The K-book
an introduction to Algebraic K-theory

Charles A. Weibel

April 4, 2013

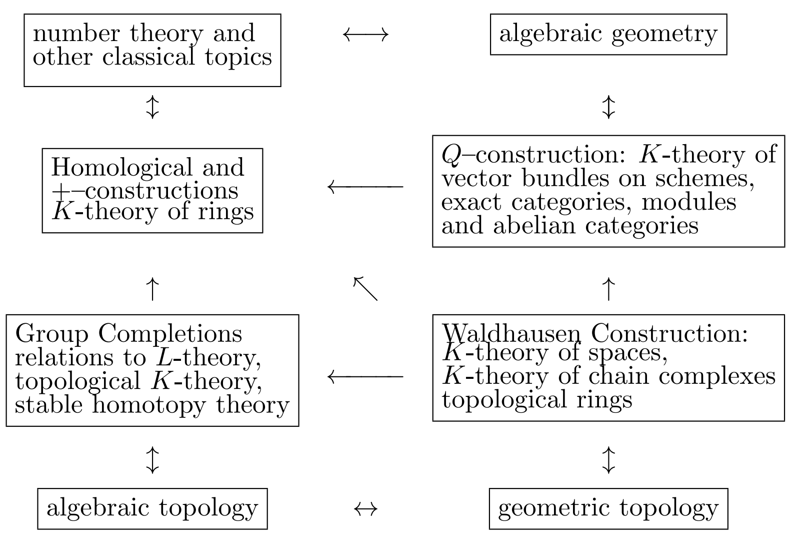
Contents

Introduction

Algebraic K-theory has two components: the classical theory which centers around the Grothendieck group K_0 of a category and uses explicit algebraic presentations, and higher algebraic K-theory which requires topological or homological machinery to define.

There are three basic versions of the Grothendieck group K_0. One involves the group completion construction, and is used for projective modules over rings, vector bundles over compact spaces and other symmetric monoidal categories. Another adds relations for exact sequences, and is used for abelian categories as well as exact categories; this is the version first used in algebraic geometry. A third adds relations for weak equivalences, and is used for categories of chain complexes and other categories with cofibrations and weak equivalences ("Waldhausen categories").

Similarly, there are four basic constructions for higher algebraic K-theory: the +–construction (for rings), the group completion constructions (for symmetric monoidal categories), Quillen's Q-construction (for exact categories), and Waldhausen's $wS.$ construction (for categories with cofibrations and weak equivalences). All these constructions give the same K-theory of a ring, but are useful in various distinct settings. These settings fit together like this:



All the constructions have one feature in common: Some category C is concocted from the given setup, and one defines a K-theory space associated to the geometric realization BC of this category. The K-theory groups are then the homotopy groups of the K-theory space. In the first chapter, we introduce the basic cast of characters: projective modules and vector bundles (over a topological space, and over a scheme). Large segments of this chapter will be familiar to many readers, but which segments are familiar will depend upon the background and interests of the reader. The unfamiliar parts of this material may be skipped at first, and referred back to when relevant. We would like to warn the complacent reader that the material on the Picard group and Chern classes for topological vector bundles is in this first chapter.

In the second chapter, we define K_0 for all the settings in the above figure, and give the basic definitions appropriate to these settings: group completions for symmetric monoidal categories, K_0 for rings and topological spaces, λ-operations, abelian and exact categories, Waldhausen categories. All definitions and manipulations are in terms of generators and relations. Our philosophy is that this algebraic beginning is the most gentle way to become acquainted with the basic ideas of higher K-theory. The material on K-theory of schemes is isolated in a separate section, so it may be skipped by those not interested in algebraic geometry.

In the third chapter we give a brief overview of the classical K-theory for K_1 and K_2 of a ring. Via the Fundamental Theorem, this leads to Bass' "negative K-theory," meaning groups K_{-1}, K_{-2}, etc. We cite Matsumoto's presentation for K_2 of a field from [131], and "Hilbert's Theorem 90 for K_2" (from chapter VI) in order to get to the main structure results. This chapter ends with a section on Milnor K-theory, including the transfer map, Izhboldin's theorem on the lack of p-torsion, the norm residue symbol and the relation to the Witt ring of a field.

In the fourth chapter we shall describe the four constructions for higher K-theory, starting with the original BGL^+ construction. In the case of $\mathbf{P}(R)$, finitely generated projective R-modules, we show that all the constructions give the same K-groups: the groups $K_n(R)$. The λ-operations are developed in terms of the $S^{-1}S$ construction. Non-connective spectra and homotopy K-theory are also presented. Very few theorems are present here, in order to keep this chapter short. We do not want to get involved in the technicalities lying just under the surface of each construction, so the key topological results we need are cited from the literature when needed.

The fundamental structural theorems for higher K-theory are presented in chapter V. This includes Additivity, Approximation, Cofinality, Resolution, Devissage and Localization (including the Thomason-Trobaugh localization theorem for schemes). As applications, we compute the K-theory and G-theory of projective spaces and Severi-Brauer varieties (§2), construct transfer maps satisfying a projection formula (§3), prove the Fundamental Theorem for G-theory (§6) and K-theory (§9). Several cases of Gersten's DVR conjecture are established in §6 and the Gersten-Quillen conjecture in §7. This is used to interpret the coniveau spectral sequence in terms of K-cohomology, and establish Bloch's

Formula that $CH^p(X) \cong H^p(X, \mathcal{K}_p)$ for regular varieties.

In chapter 6 we describe the structure of the K-theory of fields. First we handle algebraically closed fields (§1), and the real numbers $\mathbb{R}$ (§3), following Suslin and Harris-Segal. The group $K_3(F)$ can also be handled by comparison to Bloch's group $B(F)$ using these methods (§5). In order to say more, using classical invariants such as étale cohomology, we introduce the spectral sequence from Motivic Cohomology to K-theory in §4 and use it in §6–10 to describe the K-theory of local and global fields.

The Back Story:

In 1985, I started hearing a persistent rumor that I was writing a book on algebraic K-theory. This was a complete surprise to me! Someone else had started the rumor, and I never knew who. After a few years, I had heard the rumor from at least a dozen people.

It actually took a decade before the rumor had became true — like the character Topsy[1], the book project was never born, it just grew. In 1988 I wrote out a brief outline, following Quillen's paper *Higher algebraic K-theory I* [153]. It was overwhelming. I talked to Hy Bass, the author of the classic book *Algebraic K-theory* [15], about what would be involved in writing such a book. It was scary, because (in 1988) I didn't know even how to write a book.

I needed a warm-up exercise, a practice book if you will. The result, *An introduction to homological algebra* [223], took over five years to write.

By this time (1995), the K-theory landscape had changed, and with it my vision of what my K-theory book should be. Was it an obsolete idea? After all, the new developments in Motivic Cohomology were affecting our knowledge of the K-theory of fields and varieties. In addition, there was no easily accessible source for this new material. Nevertheless, I wrote early versions of Chapters I-IV during 1994-1999. The project became known as the "K-book" at this time.

In 1999, I was asked to turn a series of lectures by Voevodsky into a book. This project took over six years, in collaboration with Carlo Mazza and Vladimir Voevodsky. The result was the book *Lecture Notes on Motivic Cohomology* [122], published in 2006.

In 2004-2008, Chapters IV and V were completed. At the same time, the final steps in the proof of the Norm Residue Theorem VI.4.1 were finished. (This settles not just the Bloch-Kato Conjecture, but also the Beilinson-Lichtenbaum Conjectures and Quillen-Lichtenbaum Conjectures.) The proof of this theorem is scattered over a dozen papers and preprints, and writing it spanned over a decade of work, mostly by Rost and Voevodsky. Didn't it make sense to put this house in order? It did. I am currently collaborating with Christian Haesemeyer in writing a self-contained proof of this theorem.

Charles A. Weibel

[1] *Topsy* is a character in Harriet B. Stowe's 1852 book *Uncle Tom's Cabin*, who claimed to have never been born: "Never was born... I 'spect I grow'd. Don't think nobody never made me." (sic)

August 2012

Acknowledgements

The author is grateful to whoever started the rumor that he was writing this book. He is also grateful to the many people who have made comments on the various versions of this manuscript over the years:

R. Thomason, D. Grayson, T. Geisser, C. Haesemeyer, J.-L. Loday, M. Lorenz, J. Csirik, M. Paluch, Paul Smith, P. A. Østvær, A. Heider, J. Hornbostel, B. Calmes, G. Garkusha, P. Landweber, A. Fernandez Boix, C. Mazza J. Davis, I. Leary, C. Crissman, P. Polo, R. Brasca, O. Braeunling, F. Calegari, K. Kedlaya, D. Grinberg, P. Boavida, R. Reis, J. Levikov, O. Schnuerer, P. Pelaez, Sujatha, J. Spakula, J. Cranch, A. Asok, ...

Chapter I

Projective modules and vector bundles

The basic objects studied in algebraic K-theory are projective modules over a ring, and vector bundles over schemes. In this first chapter we introduce the cast of characters. Much of this information is standard, but collected here for ease of reference in later chapters.

Here are a few running conventions we will use. The word *ring* will always mean an associative ring with 1 ($1 \neq 0$). If R is a ring, the word *R-module* will mean right R-module unless explicitly stated otherwise.

1 Free modules, GL_n and stably free modules

If R is a field, or a division ring, then R-modules are called *vector spaces*. Classical results in linear algebra state that every vector space has a basis, and that the rank (or dimension) of a vector space is independent of the choice of basis. However, much of this fails for arbitrary rings.

As with vector spaces, a *basis* of an R-module M is a subset $\{e_i\}_{i\in I}$ such that every element of M can be expressed in a unique way as a finite sum $\sum e_i r_i$ with $r_i \in R$. We say that a module M is *free* if it has a basis. If M has a fixed ordered basis we call M a *based free module*, and define the *rank* of the based free module M to be the cardinality of its given basis. Homomorphisms between based free modules are naturally identified with matrices over R.

The canonical example of a based free module is R^n with the usual basis; it consists of n-tuples of elements of R, or "column vectors" of length n.

Unfortunately, there are rings for which $R^n \cong R^{n+t}$, $t \neq 0$. We make the following definition to avoid this pathology, referring the curious reader to the exercises for more details. (If κ is an infinite cardinal number, let $R^{(\kappa)}$ denote a free module on a basis of cardinality κ; every basis of $R^{(\kappa)}$ has cardinality κ. In particular $R^{(\kappa)}$ cannot be isomorphic to R^n for finite n. See Ch.2, 5.5 of Cohn65 [44].)

I.1.1 **Definition 1.1** (IBP)**.** We say that a ring R satisfies the (right) *invariant basis property (or IBP)* if R^m and R^n are not isomorphic for $m \neq n$. In this case, the rank of a free R-module M is an invariant, independent of the choice of basis of M.

Most of the rings we will consider satisfy the invariant basis property. For example, commutative rings satisfy the invariant basis property, and so do group rings $\mathbb{Z}[G]$. This is because a ring R must satisfy the IBP if there exists a ring map $f\colon R \to F$ from R to a field or division ring F. (If R is commutative we may take $F = R/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal of R.) To see this, note that any basis of M maps to a basis of the vector space $V = M \otimes_R F$; since $\dim V$ is independent of the choice of basis, any two bases of M must have the same cardinality.

Our choice to use right modules dictates that we write R-module homomorphisms on the left. In particular, homomorphisms $R^n \to R^m$ may be thought of as $m \times n$ matrices with entries in R, acting on the column vectors in R^n by matrix multiplication. We write $M_n(R)$ for the ring of $n \times n$ matrices, and write $GL_n(R)$ for the group of invertible $n \times n$ matrices, *i.e.,* the automorphisms of R^n. We will usually write $R^\times$ for the group $U(R) = GL_1(R)$ of *units* in R.

I.1.1.1 **Example 1.1.1.** Any finite-dimensional algebra R over a field (or division ring) F must satisfy the IBP, because the rank of a free R-module M is an invariant:

$$\operatorname{rank}(M) = \dim_F(M)/\dim_F(R).$$

For a simple artinian ring R we can say even more. Classical Artin-Wedderburn theory states that $R = M_n(F)$ for some n and F, and that every right R-module M is a direct sum of copies of the (projective) R-module V consisting of row vectors over F of length n. Moreover, the number of copies of V is an invariant of M, called its *length*; the length is also $\dim_F(M)/n$ since $\dim_F(V) = n$. In this case we also have $\operatorname{rank}(M) = \operatorname{length}(M)/n = \dim_F(M)/n^2$.

There are noncommutative rings which do not satisfy the IBP, *i.e.,* which have $R^m \cong R^n$ for some $m \neq n$. Rank is not an invariant of a free module over these rings. One example is the infinite matrix ring $\operatorname{End}_F(F^\infty)$ of endomorphisms of an infinite-dimensional vector space over a field F. Another is the cone ring $C(R)$ associated to a ring R. (See the exercises.)

Unimodular rows and stably free modules

I.1.2 **Definition 1.2.** An R-module P is called *stably free* (of rank $n-m$) if $P \oplus R^m \cong R^n$ for some m and n. (If R satisfies the IBP then the rank of a stably free module is easily seen to be independent of the choice of m and n.) Conversely, the kernel of any surjective $m \times n$ matrix $\sigma\colon R^n \to R^m$ is a stably free module, because a lift of a basis for R^m yields a decomposition $P \oplus R^m \cong R^n$.

This raises a question: when are stably free modules free? Over some rings every stably free module is free (fields, $\mathbb{Z}$ and the matrix rings $M_n(F)$ of Example I.1.1.1 1.1.1 are classical cases), but in general this is not so even if R is commutative; see example I.1.2.2 1.2.2 below.

I.1.2.1 **1.2.1.** The most important special case, at least for inductive purposes, is when $m = 1$, *i.e.,* $P \oplus R \cong R^n$. In this case σ is a row vector, and we call σ a *unimodular row.* It is not hard to see that the following conditions on a sequence $\sigma = (r_1, ..., r_n)$ of elements in R are equivalent for each n:

- σ is a unimodular row;
- $R^n \cong P \oplus R$, where $P = \ker(\sigma)$ and the projection $R^n \to R$ is σ;
- $R = r_1 R + \cdots + r_n R$;
- $1 = r_1 s_1 + \cdots + r_n s_n$ for some $s_i \in R$.

If $R^n \cong P \oplus R$ with P free, then a basis of P would yield a new basis for R^n and hence an invertible matrix g whose first row is the unimodular row $\sigma\colon R^n \to R$ corresponding to P. This gives us a general criterion: P is a free module if and only if the corresponding unimodular row may be completed to an invertible matrix. (The invertible matrix is in $GL_n(R)$ if R satisfies the IBP).

When R is commutative, every unimodular row of length 2 may be completed. Indeed, if $r_1 s_1 + r_2 s_2 = 1$, then the desired matrix is:

$$\begin{pmatrix} r_1 & r_2 \\ -s_2 & s_1 \end{pmatrix}$$

Hence $R^2 \cong R \oplus P$ implies that $P \cong R$. In §3 we will obtain a stronger result: every stably free module of rank 1 is free. The fact that R is commutative is crucial; in Ex. 1.6 we give an example of a unimodular row of length 2 which cannot be completed over $D[x, y]$, D a division ring.

I.1.2.2 **Example 1.2.2.** Here is an example of a unimodular row σ of length 3 which cannot be completed to an element of $GL_3(R)$. Hence $P = \ker(\sigma)$ is a rank 2 stably free module P which is not free, yet $P \oplus R \cong R^3$. Let σ be the unimodular row $\sigma = (x, y, z)$ over the commutative ring $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 = 1)$. Every element (f, g, h) of R^3 yields a vector field in 3-space ($\mathbb{R}^3$), and σ is the vector field pointing radially outward. Therefore an element in P yields a vector field in 3-space tangent to the 2-sphere S^2. If P were free, a basis of P would yield two tangent vector fields on S^2 which are linearly independent at every point of S^2 (because together with σ they span the tangent space of 3-space at every point). It is well known that this is impossible: you can't comb the hair on a coconut. Hence P cannot be free.

The following theorem describes a "stable range" in which stably free modules are free (see 2.3 for a stronger version). A proof, due to Bass, may be found in [15, V.3.5], using the "stable range" condition (S_n) of Ex. 1.5 below. Example 1.2.2 shows that this range is sharp.

I.1.3 **Bass Cancellation Theorem for stably free modules 1.3.** *Let R be a commutative noetherian ring of Krull dimension d. Then every stably free R-module of rank $> d$ is a free module. Equivalently, every unimodular row of length $n \geq d + 2$ may be completed to an invertible matrix.*

The study of stably free modules has a rich history, and we cannot do it justice here. An excellent source for further information is Lam's book [106].

EXERCISES

EI.1.1 **1.1.** *Semisimple rings.* A nonzero R-module M is called *simple* if it has no submodules other than 0 and M, and *semisimple* if it is the direct sum of simple modules. A ring R is called *semisimple* if R is a semisimple R-module. If R is semisimple, show that R is a direct sum of a finite (say n) number of simple modules. Then use the Jordan-Hölder Theorem, part of which states that the length of a semisimple module is an invariant, to show that every stably free module is free. In particular, this shows that semisimple rings satisfy the IBP. *Hint:* Observe that length$= n \cdot$ rank is an invariant of free R-modules.

EI.1.2 **1.2.** (P.M. Cohn) Consider the following conditions on a ring R:

(I) R satisfies the invariant basis property (IBP);

(II) For all m and n, if $R^m \cong R^n \oplus P$ then $m \geq n$;

(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.

If $R \neq 0$, show that (III) $\Rightarrow$ (II) $\Rightarrow$ (I). For examples of rings satisfying (I) but not (II), resp. (II) but not (III), see [45].

EI.1.3 **1.3.** Show that (III) and the following matrix conditions are equivalent:

(a) For all n, every surjection $R^n \to R^n$ is an isomorphism;

(b) For all n, and $f, g \in M_n(R)$, if $fg = 1_n$, then $gf = 1_n$ and $g \in GL_n(R)$.

Then show that commutative rings satisfy (b), hence (III).

EI.1.4 **1.4.** Show that right noetherian rings satisfy condition (b) of the previous exercise. Hence they satisfy (III), and have the right invariant basis property.

EI.1.5 **1.5.** *Stable Range Conditions.* We say that a ring R satisfies condition (S_n) if for every unimodular row $(r_0, r_1, ..., r_n)$ in R^{n+1} there is a unimodular row $(r'_1, ..., r'_n)$ in R^n with $r'_i = r_i - r_0 t_i$ for some $t_1, ..., t_n$ in R. The *stable range of* R, $sr(R)$, is defined to be the smallest n such that R satisfies condition (S_n). (Warning: our (S_n) is the stable range condition SR_{n+1} of [15].)

Bass' Cancellation Theorem [15, V.3.5], which is used to prove 1.3, and 2.3 below, states that $sr(R) \leq d+1$ if R is a d-dimensional commutative noetherian ring, or more generally if Max(R) is a finite union of spaces of dimension $\leq d$.

(a) (Vaserstein) Show that (S_n) holds for all $n \geq sr(R)$.

(b) If $sr(R) = n$, show that all stably free projective modules of rank $\geq n$ are free. *Hint:* compare $(r_0, ..., r_n)$, $(r_0, r'_1, ..., r'_n)$ and $(1, r'_1, ..., r'_n)$.

(c) Show that $sr(R) = 1$ for every artinian ring R. Conclude that all stably free projective R-modules are free over artinian rings.

(d) Show that if I is an ideal of R then $sr(R) \geq sr(R/I)$.

(e) (Veldkamp) If $sr(R) = n$ for some n, show that R satisfies the invariant basis property (IBP). *Hint:* Consider an isomorphism $B\colon R^N \cong R^{N+n}$, and apply (S_n) to convert B into a matrix of the form $\binom{C}{0}$.

EI.1.6 **1.6.** (Ojanguren-Sridharan) Let D be a division ring which is not a field. Choose $\alpha, \beta \in D$ such that $\alpha\beta - \beta\alpha \neq 0$, and show that $\sigma = (x+\alpha, y+\beta)$ is a unimodular row over $R = D[x,y]$. Let $P = \ker(\sigma)$ be the associated rank 1 stably free module; $P \oplus R \cong R^2$. Prove that P is not a free $D[x,y]$-module, using these steps:

(i) If $P \cong R^n$, show that $n = 1$. Thus we may suppose that $P \cong R$ with $1 \in R$ corresponding to a vector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ with $r, s \in R$.

(ii) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = c_1x + c_2y + c_3xy + c_4y^2$ and $g = d_1x + d_2y + d_3xy + d_4x^2$, $(c_i, d_i \in D)$.

(iii) Show that P cannot contain any vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with f and g linear polynomials in x and y. Conclude that the vector in (i) must be quadratic, and may be taken to be of the form given in (ii).

(iv) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = \gamma_0 + \gamma_1 y + y^2$, $g = \delta_0 + \delta_1 x - \alpha y - xy$ and $\gamma_0 = \beta u^{-1}\beta u \neq 0$. This contradicts (iii), so we cannot have $P \cong R$.

EI.1.7 **1.7.** *Direct sum rings.* A ring R (with unit) is called a *direct sum ring* if there is an R-module isomorphism $R \cong R^2$. This implies that $R \cong R^n$ for every finite n. Any homomorphism $R \to S$ makes S into a direct sum ring, so many direct sum rings exist. In this exercise and the next, we give some examples of direct sum rings.

For any ring R, let $R^\infty = R^{(\aleph_0)}$ be a fixed free R-module on a countably infinite basis. Then R^∞ is naturally a left module over the endomorphism ring $E = \mathrm{End}_R(R^\infty)$, and we identify E with the ring of infinite column-finite matrices.

If $R^\infty = V_1 \oplus V_2$ as a left R-module, show that $E = I_1 \oplus I_2$ for the right ideals $I_i = \{f \in E : f(R^\infty) \subseteq V_i\}$. Conversely, if $E = I_1 \oplus I_2$ as a right module, show that $R^\infty = V_1 \oplus V_2$, where $V_i = I_i \cdot R^\infty$. Conclude that E is a direct sum ring, and that $I \oplus J = E$ implies that $I \oplus E \cong E$ for every right ideal I of E.

EI.1.8 **1.8.** *Cone Ring.* For any ring R, the endomorphism ring $\mathrm{End}_R(R^\infty)$ of the previous exercise contains a smaller ring, namely the subring $C(R)$ consisting of row-and-column finite matrices. The ring $C(R)$ is called the *cone ring* of R.Show that $C(R)$ is a direct sum ring.

EI.1.9 **1.9.** To see why our notion of stably free module involves only finitely generated free modules, let R^∞ be the infinitely generated free module of Exercise EI.1.7 1.7. Prove that if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$. *Hint:* The image of R^m is contained in some $R^n \subseteq R^\infty$. Writing $R^\infty \cong R^n \oplus F$ and $Q = P \cap R^n$, show that $P \cong Q \oplus F$ and $F \cong F \oplus R^m$. This trick is a version of the Eilenberg swindle I.2.8 2.8 below.

EI.1.10 **1.10.** *Excision for GL_n.* If I is a ring without unit, let $\mathbb{Z} \oplus I$ be the canonical augmented ring with underlying abelian group $\mathbb{Z} \oplus I$. Let $GL_n(I)$ denote the kernel of the map $GL_n(\mathbb{Z} \oplus I) \to GL_n(\mathbb{Z})$, and let $M_n(I)$ denote the matrices with entries in I. If $g \in GL_n(I)$ then clearly $g - 1_n \in M_n(I)$.

(i) Characterize the set of all $x \in M_n(I)$ such that $1_n + x \in GL_n(I)$.

(ii) If I is an ideal in a ring R, show that $GL_n(I)$ is the kernel of $GL_n(R) \to GL_n(R/I)$, and so is independent of the choice of R.

(iii) If I is a ring with unit, show that $\mathbb{Z} \oplus I \cong \mathbb{Z} \times I$, and hence that the nonunital and unital definitions of $GL(I)$ agree. and conclude that

(iv) If $x = (x_{ij})$ is any nilpotent matrix in $M_n(I)$, such as a strictly upper triangular matrix, show that $1_n + x \in GL_n(I)$.

EI.1.11 **1.11.** (Whitehead) If $g \in GL_n(R)$, verify the following identity in $GL_{2n}(R)$:

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conclude that if $S \to R$ is a ring surjection then there is a matrix $h \in GL_{2n}(S)$ mapping to the block diagonal matrix with entries g, g^{-1} displayed above.

EI.1.12 **1.12.** *Radical Ideals.* A 2-sided ideal I in R is called a *radical ideal* if $1 + x$ is a unit of R for every $x \in I$, *i.e.,* if $(\forall x \in I)(\exists y \in I)(x + y + xy = 0)$. Every ring has a largest radical ideal, called the *Jacobson radical* of R; it is the intersection of the maximal left ideals of R.

(i) Show that every nil ideal is a radical ideal. (A *nil ideal* is an ideal in which every element is nilpotent.)

(ii) A ring R is *local* if it has a unique maximal 2-sided ideal $\mathfrak{m}$, and every element of $R - \mathfrak{m}$ is a unit. If R is local, show that $R/\mathfrak{m}$ is a field or division ring.

(iii) If I is a radical ideal of R, show that $M_n(I)$ is a radical ideal of $M_n(R)$ for every n. *Hint:* Use elementary row operations to diagonalize any matrix which is congruent to 1_n modulo I.

(iv) If I is a radical ideal, show that $GL_n(R) \to GL_n(R/I)$ is surjective for each n. That is, there is a short exact sequence of groups:

$$1 \to GL_n(I) \to GL_n(R) \to GL_n(R/I) \to 1.$$

(v) If I is a radical ideal, show that $sr(R) = sr(R/I)$, where sr is the stable range of Exercise 1.5. Conclude that $sr(R) = 1$ for every local ring R.

EI.1.13 **1.13.** A *von Neumann regular ring* is a ring R such that for every $r \in R$ there is an $x \in R$ such that $r = rxr$. It is called *unit-regular* if for every $r \in R$ there is a *unit* $x \in R$ such that $r = rxr$. If R is von Neumann regular, show that:

(a) for every $r \in R$, $R = rR \oplus (1 - rx)R$. *Hint:* $(rx)^2 = rx$.

(b) R is unit-regular $\Longleftrightarrow$ R has stable range 1 (in the sense of Exercise 1.5);

(c) If R is unit-regular then R satisfies condition (III) of Exercise 1.2. (The converse does not hold; see Example 5.10 of [71].)

A *rank function* on R is a set map $\rho\colon R \to [0,1]$ such that: (i) $\rho(0) = 0$ and $\rho(1) = 1$; (ii) $\rho(x) > 0$ if $x \neq 0$; (iii) $\rho(xy) \leq \rho(x), \rho(y)$; and (iv) $\rho(e + f) = \rho(e) + \rho(f)$ if e, f are orthogonal idempotents in A. Goodearl and Handelman proved (18.4 of [71]) that if R is a simple von Neumann ring then:

$$\text{(III) holds} \Longleftrightarrow R \text{ has a rank function.}$$

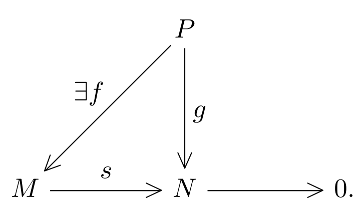
(d) Let F be a field or division ring. Show that the matrix ring $M_n(F)$ is unit-regular, and that $\rho_n(g) = \text{rank}(g)/n$ is a rank function on $M_n(F)$. Then show that the ring $\text{End}_F(F^\infty)$ is von Neumann regular but not unit-regular.

(e) Consider the union R of the matrix rings $M_{n!}(F)$, where we embed $M_{n!}(F)$ in $M_{(n+1)!}(F) \cong M_{n!}(F) \otimes M_{n+1}(F)$ as $M_{n!} \otimes 1$. Show that R is a simple von Neumann regular ring, and that the union of the ρ_n of (c) gives a rank function $\rho\colon R \to [0,1]$ with image $\mathbb{Q} \cap [0,1]$.

(f) Show that a commutative ring R is von Neumann regular if and only if it is reduced and has Krull dimension 0. These rings are called *absolutely flat rings* by Bourbaki, since every R-module is flat. Use Exercise 1.12 to conclude that every commutative 0-dimensional ring has stable range 1 (and is unit-regular).

2 Projective modules

I.2.1 **Definition 2.1.** An R-module P is called *projective* if there exists a module Q so that the direct sum $P \oplus Q$ is free. This is equivalent to saying that P satisfies the *projective lifting property:* For every surjection $s\colon M \to N$ of R-modules and every map $g\colon P \to N$ there exists a map $f\colon P \to M$ so that $g = sf$.



To see that these are equivalent, first observe that free modules satisfy this lifting property; in this case f is determined by lifting the image of a basis. To see that all projective modules satisfy the lifting property, extend g to a map from a free module $P \oplus Q$ to N and lift that. Conversely, suppose that P satisfies the projective lifting property. Choose a surjection $\pi\colon F \to P$ with F a free module; the lifting property splits π, yielding $F \cong P \oplus \ker(\pi)$.

If P is a projective module, then P is generated by n elements if and only if there is a decomposition $P \oplus Q \cong R^n$. Indeed, the generators give a surjection $\pi\colon R^n \to P$, and the lifting property yields the decomposition.

We will focus most of our attention on the category $\mathbf{P}(R)$ of finitely generated projective R-modules; the morphisms are the R-module maps. Since the direct sum of projectives is projective, $\mathbf{P}(R)$ is an additive category. We may regard $\mathbf{P}$ as a covariant functor on rings, since if $R \to S$ is a ring map then up to coherence there is an additive functor $\mathbf{P}(R) \to \mathbf{P}(S)$ sending P to $P \otimes_R S$. (Formally, there is an additive functor $\mathbf{P}'(R) \to \mathbf{P}(S)$ and an equivalence $\mathbf{P}'(R) \to \mathbf{P}(R)$; see Ex. 2.16.)

Hom and $\otimes$. If P is a projective R-module, then it is well-known that $P \otimes_R -$ is an exact functor on the category of (left) R-modules, and that $\mathrm{Hom}_R(P, -)$ is an exact functor on the category of (right) R-modules. (See [223], for example.) That is, any exact sequence $0 \to L \to M \to N \to 0$ of R-modules yields exact sequences

$$0 \to P \otimes L \to P \otimes M \to P \otimes N \to 0$$

and

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0.$$

I.2.1.1 **Example 2.1.1.** Of course free modules and stably free modules are projective.

(1) If F is a field (or a division ring) then every F-module (vector space) is free, but this is not so for all rings.

(2) Consider the matrix ring $R = M_n(F)$, $n > 1$. The R-module V of Example 1.1.1 is projective but not free, because $\mathrm{length}(V) = 1 < n = \mathrm{length}(R)$.

(3) *Componentwise free modules.* Another type of projective module arises for rings of the form $R = R_1 \times R_2$; both $P = R_1 \times 0$ and $Q = 0 \times R_2$ are projective but cannot be free because the element $e = (0, 1) \in R$ satisfies $Pe = 0$ yet $R^n e \neq 0$. We say that a module M is *componentwise free* if there is a decomposition $R = R_1 \times \cdots \times R_c$ and integers n_i such that $M \cong R_1^{n_1} \times \cdots \times R_c^{n_c}$. It is easy to see that all componentwise free modules are projective.

(4) *Topological Examples.* Other examples of nonfree projective modules come from topology, and will be discussed more in section 4 below. Consider the ring $R = C^0(X)$ of continuous functions $X \to \mathbb{R}$ on a compact topological space X. If $\eta\colon E \to X$ is a vector bundle then by Ex. 4.8 the set $\Gamma(E) = \{s\colon X \to E : \eta s = 1_X\}$ of continuous sections of η forms a projective R-module. For example, if T^n is the trivial bundle $\mathbb{R}^n \times X \to X$ then $\Gamma(T^n) = R^n$. We claim that if E is a nontrivial vector bundle then $\Gamma(E)$ cannot be a free R-module. To see this, observe that if $\Gamma(E)$ were free then the sections $\{s_1, ..., s_n\}$

in a basis would define a bundle map $f\colon T^n \to E$ such that $\Gamma(T^n) = \Gamma(E)$. Since the kernel and cokernel bundles of f have no nonzero sections they must vanish, and f is an isomorphism.

When X is compact, the category $\mathbf{P}(R)$ of finitely generated projective $C^0(X)$-modules is actually equivalent to the category of vector bundles over X; this result is called *Swan's Theorem*. (See Ex. EI.4.9 4.9 for a proof.)

I.2.1.2 **Example 2.1.2** (Idempotents)**.** An element e of a ring R is called *idempotent* if $e^2 = e$. If $e \in R$ is idempotent then $P = eR$ is projective because $R = eR \oplus (1-e)R$. Conversely, given any decomposition $R = P \oplus Q$, there are unique elements $e \in P$, $f \in Q$ such that $1 = e + f$ in R. By inspection, e and $f = 1 - e$ are idempotent, and $ef = fe = 0$. Thus idempotent elements of R are in 1-1 correspondence with decompositions $R \cong P \oplus Q$.

If $e \neq 0, 1$ and R is commutative then $P = eR$ cannot be free, because $P(1-e) = 0$ but $R(1-e) \neq 0$. The same is true for noetherian rings by Ex. EI.1.4 1.4, but obviously cannot be true for rings such that $R \cong R \oplus R$; see Ex. EI.1.2 1.2 (III).

Every finitely generated projective R-module arises from an idempotent element in a matrix ring $M_n(R)$. To see this, note that if $P \oplus Q = R^n$ then the projection-inclusion $R^n \to P \to R^n$ is an idempotent element e of $M_n(R)$. By inspection, the image $e(R^n)$ of e is P. The study of projective modules via idempotent elements can be useful, especially for rings of operators on a Banach space. (See Ro96 [162].)

If R is a Principal Ideal Domain (PID), such as $\mathbb{Z}$ or $F[x]$, F a field, then all projective R-modules are free. This follows from the Structure Theorem for modules over a PID (even for infinitely generated projectives).

Not many other classes of rings have all (finitely generated) projective modules free. A famous theorem of Quillen and Suslin states that if R is a polynomial ring (or a Laurent polynomial ring) over a field or a PID then all projective R-modules are free; a good reference for this is Lam's book Lam [106]. In particular, if G is a free abelian group then the group ring $\mathbb{Z}[G]$ is the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}, ..., z, z^{-1}]$, and has all projectives free. In contrast, if G is a nonabelian torsion-free nilpotent group, Artamanov proved in Art [4] that there are always projective $\mathbb{Z}[G]$-modules P which are stably free but not free: $P \oplus \mathbb{Z}[G] \cong (\mathbb{Z}[G])^2$.

It is an open problem to determine whether all projective $\mathbb{Z}[G]$-modules are stably free when G is a finitely presented torsion-free group. Some partial results and other examples are given in Lam [106].

For our purposes, local rings form the most important class of rings with all projectives free. A ring R is called a *local ring* if R has a unique maximal (2-sided) ideal $\mathfrak{m}$, and every element of $R - \mathfrak{m}$ is a unit; $R/\mathfrak{m}$ is either a field or a division ring by Ex. EI.1.12 1.12.

I.2.2 **Lemma 2.2.** *If R is a local ring, then every finitely generated projective R-module P is free. In fact $P \cong R^p$, where $p = \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P)$.*

Proof. We first observe that every element $u \in R$ invertible in $R/\mathfrak{m}$ is a unit of R, *i.e.*, $uv = vu = 1$ for some v. Indeed, by multiplying by a representative for the inverse of $\bar{u} \in R/\mathfrak{m}$ we may assume that $u \in 1 + \mathfrak{m}$. Since $\mathfrak{m}$ is the Jacobson radical of R, any element of $1 + \mathfrak{m}$ must be a unit of R.

Suppose that $P \oplus Q \cong R^n$. As vector spaces over $F = R/\mathfrak{m}$, $P/\mathfrak{m}P \cong F^p$ and $Q/\mathfrak{m}Q \cong F^q$ for some p and q. Since $F^p \oplus F^q \cong F^n$, $p + q = n$. Choose elements $\{e_1, ..., e_p\}$ of P and $\{e'_1, ..., e'_q\}$ of Q mapping to bases of $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$. The e_i and e'_j determine a homomorphism $R^p \oplus R^q \to P \oplus Q \cong R^n$, which may be represented by a square matrix $(r_{ij}) \in M_n(R)$ whose reduction $(\bar{r}_{ij}) \in M_n(F)$ is invertible. But every such matrix (r_{ij}) is invertible over R by Exercise 1.12. Therefore $\{e_1, ..., e_p, e'_1, ..., e'_q\}$ is a basis for $P \oplus Q$, and from this it follows that P is free on basis $\{e_1, ..., e_p\}$. $\square$

I.2.2.1 **Remark 2.2.1.** Even infinitely generated projective R-modules are free when R is local. See Kaplansky [98].

I.2.2.2 **Corollary 2.2.2.** *If $\mathfrak{p}$ is a prime ideal of a commutative ring R and P is a finitely generated projective R-module, then the localization $P_\mathfrak{p}$ is isomorphic to $(R_\mathfrak{p})^n$ for some $n \geq 0$. Moreover, there is an $s \in R - \mathfrak{p}$ such that the localization of P away from s is free:*

$$(P[\frac{1}{s}]) \cong (R[\frac{1}{s}])^n.$$

In particular, $P_{\mathfrak{p}'} \cong (R_{\mathfrak{p}'})^n$ for every other prime ideal $\mathfrak{p}'$ of R not containing s.

Proof. If $P \oplus Q = R^m$ then $P_\mathfrak{p} \oplus Q_\mathfrak{p} = (R_\mathfrak{p})^m$, so $P_\mathfrak{p}$ is a finitely generated projective $R_\mathfrak{p}$-module. Since $R_\mathfrak{p}$ is a local ring, $P_\mathfrak{p}$ is free by 2.2. Now every element of $P_\mathfrak{p}$ is of the form p/s for some $p \in P$ and $s \in R - \mathfrak{p}$. By clearing denominators, we may find an R-module homomorphism $f \colon R^n \to P$ which becomes an isomorphism upon localizing at $\mathfrak{p}$. As coker(f) is a finitely generated R-module which vanishes upon localization, it is annihilated by some $s \in R - \mathfrak{p}$. For this s, the map $f[\frac{1}{s}] \colon (R[\frac{1}{s}])^n \to P[\frac{1}{s}]$ is onto. Since $P[\frac{1}{s}]$ is projective, $(R[\frac{1}{s}])^n$ is isomorphic to the direct sum of $P[\frac{1}{s}]$ and a finitely generated $R[\frac{1}{s}]$-module M with $M_\mathfrak{p} = 0$. Since M is annihilated by some $t \in R - \mathfrak{p}$ we have

$$f[\frac{1}{st}]: \quad (R[\frac{1}{st}])^n \xrightarrow{\cong} P[\frac{1}{st}]. \quad \square$$

Suppose that there is a ring homomorphism $f \colon R \to F$ from R to a field or a division ring F. If M is any R-module (projective or not) then the *rank of M at f* is the integer $\dim_F(M \otimes_R F)$. However, the rank depends upon f, as the example $R = F \times F$, $M = F \times 0$ shows. When R is commutative, every such homomorphism factors through the field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R, so we may consider rank(M) as a function on the set Spec(R) of prime ideals in R.

