D}_{G,\star}, Y) \\
 & & \downarrow \omega \\
 & & B(\Sigma^\star \mathbb{L}_G, \mathbb{D}_{G,\star}, Y) = \mathbb{E}_G Y.
\end{array}
\tag{7.2}
$$

We shall construct the intermediate orthogonal G-spectra and maps in this zigzag and prove directly that all of the maps except ω are stable equivalences.

Recall that the homotopy groups of a pointed G-space X are $\pi_q^H(X) = \pi_q(X^H)$ and the homotopy groups of an orthogonal G-spectrum T are

$$\pi_q^H(T) = \operatorname{colim}_V \pi_q^H(\Omega^V T(V))$$

for $q \geq 0$, where the colimits are formed using the adjoint structure maps of T; our G-spectra are all connective, so that their negative homotopy groups are zero. A map $T \longrightarrow T'$ is a stable equivalence if its induced maps of homotopy groups are isomorphisms. That depends only on large V. Thus we may focus on those V that contain $\mathbb{R}^2$, so that the group completions of Theorem 6.2 are available. Applying Ω^V to the Vth spaces implicit in (7.2), we obtain a diagram of G-spaces under Y_1. By completely different proofs, both maps

$$Y_1 \longrightarrow \Omega^V \mathbb{S}_G Y(V) \quad \text{and} \quad Y_1 \longrightarrow \Omega^V \mathbb{E}_G Y(V)$$

are group completions. Therefore, once we prove that the arrows other than ω are stable equivalences, it will follow that ω is also a stable equivalence. Indeed, arranging as we may that our outputs are Ω-G-spectra and using that they are connective, ω is a stable equivalence if and only if the map ω_0 it induces on 0th G-spaces is a G-equivalence. The displayed group completions imply that ω_0 induces a homology isomorphism on fixed point spaces. Since these spaces are Hopf spaces, hence simple, it follows that ω_0 induces an isomorphism on homotopy groups, so that ω_0 is a G-equivalence.

Since wedges taken over G-sets $(\mathbf{n}, \alpha)$ play a significant role in our arguments, we introduce the following convenient notation.

Notation 7.3. For a based space A, let ${}^nA$ denote the wedge sum of n copies of A. Similarly, for a G-set $(\mathbf{n}, \alpha)$ and a based G-space A, let ${}^{(\mathbf{n},\alpha)}A$ denote the wedge sum of n copies of A with G-acting on A, but also interchanging the wedge summands.

We write $(j,a) \in {}^{(\mathbf{n},\alpha)}A$ for the element a in the jth summand. The G-action is given explicitly by $g\cdot(j,a) = (\alpha(g)(j), g\cdot a)$.

Remark 7.4. In constructing the diagram, we shall encounter an annoying but minor clash of conventions. There is a dichotomy in how one chooses to define the faces and degeneracies of the categorical bar construction. We made one choice in §3.1, but to mesh with the monadic bar construction as defined in [27, Construction 9.6], we must now make the other. Therefore, on q-simplices, we agree to replace the previous d_i and s_i by d_{q-i} and s_{q-i}, respectively. With the new convention, d_0 is given by the evaluation map of the left (contravariant) variable in the categorical two-sided bar construction, rather than the right variable.

7.3. **Construction and analysis of the map π.** Turning to the diagram (7.2), we first define the top map π. We start by defining its source orthogonal G-spectrum $B((S^\star)^\bullet, \mathscr{D}_{G,\star}, Y)$. The Vth space, as the notation indicates, is defined by plugging in V for $\star$; it is the bar construction $B((S^V)^\bullet, \mathscr{D}_{G,V}, Y)$, as defined in §3.1, namely it is the geometric realization of the simplicial space with q-simplices given by the wedge over all sequences $(\mathbf{n_q},\alpha_q),\dots,(\mathbf{n_0},\alpha_0)$ of the G-spaces

$$(S^V)^{(\mathbf{n_q},\alpha_q)} \wedge \big(\mathscr{D}_{G,V}((\mathbf{n_{q-1}},\alpha_{q-1}),(\mathbf{n_q},\alpha_q))\times\cdots\times\mathscr{D}_{G,V}((\mathbf{n_1},\alpha_1),(\mathbf{n_0},\alpha_0))\times Y(\mathbf{n_0},\alpha_0)\big)_+.$$

We have implicitly composed Y with the evident projections $\mathscr{D}_{G,V} \longrightarrow \mathscr{D}_G$ to regard Y as a $G\mathscr{T}$-functor defined on each $\mathscr{D}_{G,V}$, and we have composed the $(S^V)^\bullet$ with the composite $\mathscr{D}_{G,V} \longrightarrow \mathscr{D}_G \longrightarrow \mathscr{F}_G$ to regard the $(S^V)^\bullet$ as functors defined on $\mathscr{D}_{G,V}$. Note that $B((S^\star)^\bullet, \mathscr{D}_{G,\star}, Y)$ is not the restriction of a $\mathscr{W}_G$-G-space, but it is an $\mathscr{I}_G$-G-space. For an isometric isomorphism $V \longrightarrow V'$ in $\mathscr{I}_G$, the map

$$B((S^V)^\bullet, \mathscr{D}_{G,V}, Y) \longrightarrow B((S^{V'})^\bullet, \mathscr{D}_{G,V'}, Y)$$

is the geometric realization of the map induced at each simplicial level by the maps $\mathscr{K}_V \longrightarrow \mathscr{K}_{V'}$ and $S^V \longrightarrow S^{V'}$. Geometric realization commutes with $\wedge$, and the structure maps of the orthogonal G-spectrum $B((S^\star)^\bullet, \mathscr{D}_{G,\star}, Y)$ are geometric realizations of levelwise simplicial maps given by the maps $j\colon \mathscr{D}_{G,V} \longrightarrow \mathscr{D}_{G,V\oplus W}$ induced by the inclusions $\mathscr{K}_V \longrightarrow \mathscr{K}_{V\oplus W}$ and the maps

$$i\colon (S^V)^{(\mathbf{n},\alpha)} \wedge S^W \longrightarrow (S^{V\oplus W})^{(\mathbf{n},\alpha)} \tag{7.5}$$

defined by

$$i\big((v_1,\dots,v_n)\wedge w\big) = (v_1\wedge w,\dots,v_n\wedge w).$$

An alternative construction is to use Remark 2.18 and (3.8) in Remark 3.7 to obtain G-maps

$$B((S^V)^\bullet, \mathscr{D}_{G,V\oplus W}, Y)\wedge S^W \longrightarrow B((S^{V\oplus W})^\bullet, \mathscr{D}_{G,V\oplus W}, Y)$$

and to precompose with the G-map

$$B((S^V)^\bullet, \mathscr{D}_{G,V}, Y) \longrightarrow B((S^V)^\bullet, \mathscr{D}_{G,V\oplus W}, Y)$$

induced by $j\colon \mathscr{D}_{G,V} \longrightarrow \mathscr{D}_{G,V\oplus W}$. One can easily check that these maps do indeed give maps of bar constructions that specify the structure maps for an orthogonal G-spectrum. The projections $\mathscr{D}_{G,V} \longrightarrow \mathscr{D}_G$ induce the top map π of orthogonal G-spectra in (7.2).

Recall that $\operatorname{colim}_V \mathscr{K}_V(j) = \mathscr{K}_U(j)$, so that $\operatorname{colim}_V(\mathscr{C}_G\times\mathscr{K}_V)$ is the product $\mathscr{C}_G\times\mathscr{K}_U$, which is an E_∞ G-operad since it is the product of two such operads. Therefore the projection $(\mathscr{C}_G\times\mathscr{K}_U)(j) \longrightarrow \mathscr{C}_G(j)$ is a Λ-equivalence for all $\Lambda \in \mathbb{F}_j$

and, by Theorem 5.5, the map $\mathscr{D}_G(\mathscr{C}_G \times \mathscr{K}_U) \longrightarrow \mathscr{D}_G(\mathscr{C}_G)$ is a G-equivalence of G-COs over $\mathscr{F}_G$. The projection map

$$\pi\colon B((S^\star)^\bullet, \mathscr{D}_{G,\star}, Y) \longrightarrow B((S^\star)^\bullet, \mathscr{D}_G, Y)$$

is not a level G-equivalence, but a direct comparison of colimits shows that π is a stable equivalence. In more detail, in computing π on homotopy groups, we start from the commutative diagrams

$$\begin{array}{ccc} \Omega^V B((S^V)^\bullet, \mathscr{D}_{G,V}, Y) & \longrightarrow & \Omega^W B((S^W)^\bullet, \mathscr{D}_{G,W}, Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Omega^V B((S^V)^\bullet, \mathscr{D}_G, Y) & \longrightarrow & \Omega^W B((S^W)^\bullet, \mathscr{D}_G, Y), \end{array}$$

where $V \subset W$. We then take H-fixed points and their homotopy groups. Since the inclusions $\mathscr{D}_{G,V} \longrightarrow \mathscr{D}_{G,W}$ become isomorphisms on homotopy groups in increasing ranges of dimensions, by inspection of the homotopy types of the G-spaces comprising the Steiner operads in [12, §1.1], we conclude that π is a stable equivalence.

7.4. **The contravariant functors $I_+ \wedge (S^V)^\bullet$ on $\mathscr{D}_{G,V}$.** In the notation $I_+ \wedge (S^\star)^\bullet$ in (7.2), $\star$ is again a place holder for V, and the notation stands for $G\mathscr{T}$-functors

$$I_+ \wedge (S^V)^\bullet\colon (\mathscr{D}_{G,V})^{op} \longrightarrow \mathscr{T}_G$$

that are given on objects by sending $(\mathbf{n}, \alpha)$ to $I_+ \wedge (S^V)^{(\mathbf{n},\alpha)}$, where I is the unit interval and we have adjoined a disjoint basepoint and taken the smash product in order to have domains for based homotopies. The crux of our comparison is to specify the functors on $I_+ \wedge (S^V)^\bullet$ on morphisms in terms of homotopies that are deduced from the paths that comprise the Steiner operads.

Note that $(S^V)^{(\mathbf{n},\alpha)}$ is just $(S^V)^n$ with the G-action $\cdot_\alpha$ specified by

$$g \cdot_\alpha (x_1, \dots, x_n) = (gx_{\alpha(g)^{-1}(1)}, \dots, gx_{\alpha(g)^{-1}(n)}) = \alpha(g)_*(gx_1, \dots, gx_n).$$

Therefore, by Theorem 4.11 and Lemma 4.12, it is enough to instead define $G\mathscr{T}$-functors

$$I_+ \wedge (S^V)^\bullet\colon (\mathscr{D}_V)^{op} \longrightarrow \mathscr{T}_G$$

given on objects by sending $\mathbf{n}$ to $I_+ \wedge (S^V)^n$ and then apply the functor $\mathbb{P}$ to obtain the desired functors on $\mathscr{D}_{G,V}$. We choose to do this in order to make the definitions a little less cumbersome.

We construct the required maps on hom objects as composites

$$\mathscr{D}_V(\mathbf{m}, \mathbf{n}) \longrightarrow \mathscr{D}(\mathscr{K}_V)(\mathbf{m}, \mathbf{n}) \xrightarrow{\tilde{H}} \mathscr{T}_G(I_+ \wedge (S^V)^n, I_+ \wedge (S^V)^m).$$

The first map is the evident projection, and we shall use the same letter for maps and their composites with that projection. To define $\tilde{H}$, we shall construct a homotopy

$$H\colon I_+ \wedge (S^V)^n \wedge \mathscr{D}(\mathscr{K}_V)(\mathbf{m}, \mathbf{n}) \longrightarrow (S^V)^m \tag{7.6}$$

and then set

$$\tilde{H}(f)(t, v) = (t, H(t, v, f)), \tag{7.7}$$

where $t \in I$, $v \in (S^V)^n$, and $f \in \mathscr{D}(\mathscr{K}_V)(\mathbf{m}, \mathbf{n})$. We have written variables in the order appropriate to thinking of the homotopies H as the core of the evaluation maps of the contravariant functor $I_+ \wedge (S^V)^\bullet \colon \mathscr{D}_V \longrightarrow \mathscr{T}_G$. Note that such evaluation maps, after prolongation to $\mathscr{D}_{G,V}$, give the zeroth face operation d_0 in the simplicial G-spaces whose realizations give the central bar constructions in (7.2).

Writing H_t for H at time t, H_1 will relate to the evaluation maps of the represented functor $(S^V)^\bullet$ used in the left variable of the Segal machine and H_0 will relate to the maps that define the action of the monad $\mathbb{K}_V$ on the functor Σ^V that is used in the left variable of the operadic machine.

The following construction is the heart of the matter. Recall that a Steiner path in V is a map $h\colon I \longrightarrow R_V$ such that $h(1) = \mathrm{id}$, where R_V is the space of distance reducing embeddings $V \longrightarrow V$. The space $\mathscr{K}_V(s)$ of the Steiner operad is the space of s-tuples of Steiner paths h_r such that the $h_r(0)$ have disjoint images. We define a homotopy

$$\gamma\colon I \times S^V \times \mathscr{K}_V(s) \longrightarrow (S^V)^s$$

with coordinates γ_r by letting

$$\gamma_r(t, v, \langle h_1, \ldots, h_s\rangle) = \begin{cases} w & \text{if } h_r(t)(w) = v \\ * & \text{if } v \notin \mathrm{im}(h_r(t)). \end{cases}$$

If $t = 1$, this is just the diagonal map $S^V \longrightarrow (S^V)^s$, which is relevant to the Segal machine. If $t = 0$, this map lands in the s-fold wedge ${}^s(S^V)$ of copies of S^V since the conditions $v \in \mathrm{im}(h_r(0))$ as r varies are mutually exclusive; that is, there is at most one r such that $v \in \mathrm{im}(h_r(0))$. This is relevant to the operadic machine since the action map

$$\tilde{\alpha}\colon \Sigma^V K_V X = K_V X \wedge S^V \longrightarrow X \wedge S^V = \Sigma^V X$$

is given by

$$\tilde{\alpha}((\langle h_1, \ldots, h_s\rangle, x_1, \ldots, x_s), v) = \begin{cases} (x_r, w_r) & \text{if } h_r(0)(w_r) = v \\ * & \text{if } v \notin \mathrm{im}(h_r(0)) \text{ for } 1 \le r \le s. \end{cases}$$

Remember that we understand $\Sigma^V A$ to be $A \wedge S^V$ for a based G-space A, but we write the V coordinate on the left when looking at the evaluation maps of the functor $I_+ \wedge (S^V)^\bullet$.

We now define the homotopy H of (7.6). Recall from Definition 5.1 that

$$\mathscr{D}(\mathscr{K}_V)(\mathbf{m}, \mathbf{n}) = \coprod_{\phi\colon \mathbf{m} \longrightarrow \mathbf{n}} \prod_{1 \le j \le n} \mathscr{K}_V(s_j),$$

where $s_j = |\phi^{-1}(j)|$. Let $f = (\phi; k_1, \ldots, k_n) \in \mathscr{D}(\mathscr{K}_V)(\mathbf{m}, \mathbf{n})$, where $\phi \in \mathscr{F}(\mathbf{m}, \mathbf{n})$ and $k_j \in \mathscr{K}_V(s_j)$. For $1 \le i \le m$, define the ith coordinate H_i of H as follows. If $\phi(i) = j$, $1 \le j \le n$, and i is the rth element of $\phi^{-1}(j)$ with its natural ordering as a subset of $\mathbf{m}$, then

$$H_i(t, v_1, \ldots, v_n, f) = \gamma_r(t, v_j, k_j),$$

where γ_r is the rth coordinate of

$$\gamma\colon I \times S^V \times \mathscr{K}_V(s_j) \longrightarrow (S^V)^{s_j}.$$

If $\phi(i) = 0$, then H_i is the trivial map.

It requires some combinatorial inspection to check that these maps do indeed specify a $G\mathscr{T}$-functor $I_+ \wedge (S^V)^\bullet \colon \mathscr{D}_V^{op} \longrightarrow \mathscr{T}_G$, but we leave that to the reader. Prolonging these functors using Lemma 4.12, we obtain the $G\mathscr{T}$-functors

$$I_+ \wedge (S^V)^\bullet \colon \mathscr{D}_{G,V}^{op} \longrightarrow \mathscr{T}_G, \qquad (\mathbf{n}, \alpha) \mapsto I_+ \wedge (S^V)^{(\mathbf{n}, \alpha)},$$

needed to define the two-sided bar constructions $B(I_+ \wedge (S^V)^\bullet, \mathscr{D}_{G,V}, Y)$. Just as in §7.3, the assignment $V \mapsto B(I_+ \wedge (S^V)^\bullet, \mathscr{D}_{G,V}, Y)$ gives an $\mathscr{I}_G$-G-space, and we can construct structure maps that make it into an orthogonal G-spectrum.

We denote by $(S^V)^\bullet_1$ the restrictions of the functors $I_+ \wedge (S^V)^\bullet$ to $t = 1$. Note that the $(S^V)^\bullet_1$ are just the functors $(S^V)^\bullet$ used to define $B((S^V)^\bullet, \mathscr{D}_{G,V}, Y)$. Similarly, we denote by $(S^V)^\bullet_0$ the restrictions of the functors $I_+ \wedge (S^V)^\bullet$ to $t = 0$. For any based G-space A, let i_0 and i_1 denote the inclusions of the top and bottom copy of A into the cylinder $I_+ \wedge A$, where G acts trivially on the interval I. Note that i_0 and i_1 are G-homotopy equivalences.

The functors $(S^V)^\bullet_0$ and $(S^V)^\bullet_1$ from $\mathscr{D}^{op}_{G,V}$ to $\mathscr{T}_G$ are restrictions of $I_+ \wedge (S^V)^\bullet$, hence they commute with the face d_0, which is given by the evaluation maps of these functors. It is clear that the maps i_0 and i_1 commute with all other faces and degeneracies. Since they are levelwise G-equivalences of Reedy cofibrant simplicial G-spaces, their realizations

$$B((S^V)^\bullet_0, \mathscr{D}_{G,V}, Y) \xrightarrow{i_0} B(I_+ \wedge (S^V)^\bullet, \mathscr{D}_{G,V}, Y) \xleftarrow{i_1} B((S^V)^\bullet_1, \mathscr{D}_{G,V}, Y), \tag{7.8}$$

are G-equivalences. Therefore the maps i_0 and i_1 in (7.2) are level equivalences of orthogonal G-spectra.

