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CLASSIFYING G-SPACES AND THE SEGAL CONJECTURE

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In a previous note [11], two of us promised to describe the natural map (induced by $EG \to pt$)

(1) $$\varepsilon\colon (S_G)^G \to F(EG^+, S_G)^G \simeq F(BG^+, S) = D(BG^+)$$

in nonequivariant terms. Here S_G is the sphere G-spectrum and $(k_G)^G$ denotes the G-fixed point spectrum associated to a G-spectrum k_G. In section 1, following up and reinterpreting ideas of Adams, Gunawardena, and Miller, we shall generalize the situation and describe the natural map

(2) $$\varepsilon\colon [\Sigma_G^\infty B(G,\Pi)^+]^G \to F(EG^+, \Sigma_G^\infty B(G,\Pi)^+)^G \simeq F(BG^+, \Sigma^\infty B\Pi^+)$$

in nonequivariant terms for a finite group G and a compact Lie group Π. Here $B(G,\Pi)$ is the classifying G-space for principal (G,Π)-bundles and Σ_G^∞ denotes the suspension G-spectrum functor. (The equivalence follows from [11; Lemma 12]; see Remarks 7 below for its hypotheses.) More precisely, we shall specify an equivalence from a wedge of spectra $\Sigma^\infty BW\rho^+$ for certain groups $W\rho$ to the domain of ε and shall show that the composite of ε and this equivalence is the wedge sum of appropriate composites of transfer and classifying maps. The reader is referred to Lashof [5] for a good discussion of (G,Π)-bundles and to [8] for a comprehensive study of G-spectra and the equivariant stable category.

The completion conjecture for a G-spectrum k_G asks if $\varepsilon^*\colon k_G^*(pt) \to k_G^*(EG)$ induces an isomorphism upon completion in the $I(G)$-adic topology, where $I(G)$ is the augmentation ideal of the Burnside ring $A(G)$; see [11] for discussion. The Segal conjecture is the completion conjecture for S_G. In section 2, we shall prove the following result.

THEOREM A. *If the Segal conjecture holds for all finite groups G, then the completion conjecture holds for the G-spectra $\Sigma_G^\infty B(G,\Pi)^+$ for all finite groups G and Π.*

It would be of interest to know whether or not this result

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remains valid for general compact Lie groups Π.

By the results of [11], it suffices to prove the theorem for p-groups G, and in this case the conclusion is equivalent to the assertion that the maps ε of (2) induce equivalences upon completion at p. Given our nonequivariant interpretation of these maps, the conclusion is an observation, suggested to us by Gunawardena and also noticed by Segal, when Π is also a p-group. Indeed, by an argument that is entirely symmetrical in G and Π, one sees that if ε in (1) induces an equivalence upon completion at p for all groups Γ and $G \times \Gamma$, where Γ is a subquotient of Π, then ε in (2) induces an equivalence upon completion at p for all pairs (G,Γ).

The passage from p-groups Π to general finite groups Π is based on the following equivariant generalization of a standard result about transfer. Define a G-cover to be a G-map $\pi: E \to B$ which is also a finite cover. A G-cover π has an associated equivariant transfer map $\tau: \Sigma_G^\infty B^+ \to \Sigma_G^\infty E^+$ [8].

<u>THEOREM B</u>. Let G be a p-group and let $\pi: E \to B$ be a G-cover whose fibre has cardinality prime to p. Then the composite

$$\Sigma_G^\infty B^+ \xrightarrow{\tau} \Sigma_G^\infty E^+ \xrightarrow{\Sigma_G^\infty \pi} \Sigma_G^\infty B^+$$

induces an equivalence upon localization at p, hence $\Sigma_G^\infty B^+$ is p-locally a wedge summand of $\Sigma_G^\infty E^+$.

<u>COROLLARY C</u>. Let G be a p-group and let Λ be a p-Sylow subgroup of a finite group Π. Then $\Sigma_G^\infty B(G,\Pi)^+$ is p-locally a wedge summand of $\Sigma_G^\infty B(G,\Lambda)^+$.

There is an implied splitting of localized G-fixed point spectra, a fact that is not at all obvious from our nonequivariant description of the latter. We are very grateful to George Glauberman for proving a result in finite group theory for us that made clear that such a splitting was plausible; see Remark 14.

The second author wishes to thank Frank Adams, Jeremy Gunawardena, and Haynes Miller for discussions of the Segal conjecture, Dick Lashof for discussions of classifying G-spaces, and George Glauberman and Daniel Frohardt for background information about finite groups.

§1. THE SPLITTING OF $\Sigma_G^\infty B(G,\Pi)^+$

Our nonequivariant description of ε will be based on the following general splitting theorem for stable G-homotopy groups. Let EG be a contractible free left G-space; for a left G-space Y, write $EG\times_G Y$ for the orbit space $(EG\times Y)/G$. For $H\subset G$, let NH denote the normalizer of H in G and let $WH = NH/H$.