Recall that the set Spec(R) of prime ideals of R has the natural structure of a topological space in which the basic open sets are

$$D(s) = \{\mathfrak{p} \in \operatorname{Spec}(R) : s \notin \mathfrak{p}\} \cong \operatorname{Spec}(R[\frac{1}{s}]) \quad \text{for } s \in R.$$

I.2.2.3 **Definition 2.2.3** (Rank)**.** Let R be a commutative ring. The *rank* of a finitely generated R-module M at a prime ideal $\mathfrak{p}$ of R is $\operatorname{rank}_{\mathfrak{p}}(M) = \dim_{k(\mathfrak{p})} M \otimes_R k(\mathfrak{p})$. Since $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \cong k(\mathfrak{p})^{\operatorname{rank}_{\mathfrak{p}}(M)}$, $\operatorname{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of $M_{\mathfrak{p}}$.

If P is a finitely generated projective R-module then $\operatorname{rank}(P)\colon \mathfrak{p} \mapsto \operatorname{rank}_{\mathfrak{p}}(P)$ is a *continuous* function from the topological space Spec(R) to the discrete topological space $\mathbb{N} \subset \mathbb{Z}$, as we see from Corollary 2.2.2. In this way, we shall view rank(P) as an element of the two sets $[\operatorname{Spec}(R), \mathbb{N}]$ and $[\operatorname{Spec}(R), \mathbb{Z}]$ of continuous maps from Spec(R) to $\mathbb{N}$ and to $\mathbb{Z}$, respectively.

We say that P has *constant rank* n if $n = \operatorname{rank}_{\mathfrak{p}}(P)$ is independent of $\mathfrak{p}$. If Spec(R) is topologically connected, every continuous function $\operatorname{Spec}(R) \to \mathbb{N}$ must be constant, so every finitely generated projective R-module has constant rank. For example, suppose that R is an integral domain with field of fractions F; then Spec(R) is connected, and every finitely generated projective R-module P has constant rank: $\operatorname{rank}(P) = \dim_F(P \otimes_R F)$. Conversely, if a projective P has constant rank, then it is finitely generated; see Ex. 2.13 and 2.14.

If a module M is not projective, rank(M) need not be a continuous function on Spec(R), as the example $R = \mathbb{Z}$, $M = \mathbb{Z}/p$ shows.

I.2.2.4 **Example 2.2.4** (Componentwise free modules)**.** Every continuous function $f : \operatorname{Spec}(R) \to \mathbb{N}$ induces a decomposition of Spec(R) into the disjoint union of closed subspaces $f^{-1}(n)$. In fact, f takes only finitely many values (say $n_1, \dots, n_c$), and it is possible to write R as $R_1 \times \cdots \times R_c$ such that $f^{-1}(n_i)$ is homeomorphic to Spec(R_i). (See Ex. 2.4.) Given such a function f, form the componentwise free R-module:

$$R^f = R_1^{n_1} \times \cdots \times R_c^{n_c}.$$

Clearly R^f has constant rank n_i at every prime in Spec(R_i) and $\operatorname{rank}(R^f) = f$. For $n \geq \max\{n_i\}$, $R^f \oplus R^{n-f} = R^n$, so R^f is a finitely generated projective R-module. Hence continuous functions $\operatorname{Spec}(R) \to \mathbb{N}$ are in 1-1 correspondence with componentwise free modules.

The following variation allows us to focus on projective modules of constant rank in many arguments. Suppose that P is a finitely generated projective R-module, so that rank(P) is a continuous function. Let $R \cong R_1 \times \cdots \times R_c$ be the corresponding decomposition of R. Then each component $P_i = P \otimes_R R_i$ of P is a projective R_i-module of constant rank and there is an R-module isomorphism $P \cong P_1 \times \cdots \times P_c$.

The next theorem allows us to further restrict our attention to projective modules of rank $\leq \dim(R)$. Its proof may be found in [15, IV]. We say that two R-modules M, M' are *stably isomorphic* if $M \oplus R^m \cong M' \oplus R^m$ for some $m \geq 0$.

I.2.3 **Theorem 2.3** (Bass-Serre Cancellation). *Let R be a commutative noetherian ring of Krull dimension d, and let P be a projective R-module of constant rank $n > d$.*
(a) (Serre) $P \cong P_0 \oplus R^{n-d}$ for some projective R-module P_0 of constant rank d.
(b) (Bass) If P is stably isomorphic to P' then $P \cong P'$.
(c) (Bass) For all M, M', if $P \oplus M$ is stably isomorphic to M' then $P \oplus M \cong M'$.

I.2.3.1 **Remark 2.3.1.** If P is a projective module whose rank is not constant, then $P \cong P_1 \times \cdots \times P_c$ for some decomposition $R \cong R_1 \times \cdots \times R_c$. (See Ex. 2.4.) In this case, we can apply the results in 2.3 to each P_i individually. The reader is invited to phrase 2.3 in this generality.

I.2.4 **Lemma 2.4** (Locally Free Modules). *Let R be commutative. An R-module M is called locally free if for every prime ideal $\mathfrak{p}$ of R there is an $s \in R - \mathfrak{p}$ so that $M[\frac{1}{s}]$ is a free module. We saw in Corollary 2.2.2 that finitely generated projective R-modules are locally free. In fact, the following are equivalent:*

(1) M is a finitely generated projective R-module;

(2) M is a locally free R-module of finite rank (i.e., $\text{rank}_{\mathfrak{p}}(M) < \infty$ *for all prime ideals $\mathfrak{p}$);*

(3) M is a finitely presented R-module, and for every prime ideal $\mathfrak{p}$ of R:

$$M_{\mathfrak{p}} \text{ is a free } R_{\mathfrak{p}}\text{-module.}$$

Proof. The implication (2) $\Rightarrow$ (3) follows from the theory of faithfully flat descent; a proof is in [32, II§5.2, Thm. 1]. Nowadays we would say that M is coherent (locally finitely presented), hence finitely presented; cf. [85, II.5.4], [EGA, 0_I(1.4.3)]. To see that (3) $\Rightarrow$ (1), note that finite presentation gives an exact sequence

$$R^m \to R^n \xrightarrow{\varepsilon} M \to 0.$$

We claim that the map $\varepsilon^*\colon \text{Hom}_R(M, R^n) \to \text{Hom}_R(M, M)$ is onto. To see this, recall that being onto is a local property; locally $\varepsilon^*_{\mathfrak{p}}$ is $\text{Hom}(M_{\mathfrak{p}}, R^n_{\mathfrak{p}}) \to \text{Hom}(M_{\mathfrak{p}}, M_{\mathfrak{p}})$. This is a split surjection because $M_{\mathfrak{p}}$ is projective and $\varepsilon_{\mathfrak{p}}\colon R^n_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is a split surjection. If $s\colon M \to R^n$ is such that $\varepsilon^*(s) = \varepsilon s$ is id_M, then s makes M a direct summand of R^n, and M is a finitely generated projective module. □

I.2.5 **Open Patching Data 2.5.** It is sometimes useful to be able to build projective modules by patching free modules. The following data suffices. Suppose that $s_1, ..., s_c \in R$ form a unimodular row, *i.e.,* $s_1 R + \cdots + s_c R = R$. Then $\text{Spec}(R)$ is covered by the open sets $D(s_i) \cong \text{Spec}(R[\frac{1}{s_i}])$. Suppose we are given

$g_{ij} \in GL_n(R[\frac{1}{s_i s_j}])$ with $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ in $GL_n(R[\frac{1}{s_i s_j s_k}])$ for every i, j, k. Then

$$P = \{(x_1, ..., x_c) \in \prod_{i=1}^{c} (R[\frac{1}{s_i}])^n : g_{ij}(x_j) = x_i \text{ in } R[\frac{1}{s_i s_j}]^n \text{ for all } i, j\}$$

is a finitely generated projective R-module by 2.4, because each $P[\frac{1}{s_i}]$ is isomorphic to $R[\frac{1}{s_i}]^n$.

I.2.6 **Example 2.6** (Milnor Squares)**.** Another type of patching arises from an ideal I in R and a ring map $f\colon R \to S$ such that I is mapped isomorphically onto an ideal of S, which we also call I. In this case R is the "pullback" of S and R/I :

$$R = \{(\bar{r}, s) \in (R/I) \times S : \bar{f}(\bar{r}) = s \text{ modulo } I\};$$

the square

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

is called a *Milnor square*, because their importance for patching was emphasized by J. Milnor in [131].

One special kind of Milnor square is the *conductor square*. This arises when R is commutative and S is a finite extension of R with the same total ring of fractions. (S is often the integral closure of R). The ideal I is chosen to be the *conductor ideal*, *i.e.,* the largest ideal of S contained in R, which is just $I = \{x \in R : xS \subset R\} = \text{ann}_R(S/R)$. If S is reduced then I cannot lie in any minimal prime of R or S, so the rings R/I and S/I have lower Krull dimension.

Given a Milnor square, we can construct an R-module $M = (M_1, g, M_2)$ from the following "descent data": an S-module M_1, an R/I-module M_2 and a S/I-module isomorphism $g\colon M_2 \otimes_{R/I} S/I \cong M_1/IM_1$. In fact M is the kernel of the R-module map

$$M_1 \times M_2 \to M_1/IM_1, \qquad (m_1, m_2) \mapsto \bar{m}_1 - g(\bar{f}(m_2)).$$

We call M the R-module *obtained by patching* M_1 and M_2 together along g.

An important special case is when we patch S^n and $(R/I)^n$ together along a matrix $g \in GL_n(S/I)$. For example, R is obtained by patching S and R/I together along $g = 1$. We will return to this point when we study $K_1(R)$ and $K_0(R)$.

Here is Milnor's result.

I.2.7 **Theorem 2.7** (Milnor Patching)**.** *In a Milnor square,*

1. *If P is obtained by patching together a finitely generated projective S-module P_1 and a finitely generated projective R/I-module P_2, then P is a finitely generated projective R-module;*

2. *$P \otimes_R S \cong P_1$ and $P/IP \cong P_2$;*

3. *Every finitely generated projective R-module arises in this way;*

4. *If P is obtained by patching free modules along $g \in GL_n(S/I)$, and Q is obtained by patching free modules along g^{-1}, then $P \oplus Q \cong R^{2n}$.*

We shall prove part (3) here; the rest of the proof will be described in Exercise 2.8. If M is any R-module, the Milnor square gives a natural map from M to the R-module M' obtained by patching $M_1 = M \otimes_R S$ and $M_2 = M \otimes_R (R/I) = M/IM$ along the canonical isomorphism

$$(M/IM) \otimes_{R/I} (S/I) \cong M \otimes_R (S/I) \cong (M \otimes_R S)/I(M \otimes_R S).$$

Tensoring M with $0 \to R \to (R/I) \oplus S \to S/I \to 0$ yields an exact sequence

$$\mathrm{Tor}_1^R(M, S/I) \to M \to M' \to 0,$$

so in general M' is just a quotient of M. However, if M is projective, the Tor-term is zero and $M \cong M'$. Thus every projective R-module may be obtained by patching, as (3) asserts.

I.2.7.1 **Remark 2.7.1.** Other examples of patching may be found in [107].

I.2.8 **Example 2.8** (Eilenberg Swindle)**.** The following "swindle," discovered by Eilenberg, explains why we restrict our attention to finitely generated projective modules. Let R^∞ be an infinitely generated free module. If $P \oplus Q = R^n$, then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Moreover $R^\infty \cong R^\infty \oplus R^\infty$, and if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$ (see Ex. 1.9). Here are a few more facts about infinitely generated projective modules:

- (Bass) If R is noetherian, every infinitely generated projective module P is free, unless there is an ideal I such that P/IP has fewer generators than P;
- (Kaplansky) Every infinitely generated projective module is the direct sum of countably generated projective modules;
- (Kaplansky) There are infinitely generated projectives P whose rank is finite but $\mathrm{rank}(P)$ is not continuous on $\mathrm{Spec}(R)$. (See Ex. 2.15.)

EXERCISES

EI.2.1 **2.1.** *Radical ideals.* Let I be a radical ideal in R (Exercise 1.12). If P_1, P_2 are finitely generated projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$, show that $P_1 \cong P_2$. *Hint:* Modify the proof of 2.2, observing that $\mathrm{Hom}(P,Q) \to \mathrm{Hom}(P/I, Q/I)$ is onto.

EI.2.2 **2.2.** *Idempotent lifting.* Let I be a nilpotent ideal, or more generally an ideal that is *complete* in the sense that every Cauchy sequence $\sum_{n=1}^{\infty} x_n$ with $x_n \in I^n$ converges to a unique element of I. Show that there is a bijection between the isomorphism classes of finitely generated projective R-modules and the isomorphism classes of finitely generated projective R/I-modules. To do this, use Ex. EI.2.1 2.1 and proceed in two stages:

(i) Show that every idempotent $\bar{e} \in R/I$ is the image of an idempotent $e \in R$, and that any other idempotent lift is ueu^{-1} for some $u \in 1+I$. *Hint:* it suffices to suppose that $I^2 = 0$ (consider the tower of rings R/I^n). If r is a lift of $\bar{e}$, consider elements of the form $e = r + rxr + (1-r)y(1-r)$ and $(1+xe)e(1+xe)^{-1}$.

(ii) By applying (i) to $M_n(R)$, show that every finitely generated projective R/I-module is of the form P/IP for some finitely generated projective R-module P.

EI.2.3 **2.3.** Let e, e_1 be idempotents in $M_n(R)$ defining projective modules P and P_1. If $e_1 = geg^{-1}$ for some $g \in GL_n(R)$, show that $P \cong P_1$. Conversely, if $P \cong P_1$ show that for some $g \in GL_{2n}(R)$:

$$\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = g \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.$$

EI.2.4 **2.4.** *Rank.* If R is a commutative ring and $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ is a continuous function, show that we can write $R = R_1 \times \cdots \times R_c$ in such a way that $\mathrm{Spec}(R)$ is the disjoint union of the $\mathrm{Spec}(R_i)$, and f is constant on each of the components $\mathrm{Spec}(R_i)$ of R. To do this, proceed as follows.

(i) Show that $\mathrm{Spec}(R)$ is quasi-compact and conclude that f takes on only finitely many values, say $n_1, ..., n_c$. Each $V_i = f^{-1}(n_i)$ is a closed and open subset of $\mathrm{Spec}(R)$ because $\mathbb{Z}$ is discrete.

(ii) It suffices to suppose that R is reduced, *i.e.,* has no non-zero nilpotent elements. To see this, let $\mathfrak{N}$ be the ideal of all nilpotent elements in R, so $R/\mathfrak{N}$ is reduced. Since $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/\mathfrak{N})$, we may apply idempotent lifting (Ex. EI.2.2 2.2).

(iii) Let I_i be the ideal defining V_i, i.e., $I_i = \cap\{\mathfrak{p} : \mathfrak{p} \in V_i\}$. If R is reduced, show that $I_1 + \cdots + I_c = R$ and that for every $i \neq j$ $I_i \cap I_j = \varnothing$. Conclude using the Chinese Remainder Theorem, which says that $R \cong \prod R_i$.

EI.2.5 **2.5.** Show that the following are equivalent for every commutative ring R:

(1) $\mathrm{Spec}(R)$ is topologically connected

(2) Every finitely generated projective R-module has constant rank

(3) R has no idempotent elements except 0 and 1.

EI.2.6 **2.6.** *Dual Module.* If P is a projective R-module, show that $\check{P} = \mathrm{Hom}_R(P, R)$ (its *dual module*) is a projective R^{op}-module, where R^{op} is R with multiplication reversed.

Now suppose that R is commutative, so that $R = R^{op}$. Show that $\mathrm{rank}(P) = \mathrm{rank}(\check{P})$ as functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. The image τ_P of $\check{P} \otimes P \to R$ is called the *trace* of P; show that $\tau_P^2 = \tau_P$, and that for $\mathfrak{p} \in \operatorname{Spec}(R)$, $P_{\mathfrak{p}} \neq 0$ if and only if $\tau_P \not\subseteq \mathfrak{p}$.

EI.2.7 **2.7.** *Tensor Product.* Let P and Q be projective modules over a commutative ring R. Show that the tensor product $P \otimes_R Q$ is also a projective R-module, and is finitely generated if P and Q are. Finally, show that

$$\operatorname{rank}(P \otimes_R Q) = \operatorname{rank}(P) \cdot \operatorname{rank}(Q).$$

EI.2.8 **2.8.** *Milnor Patching.* In this exercise we prove the Milnor Patching Theorem 2.7, that any R-module obtained by patching finitely generated projective modules over S and R/I in a Milnor square is a finitely generated projective R-module. Prove the following:

(i) If $g \in GL_n(S/I)$ is the image of a matrix in either $GL_n(S)$ or $GL_n(R/I)$, the patched module $P = (S^n, g, (R/I)^n)$ is a free R-module.

(ii) Show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2) \cong (P_1 \oplus Q_1, \left(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix}\right), P_2 \oplus Q_2)$.

(iii) If $g \in GL_n(S/I)$, let M be the module obtained by patching S^{2n} and $(R/I)^{2n}$ together along the matrix $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in GL_{2n}(S/I)$. Use Ex. 1.11 to prove that $M \cong R^{2n}$. This establishes Theorem 2.7, part (4).

(iv) Given $P_1 \oplus Q_1 \cong S^n$, $P_2 \oplus Q_2 \cong (R/I)^n$ and isomorphisms $P_1/IP_1 \cong P_2 \otimes S/I$, $Q_1/IQ_1 \cong Q_2 \otimes S/I$, let P and Q be the R-modules obtained by patching the P_i and Q_i together. By (ii), $P \oplus Q$ is obtained by patching S^n and $(R/I)^n$ together along some $g \in GL_n(S/I)$. Use (iii) to show that P and Q are finitely generated projective.

(v) If $P_1 \oplus Q_1 \cong S^m$ and $P_2 \oplus Q_2 \cong (R/I)^n$, and $g\colon P_1/IP_1 \cong P_2 \otimes S/I$, show that $(Q_1 \oplus S^n) \otimes S/I$ is isomorphic to $(R/I^m \oplus Q_2) \otimes S/I$. By (iv), this proves that (P_1, g, P_2) is finitely generated projective, establishing part (1) of Theorem 2.7.

(vi) Prove part (2) of Theorem 2.7 by analyzing the above steps.

EI.2.9 **2.9.** Consider a Milnor square as in Example 2.6). Let P_1, Q_1 be finitely generated projective S-modules, and P_2, Q_2 be finitely generated projective R/I-modules such that there are isomorphisms $g\colon P_2 \otimes S/I \cong P_1/IP_1$ and $h\colon Q_2 \otimes S/I \cong Q_1/IQ_1$.

(i) If $f\colon Q_2 \otimes S/I \cong P_1/IP_1$, show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2)$ is isomorphic to $(Q_1, gf^{-1}h, P_2) \oplus (P_1, f, Q_2)$. *Hint:* Use Ex. 2.8 and the decomposition

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} gf^{-1}h & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} h^{-1}f & 0 \\ 0 & f^{-1}h \end{pmatrix}.$$

(ii) Conclude that $(S^n, g, R/I^n) \oplus (S^n, h, R/I^n) \cong (S^n, gh, R/I^n) \oplus R^n$.

EI.2.10 **2.10.** Suppose P, Q are modules over a commutative ring R such that $P \otimes Q \cong R^n$ for some $n \neq 0$. Show that P and Q are finitely generated projective R-modules. *Hint:* Find a generating set $\{p_i \otimes q_i | i = 1, ..., m\}$ for $P \otimes Q$; the $p_i \otimes q_j \otimes p_k$ generate $P \otimes Q \otimes P$. Show that $\{p_i\}$ define a split surjection $R^m \to P$.

EI.2.11 **2.11.** Let M be a finitely generated module over a commutative ring R. Show that the following are equivalent for every n:

(1) M is a finitely generated projective module of constant rank n

(2) $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ for every prime ideal $\mathfrak{p}$ of R.

Conclude that in Lemma 2.4 we may add:

(4) M is finitely generated, $M_\mathfrak{p}$ is free for every prime ideal $\mathfrak{p}$ of R, and $\text{rank}(M)$ is a continuous function on $\text{Spec}(R)$.

EI.2.12 **2.12.** If $f\colon R \to S$ is a homomorphism of commutative rings, there is a continuous map $f^*\colon \text{Spec}(S) \to \text{Spec}(R)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$. If P is a finitely generated projective R-module, show that $\text{rank}(P \otimes_R S)$ is the composition of f^* and $\text{rank}(P)$. In particular, if P has constant rank n, then so does $P \otimes_R S$.

EI.2.13 **2.13.** If P is a projective module of constant rank 1, show that P is finitely generated. *Hint:* Show that the trace $\tau_P = R$, and write $1 = \sum f_i(x_i)$.

EI.2.14 **2.14.** If P is a projective module of constant rank r, show that P is finitely generated. *Hint:* Use Ex. 2.13 to show that $\wedge^r(P)$ is finitely generated.

EI.2.15 **2.15.** (Kaplansky) Here is an example of an infinitely generated projective module whose rank is not continuous. Let R be the ring of continuous functions $f\colon [0,1] \to \mathbb{R}$ on the unit interval and I the ideal of all functions f which vanish on some neighborhood $[0, \varepsilon)$ of 0. Show that I is a projective R-module, yet $\text{rank}(I)\colon \text{Spec}(R) \to \{0,1\}$ is not continuous, so I is neither finitely generated nor free. We remark that every finitely generated projective R-module is free; this follows from Swan's Theorem, since every vector bundle on $[0,1]$ is trivial (by 4.6.1 below).

Hint: Show that the functions $f_n = \max\{0, t - \frac{1}{n}\}$ generate I, and construct a splitting to the map $R^\infty \to I$. To see that $\text{rank}(I)$ is not continuous, consider the rank of I at the primes $\mathfrak{m}_t = \{f \in R : f(t) = 0\}$, $0 \leq t \leq 1$.

EI.2.16 **2.16.** *Kleisli rectification.* Fix a small category of rings $\mathcal{R}$. By a *big* projective R-module we will mean the choice of a finitely generated projective S-module P_S for each morphism $R \to S$ in $\mathcal{R}$, equipped with an isomorphism $P_S \otimes_S T \to P_T$ for every $S \to T$ over R such that: (i) to the identity of each S we associate the identity of P_S, and (ii) to each commutative triangle of algebras we have a commutative triangle of modules. Let $\mathbf{P}'(R)$ denote the category of big projective R-modules. Show that the forgetful functor $\mathbf{P}'(R) \to \mathbf{P}(R)$ is an equivalence, and that $R \mapsto \mathbf{P}'(R)$ is a contravariant functor from $\mathcal{R}$ to exact categories. In particular, $\mathbf{P}'(R) \to \mathbf{P}(S)$ is an additive functor for each $R \to S$.

3 The Picard Group of a commutative ring

An *algebraic line bundle* L over a commutative ring R is just a finitely generated projective R-module of constant rank 1. The name comes from the fact that if R is the ring of continuous functions on a compact space X, then a topological line bundle (vector bundle which is locally $\mathbb{R} \times X \to X$) corresponds to an algebraic line bundle by Swan's Theorem (see example [I.2.1.1] 2.1.1(4) or Ex. [EI.4.9] 4.9 below).

The tensor product $L \otimes_R M \cong M \otimes_R L$ of line bundles is again a line bundle (by Ex. [EI.2.7] 2.7), and $L \otimes_R R \cong L$ for all L. Thus up to isomorphism the tensor product is a commutative associative operation on line bundles, with identity element R.

I.3.1 **Lemma 3.1.** *If L is a line bundle, then the dual module $\check{L} = \mathrm{Hom}_R(L, R)$ is also a line bundle, and $\check{L} \otimes_R L \cong R$.*

Proof. Since $\mathrm{rank}(\check{L}) = \mathrm{rank}(L) = 1$ by Ex. [EI.2.6] 2.6, $\check{L}$ is a line bundle. Consider the evaluation map $\check{L} \otimes_R L \to R$ sending $f \otimes x$ to $f(x)$. If $L \cong R$, this map is clearly an isomorphism. Therefore for every prime ideal $\mathfrak{p}$ the localization

$$(\check{L} \otimes_R L)_{\mathfrak{p}} = (L_{\mathfrak{p}})\check{} \otimes_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \to R_{\mathfrak{p}}$$

is an isomorphism. Since being an isomorphism is a local property of an R-module homomorphism, the evaluation map must be an isomorphism. □

I.3.1.1 **Definition 3.1.1.** the *Picard group* $\mathrm{Pic}(R)$ of a commutative ring R is the set of isomorphism classes of line bundles over R. As we have seen, the tensor product $\otimes_R$ endows $\mathrm{Pic}(R)$ with the structure of an abelian group, the identity element being $[R]$ and the inverse being $L^{-1} = \check{L}$.

I.3.2 **Proposition 3.2.** Pic *is a functor from commutative rings to abelian groups. That is, if $R \to S$ is a ring homomorphism then* $\mathrm{Pic}(R) \to \mathrm{Pic}(S)$ *is a homomorphism sending L to $L \otimes_R S$.*

Proof. If L is a line bundle over R, then $L \otimes_R S$ is a line bundle over S (see Ex. [EI.2.12] 2.12), so $\otimes_R S$ maps $\mathrm{Pic}(R)$ to $\mathrm{Pic}(S)$. The natural isomorphism $(L \otimes_R M) \otimes_R S \cong (L \otimes_R S) \otimes_S (M \otimes_R S)$, valid for all R-modules L and M, shows that $\otimes_R S$ is a group homomorphism. □

I.3.3 **Lemma 3.3.** *If L is a line bundle, then* $\mathrm{End}_R(L) \cong R$.

Proof. Multiplication by elements in R yields a map from R to $\mathrm{End}_R(L)$. As it is locally an isomorphism, it must be an isomorphism. □

Determinant line bundle of a projective module

If M is any module over a commutative ring R and $k \geq 0$, the k^{th} *exterior power* $\wedge^k M$ is the quotient of the k-fold tensor product $M \otimes \cdots \otimes M$ by the submodule generated by terms $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$. By convention, $\wedge^0 M = R$ and $\wedge^1 M = M$. Here are some classical facts; see [B-AC] [32, ch. 2].

(i) $\wedge^k(R^n)$ is the free module of rank $\binom{n}{k}$ generated by terms $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$. In particular, $\wedge^n(R^n) \cong R$ on $e_1 \wedge \cdots \wedge e_n$.

(ii) If $R \to S$ is a ring map, there is a natural isomorphism $(\wedge^k M) \otimes_R S \cong \wedge^k(M \otimes_R S)$, the first $\wedge^k$ being taken over R and the second taken over S. In particular, $\operatorname{rank}(\wedge^k M) = \binom{\operatorname{rank} M}{k}$ as functions from $\operatorname{Spec}(R)$ to $\mathbb{N}$.

(iii) *(Sum Formula)* There is a natural isomorphism

$$\wedge^k(P \oplus Q) \cong \bigoplus_{i=0}^{k} (\wedge^i P) \otimes (\wedge^{k-i} Q).$$

If P is a projective module of constant rank n, then $\wedge^k P$ is a finitely generated projective module of constant rank $\binom{n}{k}$, because $\wedge^k P$ is locally free: if $P[\frac{1}{s}] \cong (R[\frac{1}{s}])^n$ then $(\wedge^k P)[\frac{1}{s}] \cong (R[\frac{1}{s}])^{\binom{n}{k}}$. In particular, $\wedge^n P$ is a line bundle, and $\wedge^k P = 0$ for $k > n$. We write $\det(P)$ for $\wedge^n P$, and call it the *determinant line bundle* of P.

If the rank of a projective module P is not constant, we define the determinant line bundle $\det(P)$ componentwise, using the following recipe. From §2 we find a decomposition $R \cong R_1 \times \cdots \times R_c$ so that $P \cong P_1 \times \cdots \times P_c$ and each P_i has constant rank n_i as an R_i-module. We then define $\det(P)$ to be $(\wedge^{n_1} P_1) \times \cdots \times (\wedge^{n_c} P_c)$; clearly $\det(P)$ is a line bundle on R. If P has constant rank n, this agrees with our above definition: $\det(P) = \wedge^n P$.

As the name suggests, the determinant line bundle is related to the usual determinant of a matrix. An $n \times n$ matrix g is just an endomorphism of R^n, so it induces an endomorphism $\wedge^n g$ of $\wedge^n R^n \cong R$. By inspection, $\wedge^n g$ is multiplication by $\det(g)$.

Using the determinant line bundle, we can also take the determinant of an endomorphism g of a finitely generated projective R-module P. By the naturality of $\wedge^n$, g induces an endomorphism $\det(g)$ of $\det(P)$. By Lemma 3.3, $\det(g)$ is an element of R, acting by multiplication; we call $\det(g)$ the *determinant* of the endomorphism g.

Here is an application of the determinant construction. Let L, L' be stably isomorphic line bundles. That is, $P = L \oplus R^n \cong L' \oplus R^n$ for some n. The Sum Formula (iii) shows that $\det(P) = L$, and $\det(P) = L'$, so $L \cong L'$. Taking $L' = R$, this shows that R is the only stably free line bundle. It also gives the following slight improvement upon the Cancellation Theorem 2.3 for 1-dimensional rings:

I.3.4 **Proposition 3.4.** *Let R be a commutative noetherian 1-dimensional ring. Then all finitely generated projective R-modules are completely classified by their rank and determinant. In particular, every finitely generated projective R-module P of rank ≥ 1 is isomorphic to $L \oplus R^f$, where $L = \det(P)$ and $f = \operatorname{rank}(P) - 1$.*

Invertible Ideals

When R is a commutative integral domain (=*domain*), we can give a particularly nice interpretation of $\mathrm{Pic}(R)$, using the following concepts. Let F be the field of fractions of R; a *fractional ideal* is a nonzero R-submodule I of F such that $I \subseteq fR$ for some $f \in F$. If I and J are fractional ideals then their product $IJ = \{\sum x_i y_i : x_i \in I, y_i \in J\}$ is also a fractional ideal, and the set $\mathrm{Frac}(R)$ of fractional ideals becomes an abelian monoid with identity element R. A fractional ideal I is called *invertible* if $IJ = R$ for some other fractional ideal J; invertible ideals are sometimes called *Cartier divisors*. The set of invertible ideals is therefore an abelian group, and one writes $\mathrm{Cart}(R)$ or $\mathrm{Pic}(R, F)$ for this group.

If $f \in F^\times$, the fractional ideal $\mathrm{div}(f) = fR$ is invertible because $(fR)(f^{-1}R) = R$; invertible ideals of this form are called *principal divisors*. Since $(fR)(gR) = (fg)R$, the function $\mathrm{div}\colon F^\times \to \mathrm{Cart}(R)$ is a group homomorphism.

This all fits into the following overall picture (see Ex. EI.3.7 3.7 for a generalization).

I.3.5 **Proposition 3.5.** *If R is a commutative integral domain, every invertible ideal is a line bundle, and every line bundle is isomorphic to an invertible ideal. If I and J are fractional ideals, and I is invertible, then $I \otimes_R J \cong IJ$. Finally, there is an exact sequence of abelian groups:*

$$1 \to R^\times \to F^\times \xrightarrow{div} \mathrm{Cart}(R) \to \mathrm{Pic}(R) \to 0.$$

Proof. If I and J are invertible ideals such that $IJ \subseteq R$, then we can interpret elements of J as homomorphisms $I \to R$. If $IJ = R$ then we can find $x_i \in I$ and $y_i \in J$ so that $x_1y_1 + \cdots + x_ny_n = 1$. The $\{x_i\}$ assemble to give a map $R^n \to I$ and the $\{y_i\}$ assemble to give a map $I \to R^n$. The composite $I \to R^n \to I$ is the identity, because it sends $r \in I$ to $\sum x_i y_i r = r$. Thus I is a summand of R^n, *i.e.,* I is a finitely generated projective module. As R is an integral domain and $I \subseteq F$, $\mathrm{rank}(I)$ is the constant $\dim_F(I \otimes_R F) = \dim_F(F) = 1$. Hence I is a line bundle.

This construction gives a set map $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$; to show that it is a group homomorphism, it suffices to show that $I \otimes_R J \cong IJ$ for invertible ideals. Suppose that I is a submodule of F which is also a line bundle over R. As I is projective, $I \otimes_R -$ is an exact functor. Thus if J is an R-submodule of F then $I \otimes_R J$ is a submodule of $I \otimes_R F$. The map $I \otimes_R F \to F$ given by multiplication in F is an isomorphism because I is locally free and F is a field. Therefore the composite

$$I \otimes_R J \subseteq I \otimes_R F \xrightarrow{\text{multiply}} F$$

sends $\sum x_i \otimes y_i$ to $\sum x_i y_i$. Hence $I \otimes_R J$ is isomorphic to its image $IJ \subseteq F$. This proves the third assertion.

The kernel of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$ is the set of invertible ideals I having an isomorphism $I \cong R$. If $f \in I$ corresponds to $1 \in R$ under such an isomorphism then $I = fR = \mathrm{div}(f)$. This proves exactness of the sequence at $\mathrm{Cart}(R)$.

Clearly the units $R^\times$ of R inject into $F^\times$. If $f \in F^\times$ then $fR = R$ if and only if $f \in R$ and f is in no proper ideal, *i.e.,* if and only if $f \in R^\times$. This proves exactness at $R^\times$ and $F^\times$.

Finally, we have to show that every line bundle L is isomorphic to an invertible ideal of R. Since $\operatorname{rank}(L) = 1$, there is an isomorphism $L \otimes_R F \cong F$. This gives an injection $L \cong L \otimes_R R \subset L \otimes_R F \cong F$, *i.e.*, an isomorphism of L with an R-submodule I of F. Since L is finitely generated, I is a fractional ideal. Choosing an isomorphism $\check{L} \cong J$, Lemma 3.1 yields

$$R \cong L \otimes_R \check{L} \cong I \otimes_R J \cong IJ.$$

Hence $IJ = fR$ for some $f \in F^\times$, and $I(f^{-1}J) = R$, so I is invertible. □

Dedekind domains

Historically, the most important applications of the Picard group have been for Dedekind domains. A *Dedekind domain* is a commutative integral domain which is noetherian, integrally closed and has Krull dimension 1.

There are many equivalent definitions of Dedekind domain in the literature. Here is another: an integral domain R is Dedekind if and only if every fractional ideal of R is invertible. In a Dedekind domain every nonzero ideal (and fractional ideal) can be written uniquely as a product of prime ideals $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$. Therefore $\operatorname{Cart}(R)$ is the free abelian group on the set of (nonzero) prime ideals of R, and $\operatorname{Pic}(R)$ is the set of isomorphism classes of (actual) ideals of R.

Another property of Dedekind domains is that every finitely generated torsionfree R-module M is projective. To prove this fact we use induction on $\operatorname{rank}_0(M) = \dim_F(M \otimes F)$, the case $\operatorname{rank}_0(M) = 0$ being trivial. Set $\operatorname{rank}_0(M) = n+1$. As M is torsionfree, it is a submodule of $M \otimes F \cong F^{n+1}$. The image of M under any nonzero coordinate projection $F^{n+1} \to F$ is a fractional ideal I_0. As I_0 is invertible, the projective lifting property for I_0 shows that $M \cong M' \oplus I_0$ with $\operatorname{rank}_0(M') = n$. By induction, $M \cong I_0 \oplus \cdots \oplus I_n$ is a sum of ideals. By Propositions 3.4 and 3.5, $M \cong I \oplus R^n$ for the invertible ideal $I = \det(M) = I_0 \cdots I_n$.

Examples. Here are some particularly interesting classes of Dedekind domains.

• A *principal ideal domain* (or PID) is a domain R in which every ideal is rR for some $r \in R$. Clearly, these are just the Dedekind domains with $\operatorname{Pic}(R) = 0$. Examples of PID's include $\mathbb{Z}$ and polynomial rings $k[x]$ over a field k.

• A *discrete valuation domain* (or DVR) is a local Dedekind domain. By Lemma 2.2, a DVR is a PID R with a unique maximal ideal $M = \pi R$. Fixing π, it isn't hard to see that every ideal of R is of the form $\pi^i R$ for some $i \geq 0$. Consequently every fractional ideal of R can be written as $\pi^i R$ for a unique $i \in \mathbb{Z}$. By Proposition 3.5, $F^\times \cong R^\times \times \{\pi^i\}$. There is a (discrete) valuation ν on the field of fractions $F : \nu(f)$ is that integer i such that $fR \cong \pi^i R$.

Examples of DVR's include the p-adic integers $\hat{\mathbb{Z}}_p$, the power series ring $k[[x]]$ over a field k, and localizations $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$.

• Let F be a number field, *i.e.,* a finite field extension of $\mathbb{Q}$. An *algebraic integer* of F is an element which is integral over $\mathbb{Z}$, *i.e.,* a root of a monic polynomial $x^n + a_1x^{n-1} + \cdots + a_n$ with integer coefficients ($a_i \in \mathbb{Z}$). The set $\mathcal{O}_F$ of all algebraic integers of F is a ring—it is the integral closure of $\mathbb{Z}$ in F. A famous result in ring theory asserts that $\mathcal{O}_F$ is a Dedekind domain with field of fractions F. It follows that $\mathcal{O}_F$ is a lattice in F, *i.e.,* a free abelian group of rank $\dim_{\mathbb{Q}}(F)$.

In Number Theory, $\mathrm{Pic}(\mathcal{O}_F)$ is called the *ideal class group* of the number field F. A fundamental theorem states that $\mathrm{Pic}(\mathcal{O}_F)$ is always a finite group, but the precise structure of the ideal class group is only known for special number fields of small dimension. For example, if $\xi_p = e^{2\pi i/p}$ then $\mathbb{Z}[\xi_p]$ is the ring of algebraic integers of $\mathbb{Q}(\xi_p)$, and the class group is zero if and only if $p \leq 19$; $\mathrm{Pic}(\mathbb{Z}[\xi_{23}])$ is $\mathbb{Z}/3$. More details may be found in books on number theory, such as BSh [31].

