LARSON p-DIVISIBLE GROUPS NR, PS, TW, JP, SB, JH, FC, MW

Note Title 2/5/2010

§1 Group schemes

Def Let k be a comm ring. A (affine, finite) group scheme over k is a covar. rep'ble functor G : k-algebras $\longrightarrow$ groups given by $G(R) = \mathrm{Hom}_k(A, R)$ for finite k-alg A. (A is finitely gen'd as a k-module.)

- Write $G = \mathrm{Spec}(A)$
- Order G = dimension of A/k (minimal # of generators)

Note Maps between gp schemes are natural transformations.

Yoneda Lemma $\Rightarrow$ any map $\mathrm{Spec}\,A \longrightarrow \mathrm{Spec}\,B$ is induced by $A \longleftarrow B$.

This implies R is a Hopf algebra, i.e. the group structure is induced by a coproduct $A \to A \otimes A$ because $\mathrm{Spec}(A) \times \mathrm{Spec}(A) = \mathrm{Spec}(A \otimes_k A)$.

From now on let k be a field of char p.

Examples ① $\mu_{p^n} = \mathrm{Spec}\, k[x]/(x^{p^n}-1)$

It has order p^n

$\mu_{p^n}(R) = \{ \mu \in R \mid \mu^{p^n} = 1 \}$

= gp under mult.

$= \{1\}$ if R is a domain

② $\alpha_p = \operatorname{Spec} k[x]/x^p$ order p

$\alpha_p(R) = \{ \mu \in R : \mu^p = 0 \}$

$= \{0\}$ is R is a domain

③ $\mathbb{Z}/p\mathbb{Z} = \operatorname{Spec} k[x]/(x^p - x)$

$= \operatorname{Spec} k[x]/x(x-1)\cdots(x-p+1)$

$\cong k \times k \times \cdots \times k$ p copies of k

Def ① A gp scheme Spec(A) is étale if A is a separable algebra, i.e. $A \otimes_k \bar{k} = \bar{k} \times \cdots \times \bar{k}$.

Example $\mathbb{Z}/p\mathbb{Z}$ but not μ_{p^n}.

Spec $k[x]/(x^{p^2}-x)$ is not split $/\mathbb{F}_p$ but is split $/\mathbb{F}_{p^2}$.

Def ② A gp scheme G is connected if $G(K) = 0$ for each field K. Equivalently Spec(A) for A local

Example: α_p and μ_{p^n} are connected.
μ_l for a prime $l \neq p$ is étale.

§2 p-divisible groups

Let E/k be an elliptic curve (the motivating example).

Define $E[p^n] = \ker\{p^n : E \longrightarrow E\}$. It is a gp for all n. There is a diagram of gps

$$\begin{tikzcd} 0 \arrow[r, hook] & E[p] \arrow[r, hook] & E[p^2] \arrow[r, hook] \arrow[d, "p"] & E[p^3] \arrow[r, hook] \arrow[d, "p^2"] & E[p^4] \arrow[r] \arrow[d, "p^3"] & \cdots \\ & & E[p^2] & E[p^3] & E[p^4] & \end{tikzcd}$$

We have exact sequences

$$0 \to E[p^i] \longrightarrow E[p^{i+1}] \xrightarrow{p^i} E[p^{i+1}]$$

And (???) $|E[p^n]| = p^{2n}$ (char 0 ?)

Def A p-divisible gp of height h is a sequence of gp schemes and maps

$$N_1 \xrightarrow{i_1} N_2 \xrightarrow{i_2} N_3 \longrightarrow \cdots$$

such that $|N_n| = p^{hn}$ and

$$\operatorname{im} i_n = \ker\{N_{n+1} \xrightarrow{p^n} N_{n+1}\}.$$

Example $p=2$

$$0 \to \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/8 \xrightarrow{\cdot 2} \mathbb{Z}/16 \xrightarrow{\cdot 2} \cdots$$

(inclusion maps)

Take direct lim and get $\mathbb{Q}_2/\mathbb{Z}_2$

Example

$$\mu_{p^\infty} = \{0 \to \mu_p \to \mu_{p^2} \to \mu_{p^3} \to \cdots\}$$

$$= \mathbb{G}_m[p]$$

Def ① A hom. between p-div gps is a compatible family of maps $\varphi_n : N_n \longrightarrow N_n'$.

② A p-div gp N is connected (étale) if N_n is connected (étale) for all $n \geq 1$.

Formal gps

Note: A 1-dim formal gp Γ over k is given by a power series $F(x, y)$, i.e. a map

$$\varphi : k[[x]] \longrightarrow k[[y, z]]$$

Example $\mu_{p^n} = \operatorname{Spec} k[x]/(x^{p^n} - 1)$ form a p-div gp μ_{p^∞}. We get maps

$k[x]/(x^p-1) \leftarrow k[x]/(x^{p^2}-1) \leftarrow k[x]/(x^{p^3}-1) \leftarrow \cdots$

Let $A = \varprojlim_n k[x]/(x^{p^n}-1) = \varprojlim_n k[x]/(x-1)^{p^n}$

One can show ① $A = k[[x-1]] = k[[x]]$

② We get a hom $A \longrightarrow A \hat{\otimes}_k A$ which is the FGL $F(x,y) = x+y-xy$.

FGL's over k are equiv to p-divi gps

Divisible.

To be continued on 2/19.