LARSON P-DIVISIBLE GROUPS II

Note Title 2/19/2010

PS, SB, JP, NF, TW, OZ

$\mu_n = \mathrm{Spec}\ k[x]/(x^n - 1)$
$\alpha_p = \mathrm{Spec}\ k[x]/x^p$
$\mathbb{Z}/p = \mathrm{Spec}\ k[x]/(x^p - x)$

for k a finite field of char. p

Motivation

E/k elliptic curve

For $m \in \mathbb{Z}$, let $E[m]$ be the points of order m on E

Thm 1 a) If $(m,p)=1$ then $E[m] \simeq (\mathbb{Z}/m)^2$

b) Either $E[p^n]=0$ (supersingular case)

or $E[p^n] = \mathbb{Z}/p^n$ (ordinary case)

Thm 2 The group scheme $E(p^n)$ has order $(p^n)^2$, i.e. it is rep'd by a k-module of rank p^{2n}.

SLOGAN: The missing rank-it torsion in Thm 1 is hidden in the associated formal gp.

Ordinary E: FG has ht 1

Supersingular E: " 2

Note Given an elliptic curve E, we have
$$E(p) \hookrightarrow E(p^2) \hookrightarrow E(p^3) \hookrightarrow \cdots$$
is a p-divisible gp of height 2.

Dieudonne modules:

Let $G = \operatorname{Spec} A$ be an affine finite group scheme.

Define $A^\vee = \operatorname{Hom}_k(A, k)$.
$$G^\vee = \operatorname{Spec} A^\vee . \qquad (G^\vee)^\vee \cong G$$

This is the Cartier Dual of G. It is a contravariant functor from affine gp schemes to itself.

Examples

1. $(\mathbb{Z}/m\mathbb{Z})^\vee = \mu_m$
2. $(\alpha_p)^\vee = \alpha_p$

Frobenius

Def Let $A = \operatorname{Spec} k[x_1, \dots x_n]/(f_1, \dots f_m)$

$G = \operatorname{Spec} A$

$B = [x_1, \dots x_n]/(g_1, \dots g_m)$

$G^{(p)} = \operatorname{Spec} B$

where g_i is f_i with all coeffs raised to pth power, e.g. $g_i = f_i$ when $k = \mathbb{F}_p$

The map BA defined by $x_i \to x_i^p$ induces the **Frobenius** $G \longrightarrow G^{(p)}$.

Verschiebung

$Ver_1 := F_1^\vee : G^{(p)} \longrightarrow G_1$)

the dual of $G^\vee \xrightarrow{F_1} (G^\vee)^{(p)} = (G^{(p)})^\vee$

and $F_1 \circ Ver_1 = [p]_{G^{(p)}}$

$Ver_1 \circ F_1 = [p]_G$

Examples

1. μ_p : $F_1 = 0$, $Ver_1 = id$
2. α_p : $F_1 = Ver_1 = 0$
3. $\mathbb{Z}/p$: $F_1 = id$, $Ver_1 = 0$

With Ring $:= W(k) = W$

$W(\mathbb{F}_p) = \mathbb{Z}_p$

$W(\mathbb{F}_{p^n}) =$ valuation ring of the unique degree n unramified extension of $\mathbb{Q}_p$

$W(k) =$ the complete DVR with residue field k and maximal ideal p.

For $x \in W$, $x = (x_0, x_1, x_2, \ldots)$ where $x_i^{p^n} - x_i = 0$

$F_r(x_0, x_1, \ldots) = (x_0^p, x_1^p, x_2^p \ldots)$

$V(x_0, x_1, \ldots) = (0, x_1, x_2, x_3 \ldots)$

Let $E = W\langle F, V\rangle$ with relations

(1) $FV = VF = p$

(2) $Fx = F_r(x)\, F$

(3) $x \cdot V = V \cdot F_r(x)$

This is the Dieudonné ring

Note $E/(F) = k[V]$ and $E(V) = k[F]$

Thm: $\exists$ anti-equivalence of categories

$$D_k : \left\{\begin{array}{c}\text{finite gp}\\ \text{schemes}/k\end{array}\right\} \longrightarrow \left\{\begin{array}{c}E\text{-modules of}\\ \text{finite length}\end{array}\right\}$$

i.e. module has finite filtration with simple subquotients.

Examples ① $D_k(\mu_p) = E/(F, V-1) = k$

② $D_k(\alpha_p) = E/(F, V) = k$

③ $D_k(\mathbb{Z}/p) = E(F-1, V) = k$

④ $D_k\left(\begin{array}{l}\text{p-torsion of} \\ \text{supersingular} \\ \text{ell. curve}\end{array}\right) = E/(F^2, V^2, F+V)$

$\overset{?}{=} k\{1, F\}$

Remark $\exists$ anti-equivalence

$D_k : \{\text{p-div gps}\} \longrightarrow \left\{\begin{array}{l}\text{certain subcategory} \\ \text{of } E\text{-modules}\end{array}\right\}$

$\{\text{formal gps}\} \longrightarrow \left\{\begin{array}{l}\text{some other} \\ \text{subcategory}\end{array}\right\}$

Back to elliptic curves E over alg. closed k.

For ordinary E, $E(p) = \mu_p \times \mathbb{Z}/p\mathbb{Z}$

where μ_p is ker Fr and $\mathbb{Z}/p\mathbb{Z}$ is ker Ver

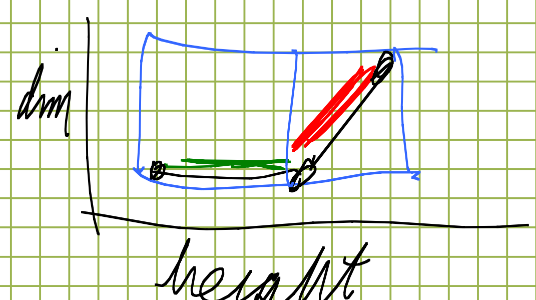
$$E(p^\infty) = \mu_{p^\infty} \times \mathbb{Z}/p^\infty$$

connected i.e. rep'd by a local ring (FGL of ht 1) dim 1

étale gp of height 1 has dim 0

Both factors are simple p-div gps. Slope = dim/ht

Newton polygon



Simple p-div gps are classified up to isogeny by their slopes and p-div gps by their Newton polygons.

Supersingular case

$$0 \to \alpha_p \to E(p) \to \alpha_p \to 0 \quad \text{(not split)}$$

$\ker F_n = \ker V_n$

Newton polygon