THEOREM 1. For a G-space Y, let ξ_H be the composite homomorphism of stable homotopy groups displayed in the following diagram:

$$\begin{array}{ccccc}
\pi_*(EWH\times_{WH}Y^H) & \xrightarrow{\zeta} & \pi_*^G(EWH\times_{WH}Y^H) & \xrightarrow{\tau} & \pi_*^G(G\times_{NH}(EWH\times Y^H)) \\
\downarrow{\scriptstyle \xi_H} & & & & \downarrow{\scriptstyle i_*} \\
\pi_*^G(Y) & \xleftarrow{\varepsilon_*} & \pi_*^G((G\times_{NH}EWH)\times Y) & \cong & \pi_*^G(G\times_{NH}(EWH\times Y))
\end{array}$$

Here ζ is induced by the unit map $S\to (S_G)^G$ and τ is the equivariant transfer associated to the natural G-cover

$$\pi\colon G\times_{NH}(EWH\times Y^H) \to EWH\times_{NH}Y^H = EWH\times_{WH}Y^H$$

with fibre G/H; i and ε are the evident inclusion and projection. Define

$$\xi = \sum_{(H)}\xi_H\colon \sum_{(H)}\pi_*(EWH\times_{WH}Y^H) \to \pi_*^G(Y),$$

where the sum runs over one group H from each conjugacy class (H) of subgroups of G. Then ξ is an isomorphism.

PROOF. Such a splitting is given by Segal [14], Kosniowski [4], and tom Dieck [1], but the present description of the splitting map is not explicit in the literature. We derive it from tom Dieck's proof and diagram chasing. Thus fix H, abbreviate $N = NH$ and $W = WH$, and consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_*(EW\times_W Y^H) & & \xrightarrow{\quad\zeta\quad} & & \pi_*^G(EW\times_W Y^H) \\
\downarrow{\scriptstyle\zeta} & \searrow{\scriptstyle\zeta} & & \swarrow{\scriptstyle\rho} & \downarrow{\scriptstyle\tau} \\
\pi_*^W(EW\times_W Y^H) & \xrightarrow{\rho} & \pi_*^N(EW\times_N Y^H) & \xrightarrow{\omega} & \pi_*^G(G/N\times(EW\times_N Y^H)) \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\
\pi_*^W(EW\times Y^H) & \xrightarrow{\rho} & \pi_*^N(EW\times Y^H) & \xrightarrow{\omega} & \pi_*^G(G\times_N(EW\times Y^H)) \\
\vdots\downarrow & & & & \downarrow{\scriptstyle i_*} \\
\pi_*^G(Y) & \xleftarrow{\varepsilon_*} & \pi_*^G((G\times_N EW)\times Y) & \cong & \pi_*^G(G\times_N(EW\times Y))
\end{array}$$

Here the ρ are restriction homomorphisms and the ω are extension of group isomorphisms due to Wirthmuller [12]; see also [8], where $\omega\rho = \tau$, $\tau\rho = \rho\tau$, and $\omega\tau = \tau\omega$ are proven. By the transitivity of transfer [8], the right vertical composite $\tau \circ \tau$ is the transfer referred to in the statement. By [11, Lemma 16], the left vertical composite $\tau \circ \zeta$ is an isomorphism. Tom Dieck [1] proved that the sum over (H) of the dotted arrow composites $\varepsilon_* i_* \omega\rho$ is an isomorphism.

REMARK 2. Except that ω requires reinterpretation and is no longer an isomorphism, the diagram in the proof commutes and tom Dieck's result remains valid for general compact Lie groups G. However, [11, Lemma 16] fails in this generality, hence so does the conclusion of the theorem. In fact, the left vertical composite $\tau \circ \zeta$ is an isomorphism if WH is finite, but the transfer here is zero if WH is not finite (see e.g. [8]). The logic becomes clear if one takes Y to be a point. Here $\pi_0^G(pt)$ is the Burnside ring and is thus the free Abelian group with one generator for each (H) such that WH is finite. However, $\pi_*^G(pt)$ is the direct sum over all (H) of the groups $\pi_*^{WH}(EWH)$, the (H) with WH infinite contributing only to the positive dimensional homotopy groups. Consideration of the completion conjecture here requires consideration of all (H), while the 0-dimensional Segal conjecture requires consideration only of those (H) with WH finite.

With the understandings that our diagram chases will be of interest only when WH is finite, that subgroups are to be closed and homomorphisms continuous, and that G-covers are to be interpreted as G-equivariant bundles, we can allow both G and Π to be arbitrary compact Lie groups until otherwise specified.

We need some notations to describe Y^H when $Y = B(G,\Pi)$.

NOTATIONS 3. Consider a subgroup H of G and a homomorphism $\rho: H \to \Pi$. Define the following associated groups:

$$\Delta\rho = \{(h,\rho(h))\} \subset G \times \Pi$$

$$N\rho = N(\Delta\rho) = \{(g,\sigma) \mid g \in NH,\ \sigma\rho(h)\sigma^{-1} = \rho(ghg^{-1}) \text{ for } h \in H\}$$

$$W\rho = W(\Delta\rho) = N\rho/\Delta\rho$$

$$M\rho = \{g \mid (g,\sigma) \in N\rho \text{ for some } \sigma \in \Pi\} \subset NH$$

$$V\rho = M\rho/H \subset WH$$

$$\Pi^{\rho} = \{\sigma \mid \sigma\rho(h) = \rho(h)\sigma \quad \text{for} \quad h \in H\}$$

Taking $g = e$, we see that Π^{ρ} is a normal subgroup of $N\rho$ with quotient $M\rho$ and a normal subgroup of $W\rho$ with quotient $V\rho$.