• If C is a smooth affine curve over a field k, then the coordinate ring R of C is a Dedekind domain. One way to construct a smooth affine curve is to start with a smooth projective curve $\bar{C}$. If $\{p_0, \dots, p_n\}$ is any nonempty set of points on $\bar{C}$, the Riemann-Roch theorem implies that $C = \bar{C} - \{p_0, \dots, p_n\}$ is a smooth affine curve.

If k is algebraically closed, $\mathrm{Pic}(R)$ is a divisible abelian group. Indeed, the points of the Jacobian variety $J(\bar{C})$ form a divisible abelian group, and $\mathrm{Pic}(R)$ is the quotient of $J(\bar{C})$ by the subgroup generated by the classes of the prime ideals of R_0 corresponding to $p_1, ..., p_n$, where $\bar{C} - \{p_0\} = \mathrm{Spec}(R_0)$.

This is best seen when $k = \mathbb{C}$, because smooth projective curves over $\mathbb{C}$ are the same as compact Riemann surfaces. If $\bar{C}$ is a compact Riemann surface of genus g, then as an abelian group the points of the Jacobian $J(\bar{C})$ form the divisible group $(\mathbb{R}/\mathbb{Z})^{2g}$. In particular, when $C = \bar{C} - \{p_0\}$ then $\mathrm{Pic}(R) \cong J(\bar{C}) \cong (\mathbb{R}/\mathbb{Z})^{2g}$.

For example, $R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x-\beta))$ is a Dedekind domain with $\mathrm{Pic}(R) \cong (\mathbb{R}/\mathbb{Z})^2$ if $\beta \neq 0, 1$. Indeed, R is the coordinate ring of a smooth affine curve C obtained by removing one point from an elliptic curve (= a projective curve of genus $g = 1$).

The Weil Divisor Class group

Let R be an integrally closed domain (= *normal domain*) with field of fractions F. If R is a noetherian normal domain, it is well-known that:

(i) $R_{\mathfrak{p}}$ is a discrete valuation ring (DVR) for all height 1 prime ideals $\mathfrak{p}$;

(ii) $R = \cap R_{\mathfrak{p}}$, the intersection being over all height 1 primes $\mathfrak{p}$ of R, each $R_{\mathfrak{p}}$ being a subring of F;

(iii) Every $r \neq 0$ in R is contained in only finitely many height 1 primes $\mathfrak{p}$.
An integral domain R satisfying (i), (ii) and (iii) is called a *Krull domain.*

Krull domains are integrally closed because every DVR $R_{\mathfrak{p}}$ is integrally closed. For a Krull domain R, the group $D(R)$ of *Weil divisors* is the free abelian group on the height 1 prime ideals of R. An *effective* Weil divisor is a divisor $D = \sum n_i[\mathfrak{p}_i]$ with all the $n_i \geq 0$.

We remark that effective divisors correspond to "divisorial" ideals of R, D corresponding to the intersection $\cap \mathfrak{p}_i^{(n_i)}$ of the symbolic powers of the $\mathfrak{p}_i$.

If $\mathfrak{p}$ is a height 1 prime of R, the $\mathfrak{p}$-adic valuation $\nu_{\mathfrak{p}}(I)$ of an invertible ideal I is defined to be that integer ν such that $I_{\mathfrak{p}} = \mathfrak{p}^{\nu} R_{\mathfrak{p}}$. By (iii), $v_{\mathfrak{p}}(I) \neq 0$ for only finitely many $\mathfrak{p}$, so $\nu(I) = \sum \nu_{\mathfrak{p}}(I)[\mathfrak{p}]$ is a well-defined element of $D(R)$. By 3.5, this gives a group homomorphism:

$$\nu\colon \operatorname{Cart}(R) \to D(R).$$

If I is invertible, $\nu(I)$ is effective if and only if $I \subseteq R$. To see this, observe that $\nu(I)$ is effective $\iff I_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}$ for all $\mathfrak{p} \iff I \subseteq \cap I_{\mathfrak{p}} \subseteq \cap R_{\mathfrak{p}} = R$. It follows that ν is an injection, because if both $\nu(I)$ and $\nu(I^{-1})$ are effective then I and I^{-1} are ideals with product R; this can only happen if $I = R$.

The *divisor class group* $Cl(R)$ of R is defined to be the quotient of $D(R)$ by the subgroup of all $\nu(fR)$, $f \in F^{\times}$. This yields a map $\operatorname{Pic}(R) \to Cl(R)$ which is evidently an injection. Summarizing, we have proven:

I.3.6 **Proposition 3.6.** *Let R be a Krull domain. Then* $\operatorname{Pic}(R)$ *is a subgroup of the class group $Cl(R)$, and there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
1 \to & R^{\times} & \longrightarrow & F^{\times} & \xrightarrow{div} & \operatorname{Cart}(R) & \longrightarrow & \operatorname{Pic}(R) & \to 0 \\
 & =\downarrow & & =\downarrow & & \nu \downarrow \mathit{into} & & \downarrow \mathit{into} & \\
1 \to & R^{\times} & \longrightarrow & F^{\times} & \xrightarrow{div} & D(R) & \longrightarrow & Cl(R) & \to 0.
\end{array}$$

I.3.6.1 **Remark 3.6.1.** The Picard group and the divisor class group of a Krull domain R are invariant under polynomial and Laurent polynomial extensions. That is, $\operatorname{Pic}(R) = \operatorname{Pic}(R[t]) = \operatorname{Pic}(R[t, t^{-1}])$ and $Cl(R) = Cl(R[t]) = Cl(R[t, t^{-1}])$. Most of this assertion is proven in [32, ch.7,§1]; the $\operatorname{Pic}[t, t^{-1}]$ part is proven in [20, 5.10].

Recall that an integral domain R is called *factorial*, or a *Unique Factorization Domain* (UFD) if every nonzero element $r \in R$ is either a unit or a product of prime elements. (This implies that the product is unique up to order and primes differing by a unit). It is not hard to see that UFD's are Krull domains; the following interpretation in terms of the class group is taken from [117, §20].

I.3.7 **Theorem 3.7.** *Let R be a Krull domain. Then R is a UFD $\iff Cl(R) = 0$.*

I.3.7.1 **Definition 3.7.1.** A noetherian ring R is called *regular* if every R-module M has a finite resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ with the P_i projective. Every localization $S^{-1}R$ of a regular ring R is also a regular ring, because $S^{-1}R$-modules are also R-modules, and a localization of an R-resolution is an $S^{-1}R$-resolution.

Now suppose that $(R, \mathfrak{m})$ is a regular local ring. It is well-known [117, §14, 19] that R is a noetherian, integrally closed domain (hence Krull), and that if $s \in \mathfrak{m} - \mathfrak{m}^2$ then sR is a prime ideal.

I.3.8 **Theorem 3.8.** *Every regular local ring is a UFD.*

Proof. We proceed by induction on $\dim(R)$. If $\dim(R) = 0$ then R is a field; if $\dim(R) = 1$ then R is a DVR, hence a UFD. Otherwise, choose $s \in \mathfrak{m} - \mathfrak{m}^2$. Since sR is prime, Ex. 3.8(b) yields $Cl(R) \cong Cl(R[\frac{1}{s}])$. Hence it suffices to show that $S = R[\frac{1}{s}]$ is a UFD. Let $\mathfrak{P}$ be a height 1 prime of S; we have to show that $\mathfrak{P}$ is a principal ideal. For every prime ideal $\mathfrak{Q}$ of S, $S_{\mathfrak{Q}}$ is a regular local ring of smaller dimension than R, so by induction $S_{\mathfrak{Q}}$ is a UFD. Hence $\mathfrak{P}_{\mathfrak{Q}}$ is principal: $xS_{\mathfrak{Q}}$ for some $x \in S$. By 2.4, $\mathfrak{P}$ is projective, hence invertible. Let $\mathfrak{p}$ be the prime ideal of R such that $\mathfrak{P} = \mathfrak{p}[\frac{1}{s}]$ and choose an R-resolution $0 \to P_n \to \cdots \to P_0 \to \mathfrak{p} \to 0$ of $\mathfrak{p}$ by finitely generated projective R-modules P_i. Since R is local, the P_i are free. Since $\mathfrak{P}$ is projective, the localized sequence $0 \to P_n[\frac{1}{s}] \to \cdots \to P_0[\frac{1}{s}] \to \mathfrak{P} \to 0$ splits. Letting E (resp. F) denote the direct sum of the odd (resp. even) $P_i[\frac{1}{s}]$, we have $\mathfrak{P} \oplus E \cong F$. Since stably free line bundles are free, $\mathfrak{P}$ is free. That is, $\mathfrak{P} = xS$ for some $x \in \mathfrak{P}$, as desired. □

I.3.8.1 **Corollary 3.8.1.** *If R is a regular domain, then* $\mathrm{Cart}(R) = D(R)$*, and hence*

$$\mathrm{Pic}(R) = Cl(R).$$

Proof. We have to show that every height 1 prime ideal $\mathfrak{P}$ of R is invertible. For every prime ideal $\mathfrak{p}$ of R we have $\mathfrak{P}_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ in the UFD $R_{\mathfrak{p}}$. By 2.4 and 3.5, $\mathfrak{P}$ is an invertible ideal. □

I.3.8.2 **Remark 3.8.2.** A ring is called *locally factorial* if $R_{\mathfrak{p}}$ is factorial for every prime ideal $\mathfrak{p}$ of R. For example, regular rings are locally factorial by 3.8. The proof of Cor. 3.8.1 shows that if R is a locally factorial Krull domain then $\mathrm{Pic}(R) = Cl(R)$.

Non-normal rings

The above discussion should make it clear that the Picard group of a normal domain is a classical object, even if it is hard to compute in practice. If R isn't normal, we can get a handle on $\mathrm{Pic}(R)$ using the techniques of the rest of this section.

For example, the next lemma allows us to restrict attention to reduced noetherian rings with finite normalization, because the quotient R_{red} of any commutative ring R by its *nilradical* (the ideal of nilpotent elements) is a reduced ring, and every commutative ring is the filtered union of its finitely generated subrings—rings having these properties.

If $\mathcal{A}$ is a small indexing category, every functor $R : \mathcal{A} \to \mathbf{Rings}$ has a colimit $\mathrm{colim}_{\alpha \in \mathcal{A}} R_\alpha$. We say that $\mathcal{A}$ is *filtered* if for every α, β there are maps $\alpha \to \gamma \leftarrow \beta$, and if for any two parallel arrows $\alpha \rightrightarrows \beta$ there is a $\beta \to \gamma$ so that the composites $\alpha \to \gamma$ agree; in this case we write $\varinjlim R_\alpha$ for $\mathrm{colim}\, R_\alpha$ and call it the *filtered direct limit*. (See [223, 2.6.13].)

I.3.9 **Lemma 3.9.** *(1)* $\mathrm{Pic}(R) = \mathrm{Pic}(R_{red})$.

(2) Pic *commutes with filtered direct limits of rings. In particular, if R is the filtered union of subrings R_α, then* $\mathrm{Pic}(R) \cong \varinjlim \mathrm{Pic}(R_\alpha)$.

Proof. Part (1) is an instance of idempotent lifting (Ex. 2.2). To prove (2), recall from 2.5 that a line bundle L over R may be given by patching data: a unimodular row $(s_1, ..., s_c)$ and units g_{ij} over the $R[\frac{1}{s_i s_j}]$. If R is the filtered direct limit of rings R_α, this finite amount of data defines a line bundle L_α over one of the R_α, and we have $L = L_\alpha \otimes_{R_\alpha} R$. If L_α and L'_α become isomorphic over R, the isomorphism is defined over some R_β, *i.e.,* L and L' become isomorphic over R_β. $\square$

If R is reduced noetherian, its normalization S is a finite product of normal domains S_i. We would like to describe $\text{Pic}(R)$ in terms of the more classical group $\text{Pic}(S) = \prod \text{Pic}(S_i)$, using the conductor square of Example 2.6. For this it is convenient to assume that S is finite over R, an assumption which is always true for rings of finite type over a field.

More generally, suppose that we are given a Milnor square as in Example 2.6:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

Given a unit β of S/I, the Milnor Patching Theorem 2.7 constructs a finitely generated projective R-module $L_\beta = (S, \beta, R/I)$ with $L_\beta \otimes_R S \cong S$ and $L_\beta / I L_\beta \cong R/I$. In fact L_β is a line bundle, because $\text{rank}(L_\beta) = 1$; every map from R to a field factors through either R/I or S (for every prime ideal $\mathfrak{p}$ of R either $I \subseteq \mathfrak{p}$ or $R_\mathfrak{p} \cong S_\mathfrak{p}$). By Ex. 2.9, $L_\alpha \oplus L_\beta \cong L_{\alpha\beta} \oplus R$; applying $\wedge^2$ yields $L_\alpha \otimes_R L_\beta \cong L_{\alpha\beta}$. Hence the formula $\partial(\beta) = [L_\beta]$ yields a group homomorphism

$$\partial \colon (S/I)^\times \to \text{Pic}(R).$$

I.3.10 **Theorem 3.10** (Units-Pic sequence)**.** *Given a Milnor square, the following sequence is exact. Here Δ denotes the diagonal map and $\pm$ denotes the difference map sending $(s, \bar{r})$ to $\bar{s} f(\bar{r})^{-1}$, resp. (L', L) to $L' \otimes_S S/I \otimes_{R/I} L^{-1}$.*

$$1 \to R^\times \xrightarrow{\Delta} S^\times \times (R/I)^\times \xrightarrow{\pm} (S/I)^\times \xrightarrow{\partial}$$
$$\text{Pic}(R) \xrightarrow{\Delta} \text{Pic}(S) \times \text{Pic}(R/I) \xrightarrow{\pm} \text{Pic}(S/I)$$

Proof. Since R is the pullback of S and R/I, exactness at the first two places is clear. Milnor Patching 2.7 immediately yields exactness at the last two places, leaving only the question of exactness at $(S/I)^\times$. Given $s \in S^\times$ and $\bar{r} \in (R/I)^\times$, set $\beta = \pm(s, \bar{r}) = \bar{s} f(\bar{r})^{-1}$, where $\bar{s}$ denotes the reduction of s modulo I. By inspection, $\lambda = (s, \bar{r}) \in L_\beta \subset S \times R/I$, and every element of L_β is a multiple of λ. It follows that $L_\beta \cong R$. Conversely, suppose given $\beta \in (S/I)^\times$ with $L_\beta \cong R$. If $\lambda = (s, \bar{r})$ is a generator of L_β we claim that s and $\bar{r}$ are units, which implies that $\beta = \bar{s} f(\bar{r})^{-1}$ and finishes the proof. If $s' \in S$ maps to $\beta \in S/I$ then $(s', 1) \in L_\beta$; since $(s', 1) = (xs, x\bar{r})$ for some $x \in R$ this implies that $\bar{r} \in (R/I)^\times$.

If $t \in S$ maps to $f(\bar{r})^{-1}\beta^{-1} \in S/I$ then $st \equiv 1$ modulo I. Now $I \subset sR$ because $I \times 0 \subset L_\beta$, so $st = 1 + sx$ for some $x \in R$. But then $s(t - x) = 1$, so $s \in S^\times$ as claimed. □

I.3.10.1 **Example 3.10.1** (Cusp)**.** Let k be a field and let R be $k[x,y]/(x^3 = y^2)$, the coordinate ring of the cusp in the plane. Setting $x = t^2$, $y = t^3$ makes R isomorphic to the subring $k[t^2, t^3]$ of $S = k[t]$. The conductor ideal from S to R is $I = t^2S$, so we get a conductor square with $R/I = k$ and $S/I = k[t]/(t^2)$. Now $\text{Pic}(k[t]) = 0$ and $(S/I)^\times \cong k^\times \times k$ with $\alpha \in k$ corresponding to $(1 + \alpha t) \in (S/I)^\times$. Hence $\text{Pic}(R) \cong k$. A little algebra shows that a nonzero $\alpha \in k$ corresponds to the invertible prime ideal $\mathfrak{p} = (1 - \alpha^2 x, x - \alpha y)R$ corresponding to the point $(x, y) = (\alpha^{-2}, \alpha^{-3})$ on the cusp.

I.3.10.2 **Example 3.10.2** (Node)**.** Let R be $k[x,y]/(y^2 = x^2 + x^3)$, the coordinate ring of the node in the plane over a field k with $\text{char}(k) \neq 2$. Setting $x = t^2 - 1$ and $y = tx$ makes R isomorphic to a subring of $S = k[t]$ with conductor ideal $I = xS$. We get a conductor square with $R/I = k$ and $S/I \cong k \times k$. Since $(S/I)^\times \cong k^\times \times k^\times$ we see that $\text{Pic}(R) \cong k^\times$. A little algebra shows that $\alpha \in k^\times$ corresponds to the invertible prime ideal $\mathfrak{p}$ corresponding to the point $(x, y) = \left(\frac{4\alpha}{(\alpha-1)^2}, \frac{4\alpha(\alpha+1)}{(\alpha-1)^3}\right)$ on the node corresponding to $t = \left(\frac{1+\alpha}{1-\alpha}\right)$.

Seminormal rings

A reduced commutative ring R is called *seminormal* if whenever $x, y \in R$ satisfy $x^3 = y^2$ there is an $s \in R$ with $s^2 = x$, $s^3 = y$. If R is an integral domain, there is an equivalent definition: R is seminormal if every s in the field of fractions satisfying $s^2, s^3 \in R$ belongs to R. Normal domains are clearly seminormal; the node (3.10.2) is not normal ($t^2 = 1 + x$), but it is seminormal (see Ex. 3.13). Arbitrary products of seminormal rings are also seminormal, because s may be found slotwise. The cusp (3.10.1) is the universal example of a reduced ring which is not seminormal.

Our interest in seminormal rings lies in the following theorem, first proven by C. Traverso for geometric rings and extended by several authors. For normal domains, it follows from Remark 3.6.1 above. Our formulation is taken from Swan's paper [197].

I.3.11 **Theorem 3.11** (Traverso)**.** *The following are equivalent for any commutative ring:*

(1) R_{red} *is seminormal;*

(2) $\text{Pic}(R) = \text{Pic}(R[t])$;

(3) $\text{Pic}(R) = \text{Pic}(R[t_1, ..., t_n])$ *for all n.*

I.3.11.1 **Remark 3.11.1.** If R is seminormal, it follows that $R[t]$ is also seminormal. By Ex. 3.11, $R[t, t^{-1}]$ and the local rings $R_\mathfrak{p}$ are also seminormal. However, the $\text{Pic}[t, t^{-1}]$ analogue of Theorem 3.11 fails. For example, if R is the node (3.10.2) then $\text{Pic}(R[t, t^{-1}]) \cong \text{Pic}(R) \times \mathbb{Z}$. For more details, see [221].

To prove Traverso's theorem, we shall need the following standard result about units of polynomial rings.

I.3.12 **Lemma 3.12.** *Let R be a commutative ring with nilradical $\mathfrak{N}$. If $r_0+r_1t+\cdots+r_nt^n$ is a unit of $R[t]$ then $r_0 \in R^\times$ and $r_1,\dots,r_n$ are nilpotent. Consequently, if $NU(R)$ denotes the subgroup $1+t\mathfrak{N}[t]$ of $R[t]^\times$ then:*

(1) $R[t]^\times = R^\times \times NU(R)$;

(2) If R is reduced then $R^\times = R[t]^\times$;

(3) Suppose that R is an algebra over a field k. If $char(k)=p$, $NU(R)$ is a p-group. If $char(k)=0$, $NU(R)$ is a uniquely divisible abelian group (= a $\mathbb{Q}$-module).

Proof of Traverso's theorem. We refer the reader to Swan's paper for the proof that (1) implies (2) and (3). By Lemma 3.9, we may suppose that R is reduced but not seminormal. Choose $x,y \in R$ with $x^3=y^2$ such that no $s\in R$ satisfies $s^2=x, s^3=y$. Then the reduced ring $S=R[s]/(s^2-x,s^3-y)_{\text{red}}$ is strictly larger than R. Since $I=xS$ is an ideal of both R and S, we have Milnor squares

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R[t] & \xrightarrow{f} & S[t] \\ \downarrow & & \downarrow \\ R/I[t] & \xrightarrow{\bar f} & S/I[t]. \end{array}$$

The Units-Pic sequence 3.10 of the first square is a direct summand of the Units-Pic sequence for the second square. Using Lemma 3.12, we obtain the exact quotient sequence

$$0 \to NU(R/I) \to NU(S/I) \xrightarrow{\partial} \frac{\operatorname{Pic}(R[t])}{\operatorname{Pic}(R)}.$$

By construction, $s\notin R$ and $\bar s \notin R/I$. Hence $\partial(1+\bar s t)$ is a nonzero element of the quotient $\operatorname{Pic}(R[t])/\operatorname{Pic}(R)$. Therefore if R isn't seminormal we have $\operatorname{Pic}(R)\neq\operatorname{Pic}(R[t])$, which is the (3) $\Rightarrow$ (2) $\Rightarrow$ (1) half of Traverso's theorem. □

EXERCISES

In these exercises, R is always a commutative ring.

EI.3.1 **3.1.** Show that the following are equivalent for every R-module L:

(a) There is a R-module M such that $L\otimes M \cong R$.

(b) L is an algebraic line bundle.

(c) L is a finitely generated R-module and $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.

Hint: Use Exercises 2.10 and 2.11.

EI.3.2 **3.2.** Show that the tensor product $P \otimes_R Q$ of two line bundles may be described using "Open Patching" 2.5 as follows. Find $s_1, ..., s_r \in R$ forming a unimodular row, such that P (resp. Q) is obtained by patching the $R[\frac{1}{s_i}]$ by units g_{ij} (resp. h_{ij}) in $R[\frac{1}{s_i s_j}]^\times$. Then $P \otimes_R Q$ is obtained by patching the $R[\frac{1}{s_i}]$ using the units $f_{ij} = g_{ij} h_{ij}$).

EI.3.3 **3.3.** Let P be a locally free R-module, obtained by patching free modules of rank n by $g_{ij} \in GL_n(R[\frac{1}{s_i s_j}])$. indexdeterminant line bundle—seeline bundle Show that $\det(P)$ is the line bundle obtained by patching free modules of rank 1 by the units $\det(g_{ij}) \in (R[\frac{1}{s_i s_j}])^\times$.

EI.3.4 **3.4.** Let P and Q be finitely generated projective modules of constant ranks m and n respectively. Show that there is a natural isomorphism $(\det P)^{\otimes n} \otimes (\det Q)^{\otimes m} \to \det(P \otimes Q)$. *Hint:* Send $(p_{11} \wedge \cdots \otimes \cdots \wedge p_{mn}) \otimes (q_{11} \wedge \cdots \otimes \cdots \wedge q_{mn})$ to $(p_{11} \otimes q_{11}) \wedge \cdots \wedge (p_{mn} \otimes q_{mn})$. Then show that this map is locally an isomorphism.

EI.3.5 **3.5.** If an ideal $I \subseteq R$ is a projective R-module and $J \subseteq R$ is any other ideal, show that $I \otimes_R J$ is isomorphic to the ideal IJ of R.

EI.3.6 **3.6.** *Excision for* Pic. If I is a commutative ring without unit, let $\mathrm{Pic}(I)$ denote the kernel of the canonical map $\mathrm{Pic}(\mathbb{Z} \oplus I) \to \mathrm{Pic}(\mathbb{Z})$. Write $I^\times$ for the group $GL_1(I)$ of Ex. 1.10. Show that if I is an ideal of R then there is an exact sequence:

$$1 \to I^\times \to R^\times \to (R/I)^\times \xrightarrow{\partial} \mathrm{Pic}(I) \to \mathrm{Pic}(R) \to \mathrm{Pic}(R/I).$$

EI.3.7 **3.7.** (Roberts-Singh) This exercise generalizes Proposition 3.5. Let $R \subseteq S$ be an inclusion of commutative rings. An R-submodule I of S is called an *invertible R-ideal of S* if $IJ = R$ for some other R-submodule J of S.

(i) If $I \subseteq S$ is an invertible R-ideal of S, show that I is finitely generated over R, and that $IS = S$.

(ii) Show that the invertible R-ideals of S form an abelian group $\mathrm{Pic}(R, S)$ under multiplication.

(iii) Show that every invertible R-ideal of S is a line bundle over R. *Hint:* use Ex. 3.5 to compute its rank. Conversely, if I is a line bundle over R contained in S and $IS = S$, then I is an R-ideal.

(iv) Show that there is a natural exact sequence:

$$1 \to R^\times \to S^\times \xrightarrow{\mathrm{div}} \mathrm{Pic}(R, S) \to \mathrm{Pic}(R) \to \mathrm{Pic}(S).$$

EI.3.8 **3.8.** *Relative Class groups.* Suppose that R is a Krull domain and that $R_S = S^{-1}R$ for some multiplicatively closed set S in R. Let $D(R, R_S)$ denote the free abelian group on the height 1 primes $\mathfrak{p}$ of R such that $\mathfrak{p} \cap S \neq \phi$. Since $D(R_S)$ is free on the remaining height 1 primes of R, $D(R) = D(R, R_S) \oplus D(R_S)$.

(a) Show that the group $\text{Pic}(R, R_S)$ of Ex. 3.7 is a subgroup of $D(R, R_S)$, and that there is an exact sequence compatible with Ex. 3.7

$$1 \to R^\times \to R_S^\times \to D(R, R_S) \to Cl(R) \to Cl(R_S) \to 0.$$

(b) Suppose that sR is a prime ideal of R. Prove that $(R[\frac{1}{s}])^\times \cong R^\times \times \mathbb{Z}$ and that $Cl(R) \cong Cl(R[\frac{1}{s}])$.

(c) Suppose that every height 1 prime $\mathfrak{p}$ of R with $\mathfrak{p} \cap S \neq \phi$ is an invertible ideal. Show that $\text{Pic}(R, R_S) = D(R, R_S)$ and that $\text{Pic}(R) \to \text{Pic}(R_S)$ is onto. (This always happens if R is a regular ring, or if the local rings R_M are unique factorization domains for every maximal ideal M of R with $M \cap S \neq \emptyset$.)

EI.3.9 **3.9.** Suppose that we are given a Milnor square with $R \subseteq S$. If $\bar{s} \in (S/I)^\times$ is the image of a nonzerodivisor $s \in S$, show that $-\partial(\bar{s}) \in \text{Pic}(R)$ is the class of the ideal $(sS) \cap R$.

EI.3.10 **3.10.** Let R be a 1-dimensional noetherian ring with finite normalization S, and let I be the conductor ideal from S to R. Show that for every maximal ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ is a line bundle $\iff I \not\subseteq \mathfrak{p}$. Using Ex. 3.9, show that these $\mathfrak{p}$ generate $\text{Pic}(R)$.

EI.3.11 **3.11.** If R is seminormal, show that every localization $S^{-1}R$ is seminormal.

EI.3.12 **3.12.** *Seminormality is a local property.* Show that the following are equivalent:
(a) R is seminormal;
(b) $R_\mathfrak{m}$ is seminormal for every maximal ideal $\mathfrak{m}$ of R;
(c) $R_\mathfrak{p}$ is seminormal for every prime ideal $\mathfrak{p}$ of R.

EI.3.13 **3.13.** If R is a pullback of a diagram of seminormal rings, show that R is seminormal. This shows that the node (3.10.2) is seminormal.

EI.3.14 **3.14.** *Normal rings.* A ring R is called *normal* if each local ring $R_\mathfrak{p}$ is an integrally closed domain. If R and R' are normal rings, so is the product $R \times R'$. Show that normal domains are normal rings, and that every reduced 0-dimensional ring is normal. Then show that every normal ring is seminormal.

EI.3.15 **3.15.** *Seminormalization.* Show that every reduced commutative ring R has an extension $R \subseteq {}^+R$ with ${}^+R$ seminormal, satisfying the following universal property: if S is seminormal, then every ring map $R \to S$ has a unique extension ${}^+R \to S$. The extension ${}^+R$ is unique up to isomorphism, and is called the *seminormalization* of R. *Hint:* First show that it suffices to construct the seminormalization of a noetherian ring R whose normalization S is finite. In that case, construct the seminormalization as a subring of S, using the observation that if $x^3 = y^2$ for $x, y \in R$, there is an $s \in S$ with $s^2 = x$, $s^3 = y$.

EI.3.16 **3.16.** An extension $R \subset R'$ is called *subintegral* if $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is a bijection, and the residue fields $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ are isomorphic. Show that the seminormalization $R \subset {}^+R$ of the previous exercise is a subintegral extension.

EI.3.17 **3.17.** Let R be a commutative ring with nilradical $\mathfrak{N}$.

(a) Show that the subgroup $1 + \mathfrak{N}[t, t^{-1}]$ of $R[t, t^{-1}]^\times$ is the product of the three groups $1 + \mathfrak{N}$, $N_tU(R) = 1 + t\mathfrak{N}[t]$, and $N_{t^{-1}}U(R) = 1 + t^{-1}\mathfrak{N}[t^{-1}]$.

(b) Show that there is a homomorphism $t\colon [\operatorname{Spec}(R), \mathbb{Z}] \to R[t, t^{-1}]^\times$ sending f to the unit t^f of $R[t, t^{-1}]$ which is t^n on the factor R_i of R where $f = n$. Here R_i is given by I.2.2.4 2.2.4 and Ex. EI.2.4 2.4.

(c) Show that there is a natural decomposition

$$R[t, t^{-1}]^\times \cong R^\times \times N_tU(R) \times N_{t^{-1}}U(R) \times [\operatorname{Spec}(R), \mathbb{Z}],$$

or equivalently, that there is a split exact sequence:

$$1 \to R^\times \to R[t]^\times \times R[t^{-1}]^\times \to R[t, t^{-1}]^\times \to [\operatorname{Spec}(R), \mathbb{Z}] \to 0.$$

EI.3.18 **3.18.** Show that the following sequence is exact:

$$1 \to \operatorname{Pic}(R) \to \operatorname{Pic}(R[t]) \times \operatorname{Pic}(R[t^{-1}]) \to \operatorname{Pic}(R[t, t^{-1}]).$$

Hint: If R is finitely generated, construct a diagram whose rows are Units-Pic sequences I.3.10 3.10, and whose first column is the naturally split sequence of Ex. EI.3.17 3.17.

EI.3.19 **3.19.** (NPic) Let $\operatorname{NPic}(R)$ denote the cokernel of the natural map $\operatorname{Pic}(R) \to \operatorname{Pic}(R[t])$. Show that $\operatorname{Pic}(R[t]) \cong \operatorname{Pic}(R) \times \operatorname{NPic}(R)$, and that $\operatorname{NPic}(R) = 0$ if and only if R_{red} is a seminormal ring. If R is an algebra over a field k, prove that:

(a) If $\operatorname{char}(k) = p > 0$ then $\operatorname{NPic}(R)$ is a p-group;

(b) If $\operatorname{char}(k) = 0$ then $\operatorname{NPic}(R)$ is a uniquely divisible abelian group.

To do this, first reduce to the case when R is finitely generated, and proceed by induction on $\dim(R)$ using conductor squares.

4 Topological Vector Bundles and Chern Classes

Because so much of the theory of projective modules is based on analogy with the theory of topological vector bundles, it is instructive to review the main aspects of the structure of vector bundles. Details and further information may be found in [135], [7] or [93]. We will work with vector spaces over $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Let X be a topological space. A *family of vector spaces* over X is a topological space E, together with a continuous map $\eta\colon E \to X$ and a finite dimensional vector space structure (over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) on each fiber $E_x = \eta^{-1}(x)$, $x \in X$. We require the vector space structure on E_x to be compatible with the topology on E. (This means scaling $F \times E \to E$ and the addition map $E \times_X E \to E$ are continuous.) By a *homomorphism* from one family $\eta\colon E \to X$ to another family $\varphi\colon F \to X$ we mean a continuous map $f\colon E \to F$ with $\eta = \varphi f$, such that each induced map $f_x\colon E_x \to F_x$ is a linear map of vector spaces. There is an evident category of families of vector spaces over X and their homomorphisms.

For example, if V is an n-dimensional vector space, the projection from $T^n = X \times V$ to X forms a "constant" family of vector spaces. We call such a family, and any family isomorphic to it, a *trivial vector bundle* over X.

If $Y \subseteq X$, we write $E|Y$ for the restriction $\eta^{-1}(Y)$ of E to Y; the restriction $\eta|Y\colon E|Y \to Y$ of η makes $E|Y$ into a family of vector spaces over Y. More generally, if $f\colon Y \to X$ is any continuous map then we can construct an induced family $f^*(\eta)\colon f^*E \to Y$ as follows. The space f^*E is the subspace of $Y \times E$ consisting of all pairs (y, e) such that $f(y) = \eta(e)$, and $f^*E \to Y$ is the restriction of the projection map. Since the fiber of f^*E at $y \in Y$ is $E_{f(y)}$, f^*E is a family of vector spaces over Y.

A *vector bundle* over X is a family of vector spaces $\eta\colon E \to X$ such that every point $x \in X$ has a neighborhood U such that $\eta|U\colon E|U \to U$ is trivial. A vector bundle is also called a *locally trivial* family of vector spaces.

The most historically important example of a vector bundle is the tangent bundle $TX \to X$ of a smooth manifold X. Another famous example is the *Möbius bundle* E over S^1; E is the open Möbius strip and $E_x \cong \mathbb{R}$ for each $x \in S^1$.

Suppose that $f\colon X \to Y$ is continuous. If $E \to Y$ is a vector bundle, then the induced family $f^*E \to X$ is a vector bundle on X. To see this, note that if E is trivial over a neighborhood U of $f(x)$ then f^*E is trivial over $f^{-1}(U)$.

The symbol $\mathbf{VB}(X)$ denotes the category of vector bundles and homomorphisms over X. If clarification is needed, we write $\mathbf{VB}_{\mathbb{R}}(X)$, $\mathbf{VB}_{\mathbb{C}}(X)$ or $\mathbf{VB}_{\mathbb{H}}(X)$. The induced bundle construction gives rise to an additive functor $f^* : \mathbf{VB}(X) \to \mathbf{VB}(Y)$.

The *Whitney sum* $E \oplus F$ of two vector bundles $\eta\colon E \to X$ and $\varphi\colon F \to X$ is the family of all the vector spaces $E_x \oplus F_x$, topologized as a subspace of $E \times F$. Since E and F are locally trivial, so is $E \oplus F$; hence $E \oplus F$ is a vector bundle. By inspection, the Whitney sum is the product in the category $\mathbf{VB}(X)$. Since there is a natural notion of the sum $f + g$ of two homomorphisms $f, g\colon E \to F$, this makes $\mathbf{VB}(X)$ into an additive category with Whitney sum the direct sum operation.

A *sub-bundle* of a vector bundle $\eta\colon E \to X$ is a subspace F of E which is a vector bundle under the induced structure. That is, each fiber F_x is a vector subspace of E_x and the family $F \to X$ is locally trivial. The *quotient bundle* E/F is the union of all the vector spaces E_x/F_x, given the quotient topology. Since F is locally a Whitney direct summand in E, we see that E/F is locally trivial, hence a vector bundle. This gives a "short exact sequence" of vector bundles in $\mathbf{VB}(X)$:

$$0 \to F \to E \to E/F \to 0.$$

A vector bundle $E \to X$ is said to be of *finite type* if there is a finite covering $U_1, \dots, U_n$ of X such that each each $E|U_i$ is a trivial bundle. Every bundle over a compact space X must be of finite type; the same is true if X is a finite-dimensional CW complex [93, §3.5], or more generally if there is an integer n such that every open cover of X has a refinement $\mathcal{V}$ such that no point of X is contained in more that n elements of $\mathcal{V}$. We will see in Exercise 4.15 that the canonical line bundle on infinite dimensional projective space $\mathbb{P}^\infty$ is an example of a vector bundle which is *not* of finite type.

Riemannian Metrics Let $E \to X$ be a real vector bundle. A *Riemannian metric* on E is a family of inner products $\beta_x\colon E_x \times E_x \to \mathbb{R}$, $x \in X$, which varies continuously with x (in the sense that β is a continuous function on the Whitney sum $E \oplus E$). The notion of *Hermitian metric* on a complex (or quaternionic) vector bundle is defined similarly. A fundamental result [93, 3.5.5 and 3.9.5] states that every vector bundle over a paracompact space X has a Riemannian (or Hermitian) metric; see Ex. 4.17 for the quaternionic case.

Dimension of vector bundles

If E is a vector bundle over X then $\dim(E_x)$ is a locally constant function on X with values in $\mathbb{N} = \{0, 1, \dots\}$. Hence $\dim(E)$ is a continuous function from X to the discrete topological space $\mathbb{N}$; it is the analogue of the rank of a projective module. An *n-dimensional vector bundle* is a bundle E such that $\dim(E) = n$ is constant; 1-dimensional vector bundles are frequently called *line bundles.* The Möbius bundle is an example of a nontrivial line bundle.

A vector bundle E is called *componentwise trivial* if we can write X as a disjoint union of (closed and open) components X_i in such a way that each $E|X_i$ is trivial. Isomorphism classes of componentwise trivial bundles are in 1-1 correspondence with the set $[X, \mathbb{N}]$ of all continuous maps from X to $\mathbb{N}$. To see this, note that any continuous map $f\colon X \to \mathbb{N}$ induces a decomposition of X into components $X_i = f^{-1}(i)$. Given such an f, let T^f denote the disjoint union

$$T^f = \coprod_{i \in \mathbb{N}} X_i \times F^i, \qquad F = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}.$$

The projection $T^f \to \amalg X_i = X$ makes T^f into a componentwise trivial vector bundle with $\dim(T^f) = f$. Conversely, if E is componentwise trivial, then

$E \cong T^{\dim(E)}$. Note that $T^f \oplus T^g \cong T^{f+g}$. Thus if f is bounded then by choosing $g = n - f$ we can make T^f into a summand of the trivial bundle T^n.

The following result, which we cite from [93, 3.5.8 and 3.9.6], illustrates some of the similarities between $\mathbf{VB}(X)$ and the category of finitely generated projective modules. It is proven using a Riemannian (or Hermitian) metric on E: $F_x^{\perp}$ is the subspace of E_x perpendicular to F_x. (A topological space is *paracompact* if it is Hausdorff and every open cover has a partition of unity subordinate to it.)