7.5. **Construction and analysis of the map ι.** To define the map of orthogonal G-spectra labeled ι in (7.2), we must look more closely at $(S^V)^\bullet_0$. Note that H_0 sends an element indexed on $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ to an element of the product over $1 \leq j \leq n$ of the wedge sums ${}^{s_j}(S^V)$, where $|\phi^{-1}(j)| = s_j$. If we restrict the domain of H_0 to ${}^n(S^V) \wedge \mathscr{D}(\mathscr{K}_V)(\mathbf{m},\mathbf{n}) \subset (S^V)^n \wedge \mathscr{D}(\mathscr{K}_V)(\mathbf{m},\mathbf{n})$, we land in ${}^m(S^V)$ since for any element $(v_1, \ldots, v_n, f)$ in the domain of H_0 such that all but one of the factors v_j is $*$, only those i such that $\phi(i) = j$ can contribute a non-basepoint image. Therefore $\tilde{H}_0$ restricts to a $G\mathscr{T}$-functor ${}^\bullet(S^V)\colon (\mathscr{D}_V)^{op} \longrightarrow \mathscr{T}_G$ that on objects sends $\mathbf{n}$ to ${}^n(S^V)$ and on morphism spaces is the adjoint of the restriction of H_0 from products to wedges. On wedges, we have the composite

$${}^n(S^V) \wedge \mathscr{D}_V(\mathbf{m},\mathbf{n}) \longrightarrow {}^n(S^V) \wedge \mathscr{D}(\mathscr{K}_V)(\mathbf{m},\mathbf{n}) \longrightarrow {}^m(S^V)$$

of projection and the evident map obtained by unravelling the definition of H. Upon applying prolongation $\mathbb{P}$, we obtain a $G\mathscr{T}$-functor

$${}^\bullet(S^V)\colon (\mathscr{D}_{G,V})^{op} \longrightarrow \mathscr{T}_G.$$

It is defined on objects by sending $(\mathbf{n},\alpha)$ to ${}^{(\mathbf{n},\alpha)}(S^V)$, and it is a subfunctor of $(S^V)^\bullet_0\colon (\mathscr{D}_{G,V})^{op} \longrightarrow \mathscr{T}_G$. Note that the map $i\colon (S^V)^n \wedge S^W \longrightarrow (S^{V\oplus W})^n$ of (7.5) restricts to the canonical identification of ${}^n(S^V) \wedge S^W$ with ${}^n(S^{V\oplus W})$, and this works just as well when the twisted action of α is taken into account. Just as in §7.3, by (3.8) in Remark 3.7 these maps give rise to the structure maps of the G-spectrum $B({}^\bullet(S^\star), \mathscr{D}_{G,\star}, Y)$. The inclusions of wedges into products give the inclusions of bar constructions that together specify the map of G-spectra labeled ι in (7.2). It is worth pausing to say what is going on philosophically before showing that ι is a stable equivalence of orthogonal G-spectra. The contravariant functor $(S^V)^\bullet$ from $\mathscr{D}_{G,V}$ to $\mathscr{T}_G$ is purely categorical since it factors through $\mathscr{F}_G$ and applies just as well to give a functor $A^\bullet$ for any A. The action of $\mathscr{F}_G$ on $(S^V)^\bullet$ does not restrict to an action on the system of subspaces ${}^\bullet(S^V)$. Use of the Steiner operad in effect gives a new and more geometric functor $(S^V)^\bullet_0$. It is again defined on products, but it depends on the geometry encoded in the Steiner operads and it does restrict to a functor defined on ${}^{\bullet}(S^V)$. That is, we have commutative diagrams

$$\begin{array}{ccc}
{}^{(\mathbf{n},\beta)}(S^V) \wedge \mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) & \longrightarrow & {}^{(\mathbf{m},\alpha)}(S^V) \\
\Big\downarrow{\scriptstyle \iota\wedge \mathrm{id}} & & \Big\downarrow{\scriptstyle \iota} \\
(S^V)^{(\mathbf{n},\beta)} \wedge \mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) & \longrightarrow & (S^V)^{(\mathbf{m},\alpha)}
\end{array}$$

where the horizontal arrows are evaluation maps of the functors ${}^{\bullet}(S^V)$, and $(S^V)_0^{\bullet}$, respectively, and the vertical arrows are given by inclusions of wedges in products. The diagram displays the essential part of the map d_0 in the simplicial bar constructions that ι compares; ι is the identity on all factors in $\mathscr{D}_{G,V}$ or Y of the G-spaces of q-simplices in these bar constructions. From here, it is routine to check that these maps of bar constructions specify a map of orthogonal G-spectra.

We claim that ι is a stable equivalence, and we spend the rest of this subsection proving this. Note that both the source and target of ι are orthogonal G-spectra which at level V are geometric realizations of simplicial G-spaces. Before passage to geometric realization, we have functors $\Delta^{op} \times \mathscr{I}_G \longrightarrow \mathscr{T}_G$. It is not hard to see that the simplicial structure commutes with the spectrum structure maps, so that we can view these as simplicial orthogonal G-spectra. Geometric realization of simplicial objects can be done in any bicomplete category that is tensored over spaces, using the usual coend definition. See for example [11, Definition X.1.1] for a discussion in the case of EKMM spectra. In the case of orthogonal G-spectra, since colimits and tensoring with spaces is done levelwise, we see that for a simplicial orthogonal G-spectrum $T_{\bullet}$, the V-th level of the geometric realization is given by

$$|T_{\bullet}|(V) = |T_{\bullet}(V)|.$$

Therefore, the source and target of ι can be viewed equivalently as geometric realizations in orthogonal G-spectra of simplicial orthogonal G-spectra. In light of this we can instead think of ι as a map of simplicial orthogonal G-spectra.

First, we claim that ι gives a stable equivalence of the qth orthogonal G-spectrum for each q. This means that ι induces isomorphisms

$$\mathrm{colim}_V\, \pi_*^H\big(\Omega^V B_q({}^{\bullet}(S^V), \mathscr{D}_{G,V}, Y)\big) \longrightarrow \mathrm{colim}_V\, \pi_*^H(\Omega^V B_q((S^V)_0^{\bullet}, \mathscr{D}_{G,V}, Y)).$$

To prove this, recall that it is standard that finite wedges are finite products in the stable category [1, Proposition III.3.11]; the same proof works equivariantly. The maps $j\colon \mathscr{D}_{G,V} \longrightarrow \mathscr{D}_{G,V\oplus W}$ of Steiner operads also induce isomorphisms on homotopy groups in increasing dimensions. The claim follows by inspection of colimits.

Next, we claim that the geometric realization of the map of simplicial orthogonal G-spectra ι is also a stable equivalence. This follows from Proposition 7.9 below, but that requires a notion of Reedy cofibrancy for simplicial orthogonal G-spectra which we now explain.

There is a definition of latching objects L_nT for simplicial objects T in any cocomplete category as a certain colimit (see, for example, [20, Definition 15.2.5], [47]), of which Definition 1.8 is a specialization. Again, since colimits of orthogonal spectra are computed levelwise, we have that $(L_nT)(V) = L_n(T(V))$ for a simplicial orthogonal G-spectrum T.

A map of orthogonal G-spectra $A \longrightarrow X$ is an *h-cofibration* if it satisfies the homotopy extension property (see [19, §A.5], [24, §I.4], [25, §5] for more details). We say that a simplicial orthogonal G-spectrum is *Reedy h-cofibrant* if for all $n \geq 0$, the latching map $L_nT \longrightarrow T_n$ is an h-cofibration of orthogonal G-spectra. We claim that the simplicial orthogonal G-spectra that are the source and target of ι are Reedy h-cofibrant. Since for each V, the simplicial G-spaces are bar constructions, they are Reedy cofibrant, so it remains to show that the homotopy extensions can be made compatibly with the orthogonal G-spectrum structure. To see this, note that the latching maps are constructed from the cofibration $* \longrightarrow \mathscr{C}_V(1) = \mathscr{C}_G(1) \times \mathscr{K}_V(1)$ that includes the identity element of the operad. That map is a cofibration because of the assumption of Remark 4.16 that $* \longrightarrow \mathscr{C}_G(1)$ is a cofibration and the fact that $\{\mathrm{id}\} \hookrightarrow \mathscr{K}_V(1)$ is the inclusion of a deformation retract. The explicit retraction from $\mathscr{K}_V(1)$ onto $\{\mathrm{id}\}$ is easily seen to be compatible with the $\mathscr{I}_G$-G-space structure and with the inclusions $j\colon \mathscr{K}_V(1) \longrightarrow \mathscr{K}_{V\oplus W}(1)$, so it is compatible with the structure maps. This in turn implies the compatibility of the retracts for the latching maps of the bar constructions as V varies.

To finish the proof, we apply the following result to the map ι in (7.2).

Proposition 7.9. *Let $f_\bullet\colon T_\bullet \longrightarrow T'_\bullet$ be a map of simplicial Reedy h-cofibrant orthogonal G-spectra that is a stable equivalence at each simplicial level. Then the map of orthogonal G-spectra $|f_\bullet|$ obtained by geometric realization is a stable equivalence.*

Proof. The proof is the same as the space level analogue, using the construction of the filtration on geometric realization via pushouts (see [45, Theorem 4.15] and [11, Theorem X.2.4]). The key facts we need about h-cofibrations of orthogonal G-spectra are that they are stable under cobase change [19, Proposition A.62], that they satisfy the analogue of [11, Lemma X.2.3] (which is proven in exactly the same way), that the gluing lemma for h-cofibrations and stable equivalences holds (see [19, Corollary B.21], [24, Theorem I.4.10 (iv)]), and that the filtered colimit along h-cofibrations of a sequence of stable equivalences is a stable equivalence [19, Proposition B.17]. □

We note that the above result holds generally in any good model category tensored over spaces. A recent treatment is offered in [47, see Corollary 10.6.]. We have not quoted that result since there is no published proof that there is a model structure on the category of orthogonal G-spectra in which the cofibrations are the h-cofibrations. We believe that the methods of [3] (and [2, §6.4]) can be applied to construct one.

7.6. **Construction of the map ω.** To construct the map ω in (7.2) and thus to complete the proof of Theorem 7.1, we must define maps

$$B({}^\bullet(S^V), \mathscr{D}_{G,V}, Y) \longrightarrow B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y).$$

Both source and target are realizations of simplicial (based) G-spaces, and we define ω as the realization of a map of simplicial G-spaces. On the spaces of 0-simplices we define

$$\omega_0\colon B({}^\bullet(S^V), \mathscr{D}_{G,V}, Y)_0 = \bigvee_{(\mathbf{n},\alpha)}{}^{(\mathbf{n},\alpha)} S^V \wedge Y(\mathbf{n},\alpha)_+ \longrightarrow \Sigma^V Y_1 = B(\Sigma^V \mathbb{L}_G, \mathbb{D}_{G,V}, Y)_0$$

to be the wedge sum of the composites of the quotient maps

$$S^V \wedge Y(\mathbf{n},\alpha)_+ \longrightarrow S^V \wedge Y(\mathbf{n},\alpha)$$

with the composites

$$ {}^{(\mathbf{n},\alpha)}S^V \wedge Y(\mathbf{n},\alpha) \xrightarrow{\mathrm{id}\wedge\delta} {}^{(\mathbf{n},\alpha)}S^V \wedge Y_1^{(\mathbf{n},\alpha)} \xrightarrow{\nu} S^V \wedge Y_1 \xrightarrow{\tau} Y_1 \wedge S^V = \Sigma^V Y_1. $$

Here, for based spaces A and B, define $\nu\colon {}^nA \wedge B^n \longrightarrow A \wedge B$ by

$$ \nu((i,a),(b_1,\ldots,b_n)) = (a,b_i) $$

where (i,a) denotes the element $a \in A$ of the ith wedge summand of ${}^nA$ and $b_j \in B$, $1 \le j \le n$. We check explicitly that ν is G-equivariant:

$$ g\cdot((i,a),(b_1,\ldots,b_n)) = \big((\alpha(g)(i), g\cdot a), (g\cdot b_{\alpha(g)^{-1}(1)},\ldots,g\cdot b_{\alpha(g)^{-1}(n)})\big) \mapsto (g\cdot a, g\cdot b_i) $$

since the $\alpha(g)(i)$ position is $\alpha(g)^{-1}\alpha(g)(i) = i$. The map $\tau\colon A\wedge B \longrightarrow B \wedge A$ is the usual twist. Then all of the maps in the definition of ω_0 are equivaraint.

Notation 7.10. For $q > 0$, we may write the space of q-simplices of $B({}^\bullet(S^V), \mathscr{D}_{G,V}, Y)$ as the wedge over pairs $(\mathbf{m},\alpha), (\mathbf{n},\beta)$ of the spaces

$$ {}^{(\mathbf{n},\beta)}S^V \wedge \big(\mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) \times Z(\mathbf{m},\alpha)\big)_+, $$

where $Z(\mathbf{m},\alpha)$ is the wedge over sequences $((\mathbf{m_0},\alpha_0),\ldots,(\mathbf{m_{q-2}},\alpha_{q-2}))$ of the spaces

$$ \mathscr{D}_{G,V}\big((\mathbf{m_{q-2}},\alpha_{q-2}),(\mathbf{m},\alpha)\big) \times \cdots \times \mathscr{D}_{G,V}\big((\mathbf{m_0},\alpha_0),(\mathbf{m_1},\alpha_1)\big) \times Y(\mathbf{m_0},\alpha_0). $$

We write $\bar{Z}(\mathbf{m},\alpha)$ for the quotient of $Z(\mathbf{m},\alpha)$ obtained by replacing $\times$ by $\wedge$ here.

Recall the definition of $(\mathbb{D}_{G,V}Y)$ from (6.6). The G-space $(\mathbb{D}_{G,V}Y)(\mathbf{n},\beta)$ is a quotient of the wedge over all $(\mathbf{m},\alpha)$ of the G-spaces

$$ \mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) \wedge Y(\mathbf{m},\alpha). $$

Therefore the space $\Sigma^V\mathbb{L}_G\mathbb{D}^q_{G,V}Y$ of q-simplices of $B(\Sigma^V\mathbb{L}_G, \mathbb{D}_{G,V}, Y)$ is a quotient of the wedge over all $(\mathbf{m},\alpha)$ of the spaces

$$ (\mathscr{D}_{G,V}((\mathbf{m},\alpha),\mathbf{1}) \wedge \bar{Z}(\mathbf{m},\alpha)) \wedge S^V. $$

Define ω_q by passage to wedges over $(\mathbf{m},\alpha)$ and to quotients from the composites

$$ \begin{array}{c} {}^{(\mathbf{n},\beta)}S^V \wedge \big(\mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) \times Z(\mathbf{m},\alpha)\big)_+ \\ \big\downarrow \\ {}^{(\mathbf{n},\beta)}S^V \wedge \mathscr{D}_{G,V}((\mathbf{m},\alpha),(\mathbf{n},\beta)) \wedge \bar{Z}(\mathbf{m},\alpha) \\ \big\downarrow {\scriptstyle \mathrm{id}\wedge\delta\wedge\mathrm{id}} \\ {}^{(\mathbf{n},\beta)}S^V \wedge \mathscr{D}_{G,V}((\mathbf{m},\alpha),\mathbf{1})^{(\mathbf{n},\beta)} \wedge \bar{Z}(\mathbf{m},\alpha) \\ \big\downarrow {\scriptstyle \nu\wedge\mathrm{id}} \\ S^V \wedge \mathscr{D}_{G,V}((\mathbf{m},\alpha),\mathbf{1}) \wedge \bar{Z}(\mathbf{m},\alpha) \\ \big\downarrow {\scriptstyle \tau} \\ \mathscr{D}_{G,V}((\mathbf{m},\alpha),\mathbf{1}) \wedge \bar{Z}(\mathbf{m},\alpha) \wedge S^V, \end{array} $$

where the first map is the evident quotient map.

Since we know that these maps are G-equivariant, to check commutative diagrams we may drop the α's and β's from the notation and only consider the underlying nonequivariant spaces. On underlying spaces, the composite above is

$$\begin{array}{c} {}^nS^V \wedge \big(\mathscr{D}_V(\mathbf{m},\mathbf{n}) \times Z_m\big)_+ \\ \big\downarrow \\ {}^nS^V \wedge \mathscr{D}_V(\mathbf{m},\mathbf{n}) \wedge \bar{Z}_m \\ \big\downarrow{\scriptstyle \mathrm{id}\wedge\delta\wedge\mathrm{id}} \\ {}^nS^V \wedge \mathscr{D}_V(\mathbf{m},\mathbf{1})^n \wedge \bar{Z}_m \\ \big\downarrow{\scriptstyle \nu\wedge\mathrm{id}} \\ S^V \wedge \mathscr{D}_V(\mathbf{m},\mathbf{1}) \wedge \bar{Z}_m \\ \big\downarrow{\scriptstyle \tau} \\ \mathscr{D}_V(\mathbf{m},\mathbf{1}) \wedge \bar{Z}_m \wedge S^V. \end{array}$$

We must show that these maps ω_q specify a map of simplicial spaces. Again recall Remark 7.4. In both the categorical and monadic bar constructions, the face maps d_i for $i > 1$ are induced by composition in $\mathscr{D}_V$ and the action of $\mathscr{D}_V$ on Y. Since the ω_q are defined using the quotient maps $Z_m \longrightarrow \bar{Z}_m$, commutation with these face maps is evident. Similarly, commutation with the degeneracy maps s_i for $i > 0$ is evident. We must show commutation with s_0, d_0, and d_1. An essential point is that the Segal maps are in Π_G, and we are taking the categorical tensor product over Π_G in the target. First consider s_0 on zero simplices. For (i,v) in ${}^nS^V$, that is, v in the ith summand, and $y \in Y_n$,

$$\begin{aligned} \omega s_0((i,v),y) &= \omega((i,v),\mathrm{id}_n,y) \\ &= (\delta_i,y)\wedge v \\ &= (\mathrm{id}_1,\delta_i(y))\wedge v \\ &= s_0\omega((i,v),y). \end{aligned}$$

Here $\mathrm{id}_n \in \mathscr{D}_V(\mathbf{n},\mathbf{n})$, the third equation uses $\delta_i = \mathrm{id}_1 \circ \delta_i$ and the equivalence relation defining $\mathbb{D}_V Y$, and the last equation uses that $\omega((i,v),y) = \delta_i(y)\wedge v$. The commutation of ω and s_0 on q-simplices for $q > 0$ is similar. The following diagrams prove the commutation of ω with d_0 and d_1 on 1-simplices, and the argument for q-simplices for $q > 1$ is similar. Remember that $\mathbb{L}_G Y = Y_1$. Again, we only write the maps of underlying spaces, dropping from the notation the indices that indicate G-actions since the G-action is not relevant to checking that the diagrams commute. The top left corners of the following two diagrams are canonically isomorphic, but they are written differently to clarify the top horizontal arrows.

$$\begin{array}{ccccc}
{}^n(S^V)\wedge \mathscr{D}_V(\mathbf{m},\mathbf{n})_+\wedge (Y_m)_+ & & \xrightarrow{\;H_0\wedge\mathrm{id}\;} & & {}^m(S^V)\wedge (Y_m)_+\\
\downarrow & & & & \downarrow\\
{}^n(S^V)\wedge \mathscr{D}_V(\mathbf{m},\mathbf{n})\wedge Y_m & & \xrightarrow{\;H_0\wedge\mathrm{id}\;} & & {}^m(S^V)\wedge Y_m\\
\downarrow{\scriptstyle \mathrm{id}\wedge\delta\wedge\mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}\wedge\delta}\\
{}^n(S^V)\wedge \mathscr{D}_V(\mathbf{m},\mathbf{1})^n\wedge Y_m & & \nearrow{\scriptstyle H_0\wedge\mathrm{id}} & & {}^m(S^V)\wedge (Y_1)^m\\
\downarrow{\scriptstyle \nu\wedge\mathrm{id}} & & & & \downarrow{\scriptstyle \nu}\\
S^V\wedge \mathscr{D}_V(\mathbf{m},\mathbf{1})\wedge Y_m & & & & S^V\wedge Y_1\\
\downarrow & & & & \downarrow\\
\Sigma^V\mathbb{L}\mathbb{D}_V Y & \xrightarrow[\cong]{} & \Sigma^V\mathbb{C}_V\mathbb{L}Y & \xrightarrow[\hat{\alpha}]{} & \Sigma^V Y_1
\end{array}$$

$$\begin{array}{ccc}
{}^n(S^V)\wedge \big(\mathscr{D}_V(\mathbf{m},\mathbf{n})\times Y_m\big)_+ & \xrightarrow{\;\mathrm{id}\wedge\mu\;} & {}^n(S^V)\wedge (Y_n)_+\\
\downarrow & & \downarrow\\
{}^n(S^V)\wedge \mathscr{D}_V(\mathbf{m},\mathbf{n})\wedge Y_m & \xrightarrow{\;\mathrm{id}\wedge\mu\;} & {}^n(S^V)\wedge Y_n\\
\downarrow{\scriptstyle \mathrm{id}\wedge\delta\wedge\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\wedge\delta}\\
{}^n(S^V)\wedge \mathscr{D}_V(\mathbf{m},\mathbf{1})^n\wedge Y_m & & {}^n(S^V)\wedge (Y_1)^n\\
\downarrow{\scriptstyle \nu\wedge\mathrm{id}} & & \downarrow{\scriptstyle \nu}\\
S^V\wedge \mathscr{D}_V(\mathbf{m},\mathbf{1})\wedge Y_m & \xrightarrow{\;\mathrm{id}\wedge\mu\;} & S^V\wedge Y_1\\
\downarrow & & \downarrow\\
\Sigma^V\mathbb{L}\mathbb{D}_V Y & \xrightarrow[\Sigma^V\mathbb{L}\mu]{} & \Sigma^V Y_1
\end{array}$$

Here μ denotes the action of $\mathbb{D}_V$ on Y, which is given by the adjoints of the G-maps $Y\colon \mathscr{D}_V(\mathbf{m},\mathbf{n}) \longrightarrow \mathscr{T}_G(Y_m, Y_n)$. Both top pieces of the diagrams commute by formal inspection, the first lower rectangle commutes by the definitions of H_0 and $\hat{\alpha}$, as recalled in the previous section, and the second lower rectangle commutes by definition. It is not hard to check that the maps ω are maps of $\mathscr{I}_G$-G-spaces and that they are compatible with the structure maps, so that they give a map of orthogonal G-spectra.