DEFINITION 4. Consider the set $R(H,\Pi)$ of Π-conjugacy classes of homomorphisms $\rho\colon H \to \Pi$, $H \subset G$. Define an action of WH on $R(H,\Pi)$ by letting $\bar{g}(\rho)$ be $(g\rho)$, where $g \in NH$ with image $\bar{g} \in WH$ and where $(g\rho)(h) = \rho(g^{-1}hg)$. It is trivial to check that this is well-defined. Observe that the isotropy group of (ρ) is $V\rho$.

PROPOSITION 5. For any subgroup H of G,

$$EWH\times_{WH}B(G,\Pi)^{H} = \coprod_{[(\rho)]} BW\rho,$$

where the union is taken over one ρ in each WH-orbit $[(\rho)]$ of Π-conjugacy classes (ρ) of homomorphisms $H \to \Pi$.

PROOF. Recall that a principal (G,Π)-bundle $p\colon E \to B$ is a principal Π-bundle and a G-map such that the actions of G and Π on E commute. We write the G action on the left and the Π action on the right. Let $p\colon E(G,\Pi) \to B(G,\Pi)$ be a universal principal (G,Π)-bundle. According to Lashof [5], p is characterized by the assertion that the $\Delta\rho$-fixed point space, $E(G,\Pi)^{\rho}$, is non-empty and contractible for all $H \subset G$ and $\rho\colon H \to \Pi$. If $y \in E(G,\Pi)^{\rho}$ and $\sigma \in \Pi$, then $y\sigma$ is fixed by the σ-conjugate $\rho\sigma$ of ρ. In particular, Π^{ρ} acts freely on the contractible space $E(G,\Pi)^{\rho}$, hence we may set

$$B\Pi^{\rho} = E(G,\Pi)^{\rho}/\Pi^{\rho}.$$

It is easily seen that $B(G,\Pi)^{H}$ is the disjoint union over $(\rho) \in R(H,\Pi)$ of the spaces $B\Pi^{\rho}$ and that $p^{-1}(B\Pi^{\rho})$ is the disjoint union of the $E(G,\Pi)^{\rho\sigma}$ as $\rho\sigma$ ranges over the distinct Π-conjugates of ρ. Under the natural action of WH on $B(G,\Pi)^{H}$, $V\rho$ fixes the space $B\Pi^{\rho}$ and

$$B(G,\Pi)^{H} = \coprod_{(\rho)} B\Pi^{\rho} \cong \coprod_{[(\rho)]} WH\times_{V\rho}B\Pi^{\rho}.$$

Therefore

$$EWH\times_{WH}B(G,\Pi)^{H} \cong \coprod_{[(\rho)]} EWH\times_{V\rho}B\Pi^{\rho}.$$

Let $W\rho$ act on EWH via the projection $W\rho \to V\rho \subset WH$ and have its natural action on $E(G,\Pi)^{\rho}$. It is easily checked that the diagonal action of $W\rho$ on $EWH \times E(G,\Pi)^{\rho}$ is free, and we may therefore set

$$EW\rho = EWH \times E(G,\Pi)^{\rho} \quad \text{and} \quad BW\rho = EW\rho/W\rho.$$

The projection $p\colon E(G,\Pi)^{\rho} \to B\Pi^{\rho}$ induces a homeomorphism $BW\rho \cong EWH\times_{V\rho}B\Pi^{\rho}$, and this proves the result.

We retain the notations of the proof in the following result.

PROPOSITION 6. When $Y = B(G,\Pi)$, the restriction ξ_{ρ} of ξ_H to $\pi_* BW\rho$ is the composite

$$\pi_* BW\rho \xrightarrow{\zeta} \pi_*^G BW\rho \xrightarrow{\tau} \pi_*^G \tilde{B}\rho \xrightarrow{\mu_*} \pi_*^G B(G,\Pi).$$

Here

$$\tilde{B}\rho = G\times_{M\rho}(EWH \times B\Pi^{\rho}) \cong [(G \times \Pi)\times_{N\rho}EW\rho]/\Pi,$$

τ is the equivariant transfer associated to the natural G-cover $\pi\colon \tilde{B}\rho \to BW\rho$ with fibre G/H, and μ is the classifying G-map of the natural principal (G,Π)-bundle $p\colon \tilde{E}\rho \to \tilde{B}\rho$, where

$$\tilde{E}\rho = G\times_{M\rho}(EWH \times p^{-1}B\Pi^{\rho}) \cong (G \times \Pi)\times_{N\rho}EW\rho.$$