I.4.1 **Theorem 4.1** (Subbundle Theorem)**.** *Let $E \to X$ be a vector bundle on a paracompact topological space X. Then:*

1. *If F is a sub-bundle of E, there is a sub-bundle $F^{\perp}$ such that $E \cong F \oplus F^{\perp}$.*
2. *E has finite type if and only if E is isomorphic to a sub-bundle of a trivial bundle. That is, if and only if there is another bundle F such that $E \oplus F$ is trivial.*

I.4.1.1 **Corollary 4.1.1.** *Suppose that X is compact, or that X is a finite-dimensional CW complex. Then every vector bundle over X is a Whitney direct summand of a trivial bundle.*

I.4.1.2 **Example 4.1.2.** If X is a smooth d-dimensional manifold, its tangent bundle $TX \to X$ is a d-dimensional real vector bundle. Embedding X in $\mathbb{R}^n$ allows us to form the *normal bundle* $NX \to X$; N_xX is the orthogonal complement of T_xX in $\mathbb{R}^n$. Clearly $TX \oplus NX$ is the trivial n-dimensional vector bundle $X \times \mathbb{R}^n \to X$ over X.

I.4.1.3 **Example 4.1.3.** Consider the canonical line bundle E_1 on projective n-space; a point x of $\mathbb{P}^n$ corresponds to a line L_x in $n+1$-space, and the fiber of E_1 at x is just L_x. In fact, E_1 is a subbundle of the trivial bundle T^{n+1}. Letting F_x be the n-dimensional hyperplane perpendicular to L_x, the family of vector spaces F forms a vector bundle such that $E_1 \oplus F = T^{n+1}$.

I.4.1.4 **Example 4.1.4** (Global sections)**.** Let $\eta\colon E \to X$ be a vector bundle. A *global section* of η is a continuous map $s\colon X \to E$ such that $\eta s = 1_X$. It is *nowhere zero* if $s(x) \neq 0$ for all $x \in X$. Every global section s determines a map from the trivial line bundle T^1 to E; if s is nowhere zero then the image is a line subbundle of E. If X is paracompact the Subbundle Theorem determines a splitting $E \cong F \oplus T^1$.

Patching vector bundles

I.4.2 **4.2.** One technique for creating vector bundles uses transition functions. The idea is to patch together a collection of vector bundles which are defined on subspaces of X. A related technique is the clutching construction discussed in 4.7 below.

Let $\eta\colon E \to X$ be an n-dimensional vector bundle on X over the field F (F is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). Since E is locally trivial, we can find an open covering $\{U_i\}$ of X, and isomorphisms $h_i : U_i \times F^n \cong E|U_i$. If $U_i \cap U_j \neq \varnothing$, the isomorphism

$$h_i^{-1}h_j : (U_i \cap U_j) \times F^n \cong \eta|U_i \cap U_j \cong (U_i \cap U_j) \times F^n$$

sends $(x, v) \in (U_i \cap U_j) \times F^n$ to $(x, g_{ij}(x)(v))$ for some $g_{ij}(x) \in GL_n(F)$.

Conversely, suppose we are given maps $g_{ij}\colon U_i \cap U_j \to GL_n(F)$ such that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. On the disjoint union of the $U_i \times F^n$, form the equivalence relation $\sim$ which is generated by the relation that $(x, v) \in U_j \times F^n$ and $(x, g_{ij}(x)(v)) \in U_i \times F^n$ are equivalent for every $x \in U_i \cap U_j$. Let E denote the quotient space $(\coprod U_i \times F^n)/\sim$. It is not hard to see that there is an induced map $\eta\colon E \to X$ making E into a vector bundle over X.

We call E the vector bundle *obtained by patching* via the transition functions g_{ij}; this patching construction is the geometric motivation for open patching of projective modules in I.2.5 2.5.

I.4.2.1 **Construction 4.2.1** (Tensor product)**.** Let E and F be real or complex vector bundles over X. There is a vector bundle $E \otimes F$ over X whose fiber over $x \in X$ is the vector space tensor product $E_x \otimes F_x$, and $\dim(E \otimes F) = \dim(E)\dim(F)$.

To construct $E \otimes F$, we first suppose that E and F are trivial bundles, *i.e.,* $E = X \times V$ and $F = X \times W$ for vector spaces V, W. In this case we let $E \otimes F$ be the trivial bundle $X \times (V \otimes W)$. In the general case, we proceed as follows. Restricting to a component of X on which $\dim(E)$ and $\dim(F)$ are constant, we may assume that E and F have constant ranks m and n respectively. Choose a covering $\{U_i\}$ and transition maps g_{ij}, g'_{ij} defining E and F by patching. Identifying $M_m(F) \otimes M_n(F)$ with $M_{mn}(F)$ gives a map $GL_m(F) \times GL_n(F) \to GL_{mn}(F)$, and the elements $g_{ij} \otimes g'_{ij}$ give transition maps for $E \otimes F$ from $U_i \cap U_j$ to $GL_{mn}(F)$. The last assertion comes from the classical vector space formula $\dim(E_x \otimes F_x) = \dim(E_x)\dim(F_x)$.

I.4.2.2 **Construction 4.2.2.** (Determinant bundle). For every n-dimensional real or complex vector bundle E, there is an associated "determinant" line bundle $\det(E) = \wedge^n E$ whose fibers are the 1-dimensional vector spaces $\wedge^n(E_x)$. In fact, $\det(E)$ is a line bundle obtained by patching, the transition functions for $\det(E)$ being the determinants $\det(g_{ij})$ of the transition functions g_{ij} of E. More generally, if E is any vector bundle then this construction may be performed componentwise to form a line bundle $\det(E) = \wedge^{\dim(E)}E$. As in §3, if L is a line bundle and $E = L \oplus T^f$, then $\det(E) = L$, so E uniquely determines L. Taking E trivial, this shows that nontrivial line bundles cannot be stably trivial.

I.4.2.3 **Orthogonal, unitary & symplectic structure groups 4.2.3.** An n-dimensional vector bundle $E \to X$ is said to have *structure group* O_n, U_n or Sp_n if the transition functions g_{ij} map $U_i \cap U_j$ to the subgroup O_n of $GL_n(\mathbb{R})$ the subgroup U_n of $GL_n(\mathbb{C})$ or the subgroup Sp_n of $GL_n(\mathbb{H})$. If X is paracompact, this can always be arranged, because then E has a (Riemannian or Hermitian) metric. Indeed, it is easy to continuously modify the isomorphisms $h_i\colon U_i \times F^n \to E|U_i$

so that on each fiber the map $F^n \cong E_x$ is an isometry. But then the fiber isomorphisms $g_{ij}(x)$ are isometries, and so belong to O_n, U_n or Sp_n. Using the same continuous modification trick, any vector bundle isomorphism between vector bundles with a metric gives rise to a metric-preserving isomorphism. If X is paracompact, this implies that $\mathbf{VB}_n(X)$ is also the set of equivalence classes of vector bundles with structure group O_n, U_n or Sp_n.

The following pair of results forms the historical motivation for the Bass-Serre Cancellation Theorem 2.3. Their proofs may be found in [93, 8.1].

I.4.3 **Real Cancellation Theorem 4.3.** *Suppose X is a d-dimensional CW complex, and $\eta\colon E \to X$ is an n-dimensional real vector bundle with $n > d$. Then*

(i) $E \cong E_0 \oplus T^{n-d}$ for some d-dimensional vector bundle E_0

(ii) If F is another bundle and $E \oplus T^k \cong F \oplus T^k$, then $E \cong F$.

I.4.3.1 **Corollary 4.3.1.** *Over a 1-dimensional CW complex, every real vector bundle E of rank ≥ 1 is isomorphic to $L \oplus T^f$, where $L = \det(E)$ and $f = \dim(E) - 1$.*

I.4.4 **Complex Cancellation Theorem 4.4.** *Suppose X is a d-dimensional CW complex, and that $\eta\colon E \to X$ is a complex vector bundle with $\dim(E) \geq d/2$.*

(i) $E \cong E_0 \oplus T^k$ for some vector bundle E_0 of dimension $\leq d/2$

(ii) If F is another bundle and $E \oplus T^k \cong F \oplus T^k$, then $E \cong F$.

I.4.4.1 **Corollary 4.4.1.** *Let X be a CW complex of dimension ≤ 3. Every complex vector bundle E of rank ≥ 1 is isomorphic to $L \oplus T^f$, where $L = \det(E)$ and $f = \dim(E) - 1$.*

There is also a cancellation theorem for a quaternionic vector bundle E with $\dim(E) \geq d/4$, $d = \dim(X)$. If $d \leq 3$ it implies that all quaternionic vector bundles are trivial; the splitting $E \cong L \oplus T^f$ occurs when $d \leq 7$.

Vector bundles are somewhat more tractable than projective modules, as the following result shows. Its proof may be found in [93, 3.4.7].

I.4.5 **Homotopy Invariance Theorem 4.5.** *If $f, g\colon Y \to X$ are homotopic maps and Y is paracompact, then $f^*E \cong g^*E$ for every vector bundle E over X.*

I.4.6 **Corollary 4.6.** *If X and Y are homotopy equivalent paracompact spaces, there is a 1–1 correspondence between isomorphism classes of vector bundles on X and Y.*

I.4.6.1 **Application 4.6.1.** If Y is a contractible paracompact space then every vector bundle over Y is trivial.

I.4.7 **Construction 4.7** (Clutching)**.** Here is an analogue for vector bundles of Milnor Patching I.2.7 2.7 for projective modules. Suppose that X is a paracompact space, expressed as the union of two closed subspaces X_1 and X_2, with $X_1 \cap X_2 = A$. Given vector bundles $E_i \to X_i$ and an isomorphism $g\colon E_1|A \to E_2|A$, we form a vector bundle $E = E_1 \cup_g E_2$ over X as follows. As a topological space E is the quotient of the disjoint union $(E_1 \coprod E_2)$ by the equivalence relation identifying $e_1 \in E_1|A$ with $g(e_1) \in E_2|A$. Clearly the natural projection $\eta\colon E \to X$ makes E a family of vector spaces, and $E|X_i \cong E_i$. Moreover, E is locally trivial over X (see Atiyah [7, p. 21]; paracompactness is needed to extend g off of A). The isomorphism $g\colon E_1|A \cong E_2|A$ is called the *clutching map* of the construction. As with Milnor patching, every vector bundle over X arises by this clutching construction. A new feature, however, is homotopy invariance: if f, g are homotopic clutching isomorphisms $E_1|A \cong E_2|A$, then $E_1 \cup_f E_2$ and $E_1 \cup_g E_2$ are isomorphic vector bundles over X.

I.4.8 **Proposition 4.8.** *Let SX denote the suspension of a paracompact space X. A choice of basepoint for X yields a 1-1 correspondence between the set $\mathbf{VB}_n(SX)$ of isomorphism classes of n-dimensional (resp. real, complex or quaternionic) vector bundles over SX and the respective set of based homotopy classes of maps*

$$[X, O_n]_*, \qquad [X, U_n]_* \quad \text{or} \quad [X, Sp_n]_*$$

from X to the orthogonal group O_n, unitary group U_n or symplectic group Sp_n.

Sketch of Proof. SX is the union of two contractible cones C_1 and C_2 whose intersection is X. As every vector bundle on the cones C_i is trivial, every vector bundle on SX is obtained from an isomorphism of trivial bundles over X via the clutching construction. Such an isomorphism is given by a homotopy class of maps from X to GL_n, or equivalently to the appropriate deformation retract (O_n, U_n or Sp_n) of GL_n. The indeterminacy in the resulting map from $[X, GL_n]$ to classes of vector bundles is eliminated by insisting that the basepoint of X map to $1 \in GL_n$. □

Vector Bundles on Spheres

I.4.9 **4.9.** Proposition I.4.8 4.8 allows us to use homotopy theory to determine the vector bundles on the sphere S^d, because S^d is the suspension of S^{d-1}. Hence n-dimensional (real, complex or symplectic) bundles on S^d are in 1-1 correspondence with elements of $\pi_{d-1}(O_n)$, $\pi_{d-1}(U_n)$ and $\pi_{d-1}(Sp_n)$, respectively. For example, every real or complex line bundle over S^d is trivial if $d \geq 3$, because the appropriate homotopy groups of $O_1 \cong S^0$ and $U_1 \cong S^1$ vanish. This is not true for $Sp_1 \cong S^3$; for example there are infinitely many symplectic line bundles on S^4 because $\pi_3 Sp_1 = \mathbb{Z}$. The classical calculation of the homotopy groups of O_n, U_n and Sp_n (see Huse [93, 7.12]) yields the following facts:

I.4.9.1 **4.9.1.** On S^1, there are $|\pi_0(O_n)| = 2$ real vector bundles of dimension n for all $n \geq 1$. The nontrivial line bundle on S^1 is the Möbius bundle. The Whitney sum of the Möbius bundle with trivial bundles yields all the other nontrivial bundles. Since $|\pi_0(U_n)| = 1$ for all n, every complex vector bundle on S^1 is trivial.

I.4.9.2 **4.9.2.** On S^2, the situation is more complicated. Since $\pi_1(O_1) = 0$ there are no nontrivial real line bundles on S^2. There are infinitely many 2-dimensional real vector bundles on S^2 (indexed by the degree d of their clutching maps), because $\pi_1(O_2) = \mathbb{Z}$. However, there is only one nontrivial n-dimensional real vector bundle for each $n \geq 3$, because $\pi_1(O_n) = \mathbb{Z}/2$. A real 2-dimensional bundle E is stably trivial (and $E \oplus T \cong T^3$) if and only if the degree d is even. The tangent bundle of S^2 has degree $d = 2$.

There are infinitely many complex line bundles L_d on S^2, indexed by the degree d (in $\pi_1(U_1) = \mathbb{Z}$) of their clutching maps. The Complex Cancellation theorem (4.4) states that every other complex vector bundle on S^2 is isomorphic to a Whitney sum $L_d \oplus T^n$, and that all the $L_d \oplus T^n$ are distinct.

I.4.9.3 **4.9.3.** Every vector bundle on S^3 is trivial. This is a consequence of the classical result that $\pi_2(G) = 0$ for every compact Lie group G, such as $G = O_n, U_n, Sp_n$.

I.4.9.4 **4.9.4.** As noted above, every real or complex line bundle on S^4 is trivial. S^4 carries infinitely many distinct n-dimensional vector bundles for $n \geq 5$ over $\mathbb{R}$, for $n \geq 2$ over $\mathbb{C}$, and for $n \geq 1$ over $\mathbb{H}$ because $\pi_3(O_n) = \mathbb{Z}$ for $n \geq 5$, $\pi_3(U_n) = \mathbb{Z}$ for $n \geq 2$ and $\pi_3(Sp_n) = \mathbb{Z}$ for $n \geq 1$. In the intermediate range, we have $\pi_3(O_2) = 0$, $\pi_3(O_3) = \mathbb{Z}$ and $\pi_3(O_4) = \mathbb{Z} \oplus \mathbb{Z}$. Every 5-dimensional real bundle comes from a unique 3-dimensional bundle but every 4-dimensional real bundle on S^4 is stably isomorphic to infinitely many other distinct 4-dimensional vector bundles.

I.4.9.5 **4.9.5.** There are no 2-dimensional real vector bundles on S^d for $d \geq 3$, because the appropriate homotopy groups of $O_2 \cong S^1 \times \mathbb{Z}/2$ vanish. This vanishing phenomenon doesn't persist though; if $d \geq 5$ the 2-dimensional complex bundles, as well as the 3-dimensional real bundles on S^d, correspond to elements of $\pi_{d-1}(O_3) \cong \pi_{d-1}(U_2) \cong \pi_{d-1}(S^3)$. This is a finite group which is rarely trivial.

Classifying Vector Bundles

One feature present in the theory of vector bundles, yet absent in the theory of projective modules, is the classification of vector bundles using Grassmannians.

If V is any finite-dimensional vector space, the set $\mathrm{Grass}_n(V)$ of all n-dimensional linear subspaces of V is a smooth manifold, called the *Grassmann manifold* of n-planes in V. If $V \subset W$, then $\mathrm{Grass}_n(V)$ is naturally a submanifold of $\mathrm{Grass}_n(W)$. The *infinite Grassmannian* Grass_n is the union of the $\mathrm{Grass}_n(V)$ as V ranges over all finite-dimensional subspaces of a fixed infinite-dimensional vector space ($\mathbb{R}^\infty$, $\mathbb{C}^\infty$ or $\mathbb{H}^\infty$); thus Grass_n is an infinite-dimensional CW complex (see [135]). For example, if $n = 1$ then Grass_1 is either $\mathbb{R}\mathbb{P}^\infty$, $\mathbb{C}\mathbb{P}^\infty$ or $\mathbb{H}\mathbb{P}^\infty$, depending on whether the vector spaces are over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

There is a canonical n-dimensional vector bundle over each $\mathrm{Grass}_n(V)$, called $E_n(V)$, whose fibre over each $x \in \mathrm{Grass}_n(V)$ is the linear subspace of V

corresponding to x. To topologize this family of vector spaces, and see that it is a vector bundle, we define $E_n(V)$ to be the sub-bundle of the trivial bundle $\mathrm{Grass}_n(V) \times V \to \mathrm{Grass}_n(V)$ having the prescribed fibers. For $n = 1$ this is just the canonical line bundle on projective space described in Example 4.1.3.

The union (as V varies) of the $E_n(V)$ yields an n-dimensional vector bundle $E_n \to \mathrm{Grass}_n$, called the n-dimensional *classifying bundle* because of the following theorem (see [93, 3.7.2]).

I.4.10 **Classification Theorem 4.10.** *Let X be a paracompact space. Then the set $\mathbf{VB}_n(X)$ of isomorphism classes of n-dimensional vector bundles over X is in 1–1 correspondence with the set $[X, \mathrm{Grass}_n]$ of homotopy classes of maps $X \to \mathrm{Grass}_n$:*

$$\mathbf{VB}_n(X) \cong [X, \mathrm{Grass}_n].$$

In more detail, every n-dimensional vector bundle $\eta\colon E \to X$ is isomorphic to $f^*(E_n)$ for some map $f\colon X \to \mathrm{Grass}_n$, and E determines f up to homotopy.

I.4.10.1 **Remark 4.10.1.** (Classifying Spaces) The Classification Theorem 4.10 states that the contravariant functor $\mathbf{VB}_n$ is representable by the infinite Grassmannian Grass_n. Because X is paracompact we may assume (by 4.2.3) that all vector bundles have structure group O_n, U_n or Sp_n, respectively. For this reason, the infinite Grassmannian Grass_n is called the *classifying space* of O_n, U_n or Sp_n (depending on the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). It is the custom to write BO_n, BU_n and BSp_n for the Grassmannians Grass_n (or any spaces homotopy equivalent to it) over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.

In fact, there are homotopy equivalences $\Omega(BG) \simeq G$ for any Lie group G. If G is O_n, U_n or Sp_n, we can deduce this from 4.8 and 4.10: for any paracompact space X we have $[X, G]_* \cong \mathbf{VB}_n(SX) \cong [SX, BG] \cong [X, \Omega(BG)]_*$. Taking X to be G and $\Omega(BG)$ yields the homotopy equivalences.

It is well-known that there are canonical isomorphisms $[X, \mathbb{R}\mathbb{P}^\infty] \cong H^1(X; \mathbb{Z}/2)$ and $[X, \mathbb{C}\mathbb{P}^\infty] \cong H^2(X; \mathbb{Z})$ respectively. Therefore the case $n = 1$ may be reformulated as follows over $\mathbb{R}$ and $\mathbb{C}$.

I.4.11 **Classification Theorem 4.11** (for line bundles)**.** *For any paracompact space X, there are natural isomorphisms:*

$$w_1 : \mathbf{VB}_{1,\mathbb{R}}(X) = \{\textit{real line bundles on } X\} \cong H^1(X; \mathbb{Z}/2)$$

$$c_1 : \mathbf{VB}_{1,\mathbb{C}}(X) = \{\textit{complex line bundles on } X\} \cong H^2(X; \mathbb{Z}).$$

I.4.11.1 **Remark 4.11.1.** Since $H^1(X)$ and $H^2(X)$ are abelian groups, it follows that the set $\mathbf{VB}_1(X)$ of isomorphism classes of line bundles is an abelian group. We can understand this group structure in a more elementary way, as follows. The tensor product $E \otimes F$ of line bundles is again a line bundle by 4.2.1, and $\otimes$ is the product in the group $\mathbf{VB}_1(X)$. The inverse of E in this group is the dual bundle $\check{E}$ of Ex. 4.3, because $\check{E} \otimes E$ is a trivial line bundle (see Ex. 4.4).

I.4.11.2 **Example 4.11.2** (Riemann Surfaces)**.** Here is a complete classification of complex vector bundles on a Riemann surface X. Recall that a Riemann surface is a compact 2-dimensional oriented manifold; the orientation gives a canonical isomorphism $H^2(X;\mathbb{Z}) = \mathbb{Z}$. If $\mathcal{L}$ is a complex line bundle, the *degree* of $\mathcal{L}$ is that integer d such that $c_1(\mathcal{L}) = d$. By Theorem 4.11, there is a unique complex line bundle $\mathcal{O}(d)$ of each degree on X. By Corollary 4.4.1, every complex vector bundle of rank r on X is isomorphic to $\mathcal{O}(d) \oplus T^{r-1}$ for some d. Therefore complex vector bundles on a Riemann surface are completely classified by their rank and degree.

For example, the tangent bundle $\mathcal{T}_X$ of a Riemann surface X has the structure of a complex line bundle, because every Riemann surface has the structure of a 1-dimensional complex manifold. The Riemann-Roch Theorem states that $\mathcal{T}_X$ has degree $2-2g$, where g is the *genus* of X. (Riemann surfaces are completely classified by their genus $g \geq 0$, a Riemann surface of genus g being a surface with g "handles.")

In contrast, there are 2^{2g} distinct real line bundles on X, classified by $H^1(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$. The Real Cancellation Theorem 4.3 shows that every real vector bundle is the sum of a trivial bundle and a bundle of dimension ≤ 2, but there are infinitely many 2-dimensional bundles over X. For example, the complex line bundles $\mathcal{O}(d)$ all give distinct oriented 2-dimensional real vector bundles on X; they are distinguished by an invariant called the *Euler class* (see [135]).

Characteristic Classes

By Theorem 4.11, the determinant line bundle $\det(E)$ of a vector bundle E yields a cohomology class: if E is a real vector bundle, it is the first Stiefel–Whitney class $w_1(E)$ in $H^1(X;\mathbb{Z}/2)$; if E is a complex vector bundle, it is the first Chern class $c_1(E)$ in $H^2(X;\mathbb{Z})$. These classes fit into a more general theory of characteristic classes, which are constructed and described in the book [135]. Here is an axiomatic description of these classes.

I.4.12 **Axioms for Stiefel-Whitney classes 4.12.** The *Stiefel–Whitney classes* of a real vector bundle E over X are elements $w_i(E) \in H^i(X;\mathbb{Z}/2)$, which satisfy the following axioms. By convention $w_0(E) = 1$.

(SW1) (Dimension) If $i > \dim(E)$ then $w_i(E) = 0$.

(SW2) (Naturality) If $Y \xrightarrow{f} X$ is continuous then $f^*\colon H^i(X;\mathbb{Z}/2) \to H^i(Y;\mathbb{Z}/2)$ sends $w_i(E)$ to $w_i(f^*E)$. If E and E' are isomorphic bundles then $w_i(E) = w_i(E')$.

(SW3) (Whitney sum formula) If E and F are bundles, then in the graded cohomology ring $H^*(X;\mathbb{Z}/2)$ we have:

$$w_n(E \oplus F) = \sum w_i(E) w_{n-i}(F) = w_n(E) + w_{n-1}(E) w_1(F) + \cdots + w_n(F).$$

(SW4) (Normalization) For the canonical line bundle E_1 over $\mathbb{R}\mathbb{P}^\infty$, $w_1(E_1)$ is the unique nonzero element of $H^1(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Axioms (SW2) and (SW4), together with the Classification Theorem 4.10, show that w_1 classifies real line bundles in the sense that it gives the isomorphism $\mathbf{VB}_1(X) \cong H^1(X;\mathbb{Z}/2)$ of Theorem 4.11. The fact that $w_1(E) = w_1(\det E)$ is a consequence of the "Splitting Principle" for vector bundles, and is left to the exercises.

Since trivial bundles are induced from the map $X \to \{*\}$, it follows from (SW1) and (SW2) that $w_i(T^n) = 0$ for every trivial bundle T^n (and $i \neq 0$). The same is true for componentwise trivial bundles; see Ex. 4.2. From (SW3) it follows that $w_i(E \oplus T^n) = w_i(E)$ for every bundle E and every trivial bundle T^n.

The *total Stiefel–Whitney class* $w(E)$ of E is defined to be the formal sum

$$w(E) = 1 + w_1(E) + \cdots + w_i(E) + \cdots$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}/2) = \prod_i H^i(X;\mathbb{Z}/2)$, which consists of all formal infinite series $a_0 + a_1 + \cdots$ with $a_i \in H^i(X;\mathbb{Z}/2)$. With this formalism, the Whitney sum formula becomes a product formula: $w(E \oplus F) = w(E)w(F)$. Now the collection U of all formal sums $1 + a_1 + \cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ forms an abelian group under multiplication (the group of units of $\hat{H}^*(X;\mathbb{Z}/2)$ if X is connected). Therefore if $E \oplus F$ is trivial we can compute $w(F)$ via the formula $w(F) = w(E)^{-1}$.

For example, consider the canonical line bundle $E_1(\mathbb{R}^n)$ over $\mathbb{RP}^n$. By axiom (SW4) we have $w(E_1) = 1 + x$ in the ring $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{F}_2[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is an n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum formula (SW3), we compute that $w(F) = 1 + x + \cdots + x^n$. Thus $w_i(F) = x^i$ for $i \leq n$ and $w_i(F) = 0$ for $i > n$.

Stiefel–Whitney classes were named for E. Stiefel and H. Whitney, who discovered the w_i independently in 1935, and used them to study the tangent bundle of a smooth manifold.

4.13 **Axioms for Chern classes 4.13.** If E is a complex vector bundle over X, the *Chern classes* of E are certain elements $c_i(E) \in H^{2i}(X;\mathbb{Z})$, with $c_0(E) = 1$. They satisfy the following axioms. Note that the natural inclusion of $S^2 \cong \mathbb{CP}^1$ in $\mathbb{CP}^\infty$ induces a canonical isomorphism $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.

(C1) (Dimension) If $i > \dim(E)$ then $c_i(E) = 0$.

(C2) (Naturality) If $f\colon Y \to X$ is continuous then $f^*\colon H^{2i}(X;\mathbb{Z}) \to H^{2i}(Y;\mathbb{Z})$ sends $c_i(E)$ to $c_i(f^*E)$. If $E \cong E'$ then $c_i(E) = c_i(E')$.

(C3) (Whitney sum formula) If E and F are bundles then

$$c_n(E \oplus F) = \sum c_i(E)c_{n-i}(F) = c_n(E) + c_{n-1}(E)c_1(F) + \cdots + c_n(F).$$

(C4) (Normalization) For the canonical line bundle E_1 over $\mathbb{C}\mathbb{P}^\infty$, $c_1(E_1)$ is the canonical generator x of $H^2(\mathbb{C}\mathbb{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

Axioms (C2) and (C4) and the Classification Theorem 4.10 imply that the first Chern class c_1 classifies complex line bundles; it gives the isomorphism $\mathbf{VB}_1(X) \cong H^2(X;\mathbb{Z})$ of Theorem 4.11. The identity $c_1(E) = c_1(\det E)$ is left to the exercises.

The total *Chern class* $c(E)$ of E is defined to be the formal sum

$$c(E) = 1 + c_1(E) + \cdots + c_i(E) + \cdots .$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}) = \prod_i H^i(X;\mathbb{Z})$. With this formalism, the Whitney sum formula becomes $c(E \oplus F) = c(E)c(F)$. As with Stiefel–Whitney classes, axioms (C1) and (C2) imply that for a trivial bundle T^n we have $c_i(T^n) = 0$ $(i \neq 0)$, and axiom (C3) implies that for all E

$$c_i(E \oplus T^n) = c_i(E).$$

For example, consider the canonical line bundle $E_1(\mathbb{C}^n)$ over $\mathbb{CP}^n$. By axiom (C4), $c(E_1) = 1+x$ in the truncated polynomial ring $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is a canonical n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum Formula (C3), we compute that $c(F) = \sum(-1)^i x^i$. Thus $c_i(F) = (-1)^i x^i$ for all $i \leq n$. Chern classes are named for S.-S. Chern, who discovered them in 1946 while studying L. Pontrjagin's 1942 construction of cohomology classes $p_i(E) \in H^{4i}(X;\mathbb{Z})$ associated to a real vector bundle E. In fact, $p_i(E)$ is $(-1)^i c_{2i}(E \otimes \mathbb{C})$, where $E \otimes \mathbb{C}$ is the complexification of E (see Ex. 4.5). However, the Whitney sum formula for Pontrjagin classes only holds up to elements of order 2 in $H^{4n}(X;\mathbb{Z})$; see Ex. 4.13.

EXERCISES

EI.4.1 **4.1.** Let $\eta\colon E \to X$ and $\varphi\colon F \to X$ be two vector bundles, and form the induced bundle η^*F over E. Show that the Whitney sum $E \oplus F \to X$ is η^*F, considered as a bundle over X by the map $\eta^*F \to E \to X$.

EI.4.2 **4.2.** Show that all of the uncountably many vector bundles on the discrete space $X = \mathbb{N}$ are componentwise trivial. Let $T^{\mathbb{N}} \to \mathbb{N}$ be the bundle with $\dim(T^{\mathbb{N}}_n) = n$ for all n. Show that every componentwise trivial vector bundle $T^f \to Y$ over every space Y is isomorphic to $f^*T^{\mathbb{N}}$. Use this to show that the Stiefel–Whitney and Chern classes vanish for componentwise trivial vector bundles.

EI.4.3 **4.3.** If E and F are vector bundles over X, show that there are vector bundles $\mathrm{Hom}(E,F)$, $\check{E}$ and $\wedge^k E$ over X whose fibers are, respectively: $\mathrm{Hom}(E_x, F_x)$, the dual space $(\check{E}_x)$ and the exterior power $\wedge^k(E_x)$. Then show that there are natural isomorphisms $(E \oplus F)\check{} \cong \check{E} \oplus \check{F}$, $\check{E} \otimes F \cong \mathrm{Hom}(E,F)$, $\wedge^1 E \cong E$ and

$$\wedge^k(E \oplus F) \cong \wedge^k E \oplus (\wedge^{k-1}E \otimes F) \oplus \cdots \oplus (\wedge^i E \otimes \wedge^{k-i} F) \oplus \cdots \oplus \wedge^k F.$$

EI.4.4 **4.4.** Show that the global sections of the bundle $\mathrm{Hom}(E,F)$ of Ex. 4.3 are in 1-1 correspondence with vector bundle maps $E \to F$. (Cf. 4.1.4.) If E is a line bundle, show that the vector bundle $\check{E} \otimes E \cong \mathrm{Hom}(E,E)$ is trivial.

EI.4.5 **4.5.** *Complexification.* Let $E \to X$ be a real vector bundle. Show that there is a complex vector bundle $E_{\mathbb{C}} \to X$ with fibers $E_x \otimes_{\mathbb{R}} \mathbb{C}$ and that there is a natural isomorphism $(E \oplus F)_{\mathbb{C}} \cong (E_{\mathbb{C}}) \oplus (F_{\mathbb{C}})$. Then show that $E_{\mathbb{C}} \to X$, considered as a real vector bundle, is isomorphic to the Whitney sum $E \oplus E$.

EI.4.6 **4.6.** *Complex conjugate bundle.* If $F \to X$ is a complex vector bundle on a paracompact space, given by transition functions g_{ij}, let $\bar{F}$ denote the complex vector bundle obtained by using the complex conjugates $\bar{g}_{ij}$ for transition functions; $\bar{F}$ is called the *complex conjugate bundle* of F. Show that F and $\bar{F}$ are isomorphic as real vector bundles, and that the complexification $F_{\mathbb{C}} \to X$ of Ex. 4.5 is isomorphic to the Whitney sum $F \oplus \bar{F}$. If $F = E_{\mathbb{C}}$ for some real bundle E, show that $F \cong \bar{F}$. Finally, show that for every complex line bundle L on X we have $\bar{L} \cong \check{L}$.

EI.4.7 **4.7.** Use the formula $\bar{L} \cong \check{L}$ of Ex. 4.6 to show that $c_1(\bar{E}) = -c_1(E)$ in $H^2(X;\mathbb{Z})$ for every complex vector bundle E on a paracompact space.

EI.4.8 **4.8.** *Global sections.* If $\eta\colon E \to X$ is a vector bundle, let $\Gamma(E)$ denote the set of all global sections of E (see 4.1.4). Show that $\Gamma(E)$ is a module over the ring $C^0(X)$ of continuous functions on X (taking values in $\mathbb{R}$ or $\mathbb{C}$). If E is an n-dimensional trivial bundle, show that $\Gamma(E)$ is a free $C^0(X)$-module of rank n.

Conclude that if X is paracompact then $\Gamma(E)$ is a locally free $C^0(X)$-module in the sense of 2.4, and that $\Gamma(E)$ is a finitely generated projective module if X is compact or if E is of finite type. This is the easy half of Swan's theorem; the rest is given in the next exercise.

EI.4.9 **4.9.** *Swan's Theorem.* Let X be a compact Hausdorff space, and write R for $C^0(X)$. Show that the functor Γ of the previous exercise is a functor from $\mathbf{VB}(X)$ to the category $\mathbf{P}(R)$ of finitely generated projective modules, and that the homomorphisms

$$\Gamma\colon \operatorname{Hom}_{\mathbf{VB}(X)}(E,F) \to \operatorname{Hom}_{\mathbf{P}(R)}(\Gamma(E),\Gamma(F)) \qquad (*)$$

are isomorphisms. This proves Swan's Theorem, that Γ is an equivalence of categories $\mathbf{VB}(X) \approx \mathbf{P}(C^0(X))$. *Hint:* First show that $(*)$ holds when E and F are trivial bundles, and then use Corollary 4.1.1.

EI.4.10 **4.10.** *Projective and Flag bundles.* If $E \to X$ is a vector bundle, consider the subspace $E_0 = E - X$ of E, where X lies in E as the zero section. The units $\mathbb{R}^\times$ (or $\mathbb{C}^\times$) act fiberwise on E_0, and the quotient space $\mathbb{P}(E)$ obtained by dividing out by this action is called the *projective bundle* associated to E. If $p\colon \mathbb{P}(E) \to X$ is the projection, the fibers $p^{-1}(x)$ are projective spaces.

(a) Show that there is a line sub-bundle L of p^*E over $\mathbb{P}(E)$. Use the Sub-bundle Theorem to conclude that $p^*E \cong E' \oplus L$.

Now suppose that $E \to X$ is an n-dimensional vector bundle, and let $\mathbb{F}(E)$ be the *flag bundle* $f\colon \mathbb{F}(E) \to X$ obtained by iterating the construction

$$\cdots \to \mathbb{P}(E'') \to \mathbb{P}(E') \to \mathbb{P}(E) \to X.$$

(b) Show that $f^*E \to \mathbb{F}(E)$ is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles.

EI.4.11 **4.11.** If E is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles, show that $\det(E) \cong L_1 \otimes \cdots \otimes L_n$. Then use the Whitney Sum formula to show that $w_1(E) = w_1(\det(E))$, resp. $c_1(E) = c_1(\det(E))$. Prove that every $w_i(E)$, resp. $c_i(E)$ is the i^{th} elementary symmetric function of the n cohomology classes $\{w_1(L_i)\}$, resp. $\{c_1(L_i)\}$.

EI.4.12 **4.12.** *Splitting Principle.* Write $H^i(X)$ for $H^i(X;\mathbb{Z}/2)$ or $H^{2i}(X;\mathbb{Z})$, depending on whether our base field is $\mathbb{R}$ or $\mathbb{C}$, and let $p : \mathbb{F}(E) \to X$ be the flag bundle of a vector bundle E over X (see Ex. 4.10). Prove that $p^* \colon H^i(X) \to H^i(\mathbb{F}(E))$ is an injection. Then use Ex. 4.11 to show that the characteristic classes $w_i(E)$ or $c_i(E)$ in $H^i(X)$ may be calculated inside $H^i(\mathbb{F}(E))$. *Hint:* For a trivial bundle this follows easily from the Künneth formula for $H^*(X \times \mathbb{F})$.

EI.4.13 **4.13.** *Pontrjagin classes.* In this exercise we assume the results of Ex. 4.6 on the conjugate bundle $\bar{F}$ of a complex bundle F. Use the Splitting Principle to show that $c_i(\bar{F}) = (-1)^i c_i(F)$. Then prove the following:

(i) The Pontrjagin classes $p_n(F)$ of F (considered as a real bundle) are

$$p_n(F) = c_n(F)^2 + 2\sum_{i=1}^{n-1} (-1)^i c_{n-i}(F)c_{n+i}(F) + (-1)^n 2c_{2n}(F).$$

(ii) If $F = E \otimes \mathbb{C}$ for some real bundle E, the odd Chern classes $c_1(F)$, $c_3(F), \ldots$ all have order 2 in $H^*(X;\mathbb{Z})$.

(iii) The Whitney sum formula for Pontrjagin classes holds modulo 2:

$$p_n(E \oplus E') - \sum p_i(E)p_{n-i}(E') \text{ has order 2 in } H^{4n}(X;\mathbb{Z}).$$

EI.4.14 **4.14.** *Disk with double origin.* The classification theorems 4.10 and 4.11 can fail for locally compact spaces which aren't Hausdorff. To see this, let D denote the closed unit disk in $\mathbb{R}^2$. The *disk with double origin* is the non-Hausdorff space X obtained from the disjoint union of two copies of D by identifying together the common subsets $D - \{0\}$. For all $n \geq 1$, show that $[X, BU_n] = [X, BO_n] = 0$, yet: $\mathbf{VB}_{n,\mathbb{C}}(X) \cong \mathbb{Z} \cong H^2(X;\mathbb{Z})$; $\mathbf{VB}_{2,\mathbb{R}}(X) \cong \mathbb{Z}$; and $\mathbf{VB}_{n,\mathbb{R}}(X) \cong \mathbb{Z}/2$ for $n \geq 3$.

EI.4.15 **4.15.** Show that the canonical line bundles E_1 over $\mathbb{R}\mathbb{P}^\infty$ and $\mathbb{C}\mathbb{P}^\infty$ do not have finite type. *Hint:* Use characteristic classes and the Subbundle Theorem, or II.3.7.2.

EI.4.16 **4.16.** Consider the suspension SX of a paracompact space X. Show that every vector bundle E over SX has finite type. *Hint:* If $\dim(E) = n$, use 4.8 and Ex. 1.11 to construct a bundle E' such that $E \oplus E' \cong T^{2n}$.