8. Proofs of technical results about the operadic machine

We prove Theorem 6.13(i) and (iii) in this section. Thus let $\mathscr{C}_G$ be a Σ-free G-operad and let $\mathbb{D}$ be the monad on Π-G-spaces associated to the category of operators $\mathscr{D} = \mathscr{D}(\mathscr{C}_G)$. Part (i) asserts that $\mathbb{D}$ preserves $\mathbb{F}_\bullet$-equivalences, and its proof is the hardest equivariant work we face. It involves a detailed combinatorical analysis of the structure of the monad $\mathbb{D}$.

8.1. **The structure of $\mathbb{D}X$.** We first discuss the structure of $\mathbb{D}X$ for a Π-G-space X. This entails combinatorial analysis of Π and $\mathscr{F}$ that will also be relevant to the technical proofs for the Segal machine in §9.1.

Fix n. Then the definition of $(\mathbb{D}X)_n$ given in (6.5) implies that it is the quotient

$$\Big(\bigvee_q \mathscr{D}(\mathbf{q},\mathbf{n}) \wedge X_q\Big)/(\sim)$$

where $\sim$ is the equivalence relation specified by

$$(\psi^* d; x) \sim (d; \psi_* x)$$

for $d \in \mathscr{D}(\mathbf{q},\mathbf{n})$, $\psi \in \Pi(\mathbf{p},\mathbf{q})$ and $x \in X_p$. By Lemma 1.16, we can replace wedges and smash products by disjoint unions and products, that is, $(\mathbb{D}X)_n$ is the quotient

$$\Big(\coprod_q \mathscr{D}(\mathbf{q},\mathbf{n}) \times X_q\Big)/(\sim).$$

Recall that the morphism space $\mathscr{D}(\mathbf{q},\mathbf{n})$ is given by the disjoint union of components indexed on all $\phi\colon \mathbf{q} \longrightarrow \mathbf{n}$ in $\mathscr{F}$

$$\mathscr{D}(\mathbf{q},\mathbf{n}) = \coprod_{\phi\in\mathscr{F}(\mathbf{q},\mathbf{n})} \prod_{1\le j\le n} \mathscr{C}_G(j_\phi),$$

where $j_\phi = |\phi^{-1}(j)|$. The basepoint is the component indexed on $\phi = 0_{q,n}$. We write a non-basepoint morphism as $(\phi; c)$, where $c = (c_1, \ldots, c_n)$ with $c_j \in \mathscr{C}_G(j_\phi)$. For a morphism $\psi\colon \mathbf{p} \longrightarrow \mathbf{q}$ in Π, write

$$\psi^*\colon \prod_{1\le j\le n} \mathscr{C}_G(j_\phi) \longrightarrow \prod_{1\le j\le n} \mathscr{C}_G(j_{\phi\psi})$$

for the map $\mathscr{D}(\mathbf{q},\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{p},\mathbf{n})$ induced by ψ from the component of ϕ to the component of $\phi\circ\psi$, and write $\psi_*\colon X_p \longrightarrow X_q$ for the induced morphism giving the covariant functoriality of X. Then the equivalence relation $\sim$ takes the form

$$(\phi\circ\psi; \psi^* c; x) \sim (\phi; c; \psi_* x) \tag{8.1}$$

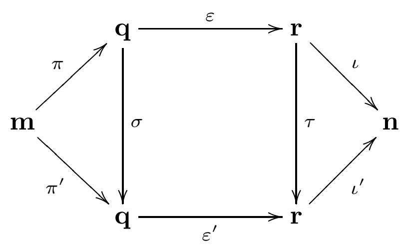
for $(\phi; c) \in \mathscr{D}(\mathbf{q},\mathbf{n})$ and $x \in X_p$. We shall use the identifications induced by $\sim$ to cut down on the number of components that need be considered. To this end, we first describe the structure of Π and $\mathscr{F}$, partially following [38, §5].

Definition 8.2. Recall that Π is the subcategory of $\mathscr{F}$ with the same objects and those maps $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $|\phi^{-1}(j)| \le 1$ for $1 \le j \le n$. A map $\pi \in \Pi$ is a *projection* if $|\pi^{-1}(j)| = 1$ for $1 \le j \le n$. A map $\iota \in \Pi$ is an *injection* if $\iota^{-1}(0) = \{0\}$. The permutations are the maps in Π that are both injections and projections. A projection or injection is *proper* if it is not a permutation. Recall that $\mathcal{I}$ is the subcategory of injections in Π.

Definition 8.3. A map $\phi \in \mathscr{F}$ is *ordered* (or more accurately monotonic) if $i < j$ implies $\phi(i) \le \phi(j)$; note that this does not restrict the ordering of those i such that $\phi(i) = k$ for some fixed k. A map $\varepsilon \in \mathscr{F}$ is *effective* if $\varepsilon^{-1}(0) = \{0\}$, and an effective map ε is *essential* if it is surjective, that is, if $j_\varepsilon \ge 1$ for $1 \le j \le n$.

Observe that every morphism of Π is a composite of proper projections, proper injections, and permutations, and that $\mathscr{F}$ is generated under wedge sum and composition by Π and the single product morphism $\phi_2\colon \mathbf{2} \longrightarrow \mathbf{1}$.

Lemma 8.4. *A map $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathscr{F}$ factors as the composite $\iota \circ \varepsilon \circ \pi$ of a projection π, an essential map ε and an injection ι, uniquely up to permutation. That is, given two such decompositions of ϕ, there are permutations σ and τ making the following diagram commute.*



Proof. The projection π is determined up to order by which $i \geq 1$ in $\mathbf{m}$ are mapped to 0 in $\mathbf{n}$. The injection ι is determined up to order by which $j \geq 1$ in $\mathbf{n}$ are not in the image of ϕ. Up to order, ε is the wedge sum in $\mathscr{F}$ of the product maps $\phi_{s_j} : \mathbf{s_j} \longrightarrow \mathbf{1}$, where $s_j = |\phi^{-1}(j)|$ for those j such that $1 \leq j \leq n$ and $\phi^{-1}(j)$ is nonempty. Up to permutation, these s_j run through the numbers $|\varepsilon^{-1}(j)|$, $1 \leq j \leq r$. $\square$

Remark 8.5. We remark that Π and $\mathscr{F}$ are dualizable Reedy categories, as defined by Berger and Moerdijk [4, Definition 1.1 and Example 1.9(b)]. They write $\mathscr{F}^+$ for the monomorphisms and $\mathscr{F}_-$ for the epimorphisms in $\mathscr{F}$. We have factored epimorphisms into composites of projections and essential maps to make the structure clearer. We say that an $\mathscr{F}$-G-space is Reedy cofibrant if its underlying Π-G-space is so (see Definition 6.11). In discarded drafts, we proved that all bar construction $\mathscr{F}$-G-spaces used in the Segal machine are Reedy cofibrant.

A map $\phi\colon \mathbf{q} \longrightarrow \mathbf{n}$ in $\mathscr{F}$ is ineffective if and only if it factors as a composite

$$\mathbf{q} \xrightarrow{\pi} \mathbf{p} \xrightarrow{\zeta} \mathbf{n},$$

where $p = q - 0_\phi$, π is the proper ordered projection such that $\pi(k) = 0$ if and only if $\phi(k) = 0$, and ζ is an effective morphism. Then $j_\zeta = j_\phi$ for $j \geq 1$ and, as in Observation 5.2, $\pi^*(c) = c$ for any $c \in \prod_j \mathscr{C}_G(j_\zeta)$. Therefore

$$(\phi; c; x) = (\zeta\pi; \pi^*(c); x) \sim (\zeta; c; \pi_*(x)).$$

This says both that we may restrict to those wedge summands that are indexed on the effective morphisms of $\mathscr{F}$, ignoring the ineffective ones, and that we can ignore the proper projections in Π, restricting further analysis to $\sim$ applied to morphisms of $\mathcal{I} \subset \Pi$. Here we must start paying attention to permutations.

Lemma 8.6. *If $\varepsilon\colon \mathbf{p} \longrightarrow \mathbf{n}$ is an effective morphism in $\mathscr{F}$, there is a permutation $\nu \in \Sigma_p$ such that $\varepsilon \circ \nu$ is ordered; ν is not unique, but the ordered morphism $\varepsilon \circ \nu$ is.*

Applying $\sim$ to the permutations ν, we can further restrict to components indexed on ordered effective morphisms. We abbreviate notation.

Notation 8.7. We say that an ordered effective morphism in $\mathscr{F}$ is an OE-function. If ε is effective and $\varepsilon \circ \nu$ is ordered, we call it the OE-function associated to ε. We let $\mathscr{E}(\mathbf{p}, \mathbf{n})$ denote the set of all OE-functions $\mathbf{p} \longrightarrow \mathbf{n}$.

Definition 8.8. Let $\varepsilon\colon \mathbf{p} \to \mathbf{n}$ be an OE-function. Note that the sum of the j_ε is p and define

$$\Sigma(\varepsilon) = \Sigma_{1_\varepsilon} \times \cdots \times \Sigma_{n_\varepsilon} \subset \Sigma_p,$$

where the inclusion is determined by identifying $\mathbf{p}$ with $\mathbf{1}_\varepsilon \vee \cdots \vee \mathbf{n}_\varepsilon$. In other words, we partition $\{1, \dots, p\}$ into n blocks of letters, as dictated by ε.

Lemma 8.9. *If $\varepsilon\colon \mathbf{p} \to \mathbf{n}$ is an OE-function and $\nu \in \Sigma_p$, then $\varepsilon \circ \nu$ is ordered (and hence equal to ε) if and only ν is in the subgroup $\Sigma(\varepsilon)$.*

The equivalence relation $\sim$ is defined in terms of precomposition of morphisms of $\mathscr{F}$ with morphisms of Π, while the action of Π on $\mathbb{D}X$ is defined in terms of postcomposition of morphisms of $\mathscr{F}$ with morphisms of Π. Especially for permutations, these are related. We discuss composition with permutations on both sides in the following three remarks.

Remark 8.10. Let $\varepsilon\colon \mathbf{p} \longrightarrow \mathbf{n}$ be an OE-function and let $\sigma \in \Sigma_n$. Define $\tau(\sigma) \in \Sigma_p$ to be the permutation that permutes the n blocks of letters $\{1_\varepsilon, \dots, n_\varepsilon\}$ as σ permutes n letters. Then $\sigma \circ \varepsilon \circ \tau(\sigma)^{-1}$ is again ordered, and it is the OE-function associated to $\sigma \circ \varepsilon$. Moreover, the function $\tau = \tau_\varepsilon\colon \Sigma_n \longrightarrow \Sigma_p$ is a homomorphism. Inspecting our identifications and using Observation 5.2, we see that postcomposition with σ sends a point with representative $(\varepsilon; c_1, \dots, c_n; x)$ to the point with representative

$$\big(\sigma\varepsilon\tau(\sigma)^{-1}; c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; \tau(\sigma)_* x\big). \tag{8.11}$$

Remark 8.12. One can think of an OE-function $\varepsilon\colon \mathbf{p} \longrightarrow \mathbf{n}$ as an ordered partition of the set $\{1, \dots, p\}$ into n ordered subsets. Observe that ε is essential if and only if $j_\varepsilon > 0$ for $1 \le j \le n$, so that our n subsets are all nonempty. Define the signature of ε to be the unordered set of numbers $\{1_\varepsilon, \dots, n_\varepsilon\}$. The action of Σ_n permutes partitions with the same signature (and thus the same p). The OE-functions ε and $\sigma \circ \varepsilon \circ \tau(\sigma)^{-1}$ have the same signature, and any two ordered partitions with the same signature are connected this way.

Remark 8.13. If $\varepsilon\colon \mathbf{p} \longrightarrow \mathbf{n}$ is an OE-function and $\rho \in \Sigma_p$, then

$$(\varepsilon \circ \rho; c; x) \sim (\varepsilon; (\rho^{-1})^* c; \rho_*^{-1} x).$$

We have now accounted for $\sim$ applied to all proper projections and to all permutations. It remains to consider proper injections ι. For any such $\iota\colon \mathbf{p} \longrightarrow \mathbf{q}$, there is a permutation $\nu \in \Sigma_p$ such that $\iota \circ \nu$ is ordered. Recall that we have the ordered injections $\sigma_i\colon \mathbf{p-1} \longrightarrow \mathbf{p}$ that skip i, $1 \le i \le p$. Every proper ordered injection is a composite of such σ_i, so it remains to account for $\sim$ applied to the σ_i. These give an equivalence relation on

$$\coprod_q \coprod_{\varepsilon \in \mathscr{E}(\mathbf{q},\mathbf{n})} \Big(\prod_{1 \le j \le n} \mathscr{C}_G(j_\varepsilon) \Big) \times_{\Sigma(\varepsilon)} X_q$$

whose quotient is $(\mathbb{D}X)_n$. The component with $q = 0$ gives the basepoint.

The monad $\mathbb{D}$, like the monad $\mathbb{C}_G$, is filtered. Its pth filtration at level n, denoted $F_p(\mathbb{D}X)_n$, is the image of the components indexed on $q \le p$. We can think of the quotient as given by filtration-lowering "basepoint identifications", namely

$$(\varepsilon; c_1, \dots, c_n; (\sigma_i)_* x) \sim (\varepsilon \circ \sigma_i; c_1, \dots, c_{i-1}, \sigma_r c_i, c_{i+1}, \dots, c_n; x), \tag{8.14}$$

for some $i = 1, \ldots, q$. Here r is the position of i within its block of j_ε letters, where $j = \varepsilon(i)$, and $\sigma_r \colon \mathscr{C}_G(j_\varepsilon) \longrightarrow \mathscr{C}_G(j_\varepsilon - 1)$ is the map from (6.1).[19] In equivalent abbreviated notation, we write this as

$$(\varepsilon; c; (\sigma_i)_* x) \sim (\varepsilon \circ \sigma_i; \sigma_i^* c; x) \tag{8.15}$$

Definition 8.16. Fixing an $\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})$, write $[c; x]$ for an element of

$$\prod_{1 \le j \le n} \mathscr{C}_G(j_\varepsilon) \times_{\Sigma(\varepsilon)} X_p,$$

meaning that $(c; x)$ is a representative element for an orbit $[c; x]$ under the action of $\Sigma(\varepsilon)$. Recall from Definition 6.11 that x is degenerate if $x \in L_p X$, that is, if $x = (\sigma_i)_* y$ for some $y \in X_{p-1}$ and some i. Say that $(\varepsilon; [c; x])$ is degenerate if x is degenerate. Since $\tau \circ \sigma_i$ is a proper injection for any $\tau \in \Sigma(\varepsilon)$ and any i, the condition of being degenerate is independent of the orbit representative $(c; x)$.

By use of (8.15), we reach the following description of the elements of $(\mathbb{D}X)_n$.

Lemma 8.17. *A point of $(\mathbb{D}X)_n$ has a unique nondegenerate representative $(\varepsilon; [c; x])$.*

Just as nonequivariantly ([38, p. 218]), we have pushouts of $(G \times \Sigma_n)$-spaces

$$\begin{array}{ccc} \coprod_{\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})} \big(\prod_{1 \le j \le n} \mathscr{C}_G(j_\varepsilon)\big) \times_{\Sigma(\varepsilon)} L_p X & \xrightarrow{\nu} & F_{p-1}(\mathbb{D}X)_n \\ \big\downarrow & & \big\downarrow \\ \coprod_{\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})} \big(\prod_{1 \le j \le n} \mathscr{C}_G(j_\varepsilon)\big) \times_{\Sigma(\varepsilon)} X_p & \longrightarrow & F_p(\mathbb{D}X)_n \end{array} \tag{8.18}$$

With notation as in (8.15), the map ν sends a point with orbit representative $(\varepsilon; c; (\sigma_i)_* x)$ to the point with orbit representative $(\varepsilon \circ \sigma_i; \sigma_i^* c; x)$.

Recall that X is Reedy cofibrant if the inclusion $L_p X \longrightarrow X_p$ is a $(G \times \Sigma_p)$-cofibration for each p. When X is Reedy cofibrant, each component of the left vertical map is a $(G \times \Sigma_p)$-cofibration before taking the quotient by $\Sigma(\varepsilon)$, so the map on $\Sigma(\varepsilon)$ quotients is a G-cofibration by [7, Lemma A.2.3]. Since Σ_n acts by permuting components and the factors of the displayed products, it follows that the left vertical map is a $(G \times \Sigma_n)$-cofibration, hence so is the right vertical arrow.