PROOF. Notice that $\pi\colon \tilde{B}\rho \to BW\rho$ is a restriction of

$$\pi\colon G\times_{NH}(EWH \times B(G,\Pi)^H) \to EWH\times_{WH}B(G,\Pi)^H$$

and that the following squares are pullbacks:

$$\begin{array}{ccccc}
G\times_{M\rho}(EWH\times p^{-1}B\Pi^{\rho}) & \xrightarrow{i} & G\times_{NH}(EWH\times E(G,\Pi)) \cong (G\times_{NH}EWH)\times E(G,\Pi) & \xrightarrow{\varepsilon} & E(G,\Pi) \\
\downarrow & & \downarrow \qquad\qquad\qquad \downarrow & & \downarrow \\
G\times_{M\rho}(EWH\times B\Pi^{\rho}) & \xrightarrow{i} & G\times_{NH}(EWH\times B(G,\Pi)) \cong (G\times_{NH}EWH)\times B(G,\Pi) & \xrightarrow{\varepsilon} & B(G,\Pi)
\end{array}$$

The vertical arrows are all induced by p. The conclusion follows by a simple diagram chase.

REMARKS 7. (i) When $\rho\colon H \to \Pi$ is the trivial homomorphism, $\Pi^{\rho} = \Pi$ and $W\rho = WH \times \Pi$. In particular, with $H = e$, we see that $B(G,\Pi) = B\Pi$ as a nonequivariant space and, with $H = G$, we see that there is an inclusion $\zeta\colon B\Pi \to B(G,\Pi)^G$ whose composite with the inclusion of $B(G,\Pi)^G$ in $B(G,\Pi)$ is a nonequivariant homotopy equivalence. It follows that $\Sigma_G^{\infty}B(G,\Pi)^+$ is a split G-spectrum in the sense of [11, Remarks 11] with splitting map

$$\zeta\colon \Sigma^\infty B\Pi^+ = S \wedge B\Pi^+ \xrightarrow{\zeta\wedge\zeta} (S_G)^G \wedge [B(G,\Pi)^+]^G \to [\Sigma_G^\infty B(G,\Pi)^+]^G.$$

The unlabeled arrow is an instance of the natural map $(k_G)^G \wedge X^G \to (k_G \wedge X)^G$ for G-spectra k_G and based G-spaces X (which need not be an equivalence; see [8]).

(ii) Up to G-homotopy, $B(G,\Pi)$ is a product - preserving functor of Π. If Π is Abelian, this implies that $B(G,\Pi)$ is a Hopf G-space. In fact, for any compact Lie group G, the topological Abelian G-group $F(EG^+, B\Pi)$ is here a model for $B(G,\Pi)$; see Lashof, May, and Segal [6]. From this, we easily deduce that $\Sigma_G^\infty B(G,\Pi)^+$ is a split ring G-spectrum and thus that [11, Proposition 15] applies (its finite generation and $\lim^1$ vanishing hypotheses also being easy to check).

The various homomorphisms of which the ξ_H for general Y and their restrictions ξ_ρ for $Y = B(G,\Pi)$ are composites are all induced by maps in the stable category, and we also write ξ_H and

$$\xi_\rho\colon \Sigma^\infty BW\rho^+ \to [\Sigma_G^\infty B(G,\Pi)^+]^G$$

for the resulting composite maps. Thus the wedge sum of the ξ_ρ is an equivalence (for G finite)

$$\text{(3)} \qquad \xi\colon \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty BW\rho^+ \to [\Sigma_G^\infty B(G,\Pi)^+]^G.$$

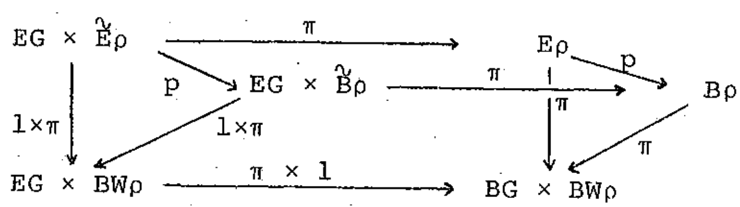
We can now compute the composite of ξ and the map ε of (2) in nonequivariant terms.

<u>THEOREM 8</u>. The adjoint of $\varepsilon\circ\xi_\rho$ is the following composite:

$$\Sigma^\infty BG^+ \wedge \Sigma^\infty BW\rho^+ \cong \Sigma^\infty(BG \times BW\rho)^+ \xrightarrow{\tau} \Sigma^\infty B\rho^+ \xrightarrow{\Sigma^\infty\mu} \Sigma^\infty B\Pi^+.$$

Here $B\rho = EG\times_G \tilde{B}\rho$, τ is the transfer associated to the natural cover $B\rho \to BG \times BW\rho$, and μ is the classifying map of the natural principal Π-bundle $p\colon E\rho \to B\rho$, where $E\rho = EG\times_G\tilde{E}\rho$.