EI.4.17 **4.17.** Let V be a complex vector space. A quaternionic *structure map* on V is a complex conjugate-linear automorphism j satisfying $j^2 = -1$. A (complex) Hermitian metric β on V is said to be *quaternionic* if $\beta(jv, jw) = \overline{\beta(v, w)}$.

(a) Show that structure maps on V are in 1-1 correspondence with underlying $\mathbb{H}$-vector space structures on V in which $j \in \mathbb{H}$ acts as j.

(b) Given a structure map and a complex Hermitian metric β on V, show that the Hermitian metric $\frac{1}{2}\big(\beta(v, w) + \overline{\beta(jv, jw)}\big)$ is quaternionic. Conclude that every quaternionic vector bundle over a paracompact space has a quaternionic Hermitian metric.

(c) If V is a vector space over $\mathbb{H}$, show that its dual $\check{V} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is also a vector space over $\mathbb{H}$. If E is a quaternionic vector bundle, show that there is a quaternionic vector bundle $\check{E}$ whose fibers are $\check{E}_x$. *Hint:* If V is a right $\mathbb{H}$-module, first construct $\check{V}$ as a left $\mathbb{H}$-module using Ex. EI.2.6 2.6 and then use $\mathbb{H} \cong \mathbb{H}^{op}$ to make it a right module.

EI.4.18 **4.18.** Let E be a quaternionic vector bundle, and uE its underlying real vector bundle. If F is any real bundle, show that $\mathbb{H}^m \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{H}^{mn}$ endows the real bundle $uE \otimes F$ with the natural structure of a quaternionic vector bundle, which we write as $E \otimes F$. Then show that $(E \otimes F_1) \otimes F_2 \cong E \otimes (F_1 \otimes F_2)$.

EI.4.19 **4.19.** If E and F are quaternionic vector bundles over X, show that there are real vector bundles $E \otimes_{\mathbb{H}} F$ and $\mathrm{Hom}_{\mathbb{H}}(E, F)$, whose fibers are, respectively: $E_x \otimes_{\mathbb{H}} F_x$ and $\mathrm{Hom}_{\mathbb{H}}(E_x, F_x)$. Then show that $\mathrm{Hom}_{\mathbb{H}}(E, F) \cong \check{E} \otimes_{\mathbb{H}} F$.

5 Algebraic Vector Bundles

Modern Algebraic Geometry studies sheaves of modules over schemes. This generalizes modules over commutative rings, and has many features in common with the topological vector bundles that we considered in the last section. In this section we discuss the main aspects of the structure of algebraic vector bundles.

We will assume the reader has some rudimentary knowledge of the language of schemes, in order to get to the main points quickly. Here is a glossary of the basic concepts; details for most things may be found in Hartshorne's book [85], but the ultimate source is [EGA].

A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$; it is a *locally ringed space* if each $\mathcal{O}_X(U)$ is a commutative ring, and if for every $x \in X$ the stalk ring $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$ is a local ring. By definition, an *affine scheme* is a locally ringed space isomorphic to $(\mathrm{Spec}(R), \tilde{R})$ for some commutative ring R (where $\tilde{R}$ is the canonical structure sheaf), and a *scheme* is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets U_i such that each $(U_i, \mathcal{O}_X|U_i)$ is an affine scheme.

An $\mathcal{O}_X$-*module* is a sheaf $\mathcal{F}$ on X such that (i) for each open $U \subseteq X$ the set $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and (ii) if $V \subset U$ then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures. A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a sheaf map such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The category $\mathcal{O}_X$-**mod** of all $\mathcal{O}_X$-modules is an abelian category.

A *global section* of an $\mathcal{O}_X$-module $\mathcal{F}$ is an element e_i of $\mathcal{F}(X)$. We say that $\mathcal{F}$ is *generated by global sections* if there is a set $\{e_i\}_{i \in I}$ of global sections of $\mathcal{F}$ whose restrictions $e_i|_U$ generate $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module for every open $U \subseteq X$. We can reinterpret these definitions as follows. Giving a global section e of $\mathcal{F}$ is equivalent to giving a morphism $\mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules, and to say that $\mathcal{F}$ is generated by the global sections $\{e_i\}$ is equivalent to saying that the corresponding morphism $\oplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$ is a surjection.

Free modules We say that $\mathcal{F}$ is a *free $\mathcal{O}_X$-module* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. A set $\{e_i\} \subset \mathcal{F}(X)$ is called a *basis* of $\mathcal{F}$ if the restrictions $e_i|_U$ form a basis of each $\mathcal{F}(U)$, *i.e.,* if the e_i provide an explicit isomorphism $\oplus \mathcal{O}_X \cong \mathcal{F}$.

The rank of a free $\mathcal{O}_X$-module $\mathcal{F}$ is not well-defined over all ringed spaces. For example, if X is a 1-point space then $\mathcal{O}_X$ is just a ring R and an $\mathcal{O}_X$-module is just an R-module, so our remarks in §1 about the invariant basis property (IBP) apply. There is no difficulty in defining the rank of a free $\mathcal{O}_X$-module when $(X, \mathcal{O}_X)$ is a scheme, or a locally ringed space, or even more generally when any of the rings $\mathcal{O}_X(U)$ satisfy the IBP. We shall avoid these difficulties by assuming henceforth that $(X, \mathcal{O}_X)$ is a locally ringed space.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_U$-module. The *rank* of a locally free module $\mathcal{F}$ is defined at each point x of X: $\mathrm{rank}_x(\mathcal{F})$ is the rank of the free $\mathcal{O}_U$-module $\mathcal{F}|_U$, where U is a neighborhood of x on which $\mathcal{F}|_U$ is free. Since the function

$x \mapsto \mathrm{rank}_x(\mathcal{F})$ is locally constant, $\mathrm{rank}(\mathcal{F})$ is a continuous function on X. In particular, if X is a connected space then every locally free module has constant rank.

I.5.1 **Definition 5.1** (Vector Bundles)**.** A *vector bundle* over a ringed space X is a locally free $\mathcal{O}_X$-module whose rank is finite at every point. We will write $\mathbf{VB}(X)$ or $\mathbf{VB}(X, \mathcal{O}_X)$ for the category of vector bundles on $(X, \mathcal{O}_X)$; the morphisms in $\mathbf{VB}(X)$ are just morphisms of $\mathcal{O}_X$-modules. Since the direct sum of locally free modules is locally free, $\mathbf{VB}(X)$ is an additive category.

A *line bundle* $\mathcal{L}$ is a locally free module of constant rank 1. A line bundle is also called an *invertible sheaf* because as we shall see in 5.3 there is another sheaf $\mathcal{L}'$ such that $\mathcal{L} \otimes \mathcal{L}' = \mathcal{O}_X$.

These notions are the analogues for ringed spaces of finitely generated projective modules and algebraic line bundles, as can be seen from the discussion in 2.4 and §3. However, the analogy breaks down if X is not locally ringed; in effect locally projective modules need not be locally free.

I.5.1.1 **Example 5.1.1** (Topological spaces)**.** Fix a topological space X. Then $X_{\mathrm{top}} = (X, \mathcal{O}_{\mathrm{top}})$ is a locally ringed space, where $\mathcal{O}_{\mathrm{top}}$ is the sheaf of ($\mathbb{R}$ or $\mathbb{C}$-valued) continuous functions on X: $\mathcal{O}_{\mathrm{top}}(U) = C^0(U)$ for all $U \subseteq X$. The following constructions give an equivalence between the category $\mathbf{VB}(X_{\mathrm{top}})$ of vector bundles over the ringed space X_{top} and the category $\mathbf{VB}(X)$ of (real or complex) topological vector bundles over X in the sense of §4. Thus our notation is consistent with the notation of §4.

If $\eta \colon E \to X$ is a topological vector bundle, then the sheaf $\mathcal{E}$ of continuous sections of E is defined by $\mathcal{E}(U) = \{s \colon U \to E : \eta s = 1_U\}$. By Ex. 4.8 we know that $\mathcal{E}$ is a locally free $\mathcal{O}_{\mathrm{top}}$-module. Conversely, given a locally free $\mathcal{O}_{\mathrm{top}}$-module $\mathcal{E}$, choose a cover $\{U_i\}$ and bases for the free $\mathcal{O}_{\mathrm{top}}$-modules $\mathcal{E}|U_i$; the base change isomorphisms over the $U_i \cap U_j$ are elements g_{ij} of $GL_n(C^0(U_i \cap U_j))$. Interpreting the g_{ij} as maps $U_i \cap U_j \to GL_n(\mathbb{C})$, they become transition functions for a topological vector bundle $E \to X$ in the sense of 4.2.

I.5.1.2 **Example 5.1.2** (Affine schemes)**.** Suppose $X = \mathrm{Spec}(R)$. Every R-module M yields an $\mathcal{O}_X$-module $\tilde{M}$, and $\tilde{R} = \mathcal{O}_X$. Hence every free $\mathcal{O}_X$-module arises as $\tilde{M}$ for a free R-module M. The $\mathcal{O}_X$-module $\mathcal{F} = \tilde{P}$ associated to a finitely generated projective R-module P is locally free by 2.4, and the two rank functions agree: $\mathrm{rank}(P) = \mathrm{rank}(\mathcal{F})$. Conversely, if $\mathcal{F}$ is locally free $\mathcal{O}_X$-module, it can be made trivial on a covering by open sets of the form $U_i = D(s_i)$, *i.e.,* there are free modules M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$. The isomorphisms between the restrictions of $\tilde{M}_i$ and $\tilde{M}_j$ to $U_i \cap U_j$ amount to open patching data defining a projective R-module P as in 2.5. In fact it is not hard to see that $\mathcal{F} \cong \tilde{P}$. Thus vector bundles on $\mathrm{Spec}(R)$ are in 1-1 correspondence with finitely generated projective R-modules. And it is no accident that the notion of an algebraic line bundle over a ring R in §3 corresponds exactly to the notion of a line bundle over the ringed space $(\mathrm{Spec}(R), \tilde{R})$.

More is true: the categories $\mathbf{VB}(X)$ and $\mathbf{P}(R)$ are equivalent when $X = \mathrm{Spec}(R)$. To see this, recall that an $\mathcal{O}_X$-module is called *quasicoherent* if it is isomorphic to some $\tilde{M}$ ([85, II.5.4]). The above correspondence shows that every vector bundle is quasicoherent. It turns out that the category $\mathcal{O}_X$-$\mathbf{mod}_{qcoh}$ of quasicoherent $\mathcal{O}_X$-modules is equivalent to the category R-**mod** of all R-modules (see [85, II.5.5]). Since the subcategories $\mathbf{VB}(\operatorname{Spec} R)$ and $\mathbf{P}(R)$ correspond, they are equivalent.

Definition (Coherent modules)**.** Suppose that X is any scheme. We say that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasicoherent* if X may be covered by affine opens $U_i = \operatorname{Spec}(R_i)$ such that each $\mathcal{F}|U_i$ is $\tilde{M}_i$ for an R_i-module M_i. (If X is affine, this agrees with the definition of quasicoherent in Example 5.1.2 by [85, II.5.4].) We say that $\mathcal{F}$ is *coherent* if moreover each M_i is a finitely presented R_i-module.

The category of quasicoherent $\mathcal{O}_X$-modules is abelian; if X is noetherian then so is the category of coherent $\mathcal{O}_X$-modules.

If X is affine then $\mathcal{F} = \tilde{M}$ is coherent if and only if M is a finitely presented R-module, by [EGA, I(1.4.3)]. In particular, if R is noetherian then "coherent" is just a synonym for "finitely generated." If X is a noetherian scheme, our definition of coherent module agrees with [85] and [EGA]. For general schemes, our definition is slightly stronger than in Hartshorne [85], and slightly weaker than in [EGA, $0_I(5.3.1)$]; $\mathcal{O}_X$ is always coherent in our sense, but not in the sense of [EGA].

The equivalent conditions for locally free modules in 2.4 translate into:

I.5.1.3 **Lemma 5.1.3.** *For every scheme X and $\mathcal{O}_X$-module $\mathcal{F}$, the following conditions are equivalent:*

1. *$\mathcal{F}$ is a vector bundle (i.e., is locally free of finite rank);*
2. *$\mathcal{F}$ is quasicoherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules of finite rank;*
3. *$\mathcal{F}$ is coherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules;*
4. *For every affine open $U = \operatorname{Spec}(R)$ in X, $\mathcal{F}|_U$ is the sheaf of a finitely generated projective R-module.*

I.5.1.4 **Example 5.1.4** (Analytic spaces)**.** Analytic spaces form another family of locally ringed spaces. To define them, one proceeds as follows. On the topological space $\mathbb{C}^n$, the subsheaf $\mathcal{O}_{\text{an}}$ of $\mathcal{O}_{\text{top}}$ consisting of analytic functions makes $(\mathbb{C}^n, \mathcal{O}_{\text{an}})$ into a locally ringed space. A *basic analytic set* in an open subset U of $\mathbb{C}^n$ is the zero locus V of a finite number of holomorphic functions, made into a locally ringed space $(V, \mathcal{O}_{V,\text{an}})$ as follows. If $\mathcal{I}_V$ is the subsheaf of $\mathcal{O}_{U,an}$ consisting of functions vanishing on V, the quotient sheaf $\mathcal{O}_{V,\text{an}} = \mathcal{O}_{U,\text{an}}/\mathcal{I}_V$ is supported on V, and is a subsheaf of the sheaf $\mathcal{O}_{V,\text{top}}$. By definition, a (reduced) *analytic space* $X_{\text{an}} = (X, \mathcal{O}_{\text{an}})$ is a ringed space which is locally isomorphic to a basic analytic set. A good reference for (reduced) analytic spaces is [79]; the original source is Serre's [GAGA].

Let X_{an} be an analytic space. For clarity, a vector bundle over X_{an} (in the sense of Definition 5.1) is sometimes called an *analytic vector bundle*. Since finitely generated $\mathcal{O}_{\text{an}}(U)$-modules are finitely presented, there is also good a notion of coherence on an analytic space: an $\mathcal{O}_{\text{an}}$-module $\mathcal{F}$ is called *coherent* if it is locally finitely presented in the sense that in a neighborhood U of any point it is presented as a cokernel:

$$\mathcal{O}^n_{U,\text{an}} \to \mathcal{O}^m_{U,\text{an}} \to \mathcal{F}|_U \to 0.$$

One special class of analytic spaces is the class of *Stein spaces*. It is known that analytic vector bundles are the same as topological vector bundles over a Stein space. For example, any analytic subspace of $\mathbb{C}^n$ is a Stein space. See [73].

Morphisms of ringed spaces

I.5.2 **5.2.** Here are two basic ways to construct new ringed spaces and morphisms:

(1) If $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, $(X, \mathcal{A})$ is a ringed space;

(2) If $f\colon Y \to X$ is a continuous map and $(Y, \mathcal{O}_Y)$ is a ringed space, the direct image sheaf $f_*\mathcal{O}_Y$ is a sheaf of rings on X, so $(X, f_*\mathcal{O}_Y)$ is a ringed space.

A *morphism of ringed spaces* $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a continuous map $f\colon Y \to X$ together with a map $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves of rings on X. In case (1) there is a morphism $i\colon (X, \mathcal{A}) \to (X, \mathcal{O}_X)$; in case (2) the morphism is $(Y, \mathcal{O}_Y) \to (X, f_*\mathcal{O}_Y)$; in general, every morphism factors as $(Y, \mathcal{O}_Y) \to (X, f_*\mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

A morphism of ringed spaces $f\colon X \to Y$ between two locally ringed spaces is a *morphism of locally ringed spaces* if in addition for each point $y \in Y$ the map of stalk rings $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ sends the maximal ideal $\mathfrak{m}_{f(y)}$ into the maximal ideal $\mathfrak{m}_y$. A *morphism of schemes* is a morphism of locally ringed spaces $f\colon Y \to X$ between schemes.

If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then the direct image sheaf $f_*\mathcal{F}$ is an $f_*\mathcal{O}_Y$-module, and hence also an $\mathcal{O}_X$-module. Thus f_* is a functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules, making $\mathcal{O}_X$-**mod** covariant in X. If $\mathcal{F}$ is a vector bundle over Y then $f_*\mathcal{F}$ is a vector bundle over $(X, f_*\mathcal{O}_Y)$. However, $f_*\mathcal{F}$ will not be a vector bundle over $(X, \mathcal{O}_X)$ unless $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of finite rank, which rarely occurs.

If $f\colon Y \to X$ is a *proper* morphism between noetherian schemes then Serre's "Theorem B" states that if $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module then the direct image $f_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. (See [EGA, III(3.2.2)] or [85, III.5.2 and II.5.19].)

I.5.2.1 **Example 5.2.1** (Projective Schemes)**.** When Y is a projective scheme over a field k, the structural map $\pi\colon Y \to \operatorname{Spec}(k)$ is proper. In this case the direct image $\pi_*\mathcal{F} = H^0(Y, \mathcal{F})$ is a finite-dimensional vector space over k. Indeed, every coherent k-module is finitely generated. Not surprisingly, $\dim_k H^0(Y, \mathcal{F})$ gives an important invariant for coherent modules (and vector bundles) over projective schemes.

The functor f_* has a left adjoint f^* (from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules):

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, f_*\mathcal{F})$$

for every $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{F}$. The explicit construction is given in [85, II.5], and shows that f^* sends free $\mathcal{O}_X$-modules to free $\mathcal{O}_Y$-modules, with $f^*\mathcal{O}_X \cong \mathcal{O}_Y$. If $i\colon U \subset X$ is the inclusion of an open subset then $i^*\mathcal{E}$ is just $\mathcal{E}|U$; it follows that if $\mathcal{E}|U$ is free then $(f^*\mathcal{E})|f^{-1}(U)$ is free. Thus f^* sends locally free $\mathcal{O}_X$-modules to locally free $\mathcal{O}_Y$-modules, and yields a functor $f^*\colon \mathbf{VB}(X) \to \mathbf{VB}(Y)$, making $\mathbf{VB}(X)$ contravariant in the ringed space X.

I.5.2.2 **Example 5.2.2.** If R and S are commutative rings then ring maps $f^\#\colon R \to S$ are in 1–1 correspondence with morphisms $f\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of ringed spaces. The direct image functor f_* corresponds to the forgetful functor from S-modules to R-modules, and the functor f^* corresponds to the functor $\otimes_R S$ from R-modules to S-modules.

I.5.2.3 **Example 5.2.3** (Associated analytic and topological bundles)**.** Let X be a scheme of finite type over $\mathbb{C}$, such as a subvariety of either $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{A}^n_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n])$. The closed points $X(\mathbb{C})$ of X have the natural structure of an analytic space; in particular it is a locally compact topological space. Indeed, $X(\mathbb{C})$ is covered by open sets $U(\mathbb{C})$ homeomorphic to analytic subspaces of $\mathbb{A}^n(\mathbb{C})$, and $\mathbb{A}^n(\mathbb{C}) \cong \mathbb{C}^n$. Note that if X is a projective variety then $X(\mathbb{C})$ is compact, because it is a closed subspace of the compact space $\mathbb{P}^n(\mathbb{C}) \cong \mathbb{CP}^n$.

Considering $X(\mathbb{C})$ as topological and analytic ringed spaces as in Examples 5.1.1 and 5.1.4, the evident continuous map $\tau\colon X(\mathbb{C}) \to X$ induces morphisms of ringed spaces $X(\mathbb{C})_{\mathrm{top}} \to X(\mathbb{C})_{\mathrm{an}} \to X$. This yields functors from $\mathbf{VB}(X, \mathcal{O}_X)$ to $\mathbf{VB}(X(\mathbb{C})_{\mathrm{an}})$, and from $\mathbf{VB}(X_{\mathrm{an}})$ to $\mathbf{VB}(X(\mathbb{C})_{\mathrm{top}}) \cong \mathbf{VB}_{\mathbb{C}}(X(\mathbb{C}))$. Thus every vector bundle $\mathcal{E}$ over the scheme X has an associated analytic vector bundle $\mathcal{E}_{\mathrm{an}}$, as well as an associated complex vector bundle $\tau^*\mathcal{E}$ over $X(\mathbb{C})$. In particular, every vector bundle $\mathcal{E}$ on X has topological Chern classes $c_i(\mathcal{E}) = c_i(\tau^*\mathcal{E})$ in the group $H^{2i}(X(\mathbb{C}); \mathbb{Z})$.

The main theorem of [GAGA] is that if X is a projective algebraic variety over $\mathbb{C}$ then there is an equivalence between the categories of coherent modules over X and over X_{an}. In particular, the categories of vector bundles $\mathbf{VB}(X)$ and $\mathbf{VB}(X_{\mathrm{an}})$ are equivalent.

A similar situation arises if X is a scheme of finite type over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the closed points of X with residue field $\mathbb{R}$; it too is a locally compact space. We consider $X(\mathbb{R})$ as a ringed space, using $\mathbb{R}$-valued functions as in Example 5.1.1. There is a morphism of ringed spaces $\tau\colon X(\mathbb{R}) \to X$, and the functor τ^* sends $\mathbf{VB}(X)$ to $\mathbf{VB}_{\mathbb{R}}(X(\mathbb{R}))$. That is, every vector bundle $\mathcal{F}$ over X has an associated real vector bundle $\tau^*\mathcal{F}$ over $X(\mathbb{R})$; in particular, every vector bundle $\mathcal{F}$ over X has Stiefel–Whitney classes $w_i(\mathcal{F}) = w_i(\tau^*\mathcal{F}) \in H^i(X(\mathbb{R}); \mathbb{Z}/2)$.

I.5.3 **5.3** (Patching and Operations)**.** Just as we built up projective modules by patching in 2.5, we can obtain a locally free sheaf $\mathcal{F}$ by patching (or *glueing*) locally free sheaves $\mathcal{F}_i$ of $\mathcal{O}_{U_i}$-modules via isomorphisms g_{ij} between $\mathcal{F}_j|U_i \cap U_j$ and $\mathcal{F}_i|U_i \cap U_j$, as long as $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k.

The patching process allows us to take any natural operation on free modules and extend it to locally free modules. For example, if $\mathcal{O}_X$ is commutative we can construct tensor products $\mathcal{F} \otimes \mathcal{G}$, Hom-modules $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, dual modules $\check{\mathcal{F}}$ and exterior powers $\wedge^i \mathcal{F}$ using $P \otimes_R Q$, $\mathrm{Hom}_R(P, Q)$, $\check{P}$ and $\wedge^i P$. If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles, then so are $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, $\check{\mathcal{F}}$ and $\wedge^i \mathcal{F}$. All of the natural isomorphisms such as $\check{\mathcal{F}} \otimes \mathcal{G} \cong \mathcal{H}om(\mathcal{F}, \mathcal{G})$ hold for locally free modules, because a sheaf map is an isomorphism if it is locally an isomorphism.

The Picard group and determinant bundles

If $(X, \mathcal{O}_X)$ is a commutative ringed space, the set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles forms a group, called the *Picard group* of X. To see this, we modify the proof in §3: the dual $\check{\mathcal{L}}$ of a line bundle $\mathcal{L}$ is again a line bundle and $\check{\mathcal{L}} \otimes \mathcal{L} \cong \mathcal{O}_X$ because by Lemma 3.1 this is true locally. Note that if X is $\mathrm{Spec}(R)$, we recover the definition of §3: $\mathrm{Pic}(\mathrm{Spec}(R)) = \mathrm{Pic}(R)$.

If $\mathcal{F}$ is locally free of rank n, then $\det(\mathcal{F}) = \wedge^n(\mathcal{F})$ is a line bundle. Operating componentwise as in §3, every locally free $\mathcal{O}_X$-module $\mathcal{F}$ has an associated determinant line bundle $\det(\mathcal{F})$. The natural map $\det(\mathcal{F}) \otimes \det(\mathcal{G}) \to \det(\mathcal{F} \oplus \mathcal{G})$ is an isomorphism because this is true locally by the Sum Formula in §3 (see Ex. 5.4 for a generalization). Thus det is a useful invariant of a locally free $\mathcal{O}_X$-module. We will discuss $\mathrm{Pic}(X)$ in terms of divisors at the end of this section.

Projective schemes

If X is a projective variety, maps between vector bundles are most easily described using graded modules. Following [85, II.2] this trick works more generally if X is $\mathrm{Proj}(S)$ for a commutative graded ring $S = S_0 \oplus S_1 \oplus \cdots$. By definition, the scheme $\mathrm{Proj}(S)$ is the union of the affine open sets $D_+(f) = \mathrm{Spec}\, S_{(f)}$, where $f \in S_n$ $(n \geq 1)$ and $S_{(f)}$ is the degree 0 subring of the $\mathbb{Z}$-graded ring $S[\frac{1}{f}]$. To cover $\mathrm{Proj}(S)$, it suffices to use $D_+(f)$ for a family of f's generating the ideal $S_+ = S_1 \oplus S_2 \oplus \cdots$ of S. For example, *projective n-space* over R is $\mathbb{P}^n_R = \mathrm{Proj}(R[X_0, ..., X_n])$; it is covered by the $D_+(X_i)$ and if $x_j = X_j/X_i$ then $D_+(X_i) = \mathrm{Spec}(R[x_1, ..., x_n])$.

If $M = \oplus_{i \in \mathbb{Z}} M_i$ is a graded S-module, there is an associated $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \mathrm{Proj}(S)$. The restriction of $\widetilde{M}$ to $D_+(f)$ is the sheaf associated to $M_{(f)}$, the $S_{(f)}$-module which constitutes the degree 0 submodule of $M[\frac{1}{f}]$; more details of the construction of $\widetilde{M}$ are given in [85, II.5.11]. Clearly $\tilde{S} = \mathcal{O}_X$. The functor $M \mapsto \widetilde{M}$ is exact, and has the property that $\widetilde{M} = 0$ whenever $M_i = 0$ for large i.

I.5.3.1 **Example 5.3.1** (Twisting Line Bundles)**.** The most important example of this construction is when M is $S(n)$, the module S regraded so that the degree i part is S_{n+i}; the associated sheaf $\tilde{S}(n)$ is written as $\mathcal{O}_X(n)$. If $f \in S_1$ then $S(n)_{(f)} \cong S_{(f)}$, so if S is generated by S_1 as an S_0-algebra then $\mathcal{O}_X(n)$ is a line bundle on $X = \mathrm{Proj}(S)$; it is called the n^{th} *twisting line bundle*. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \mathcal{O}_X(n)$, and call it "$\mathcal{F}$ twisted n times."

We will usually assume that S is generated by S_1 as an S_0-algebra, so that the $\mathcal{O}_X(n)$ are line bundles. This hypothesis ensures that every quasicoherent $\mathcal{O}_X$-module has the form $\widetilde{M}$ for some M (Hart [85, II.5.15]). It also ensures that the canonical maps $\widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to (M \otimes_S N)\tilde{}$ are isomorphisms, so if $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}(n)$ is the $\mathcal{O}_X$-module associated to $M(n) = M \otimes_S S(n)$. Since $S(m) \otimes S(n) \cong S(m+n)$ we have the formula

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).$$

Thus there is a homomorphism from $\mathbb{Z}$ to $\mathrm{Pic}(X)$ sending n to $\mathcal{O}_X(n)$. Operating componentwise, the same formula yields a homomorphism $[X, \mathbb{Z}] \to \mathrm{Pic}(X)$.

Here is another application of twisting line bundles. An element $x \in M_n$ gives rise to a graded map $S(-n) \to M$ and hence a sheaf map $\mathcal{O}_X(-n) \to \widetilde{M}$. Taking the direct sum over a generating set for M, we see that for every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a surjection from a locally free module $\oplus\mathcal{O}_X(-n_i)$ onto $\mathcal{F}$. In contrast, there is a surjection from a free $\mathcal{O}_X$-module onto $\mathcal{F}$ if and only if $\mathcal{F}$ can be generated by global sections, which is not always the case.

If P is a graded finitely generated projective S-module, the $\mathcal{O}_X$-module $\tilde{P}$ is a vector bundle over $\mathrm{Proj}(S)$. To see this, suppose the generators of P lie in degrees $n_1, ..., n_r$ and set $F = S(-n_1) \oplus \cdots \oplus S(-n_r)$. The kernel Q of the surjection $F \to P$ is a graded S-module, and that the projective lifting property implies that $P \oplus Q \cong F$. Hence $\tilde{P} \oplus \tilde{Q}$ is the direct sum $\tilde{F}$ of the line bundles $\mathcal{O}_X(-n_i)$, proving that $\tilde{P}$ is a vector bundle.

I.5.4 **Example 5.4** (No vector bundles are projective). Consider the projective line $\mathbb{P}^1_R = \mathrm{Proj}(S)$, $S = R[x, y]$. Associated to the "Koszul" exact sequence of graded S-modules

$$0 \to S(-2) \xrightarrow{(y,-x)} S(-1) \oplus S(-1) \xrightarrow{(x,y)} S \to R \to 0 \qquad (5.4.1)$$ I.5.4.1

is the short exact sequence of vector bundles over $\mathbb{P}^1_R$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \qquad (5.4.2)$$ I.5.4.2

The sequence (I.5.4.2 5.4.2) cannot split, because there are no nonzero maps from $\mathcal{O}_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see Ex. EI.5.2 5.2). This shows that the projective lifting property of §2 fails for the free module $\mathcal{O}_{\mathbb{P}^1}$. In fact, the projective lifting property fails for every vector bundle over $\mathbb{P}^1_R$; the category of $\mathcal{O}_{\mathbb{P}^1}$-modules has no "projective objects." This failure is the single biggest difference between the study of projective modules over rings and vector bundles over schemes.

The strict analogue of the Cancellation Theorem I.2.3 2.3 does not hold for projective schemes. To see this, we cite the following result from Atiy56 [5]. A vector bundle is called *indecomposable* if it cannot be written as the sum of two proper sub-bundles. For example, every line bundle is indecomposable.

I.5.5 **Krull-Schmidt Theorem 5.5.** *(Atiyah) Let X be a projective scheme over a field k. Then the Krull-Schmidt theorem holds for vector bundles over X. That is, every vector bundle over X can be written uniquely (up to reordering) as a direct sum of indecomposable vector bundles.*

In particular, the direct sums of line bundles $\mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r)$ are all distinct whenever $\dim(X) \neq 0$, because then all the $\mathcal{O}_X(n_i)$ are distinct.

I.5.5.1 **Example 5.5.1.** If X is a smooth projective curve over $\mathbb{C}$, then the associated topological space $X(\mathbb{C})$ is a Riemann surface. We saw in 4.11.2 that every topological line bundle on $X(\mathbb{C})$ is completely determined by its topological degree, and that every topological vector bundle is completely determined by its rank and degree. Now it is not hard to show that the twisting line bundle $\mathcal{O}_X(d)$ has degree d. Hence every topological vector bundle $\mathcal{E}$ of rank r and degree d is isomorphic to the direct sum $\mathcal{O}_X(d) \oplus T^{r-1}$. Moreover, the topological degree of a line bundle agrees with the usual algebraic degree one encounters in Algebraic Geometry.

The Krull-Schmidt Theorem shows that for each $r \geq 2$ and $d \in \mathbb{Z}$ there are infinitely many vector bundles over X with rank r and degree d. Indeed, there are infinitely many ways to choose integers $d_1, \dots, d_r$ so that $\sum d_i = d$, and these choices yield the vector bundles $\mathcal{O}_X(d_1) \oplus \cdots \oplus \mathcal{O}_X(d_r)$, which are all distinct with rank r and degree d.

For $X = \mathbb{P}^1_k$, the only indecomposable vector bundles are the line bundles $\mathcal{O}(n)$. This is a theorem of A. Grothendieck, proven in [80]. Using the Krull-Schmidt Theorem, we obtain the following classification.

I.5.6 **Theorem 5.6** (Classification of Vector Bundles over $\mathbb{P}^1_k$). *Let k be an algebraically closed field. Every vector bundle $\mathcal{F}$ over $X = \mathbb{P}^1_k$ is a direct sum of the line bundles $\mathcal{O}_X(n)$ in a unique way. That is, $\mathcal{F}$ determines a finite decreasing family of integers $n_1 \geq \cdots \geq n_r$ such that*

$$\mathcal{F} \cong \mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r).$$

The classification over other spaces is much more complicated than it is for $\mathbb{P}^1$. The following example is taken from [6]. Atiyah's result holds over any algebraically closed field k, but we shall state it for $k = \mathbb{C}$ because we have not yet introduced the notion on the degree of a line bundle. (Using the Riemann-Roch theorem, we could define the degree of a line bundle $\mathcal{L}$ over an elliptic curve as the integer $\dim H^0(X, \mathcal{L}(n)) - n$ for $n \gg 0$.)

I.5.7 **Example 5.7** (Classification of vector bundles over elliptic curves). Let X be a smooth elliptic curve over $\mathbb{C}$. Every vector bundle $\mathcal{E}$ over X has two integer invariants: its rank, and its *degree*, which we saw in 5.5.1 is just the Chern class $c_1(E) \in H^2(X(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$ of the associated topological vector bundle over the Riemann surface $X(\mathbb{C})$ of genus 1, defined in 5.2.3. Let $\mathbf{VB}^{ind}_{r,d}(X)$ denote the set of isomorphism classes of *indecomposable* vector bundles over X having rank r and degree d. Then for all $r \geq 1$ and $d \in \mathbb{Z}$:

(1) All the vector bundles in the set $\mathbf{VB}^{ind}_{r,d}(X)$ yield the same topological vector bundle E over $X(\mathbb{C})$. This follows from Example I.5.5.1 5.5.1.

(2) There is a natural identification of each $\mathbf{VB}^{ind}_{r,d}(X)$ with the set $X(\mathbb{C})$; in particular, there are uncountably many indecomposable vector bundles of rank r and degree d.

(3) Tensoring with the twisting bundle $\mathcal{O}_X(d)$ induces a bijection between $\mathbf{VB}^{ind}_{r,0}(X)$ and $\mathbf{VB}^{ind}_{r,d}(X)$.

(4) The r^{th} exterior power $\wedge^r$ maps $\mathbf{VB}^{ind}_{r,d}(X)$ onto $\mathbf{VB}^{ind}_{1,d}(X)$. This map is a bijection if and only if r and d are relatively prime. If $(r,d)=h$ then for each line bundle $\mathcal{L}$ of degree d there are h^2 vector bundles $\mathcal{E}$ with rank r and determinant $\mathcal{L}$.

I.5.8 **Construction 5.8** (Projective bundles)**.** If $\mathcal{E}$ is an vector bundle over a scheme X, we can form a *projective space bundle* $\mathbb{P}(\mathcal{E})$, which is a scheme equipped with a map $\pi\colon \mathbb{P}(\mathcal{E})\to X$ and a canonical line bundle $\mathcal{O}(1)$. To do this, we first construct $\mathbb{P}(\mathcal{E})$ when X is affine, and then glue the resulting schemes together.

If M is any module over a commutative ring R, the i^{th} *symmetric product* Sym^iM is the quotient of the i-fold tensor product $M\otimes\cdots\otimes M$ by the permutation action of the symmetric group, identifying $m_1\otimes\cdots\otimes m_i$ with $m_{\sigma(1)}\otimes\cdots\otimes m_{\sigma(i)}$ for every permutation σ. The obvious concatenation product $(Sym^iM)\otimes_R(Sym^jM)\to Sym^{i+j}M$ makes $Sym(M)=\oplus Sym^i(M)$ into a graded commutative R-algebra, called the *symmetric algebra* of M. As an example, note that if $M=R^n$ then $Sym(M)$ is the polynomial ring $R[x_1,...,x_n]$. This construction is natural in R: if $R\to S$ is a ring homomorphism, then $Sym(M)\otimes_R S\cong Sym(M\otimes_R S)$.

If E is a finitely generated projective R-module, let $\mathbb{P}(E)$ denote the scheme $\mathrm{Proj}(Sym(E))$. This scheme comes equipped with a map $\pi\colon\mathbb{P}(E)\to\mathrm{Spec}(R)$ and a canonical line bundle $\mathcal{O}(1)$; the scheme $\mathbb{P}(E)$ with this data is called the *projective space bundle* associated to E. If $E=R^n$, then $\mathbb{P}(E)$ is just the projective space $\mathbb{P}^{n-1}_R$. In general, the fact that E is locally free implies that $\mathrm{Spec}(R)$ is covered by open sets $D(s)=\mathrm{Spec}(R[\frac{1}{s}])$ on which E is free. If $E[\frac{1}{s}]$ is free of rank n then the restriction of $\mathbb{P}(E)$ to $D(s)$ is

$$\mathbb{P}(E[\frac{1}{s}])\cong\mathrm{Proj}(R[\frac{1}{s}][x_1,...,x_n])=\mathbb{P}^{n-1}_{D(s)}.$$

Hence $\mathbb{P}(E)$ is locally just a projective space over $\mathrm{Spec}(R)$. The vector bundles $\mathcal{O}(1)$ and $\pi^*\tilde{E}$ on $\mathbb{P}(E)$ are the sheaves associated to the graded S-modules $S(1)$ and $E\otimes_R S$, where S is $Sym(E)$. The concatenation $E\otimes Sym^j(E)\to Sym^{1+j}(E)$ yields an exact sequence of graded modules,

$$0\to E_1\to E\otimes_R S\to S(1)\to R(-1)\to 0 \qquad (5.8.1)$$ I.5.8.1

hence a natural short exact sequence of $\mathbb{P}(E)$-modules

$$0\to\mathcal{E}_1\to\pi^*\tilde{E}\to\mathcal{O}(1)\to 0. \qquad (5.8.2)$$ I.5.8.2

Since $\pi^*\tilde{E}$ and $\mathcal{O}(1)$ are locally free, $\mathcal{E}_1$ is locally free and $\text{rank}(\mathcal{E}_1) = \text{rank}(E)-1$. For example, if $E = R^2$ then $\mathbb{P}(E)$ is $\mathbb{P}^1_R$ and $\mathcal{E}_1$ is $\mathcal{O}(-1)$ because (5.8.1) is the sequence (5.4.1) tensored with $S(1)$. That is, (5.8.2) is just (5.4.2):

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(+1) \to 0.$$

Having constructed $\mathbb{P}(E)$ over affine schemes, we now suppose that $\mathcal{E}$ is a vector bundle over any scheme X. We can cover X by affine open sets U and construct the projective bundles $\mathbb{P}(\mathcal{E}|U)$ over each U. By naturality of the construction of $\mathbb{P}(\mathcal{E}|U)$, the restrictions of $\mathbb{P}(\mathcal{E}|U)$ and $\mathbb{P}(\mathcal{E}|V)$ to $U \cap V$ may be identified with each other. Thus we can glue the $\mathbb{P}(\mathcal{E}|U)$ together to obtain a projective space bundle $\mathbb{P}(\mathcal{E})$ over X; a patching process similar to that in 5.3 yields a canonical line bundle $\mathcal{O}(1)$ over $\mathbb{P}(\mathcal{E})$.