8.2. **The proof that $\mathbb{D}X$ is Reedy cofibrant.** Let $T \subset \mathbf{n} \setminus \{0\}$. We use the notation Σ_T for the subgroup of Σ_n of those permutations τ such that $\tau(T) = T$. Note that Σ_T consists of those permutations that act separately within T and its complement. As a group, Σ_T it is isomorphic to $\Sigma_{|T|} \times \Sigma_{n-|T|}$.

Notation 8.19. We denote by $\sigma_T \colon \mathbf{n} - |\mathbf{T}| \longrightarrow \mathbf{n}$ the ordered injection that misses the elements of T. It can be written as $\sigma_T = \sigma_{i_k} \cdots \sigma_{i_1}$ where $i_1 < \cdots < i_k$ are the elements of T. If T is empty we use the convention that $\sigma_T = \mathrm{id}$ (which makes sense as the empty composition). Note that for a Π-G-space X,

$$\sigma_T X_{n-|T|} = \bigcap_{i \in T} \sigma_i X_{n-1}.$$

For the case $T = \emptyset$ this matches the intuition that an empty intersection of subsets of X_n should be X_n.

[19]For consistency of notation here, we might have written σ_r^* instead of σ_r, as appears in (6.1).

Note that any ordered injection is of the form σ_T. By Remark 6.12, it suffices to show that for all subsets T, the maps

$$\sigma_T\colon (\mathbb{D}X)_{n-|T|} \longrightarrow (\mathbb{D}X)_n \tag{8.20}$$

are $(G\times\Sigma_T)$-cofibrations. Note that the action of Σ_T on $(\mathbb{D}X)_{n-|T|}$ is given by restricting to the action on the block $\mathbf{n}\setminus T$ and identifying it with the set $\mathbf{n}-|\mathbf{T}|$.

Consider the cube obtained by mapping the pushout square of (8.18) for $(\mathbb{D}X)_{n-|T|}$ to the pushout square (8.18) for $(\mathbb{D}X)_n$. Write σ_T for the maps from the four corners of the first square to the four corners of the second square. We will prove by induction on p that the map

$$\sigma_T\colon F_p(\mathbb{D}X)_{n-|T|} \longrightarrow F_p(\mathbb{D}X)_n \tag{8.21}$$

is a $(G\times\Sigma_T)$-cofibration, and we assume this for the map with p replaced by $p-1$. The vertical maps in the diagram (8.18) for $\mathbb{D}X_{n-|T|}$ are $(G\times\Sigma_{n-|T|})$-cofibrations, so they are $(G\times\Sigma_T)$-cofibrations via the action described above. The vertical maps in the diagram (8.18) for $\mathbb{D}X_n$ are $(G\times\Sigma_n)$-cofibrations, so in particular they are also $(G\times\Sigma_T)$-cofibrations.

The map σ_T on the left corners of the diagram is given by

$$(\varepsilon,[(c_1,\dots,c_{n-|T|}),x)]\mapsto(\sigma_T\circ\varepsilon,[(d,x)]),$$

where d is the n-tuple given by

$$d_j=\begin{cases}0\in\mathscr{C}_G(0) & \text{if } j\in T\\ c_{\sigma_T^{-1}(j)} & \text{if } j\notin T\end{cases}$$

It is not hard to see that for the left entries of the pushout diagram, the map σ_T is the inclusion of those components labeled by maps $\varepsilon\colon\mathbf{p}\longrightarrow\mathbf{n}$ that miss the elements of T. The groups $\Sigma_{n-|T|}$ and Σ_n act on the source and target, respectively, as stated in Remark 8.10. In particular, both actions shuffle components, hence the inclusion of components is a $(G\times\Sigma_T)$-cofibration. By the induction hypothesis, the map connecting the top right corners of the cube is also a $(G\times\Sigma_T)$-cofibration.

It follows from Proposition 10.1 that the map (8.21) connecting the bottom right corners of the cube is a $(G\times\Sigma_T)$-cofibration, as claimed, noting that

$$\sigma_T\left(\coprod_{\varepsilon\in\mathscr{E}(\mathbf{p},\mathbf{n}-|\mathbf{T}|)}(\prod_{j=1}^{n-|T|}\mathscr{C}_G(j_\varepsilon))\times_{\Sigma(\varepsilon)}X_p\right)\cap\left(\coprod_{\varepsilon\in\mathscr{E}(\mathbf{p},\mathbf{n})}(\prod_{j=1}^{n}\mathscr{C}_G(j_\varepsilon))\times_{\Sigma(\varepsilon)}L_pX\right)$$

is equal to

$$\sigma_T\left(\coprod_{\varepsilon\in\mathscr{E}(\mathbf{p},\mathbf{n}-|\mathbf{T}|)}(\prod_{j=1}^{n-|T|}\mathscr{C}_G(j_\varepsilon))\times_{\Sigma(\varepsilon)}L_pX\right).$$

To complete the proof that the map (8.20) is a $(G\times\Sigma_T)$-cofibration, we use Proposition 10.2 to conclude that the map of colimits

$$\sigma_T\colon(\mathbb{D}X)_{n-|T|}=\operatorname{colim}_p F_p(\mathbb{D}X)_{n-|T|}\longrightarrow\operatorname{colim}_p F_p(\mathbb{D}X)_n=(\mathbb{D}X)_n$$

is a $(G\times\Sigma_T)$-cofibration. We must check that the intersection condition

$$\sigma_T(F_p(\mathbb{D}X)_{n-|T|})\cap F_{p-1}(\mathbb{D}X)_n=\sigma_T(F_{p-1}(\mathbb{D}X)_{n-|T|})$$

of Proposition 10.2 is satisfied. One inclusion is obvious. For the other, take an element $(\varepsilon,[c,x])\in F_p(\mathbb{D}X)_{n-|T|}\setminus F_{p-1}(\mathbb{D}X)_{n-|T|}$; in particular, $x\in X_p\setminus L_pX$. Then $\sigma_T(\varepsilon,[c,x])=(\sigma_T\varepsilon,[\sigma_T^*c,x])\in F_p(\mathbb{D}X)_n\setminus F_{p-1}(\mathbb{D}X)_n$, as required.

8.3. **The proof that $\mathbb{D}$ preserves $\mathbb{F}_\bullet$-equivalences.** We assume given an $\mathbb{F}_\bullet$-equivalence $f\colon X \longrightarrow Y$ between Reedy cofibrant Π-G-spaces. Theorem 6.13(i) says that $\mathbb{D}f\colon \mathbb{D}X \longrightarrow \mathbb{D}Y$ is an $\mathbb{F}_\bullet$-equivalence. We shall prove it by proving by induction on p that f induces an $\mathbb{F}_n$-equivalence $F_p(\mathbb{D}X)_n \longrightarrow F_p(\mathbb{D}Y)_n$ for each n and each $p \geq 0$, there being nothing to prove when $p = 0$. By the usual gluing lemma on pushouts, proven equivariantly in [6, Theorem A.4.4] (but also a model theoretic formality), it suffices to prove that the maps induced by f on the source and target of the left vertical arrow in (8.18) induce equivalences on Λ-fixed point spaces, where $\Lambda \subset G \times \Sigma_n$ and $\Lambda \cap \Sigma_n = \{e\}$. We have $\Lambda = \{(h, \alpha(h)) \mid h \in H\}$ for some subgroup H of G and homomorphism $\alpha\colon H \longrightarrow \Sigma_n$, and we regard $\mathbf{n}$ as a based H-set via α. Fixing Λ for the rest of the section, we shall prove Theorem 6.13(i) by analyzing Λ-fixed points.

We first consider the target, that is the lower left corner of the diagram. To clarify the argument, we separate out some of its combinatorics before proceeding.

Definition 8.22. Let $\varepsilon\colon \mathbf{p} \longrightarrow \mathbf{n}$ be an OE-function, let $\tau\colon \Sigma_n \longrightarrow \Sigma_p$ be the homomorphism determined by ε as defined in Remark 8.10, and define $\beta\colon H \longrightarrow \Sigma_p$ to be the composite homomorphism $\tau\alpha$. Say that ε is Λ-fixed if $\alpha(h)\varepsilon = \varepsilon\beta(h)$ for all $h \in H$. Note that this implies that $j_\varepsilon = k_\varepsilon$ if j and k are in the same H-orbit.

Define $\mathscr{E}(\mathbf{p}, \mathbf{n})^\Lambda$ to be the set of all Λ-fixed OE-functions $\mathbf{p} \longrightarrow \mathbf{n}$. Fix $\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})^\Lambda$. Say that a function

$$\gamma = (\gamma_1, \ldots, \gamma_n)\colon H \longrightarrow \Sigma(\varepsilon)$$

is *admissible*, or admissible with respect to α, if

$$\gamma_j(hk) = \gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(k) \tag{8.23}$$

for $h, k \in H$ and $1 \leq j \leq n$. For any function $\gamma\colon H \longrightarrow \Sigma(\varepsilon)$, define a function[20] $\gamma \cdot \beta\colon H \longrightarrow \Sigma_p$ by $(\gamma \cdot \beta)(h) = \gamma(h)\beta(h)$.

We leave the combinatorial proof of the following lemma to the reader. When the action of H on $\mathbf{n} \setminus \mathbf{0}$ has a single orbit, there is a conceptual rather than combinatorial proof using wreath products.

Lemma 8.24. *Fix $\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})^\Lambda$. A function $\gamma\colon H \longrightarrow \Sigma(\varepsilon)$ is admissible with respect to α if and only if $\gamma \cdot \beta$ is a homomorphism $H \longrightarrow \Sigma_p$.*

The following result is the central step of the proof of Theorem 6.13(i). It identifies the Λ-fixed points of the bottom left corner of the pushout diagram (8.18).

Proposition 8.25. *Let $\Lambda = \{(h, \alpha(h))\}$ and assume that the action of H on $\mathbf{n}\setminus\{0\}$ defined by α is transitive. Then there is a natural homeomorphism*

$$\Big(\textstyle\coprod_{\varepsilon \in \mathscr{E}(\mathbf{p},\mathbf{n})} \big(\big(\prod_{1 \leq j \leq n} \mathscr{C}_G(j_\varepsilon)\big) \times_{\Sigma(\varepsilon)} X_p\big)\Big)^\Lambda$$

$$\Big\downarrow \omega$$

$$\textstyle\coprod_{\varepsilon \in \mathscr{E}(\mathbf{p},\mathbf{n})^\Lambda} \Big(\coprod_{\gamma\colon H \longrightarrow \Sigma(\varepsilon)} \mathscr{C}_G(1_\varepsilon)^{\Lambda_\gamma^1} \times X_p^{\Lambda_\gamma}\Big)/\Sigma(\varepsilon).$$

In the target, the second wedge runs over all admissible functions

$$\gamma = (\gamma_1, \ldots, \gamma_n)\colon H \longrightarrow \Sigma(\varepsilon);$$

[20] We use the notation $\cdot$ since we often use juxtaposition to mean composition in this section.

the groups Λ_γ and Λ^1_γ are specified by

$$\Lambda_\gamma = \{(h, (\gamma \cdot \beta)(h)) | h \in H\} \subset G \times \Sigma_p$$

and

$$\Lambda^1_\gamma = \{(k, \gamma_1(k)) | k \in K\} \subset G \times \Sigma_{1_\varepsilon},$$

where $K \subset H$ is the isotropy group of 1 under the action of H on $\mathbf{n}$ given by α.

Proof. Since $\alpha(k)(1) = 1$ for $k \in K$, γ_1 is a homomorphism $K \longrightarrow \Sigma_{1_\varepsilon}$ by specialization of (8.23). In the target, we pass to orbits from the $\Sigma(\varepsilon)$-action defined on the term in parentheses by

$$\rho(\gamma; c; x) = (\rho * \gamma; c\rho_1^{-1}; \rho_* x),$$

Here $\rho = (\rho_1, \dots, \rho_n)$ is in $\Sigma(\varepsilon)$, γ is admissible, $c \in \mathscr{C}_G(1_\varepsilon)^{\Lambda^1_\gamma}$, and $x \in X_p^{\Lambda_\gamma}$. The jth coordinate of $\rho * \gamma$ is defined by

$$(\rho * \gamma)_j(h) = \rho_j \gamma_j(h) \rho^{-1}_{\alpha(h)^{-1}(j)}. \tag{8.26}$$

A quick check of definitions shows that

$$(\rho * \gamma) \cdot \beta = \rho(\gamma \cdot \beta)\rho^{-1},$$

which also implies that $\rho * \gamma$ is admissible since γ is admissible. Similarly, $c\rho_1^{-1}$ is fixed by $\Lambda^1_{\rho*\gamma}$ since c is fixed by Λ^1_γ and $\rho_*(x)$ is fixed by $\Lambda_{\rho*\gamma}$ since x is fixed by Λ_γ. Thus the action makes sense. Moreover, as we shall need later, this action is free. If $\rho(\gamma; c; x) = (\gamma; c; x)$, then $c\rho_1^{-1} = c$ and thus $\rho_1 = 1$ since Σ_{1_ε} acts freely on $\mathscr{C}_G(1_\varepsilon)$. Also, $\rho * \gamma = \gamma$ and thus $\rho_j^{-1}\gamma_j(h)\rho_{\alpha(h)^{-1}(j)} = \gamma_j(h)$ for all h. Taking $j = 1$, this implies that $\rho_{\alpha(h)^{-1}(1)} = 1$ for all h. Since we are assuming the action of H induced by α on $\mathbf{n} \setminus \{0\}$ is transitive, this implies that $\rho = 1 \in \Sigma(\varepsilon)$.

We turn to the promised homeomorphism. By (8.11), for a point z represented by $(\varepsilon; c_1, \dots, c_n; x)$, $c_j \in \mathscr{C}_G(j_\varepsilon)$ and $x \in X_p$, $(h, \alpha(h))z$ is represented by

$$\big(\alpha(h)\varepsilon\beta(h)^{-1}; hc_{\alpha(h)^{-1}(1)}, \dots, hc_{\alpha(h)^{-1}(n)}; \beta(h)_*(hx)\big)$$

where, as before, $\beta = \tau\alpha$. Assume that z is fixed by Λ. Then we must have $\alpha(h)\varepsilon\beta(h)^{-1} = \varepsilon$, so that $\alpha(h)\varepsilon = \varepsilon\beta(h)$ and thus $\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})^\Lambda$. We must also have

$$(c_1, \dots, c_n; x) \sim \big(hc_{\alpha(h)^{-1}(1)}, \dots hc_{\alpha(h)^{-1}(n)}; \beta(h)_*(hx)\big),$$

so that for each $h \in H$ there exists $\gamma(h) = \gamma_1(h) \times \cdots \times \gamma_n(h) \in \Sigma(\varepsilon)$ such that

$$c_j\gamma_j(h) = hc_{\alpha(h)^{-1}(j)} \quad \text{and} \quad x = \gamma(h)_*\beta(h)_*(hx). \tag{8.27}$$

Note that for any given n-tuple $(c_1, \dots, c_n)$, $\gamma(h)$ is unique since the action of $\Sigma(\varepsilon)$ on $\prod_{1 \le j \le n} \mathscr{C}_G(j_\varepsilon)$ is free. For $h, k \in H$ and $1 \le j \le n$,

$$\begin{aligned} c_j\gamma_j(hk) &= (hk)c_{\alpha(hk)^{-1}(j)} \\ &= h(kc_{\alpha(k)^{-1}(\alpha(h)^{-1}(j))}) \\ &= hc_{\alpha(h)^{-1}(j)}\gamma_{\alpha(h)^{-1}(j)}(k) \\ &= c_j\gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(k). \end{aligned}$$

Since the action of Σ_{j_ε} on $\mathscr{C}_G(j_\varepsilon)$ is free, this implies that (8.23) holds, so that γ is admissible. Note that we have not yet used that the action given by α is transitive.

Now the map ω is defined by

$$\omega(\varepsilon; c_1, \dots, c_n; x) = (\varepsilon; \gamma; c_1; x).$$

We see from (8.27) that x is in $X_p^{\Lambda_\gamma}$ and that c_1 is in $\mathscr{C}_G(1_\varepsilon)^{\Lambda^1_\gamma}$, the latter using the fact that K is the isotropy group of $1 \in \mathbf{n} \setminus \{0\}$. We must check that our map is well-defined. Thus suppose that

$$(\varepsilon; c_1, \ldots, c_n; x) \sim (\varepsilon; d_1, \ldots, d_n; y).$$

Then there exists $\rho \in \Sigma(\varepsilon)$ such that $c_j = d_j\rho_j$ and $y = \rho_* x$. Using (8.26) and (8.27),

$$\begin{aligned} hd_{\alpha(h)^{-1}(j)} &= hc_{\alpha(h)^{-1}(j)}\rho^{-1}_{\alpha(h)^{-1}(j)} \\ &= c_j\gamma_j(h)\rho^{-1}_{\alpha(h)^{-1}(j)} \\ &= d_j\rho_j\gamma_j(h)\rho^{-1}_{\alpha(h)^{-1}(j)} \\ &= d_j(\rho * \gamma(h))_j \end{aligned}$$

Thus, comparing with (8.27) for $(d_1, \ldots, d_n; y)$, and using the freeness of the action, we see that

$$\omega(\varepsilon; d_1, \ldots, d_n; y) = (\varepsilon; \rho * \gamma, d_1, y) = (\varepsilon; \rho * \gamma, c_1\rho_1^{-1}, \rho_* x) = \rho(\varepsilon; \gamma; c_1; x),$$

so that the targets of our equivalent elements are equivalent.

Clearly ω is continuous since it is obtained by passage to orbits from a (disconnected) cover by restriction to subspaces of the projection that forgets the coordinates $(c_2, \ldots, c_n)$.