<u>PROOF</u>: Passage to orbits over G gives maps of covers



in which the top rectangle is a map of principal Π-bundles. With

G acting trivially on the right hand triangle, we may view these as maps of G-covers in which the top rectangle is a map of principal (G,Π)-bundles. By the naturality of the equivariant transfer and its behavior on products, this implies the following commutative diagram of G-spectra:

$$\begin{array}{ccccc}
EG^+ \wedge \Sigma_G^\infty BW\rho^+ & \cong & \Sigma_G^\infty(EG \times BW\rho)^+ & \xrightarrow{\Sigma_G^\infty(\pi\times 1)} & \Sigma_G^\infty(BG \times BW\rho)^+ \\
\big\downarrow{\scriptstyle 1\wedge\tau} & & \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau} \\
EG^+ \wedge \Sigma_G^\infty \tilde{B}\rho & \cong & \Sigma_G^\infty(EG \times \tilde{B}\rho)^+ & \xrightarrow{\Sigma_G^\infty \pi} & \Sigma_G^\infty B\rho^+
\end{array}$$

Direct inspection of definitions shows that the nonequivariant transfer of a cover $\pi\colon E \to B$ and the equivariant transfer of π regarded as a G-trivial G-cover are related by the commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty B^+ & \xrightarrow{\zeta} & (\Sigma_G^\infty B^+)^G \\
\big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau^G} \\
\Sigma^\infty E^+ & \xrightarrow{\zeta} & (\Sigma_G^\infty E^+)^G
\end{array}$$

With these preliminaries, we turn to the proof. By Proposition 6 and the definition of ε in [11], we see that the following diagram of spectra must be shown to commute:

$$\begin{array}{ccccccc}
\Sigma^\infty BW\rho^+ & \xrightarrow{\mathrm{adj}(\Sigma^\infty\mu\circ\tau)} & F(BG^+, \Sigma^\infty B\Pi^+) & & \xrightarrow{F(1,\zeta)} & & F(BG^+, \Sigma_G^\infty B(G,\Pi)^+)^G \\
\big\downarrow{\scriptstyle \zeta} & & \big\uparrow{\scriptstyle \varepsilon} & \nwarrow{\scriptstyle \cong} & & & \big\downarrow{\scriptstyle F(\pi,1)^G} \\
(\Sigma_G^\infty BW\rho^+)^G & \xrightarrow{(\Sigma^\infty\mu\circ\tau)^G} & [\Sigma_G^\infty B(G,\Pi)^+]^G & & \xrightarrow{F(\varepsilon,1)} & & F(EG^+, \Sigma_G^\infty B(G,\Pi)^+)^G
\end{array}$$

At the top right, we have used that $F(X,E^G) = F(X,E)^G$ for a space X and G-spectrum E, $F(X,E)$ being the function G-spectrum; the equality is immediate from the definitions in [8]. The composite of $F(\pi,1)^G$ and $F(1,\zeta)$ is an equivalence by [11, Lemma 12] and, by definition, ε is the composite of $F(\varepsilon,1)$ and the inverse of this equivalence. The fixed point spectrum functor from G-spectra to spectra has a left adjoint, which we shall denote by Φ. It may be viewed as assigning trivial G action to a spectrum. (This is not strictly accurate since passage to fixed point spectra involves neglect of indexing representations, hence Φ must resurrect such indexing; see [8].) Moreover, Φ commutes with smash products. On passage to adjoints, the outer rectangle of the previous diagram transforms to the outer part of the following diagram of G-spectra; the maps $\tilde{\zeta}$ are adjoint to ζ and are equivalences.

$$
\begin{array}{ccccccccc}
EG^+\wedge\Phi\Sigma^\infty BW\rho^+ & \xrightarrow{\pi\wedge 1} & BG^+\wedge\Phi\Sigma^\infty BW\rho^+ & \cong & \Phi\Sigma^\infty(BG\times BW\rho)^+ & \xrightarrow{\Phi\tau} & \Phi\Sigma^\infty B\rho^+ & & \\
\Big\downarrow{\scriptstyle 1\wedge\tilde{\zeta}} & & \Big\downarrow{\scriptstyle 1\wedge\tilde{\zeta}} & & \Big\downarrow{\scriptstyle \tilde{\zeta}} & & \Big\downarrow{\scriptstyle \tilde{\zeta}} & \searrow{\scriptstyle \Phi\Sigma^\infty\mu} & \\
EG^+\wedge\Sigma^\infty_G BW\rho^+ & \xrightarrow{\pi\wedge 1} & BG^+\wedge\Sigma^\infty_G BW\rho^+ & \cong & \Sigma^\infty_G(BG\times BW\rho)^+ & \xrightarrow{\tau} & \Sigma^\infty_G B\rho^+ & & \Phi\Sigma^\infty B\Pi^+ \\
\Big\downarrow{\scriptstyle 1\wedge\tau} & & & \nearrow{\scriptstyle \Sigma^\infty_G\pi} & & & & \searrow{\scriptstyle \Sigma^\infty_G\mu} & \Big\downarrow{\scriptstyle \tilde{\zeta}} \\
EG^+\wedge\Sigma^\infty_G\tilde{B}\rho^+ \cong \Sigma^\infty_G(EG\times\tilde{B}\rho)^+ & \xrightarrow{\Sigma^\infty_G(1\times\mu)} & \Sigma^\infty_G(EG\times B(G,\Pi))^+ & & & \xrightarrow{\Sigma^\infty_G(\varepsilon\times 1)} & & & \Sigma^\infty_G B(G,\Pi)^+
\end{array}
$$

The parts of the diagram involving transfer maps have already been observed to commute and the parts involving classifying maps commute in view of the first diagram of the proof.