By naturality of $E \otimes_R Sym(E) \to Sym(E)(1)$, we have a natural short exact sequence of vector bundles on $\mathbb{P}(\mathcal{E})$, which is locally the sequence (5.8.2):

$$0 \to \mathcal{E}_1 \to \pi^*\mathcal{E} \to \mathcal{O}(1) \to 0. \qquad (5.8.3)$$ I.5.8.3

Let ρ denote the projective space bundle $\mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$ and let $\mathcal{E}_2$ denote the kernel of $\rho^*\mathcal{E}_1 \to \mathcal{O}(1)$. Then $(\pi\rho)^*\mathcal{E}$ has a filtration $\mathcal{E}_2 \subset \rho^*\mathcal{E}_1 \subset (\rho\pi)^*\mathcal{E}$ with filtration quotients $\mathcal{O}(1)$ and $\rho^*\mathcal{O}(1)$. This yields a projective space bundle $\mathbb{P}(\mathcal{E}_2) \to \mathbb{P}(\mathcal{E}_1)$. As long as $\mathcal{E}_i$ has rank ≥ 2 we can iterate this construction, forming a new projective space bundle $\mathbb{P}(\mathcal{E}_i)$ and a vector bundle $\mathcal{E}_{i+1}$. If $\text{rank}\,\mathcal{E} = r$, $\mathcal{E}_{r-1}$ will be a line bundle. We write $\mathbb{F}(\mathcal{E})$ for $\mathbb{P}(\mathcal{E}_{r-2})$, and call it the *flag bundle* of $\mathcal{E}$. We may summarize the results of this construction as follows.

I.5.9 **Theorem 5.9** (Splitting Principle)**.** *Given a vector bundle $\mathcal{E}$ of rank r on a scheme X, there exists a morphism $f\colon \mathbb{F}(\mathcal{E}) \to X$ such that $f^*\mathcal{E}$ has a filtration*

$$f^*\mathcal{E} = \mathcal{E}'_0 \supset \mathcal{E}'_1 \supset \cdots \supset \mathcal{E}'_r = 0$$

by sub-vector bundles whose successive quotients $\mathcal{E}'_i/\mathcal{E}'_{i+1}$ are all line bundles.

Cohomological classification of vector bundles

The formation of vector bundles via the patching process in 5.3 may be codified into a classification of rank n vector bundles via a Čech cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$ which is associated to the sheaf of groups $\mathcal{G} = GL_n(\mathcal{O}_X)$. This cohomology set is defined more generally for any sheaf of groups $\mathcal{G}$ as follows. A *Čech 1-cocycle* for an open cover $\mathcal{U} = \{U_i\}$ of X is a family of elements g_{ij} in $\mathcal{G}(U_i \cap U_j)$ such that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k. Two 1-cocycles $\{g_{ij}\}$ and $\{h_{ij}\}$ are said to be *equivalent* if there are $f_i \in \mathcal{G}(U_i)$ such that $h_{ij} = f_i g_{ij} f_j^{-1}$. The equivalence classes of 1-cocycles form the set $\check{H}^1(\mathcal{U}, \mathcal{G})$. If $\mathcal{V}$ is a refinement of a cover $\mathcal{U}$, there is a set map from $\check{H}^1(\mathcal{U}, \mathcal{G})$ to $\check{H}^1(\mathcal{V}, \mathcal{G})$. The cohomology set $\check{H}^1(X, \mathcal{G})$ is defined to be the direct limit of the $\check{H}^1(\mathcal{U}, \mathcal{G})$ as $\mathcal{U}$ ranges over the system of all open covers of X.

We saw in 5.3 that every rank n vector bundle arises from patching, using a 1-cocycle for $\mathcal{G} = GL_n(\mathcal{O}_X)$. It isn't hard to see that equivalent cocycles give isomorphic vector bundles. From this, we deduce the following result.

I.5.10 **Classification Theorem 5.10.** *For every ringed space X, the set $\mathbf{VB}_n(X)$ of isomorphism classes of vector bundles of rank n over X is in 1-1 correspondence with the cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$:*

$$\mathbf{VB}_n(X) \cong \check{H}^1(X, GL_n(\mathcal{O}_X)).$$

When $\mathcal{G}$ is an abelian sheaf of groups, such as $\mathcal{O}_X^\times = GL_1(\mathcal{O}_X)$, it is known that the Čech set $\check{H}^1(X,\mathcal{G})$ agrees with the usual sheaf cohomology group $H^1(X,\mathcal{G})$ (see Ex. III.4.4 of [85]). In particular, each $\check{H}^1(X,\mathcal{G})$ is an abelian group. A little work, detailed in [EGA, 0_I(5.6.3)]) establishes:

I.5.10.1 **Corollary 5.10.1.** *For every locally ringed space X the isomorphism of Theorem 5.10 is a group isomorphism:*

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times).$$

As an application, suppose that X is the union of two open sets V_1 and V_2. Write $U(X)$ for the group $H^0(X,\mathcal{O}_X^\times) = \mathcal{O}_X^\times(X)$ of global units on X. The cohomology Mayer-Vietoris sequence translates to the following exact sequence.

$$\begin{aligned} 1 \to U(X) \to U(V_1)\times U(V_2) \to U(V_1\cap V_2) \xrightarrow{\partial} \\ \xrightarrow{\partial} \mathrm{Pic}(X) \to \mathrm{Pic}(V_1)\times \mathrm{Pic}(V_2) \to \mathrm{Pic}(V_1\cap V_2). \end{aligned} \tag{5.10.2}$$ I.5.10.2

To illustrate how this sequence works, consider the standard covering of $\mathbb{P}^1_R$ by Spec$(R[t])$ and Spec$(R[t^{-1}])$. Their intersection is Spec$(R[t,t^{-1}])$. Comparing (5.10.2) with the sequences of Ex. 3.17 and Ex. 3.18 yields

I.5.11 **Theorem 5.11.** *For any commutative ring R,*

$$U(\mathbb{P}^1_R) = U(R) = R^\times \textit{ and } \quad \mathrm{Pic}(\mathbb{P}^1_R) \cong \mathrm{Pic}(R)\times[\mathrm{Spec}(R),\mathbb{Z}].$$

As in 5.3.1, the continuous function Spec$(R) \xrightarrow{n} \mathbb{Z}$ *corresponds to the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1_R$ obtained by patching $R[t]$ and $R[t^{-1}]$ together via $t^n \in R[t,t^{-1}]^\times$.*

Here is an application of Corollary 5.10.1 to nonreduced schemes. Suppose that $\mathcal{I}$ is a sheaf of nilpotent ideals, and let X_0 denote the ringed space $(X,\mathcal{O}_X/\mathcal{I})$. Writing $\mathcal{I}^\times$ for the sheaf $GL_1(\mathcal{I})$ of Ex. 1.10, we have an exact sequence of sheaves of abelian groups:

$$1 \to \mathcal{I}^\times \to \mathcal{O}_X^\times \to \mathcal{O}_{X_0}^\times \to 1.$$

The resulting long exact cohomology sequence starts with global units:

$$U(X) \to U(X_0) \to H^1(X,\mathcal{I}^\times) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_0) \to H^2(X,\mathcal{I}^\times)\cdots. \tag{5.11.1}$$ I.5.11.1

Thus $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ may not be an isomorphism, as it is in the affine case (Lemma 3.9).

Invertible ideal sheaves

Suppose that X is an integral scheme, *i.e.,* that each $\mathcal{O}_X(U)$ is an integral domain. The function field $k(X)$ of X is the common quotient field of the integral domains $\mathcal{O}_X(U)$. Following the discussion in §3, we use $\mathcal{K}$ to denote the constant sheaf $U \mapsto k(X)$ and consider $\mathcal{O}_X$-submodules of $\mathcal{K}$. Those that lie in some $f\mathcal{O}_X$ we call *fractional*; a fractional ideal $\mathcal{I}$ is called *invertible* if $\mathcal{I}\mathcal{J} = \mathcal{O}_X$ for some $\mathcal{J}$. As in Proposition 3.5, invertible ideals are line bundles and $\mathcal{I} \otimes \mathcal{J} \cong \mathcal{I}\mathcal{J}$. The set $\mathrm{Cart}(X)$ of invertible ideals in $\mathcal{K}$ is therefore an abelian group.

I.5.12 **Proposition 5.12.** *If X is an integral scheme, there is an exact sequence*

$$1 \to U(X) \to k(X)^\times \to \mathrm{Cart}(X) \to \mathrm{Pic}(X) \to 1. \tag{5.12.1}$$ I.5.12.1

Proof. The proof of 3.5 goes through to prove everything except that every line bundle $\mathcal{L}$ on X is isomorphic to an invertible ideal. On any affine open set U we have $(\mathcal{L} \otimes \mathcal{K})|U \cong \mathcal{K}|U$, a constant sheaf on U. This implies that $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$, because over an irreducible scheme like X any locally constant sheaf must be constant. Thus the natural inclusion of $\mathcal{L}$ in $\mathcal{L} \otimes \mathcal{K}$ expresses $\mathcal{L}$ as an $\mathcal{O}_X$-submodule of $\mathcal{K}$, and the rest of the proof of 3.5 goes through. $\square$

Here is another way to understand $\mathrm{Cart}(X)$. Let $\mathcal{K}^\times$ denote the constant sheaf of units of $\mathcal{K}$; it contains the sheaf $\mathcal{O}_X^\times$. Associated to the exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}^\times \to \mathcal{K}^\times/\mathcal{O}_X^\times \to 1$$

is a long exact cohomology sequence. Since X is irreducible and $\mathcal{K}^\times$ is constant, we have $H^0(X, \mathcal{K}^\times) = k(X)^\times$ and $H^1(X, \mathcal{K}^\times) = 0$. Since $U(X) = H^0(X, \mathcal{O}_X^\times)$ we get the exact sequence

$$1 \to U(X) \to k(X)^\times \to H^0(X, \mathcal{K}^\times/\mathcal{O}_X^\times) \to \mathrm{Pic}(X) \to 1. \tag{5.12.2}$$ I.5.12.2

Motivated by this sequence, we use the term *Cartier divisor* for a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}_X^\times$. A Cartier divisor can be described by giving an open cover $\{U_i\}$ of X and $f_i \in k(X)^\times$ such that f_i/f_j is in $\mathcal{O}_X^\times(U_i \cap U_j)$ for each i and j.

I.5.13 **Lemma 5.13.** *Over every integral scheme X, there is a 1-1 correspondence between Cartier divisors on X and invertible ideal sheaves. Under this identification the sequences (5.12.1) and (5.12.2) are the same.*

Proof. If $\mathcal{I} \subset \mathcal{K}$ is an invertible ideal, there is a cover $\{U_i\}$ on which $\mathcal{I}$ is trivial, *i.e.,* $\mathcal{I}|U_i \cong \mathcal{O}_{U_i}$. Choosing $f_i \in \mathcal{I}(U_i) \subseteq k(X)$ generating $\mathcal{I}|U_i$ gives a Cartier divisor. This gives a set map $\mathrm{Cart}(X) \to H^0(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$; it is easily seen to be a group homomorphism compatible with the map from $k(X)^\times$, and with the map to $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$. This gives a map between the sequences (5.12.1) and (5.12.2); the 5-lemma implies that $\mathrm{Cart}(X) \cong H^0(X, \mathcal{K}^\times/\mathcal{O}_X^\times)$. $\square$

I.5.13.1 **Variant 5.13.1.** Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Historically, the invertible ideal sheaf associated to D is the subsheaf $\mathcal{L}(D)$ of $\mathcal{K}$ defined by letting $\mathcal{L}(D)|U_i$ be the submodule of $k(X)$ generated by f_i^{-1}. Since f_i/f_j is a unit on $U_i \cap U_j$, these patch to yield an invertible ideal. If $\mathcal{I}$ is invertible and D is the Cartier divisor attached to $\mathcal{I}$ by 5.13, then $\mathcal{L}(D)$ is $\mathcal{I}^{-1}$. Under the correspondence $D \leftrightarrow \mathcal{L}(D)$ the sequences (5.12.1) and (5.12.2) differ by a minus sign.

For example if $X = \mathbb{P}^1_R$, let D be the Cartier divisor given by t^n on $\mathrm{Spec}(R[t])$ and 1 on $\mathrm{Spec}(R[t^{-1}])$. The correspondence of Lemma 5.13 sends D to $\mathcal{O}(n)$, but $\mathcal{L}(D) \cong \mathcal{O}(-n)$.

Weil divisors

There is a notion of Weil divisor corresponding to that for rings (see 3.6). We say that a scheme X is *normal* if all the local rings $\mathcal{O}_{X,x}$ are normal domains (if X is affine this is the definition of Ex. 3.14), and *Krull* if it is integral, separated and has an affine cover $\{\mathrm{Spec}(R_i)\}$ with the R_i Krull domains. For example, if X is noetherian, integral and separated, then X is Krull if and only if it is normal.

A *prime divisor* on X is a closed integral subscheme Y of codimension 1; this is the analogue of a height 1 prime ideal. A *Weil divisor* is an element of the free abelian group $D(X)$ on the set of prime divisors of X; we call a Weil divisor $D = \sum n_i Y_i$ *effective* if all the $n_i \geq 0$.

Let $k(X)$ be the function field of X. Every prime divisor Y yields a discrete valuation on $k(X)$, because the local ring $\mathcal{O}_{X,y}$ at the generic point y of Y is a DVR. Conversely, each discrete valuation on $k(X)$ determines a unique prime divisor on X, because X is separated [85, Ex. II(4.5)]. Having made these observations, the discussion in §3 applies to yield group homomorphisms $\nu\colon k(X)^\times \to D(X)$ and $\nu\colon \mathrm{Cart}(X) \to D(X)$. We define the *divisor class group* $Cl(X)$ to be the quotient of $D(X)$ by the subgroup of all Weil divisors $\nu(f)$, $f \in k(X)^\times$. The proof of Proposition 3.6 establishes the following result.

I.5.14 **Proposition 5.14.** *Let X be Krull. Then* $\mathrm{Pic}(X)$ *is a subgroup of the divisor class group $Cl(X)$, and there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 1 \to & U(X) & \longrightarrow & k(X)^\times & \longrightarrow & \mathrm{Cart}(X) & \longrightarrow & \mathrm{Pic}(X) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \nu} & & \cap & \\ 1 \to & U(X) & \longrightarrow & k(X)^\times & \longrightarrow & D(X) & \longrightarrow & Cl(X) & \to 1. \end{array}$$

A scheme X is called *regular* (resp. *locally factorial*) if the local rings $\mathcal{O}_{X,x}$ are all regular local rings (resp. UFD's). By 3.8, regular schemes are locally factorial. Suppose that X is locally factorial and Krull. If $\mathcal{I}_Y$ is the ideal of a prime divisor Y and $U = \mathrm{Spec}(R)$ is an affine open subset of X, $\mathcal{I}_Y|U$ is invertible by Corollary 3.8.1. Since $\nu(\mathcal{I}_Y) = Y$, this proves that $\nu : \mathrm{Cart}(X) \to D(X)$ is onto. Inspecting the diagram of Proposition 5.14, we have:

I.5.15 **Proposition 5.15.** *Let X be an integral, separated and locally factorial scheme. Then*

$$\mathrm{Cart}(X) \cong D(X) \qquad \textit{and} \qquad \mathrm{Pic}(X) \cong Cl(X).$$

I.5.15.1 **Example 5.15.1.** ([85, II(6.4)]). If X is the projective space $\mathbb{P}^n_k$ over a field k, then $\mathrm{Pic}(\mathbb{P}^n_k) \cong Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$. By Theorem 5.11, $\mathrm{Pic}(\mathbb{P}^n)$ is generated by $\mathcal{O}(1)$. The class group $Cl(\mathbb{P}^n)$ is generated by the class of a hyperplane H, whose corresponding ideal sheaf $\mathcal{I}_H$ is isomorphic to $\mathcal{O}(1)$. If Y is a hypersurface defined by a homogeneous polynomial of degree d, we say $\deg(Y) = d$; $Y \sim dH$ in $D(\mathbb{P}^n)$.

The *degree* of a Weil divisor $D = \sum n_i Y_i$ is defined to be $\sum n_i \deg(Y_i)$; the degree function $D(\mathbb{P}^n) \to \mathbb{Z}$ induces the isomorphism $Cl(\mathbb{P}^n) \cong \mathbb{Z}$. We remark that when $k = \mathbb{C}$ the degree of a Weil divisor agrees with the topological degree of the associated line bundle in $H^2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$, defined by the first Chern class as in Example 4.11.2.

I.5.15.2 **Example 5.15.2** (Blowing Up)**.** Let X be a smooth variety over an algebraically closed field, and let Y be a smooth subvariety of codimension ≥ 2. If the ideal sheaf of Y is $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on Y. The *blowing up of X along Y* is a nonsingular variety $\tilde{X}$, containing a prime divisor $\tilde{Y} \cong \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, together with a map $\pi\colon \tilde{X} \to X$ such that $\pi^{-1}(Y) = \tilde{Y}$ and $\tilde{X} - \tilde{Y} \cong X - Y$ (see [85, II.7]). For example, the blowing up of a smooth surface X at a point x is a smooth surface $\tilde{X}$, and the smooth curve $\tilde{Y} \cong \mathbb{P}^1$ is called the *exceptional divisor*.

The maps $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X})$ sending n to $n[\tilde{Y}]$ give rise to an isomorphism (see [85, Ex. II.8.5 or V.3.2]):

$$\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}.$$

I.5.15.3 **Example 5.15.3.** Consider the rational ruled surface S in $\mathbb{P}^1 \times \mathbb{P}^2$, defined by $X_i Y_j = X_j Y_i$ $(i, j = 1, 2)$, and the smooth quadric surface Q in $\mathbb{P}^3$, defined by $xy = zw$. Now S is obtained by blowing up $\mathbb{P}^2_k$ at a point [85, V.2.11.5], while Q is obtained from $\mathbb{P}^2_k$ by first blowing up two points, and then blowing down the line between them [85, Ex. V.4.1]. Thus $\mathrm{Pic}(S) = Cl(S)$ and $\mathrm{Pic}(Q) = Cl(Q)$ are both isomorphic to $\mathbb{Z} \times \mathbb{Z}$. For both surfaces, divisors are classified by a pair (a, b) of integers (see [85, II.6.6.1]).

I.5.16 **Example 5.16.** Let X be a smooth projective curve over an algebraically closed field k. In this case a Weil divisor is a formal sum of closed points on X: $D = \sum n_i x_i$. The *degree* of D is defined to be $\sum n_i$; a point has degree 1. Since the divisor of a function has degree 0 [85, II(6.4)], the degree induces a surjective homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}$. Writing $\mathrm{Pic}^0(X)$ for the kernel, the choice of a basepoint $\infty \in X$ determines a splitting $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(X)$. The group $\mathrm{Pic}^0(X)$ is divisible and has the same cardinality as k; its torsion subgroup is $(\mathbb{Q}/\mathbb{Z})^{2g}$ if $\mathrm{char}(k) = 0$. If k is perfect of characteristic $p > 0$, the torsion subgroup lies between $(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])^{2g}$ and $(\mathbb{Q}/\mathbb{Z})^{2g}$. These facts are established in [140, II].

If X has genus 0, then $X \cong \mathbb{P}^1$ and $\mathrm{Pic}^0(X) = 0$. If X has genus 1, the map $x \mapsto x - \infty$ gives a canonical bijection $X(k) \cong \mathrm{Pic}^0(X)$. In general, if X has genus g there is an abelian variety $J(X)$ of dimension g, called the *Jacobian variety* of X [85, IV.4.10] such that the closed points of $J(X)$ are in 1–1 correspondence with the elements of $\mathrm{Pic}^0(X)$. The Jacobian variety is a generalization of the Picard variety of Exercise 5.9 below.

I.5.17 **Example 5.17.** Let X be a smooth projective curve over a finite field $\mathbb{F} = \mathbb{F}_q$. As observed in [85, IV.4.10.4], the elements of the kernel $\mathrm{Pic}^0(X)$ of the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$ are in 1–1 correspondence with the set $J(X)(\mathbb{F})$ of closed points of the Jacobian variety $J(X)$ whose coordinates belong to $\mathbb{F}$. Since $J(X)$ is contained in some projective space, the set $J(X)(\mathbb{F})$ is finite. Thus $\mathrm{Pic}(X)$ is the direct sum of $\mathbb{Z}$ and a finite group. We may assume that $H^0(X, \mathcal{O}_X) = \mathbb{F}$, so that $U(X) = \mathbb{F}^\times$.

Now let S be any nonempty set of closed points of X and consider the affine curve $X - S$; the coordinate ring R of $X - S$ is called the *ring of S-integers* in the function field $\mathbb{F}(X)$. Comparing the sequences of Proposition 5.14 for X and $X - S$ yields the exact sequence

$$1 \to \mathbb{F}^\times \to R^\times \to \mathbb{Z}^S \to \mathrm{Pic}(X) \to \mathrm{Pic}(R) \to 0. \qquad (5.17.1)$$ I.5.17.1

(See Ex. 5.12.) The image of the map $\mathbb{Z}^S \to \mathrm{Pic}(X)$ is the subgroup generated by the lines bundles associated to the points of S via the identification of Weil divisors with Cartier divisors given by Proposition 5.15 (compare with Ex. 3.8.)

When $S = \{s\}$ is a single point, the map $\mathbb{Z} \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$ is multiplication by the degree of the field extension $[\mathbb{F}(s) : \mathbb{F}]$, so $\mathrm{Pic}(R)$ is a finite group; it is an extension of $\mathrm{Pic}^0(X)$ by a cyclic group of order $[k(s) : \mathbb{F}]$. (See Ex. 5.12(b).) From the exact sequence, we see that $R^\times = \mathbb{F}^\times$. By induction on $|S|$, it follows easily from (5.17.1) that $\mathrm{Pic}(R)$ is finite and $R^\times \cong \mathbb{F}^\times \oplus \mathbb{Z}^{|S|-1}$.

I.5.18 **Remark 5.18** (Historical Note)**.** The term "Picard group" (of a scheme or commutative ring), and the notation $\mathrm{Pic}(X)$, was introduced by Grothendieck around 1960. Of course the construction itself was familiar to the topologists of the early 1950's, and the connection to invertible ideals was clear from to framework of Serre's 1954 paper "Faisceaux algébriques cohérents," [166], but had not been given a name.

Grothendieck's choice of terminology followed André Weil's usage of the term *Picard variety* in his 1950 paper *Variétés Abéliennes*. Weil says that, "accidentally enough," his choice coincided with the introduction by Castelnuovo in 1905 of the "Picard variety associated with continuous systems of curves" on a surface X (*Sugli integrali semplici appartenenti ad una superficie irregolare*, Rend. Accad. dei Lincei, vol XIV, 1905). In turn, Castelnuovo named it in honor of Picard's paper *Sur la théorie des groupes et des surfaces algébriques* (Rend. Circolo Mat. Palermo, IX, 1895), which studied the number of integrals of the first kind attached to algebraic surfaces. (I am grateful to Serre and Pedrini for the historical information.)

EXERCISES

EI.5.1 **5.1.** Give an example of a ringed space $(X, \mathcal{O}_X)$ such that the rank of $\mathcal{O}_X(X)$ is well-defined, but such that the rank of $\mathcal{O}_X(U)$ is not well-defined for any proper open $U \subseteq X$.

EI.5.2 **5.2.** Show that the global sections of the vector bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ are in 1-1 correspondence with vector bundle maps $\mathcal{E} \to \mathcal{F}$. Conclude that there is a non-zero map $\mathcal{O}(m) \to \mathcal{O}(n)$ over $\mathbb{P}^1_R$ only if $m \le n$.

EI.5.3 **5.3.** *Projection Formula.* If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, show that there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

EI.5.4 **5.4.** Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves. Show that each $\wedge^n \mathcal{F}$ has a finite filtration

$$\wedge^n \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n+1} = 0$$

with successive quotients $F^i/F^{i+1} \cong (\wedge^i \mathcal{E}) \otimes (\wedge^{n-i} \mathcal{G})$. In particular, show that $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$.

EI.5.5 **5.5.** Let S be a graded ring generated by S_1 and set $X = \mathrm{Proj}(S)$. Show that $\mathcal{O}_X(n)\check{} \cong \mathcal{O}_X(-n)$ and $\mathcal{H}om(\mathcal{O}_X(m), \mathcal{O}_X(n)) \cong \mathcal{O}_X(n-m)$.

EI.5.6 **5.6.** *Serre's "Theorem A."* Suppose that X is Proj(S) for a graded ring S which is finitely generated as an S_0-algebra by S_1. Recall from 5.3.1 (or [85, II.5.15]) that every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some graded S-module M. In fact, we can take M_n to be $H^0(X, \mathcal{F}(n))$.

(a) If M is generated by M_0 and the M_i with $i < 0$, show that the sheaf $\widetilde{M}$ is generated by global sections. *Hint:* consider $M_0 \oplus M_1 \oplus \cdots$.

(b) By (a), $\mathcal{O}_X(n)$ is generated by global sections if $n \ge 0$. Is the converse true?

(c) If M is a finitely generated S-module, show that $\widetilde{M}(n)$ is generated by global sections for all large n (*i.e.,* for all $n \ge n_0$ for some n_0).

(d) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, show that $\mathcal{F}(n)$ is generated by global sections for all large n. This result is known as Serre's "Theorem A," and it implies that $\mathcal{O}_X(1)$ is an *ample line bundle* in the sense of [EGA, II(4.5.5)].

EI.5.7 **5.7.** Let X be a d-dimensional quasi-projective variety, *i.e.,* a locally closed integral subscheme of some $\mathbb{P}^n_k$, where k is an algebraically closed field.

(a) Suppose that $\mathcal{E}$ is a vector bundle generated by global sections. If rank$(\mathcal{E}) > d$, Bertini's Theorem implies that $\mathcal{E}$ has a global section s such that $s_x \notin \mathfrak{m}_x\mathcal{E}_x$ for each $x \in X$. Establish the analogue of the Serre Cancellation Theorem 2.3(a), that there is a short exact sequence of vector bundles

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{F} \to 0.$$

(b) Now suppose that X is a curve. Show that every vector bundle $\mathcal{E}$ is a successive extension of invertible sheaves in the sense that there is a filtration of $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_r = 0.$$

by sub-bundles such that each $\mathcal{E}_i/\mathcal{E}_{i+1}$ is a line bundle. *Hint*: by Ex. 5.6(d), $\mathcal{E}(n)$ is generated by global sections for large n.

EI.5.8 **5.8.** *Complex analytic spaces.* Recall from Example 5.1.4 that a *complex analytic space* is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a basic analytic subset of $\mathbb{C}^n$.

(a) Use Example 5.2.3 to show that every analytic vector bundle on $\mathbb{C}^n$ is free, *i.e.,* $\mathcal{O}^r_{\rm an}$ for some r. What about $\mathbb{C}^n - 0$?

(b) Let X be the complex affine node defined by the equation $y^2 = x^3 - x^2$. We saw in 3.10.2 that $\mathrm{Pic}(X) \cong \mathbb{C}^\times$. Use (4.9.1) to show that $\mathrm{Pic}(X(\mathbb{C})_{\rm an}) = 0$.

(c) (Serre) Let X be the scheme $\mathrm{Spec}(\mathbb{C}[x, y]) - \{0\}$, 0 being the origin. For the affine cover of X by $D(x)$ and $D(y)$, show that $\mathrm{Pic}(X) = 0$ but $\mathrm{Pic}(X_{\rm an}) \neq 0$.

EI.5.9 **5.9.** *Picard Variety.* Let X be a scheme over $\mathbb{C}$ and $X_{\rm an} = X(\mathbb{C})_{\rm an}$ the associated complex analytic space of Example 5.1.4. There is an exact sequence of sheaves of abelian groups on the topological space $X(\mathbb{C})$ underlying $X_{\rm an}$:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{X_{\rm an}} \xrightarrow{\exp} \mathcal{O}^\times_{X_{\rm an}} \to 0, \qquad (*)$$

where $\mathbb{Z}$ is the constant sheaf on $X(\mathbb{C})$.

(a) Show that the Chern class $c_1 \colon \mathrm{Pic}(X_{\rm an}) \to H^2(X(\mathbb{C})_{\rm top}; \mathbb{Z})$ of Example 5.2.3 is naturally isomorphic to the composite map

$$\mathrm{Pic}(X_{\rm an}) \cong H^1(X_{\rm an}, \mathcal{O}^\times_{X_{\rm an}}) \cong H^1(X(\mathbb{C})_{\rm top}; \mathcal{O}^\times_{X_{\rm top}}) \xrightarrow{\partial} H^2(X(\mathbb{C})_{\rm top}; \mathbb{Z})$$

coming from Corollary 5.10.1, the map $X_{\rm an} \to X(\mathbb{C})_{\rm top}$ of Example 5.1.4, and the boundary map of $(*)$.

Now suppose that X is projective. The image of $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_{\rm an})$ in $H^2(X(\mathbb{C}); \mathbb{Z})$ is called the *Néron-Severi group* $NS(X)$ and the kernel of $\mathrm{Pic}(X) \to NS(X)$ is written as $\mathrm{Pic}^0(X)$. Since $H^2(X(\mathbb{C}); \mathbb{Z})$ is a finitely generated abelian group, so is $NS(X)$. It turns out that $H^1(X(\mathbb{C}), \mathcal{O}_{X_{\rm an}}) \cong \mathbb{C}^n$ for some n, and that $H^1(X(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}^{2n}$ is a lattice in $H^1(X, \mathcal{O}_{X_{\rm an}})$.

(b) Show that $\mathrm{Pic}^0(X)$ is isomorphic to $H^1(X, \mathcal{O}_X)/H^1(X(\mathbb{C}); \mathbb{Z})$. Thus $\mathrm{Pic}^0(X)$ is a complex analytic torus; in fact it is the set of closed points of an abelian variety, called the *Picard variety* of X.

EI.5.10 **5.10.** If E and F are finitely generated projective R-modules, show that their projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(F)$ are isomorphic as schemes over R if and only if $E \cong F \otimes_R L$ for some line bundle L on R.

EI.5.11 **5.11.** Let X be a Krull scheme and Z an irreducible closed subset with complement U. Define a map $\rho\colon Cl(X) \to Cl(U)$ of class groups by sending the Weil divisor $\sum n_i Y_i$ to $\sum n_i(Y_i \cap U)$, ignoring terms $n_i Y_i$ for which $Y_i \cap U = \emptyset$. (Cf. Ex. EI.3.8 3.8.)

(a) If Z has codimension ≥ 2, show that $\rho\colon Cl(X) \cong Cl(U)$.

(b) If Z has codimension 1, show that there is an exact sequence

$$\mathbb{Z} \xrightarrow{[Z]} Cl(X) \xrightarrow{\rho} Cl(U) \to 0.$$

EI.5.12 **5.12.** Let X be a smooth curve over a field k, and let S be a finite nonempty set of closed points in X. By Riemann-Roch, the complement $U = X - S$ is affine; set $R = H^0(U, \mathcal{O})$ so that $U = \mathrm{Spec}(R)$.

(a) Using Propositions I.5.12 5.12 and I.5.14 5.14, show that there is an exact sequence

$$1 \to H^0(X, \mathcal{O}_X^\times) \to R^\times \to \mathbb{Z}^S \to \mathrm{Pic}(X) \to \mathrm{Pic}(R) \to 0.$$

(b) If X is a smooth projective curve over k and $s \in X$ is a closed point, show that the map $\mathbb{Z} \xrightarrow{[s]} \mathrm{Pic}(X) = Cl(X)$ in (a) is injective. (It is the map of Ex. EI.5.11 5.11(b).) If $k(x) = k$, conclude that $\mathrm{Pic}(X) \cong \mathrm{Pic}(U) \times \mathbb{Z}$. What happens if $k(x) \neq k$?

Chapter II

The Grothendieck Group K_0

There are several ways to construct the "Grothendieck group" of a mathematical object. We begin with the group completion version, because it has been the most historically important. After giving the applications to rings and topological spaces, we discuss λ-operations in §4. In sections 6 and 7 we describe the Grothendieck group of an "exact category," and apply it to the K-theory of schemes in §8. This construction is generalized to the Grothendieck group of a "Waldhausen category" in §9.

1 The Group Completion of a monoid

Both $K_0(R)$ and $K^0(X)$ are formed by taking the group completion of an abelian monoid—the monoid $\mathbf{P}(R)$ of finitely generated projective R-modules and the monoid $\mathbf{VB}(X)$ of vector bundles over X, respectively. We begin with a description of this construction.

Recall that an *abelian monoid* is a set M together with an associative, commutative operation $+$ and an "additive" identity element 0. A monoid map $f\colon M \to N$ is a set map such that $f(0) = 0$ and $f(m + m') = f(m) + f(m')$. The most famous example of an abelian monoid is $\mathbb{N} = \{0, 1, 2, ...\}$, the natural numbers with additive identity zero. If A is an abelian group then not only is A an abelian monoid, but so is any additively closed subset of A containing 0.

The *group completion* of an abelian monoid M is an abelian group $M^{-1}M$, together with a monoid map $[\ ]\colon M \to M^{-1}M$ which is universal in the sense that, for every abelian group A and every monoid map $\alpha\colon M \to A$, there is a unique abelian group homomorphism $\tilde{\alpha}\colon M^{-1}M \to A$ such that $\tilde{\alpha}([m]) = \alpha(m)$ for all $m \in M$.

For example, the group completion of $\mathbb{N}$ is $\mathbb{Z}$. If A is an abelian group then clearly $A^{-1}A = A$; if M is a submonoid of A (additively closed subset containing 0), then $M^{-1}M$ is the subgroup of A generated by M.

Every abelian monoid M has a group completion. One way to contruct it is to form the free abelian group $F(M)$ on symbols $[m]$, $m \in M$, and then factor out by the subgroup $R(M)$ generated by the relations $[m+n]-[m]-[n]$. By universality, if $M \to N$ is a monoid map, the map $M \to N \to N^{-1}N$ extends uniquely to a homomorphism from $M^{-1}M$ to $N^{-1}N$. Thus group completion is a functor from abelian monoids to abelian groups. A little decoding shows that in fact it is left adjoint to the forgetful functor, because of the natural isomorphism

$$\mathrm{Hom}_{\substack{\text{abelian}\\ \text{monoids}}}(M, A) \cong \mathrm{Hom}_{\substack{\text{abelian}\\ \text{groups}}}(M^{-1}M, A).$$

II.1.1 **Proposition 1.1.** *Let M be an abelian monoid. Then:*

(a) Every element of $M^{-1}M$ is of the form $[m]-[n]$ for some $m, n \in M$;

(b) If $m, n \in M$ then $[m] = [n]$ in $M^{-1}M$ if and only if $m + p = n + p$ for some $p \in M$;

(c) The monoid map $M \times M \to M^{-1}M$ sending (m, n) to $[m]$–$[n]$ is surjective.

(d) Hence $M^{-1}M$ is the set-theoretic quotient of $M \times M$ by the equivalence relation generated by $(m, n) \sim (m + p, n + p)$.

Proof. Every element of a free abelian group is a difference of sums of generators, and in $F(M)$ we have $([m_1] + [m_2] + \cdots) \equiv [m_1 + m_2 + \cdots]$ modulo $R(M)$. Hence every element of $M^{-1}M$ is a difference of generators. This establishes (a) and (c). For (b), suppose that $[m]-[n] = 0$ in $M^{-1}M$. Then in the free abelian group $F(M)$ we have

$$[m] - [n] = \sum ([a_i + b_i] - [a_i] - [b_i]) - \sum ([c_j + d_j] - [c_j] - [d_j]) .$$

Translating negative terms to the other side yields the following equation:

$$[m] + \sum([a_i] + [b_i]) + \sum[c_j + d_j] = [n] + \sum[a_i + b_i] + \sum([c_j] + [d_j]). \quad (1.1.1)$$ II.1.1.1

Now in a free abelian group two sums of generators $\sum[x_i]$ and $\sum[y_j]$ can only be equal if they have the same number of terms, and the generators differ by a permutation σ in the sense that $y_i = x_{\sigma(i)}$. Hence the generators on the left and right of (II.1.1.1 1.1.1) differ only by a permutation. This means that in M the sum of the terms on the left and right of (II.1.1.1 1.1.1) are the same, *i.e.,*

$$m + \sum(a_i + b_i) + \sum(c_j + d_j) = n + \sum(a_i + b_i) + \sum(c_j + d_j)$$

in M. This yields (b), and part (d) follows from (a) and (b). $\square$

The two corollaries below are immediate from Proposition II.1.1 1.1, given the following definitions. An (abelian) *cancellation monoid* is an abelian monoid M such that for all $m, n, p \in M$, $m+p = n+p$ implies $m = n$. A submonoid L of an abelian monoid M is called *cofinal* if for every $m \in M$ there is an $m' \in M$ so that $m + m' \in L$.

II.1.2 **Corollary 1.2.** *M injects into $M^{-1}M$ if and only if M is a cancellation monoid.*

II.1.3 **Corollary 1.3.** *If L is cofinal in an abelian monoid M, then:*

(a) $L^{-1}L$ is a subgroup of $M^{-1}M$;

(b) Every element of $M^{-1}M$ is of the form $[m] - [\ell]$ for some $m \in M$, $\ell \in L$;

(c) If $[m] = [m']$ in $M^{-1}M$ then $m + \ell = m' + \ell$ for some $\ell \in L$.

A *semiring* is an abelian monoid $(M, +)$, together with an associative product $\cdot$ which distributes over $+$, and a 2-sided multiplicative identity element 1. That is, a semiring satisfies all the axioms for a ring except for the existence of subtraction. The prototype semiring is $\mathbb{N}$.

The group completion $M^{-1}M$ (with respect to $+$) of a semiring M is a ring, the product on $M^{-1}M$ being extended from the product on M using II.1.1 1.1. If $M \to N$ is a semiring map, then the induced map $M^{-1}M \to N^{-1}N$ is a ring homomorphism. Hence group completion is also a functor from semirings to rings, and from commutative semirings to commutative rings.

II.1.4 **Example 1.4.** Let X be a topological space. The set $[X, \mathbb{N}]$ of continuous maps $X \to \mathbb{N}$ is a semiring under pointwise $+$ and $\cdot$. The group completion of $[X, \mathbb{N}]$ is the ring $[X, \mathbb{Z}]$ of all continuous maps $X \to \mathbb{Z}$.