To define ω^{-1}, first choose coset representatives for H/K where K is the isotropy group of 1, that is, choose $h_j \in H$ such that $\alpha(h_j)(1) = j$ for $1 \le j \le n$, taking $h_1 = e$. Then define ω^{-1} by

$$\omega^{-1}(\varepsilon; \gamma; c; x) = (\varepsilon; c_1, \ldots, c_n; x)$$

where $c_j = h_j c\gamma_j(h_j)^{-1}$. Note that $c_1 = c$ and that the map does not depend on the choice of coset representatives. Here, ε is Λ-fixed, $\gamma\colon H \longrightarrow \Sigma(\varepsilon)$ is admissible, $c \in \mathscr{C}_G(1_\varepsilon)^{\Lambda^1_\gamma}$ and $x \in X_p^{\Lambda_\gamma}$. We must show that $\omega^{-1}(\varepsilon; \gamma; c; x)$ is fixed by Λ. First, note that $(h, \alpha(h))$ sends ε to $\alpha(h)\varepsilon\beta(h)^{-1} = \varepsilon$, since $\varepsilon \in \mathscr{E}(\mathbf{p}, \mathbf{n})^\Lambda$. Omitting ε from the notation for readability,

$$\begin{aligned} (h, \alpha(h))(c_1, \ldots, c_n, x) &= (hc_{\alpha(h)^{-1}(1)},\ \ldots,\ hc_{\alpha(h)^{-1}(n)}, \beta(h)_*(hx)) \\ &= (hc_{\alpha(h)^{-1}(1)},\ \ldots,\ hc_{\alpha(h)^{-1}(n)}, \gamma(h)_*^{-1}(x)) \\ &\sim (hc_{\alpha(h)^{-1}(1)}\gamma_1(h)^{-1},\ \ldots,\ hc_{\alpha(h)^{-1}(n)}\gamma_n(h)^{-1}, x). \end{aligned}$$

We claim that $hc_{\alpha(h)^{-1}(j)}\gamma_j(h)^{-1} = c_j$. The definition of $c_{\alpha(h)^{-1}(j)}$ gives us the following identification.

$$hc_{\alpha(h)^{-1}(j)}\gamma_j(h)^{-1} = h\big(h_{\alpha(h)^{-1}(j)}c\gamma_{\alpha(h)^{-1}(j)}(h_{\alpha(h)^{-1}(j)})^{-1}\big)\gamma_j(h)^{-1}$$

Now note that

$$\alpha(hh_{\alpha(h)^{-1}(j)})(1) = \alpha(h)(\alpha(h)^{-1}(j)) = j,$$

thus $hh_{\alpha(h)^{-1}(j)}$ is in the coset represented by h_j. So there exists a $k \in K$ such that $hh_{\alpha(h)^{-1}(j)} = h_j k$. Since γ satisfies equation (8.23), we get the following:

$$\begin{aligned} \gamma_j(h)\gamma_{\alpha(h)^{-1}(j)}(h_{\alpha(h)^{-1}(j)}) &= \gamma_j(hh_{\alpha(h)^{-1}}(j)) \\ &= \gamma_j(h_j k) \\ &= \gamma_j(h_j)\gamma_{\alpha(h_j)^{-1}(j)}(k) \\ &= \gamma_j(h_j)\gamma_1(k) \end{aligned}$$

Thus

$$hc_{\alpha(h^{-1})(j)}\gamma_j(h)^{-1} = h_j kc\gamma_1(k)^{-1}\gamma_j(h_j)^{-1} = h_j c\gamma_j(h_j)^{-1} = c_j$$

as claimed. Thus the map really does land in the Λ-fixed points.

To show that ω^{-1} is well-defined, note that if $(\varepsilon;\gamma;c;x) \sim (\varepsilon, \rho * \gamma; c\rho_1^{-1}, \rho_* x)$ for some $\rho \in \Sigma(\varepsilon)$, we have that ω^{-1} sends the latter to $(\varepsilon; d_1, \dots, d_n; \rho_* x)$, where

$$\begin{aligned} d_j =& h_j c\rho_1^{-1}(\rho_j\gamma_j(h_j)\rho^{-1}_{\alpha(h_j)^{-1}(j)})^{-1} \\ =& h_j c\rho_1^{-1}(\rho_j\gamma_j(h_j)\rho_1^{-1})^{-1} \\ =& h_j c\rho_1^{-1}\rho_1\gamma_j(h_j)^{-1}\rho_j^{-1} \\ =& c_j\rho_j^{-1} \end{aligned}$$

Thus

$$(d_1, \dots, d_n, \rho_* x) = \rho \cdot (c_1, \dots, c_n, x),$$

so the map is well-defined. This map is clearly continuous.

It is easy to see that the map forward and the map backward composed in either order are the identity, hence we get the claimed homeomorphism. □

The restriction to transitive action by α in the previous result serves only to simplify the combinatorics. The following remark indicates the changes that are needed to deal with the general case.

Remark 8.28. When the action of H on $\mathbf{n} \setminus \{0\}$ is not transitive, we argue analogously to Lemma 2.7 and Theorem 5.5 to obtain an anaologous homeomorphism. We break the H-set $\mathbf{n} \setminus \{0\}$ given by α into a disjoint union of orbits H/K_a of size $n_a = |H/K_a|$, where $\sum_a n_a = n$ and K_a is the isotropy group of the initial element, denoted 1_a, in its orbit in $\mathbf{n} \setminus \{0\}$. That breaks $\mathbf{n}$ into the wedge of subsets $\mathbf{n}_a$ and breaks $\Sigma(\varepsilon)$ into a product of subgroups $\Sigma(\varepsilon(a)) = \prod_{j\in H/K_a} \Sigma(j_\varepsilon)$. Paying attention to the ordering, the product of the $\mathscr{C}_G(j_\varepsilon)$ in the source of ω breaks into the product over a of those $\mathscr{C}_G(j_\varepsilon)$ such that j is in the ath orbit of $\mathbf{n} \setminus \{0\}$. To generalize the target of ω accordingly, define subgroups

$$\Lambda^{1_a}_\gamma = \{(k, \gamma_{1_a}(k)) | k \in K_a\} \subset G \times \Sigma_{(1_a)_\varepsilon}$$

and replace $\mathscr{C}_G(1_\varepsilon)^{\Lambda^1_\gamma}$ by the product over a of the $\mathscr{C}_G((1_a)_\varepsilon)^{\Lambda^{1_a}_\gamma}$. With these changes of source and target and just a bit of extra bookkeeping, it is straightforward to state and prove the general analogue of Proposition 8.25.

In the single orbit case, observe that if $f\colon X_p \longrightarrow Y_p$ is a Λ_γ-equivalence, it induces an equivalence on the target of ω before passage to quotients under the action of $\Sigma(\varepsilon)$. Since the $\Sigma(\varepsilon)$ action is free, the equivalence passes to the quotients. Generalizing to the multi-orbit case, this concludes our proof that we have a Λ-equivalence in the lower left corner of the pushout diagram (8.18).

We next consider the upper left corner of (8.18). Precisely the same argument as that just given, but with X_p replaced by L_pX, identifies the Λ-fixed subspace of the upper left corner in terms of appropriate fixed point subspaces of L_pX. Therefore the same argument as that just given shows that the following result implies that f induces an equivalence on the upper left corner, as needed to complete the proof of Theorem 6.13(i).

Proposition 8.29. *If $f\colon X \longrightarrow Y$ is an $\mathbb{F}_\bullet$-equivalence of Reedy cofibrant Π-G-spaces, then $f\colon L_nX \longrightarrow L_nY$ is an $\mathbb{F}_n$-equivalence for each n.*

To prove this, we need more combinatorics to identify $(L_nX)^\Lambda$, where $\Lambda \in \mathbb{F}_n$. We again take $\Lambda = \{(h, \alpha(h)) \mid \alpha\colon H \longrightarrow \Sigma_n\} \subset G \times \Sigma_n$ and again view $\mathbf{n} \setminus \{0\}$ as an H-set via α. For a subset U of $\mathbf{n}\setminus\{0\}$ with u elements, recall from Notation 8.19 that $\sigma_U : \mathbf{n} - \mathbf{u} \longrightarrow \mathbf{n}$ denotes the ordered injection that skips the elements in U. It is a composite of degeneracies $\sigma_U = \sigma_{i_k} \cdots \sigma_{i_1}$ where $i_1 < \cdots < i_k$ are the elements of U. Given $1 \leq i \leq n$, let $\pi_i\colon \mathbf{n} \longrightarrow \mathbf{n-1}$ be the ordered projection that sends i to 0. Similarly, define $\pi_U\colon \mathbf{n} \longrightarrow \mathbf{n} - \mathbf{u}$ to be the ordered projection that sends the elements of U to 0; explicitly, $\pi_U = \pi_{i_1} \cdots \pi_{i_k}$. Note that $\pi_U\sigma_U$ is the identity, and

$$\sigma_U\pi_U(i) = \begin{cases} i & \text{if } i \notin U \\ 0 & \text{if } i \in U. \end{cases}$$

Remark 8.30. The maps σ_i correspond to the degeneracies in Δ^{op} via the inclusion $F\colon \Delta^{op} \longrightarrow \mathscr{F}$, except there is a shift since we are indexing on the non-zero elements of $\mathbf{n}$. The maps π_i are mostly invisible to Δ. The collection of maps $\{\sigma, \pi\}$ satisfies the following subset of the simplicial relations, as can be easily checked.

$$\begin{aligned} \pi_i\pi_j &= \pi_{j-1}\pi_i \quad \text{if } i < j \\ \sigma_j\sigma_i &= \sigma_i\sigma_{j-1} \quad \text{if } i < j \\ \pi_i\sigma_j &= \sigma_{j-1}\pi_i \quad \text{if } i < j \\ \pi_i\sigma_i &= \mathrm{id}\,. \end{aligned}$$

Now assume that $U \subset \mathbf{n}\setminus\{0\}$ is a H-subset of $\mathbf{n}\setminus\{0\}$ and note that its complement is also a H-subset of $\mathbf{n} \setminus \{0\}$. For $h \in H$, define

$$\alpha_U(h) = \pi_U\alpha(h)\sigma_U : \mathbf{n} - \mathbf{u} \longrightarrow \mathbf{n} - \mathbf{u}.$$

This is essentially the restriction of $\alpha(h)$ to $\mathbf{n} \setminus U$, but using the ordered inclusion σ_U to identify that set with $\mathbf{n} - \mathbf{u}$. Note that $\alpha_U(e) = \mathrm{id}$ and that

$$\alpha(h)\sigma_U = \sigma_U\pi_U\alpha(h)\sigma_U = \sigma_U\alpha_U(h) \tag{8.31}$$

since $\sigma_U\pi_U$ is the identity on $\mathbf{n} \setminus U$ and 0 on U and since $\alpha(h)\sigma_U$ is 0 on U and takes $\mathbf{n} \setminus U$ to itself. This implies that α_U is a homomorphism $H \longrightarrow \Sigma_{n-u}$ since

$$\alpha_U(h)\alpha_U(k) = \pi_U\alpha(h)\sigma_U\pi_U\alpha(k)\sigma_U = \pi_U\alpha(h)\alpha(k)\sigma_U = \pi_U\alpha(hk)\sigma_U = \alpha_U(hk),$$

where the second equality uses (8.31) with h replaced by k. Thus we can define

$$\Lambda_U = \{(h, \alpha_U(h)) \mid h \in H\}.$$

We have the following identification of $(L_nX)^\Lambda$. Henceforward we abbreviate notation for the action of $\mathscr{F}$ on X, writing σ_U for σ_{U*} and so forth.

Lemma 8.32.

$$(L_nX)^\Lambda = \bigcup \sigma_U\left((X_{n-u})^{\Lambda_U}\right)$$

where the union runs over the H-orbits $U \subset \mathbf{n} \setminus \{0\}$.

Proof. For $U \subset \mathbf{n} \setminus \{0\}$ and $z \in (X_{n-u})^{\Lambda_U}$, $\sigma_U z$ is a Λ-fixed point since

$$(h, \alpha(h)) \cdot (\sigma_U z) = \alpha(h)(h\sigma_U z) = \alpha(h)\sigma_U(hz) = \sigma_U\alpha_U(h)(hz) = \sigma_U z,$$

for $h \in H$; the next to last equality holds by (8.31) and the last holds since z is a Λ_U-fixed point. This gives one inclusion.

For the other inclusion, we first note that the action of Σ_n on L_nX can be expressed as follows. Let $\rho \in \Sigma_n$ and $x \in L_nX$, so that $x = \sigma_i y$ for some i, $1 \le i \le n$, and some $y \in X_{n-1}$. Then

$$\rho x = \rho\sigma_i y = \sigma_{\rho(i)}\tilde{\rho} y$$

where $\tilde{\rho} = \pi_{\rho(i)}\rho\sigma_i$ is a permutation in Σ_{n-1}, as is easily checked. Now suppose that x is a Λ-fixed point and let U be the H-orbit of i in $\mathbf{n}\backslash\{0\}$. Then $x \in \sigma_j(X_{n-1})$ for all $j \in U$ since

$$x = (h, \alpha(h)) \cdot x = \alpha(h)(hx) = \alpha(h)(h\sigma_i y) = \alpha(h)\sigma_i(hy) = \sigma_{\alpha(h)(i)}\widetilde{\alpha(h)}(hy)$$

for $h \in H$. It follows that $x = \sigma_U z$, where $z = \pi_U x \in X_{n-u}$. We claim that z is a Λ_U-fixed point. Indeed

$$\sigma_U z = x = \alpha(h)(hx) = \alpha(h)\sigma_U(hz) = \sigma_U \alpha_U(h)(hz),$$

and the claim follows since σ_U is injective. This gives the other inclusion. □

Next we consider the intersection of the subspaces corresponding to two such subsets U.

Lemma 8.33. *Let U and V be disjoint H-subsets of the action of H on $\mathbf{n} \setminus \{0\}$ given by α. Let U have u elements and V have v elements. Then*

$$\begin{aligned}\sigma_U\Big((X_{n-u})^{\Lambda_U}\Big) \cap \sigma_V\Big((X_{n-v})^{\Lambda_V}\Big) &= \sigma_{U\cup V}\Big((X_{n-u-v})^{\Lambda_{U\cup V}}\Big)\\ &= \sigma_U\sigma_{\tilde{V}}\Big((X_{n-u-v})^{(\Lambda_U)_{\tilde{V}}}\Big),\end{aligned}$$

where $\widetilde{V}$, also with v elements, is the subset of $\mathbf{n} \setminus (U \cup \{0\})$ that σ_U maps onto the subset V of $\mathbf{n} \setminus \{0\}$.

Proof. The notation obscures the fact that $\tilde{V}$ depends on U, but we always use it directly following the U to which it pertains. Note that the last equality follows from the facts that $\sigma_{U\cup V} = \sigma_U\sigma_{\tilde{V}}$ and $\Lambda_{U\cup V} = (\Lambda_U)_{\tilde{V}}$. Suppose that x is in the intersection. Then $x = \sigma_i y_i$ for all $i \in U$ and also for all $i \in V$, hence $x = \sigma_{U\cup V} z$, where $z = \pi_{U\cup V} x \in X_{n-|U\cup V|}$. By the same argument as in the proof of the previous lemma, z is a $\Lambda_{U\cup V}$-fixed point.

For the opposite inclusion, let $x = \sigma_{U\cup V} z$, where $z \in (X_{n-|U\cup V|})^{\Lambda_{U\cup V}}$. Then $x = \sigma_U y_U$ where $y_U = \sigma_{\tilde{V}} z$ is a Λ_U-fixed point by the same argument as in the previous proof. Indeed, its first part works for all H-subsets, not just orbits, to give that x is a Λ-fixed point, and then its second part gives that y_U is a Λ_U-fixed point. The symmetric argument gives that $x = \sigma_V y_V$ where y_V is a Λ_V-fixed point. □

Finally, we use these lemmas to prove Proposition 8.29.

Proof of Proposition 8.29. We first observe that an argument similar to the proof of Lemma 8.32 shows that for all orbits (in fact, all H-subsets) U of $\mathbf{n}$,

$$\sigma_U\big((X_{n-u})^{\Lambda_U}\big) = \big(\sigma_U(X_{n-u})\big)^{\Lambda} = \sigma_U(X_{n-u}) \cap X_n^{\Lambda}.$$

Since σ_U is a closed inclusion, this shows that $\sigma_U\big((X_{n-u})^{\Lambda_U}\big)$ is closed in X_n^{Λ}. Let $U_1, \ldots, U_m$ be the orbits of the H-set $\mathbf{n} \setminus \{0\}$, with corresponding cardinalities

$u_1, \ldots, u_m$. For $1 \le k \le m$, we have

$$\begin{aligned} \sigma_{U_k}(X^{\Lambda_{U_k}}_{n-u_k}) \cap \Big(\bigcup_{i=1}^{k-1} \sigma_{U_i}(X^{\Lambda_{U_i}}_{n-u_i}) \Big) &= \bigcup_{i=1}^{k-1} \sigma_{U_k}(X^{\Lambda_{U_k}}_{n-u_k}) \cap \sigma_{U_i}(X^{\Lambda_{U_i}}_{n-u_i}) \\ &= \bigcup_{i=1}^{k-1} \sigma_{U_k}\sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i}) \\ &= \sigma_{U_k}\big(\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i})\big) \end{aligned}$$

where the next to last equality holds by Lemma 8.33 and the others are standard set manipulations. We therefore have inclusions which give the following pushout diagram.

$$\begin{array}{ccc} \sigma_{U_k}\big(\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i})\big) & \longrightarrow & \bigcup_{i=1}^{k-1} \sigma_{U_i}(X^{\Lambda_{U_i}}_{n-u_i}) \\ \downarrow & & \downarrow \\ \sigma_{U_k}(X^{\Lambda_{U_k}}_{n-u_k}) & \longrightarrow & \bigcup_{i=1}^{k} \sigma_{U_i}(X^{\Lambda_{U_i}}_{n-u_i}) \end{array} \tag{8.34}$$

This diagram is clearly a pushout of sets. It is a pushout of spaces since the lower horizontal and the right vertical arrows are closed inclusions by our first observation, so that their target has the topology of the union. By Lemma 8.32, the lower right corner is $(L_nX)^\Lambda$ when $k = m$.

We claim that the left vertical arrow and therefore the right vertical arrow is a cofibration for each $k \le m$, the assertion being vacuous if $k = 1$. We prove this by induction on n. Thus suppose it holds for all values less than n. In particular, assume that it holds for each $n - u_k$. Note that the orbits of $\mathbf{n} \setminus (U_k \cup \{0\})$ are $\tilde{U}_1, \ldots, \tilde{U}_{k-1}, \tilde{U}_{k+1}, \ldots, \tilde{U}_m$. Since σ_{U_k} (or any restriction of it to a subspace) is a homeomorphism onto its image, it suffices to prove that the left vertical map is a cofibration before application of σ_{U_k}. With n replaced by $n - u_k$ and with each $\tilde{U}_i$ referring to U_k, the induction hypothesis applied to right vertical arrows gives that

$$\bigcup_{i=1}^{k-1} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i}) \longrightarrow \bigcup_{i=1,\ldots,k-1,k+1} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i})$$

and each map

$$\bigcup_{i=1,\ldots,k-1,k+1,\ldots,k+j-1} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i}) \longrightarrow \bigcup_{i=1,\ldots,k-1,k+1,\ldots,k+j} \sigma_{\tilde{U}_i}(X^{(\Lambda_{U_k})_{\tilde{U}_i}}_{n-u_k-u_i}),$$

$2 \le j \le m$, is a cofibration. When $j = m$, the target of the last map is $(L_{n-u_k}X)^{\Lambda_{U_k}}$, and

$$(L_{n-u_k}X)^{\Lambda_{U_k}} \longrightarrow (X_{n-u_k})^{\Lambda_{U_k}}$$

is a cofibration since X is Reedy cofibrant. Application of σ_{U_k} to the composite of these cofibrations gives the left vertical arrow, completing the proof of our claim.

This allows us to prove by induction on k that we have a weak equivalence

$$\bigcup_{i=1}^{k} \sigma_{U_i}(X^{\Lambda_{U_i}}_{n-u_i}) \longrightarrow \bigcup_{i=1}^{k} \sigma_{U_i}(Y^{\Lambda_{U_i}}_{n-u_i})$$

for any $k \leq m$. Since the σ_{U_i} are homeomorphisms onto their images, the base case $k = 1$ holds by our assumption on f. The inductive step is an application of the gluing lemma to the pushout diagram (8.34). For $k = m$, this gives that

$$(L_n X)^\Lambda \longrightarrow (L_n Y)^\Lambda$$

is a weak equivalence, as required. □

9. Proofs of technical results about the Segal machine

We prove Propositions 2.22 and 2.23 in §9.1, focusing on the combinatorial analysis of the simplicial, conceptual, and homotopical versions of the Segal machine. We prove the group completion property, Proposition 2.12, in §9.2. We prove the positive linearity property, Theorem 3.18, in §9.3. The main step is to verify that the relevant $\mathscr{W}_G$-G-spaces satisfy the wedge axiom formulated in Definition 9.9, and we prove that in §9.4.