REMARKS 9. (i) Everything we have done so far is valid for G finite and Π any compact Lie group; compare Remark 2.

(ii) Unraveling notations, we see that

$$E\rho = EG\times_G[(G\times\Pi)\times_{N\rho}EW\rho]$$

and $B\rho = E\rho/\Pi$. There are alternative descriptions, such as

$$E\rho \cong (G\times\Pi/\Delta\rho)\times_{G\times W\rho}(EG\times EW\rho),$$

$W\rho$ being regarded as the group of automorphisms of the transitive $(G\times\Pi)$-set $G\times\Pi/\Delta\rho$.

(iii) To specialize to the situation of the ordinary Segal conjecture, let Π be the trivial group and write

$$CH = EG\times_G(G\times_{NH}EWH) \cong EG\times_{NH}EWH.$$

(Thus $CH = E\rho = B\rho$ for the trivial homomorphism ρ.) Here μ is just the trivial map $CH \to pt$, hence $\Sigma^\infty\mu^+ : \Sigma^\infty CH^+ \to S$ is the unit $1 \in \pi^0(CH)$ of the stable cohomotopy ring of CH. We conclude that $\varepsilon\circ\xi$ is the sum over (H) of the adjoints

$$\Sigma^\infty BWH^+ \to F(BG^+,S) = D(BG^+)$$

of the elements $\tau(1) \in \pi^0(BG\times BWH)$.

(iv) With the description in (iii), the work of Lin, Gunawardena, and Ravenel shows that $\varepsilon\circ\xi$ induces an equivalence upon completion at p when G is a cyclic p-group. See Gunawardena [2,1.5.3] and the appendix by Miller to Ravenel's paper [13]. When G is an elementary Abelian 2-group, unpublished work of Adams, Gunawardena, and Miller gives the same conclusion. Of course, by Theorem 8 and the results of [11], we are entitled to conclude the equivariant form of the Segal conjecture for all finite groups all of whose p-Sylow subgroups are either cyclic or, if $p = 2$, elementary Abelian. Maunder's paper [10] is devoted to a different proof of this passage from the nonequivariant to the equivariant version of the Segal conjecture in the special case $G = \mathbb{Z}_2$.

§2. THE PROOFS OF THEOREMS A AND B

By definition, $BW\rho$ is just $BW(\Delta\rho)$ for the subgroup $\Delta\rho$ of $G\times\Pi$. Thus $[\Sigma_G^\infty B(G,\Pi)^+]^G$ is a wedge summand of $(S_{G\times\Pi})^{G\times\Pi}$. It is simple group theoretical bookkeeping to identify the complementary wedge summand.

PROPOSITION 10. There is an identification

$$\bigvee_{(K)} \Sigma^\infty BWK^+ = \bigvee_{(\Lambda)} \ \bigvee_{(H)} \ \bigvee_{[(\rho)]} \Sigma^\infty BW\rho^+,$$

where the wedge on the left is taken over one K in each conjugacy class of subgroups of $G\times\Pi$ and the wedge on the right is taken over one Λ in each conjugacy class of subgroups of Π, one H in each conjugacy class of subgroups of G, and one ρ in each WH-orbit of $W\Lambda$-conjugacy classes of homomorphisms $H \to W\Lambda$. Therefore (for G finite)

$$(S_{G\times\Pi})^{G\times\Pi} \simeq \bigvee_{(\Lambda)} [\Sigma_G^\infty B(G,W\Lambda)^+]^G.$$

PROOF. The second statement will follow from the equivalences of (3). For the first statement, associate $\Lambda \subset \Pi$, $H \subset G$, and $\rho\colon H \to W\Lambda$ to a subgroup $K \subset G\times\Pi$ by

$$\Lambda = \{\lambda \mid (e,\lambda) \in K\}$$

$$H = \{h \mid (h,\sigma) \in K \text{ for some } \sigma \in \Pi\},$$

and, with the observation that $\sigma \in N\Lambda$ if $(h,\sigma) \in K$,

$$\rho(h) = \sigma\Lambda, \text{ where } (h,\sigma) \in K.$$

Conversely, associate a subgroup $K \subset G\times\Pi$ to a triple (Λ,H,ρ) by

$$K = \{(h,\sigma) \mid h \in H \text{ and } \sigma \in \rho(h)\},$$

where the cosets $\rho(h)$ are regarded as subsets of Π. It is easily checked that these are inverse bijective correspondences such that K runs over a set of conjugacy class representatives as (Λ,H,ρ) runs over a set of representatives as in the statement. Another easy check shows that

$$NK = \{(g,\sigma) \mid g \in NH,\ \sigma \in N\Lambda,\ \sigma\rho(h)\sigma^{-1} = \rho(ghg^{-1})\}.$$

Visibly the quotient homomorphism $N\Lambda \to W\Lambda$ induces an isomorphism

$$WK = NK/K \to N\rho/\Delta\rho = W\rho.$$

REMARK 11. What really seems to be going on here is that there is an equivalence of G-spectra

$$(S_{G\times\Pi})^{\Pi} \simeq \bigvee_{(\Lambda)} \Sigma^{\infty}_{G} B(G,W\Lambda)^{+}.$$

The second author has an outline proof which, if correct, will appear in [3] since it depends on equivariant infinite loop space theory for the construction of a map.