If X is (quasi-)compact, $[X, \mathbb{Z}]$ is a free abelian group. Indeed, $[X, \mathbb{Z}]$ is a subgroup of the group S of all bounded set functions from X to $\mathbb{Z}$, and S is a free abelian group (S is a "Specker group"; see Fuchs [57]).

II.1.5 **Example 1.5** (Burnside Ring)**.** Let G be a finite group. The set M of (isomorphism classes of) finite G-sets is an abelian monoid under disjoint union, '0' being the empty set $\emptyset$. Suppose there are c distinct G-orbits. Since every G-set is a disjoint union of orbits, M is the free abelian monoid $\mathbb{N}^c$, a basis of M being the classes of the c distinct orbits of G. Each orbit is isomorphic to a coset G/H, where H is the stabilizer of an element, and $G/H \cong G/H'$ if and only if H and H' are conjugate subgroups of G, so c is the number of conjugacy classes of subgroups of G. Therefore the group completion $A(G)$ of M is the free abelian group $\mathbb{Z}^c$, a basis being the set of all c coset spaces $[G/H]$.

The direct product of two G-sets is again a G-set, so M is a semiring with '1' the 1-element G-set. Therefore $A(G)$ is a commutative ring; it is called the *Burnside ring* of G. The forgetful functor from G-sets to sets induces a map $M \to \mathbb{N}$ and hence an augmentation map $\epsilon\colon A(G) \to \mathbb{Z}$. For example, if G is cyclic of prime order p, then $A(G)$ is the ring $\mathbb{Z}[x]/(x^2 = px)$ and $x = [G]$ has $\epsilon(x) = p$.

II.1.6 **Example 1.6** (Representation ring). Let G be a finite group. The set $Rep_{\mathbb{C}}(G)$ of finite-dimensional representations $\rho\colon G \to GL_n(\mathbb{C})$ (up to isomorphism) is an abelian monoid under $\oplus$. By Maschke's Theorem, $\mathbb{C}[G]$ is semisimple and $Rep_{\mathbb{C}}(G) \cong \mathbb{N}^r$, where r is the number of conjugacy classes of elements of G. Therefore the group completion $R(G)$ of $Rep_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{Z}^r$ as an abelian group.

The tensor product $V \otimes_{\mathbb{C}} W$ of two representations is also a representation, so $Rep_{\mathbb{C}}(G)$ is a semiring (the element 1 is the 1-dimensional trivial representation). Therefore $R(G)$ is a commutative ring; it is called the *Representation ring* of G. For example, if G is cyclic of prime order p then $R(G)$ is isomorphic to the group ring $\mathbb{Z}[G]$, a subring of $\mathbb{Q}[G] = \mathbb{Q} \times \mathbb{Q}(\zeta)$, $\zeta^p = 1$.

Every representation is determined by its character $\chi\colon G \to \mathbb{C}$, and irreducible representations have linearly independent characters. Therefore $R(G)$ is isomorphic to the ring of all complex characters $\chi\colon G \to \mathbb{C}$, a subring of the ring $Map(G, \mathbb{C})$ of all functions $G \to \mathbb{C}$.

Definition. A (connected) *partially ordered abelian group* (A, P) is an abelian group A, together with a submonoid P of A which generates A (so $A = P^{-1}P$) and $P \cap (-P) = \{0\}$. This structure induces a translation-invariant partial ordering $\geq$ on A: $a \geq b$ if $a - b \in P$. Conversely, given a translation-invariant partial order on A, let P be $\{a \in A : a \geq 0\}$. If $a, b \geq 0$ then $a + b \geq a \geq 0$, so P is a submonoid of A. If P generates A then (A, P) is a partially ordered abelian group.

If M is an abelian monoid, $M^{-1}M$ need not be partially ordered (by the image of M), because we may have $[a] + [b] = 0$ for $a, b \in M$. However, interesting examples are often partially ordered. For example, the Burnside ring $A(G)$ and Representation ring $R(G)$ are partially ordered (by G-sets and representations).

When it exists, the ordering on $M^{-1}M$ is an extra piece of structure. For example, $\mathbb{Z}^r$ is the group completion of both $\mathbb{N}^r$ and $M = \{0\} \cup \{(n_1, ..., n_r) \in \mathbb{N}^r : n_1, ..., n_r > 0\}$. However, the two partially ordered structures on $\mathbb{Z}^r$ are different.

EXERCISES

EII.1.1 **1.1.** The group completion of a non-abelian monoid M is a group $\widehat{M}$, together with a monoid map $M \to \widehat{M}$ which is universal for maps from M to groups. Show that every monoid has a group completion in this sense, and that if M is abelian then $\widehat{M} = M^{-1}M$. If M is the free monoid on a set X, show that the group completion of M is the free group on the set X.

Note: The results in this section fail for non-abelian monoids. Proposition II.1.1 1.1 fails for the free monoid on X. Corollary II.1.2 1.2 can also fail: an example of a cancellation monoid M which does not inject into $\widehat{M}$ was given by Mal'cev in 1937.

EII.1.2 **1.2.** If $M = M_1 \times M_2$, show that $M^{-1}M$ is the product group $(M_1^{-1}M_1) \times (M_2^{-1}M_2)$.

EII.1.3 **1.3.** If M is the filtered colimit of abelian monoids M_α, show that $M^{-1}M$ is the filtered colimit of the abelian groups $M_\alpha^{-1}M_\alpha$.

EII.1.4 **1.4.** *Mayer-Vietoris for group completions.* Suppose that a sequence $L \to M_1 \times M_2 \to N$ of abelian monoids is "exact" in the sense that whenever $m_1 \in M_1$ and $m_2 \in M_2$ agree in N then m_1 and m_2 are the images of a common $\ell \in L$. If L is cofinal in M_1, M_2 and N, show that there is an exact sequence of groups $L^{-1}L \to (M_1^{-1}M_1) \oplus (M_2^{-1}M_2) \to N^{-1}N$, where the first map is the diagonal inclusion and the second map is the difference map $(m_1, m_2) \mapsto \bar{m}_1 - \bar{m}_2$.

EII.1.5 **1.5.** Classify all abelian monoids which are quotients of $\mathbb{N} = \{0, 1, \dots\}$ and show that they are all finite. How many quotient monoids $M = \mathbb{N}/\sim$ of $\mathbb{N}$ have m elements and group completion $\widehat{M} = \mathbb{Z}/n\mathbb{Z}$?

EII.1.6 **1.6.** Here is another description of the Burnside ring $A(G)$ of a finite group G. For each subgroup H, and finite G-set X, let $\chi_H(X)$ denote the cardinality of X^H.

(a) Show that χ_H defines a ring homomorphism $A(G) \to \mathbb{Z}$, and $\epsilon = \chi_1$.

(b) Deduce that the product χ of the χ_H (over the c conjugacy classes of subgroups) induces an injection of $A(G)$ into the product ring $\prod_1^c \mathbb{Z}$.

(c) Conclude that $A(G) \otimes \mathbb{Q} \cong \prod_1^c \mathbb{Q}$.

EII.1.7 **1.7** (T-Y Lam)**.** Let $\phi : G \to H$ be a homomorphism of finite groups. Show that the restriction functor from H-sets to G-sets ($gx = \phi(g)x$) induces a ring homomorphism $\phi^* : A(H) \to A(G)$. If X is a G-set, we can form the H-set $H \times_G X = H \times X/\{(h, gx) \sim (h\phi(g), x))\}$. Show that $H \times_G$ induces a group homomorphism $\phi_* : A(G) \to A(H)$. If ϕ is an injection, show that the *Frobenius Reciprocity* formula holds: $\phi_*(\phi^*(x) \cdot y) = x \cdot \phi_*(y)$ for all $x \in A(H)$, $y \in A(G)$.

2 K_0 of a ring

Let R be a ring. The set $\mathbf{P}(R)$ of isomorphism classes of finitely generated projective R-modules, together with direct sum $\oplus$ and identity 0, forms an abelian monoid. The *Grothendieck group of* R, $K_0(R)$, is the group completion $\mathbf{P}^{-1}\mathbf{P}$ of $\mathbf{P}(R)$.

When R is commutative, $K_0(R)$ is a commutative ring with $1 = [R]$, because the monoid $\mathbf{P}(R)$ is a commutative semiring with product $\otimes_R$. This follows from the following facts: $\otimes$ distributes over $\oplus$; $P \otimes_R Q \cong Q \otimes_R P$ and $P \otimes_R R \cong P$; if P, Q are finitely generated projective modules then so is $P \otimes_R Q$ (by Ex. I.2.7).

For example, let k be a field or division ring. Then the abelian monoid $\mathbf{P}(k)$ is isomorphic to $\mathbb{N} = \{0, 1, 2, \dots\}$, so $K_0(k) = \mathbb{Z}$. The same argument applies to show that $K_0(R) = \mathbb{Z}$ for every local ring R by I.2.2, and also for every PID (by the Structure Theorem for modules over a PID). In particular, $K_0(\mathbb{Z}) = \mathbb{Z}$.

The *Eilenberg Swindle* I.2.8 shows why we restrict to finitely generated projectives. If we had included the module R^∞ (defined in Ex. I.1.7), then the formula $P \oplus R^\infty \cong R^\infty$ would imply that $[P] = 0$ for every finitely generated projective R-module, and we would have $K_0(R) = 0$.

K_0 is a functor from rings to abelian groups, and from commutative rings to commutative rings. To see this, suppose that $R \to S$ is a ring homomorphism. The functor $\otimes_R S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ (sending P to $P \otimes_R S$) yields a monoid map $\mathbf{P}(R) \to \mathbf{P}(S)$, hence a group homomorphism $K_0(R) \to K_0(S)$. If R, S are commutative rings then $\otimes_R S \colon K_0(R) \to K_0(S)$ is a ring homomorphism, because $\otimes_R S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ is a semiring map:

$$(P \otimes_R Q) \otimes_R S \cong (P \otimes_R S) \otimes_S (Q \otimes_R S).$$

The free modules play a special role in understanding $K_0(R)$ because they are cofinal in $\mathbf{P}(R)$. By Corollary 1.3 every element of $K_0(R)$ can be written as $[P] - [R^n]$ for some P and n. Moreover, $[P] = [Q]$ in $K_0(R)$ if and only if P, Q are stably isomorphic: $P \oplus R^m \cong Q \oplus R^m$ for some m. In particular, $[P] = [R^n]$ if and only if P is stably free. The monoid L of isomorphism classes of free modules is $\mathbb{N}$ if and only if R satisfies the Invariant Basis Property of Chapter I, §1. This yields the following information about $K_0(R)$.

II.2.1 **Lemma 2.1.** *The monoid map $\mathbb{N} \to \mathbf{P}(R)$ sending n to R^n induces a group homomorphism $\mathbb{Z} \to K_0(R)$. We have:*

(1) $\mathbb{Z} \to K_0(R)$ is injective if and only if R satisfies the Invariant Basis Property (IBP);

(2) Suppose that R satisfies the IBP (e.g., R is commutative). Then

$$K_0(R) \cong \mathbb{Z} \iff \text{every finitely generated projective } R\text{-module is stably free.}$$

II.2.1.1 **Example 2.1.1.** Suppose that R is commutative, or more generally that there is a ring map $R \to F$ to a field F. In this case $\mathbb{Z}$ is a direct summand of $K_0(R)$, because the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ takes $[R]$ to 1. A ring with $K_0(R) = \mathbb{Q}$ is given in Exercise 2.12 below.

II.2.1.2 **Example 2.1.2** (Simple rings)**.** Consider the matrix ring $R = M_n(F)$ over a field F. We saw in Example I.1.1.1 that every R-module is projective (because it is a sum of copies of the projective module $V \cong F^n$), and that length is an invariant of finitely generated R-modules. Thus *length* is an abelian group isomorphism $K_0(M_n(F)) \xrightarrow{\cong} \mathbb{Z}$ sending $[V]$ to 1. Since R has length n, the subgroup of $K_0(R) \cong \mathbb{Z}$ generated by the free modules has index n. In particular, the inclusion $\mathbb{Z} \subset K_0(R)$ of Lemma 2.1 does not split.

II.2.1.3 **Example 2.1.3.** (Karoubi) We say a ring R is *flasque* if there is an R-bimodule M, finitely generated projective as a right module, and a bimodule isomorphism $\theta : R \oplus M \cong M$. If R is flasque then $K_0(R) = 0$. This is because for every P we have a natural isomorphism $P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong (P \otimes_R M)$.

If R is flasque and the underlying right R-module structure on M is R, we say that R is an *infinite sum ring.* The right module isomorphism $R^2 \cong R$ underlying θ makes R a direct sum ring (Ex. I.1.7). The cone rings of Ex. I.1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.1.7, are examples of infinite sum rings, and hence flasque rings; see Exercise 2.15.

If $R = R_1 \times R_2$ then $\mathbf{P}(R) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$. As in Exercise 1.2, this implies that $K_0(R) \cong K_0(R_1) \times K_0(R_2)$. Thus K_0 may be computed componentwise.

II.2.1.4 **Example 2.1.4** (Semisimple rings)**.** Let R be a semisimple ring, with simple modules $V_1, ..., V_r$ (see Ex. I.1.1). Schur's Lemma states that each $D_i = \mathrm{Hom}_R(V_i, V_i)$ is a division ring; the Artin-Wedderburn Theorem states that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $\dim_{D_i}(V_i) = n_i$. By 2.1.2, $K_0(R) \cong \prod K_0(M_{n_i}(D_i)) \cong \mathbb{Z}^r$.

Another way to see that $K_0(R) \cong \mathbb{Z}^r$ is to use the fact that $\mathbf{P}(R) \cong \mathbb{N}^r$: the Krull-Schmidt Theorem states that every finitely generated (projective) module M is $V_1^{\ell_1} \times \cdots \times V_r^{\ell_r}$ for well-defined integers $\ell_1, ..., \ell_r$.

II.2.1.5 **Example 2.1.5** (Von Neumann regular rings)**.** A ring R is said to be *von Neumann regular* if for every $r \in R$ there is an $x \in R$ such that $rxr = r$. Since $rxrx = rx$, the element $e = rx$ is idempotent, and the ideal $rR = eR$ is a projective module. In fact, every finitely generated right ideal of R is of the form eR for some idempotent, and these form a lattice. Declaring $e \simeq e'$ if $eR = e'R$, the equivalence classes of idempotents in R form a lattice: $(e_1 \wedge e_2)$ and $(e_1 \vee e_2)$ are defined to be the idempotents generating $e_1R + e_2R$ and $e_1R \cap e_2R$, respectively. Kaplansky proved in [98] that every projective R-module is a direct sum of the modules eR. It follows that $K_0(R)$ is determined by the lattice of idempotents (modulo $\simeq$) in R. We will see several examples of von Neumann regular rings in the exercises.

Many von Neumann regular rings do not satisfy the (IBP), the ring $\mathrm{End}_F(F^\infty)$ of Ex. I.1.7 being a case in point.

We call a ring R *unit-regular* if for every $r \in R$ there is a unit $x \in R$ such that $rxr = r$. Every unit-regular ring is von Neumann regular, has stable range 1, and satisfies the (IBP) (Ex. I.1.13). In particular, $\mathbb{Z} \subseteq K_0(R)$. It is unknown whether for every simple unit-regular ring R the group $K_0(R)$ is *strictly unperforated*, meaning that whenever $x \in K_0(R)$ and $nx = [Q]$ for some Q, then $x = [P]$ for some P. Goodearl [72] has given examples of simple unit-regular rings R in which the group $K_0(R)$ is strictly unperforated, but has torsion.

An example of a von Neumann regular ring R having the IBP, stable range 2 and $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}/n$ is given in [123].

II.2.1.6 **2.1.6.** Suppose that R is the direct limit of a filtered system $\{R_i\}$ of rings. Then every finitely generated projective R-module is of the form $P_i \otimes_{R_i} R$ for some i and some finitely generated projective R_i-module P_i. Any isomorphism $P_i \otimes_{R_i} R \cong P'_i \otimes_{R_i} R$ may be expressed using finitely many elements of R, and hence $P_i \otimes_{R_i} R_j \cong P'_i \otimes_{R_i} R_j$ for some j. That is, $\mathbf{P}(R)$ is the filtered colimit of the $\mathbf{P}(R_i)$. By Ex. 1.3 we have

$$K_0(R) \cong \varinjlim K_0(R_i).$$

This observation is useful when studying $K_0(R)$ of a commutative ring R, because R is the direct limit of its finitely generated subrings. As finitely generated commutative rings are noetherian with finite normalization, properties of $K_0(R)$ may be deduced from properties of K_0 of these nice subrings. If R is integrally closed we may restrict to finitely generated normal subrings, so $K_0(R)$ is determined by K_0 of noetherian integrally closed domains.

Here is another useful reduction; it follows immediately from the observation that if I is nilpotent (or complete) then idempotent lifting (Ex. I.2.2) yields a monoid isomorphism $\mathbf{P}(R) \cong \mathbf{P}(R/I)$. Recall that an ideal I is said to be *complete* if every Cauchy sequence $\sum_{n=1}^{\infty} x_n$ with $x_n \in I^n$ converges to a unique element of I.

II.2.2 **Lemma 2.2.** *If I is a nilpotent ideal of R, or more generally a complete ideal, then*

$$K_0(R) \cong K_0(R/I).$$

In particular, if R is commutative then $K_0(R) \cong K_0(R_{red})$.

II.2.2.1 **Example 2.2.1** (0-dimensional commutative rings)**.** Let R be a commutative ring. It is elementary that R_{red} is Artinian if and only if $\mathrm{Spec}(R)$ is finite and discrete. More generally, it is known (see Ex. I.1.13 and [8, Ex. 3.11]) that the following are equivalent:

(i) R_{red} is a commutative von Neumann regular ring (2.1.5);

(ii) R has Krull dimension 0;

(iii) $X = \mathrm{Spec}(R)$ is compact, Hausdorff and totally disconnected.

(For example, to see that a commutative von Neumann regular R must be reduced, observe that if $r^2 = 0$ then $r = rxr = 0$.)

When R is a commutative von Neumann regular ring, the modules eR are componentwise free; Kaplansky's result states that every projective module is componentwise free. By §I.2, the monoid $\mathbf{P}(R)$ is just $[X, \mathbb{N}]$, $X = \mathrm{Spec}(R)$. By 1.4 this yields $K_0(R) = [X, \mathbb{Z}]$. By Lemma 2.2, this proves

II.2.2.2 **Pierce's Theorem 2.2.2.** *For every 0-dimensional commutative ring R:*

$$K_0(R) = [\mathrm{Spec}(R), \mathbb{Z}].$$

II.2.2.3 **Example 2.2.3** (K_0 does not commute with infinite products)**.** Suppose that $R = \prod F_i$ is an infinite product of fields. Then R is von Neumann regular, so $X = \mathrm{Spec}(R)$ is an uncountable totally disconnected compact Hausdorff space. By Pierce's Theorem, $K_0(R) \cong [X, \mathbb{Z}]$. This is contained in but not equal to the product $\prod K_0(F_i) \cong \prod \mathbb{Z}$.

Rank and H_0

Definition. When R is commutative, we write $H_0(R)$ for $[\mathrm{Spec}(R), \mathbb{Z}]$, the ring of all continuous maps from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Since $\mathrm{Spec}(R)$ is quasi-compact, we know by 1.4 that $H_0(R)$ is always a free abelian group. If R is a noetherian ring, then $\mathrm{Spec}(R)$ has only finitely many (say c) components, and $H_0(R) \cong \mathbb{Z}^c$. If R is a domain, or more generally if $\mathrm{Spec}(R)$ is connected, then $H_0(R) = \mathbb{Z}$.

$H_0(R)$ is a subring of $K_0(R)$. To see this, consider the submonoid L of $\mathbf{P}(R)$ consisting of componentwise free modules R^f. Not only is L cofinal in $\mathbf{P}(R)$, but $L \to \mathbf{P}(R)$ is a semiring map: $R^f \otimes R^g \cong R^{fg}$; by 1.3, $L^{-1}L$ is a subring of $K_0(R)$. Finally, L is isomorphic to $[\mathrm{Spec}(R), \mathbb{N}]$, so as in 1.4 we have $L^{-1}L \cong H_0(R)$. For example, Pierce's theorem 2.2.2 states that if $\dim(R) = 0$ then $K_0(R) \cong H_0(R)$.

Recall from §I.2 that the rank of a projective module gives a map from $\mathbf{P}(R)$ to $[\mathrm{Spec}(R), \mathbb{N}]$. Since $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$ and $\mathrm{rank}(P \otimes Q) = \mathrm{rank}(P)\,\mathrm{rank}(Q)$ (by Ex. I.2.7, this is a semiring map. As such it induces a ring map

$$\mathrm{rank}\colon K_0(R) \to H_0(R).$$

Since $\mathrm{rank}(R^f) = f$ for every componentwise free module, the composition $H_0(R) \subset K_0(R) \to H_0(R)$ is the identity. Thus $H_0(R)$ is a direct summand of $K_0(R)$.

II.2.3 **Definition 2.3.** The ideal $\widetilde{K}_0(R)$ of the ring $K_0(R)$ is defined as the kernel of the rank map. By the above remarks, there is a natural decomposition

$$K_0(R) \cong H_0(R) \oplus \widetilde{K}_0(R).$$

We will see later (in 4.6.1) that $\widetilde{K}_0(R)$ is a nil ideal. Since $H_0(R)$ is visibly a reduced ring, $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$.

II.2.3.1 **Lemma 2.3.1.** *If R is commutative, let $\mathbf{P}_n(R)$ denote the subset of all modules in $\mathbf{P}(R)$ consisting of projective modules of constant rank n. There is a map $\mathbf{P}_n(R) \to K_0(R)$ sending P to $[P] - [R^n]$. This map is compatible with the stabilization map $\mathbf{P}_n(R) \to \mathbf{P}_{n+1}(R)$ sending P to $P \oplus R$, and the induced map is an isomorphism:*

$$\varinjlim \mathbf{P}_n(R) \cong \widetilde{K}_0(R).$$

Proof. This follows easily from Corollary II.1.3 1.3. □

II.2.3.2 **Corollary 2.3.2.** *Let R be a commutative noetherian ring of Krull dimension d — or more generally any commutative ring of stable range $d+1$ (Ex. I.EI.1.5 1.5). For every $n > d$ the above maps are bijections: $\mathbf{P}_n(R) \cong \widetilde{K}_0(R)$.*

Proof. If P and Q are finitely generated projective modules of rank $> d$, then by Bass Cancellation (I.I.2.3 2.3b) we may conclude that

$$[P] = [Q] \text{ in } K_0(R) \quad \text{if and only if} \quad P \cong Q. \qquad \square$$

Here is another interpretation of $\widetilde{K}_0(R)$: it is the intersection of the kernels of $K_0(R) \to K_0(F)$ over all maps $R \to F$, F a field. This follows from naturality of rank and the observation that $\widetilde{K}_0(F) = 0$ for every field F.

This motivates the following definition for a noncommutative ring R: let $\widetilde{K}_0(R)$ denote the intersection of the kernels of $K_0(R) \to K_0(S)$ over all maps $R \to S$, where S is a simple artinian ring. If no such map $R \to S$ exists, we set $\widetilde{K}_0(R) = K_0(R)$. We define $H_0(R)$ to be the quotient of $K_0(R)$ by $\widetilde{K}_0(R)$. When R is commutative, this agrees with the above definitions of H_0 and $\widetilde{K}_0$, because the maximal commutative subrings of a simple artinian ring S are finite products of 0-dimensional local rings.

$H_0(R)$ is a torsionfree abelian group for every ring R. To see this, note that there is a set X of maps $R \to S_x$ through which every other $R \to S'$ factors. Since each $K_0(S_x) \to K_0(S')$ is an isomorphism, $\widetilde{K}_0(R)$ is the intersection of the kernels of the maps $K_0(R) \to K_0(S_x)$, $x \in X$. Hence $H_0(R)$ is the image of $K_0(R)$ in the torsionfree group $\prod_{x \in X} K_0(S_x) \cong \prod_x \mathbb{Z} \cong Map(X, \mathbb{Z})$.

II.2.4 **Example 2.4** (Whitehead group Wh_0)**.** If R is the group ring $\mathbb{Z}[G]$ of a group G, Whitehead group—Wh_0 the (zero-th) *Whitehead group* $Wh_0(G)$ is the quotient of $K_0(\mathbb{Z}[G])$ by the subgroup $K_0(\mathbb{Z}) = \mathbb{Z}$. The augmentation map $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ sending G to 1 induces a decomposition $K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \oplus Wh_0(G)$, and clearly $\widetilde{K}_0(\mathbb{Z}[G]) \subseteq Wh_0(G)$. It follows from a theorem of Swan ([Bass, XI(5.2)]) that if G is finite then $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ and $H_0(\mathbb{Z}G) = \mathbb{Z}$. The author does not know whether $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ for every group.

The group $Wh_0(G)$ arose in topology via the following result of C.T.C. Wall. We say that a CW complex X is *dominated* by a complex K if there is a map $f\colon K \to X$ having a right homotopy inverse; this says that X is a retract of K in the homotopy category.

II.2.4.1 **Theorem 2.4.1** (Wall's Finiteness Obstruction)**.** *Whitehead group—Wh_0 Suppose that X is dominated by a finite CW complex, with fundamental group $G = \pi_1(X)$. This data determines an element $w(X)$ of $Wh_0(G)$ such that $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.*

Hattori-Stallings trace map

For any associative ring R, let $[R, R]$ denote the subgroup of R generated by the elements $[r, s] = rs - sr$, $r, s \in R$.

For each n, the trace of an $n\times n$ matrix provides an additive map from $M_n(R)$ to $R/[R, R]$ invariant under conjugation; the inclusion of $M_n(R)$ in $M_{n+1}(R)$ via $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is compatible with the trace map. It is not hard to show that the trace $M_n(R) \to R/[R, R]$ induces an isomorphism:

$$M_n(R)/[M_n(R), M_n(R)] \cong R/[R, R].$$

If P is a finitely generated projective module, choosing an isomorphism $P \oplus Q \cong R^n$ yields an idempotent e in $M_n(R)$ such that $P = e(R^n)$ and $\mathrm{End}(P) = eM_n(R)e$. By Ex. I.2.3, any other choice yields an e_1 which is conjugate to e in some larger $M_m(R)$. Therefore the trace of an endomorphism of P is a well-defined element of $R/[R, R]$, independent of the choice of e. This gives the *trace map* $\mathrm{End}(P) \to R/[R, R]$. In particular, the trace of the identity map of P is the trace of e; we call it the *trace* of P.

If P' is represented by an idempotent matrix f then $P \oplus P'$ is represented by the idempotent matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$ so the trace of $P \oplus P'$ is $\mathrm{trace}(P) + \mathrm{trace}(P')$. Therefore the trace is an additive map on the monoid $\mathbf{P}(R)$. The map $K_0(R) \to R/[R, R]$ induced by universality is called the *Hattori-Stallings trace map*, after the two individuals who first studied it.

When R is commutative, we can provide a direct description of the ring map $H_0(R) \to R$ obtained by restricting the trace map to the subring $H_0(R)$ of $K_0(R)$. Any continuous map $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ induces a decomposition $R = R_1 \times \cdots \times R_c$ by Ex. I.2.4; the coordinate idempotents $e_1, ..., e_c$ are elements of R. Since $\mathrm{trace}(e_iR)$ is e_i, it follows immediately that $\mathrm{trace}(f)$ is $\sum f(i)e_i$. The identity $\mathrm{trace}(fg) = \mathrm{trace}(f)\mathrm{trace}(g)$ which follows immediately from this formula shows that trace is a ring map.

II.2.5 **Proposition 2.5.** *If R is commutative, the Hattori-Stallings trace factors as*

$$K_0(R) \xrightarrow{\text{rank}} H_0(R) \to R.$$

Proof. The product over all $\mathfrak{p}$ in $\mathrm{Spec}(R)$ yields the commutative diagram:

$$\begin{array}{ccc} K_0(R) & \longrightarrow & \prod K_0(R_{\mathfrak{p}}) \\ \text{trace}\downarrow & & \downarrow\text{trace} \\ R & \xrightarrow[\text{inclusion}]{\text{diagonal}} & \prod R_{\mathfrak{p}}. \end{array}$$

The kernel of the top arrow is $\widetilde{K}_0(R)$, so the left arrow factors as claimed. $\square$

II.2.5.1 **Example 2.5.1** (Group rings)**.** Let k be a commutative ring, and suppose that R is the group ring $k[G]$ of a group G. If g and h are conjugate elements of G then $h - g \in [R, R]$ because $xgx^{-1} - g = [xg, x^{-1}]$. From this it is not hard to see that $R/[R, R]$ is isomorphic to the free k-module $\oplus k[g]$ whose basis is the set $G/\sim$ of conjugacy classes of elements of G. Relative to this basis, we may write

$$\mathrm{trace}(P) = \sum r_P(g)[g].$$

Clearly, the coefficients $r_P(g)$ of trace(P) are functions of the set $G/\sim$ for each P.

If G is finite, then any finitely generated projective k[G]-module P is also a projective k-module, and we may also form the trace map $\mathrm{End}_k(P) \to k$ and hence the "character" $\chi_P\colon G \to k$ by the formula $\chi_P(g) = \mathrm{trace}(g)$. Hattori proved that if $Z_G(g)$ denotes the centralizer of $g \in G$ then *Hattori's formula* holds (see Bass76 [16, 5.8]):

$$\chi_P(g) = |Z_G(g)|\ r_P(g^{-1}). \qquad (2.5.2)$$ II.2.5.2

II.2.5.3 **Corollary 2.5.3.** *If G is a finite group, the ring $\mathbb{Z}[G]$ has no idempotents except 0 and 1.*

Proof. Let e be an idempotent element of $\mathbb{Z}[G]$. $\chi_P(1)$ is the rank of the $\mathbb{Z}$-module $P = e\,\mathbb{Z}[G]$, which must be less than the rank $|G|$ of $\mathbb{Z}[G]$. Since $r_P(1) \in \mathbb{Z}$, this contradicts Hattori's formula $\chi_P(1) = |G|\, r_P(1)$. □

Bass has conjectured that for every group G and every finitely generated projective $\mathbb{Z}[G]$-module P we have $r_P(g) = 0$ for $g \neq 1$ and $r_P(1) = \mathrm{rank}_{\mathbb{Z}}(P \otimes_{\mathbb{Z}[G]} \mathbb{Z})$. For G finite, this follows from Hattori's formula and Swan's theorem (cited in II.2.4 2.4) that $\widetilde{K}_0 = Wh_0$. See Bass76 [16].

II.2.5.4 **Example 2.5.4.** Suppose that k is a field of characteristic 0 and $kG = k[G]$ is the group ring of a finite group G with c conjugacy classes. By Maschke's theorem, kG is a product of simple k-algebras: $S_1 \times \cdots \times S_c$, so $kG/[kG, kG]$ is k^c. By II.2.1.4 2.1.4 $K_0(kG) \cong \mathbb{Z}^c$. Hattori's formula (and some classical representation theory) shows that the trace map from $K_0(kG)$ to $kG/[kG, kG]$ is isomorphic to the natural inclusion of $\mathbb{Z}^c$ in k^c.

Determinant

Suppose now that R is a commutative ring. Recall from §I.3 that the determinant of a finitely generated projective module P is an element of the Picard group Pic(R).

II.2.6 **Proposition 2.6.** *The determinant induces a surjective group homomorphism*

$$\det\colon K_0(R) \to \mathrm{Pic}(R)$$

Proof. By the universal property of K_0, it suffices to show that $\det(P \oplus Q) \cong \det(P) \otimes_R \det(Q)$. We may assume that P and Q have constant rank m and n, respectively. Then $\wedge^{m+n}(P \oplus Q)$ is the sum over all i, j such that $i + j = m + n$ of $(\wedge^i Q) \otimes (\wedge^j P)$. If $i > m$ or $j > n$ we have $\wedge^i P = 0$ or $\wedge^j Q = 0$, respectively. Hence $\wedge^{m+n}(P \oplus Q) = (\wedge^m P) \otimes (\wedge^n Q)$, as asserted. □

II.2.6.1 **Definition 2.6.1.** Let $SK_0(R)$ denote the subset of $K_0(R)$ consisting of the classes $x = [P] - [R^m]$, where P has constant rank m and $\wedge^m P \cong R$. This is the kernel of $\det\colon \widetilde{K}_0(R) \to \operatorname{Pic}(R)$, by Lemma 2.3.1 and Proposition 2.6.

$SK_0(R)$ is an ideal of $K_0(R)$. To see this, we use Exercise I.3.4: if $x = [P] - [R^m]$ is in $SK_0(R)$ and Q has rank n then $\det(x \cdot Q)$ equals $(\det P)^{\otimes n}(\det Q)^{\otimes m}(\det Q)^{\otimes -m} = R$.

II.2.6.2 **Corollary 2.6.2.** *For every commutative ring R, $H_0(R) \oplus \operatorname{Pic}(R)$ is a ring with square-zero ideal $\operatorname{Pic}(R)$, and there is a surjective ring homomorphism with kernel $SK_0(R)$:*

$$\operatorname{rank} \oplus \det\colon K_0(R) \to H_0(R) \oplus \operatorname{Pic}(R)$$

II.2.6.3 **Corollary 2.6.3.** *If R is a 1-dimensional commutative noetherian ring, then the classification of finitely generated projective R-modules in I.3.4 induces an isomorphism:*

$$K_0(R) \cong H_0(R) \oplus \operatorname{Pic}(R).$$

Morita Equivalence

We say that two rings R and S are *Morita equivalent* if **mod-**R and **mod-**S are equivalent as abelian categories, that is, if there exist additive functors T and U

$$\textbf{mod-}R \underset{U}{\overset{T}{\rightleftarrows}} \textbf{mod-}S$$

such that $UT \cong \mathrm{id}_R$ and $TU \cong \mathrm{id}_S$. This implies that T and U preserve filtered colimits. Set $P = T(R)$ and $Q = U(S)$; P is an R-S bimodule and Q is a S–R bimodule via the maps $R = \operatorname{End}_R(R) \xrightarrow{T} \operatorname{End}_S(P)$ and $S = \operatorname{End}_S(S) \xrightarrow{U} \operatorname{End}_R(Q)$. Since $T(\oplus R) = \oplus P$ and $U(\oplus S) = \oplus Q$ it follows that we have $T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$ for all M, N. Both $UT(R) \cong P \otimes_S Q \cong R$ and $TU(S) \cong Q \otimes_R P \cong S$ are bimodule isomorphisms. Here is the main structure theorem, taken from [15, II.3].

II.2.7 **Theorem 2.7** (Structure Theorem for Morita Equivalence)**.** *If R and S are Morita equivalent, and P, Q are as above, then:*

(a) P and Q are finitely generated projective, both as R-modules and as S-modules;

(b) $End_S(P) \cong R \cong End_S(Q)^{op}$ and $End_R(Q) \cong S \cong End_R(P)^{op}$;

(c) P and Q are dual S-modules: $P \cong \operatorname{Hom}_S(Q, S)$ and $Q \cong \operatorname{Hom}_S(P, S)$;

(d) $T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$ for every M and N;

(e) P is a "faithful" S-module in the sense that the functor $\operatorname{Hom}_S(P, -)$ from **mod-**S *to abelian groups is a faithful functor. (If S is commutative then P is faithful if and only if $\operatorname{rank}(P) \geq 1$.) Similarly, Q is a "faithful" R-module.*

Since P and Q are finitely generated projective, the Morita functors T and U also induce an equivalence between the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$. This implies the following:

II.2.7.1 **Corollary 2.7.1.** *If R and S are Morita equivalent then $K_0(R) \cong K_0(S)$.*

II.2.7.2 **Example 2.7.2.** $R = M_n(S)$ is always Morita equivalent to S; P is the bimodule S^n of "column vectors" and Q is the bimodule $(S^n)^t$ of "row vectors." More generally suppose that P is a "faithful" finitely generated projective S-module. Then $R = \mathrm{End}_S(P)$ is Morita equivalent to S, the bimodules being P and $Q = \mathrm{Hom}_S(P, S)$. By 2.7.1, we see that $K_0(S) \cong K_0(M_n(S))$.

II.2.8 **Remark 2.8** (Additive Functors)**.** Any R-S bimodule P which is finitely generated projective as a right S-module, induces an additive (hence exact) functor $T(M) = M \otimes_R P$ from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, and therefore induces a map $K_0(R) \to K_0(S)$. If all we want is an additive functor T from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, we do not need the full strength of Morita equivalence. Given T, set $P = T(R)$. By additivity we have $T(R^n) = P^n \cong R^n \otimes_R P$; from this it is not hard to see that $T(M) \cong M \otimes_R P$ for every finitely generated projective M, and that T is isomorphic to $- \otimes_R P$. See Ex. 2.14 for more details.

A bimodule map (resp., isomorphism) $P \to P'$ induces an additive natural transformation (resp., isomorphism) $T \to T'$. This is the case, for example, with the bimodule isomorphism $R \oplus M \cong M$ defining a flasque ring (2.1.3).

II.2.8.1 **Example 2.8.1** (Base change and Transfer maps)**.** Suppose that $R \xrightarrow{f} S$ is a ring map. Then S is an R–S bimodule, and it represents the *base change functor* $f^*\colon K_0(R) \to K_0(S)$ sending P to $P \otimes_R S$. If in addition S is finitely generated projective as a right R-module then there is a forgetful functor from $\mathbf{P}(S)$ to $\mathbf{P}(R)$; it is represented by S as a S–R bimodule because it sends Q to $Q \otimes_S S$. The induced map $f_*\colon K_0(S) \to K_0(R)$ is called the *transfer map*. We will return to this point in 7.9 below, explaining why we have selected the contravariant notation f^* and f_*.

Mayer-Vietoris sequences

For any ring R with unit, we can include $GL_n(R)$ in $GL_{n+1}(R)$ as the matrices $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. The group $GL(R)$ is the union of the groups $GL_n(R)$. Now suppose we are given a Milnor square of rings, as in §I.2:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

Define $\partial_n\colon GL_n(S/I) \to K_0(R)$ by Milnor patching: $\partial_n(g)$ is $[P] - [R^n]$, where P is the projective R-module obtained by patching free modules along g as in I.2.6.

The formulas of Ex. I.2.9 imply that $\partial_n(g) = \partial_{n+1}\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\partial_n(g) + \partial_n(h) = \partial_n(gh)$. Therefore the $\{\partial_n\}$ assemble to give a group homomorphism ∂ from $GL(S/I)$ to $K_0(R)$. The following result now follows from I.2.6 and Ex. I.4.