9.1. **Combinatorial analysis of $A^\bullet \otimes_{\mathscr{F}} X$.** Let X be an $\mathscr{F}$-G-space. We first compare the functor $A^\bullet \otimes_{\mathscr{F}} X$ with geometric realization. Recall that the objects of Δ are the ordered finite sets $[n] = \{0, 1, \ldots, n\}$ and its morphisms are the non-decreasing functions. As in §2.2, let F denote the simplicial circle $S^1_s = \Delta[1]/\partial\Delta[1]$ viewed as a functor $\Delta^{op} \longrightarrow \mathscr{F}$. Take the topological circle to be $S^1 = I/\partial I$.

Remark 9.1. The functor F sends the ordered set $[n]$ to the based set $\mathbf{n}$. For a map $\phi\colon [n] \longrightarrow [m]$ in Δ and $1 \leq j \leq n$, $F\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ sends i to j if $\phi(j-1) < i \leq \phi(j)$ and sends i to 0 if there is no such j. Thus

$$(F\phi)^{-1}(j) = \{i \mid \phi(j-1) < i \leq \phi(j)\} \quad \text{for} \quad 1 \leq j \leq n.$$

If $\delta_i\colon [n-1] \longrightarrow [n]$ and $\sigma_i\colon [n+1] \longrightarrow [n]$, $0 \leq i \leq n$, are the standard face and degeneracy maps that skip or repeat i in the target, then $F\delta_i = d_i\colon \mathbf{n} \longrightarrow \mathbf{n-1}$ is the ordered surjection that repeats i for $i < n$ but sends n to 0 if $i = n$, and $F\sigma_i = s_i\colon \mathbf{n} \longrightarrow \mathbf{n+1}$ is the ordered injection[21] that skips $i+1$. Note in particular that $F\delta_1 = \phi_2\colon \mathbf{2} \longrightarrow \mathbf{1}$, which sends 1 and 2 to 1. In $\mathscr{F}$, we also have ordered projections $\pi_i\colon \mathbf{n} \longrightarrow \mathbf{n-1}$, used in §8.3, that are mostly invisible to Δ. The map π_i sends i to 0 and it sends j to j if $j < i$ and to $j-1$ if $j > i$.

To prove Proposition 2.23, we must compare

$$|X| = X \otimes_\Delta \Delta \quad \text{with} \quad X(A) := \mathbb{P}(X)(A) = A^\bullet \otimes_{\mathscr{F}} X$$

when $A = S^1$. To aid in the comparison, we rewrite $|X|$ as $\Delta \otimes_{\Delta^{op}} X$. Here Δ on the left is the covariant functor $\Delta \longrightarrow \mathscr{U}$ that sends $[n]$ to the topological simplex

$$\Delta_n = \{(t_1, \ldots, t_n) \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}.$$

Nowadays it is more usual to use tuples $(s_0, s_1, \ldots, s_n)$ such that $0 \leq s_i \leq 1$ and $\sum_i s_i = 1$, but the formulae $s_i = t_{i+1} - t_i$ and $t_i = s_0 + \cdots + s_{i-1}$ translate between the two descriptions. For $0 \leq i \leq n$, the face map $\delta_i\colon \Delta_{n-1} \longrightarrow \Delta_n$ and the degeneracy map $\sigma_i\colon \Delta_{n+1} \longrightarrow \Delta_n$ are given by

$$\delta_i(t_1, \ldots, t_{n-1}) = \begin{cases} (0, t_1, \ldots, t_{n-1}) & \text{if } i = 0 \\ (t_1, \ldots t_{i-1}, t_i, t_i, t_{i+1}, \ldots t_{n-1}) & \text{if } 0 < i < n \\ (t_1, \ldots, t_{n-1}, 1) & \text{if } i = n \end{cases}$$

$$\sigma_i(t_1, \ldots, t_{n+1}) = (t_1, \ldots, t_i, t_{i+2}, \ldots, t_{n+1}).$$

[21] In §6.2 and §8 it was denoted σ_{i+1} as a map in the category $\mathcal{I}$ of finite sets and injections.

Map Δ_n to $(S^1)^n$ by sending $(t_1, \ldots, t_n) \in \Delta_n$ to $(t_1, \ldots, t_n) \in (S^1)^n$. Looking at the definition of the functor F, we see that this defines a map $\xi\colon \Delta \longrightarrow (S^1)^\bullet$ of cosimplicial spaces,[22] where $(S^1)^\bullet$ is a cosimplicial space by pullback along F. Therefore ξ induces a natural map

$$\xi_*\colon |X| = \Delta \otimes_{\Delta^{op}} X \longrightarrow (S^1)^\bullet \otimes_{\mathscr{F}} X = X(S^1).$$

Recall that every point of $|X|$ is represented by a unique point (u, x) such that $u \in \Delta_p$ is an interior point and $x \in X_p$ is a nondegenerate point [39, Lemma 14.2]. Said another way, $|X|$ is filtered with strata

$$F_p|X| \setminus F_{p-1}|X| = (\Delta_p \setminus \partial\Delta_p) \times (X_p \setminus L_pX),$$

where L_pX, the pth latching space, is the union of the subspaces $s_i(X_{p-1})$ (see Definition 1.8). The construction of $F_p|X|$ from $F_{p-1}|X|$ is summarized by the concatenated pushout diagrams

(9.2)
$$\begin{array}{ccccc}
\partial\Delta_p \times L_pX & \longrightarrow & \Delta_p \times L_pX & & \\
\downarrow & & \downarrow & & \\
\partial\Delta_p \times X_p & \longrightarrow & \Delta_p \times L_pX \cup \partial\Delta_p \times X_p & \longrightarrow & F_{p-1}|X| \\
 & & \downarrow & & \downarrow \\
 & & \Delta_p \times X_p & \longrightarrow & F_p|X|
\end{array}$$

We shall describe $X(A)$ similarly for all $A \in G\mathscr{W}$, and we shall specialize to $A = S^1$ to see that ξ_* is a natural homeomorphism, using results about the structure of $\mathscr{F}$ recorded in §8.1.

Remark 9.3. The functor F is a map of generalized Reedy categories in the sense of [4]. Recall that the latching G-space $L_pX \subset X_p$ of an $\mathscr{F}$-G-space X is defined to be the latching space of its underlying Π-G-space, as defined in Definition 6.11. The $\mathscr{F}$-G-space X also has a latching space when regarded as a simplicial G-space via F. Direct comparison of definitions shows that these two latching spaces are the same.

By Lemma 1.16, the G-space $X(A)$ is the quotient of $\coprod_{n\geq 0} A^n \times X_n$ obtained by identifying $(\phi^*(a), x)$ with $(a, \phi_*(x))$ for all $\phi\colon \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathscr{F}$, $a \in A^n$, and $x \in X_m$. Here $\phi^*(a_1, \ldots, a_n) = (b_1, \ldots, b_m)$ where $b_i = a_{\phi(i)}$, with $b_i = *$ if $\phi(i) = 0$, and $\phi_*(x)$ is given by the covariant functoriality of X. The image of $\coprod_{n\leq p} A^n \times X_n$ is topologized as a quotient and denoted $F_pX(A)$, and $X(A)$ is given the topology of the union of the $F_pX(A)$.

Notation 9.4. For an unbased G-space U, the configuration space $\mathbf{Conf}(U, p)$ is the G-subspace of X^p of points $(u_1, \ldots, u_p)$ such that $u_i \neq u_j$ for $i \neq j$. For a based G-space A, the *based fat diagonal* $\delta A^p \subset A^p$ is the G-subspace of points $(a_1, \ldots, a_p)$ such that either some a_i is the basepoint or $a_i = a_j$ for some $i \neq j$. Observe that

$$A^p \setminus \delta A^p = \mathbf{Conf}(A \setminus \{*\}, p).$$

Lemma 9.5. $F_0X(A) = * \times X_0$. *For* $p \geq 1$, *the stratum*

$$\mathbf{Conf}(A \setminus \{*\}, p) \times_{\Sigma_p} (X_p \setminus L_pX).$$

[22] Warning: we are thinking of both source and target as cosimplicial *unbased* spaces.

The construction of $F_pX(A)$ from $F_{p-1}X(A)$ is summarized by the concatenated pushout diagrams

$$\begin{array}{ccccc}
\delta A^p \times_{\Sigma_p} L_pX & \longrightarrow & A^p \times_{\Sigma_p} L_pX & & \\
\downarrow & & \downarrow & & \\
\delta A^p \times_{\Sigma_p} X_p & \longrightarrow & A^p \times_{\Sigma_p} L_pX \cup \delta A^p \times_{\Sigma_p} X_p & \longrightarrow & F_{p-1}X(A) \\
 & & \downarrow & & \downarrow \\
 & & A^p \times_{\Sigma_p} X_p & \longrightarrow & F_pX(A)
\end{array}$$

Proof. Using projections in $\mathscr{F}$, every point of $\coprod_{n\geq 1} A^n \times X_n$ is equivalent to a point (a,x) such that either $n=0$ or no coordinate of a is the basepoint of A. Using permutations and canonical maps $\phi_i\colon \mathbf{i} \longrightarrow \mathbf{1}$ when i coordinates of a are equal, every point is equivalent to a point (a,x) such that a has no repeated coordinates. We must take orbits under the action of Σ_p as stated to avoid double counting of elements. Using injections, every point is equivalent to a point (a,x) such that x is nondegenerate. Taking care of the order in which the cited operations are taken, using Lemma 8.4, the conclusion follows. □

It is now easy to see that $\xi\colon |X| \longrightarrow X(S^1)$ is a homeomorphism.

Proof of Proposition 2.23. As noted in Remark 9.3, the latching subspaces L_pX for X as a Δ^{op}-G-space and as an $\mathscr{F}$-G-space agree. We consider the strata on the filtrations for $|X|$ and $X(S^1)$ and find that ξ defines homeomorphisms

$$(\Delta_p \setminus \partial\Delta_p) \times (X_p \setminus L_pX) \longrightarrow \mathbf{Conf}(S^1 \setminus \{*\}, p) \times_{\Sigma_p} (X_p \setminus L_pX).$$

To see this, identify $\mathbf{Conf}(S^1 \setminus \{*\}, p)$ with $\mathbf{Conf}(I \setminus \{0,1\}, p)$. Then ξ sends a point $((t_1,\dots,t_p),x)$ in the domain with $0 < t_1 < \cdots < t_p < 1$ to the equivalence class of $((t_1,\dots,t_p),x)$ in the target. Note that $((t_1,\dots,t_p),x)$ is the unique representative of its class such that the coordinates t_i are in increasing order. □

Proof of Proposition 2.22. Recall that we have the classifying $\mathscr{F}$-G-space $\mathbb{B}X$ with qth space $|X[q]|$ and that $\mathbb{S}_nX = (\mathbb{B}^nX)_1$. We must prove that $(\mathbb{B}^nX)_1$ is homeomorphic to $X(S^n)$, and Proposition 2.23 shows that $|X[q]| \cong X[q](S^1)$. For any $A \in G\mathscr{W}$, we have an $\mathscr{F}$-G-space $A \otimes X$ with q-th space $(X[q])(A)$. Thus $\mathbb{B}X \cong S^1 \otimes X$. We claim that $S^n \otimes X$ is isomorphic to $\mathbb{B}^nX$; evaluating at $q=1$, this gives the desired homeomorphism. Since $S^n = S^1 \wedge S^{n-1}$, the claim is an immediate induction from the following result, which is essentially Segal's [51, Lemma 3.7]. □

Proposition 9.6. *For $A, B \in G\mathscr{W}$, $A \otimes (B \otimes X)$ and $(A \wedge B) \otimes X$ are naturally isomorphic.*

Proof. Recall that $X[q]$ has jth space X_{jq}, that is $X(\mathbf{j} \wedge \mathbf{q})$. Thus $X[q](B)$ is a quotient of $\coprod_j B^j \times X(\mathbf{j} \wedge \mathbf{q})$. We can write it schematically as $B^\bullet \otimes_{\mathscr{F}} X(\bullet \wedge \mathbf{q})$. Writing $\star$ for another schematic variable, we can write the q-th space of $A \otimes (B \otimes X)$ as

$$A^\star \otimes_{\mathscr{F}} (B^\bullet \otimes_{\mathscr{F}} X(\bullet \wedge \star \wedge \mathbf{q})).$$

It is a quotient of $\coprod_{i,j} A^i \times B^j \times X_{jiq}$. We define a map $A \otimes (B \otimes X) \longrightarrow (A \wedge B) \otimes X$ by passage to coequalizers from the maps that send

$$((a_1, \dots, a_i), (b_1, \dots, b_j), x) \quad \text{to} \quad ((a_r \wedge b_s), x)$$

where the a_r's are in A, the b_s's are in B, and $x \in X_{jiq}$. Here $(a_r \wedge b_s)$ means the set of $a_r \wedge b_s$ in reverse lexicographic order. Indeed, since $\mathbf{j} \wedge \mathbf{i}$ is ordered lexicographically, we must order the $a_r \wedge b_s$ to match that. However, r runs through indices in $\mathbf{i}$ and s runs through indices in $\mathbf{j}$, so the reverse lexicographical order is required. In the other direction, given $a_t \wedge b_t$ for $1 \le t \le k$ and $x \in X_{kq}$ we map

$$((a_1 \wedge b_1, \dots, a_k \wedge b_k), x) \quad \text{to} \quad ((a_1, \dots, a_k), (b_1, \dots, b_k), \Delta_*(x)),$$

where $\Delta\colon \mathbf{k} \wedge \mathbf{q} \longrightarrow \mathbf{k} \wedge \mathbf{k} \wedge \mathbf{q}$ is $\Delta \wedge \mathrm{id}$. Following Segal [51, p. 304], "we shall omit the verification that the two maps are well-defined and inverse to each other". It can be seen in terms of the explicit description of the filtration strata in Lemma 9.5. □

9.2. **The proof of the group completion property.** Let X be a special $\mathscr{F}$-G-space, where G is any topological group. Then the Segal maps

$$\delta^H\colon X_n^H \longrightarrow (X_1^n)^H = (X_1^H)^n$$

are weak equivalences and X^H is a nonequivariant special $\mathscr{F}$-space. We emphasize that we only need this naive condition: we do not require X to be $\mathbb{F}_\bullet$-special.

It is convenient but not essential to modify the definition of a special $\mathscr{F}$-G-space by requiring the Segal maps δ to be G-homotopy equivalences rather than just weak G-equivalences, and then their fixed point maps δ^H are also homotopy equivalences. We can make this assumption without loss of generality since we are free to replace X by ΓX, where Γ is a cofibrant approximation functor on G-spaces. We give X_1 a Hopf G-space structure by choosing a G-homotopy inverse to δ when $n = 2$ and using ϕ_2. Then X_1 and each X_1^H are homotopy associative and commutative, as in our standing conventions about Hopf G-spaces in §1.1. We could instead work with weak Hopf G-spaces, but doing so explicitly only obscures the exposition.

We must prove that the canonical map $\eta\colon X_1 \longrightarrow \Omega|X|$ is a group completion in the sense of Definition 1.6. Passage to H-fixed point spaces commutes with realization, as we see by inspection of elements of $|X|^H$ in nondegenerate form $|x, u|$ where x is a nondegenerate n-simplex and u is an interior point of Δ_n for some n: x must be H-fixed. It also commutes with taking loops since G acts trivially on S^1. Thus the equivariant case of Proposition 2.12 follows directly from the nonequivariant case. We therefore take $G = e$ and ignore equivariance in the rest of this section.

If M is a topological monoid, we use its product to define a simplicial space $B_\bullet M$ with $B_n M = M^n$. Then $|B_\bullet M|$ is just the classical classifying space BM. When M is commutative, $B_\bullet M$ is the simplicial space obtained by pullback of the evident special $\mathscr{F}$-space with nth space M^n. When X is a special $\mathscr{F}$-space its first space X_1 plays the role of M. Since $X_0 = *$, X_1 has a specified unit element e. Spaces of the form X_1 for a special $\mathscr{F}$-space X give the Segal version of an E_∞-space.

It makes sense to ask that a simplicial space X be reduced and special, since we can use iterated face maps to define Segal maps $X_n \longrightarrow X_1^n$. The Segal maps of $\mathscr{F}$-spaces are the images under F of these more general Segal maps. Then X_1 is a Hopf space with product induced by a homotopy inverse to the second Segal map and the map $d_1\colon X_2 \longrightarrow X_1$. When X is an $\mathscr{F}$-space, $\phi_2 = Fd_1$. Spaces of the form X_1 for a reduced special simplicial space X give the Segal version of an A_∞-space. The group completion theorem for (reduced) special $\mathscr{F}$-spaces is a special case of the following more general group completion theorem.

Theorem 9.7. *If X is a reduced special simplicial space such that X_1 and $\Omega|X|$ are homotopy associative and commutative, then $\eta\colon X_1 \longrightarrow \Omega|X|$ is a group completion.*

Just as for classical A_∞-spaces, one can prove that X is equivalent as a Hopf space to a topological monoid. Then the theorem can be derived from the result for topological monoids. However, as we indicate briefly, the result as stated, with homotopy commutativity weakened to the assumption that left and right multiplication by any element are homotopic, was proven but not stated in [40, §15]. The result actually stated there, [40, Theorem 15.1], is the case when $X_n = G^n$ for a topological monoid G but the proof is given in the generality of Theorem 9.7. We summarize the argument, referring to [40] for details.

Since $\pi_1(|X|) = \pi_0(\Omega|X|)$ is an abelian group, it is isomorphic to $H_1(|X|;\mathbb{Z})$. Using the Künneth theorem and inspection, we can check that $\pi_0(\Omega|X|)$ is the group completion of $\pi_0(X_1)$ by an easy chain level argument given in [40, Lemma 15.2]. We just replace G and BG there with X_1 and $\Omega|X|$ here.

For the rest, the proof of the homological part of the group completion property is the same as in [40, §15]. The reader may appreciate a quick sketch since the strategy becomes a good deal clearer without the details, but the details are there. The proof starts from a based variant of the standard adjunction between topological spaces and simplicial sets. By [40, Definition 15.3], there is an adjunction (T,S) relating the category of reduced special simplicial spaces, denoted $\mathscr{S}^+\mathscr{T}$ there, to the category $\mathscr{T}$. The functor $T = |-|$ is geometric realization. The functor S is a reduced version of the total singular complex. For a based space K, S_pK is the set of p-simplices $\Delta_p \longrightarrow K$ that map all vertices to the basepoint. In particular, $S_1K = \Omega K$. Let $\phi\colon TSK \longrightarrow K$ and $\psi\colon X \longrightarrow STX$ be the counit and unit of the adjunction. Then [40, Proposition 15.5] gives the following result.