We use the previous proposition to relate the maps ε of (1) and (2). Henceforward, G and Π are both to be finite.

PROPOSITION 12. The following diagram commutes:

$$\begin{array}{ccccc}
(S_{G\times\Pi})^{G\times\Pi} & \cong & \bigvee_{(\Lambda)} [\Sigma^{\infty}_{G}B(G,W\Lambda)^{+}]^{G} & \xrightarrow{\vee\varepsilon} & \bigvee_{(\Lambda)} F(BG^{+},\Sigma^{\infty}BW\Lambda^{+}) \\
\downarrow{\scriptstyle\varepsilon} & & & & \wr\| \\
F((BG\times B\Pi)^{+},S) & \cong & F(BG^{+},F(B\Pi^{+},S)) & \xleftarrow{F(1,\varepsilon)} & F(BG^{+},(S_{\Pi})^{\Pi})
\end{array}$$

PROOF. By Theorems 1 and 8, the following commutative diagram implies the result upon passage to adjoints. Let K correspond to (Λ,H,ρ) as in the previous proof.

$$\begin{array}{ccccccc}
B\Pi^{+}\wedge\Sigma^{\infty}(BG\times BW\rho)^{+} & & & = & & & \Sigma^{\infty}(BG\times B\Pi\times BWK)^{+} \\
\downarrow{\scriptstyle\tau} & & & & & & \downarrow{\scriptstyle\tau} \\
B\Pi^{+}\wedge\Sigma^{\infty}B\rho^{+} & \cong & \Sigma^{\infty}(B\Pi\times B\rho)^{+} & \xrightarrow{\tau} & \Sigma^{\infty}(E\Pi\times_{N\Lambda}E\rho)^{+} & \cong & \Sigma^{\infty}CK^{+} \\
\downarrow{\scriptstyle 1\wedge\Sigma^{\infty}\mu} & & \downarrow{\scriptstyle\Sigma^{\infty}(1\times\mu)} & & \downarrow{\scriptstyle\Sigma^{\infty}(1\times\tilde{\mu})} & & \downarrow{\scriptstyle\Sigma^{\infty}\mu} \\
B\Pi^{+}\wedge\Sigma^{\infty}BW\Lambda^{+} & \cong & \Sigma^{\infty}(B\Pi\times BW\Lambda)^{+} & \xrightarrow{\tau} & \Sigma^{\infty}C\Lambda^{+} & \xrightarrow{\Sigma^{\infty}\mu} & S
\end{array}$$

Here $C\Lambda = E\Pi\times_{N\Lambda}EW\Lambda$ and $\tilde{\mu}: E\rho \to EW\Lambda$ covers μ. The bottom middle square commutes by the naturality of transfer. We easily obtain a homeomorphism $E\Pi\times_{N\Lambda}E\rho \cong CK$ from

$$E\rho = EG\times_{G}[(G\times W\Lambda)\times_{N\rho}EW\rho],\quad CK = (EG\times E\Pi)\times_{NK}EWK,$$

and $W\rho \cong WK$. The bottom right square commutes trivially and the top rectangle commutes by the transitivity of transfer.

Of course, $W(e) = \Pi$ while all other $W\Lambda$ are proper subquotients of Π. To prove Theorem A when G and Π are p-groups, we proceed by induction on the order of Π. Upon completing the diagram of the previous proposition at p, we find in this case that the Segal conjecture for the groups Π and $G\times\Pi$ and the completion conjecture for the G-spectra $\Sigma_G^\infty B(G,W\Lambda)^+$, $\Lambda \neq e$, implies the completion conjecture for $\Sigma_G^\infty B(G,\Pi)^+$.

Because of the different topologies involved, this argument fails hopelessly when G and Π are not p-groups. We use the following observation to prove Theorem A when G is but Π is not a p-group.

<u>LEMMA 13</u>. The following diagram commutes for $i: \Lambda \subset \Pi$:

$$\begin{array}{ccccc}
[\Sigma_G^\infty B(G,\Pi)^+]^G & \xrightarrow{\tau^G} & [\Sigma_G^\infty B(G,\Lambda)^+]^G & \xrightarrow{(\Sigma_G^\infty Bi)^G} & [\Sigma_G^\infty B(G,\Pi)^+]^G \\
\downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \varepsilon \\
F(BG^+,\Sigma^\infty B\Pi^+) & \xrightarrow{F(1,\tau)} & F(BG^+,\Sigma^\infty B\Lambda^+) & \xrightarrow{F(1,\Sigma^\infty Bi)} & F(BG^+,\Sigma^\infty B\Pi^+)
\end{array}$$

<u>PROOF</u>. Clearly $E(G,\Pi)/\Lambda$ is a model for $B(G,\Lambda)$, hence we have a G-cover $Bi: B(G,\Lambda) \to B(G,\Pi)$ with fibre Π/Λ. The right square is an obvious naturality diagram. The left square relates the equivariant transfer of Bi to the nonequivariant transfer of $Bi: B\Lambda \to B\Pi$ and commutes by naturality and the third diagram in the proof of Theorem 8.