Proposition 9.8. *If $K \in \mathscr{T}$ is path connected, then $\phi\colon TSK \longrightarrow K$ is a weak equivalence. For any $X \in \mathscr{S}^+\mathscr{T}$, $T\psi\colon TX \longrightarrow TSTX$ is a weak equivalence.*

From here, the main tool is the standard homology spectral sequence of the filtered space $TX = |X|$. We take coefficients in a field R. Then, using the Künneth theorem and the fact that X is special, we see that E^1X is the algebraic bar construction on $H_*(X_1)$, so that $E^2_{p,q}X = \mathrm{Tor}^{H_*(X_1)}_{p,q}(R,R)$. Clearly $E^2_{0,0}X = R$ and $E^2_{0,q}X = 0$ for $q > 0$. The spectral sequence converges to $H_*(|X|)$. We have the analogous spectral sequence for STX. The idea is to apply an appropriate version of the comparison theorem for spectral sequences, [40, Lemma 15.6], to the map of spectral sequences induced by the map of simplicial spaces $\psi\colon X \longrightarrow STX$. On 1-simplices, $\psi_1 = \eta\colon X_1 \longrightarrow \Omega|X|$ and therefore $E^2\psi = \mathrm{Tor}^{\eta_*}(\mathrm{id},\mathrm{id})$. The map $\{E^r\psi\}$ of spectral sequences converges to the weak equivalence $TX \longrightarrow TSTX$. Therefore $E^\infty\psi$ is an isomorphism.

Write $A = H_*(X_1)$ and let $\iota\colon A \longrightarrow \overline{A}$ be its localization at the monoid $\pi_0(X_1)$. Write $B = H_*(\Omega|X|)$ and let $\zeta\colon \overline{A} \longrightarrow B$ be the map of R-algebras such that $\zeta\circ\iota = \eta_*$; it is given by the universal property of localization. We must prove that ζ is an isomorphism. It is a classical algebraic result [9, Proposition VI.4.1.1] that

$$\mathrm{Tor}^{\iota}(\mathrm{id},\mathrm{id})\colon\ \mathrm{Tor}^{A}(R,R) \longrightarrow \mathrm{Tor}^{\overline{A}}(R,R)$$

is an isomorphism in our situation. Therefore we can identify $E^2\psi$ with

$$\mathrm{Tor}^{\zeta}(\mathrm{id},\mathrm{id})\colon\ \mathrm{Tor}^{\overline{A}}(R,R) \longrightarrow \mathrm{Tor}^{B}(R,R).$$

The rest of the argument is given in detail in [40, p. 93]. Both $\overline{A}$ and B are the tensor products of their identity components with the group ring $R[\pi_0(\Omega|X|)]$. The cited version of the comparison theorem shows how to prove that $E^2_{1,*}\psi$ is an isomorphism and $E^2_{2,*}\psi$ is an epimorphism. For connected graded algebras A, $\mathrm{Tor}^A_{1,*}$ and $\mathrm{Tor}^A_{2,*}$ compute the generators and relations of A. Now the detailed argument of [40, p. 93] proves by induction on degree that ζ is an isomorphism between the identity components of A and B and is therefore an isomorphism.

9.3. **The positive linearity theorem.** We prove Theorem 3.18 in this section and the next. In both, let G be any topological group and let X be an $\mathscr{F}$-G-space. For a based G-CW complex A, write $Y(A)$ ambiguously for either $B(A^\bullet,\mathscr{F},X)$, where X is special, or $B(A^\bullet,\mathscr{F}_G,\mathbb{P}X)$, where X is $\mathbb{F}_\bullet$-special; the latter case is equivalent to $B(A^\bullet,\mathscr{F}_G,Z)$, where Z is a special $\mathscr{F}_G$-G-space. Under either assumption, write $B_\bullet(A;X)$ for the simplicial bar construction whose realization is $Y(A)$. These notations remain fixed throughout these two sections. Recall §1.2. As asserted in Remark 1.12 and is easily checked using that the degeneracy maps are given by identity maps of $\mathscr{F}$ and $\mathscr{F}_G$, $B_\bullet(A;X)$ is Reedy cofibrant in the simplicial sense.

To prove that the $\mathscr{W}_G$-G-space Y is positive linear, as defined in Definition 3.17, we first isolate properties that together imply positive linearity.

Definition 9.9. A $\mathscr{W}_G$-G-space Z satisfies the wedge axiom if for all A and B in $G\mathscr{W}$ the natural map

$$\pi\colon Z(A\vee B) \longrightarrow Z(A)\times Z(B)$$

induced by the canonical G-maps $A\vee B \longrightarrow A$ and $A\vee B \longrightarrow B$ is a weak G-equivalence.

Definition 9.10. Let Z be a $\mathscr{W}_G$-G-space and consider simplicial based G-CW complexes $A_\bullet$.

(i) Z commutes with geometric realization if the natural G-map

$$|Z(A_\bullet)| \longrightarrow Z(|A_\bullet|)$$

is a homeomorphism.

(ii) Z preserves Reedy cofibrancy if the simplicial G-space $Z(A_\bullet)$ is Reedy cofibrant when $A_\bullet$ is Reedy cofibrant.

Definition 9.11. A $\mathscr{W}_G$-G-space Z preserves connectivity if $Z(A)$ is G-connected when A is G-connected.

Our $\mathscr{W}_G$-G-space Y satisfies all of the properties above. We record the results here, with the proofs delayed to later.

Proposition 9.12. *The $\mathscr{W}_G$-G-space Y satisfies the wedge axiom.*

Proposition 9.13. *The $\mathscr{W}_G$-G-space Y commutes with realization and preserves Reedy cofibrancy.*

Lemma 9.14. *The $\mathscr{W}_G$-G-space Y preserves connectivity.*

Granting these results, Theorem 3.18 is an application of the following theorem.

Theorem 9.15. *Let Z be a $\mathscr{W}_G$-G-space that satisfies the wedge axiom, commutes with geometric realization, preserves Reedy cofibrancy, and preserves connectivity. Then Z is positive linear.*

The proof centers around the following construction of cofiber sequences in terms of wedges and geometric realizations; it is a corrected version and equivariant generalization of a construction due to Woolfson [59].

Construction 9.16. Let $f\colon A \longrightarrow B$ be a map in $G\mathscr{W}$ with cofiber $i\colon B \longrightarrow Cf$, where A is G-connected. We give an elementary simplicial description of Cf in terms of wedges. Let ${}^qA$ denote the wedge of q copies of A, labelling the ith wedge summand as A_i and setting ${}^0A = *$. We define a simplicial G-space $W_\bullet(B, A)$ whose space of q-simplices $W_q(B, A)$ is $B \vee {}^qA$. Define face and degeneracy operators d_i and s_i with domain $W_q(B, A)$ for $0 \le i \le q$ as follows.

- All d_i and s_i map B onto B by the identity map.
- d_0 maps A_1 to B via f and maps A_j to A_{j-1} by the identity map if $j > 1$.
- d_i, $0 < i < q$, maps A_j to A_j by the identity map if $j < i$ and maps A_j to A_{j-1} by the identity map if $j > i$.
- d_q maps A_j to A_j by the identity map if $j < q$ and maps A_q to $*$.
- s_i maps A_j to A_j by the identity map if $j \le i$ and maps A_j to A_{j+1} by the identity map if $j > i$.

The simplicial identities are easily checked.[23] Note that there is not much choice: the s_i must be inclusions with $s_i s_j = s_{j+1} s_i$ for $i \le j$ and they must satisfy $d_i s_i = \mathrm{id} = d_{i+1} s_i$. If we specify the s_i as stated, then the d_i must be as stated except in the exceptional cases noted for d_0 and d_q.

Lemma 9.17. *The realization $|W_\bullet(B, A)|$ is homeomorphic to Cf. In particular, the realization $|W_\bullet(*, A)|$ is homeomorphic to ΣA.*

Proof. Clearly every point of $W_q(B, A)$ for $q \ge 2$ is degenerate. Therefore, identifying Δ_1 with I, the realization is the quotient of $B \times \{*\} \amalg (B \vee A) \times I$ obtained by the identifications

$$(b, t) \sim (b, *) \quad \text{for} \quad b \in B \quad \text{and} \quad t \in I \quad \text{since} \quad s_0 b = b$$

$$(a, 0) \sim (f(a), *) \quad \text{for} \quad a \in A \quad \text{since} \quad d_0(a) = f(a)$$

$$(a, 1) \sim (*, *) \quad \text{for} \quad a \in A \quad \text{since} \quad d_1(a) = *$$

It is simple to verify that the result is homeomorphic to $Cf = B \cup_f (A \wedge I)$. $\square$

We record the following result, which is the equivariant generalization of [27, Theorem 12.7]. It is proven by applying that result to H-fixed point simplicial spaces for all closed subgroups H of G. A G-map f is a G-quasifibration if each f^H is a quasifibration. Similarly, a map p of simplicial based G-spaces is a simplicial based Hurewicz G-fibration if each p^H is a simplicial based Hurewicz fibration in the sense of [27, Definition 12.5]).

Theorem 9.18. *Let $E_\bullet$ and $B_\bullet$ be simplicial based G-spaces and let $p_\bullet\colon E_\bullet \longrightarrow B_\bullet$ a simplicial based Hurewicz G-fibration with fiber $F_\bullet = p_\bullet^{-1}(*)$. If each B_q is G-connected and $B_\bullet$ is Reedy cofibrant, then the realization $|p_\bullet|\colon |E_\bullet| \longrightarrow |B_\bullet|$ is a G-quasifibration with fiber $|F_\bullet|$.*

[23] They fail with the erroneous specification of faces in [59, Lemma 1.10].

Proof of Theorem 9.15. Let $f\colon A \longrightarrow B$ be a map in $G\mathscr{W}$, where A is G-connected. We must show that application of Z to the cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{i} Cf$$

gives a fiber sequence.

Let $p_\bullet\colon W_\bullet(B,A) \longrightarrow W_\bullet(*,A)$ be the map of simplicial G-spaces given by sending B to the basepoint and let $i_\bullet\colon B_\bullet \longrightarrow W_\bullet(B,A)$ be the inclusion, where $B_\bullet$ denotes the constant simplicial space at B. On passage to realization, these give the canonical maps

$$B \xrightarrow{i} Cf \xrightarrow{p} \Sigma A\,.$$

We apply our given $\mathscr{W}_G$-G-space Z to these maps to obtain the sequence

$$Z(B) \xrightarrow{Z(i)} Z(Cf) \xrightarrow{Z(p)} Z(\Sigma A)\,.$$

We claim that it is a fiber sequence. Since Z commutes with realization and since $Z(B) \cong |Z(B_\bullet)|$, this sequence is G-homeomorphic to the sequence

$$|Z(B_\bullet)| \xrightarrow{|Z(i_\bullet)|} |Z(W_\bullet(B,A))| \xrightarrow{|Z(p_\bullet)|} |Z(W_\bullet(*,A))|.$$

On q-simplices, before realization, we have the commutative diagram

$$\begin{array}{ccccc} Z(B) & \longrightarrow & Z(B \vee {}^qA) & \longrightarrow & Z({}^qA) \\ \| & & \downarrow & & \downarrow \\ Z(B) & \longrightarrow & Z(B) \times Z(A)^q & \longrightarrow & Z(A)^q \end{array}$$

where the horizontal arrows are given by the evident inclusions and projections. The vertical arrows are the canonical maps and are G-equivalences by the wedge axiom. The simplicial G-spaces $W_\bullet(B,A)$ and $W_\bullet(*,A)$ are trivially Reedy cofibrant, hence so are $Z(W_\bullet(B,A))$ and $Z(W_\bullet(*,A))$. The nondegeneracy of basepoints implies that $Z(B) \times Z(A)^\bullet$ and $Z(A)^\bullet$ are also Reedy cofibrant. Therefore the vertical arrows become G-equivalences after passage to realization. Since A and therefore $Z(A)$ is G-connected, Theorem 9.18 applies to show that the realization of the bottom row is a fibration sequence up to homotopy. Thus we have the fiber sequence

$$Z(B) \longrightarrow Z(Cf) \longrightarrow Z(\Sigma A)$$

and therefore also the fiber sequence

$$\Omega Z(\Sigma A) \longrightarrow Z(B) \longrightarrow Z(Cf).$$

Specializing to the map $\mathrm{id}\colon A \longrightarrow A$, we also have a fiber sequence

$$Z(A) \longrightarrow Z(CA) \longrightarrow Z(\Sigma A).$$

Since $Z(CA)$ is G-contractible, we therefore have a G-equivalence $Z(A) \longrightarrow \Omega Z(\Sigma A)$ and therefore the desired fiber sequence

$$Z(A) \longrightarrow Z(B) \longrightarrow Z(Cf). \qquad \square$$

Proof of Lemma 9.14. Let $A \in G\mathscr{W}$ be G-connected, so that A^H is connected for all H. Using that passage to fixed points commutes with pushouts one leg of which is a closed inclusion, it is easily checked that geometric realization and the bar construction commute with passage to H-fixed points. Using this and Remark 3.6, we see that $Y(A)^H$ is the geometric realization of a simplicial space with 0-simplices given by

$$\bigvee_n (A^H)^n \wedge (X_n^H)_+ \tag{9.19}$$

or

$$\bigvee_{(\mathbf{n},\alpha)} (A^{(\mathbf{n},\alpha)})^H \wedge ((\mathbb{P}X)(\mathbf{n},\alpha)^H)_+, \tag{9.20}$$

depending on whether $Y(A)$ is $B(A^\bullet, \mathscr{F}, X)$ or $B(A^\bullet, \mathscr{F}_G, \mathbb{P}X)$ for an $\mathscr{F}$-G-space X. We claim that the space of 0-simplices in either case is connected, so that the geometric realization $Y(A)^H$ is also connected. In the first case, it is clear that the space (9.19) is connected since we assume that A^H is connected, and in the second case, the space (9.20) is connected because $(A^{(\mathbf{n},\alpha)})^H$ is connected. Indeed, note that $(A^{(\mathbf{n},\alpha)})^H \cong (A^n)^\Lambda$ where $\Lambda = \{(h, \alpha(h)) \mid h \in H\} \subset G \times \Sigma_n$, thus by Lemma 2.7,

$$(A^{(\mathbf{n},\alpha)})^H \cong \prod A^{K_i},$$

where the product is taken over the orbits of the H-set $(\mathbf{n}, \alpha|_H)$ and the $K_i \subset H$ are the stabilizers of elements in the corresponding orbit. Again, by our assumption that the A^{K_i} are connected it follows that $(A^{(\mathbf{n},\alpha)})^H$ is connected. □

The following result will apply to show that Y commutes with realization.

Lemma 9.21. *Let Z be a $\mathscr{W}_G$-G-space such that $Z(A)$ is naturally isomorphic to $|Z_\bullet(A)|$ for some functor $Z_\bullet$ from $\mathscr{W}_G$ to simplicial based G-spaces. If the natural G-map*

$$|Z_q(A_\bullet)| \longrightarrow Z_q(|A_\bullet|)$$

is a G-homeomorphism for all simplicial based G-spaces $A_\bullet$ and all $q \geq 0$, then the natural map

$$|Z(A_\bullet)| \longrightarrow Z(|A_\bullet|)$$

is a G-homeomorphism.

Proof. For a bisimplicial G-space, realizing first in one direction and then the other gives a space that is G-homeomorphic to the one obtained by realizing in the opposite order. Let $A_\bullet$ be a simplicial $G\mathscr{W}$-space. Then $Z(A_p) = |q \mapsto Z_q(A_p)|$, so

$$|Z(A_\bullet)| = |p \mapsto |q \mapsto Z_q(A_p)||.$$

By the assumption on Z_q,

$$Z(|A_\bullet|) = |q \mapsto Z_q(|p \mapsto A_p|)| \cong |q \mapsto |p \mapsto Z_q(A_p)||.$$

The result follows. □

Proof of Proposition 9.13. We first prove that Y commutes with realization. Let $A_\bullet$ be a simplicial $G\mathscr{W}$-space. In view of Lemma 9.21, it suffices to prove that the natural map

$$|Y_q(A_\bullet)| \longrightarrow Y_q(|A_\bullet|)$$

is a G-homeomorphism for all $q \geq 0$. Using the description of Y_q in Remark 3.6 and the commutation of realization with products and half-smash products, this follows from the definition of Y.

Now assume that $A_\bullet$ is a Reedy cofibrant simplicial G-space. We must prove that $Y(A_\bullet)$ is Reedy cofibrant. Let $Y = B((-)^\bullet, \mathscr{F}, X)$; the proof in the case

$Y = B((-)^\bullet, \mathscr{F}_G, \mathbb{P}X)$ is the same. By Lemma 1.9, it suffices to show that all of the degeneracy maps

$$S_i\colon B((A_{n-1})^\bullet, \mathscr{F}, X) \xrightarrow{B((s_i)^\bullet, \mathrm{id}, \mathrm{id})} B((A_n)^\bullet, \mathscr{F}, X),$$

are G-cofibrations. Using a shorthand notation, the maps S_i are geometric realizations of maps of Reedy cofibrant simplicial G-spaces that are given on q-simplices by the G-cofibrations

$$(s_i)^{n_q} \wedge (\mathrm{id})\colon \bigvee_{n_0,\ldots,n_q} (A_{n-1})^{n_q} \wedge (-) \longrightarrow \bigvee_{n_0,\ldots,n_q} (A_n)^{n_q} \wedge (-).$$

By Theorem 1.11, the S_i are therefore G-cofibrations. □

9.4. **The proof that the wedge axiom holds.** We must prove Proposition 9.12, which says that the $\mathscr{W}_G$-G-space Y satisfies the wedge axiom.

Proof of Proposition 9.12. We write the proof for $Y(A) = B(A^\bullet, \mathscr{F}_G, \mathbb{P}X)$, where X is $\mathbb{F}_\bullet$-special. The proof for $Y(A) = B(A^\bullet, \mathscr{F}, X)$, where X is (naively) special, is essentially the same, but simpler since it is simpler to keep track of equivariance in that case. For convenience of notation, we write a, b, c, d, e, f for based finite G-sets, that is objects of $\mathscr{F}_G$. We write $X(a)$ for the value of $\mathbb{P}X$ on a and A^a for the based G-space $\mathscr{T}_G(a, A)$, with G acting by conjugation.

Recall from §3.1 that $B(A^\bullet, \mathscr{F}_G, \mathbb{P}X) = B^\times(A^\bullet, \mathscr{F}_G, \mathbb{P}X)/B^\times(*, \mathscr{F}_G, \mathbb{P}X)$, where $B^\times$ is the usual categorical bar construction defined using the cartesian product. Since the map

$$B^\times(A^\bullet, \mathscr{F}_G, \mathbb{P}X) \longrightarrow B(A^\bullet, \mathscr{F}_G, \mathbb{P}X)$$

of (3.4) is a G-equivalence, it suffices to prove the result for $Y(A) = B^\times(A^\bullet, \mathscr{F}_G, \mathbb{P}X)$ instead.[24]

To do so, we shall construct a G-homotopy commutative diagram of G-spaces

(9.22)
$$\begin{array}{ccc} & Z(A,B) & \\ {}^{F}\swarrow & & \searrow^{Q} \\ Y(A\vee B) & \xrightarrow[P]{} & Y(A)\times Y(B) \end{array}$$

in which F and Q are weak G-equivalences and P is the canonical map induced by the projections $\pi_A\colon A \vee B \longrightarrow A$ and $\pi_B\colon A \vee B \longrightarrow B$. This will prove that P is a weak G-equivalence.