Now let G be a p-group, let Λ be a p-Sylow subgroup of Π, and complete the diagram of the lemma at p. The bottom composite is then clearly an equivalence, and Theorem B will imply that the top composite is also an equivalence. It follows that ε for Π is an equivalance if ε for Λ is an equivalence, and this completes the proof of Theorem A.

<u>REMARK 14</u>. Under the hypotheses of the previous paragraph, we may consider $(\Sigma_G^\infty Bi)^G$ as a map

$$\bigvee_{(H)} \bigvee_{[(\sigma)]} \Sigma^\infty BW\sigma^+ \to \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty BW\rho^+,$$

$\sigma: H \to \Lambda$ and $\rho: H \to \Pi$, and Theorem B implies that this map is a p-local retraction. Clearly the wedge summand indexed on $[(\sigma)]$ maps to the wedge summand indexed on $[(i\sigma)]$, but many $[(\sigma)]$

may map to the same $[(\rho)]$. Glauberman has proven that the number of $[(\sigma)]$ which map to a given $[(\rho)]$ and are such that $W(\sigma)$ is conjugate to a p-Sylow subgroup of $W(\rho)$ is prime to p. This provides a plausibility argument for the cited retraction property.

It remains to prove Theorem B. This is an easy exercise in the use of the ordinary RO(G)-graded cohomology theories on G-spectra which we introduced in [7] and will study in detail in [9].

A map $f: X \to Y$ of connective G-spectra induces an equivalence upon localization at p if and only if

$$f^*: H_G^*(Y;M) \to H_G^*(X;M)$$

is an isomorphism for all p-local Mackey functors M. Compare [12], where the analogous G-space level statement is proven.

Let $\pi: E \to B$ be a G-cover with fibre F and consider the composite

$$f: \Sigma_G^\infty B^+ \xrightarrow{\tau} \Sigma_G^\infty E^+ \xrightarrow{\Sigma_G^\infty \pi} \Sigma_G^\infty B^+.$$

If k_G^* is a ring valued RO(G)-graded cohomology theory and m_G^* is a k_G^*-module valued RO(G)-graded cohomology theory, then $f^*: m_G^*(B) \to m_G^*(B)$ is multiplication by $\tau(1) \in k_G^0(B)$, where $1 \in k_G^0(E)$ is the identity element.

For any Mackey functor M, $H_G^*(?;M)$ is module-valued over $H_G^*(?;\underline{A})$, where $\underline{A}$ is the Burnside ring Green functor, $\underline{A}(G/H) = A(H)$. The same holds for p-local M and the localization $\underline{A}_p$ of $\underline{A}$ at p. To prove Theorem B, it suffices to show that $\tau(1) \in H_G^0(B;\underline{A}_p)$ is a unit when G is a p-group and $|F|$ is prime to p.

By G-CW approximation and an easy colimit argument, we may assume without loss of generality that B is a G-CW complex with finite 0-skeleton B^0. The inclusion of B^0 in B induces a monomorphism $H_G^0(B;M) \to H_G^0(B^0;M)$ for any M. Our finiteness assumption ensures that $H_G^0(B^0;\underline{A}_p)$ is an integral extension of $H_G^0(B^0;\underline{A}_p)$. Thus it suffices to prove that $\tau(1) \in H_G^0(B^0;\underline{A}_p)$ is a unit, where τ is the transfer associated to the restriction $\pi: \pi^{-1}(B^0) \to B^0$.

Since B^0 is a (finite) disjoint union of orbits G/H, $\pi^{-1}(B^0) \to B^0$ breaks up into a disjoint union of G-covers $\pi^{-1}(G/H) \to G/H$ with fibre F. It suffices to show that

$$\tau(1) \in H^0_G(G/H;\underline{A}_p) = A(H)_p$$

is a unit for each orbit G/H. Now $\pi^{-1}(G/H) \to G/H$ has the form $G\times_H F \to G/H$ for some action of H on F. We may regard F as a finite H-set and thus as an element of $A(H)$ via this action. A check of definitions shows that $\tau(1) = F$.

Since H is a p-group, $|F^K| \equiv |F| \not\equiv 0 \mod p$ for $K \subset H$. It follows that the image of F under the natural embedding of rings $\chi\colon A(H)_p \to \times_{(K)} Z_{(p)}$, $\chi_K(S) = |S^K|$ for an H-set S, is a unit. Since χ is an integral extension, F is a unit in $A(H)_p$ and the proof is complete.

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