In this section, we abbreviate notation by writing $\mathscr{C}(A;\ X)$ for the category internal to $G\mathscr{U}$ whose nerve is $B^\times_\bullet(A, \mathscr{F}_G, \mathbb{P}X)$ and whose classifying G-space is therefore $Y(A)$. Recall from §3.1 that the object and morphism G-spaces of $\mathscr{C}(A; X)$ are

$$\coprod_a A^a \times X(a)$$

and

$$\coprod_{a,c} A^c \times \mathscr{F}_G(a, c) \times X(a).$$

Its source and target G-maps S and T are induced from the evaluation maps of the contravariant $G\mathscr{T}$-functor $A^\bullet$ and the covariant $G\mathscr{T}$-functor X from $\mathscr{F}_G$ to

[24] It is irrelevant here that the new Y is not a $\mathscr{W}_G$-G-space since it is not a $G\mathscr{T}$-functor, as discussed in §3.1.

$\mathscr{T}_G$. The identity and composition G-maps I and C are induced from identity morphisms and composition in $\mathscr{F}_G$.

Analogously, we define $Z(A,B)$ to be the classifying G-space of the category internal to $G\mathscr{U}$ $\mathscr{C}(A,B;X)$ whose nerve is the simplicial bar construction

$$B^{\times}_{\bullet}(A^{\bullet}\times B^{\star};\mathscr{F}_G\times\mathscr{F}_G;(\mathbb{P}X)(\bullet\vee\star)).$$

Its object and morphism G-spaces are

$$\coprod_{a,b}(A^a\times B^b)\times X(a\vee b)$$

and

$$\coprod_{a,b,c,d}(A^c\times B^d)\times\mathscr{F}_G(a,c)\times\mathscr{F}_G(b,d)\times X(a\vee b).$$

Here S and T are induced from the evaluation maps of $A^{\bullet}\times B^{\star}$, and the composite of $\vee\colon\mathscr{F}_G\times\mathscr{F}_G\longrightarrow\mathscr{F}_G$ with the evaluation maps of $\mathbb{P}X$. Again, identity and composition maps are induced from identity morphisms and composition in $\mathscr{F}_G$.

Using categories, functors, and natural transformations to mean these notions internal to $G\mathscr{U}$ in what follows, we shall define functors giving the following diagram of categories and shall prove that it is commutative up to natural transformation.

(9.23)
$$\begin{array}{ccc} & \mathscr{C}(A,B;X) & \\ \swarrow^{F} & & \searrow^{Q} \\ \mathscr{C}(A\vee B;X) & \xrightarrow[P]{\qquad} & \mathscr{C}(A;X)\times\mathscr{C}(B;X)\end{array}$$

Passing to classifying G-spaces, this will give the diagram (9.22).

To define F and Q, it is convenient to write elements of A^a as based maps $\mu\colon a\longrightarrow A$, with G acting by conjugation on maps. The functor F sends an object (μ,ν,x) to $(\mu\vee\nu,x)$ and sends a morphism (μ,ν,ϕ,ψ,x) to $(\mu\vee\nu,\phi\vee\psi,x)$. The functor Q sends an object (μ,ν,x) to $(\mu,x_a)\times(\nu,x_b)$ where x_a and x_b are obtained from $x\in X(a\vee b)$ by using the G-maps induced by the projections $\pi_a\colon a\vee b\longrightarrow a$ and $\pi_b\colon a\vee b\longrightarrow b$. It sends a morphism (μ,ν,ϕ,ψ,x) to the morphism $(\mu,\phi,x_a)\times(\nu,\psi,x_b)$. As in (9.22), P is induced by the projections π_A and π_B. Noting that π_a and π_b are G-fixed morphisms of $\mathscr{F}_G$ and that $\pi_A\circ(\mu\vee\nu)=\mu\circ\pi_a$ and $\pi_B\circ(\mu\vee\nu)=\nu\circ\pi_b$, we see that the morphisms

$$(\mu,\pi_a,x)\times(\nu,\pi_b,x)\colon(\pi_A\circ(\mu\vee\nu),x)\times(\pi_B\circ(\mu\vee\nu),x)\longrightarrow(\mu,x_a)\times(\nu,x_b)$$

give a natural transformation $P\circ F\longrightarrow Q$ in diagram (9.23) that induces a G-homotopy $P\circ F\longrightarrow Q$ in diagram (9.22).

While Q need not be an equivalence of categories of any sort, we see from the assumption that $\mathbb{P}X$ is special and our use of the projections π_a and π_b that Q gives a level weak equivalence of simplicial G-spaces on passage to nerves. Reedy cofibrancy of the bar constructions then implies that the induced map Q in (9.22) is a weak equivalence of classifying G-spaces. To complete the proof, we shall construct a functor $F^{-1}\colon\mathscr{C}(A\vee B;X)\longrightarrow\mathscr{C}(A,B;X)$ and natural transformations $\mathrm{Id}\longrightarrow F^{-1}\circ F$ and $\mathrm{Id}\longrightarrow F\circ F^{-1}$. Passing to classifying G-spaces, this will imply that F in (9.22) is a G-homotopy equivalence.

For $\omega\colon f \longrightarrow A \vee B$, define $\omega_A = \pi_A \circ \omega$ and $\omega_B = \pi_B \circ \omega$. Define $\sigma_\omega\colon f \longrightarrow f \vee f$, called the splitting of ω, by

$$\sigma_\omega(j) = \begin{cases} j & \text{in the first copy of } f \text{ if } \omega(j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } \omega(j) \in B \setminus * \\ * & \text{if } \omega(j) = * \end{cases}$$

Observe that ω factors as the composite

$$f \xrightarrow{\sigma_\omega} f \vee f \xrightarrow{\omega_A \vee \omega_B} A \vee B.$$

Define F^{-1} on objects by

$$F^{-1}(\omega, x) = (\omega_A, \omega_B, (\sigma_\omega)_*(x)).$$

For a morphism (ω, ϕ, x), $\omega\colon f \longrightarrow A \vee B$, $\phi\colon e \longrightarrow f$, and $x \in X(e)$, observe that

$$(\omega \circ \phi)_A = \omega_A \circ \phi \quad \text{and} \quad (\omega \circ \phi)_B = \omega_B \circ \phi.$$

Define F^{-1} on morphisms by

$$F^{-1}(\omega, \phi, x) = (\omega_A, \omega_B, \phi, \phi, (\sigma_{\omega\circ\phi})_*(x).$$

A check of definitions using the commutative diagram

$$\begin{array}{ccccc} e & \xrightarrow{\phi} & f & \xrightarrow{\omega} & A \vee B \\ {\scriptstyle \sigma_{\omega\circ\phi}}\downarrow & & {\scriptstyle \sigma_\omega}\downarrow & \nearrow {\scriptstyle \omega_A \vee \omega_B} & \\ e \vee e & \xrightarrow[\phi\vee\phi]{} & f \vee f & & \end{array}$$

shows that $S \circ F^{-1} = F^{-1} \circ S$ and $T \circ F^{-1} = F^{-1} \circ T$, and F^{-1} is clearly compatible with composition and identities. It is easily checked that F^{-1} is continuous on object and morphism G-spaces, but equivariance is a little more subtle. We first claim that $\sigma_{g\cdot\omega} = g \cdot \sigma_\omega$. Since $(g \cdot \omega)(j) = g\omega(g^{-1}j)$,

$$\sigma_{g\cdot\omega}(j) = \begin{cases} j & \text{in the first copy of } f \text{ if } g\omega(g^{-1}j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } g\omega(g^{-1}j) \in B \setminus * \\ * & \text{if } g\omega(g^{-1}j) = *. \end{cases}$$

On the other hand, using the definition of σ_ω and the fact that $gg^{-1} = 1$,

$$g \cdot \sigma_\omega(j) = g\sigma_\omega(g^{-1}j) = \begin{cases} j & \text{in the first copy of } f \text{ if } \omega(g^{-1}j) \in A \setminus * \\ j & \text{in the second copy of } f \text{ if } \omega(g^{-1}j) \in B \setminus * \\ g* = * & \text{if } \omega(g^{-1}j) = *. \end{cases}$$

Observing that $gz \in A \setminus *$ if and only if $z \in A \setminus *$ and similarly for B, we see that these agree, proving the claim. Since π_A is a G-map, we also have

$$g \cdot \omega_A = g \cdot (\pi_A \circ \omega) = g\pi_A\omega g^{-1} = \pi_A g\omega g^{-1} = \pi_A \circ g \cdot \omega = (g \cdot \omega)_A,$$

and similarly for B. Putting these together gives

$$\begin{aligned} F^{-1}(g \cdot (\omega, x)) = F^{-1}(g \cdot \omega, gx) &= \big((g \cdot \omega)_A, (g \cdot \omega)_B, (\sigma_{g\cdot\omega})_*(gx)\big) \\ &= \big(g \cdot \omega_A, g \cdot \omega_B, (g \cdot \sigma_\omega)_*(gx)\big). \end{aligned}$$

Since the evaluation map $\mathscr{F}_G(a, b) \wedge X(a) \longrightarrow X(b)$ is a G-map, this is equal to

$$\big(g \cdot \omega_A, g \cdot \omega_B, g \cdot ((\sigma_\omega)_* x)\big) = g \cdot F^{-1}(\omega, x).$$

This shows that F^{-1} is a G-map on objects. Following the same argument and using that $g \cdot (\omega \circ \phi) = (g \cdot \omega) \circ (g \cdot \phi)$, we see that F^{-1} is a G-map on morphisms.

Now consider the composite $F \circ F^{-1}$. It sends the object (ω, x) in $\mathscr{C}(A \vee B; X)$ to the object $(\omega_A \vee \omega_B, (\sigma_\omega)_* x)$. Here $(\omega_A \vee \omega_B, \sigma_\omega, x)$ is a morphism from (ω, x) to $(\omega_A \vee \omega_B, (\sigma_\omega)_* x)$. We claim that this morphism is the component at (ω, x) of a natural transformation $\mathrm{Id} \longrightarrow F \circ F^{-1}$. These morphisms clearly give a continuous map from the object G-space of $\mathscr{C}(A \vee B; X)$ to its morphism G-space, and it is not hard to check naturality using the diagram just above. To show equivariance, if $g \in G$ then

$$g \cdot (\omega_A \vee \omega_B, \sigma_\omega, x) = (g \cdot (\omega_A \vee \omega_B), g \cdot \sigma_\omega, gx) = ((g \cdot \omega)_A \vee (g \cdot \omega)_B, \sigma_{g \cdot \omega}, gx),$$

which is precisely the component at $g \cdot (\omega, x) = (g\omega, gx)$.

The composite $F^{-1} \circ F$ sends an object (μ, ν, x) in $\mathscr{C}(A, B; X)$ to the object

$$(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), (\sigma_{\mu \vee \nu})_* x) = (\mu \circ \pi_a, \nu \circ \pi_b, (\sigma_{\mu \vee \nu})_* x).$$

Note that $\sigma_{\mu \vee \nu}$ is given by $\tilde{\sigma}_\mu \vee \tilde{\sigma}_\nu$, where $\tilde{\sigma}_\mu \colon a \longrightarrow a \vee b$ is the inclusion, except that if $\mu(j) = *$, then $\tilde{\sigma}_\mu(j) = *$, and similarly for $\tilde{\sigma}_\nu \colon b \longrightarrow a \vee b$. Here

$$\big(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), \tilde{\sigma}_\mu, \tilde{\sigma}_\nu, x\big)$$

is a morphism in $\mathscr{C}(A, B; X)$ from (μ, ν, x) to $\big(\pi_A \circ (\mu \vee \nu), \pi_B \circ (\mu \vee \nu), (\sigma_{\mu \vee \nu})_* x\big)$. To see that the source of this morphism is as claimed, observe that $\mu \circ \pi_a \circ \tilde{\sigma}_\mu = \mu$ since $\pi_a \circ \tilde{\sigma}_\mu = \mathrm{id}$ except on those j such that $\mu(j) = *$, and similarly for ν. The naturality follows from the equations

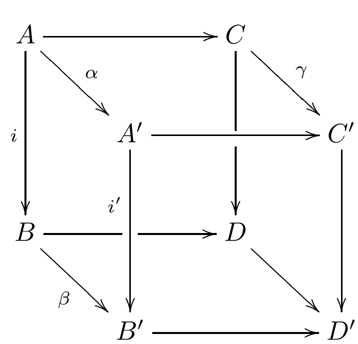
$$\tilde{\sigma}_\mu \circ \phi = (\phi \vee \psi) \circ \tilde{\sigma}_{\mu \circ \phi} \quad \text{and} \quad \tilde{\sigma}_\nu \circ \psi = (\phi \vee \psi) \circ \tilde{\sigma}_{\nu \circ \psi},$$

which are easily checked. The continuity of the assignment is also easily verified. For $g \in G$, a verification similar to that for $F \circ F^{-1}$ shows that g acting on the component of our natural transformation at (μ, ν, x) is the component of the transformation at $g \cdot (\mu, \nu, x) = (g\mu, g\nu, gx)$. □

10. Appendix: Levelwise realization of G-cofibrations

In this section we prove Theorem 1.11. We will need the following two standard results about G-cofibrations. Recall that we are working in the category of compactly generated weak Hausdorff spaces, so all cofibrations are closed inclusions. Throughout this section we use the convention of identifying the domain of a cofibration with its image.

Proposition 10.1. *Consider the following diagram in G-spaces.*



Assume that the front and back faces are pushouts, α, β, γ, i and i' are G-cofibrations, and $A' \cap B = A$ (as subsets of B'). Then the map $D \longrightarrow D'$ is also a G-cofibration.

Proof. The nonequivariant result is [22, Proposition 2.5] and, as pointed out there, the equivariant proof goes through the same way. □

The following result is given in [7, Proposition A.4.9] and [22, Lemma 3.2.(a)].

Proposition 10.2. *Let $A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$ and $B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots$ be diagrams of G-cofibrations and let $f_i \colon A_i \longrightarrow B_i$ be a map of diagrams such that each f_i is a G-cofibration. Assume moreover that for every $i \geq 1$, $A_{i-1} = A_i \cap B_{i-1}$. Then the induced map $\operatorname{colim}_i A_i \longrightarrow \operatorname{colim}_i B_i$ is a G-cofibration.*

The following observation is the key to using these results to prove Theorem 1.11.

Lemma 10.3. *Let $f_\bullet \colon X_\bullet \longrightarrow Y_\bullet$ be a map of simplicial G-spaces that is levelwise injective. Then for all $n \geq 1$ and all $0 \leq i \leq n-1$,*

$$s_i(Y_{n-1}) \cap f_n(X_n) = f_n s_i(X_{n-1})$$

and

$$L_n Y \cap f_n(X_n) = f_n(L_n X).$$

Proof. For the first statement, one of the inclusions is obvious. For the other inclusion, take $y = s_i(y') = f_n(x)$. Then $y' = d_i s_i(y') = d_i f_n(x) = f_{n-1} d_i(x)$. Thus, $y = s_i f_{n-1} d_i(x) = f_n s_i d_i(x) \in f_n s_i(X_{n-1})$, as wanted. The second statement is obtained from the first by taking the union over all i. □

Proposition 10.4. *Let $X_\bullet$ and $Y_\bullet$ be Reedy cofibrant simplicial G-spaces, and let $f_\bullet \colon X_\bullet \longrightarrow Y_\bullet$ be a level G-cofibration. Then for all $n \geq 1$, $L_n X \longrightarrow L_n Y$ is a G-cofibration.*

Proof. We proceed by induction on n. Note that when $n = 1$, $L_1 X = s_0 X_0$ which is homeomorphic to X_0, so there is nothing to prove. For $n > 1$, let $k \leq n-1$. Just as nonequivariantly, we have a pushout square

$$\begin{array}{ccc} s_k\left(\bigcup_{i=0}^{k-1} s_i(X_{n-2})\right) & \longrightarrow & \bigcup_{i=0}^{k-1} s_i(X_{n-1}) \\ \downarrow & & \downarrow \\ s_k(X_{n-1}) & \longrightarrow & \bigcup_{i=0}^{k} s_i(X_{n-1}). \end{array} \tag{10.5}$$

We show by induction on n and k that the right vertical map is a G-cofibration. By the inductive hypothesis for $n-1$ and the fact that $X_\bullet$ is Reedy cofibrant, we have that the composite

$$\bigcup_{i=0}^{k-1} s_i(X_{n-2}) \longrightarrow \bigcup_{i=0}^{k} s_i(X_{n-2}) \longrightarrow \ldots \longrightarrow \bigcup_{i=0}^{n-2} s_i(X_{n-2}) = L_{n-1}X \longrightarrow X_{n-1}$$

is a G-cofibration. Since s_k is a G-homeomorphism onto its image, the left hand map in the square is a G-cofibration. Therefore, the righthand map in (10.5) is also a G-cofibration by cobase change.

We apply Proposition 10.1 to the pushout in (10.5) to deduce that the maps

$$\bigcup_{i=0}^{k} s_i(X_{n-1}) \longrightarrow \bigcup_{i=0}^{k} s_i(Y_{n-1})$$

are G-cofibrations. In particular, by induction on n and k, $L_nX \longrightarrow L_nY$ is a G-cofibration. The required intersection condition follows from Lemma 10.3. □

We are now ready to prove Theorem 1.11.

Proof of Theorem 1.11. Recall from §9.1 that $|X_\bullet|$ is the colimit of its filtration pieces $F_p|X|$, and that these can be built by the iterated pushout squares (9.2). Note that all of the vertical maps in diagram (9.2) are G-cofibrations. There is a similar diagram for $|Y_\bullet|$ and a map of diagrams induced by $f_\bullet$. The maps between the three corners of the upper squares are G-cofibrations. By Proposition 10.1 and Lemma 10.3, the map between the pushouts of the upper squares is a G-cofibration.

We assume by induction that the map $F_{p-1}|X| \longrightarrow F_{p-1}|Y|$ is a G-cofibration, so again we have that all three maps between the corners of the lower pushout square for X and the one for Y are G-cofibrations. Lemma 10.3 implies again that the intersection condition necessary to apply Proposition 10.1 holds, and we can deduce by induction that the map $F_p|X| \longrightarrow F_p|Y|$l is a G-cofibration.

Lastly, we check the intersection condition of Proposition 10.2, namely that $f(F_p|X|) \cap F_{p-1}|Y| = f(F_{p-1}|X|)$. One inclusion is obvious. To see the other inclusion, take an element (u, f_px) in $f(F_p|X|) \setminus f(F_{p-1}|X|) = (\Delta_p \setminus \partial\Delta_p) \times f_p(X_p \setminus L_pX)$. Since f_p is injective, we can again use Lemma 10.3 to see that $(u, f_px) \in F_pY \setminus F_{p-1}Y$. Thus an element in the intersection must be in $F_{p-1}|X|$. By Proposition 10.2, we conclude that the map

$$|f_\bullet|\colon |X_\bullet| \longrightarrow |Y_\bullet|$$

is a G-cofibration.

□

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