II.2.9 **Theorem 2.9** (Mayer-Vietoris)**.** *Given a Milnor square as above, the sequence*

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I)$$

is exact. The image of ∂ *is the double coset space*

$$GL(S)\backslash GL(S/I)/GL(R/I) = GL(S/I)/\sim$$

where $x \sim gxh$ *for* $x \in GL(S/I), g \in GL(S)$ *and* $h \in GL(R/I)$*.*

II.2.9.1 **Example 2.9.1.** If R is the coordinate ring of the node over a field k (I.3.10.2) then $K_0(R) \cong \mathbb{Z} \oplus k^\times$. If R is the coordinate ring of the cusp over k (I.3.10.1) then $K_0(R) \cong \mathbb{Z} \oplus k$. Indeed, the coordinate rings of the node and the cusp are 1-dimensional noetherian rings, so 2.6.3 reduces the Mayer-Vietoris sequence to the Units-Pic sequence I.3.10.

We conclude with a useful construction, anticipating several later developments.

II.2.10 **Definition 2.10.** Let $T : \mathbf{P}(R) \to \mathbf{P}(S)$ be an additive functor, such as the base change or transfer of 2.8.1. $\mathbf{P}(T)$ is the category whose objects are triples (P, α, Q), where $P, Q \in \mathbf{P}(R)$ and $\alpha : T(P) \to T(Q)$ is an isomorphism. A morphism $(P, \alpha, Q) \to (P', \alpha', Q')$ is a pair of R-module maps $p : P \to P'$, $q : Q \to Q'$ such that $\alpha' T(p) = T(q)\alpha$. An *exact sequence* in $\mathbf{P}(T)$ is a sequence

$$0 \to (P', \alpha', Q') \to (P, \alpha, Q) \to (P'', \alpha'', Q'') \to 0 \qquad (2.10.1)$$ II.2.10.1

whose underlying sequences $0 \to P' \to P \to P'' \to 0$ and $0 \to Q' \to Q \to Q'' \to 0$ are exact. We define $K_0(T)$ to be the abelian group with generators the objects of $\mathbf{P}(T)$ and relations:

(a) $[(P, \alpha, Q)] = [(P', \alpha', Q')] + [(P'', \alpha'', Q'')]$ for every exact sequence (2.10.1);

(b) $[(P_1, \alpha, P_2)] + [(P_2, \beta, P_3)] = [(P_1, \beta\alpha, P_3)]$.

If T is the base change f^*, we write $K_0(f)$ for $K_0(T)$.

It is easy to see that there is a map $K_0(T) \to K_0(R)$ sending $[(P, \alpha, Q)]$ to $[P] - [Q]$. If T is a base change functor f^* associated to $f : R \to S$, or more generally if the $T(R^n)$ are cofinal in $\mathbf{P}(S)$, then there is an exact sequence:

$$GL(S) \xrightarrow{\partial} K_0(T) \to K_0(R) \to K_0(S). \qquad (2.10.2)$$ II.2.10.2

The construction of ∂ and verification of exactness is not hard, but lengthy enough to relegate to Exercise 2.17. If $f : R \to R/I$ then $K_0(f^*)$ is the group $K_0(I)$ of Ex. 2.4; see Ex. 2.4(e).

EXERCISES

EII.2.1 **2.1.** Let R be a commutative ring. If A is an R-algebra, show that the functor $\otimes_R\colon \mathbf{P}(A)\times\mathbf{P}(R)\to\mathbf{P}(A)$ yields a map $K_0(A)\otimes_{\mathbb{Z}} K_0(R)\to K_0(A)$ making $K_0(A)$ into a $K_0(R)$-module. If $A\to B$ is an algebra map, show that $K_0(A)\to K_0(B)$ is a $K_0(R)$-module homomorphism.

EII.2.2 **2.2.** *Projection Formula.* Let R be a commutative ring, and A an R-algebra which as an R-module is finitely generated projective of rank n. By Ex. 2.1, $K_0(A)$ is a $K_0(R)$-module, and the base change map $f^*\colon K_0(R)\to K_0(A)$ is a module homomorphism. We shall write $x\cdot f^*y$ for the product in $K_0(A)$ of $x\in K_0(A)$ and $y\in K_0(R)$; this is an abuse of notation when A is noncommutative.

(a) Show that the transfer map $f_*\colon K_0(A)\to K_0(R)$ of Example 2.8.1 is a $K_0(R)$-module homomorphism, *i.e.,* that the *projection formula* holds:

$$f_*(x\cdot f^*y)=f_*(x)\cdot y \qquad \text{for every } \ x\in K_0(A),\ y\in K_0(R).$$

(b) Show that both compositions f^*f_* and f_*f^* are multiplication by $[A]$.

(c) Show that the kernels of f^*f_* and f_*f^* are annihilated by a power of n.

EII.2.3 **2.3.** *Excision for K_0.* If I is an ideal in a ring R, form the augmented ring $R\oplus I$ and let $K_0(I)=K_0(R,I)$ denote the kernel of $K_0(R\oplus I)\to K_0(R)$.

(a) If $R\to S$ is a ring map sending I isomorphically onto an ideal of S, show that $K_0(R,I)\cong K_0(S,I)$. Thus $K_0(I)$ is independent of R. *Hint.* Show that $GL(S)/GL(S\oplus I)=1$.

(b) If $I\cap J=0$, show that $K_0(I+J)\cong K_0(I)\oplus K_0(J)$.

(c) *Ideal sequence.* Show that there is an exact sequence

$$GL(R)\to GL(R/I)\xrightarrow{\partial} K_0(I)\to K_0(R)\to K_0(R/I).$$

(d) If R is commutative, use Ex. I.3.6 to show that there is a commutative diagram with exact rows, the vertical maps being determinants:

$$\begin{array}{ccccccccc}
GL(R) & \longrightarrow & GL(R/I) & \xrightarrow{\partial} & K_0(I) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R/I)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
R^\times & \longrightarrow & (R/I)^\times & \xrightarrow{\partial} & \mathrm{Pic}(I) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(R/I).
\end{array}$$

EII.2.4 **2.4.** $K_0(I)$. If I is a ring without unit, we define $K_0(I)$ as follows. Let R be a ring with unit acting upon I, form the augmented ring $R \oplus I$, and let $K_0(I)$ be the kernel of $K_0(R \oplus I) \to K_0(R)$. Thus $K_0(R \oplus I) \cong K_0(R) \oplus K_0(I)$ by definition.

(a) If I has a unit, show that $R \oplus I \cong R \times I$ as rings with unit. Since $K_0(R \times I) = K_0(R) \times K_0(I)$, this shows that the definition of Ex. 2.3 agrees with the usual definition of $K_0(I)$.

(b) Show that a map $I \to J$ of rings without unit induces a map $K_0(I) \to K_0(J)$

(c) Let $M_\infty(R)$ denote the union $\cup M_n(R)$ of the matrix groups, where $M_n(R)$ is included in $M_{n+1}(R)$ as the matrices $\left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right)$. $M_\infty(R)$ is a ring without unit. Show that the inclusion of $R = M_1(R)$ in $M_\infty(R)$ induces an isomorphism

$$K_0(R) \cong K_0(M_\infty(R)).$$

(d) If k is a field, show that $R = k \oplus M_\infty(k)$ is a von Neumann regular ring. Then show that $H_0(R) = \mathbb{Z}$ and $K_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(e) If $f : R \to R/I$, show that $K_0(I)$ is the group $K_0(f)$ of 2.10. *Hint:* Use $f_0 : R \oplus I \to R$ and Ex. 2.3(c).

EII.2.5 **2.5.** *Radical ideals.* Let I be a radical ideal in a ring R (see Ex. I.1.12, I.2.1).

(a) Show that $K_0(I) = 0$, and that $K_0(R) \to K_0(R/I)$ is an injection.

(b) If I is a complete ideal, $K_0(R) \cong K_0(R/I)$ by Lemma 2.2. If R is a semilocal but not local domain, show that $K_0(R) \to K_0(R/I)$ is not an isomorphism when I is the Jacobson radical.

EII.2.6 **2.6.** *Semilocal rings.* A ring R is called *semilocal* if R/J is semisimple for some radical ideal J. Show that if R is semilocal then $K_0(R) \cong \mathbb{Z}^n$ for some $n > 0$.

EII.2.7 **2.7.** Show that if $f : R \to S$ is a map of commutative rings, then:

$$\ker(f) \text{ contains no idempotents}(\neq 0) \Leftrightarrow H_0(R) \to H_0(S) \text{ is an injection.}$$

Conclude that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

EII.2.8 **2.8.** Consider the following conditions on a ring R (cf. Ex. I.1.2):

(IBP) R satisfies the Invariant Basis Property (IBP);
(PO) $K_0(R)$ is a partially ordered abelian group (see §1);
(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.

Show that (III) $\Rightarrow$ (PO) $\Rightarrow$ (IBP). This implies that $K_0(R)$ is a partially ordered abelian group if R is either commutative or noetherian. (See Ex. I.1.4.)

EII.2.9 **2.9.** *Rim squares.* Let G be a cyclic group of prime order p, and $\zeta = e^{2\pi i/p}$ a primitive p^{th} root of unity. Show that the map $\mathbb{Z}[G] \to \mathbb{Z}[\zeta]$ sending a generator of G to ζ induces an isomorphism $K_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[\zeta])$ and hence $Wh_0(G) \cong \mathrm{Pic}(\mathbb{Z}[\zeta])$. *Hint:* Form a Milnor square with $\mathbb{Z}[G]/I = \mathbb{Z}$, $\mathbb{Z}[\zeta]/I = \mathbb{F}_p$, and consider the cyclotomic units $u = \frac{\zeta^i - 1}{\zeta - 1}$, $1 \le i < p$.

EII.2.10 **2.10.** Let R be a commutative ring. Prove that

(a) If $\mathrm{rank}(x) > 0$ for some $x \in K_0(R)$, then there is an $n > 0$ and a finitely generated projective module P so that $nx = [P]$. (This says that the partially ordered group $K_0(R)$ is "unperforated" in the sense of Gdearl [71].)

(b) If P, Q are finitely generated projectives such that $[P] = [Q]$ in $K_0(R)$, then there is an $n > 0$ such that $P \oplus \cdots \oplus P \cong Q \oplus \cdots \oplus Q$ (n copies of P, n copies of Q).

Hint: First assume that R is noetherian of Krull dimension $d < \infty$, and use Bass-Serre Cancellation. In the general case, write R as a direct limit.

EII.2.11 **2.11.** A (normalized) *dimension function* for a von Neumann regular ring R is a group homomorphism $d : K_0(R) \to \mathbb{R}$ so that $d(R^n) = n$ and $d(P) > 0$ for every nonzero finitely generated projective P.

(a) If $P \subseteq Q$, show that any dimension function must have $d(P) \le d(Q)$

(b) If R has a dimension function, show that the formula $\rho(r) = d(rR)$ defines a rank function $\rho\colon R \to [0, 1]$ in the sense of Ex. EI.1.13 I.1.13. Then show that this gives a 1-1 correspondence between rank functions on R and dimension functions on $K_0(R)$.

EII.2.12 **2.12.** Let R be the union of the matrix rings $M_{n!}(F)$ constructed in Ex. EI.1.13 I.1.13. Show that the inclusion $\mathbb{Z} \subset K_0(R)$ extends to an isomorphism $K_0(R) \cong \mathbb{Q}$.

EII.2.13 **2.13.** Let R be the infinite product of the matrix rings $M_i(\mathbb{C})$, $i = 1, 2, ...$

(a) Show that every finitely generated projective R-module P is componentwise trivial in the sense that $P \cong \prod P_i$, the P_i being finitely generated projective $M_i(\mathbb{C})$-modules.

(b) Show that the map from $K_0(R)$ to the group $\prod K_0(M_i(\mathbb{C})) = \prod \mathbb{Z}$ of infinite sequences $(n_1, n_2, ...)$ of integers is an injection, and that $K_0(R) = H_0(R)$ is isomorphic to the group of bounded sequences.

(c) Show that $K_0(R)$ is not a free abelian group, even though it is torsionfree. *Hint:* Consider the subgroup S of sequences $(n_1, ...)$ such that the power of 2 dividing n_i approaches ∞ as $i \to \infty$; show that S is uncountable but that $S/2S$ is countable.

EII.2.14 **2.14.** *Bivariant* K_0. If R and R' are rings, let $Rep(R,R')$ denote the set of isomorphism classes of R–R' bimodules M such that M is finitely generated projective as a right R'-module. Each M gives a functor $\otimes_R M$ from $\mathbf{P}(R)$ to $\mathbf{P}(R')$ sending P to $P\otimes_R M$. This induces a monoid map $\mathbf{P}(R)\to\mathbf{P}(R')$ and hence a homomorphism from $K_0(R)$ to $K_0(R')$. For example, if $f\colon R\to R'$ is a ring homomorphism and R' is considered as an element of $Rep(R,R')$, we obtain the map $\otimes_R R'$. Show that:

(a) Every additive functor $\mathbf{P}(R)\to\mathbf{P}(R')$ is induced from an M in $Rep(R,R')$;

(b) If $K_0(R,R')$ denotes the group completion of $Rep(R,R')$, then $M\otimes_{R'}N$ induces a bilinear map from $K_0(R,R')\otimes K_0(R',R'')$ to $K_0(R,R'')$;

(c) $K_0(\mathbb{Z},R)$ is $K_0(R)$, and if $M\in Rep(R,R')$ then the map $\otimes_R M : K_0(R)\to K_0(R')$ is induced from the product of (b).

(d) If R and R' are Morita equivalent, and P is the R-R' bimodule giving the isomorphism **mod**-$R\cong$ **mod**-R', the class of P in $K_0(R,R')$ gives the Morita isomorphism $K_0(R)\cong K_0(R')$.

EII.2.15 **2.15.** In this exercise, we connect the definition of infinite sum ring in Example II.2.1.3 2.1.3 with a more elementary description due to Wagoner. If R is a direct sum ring, the isomorphism $R^2\cong R$ induces a ring homomorphism $\oplus : R\times R\subset \mathrm{End}_R(R^2)\cong\mathrm{End}_R(R)=R$.

(a) Suppose that R is an infinite sum ring with bimodule M, and write $r\mapsto r^\infty$ for the ring homomorphism $R\to\mathrm{End}_R(M)\cong R$ arising from the left action of R on the right R-module M. Show that $r\oplus r^\infty=r^\infty$ for all $r\in R$.

(b) Conversely, suppose that R is a direct sum ring, and $R\xrightarrow{\infty}R$ is a ring map so that $r\oplus r^\infty=r^\infty$ for all $r\in R$. Show that R is an infinite sum ring.

(c) (Wagoner) Show that the cone rings of Ex. I.EI.1.8 1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.EI.1.7 1.7, are infinite sum rings. *Hint:* $R^\infty\cong\coprod_{i=1}^\infty R^\infty$, so a version of the Eilenberg Swindle I.I.2.8 2.8 applies.

EII.2.16 **2.16.** For any ring R, let J be the (nonunital) subring of $E=\mathrm{End}_R(R^\infty)$ of all f such that $f(R^\infty)$ is finitely generated (Ex. I.EI.1.7 1.7). Show that $M_\infty(R)\subset J_n$ induces an isomorphism $K_0(R)\cong K_0(J)$. *Hint:* For the projection $e_n: R^\infty\to R^n$, $J_n=e_nE$ maps onto $M_n(R)=e_nEe_n$ with nilpotent kernel. But $J=\cup J_n$.

EII.2.17 **2.17.** This exercise shows that there is an exact sequence (II.2.10.2 2.10.2) when T is cofinal.

(a) Show that $[(P,\alpha,Q)]+[(Q,-\alpha^{-1},P)]=0$ and $[(P,T(\gamma),Q)]=0$ in $K_0(T)$.

(b) Show that every element of $K_0(T)$ has the form $[(P,\alpha,R^n)]$.

(c) Use cofinality and the maps $\mathrm{Aut}(TR^n)\xrightarrow{\partial}K_0(T)$, $\partial(\alpha)=[(R^n,\alpha,R^n)]$ of (II.2.10.2 2.10.2) to show that there is a homomorphism $\partial: GL(S)\to K_0(T)$.

(d) Use (a), (b) and (c) to show that (II.2.10.2 2.10.2) is exact at $K_0(T)$.

(e) Show that (II.2.10.2 2.10.2) is exact at $K_0(R)$.

3 $K(X)$, $KO(X)$ and $KU(X)$ of a topological space

Let X be a paracompact topological space. The sets $\mathbf{VB}_{\mathbb{R}}(X)$ and $\mathbf{VB}_{\mathbb{C}}(X)$ of isomorphism classes of real and complex vector bundles over X are abelian monoids under Whitney sum. By Construction I.4.2.1, they are commutative semirings under $\otimes$. Hence the group completions $KO(X)$ of $\mathbf{VB}_{\mathbb{R}}(X)$ and $KU(X)$ of $\mathbf{VB}_{\mathbb{C}}(X)$ are commutative rings with identity $1 = [T^1]$. If the choice of $\mathbb{R}$ or $\mathbb{C}$ is understood, we will just write $K(X)$ for simplicity.

Similarly, the set $\mathbf{VB}_{\mathbb{H}}(X)$ is an abelian monoid under $\oplus$, and we write $KSp(X)$ for its group completion. Although it has no natural ring structure, the construction of Ex. I.4.18 endows $KSp(X)$ with the structure of a module over the ring $KO(X)$.

For example if $*$ denotes a 1-point space then $K(*) = \mathbb{Z}$. If X is contractible, then $KO(X) = KU(X) = \mathbb{Z}$ by I.4.6.1. More generally, $K(X) \cong K(Y)$ whenever X and Y are homotopy equivalent by I.4.6.

The functor $K(X)$ is contravariant in X. Indeed, if $f\colon Y \to X$ is continuous, the induced bundle construction $E \mapsto f^*E$ yields a function $f^*\colon \mathbf{VB}(X) \to \mathbf{VB}(Y)$ which is a morphism of monoids and semirings; hence it induces a ring homomorphism $f^*\colon K(X) \to K(Y)$. By the Homotopy Invariance Theorem I.4.5, the map f^* depends only upon the homotopy class of f in $[Y, X]$.

For example, the universal map $X \to *$ induces a ring map from $\mathbb{Z} = K(*)$ into $K(X)$, sending $n > 0$ to the class of the trivial bundle T^n over X. If $X \neq \emptyset$ then any point of X yields a map $* \to X$ splitting the universal map $X \to *$. Thus the subring $\mathbb{Z}$ is a direct summand of $K(X)$ when $X \neq \emptyset$. (But if $X = \emptyset$ then $K(\emptyset) = 0$.) For the rest of this section, we will assume $X \neq \emptyset$ in order to avoid awkward hypotheses.

The trivial vector bundles T^n and the componentwise trivial vector bundles T^f form sub-semirings of $\mathbf{VB}(X)$, naturally isomorphic to $\mathbb{N}$ and $[X, \mathbb{N}]$, respectively. When X is compact, the semirings $\mathbb{N}$ and $[X, \mathbb{N}]$ are cofinal in $\mathbf{VB}(X)$ by the Subbundle theorem I.4.1, so by Corollary 1.3 we have subrings

$$\mathbb{Z} \subset [X, \mathbb{Z}] \subset K(X).$$

More generally, it follows from Construction I.4.2.1 that $\dim\colon \mathbf{VB}(X) \to [X, \mathbb{N}]$ is a semiring map splitting the inclusion $[X, \mathbb{N}] \subset \mathbf{VB}(X)$. Passing to Group Completions, we get a natural ring map

$$\dim\colon K(X) \to [X, \mathbb{Z}]$$

splitting the inclusion of $[X, \mathbb{Z}]$ in $K(X)$.

The kernel of dim will be written as $\widetilde{K}(X)$, or as $\widetilde{KO}(X)$ or $\widetilde{KU}(X)$ if we wish to emphasize the choice of $\mathbb{R}$ or $\mathbb{C}$. Thus $\widetilde{K}(X)$ is an ideal in $K(X)$, and there is a natural decomposition

$$K(X) \cong \widetilde{K}(X) \oplus [X, \mathbb{Z}].$$

Warning. If X is not connected, our group $\widetilde{K}(X)$ differs slightly from the notation in the literature. However, most applications will involve connected spaces, where the notation is the same. This will be clarified by Theorem 3.2 below.

Consider the set map $\mathbf{VB}_n(X) \to \widetilde{K}(X)$ sending E to $[E] - n$. This map is compatible with the stabilization map $\mathbf{VB}_n(X) \to \mathbf{VB}_{n+1}(X)$ sending E to $E \oplus T$, giving a map

$$\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X). \tag{3.1.0}$$ II.3.1.0

We can interpret this in terms of maps between the infinite Grassmannian spaces Grass_n ($= BO_n$, BU_n or BSp_n) as follows. Recall from the Classification Theorem I.4.10 that the set $\mathbf{VB}_n(X)$ is isomorphic to the set $[X, \mathrm{Grass}_n]$ of homotopy classes of maps. Adding a trivial bundle T to the universal bundle E_n over Grass_n gives a vector bundle over Grass_n, so again by the Classification Theorem there is a map $i_n\colon \mathrm{Grass}_n \to \mathrm{Grass}_{n+1}$ such that $E_n \oplus T \cong i_n^*(E_{n+1})$. By Cellular Approximation there is no harm in assuming i_n is cellular. Using I.4.10.1, the map $\Omega i_n\colon \Omega\,\mathrm{Grass}_n \to \Omega\,\mathrm{Grass}_{n+1}$ is homotopic to the standard inclusion $O_n \hookrightarrow O_{n+1}$ (resp. $U_n \hookrightarrow U_{n+1}$ or $Sp_n \hookrightarrow Sp_{n+1}$), which sends an $n \times n$ matrix g to the $n+1 \times n+1$ matrix $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. By construction, the resulting map $i_n\colon [X, \mathrm{Grass}_n] \to [X, \mathrm{Grass}_{n+1}]$ corresponds to the stabilization map. The direct limit $\varinjlim [X, \mathrm{Grass}_n]$ is then in 1-1 correspondence with the direct limit $\varinjlim \mathbf{VB}_n(X)$ of (3.1.0).

II.3.1 **Stabilization Theorem 3.1.** *Let X be either a compact space or a finite dimensional connected CW complex. Then the map (3.1.0) induces an isomorphism $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}(X) \cong \varinjlim [X, \mathrm{Grass}_n]$. In particular,*

$$\widetilde{KO}(X) \cong \varinjlim [X, BO_n],\ \ \widetilde{KU}(X) \cong \varinjlim [X, BU_n] \text{ and } \widetilde{KSp}(X) \cong \varinjlim [X, BSp_n].$$

Proof. We argue as in Lemma 2.3.1. Since the monoid of (componentwise) trivial vector bundles T^f is cofinal in $\mathbf{VB}(X)$ (I.4.1), we see from Corollary 1.3 that every element of $\widetilde{K}(X)$ is represented by an element $[E] - n$ of some $\mathbf{VB}_n(X)$, and if $[E] - n = [F] - n$ then $E \oplus T^\ell \cong F \oplus T^\ell$ in some $\mathbf{VB}_{n+\ell}(X)$. Thus $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}_n(X)$, as claimed. ☐

II.3.1.1 **Example 3.1.1** (Spheres)**.** From I(4.9) we see that $KO(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ but $KU(S^1) = \mathbb{Z}$, $KO(S^2) = \mathbb{Z} \oplus \mathbb{Z}/2$ but $KU(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, $KO(S^3) = KU(S^3) = \mathbb{Z}$ and $KO(S^4) \cong KU(S^4) = \mathbb{Z} \oplus \mathbb{Z}$.

By Prop. I.4.8, the n-dimensional ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) vector bundles on S^d are classified by the homotopy groups $\pi_{d-1}(O_n)$, $\pi_{d-1}(U_n)$ and $\pi_{d-1}(Sp_n)$, respectively. By the Stabilization Theorem, $\widetilde{KO}(S^d) = \lim_{n\to\infty} \pi_{d-1}(O_n)$ and $\widetilde{KU}(S^d) = \lim_{n\to\infty} \pi_{d-1}(U_n)$.

Now Bott Periodicity says that the homotopy groups of O_n, U_n and Sp_n stabilize for $n \gg 0$. Moreover, if $n \geq d/2$ then $\pi_{d-1}(U_n)$ is 0 for d odd and $\mathbb{Z}$ for d even. Thus $KU(S^d) = \mathbb{Z} \oplus \widetilde{KU}(S^d)$ is periodic of order 2 in $d > 0$: the ideal $\widetilde{KU}(S^d)$ is 0 for d odd and $\mathbb{Z}$ for d even, $d \neq 0$.

Similarly, the $\pi_{d-1}(O_n)$ and $\pi_{d-1}(Sp_n)$ stabilize for $n \geq d+1$ and $n \geq d/4$; both are periodic of order 8. Thus $KO(S^d) = \mathbb{Z} \oplus \widetilde{KO}(S^d)$ and $KSp(S^d) = \mathbb{Z} \oplus \widetilde{KSp}(S^d)$ are periodic of order 8 in $d > 0$, with the groups $\widetilde{KO}(S^d) = \pi_{d-1}(O)$ and $\widetilde{KSp}(S^d) = \pi_{d-1}(Sp)$ being tabulated in the following table.

d (mod 8)	1	2	3	4	5	6	7	8
$\widetilde{KO}(S^d)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\widetilde{KSp}(S^d)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Both of the ideals $\widetilde{KO}(S^d)$ and $\widetilde{KU}(S^d)$ are of square zero.

II.3.1.2 **Remark 3.1.2.** The complexification maps $\mathbb{Z} \cong \widetilde{KO}(S^{4k}) \to \widetilde{KU}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. (The forgetful maps $\widetilde{KU}(S^{4k}) \to \widetilde{KO}(S^{4k})$ have the opposite parity in k.) Similarly, the maps $\mathbb{Z} \cong \widetilde{KU}(S^{4k}) \to \widetilde{KSp}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. (The forgetful maps $\widetilde{KSp}(S^{4k}) \to \widetilde{KU}(S^{4k})$ have the opposite parity in k.) These calculations are taken from [136], IV.5.12 and IV.6.1.

Let BO (resp. BU, BSp) denote the direct limit of the Grassmannians Grass_n. As noted after (3.1.0) and in I.4.10.1, the notation reflects the fact that $\Omega\,\mathrm{Grass}_n$ is O_n (resp. U_n, Sp_n), and the maps in the direct limit correspond to the standard inclusions, so that we have $\Omega BO \simeq O = \bigcup O_n$, $\Omega BU \simeq U = \bigcup U_n$ and $\Omega BSp \simeq Sp = \bigcup Sp_n$.

II.3.2 **Theorem 3.2.** *For every compact space X:*

$$KO(X) \cong [X, \mathbb{Z} \times BO] \quad \text{and} \quad \widetilde{KO}(X) \cong [X, BO];$$
$$KU(X) \cong [X, \mathbb{Z} \times BU] \quad \text{and} \quad \widetilde{KU}(X) \cong [X, BU];$$
$$KSp(X) \cong [X, \mathbb{Z} \times BSp] \quad \text{and} \quad \widetilde{KSp}(X) \cong [X, BSp].$$

In particular, the homotopy groups $\pi_n(BO) = \widetilde{KO}(S^n)$, $\pi_n(BU) = \widetilde{KU}(S^n)$ and $\pi_n(BSp) = \widetilde{KSp}(S^n)$ are periodic and given in Example 3.1.1.

Proof. If X is compact then we have $[X, BO] = \varinjlim [X, BO_n]$ and similarly for $[X, BU]$ and $[X, BSp]$. The result now follows from Theorem 3.1 for connected X. For non-connected compact spaces, we only need to show that the maps $[X, BO] \to \widetilde{KO}(X)$, $[X, BU] \to \widetilde{KU}(X)$ and $[X, BSp] \to \widetilde{KSp}(X)$ of Theorem 3.1 are still isomorphisms.

Since X is compact, every continuous map $X \to \mathbb{Z}$ is bounded. Hence the rank of every vector bundle E is bounded, say $\operatorname{rank} E \leq n$ for some $n \in \mathbb{N}$. If $f = n - \operatorname{rank} E$ then $F = E \oplus T^f$ has constant rank n, and $[E] - \operatorname{rank} E = [F] - n$. Hence every element of $\widetilde{K}(X)$ comes from some $\mathbf{VB}_n(X)$.

To see that these maps are injective, suppose that $E, F \in \mathbf{VB}_n(X)$ are such that $[E] - n = [F] - n$. By 1.3 we have $E \oplus T^f = F \oplus T^f$ in $\mathbf{VB}_{n+f}(X)$ for some $f \in [X, \mathbb{N}]$. If $f \leq p$, $p \in \mathbb{N}$, then adding T^{p-f} yields $E \oplus T^p = F \oplus T^p$. Hence E and F agree in $\mathbf{VB}_{n+p}(X)$. $\square$

II.3.2.1 **Definition 3.2.1** (K^0)**.** For every paracompact X we write $KO^0(X)$ for $[X, \mathbb{Z} \times BO]$, $KU^0(X)$ for $[X, \mathbb{Z} \times BU]$ and $KSp^0(X)$ for $[X, \mathbb{Z} \times BSp]$. By Theorem 3.2, we have $KO^0(X) \cong KO(X)$, $KU^0(X) \cong KU(X)$ and $KSp^0(X) \cong KSp(X)$ for every compact X. Similarly, we shall write $\widetilde{KO}^0(X)$, $\widetilde{KU}^0(X)$ and $\widetilde{KSp}^0(X)$ for $[X, BO]$, $[X, BU]$ and $[X, BSp]$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we will just write $K^0(X)$ and $\widetilde{K}^0(X)$.

If Y is a subcomplex of X, we define relative groups $K^0(X, Y) = K^0(X/Y)/\mathbb{Z}$ and $\widetilde{K}^0(X, Y) = \widetilde{K}^0(X/Y)$.

When X is paracompact but not compact, $\tilde{K}^0(X)$ and $\tilde{K}(X)$ are connected by stabilization and the map (3.1.0):

$$\widetilde{KO}(X) \longleftarrow \varinjlim \mathbf{VB}_n(X) \cong \varinjlim [X, BO_n] \longrightarrow [X, BO] = \widetilde{KO}^0(X)$$

and similarly for $\widetilde{KU}(X)$ and $\widetilde{KSp}(X)$. We will see in Example 3.7.2 and Ex. 3.2 that the left map need not be an isomorphism. Here are two examples showing that the right map need not be an isomorphism either.

II.3.2.2 **Example 3.2.2** (McGibbon)**.** Let X be $S^3 \vee S^5 \vee S^7 \vee \cdots$, an infinite bouquet of odd-dimensional spheres. By homotopy theory, there is a map $f\colon X \to BO_3$ whose restriction to S^{2p+1} is essential of order p for each odd prime p. If E denotes the 3-dimensional vector bundle f^*E_3 on X, then the class of f in $\varinjlim [X, BO_n]$ corresponds to $[E] - 3 \in KO(X)$. In fact, since X is a suspension, we have $\varinjlim [X, BO_n] \cong \widetilde{KO}(X)$ by Ex. 3.8.

Each $(n+3)$-dimensional vector bundle $E \oplus T^n$ is nontrivial, since its restriction to S^{2p+1} is nontrivial whenever $2p > n + 3$ (again by homotopy theory). Hence $[E] - 3$ is a nontrivial element of $\widetilde{KO}(X)$. However, the corresponding element in $\widetilde{KO}^0(X) = [X, BO]$ is zero, because the homotopy groups of BO have no odd torsion.

II.3.2.3 **Example 3.2.3.** If X is the union of compact CW complexes X_i, Milnor showed in [129] that (in the notation of 3.5 below) there is an exact sequence for each n

$$0 \to \varprojlim{}^1 K^{n-1}(X_i) \to K^n(X) \to \varprojlim K^n(X_i) \to 0.$$

In particular, $KU^0(\mathbb{C}\mathbb{P}^\infty) \cong \mathbb{Z}[[x]]$ by Ex. 3.7.

II.3.3 **Proposition 3.3.** *If Y is a subcomplex of a CW complex X, the following sequences are exact:*

$$\widetilde{K}^0(X/Y) \to \widetilde{K}^0(X) \to \widetilde{K}^0(Y),$$

$$K^0(X, Y) \to K^0(X) \to K^0(Y).$$

Proof. Since $Y \subset X$ is a cofibration, we have an exact sequence $[X/Y, B] \to [X, B] \to [Y, B]$ for every connected space B; see III(6.3) in [228]. This yields the first sequence (B is BO, BU or BSp). The second follows from this and the classical exact sequence $\widetilde{H}^0(X/Y; \mathbb{Z}) \to H^0(X; \mathbb{Z}) \to H^0(Y; \mathbb{Z})$. □

II.3.4 **Example 3.4** (Change of structure field)**.** If X is any space, the monoid (or semiring) map $\mathbf{VB}_{\mathbb{R}}(X) \to \mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to $[E \otimes \mathbb{C}]$ (see Ex. I.4.5) extends by universality to a ring homomorphism $KO(X) \to KU(X)$. For example, $KO(S^{8n}) \to KU(S^{8n})$ is an isomorphism but $\widetilde{KO}(S^{8n+4}) \cong \mathbb{Z}$ embeds in $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ as a subgroup of index 2.

Similarly, the forgetful map $\mathbf{VB}_{\mathbb{C}}(X) \to \mathbf{VB}_{\mathbb{R}}(X)$ extends to a group homomorphism $KU(X) \to KO(X)$. As $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{C}}(V)$, the summand $[X, \mathbb{Z}]$ of $KU(X)$ embeds as $2[X, \mathbb{Z}]$ in the summand $[X, \mathbb{Z}]$ of $KO(X)$. Since $E \otimes \mathbb{C} \cong E \oplus E$ as real vector bundles (by Ex. I.4.5), the composition $KO(X) \to KU(X) \to KO(X)$ is multiplication by 2. The composition in the other direction is more complicated; see Exercise 3.1. For example, it is the zero map on $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ but is multiplication by 2 on $\widetilde{KU}(S^{8n}) \cong \mathbb{Z}$.

There are analogous maps $KU(X) \to KSp(X)$ and $KSp(X) \to KU(X)$, whose properties we leave to the exercises.

Higher Topological K-theory

Once we have a representable functor such as K^0, standard techniques in infinite loop space theory allow us to expand it into a generalized cohomology theory. Rather than get distracted by infinite loop spaces now, we choose to adopt a rather pedestrian approach, ignoring the groups K^n for $n > 0$. For this we use the suspensions S^nX of X, which are all connected paracompact spaces.

II.3.5 **Definition 3.5.** For each integer $n > 0$, we define $\widetilde{KO}^{-n}(X)$ and $KO^{-n}(X)$ by:

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}^0(S^nX) = [S^nX, BO]; \quad KO^{-n}(X) = \widetilde{KO}^{-n}(X) \oplus \widetilde{KO}(S^n).$$

Replacing 'O' by 'U' yields definitions $\widetilde{KU}^{-n}(X) = \widetilde{KU}^0(S^nX) = [S^nX, BU]$ and $KU^{-n}(X) = \widetilde{KU}^{-n}(X) \oplus \widetilde{KU}(S^n)$; replacing '$O$' by '$Sp$' yields definitions for $\widetilde{KSp}^{-n}(X)$ and $KSp^{-n}(X)$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we shall drop the 'O,' 'U' and 'Sp,', writing simply $\widetilde{K}^{-n}(X)$ and $K^{-n}(X)$.

We shall also define relative groups as follows. If Y is a subcomplex of X, and $n > 0$, we set $K^{-n}(X, Y) = \widetilde{K}^{-n}(X/Y)$.

II.3.5.1 **Based Maps 3.5.1.** Note that our definitions do not assume X to have a basepoint. If X has a nondegenerate basepoint and Y is an H-space with homotopy inverse (such as BO, BU or BSp), then the group $[X, Y]$ is isomorphic to the group $\pi_0(Y) \times [X, Y]_*$, where the second term denotes homotopy classes of *based* maps from X to Y; see pp. 100 and 119 of [228]. For such spaces X we can interpret the formulas for $KO^{-n}(X)$, $KU^{-n}(X)$ and $KSp^{-n}(X)$ in terms of based maps, as is done in Atiyah [7, p.68].

If X_* denotes the disjoint union of X and a basepoint $*$, then we have the usual formula for an unreduced cohomology theory: $K^{-n}(X) = \widetilde{K}(S^n(X_*))$. This easily leads (see Ex. 3.11) to the formulas for $n \geq 1$:

$$KO^{-n}(X) \cong [X, \Omega^n BO], \quad KU^{-n}(X) \cong [X, \Omega^n BU]$$

and $KSp^{-n}(X) \cong [X, \Omega^n BSp]$.

II.3.6 **Theorem 3.6.** *If Y is a subcomplex of a CW complex X, we have the exact sequences (infinite to the left):*

$$\cdots \widetilde{K}^{-2}(Y) \to \widetilde{K}^{-1}(X/Y) \to \widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(Y) \to \widetilde{K}^{0}(X/Y) \to \widetilde{K}^{0}(X) \to \widetilde{K}^{0}(Y),$$

$$\cdots K^{-2}(Y) \to K^{-1}(X,Y) \to K^{-1}(X) \to K^{-1}(Y) \to K^{0}(X,Y) \to K^{0}(X) \to K^{0}(Y).$$

Proof. Exactness at $K^0(X)$ was proven in Proposition II.3.3 3.3. The mapping cone cone(i) of $i\colon Y \subset X$ is homotopy equivalent to X/Y, and $j\colon X \subset \text{cone}(i)$ induces cone$(i)/X \simeq SY$. This gives exactness at $K^0(X,Y)$. Similarly, cone$(j) \simeq SY$ and cone$(j)/\text{cone}(i) \simeq SX$, giving exactness at $K^{-1}(Y)$. The long exact sequences follows by replacing $Y \subset X$ by $SY \subset SX$. □

II.3.7 **Characteristic Classes 3.7.** The total Stiefel–Whitney class $w(E)$ of a real vector bundle E was defined in chapter I, §4. By (SW3) it satisfies the product formula: $w(E \oplus F) = w(E)w(F)$. Therefore if we interpret $w(E)$ as an element of the abelian group U of all formal sums $1 + a_1 + \cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ we get a group homomorphism $w\colon KO(X) \to U$. It follows that each Stiefel–Whitney class induces a well-defined set map $w_i\colon KO(X) \to H^i(X;\mathbb{Z}/2)$. In fact, since w vanishes on each componentwise trivial bundle T^f it follows that $w([E] - [T^f]) = w(E)$. Hence each Stiefel–Whitney class w_i factors through the projection $KO(X) \to \widetilde{KO}(X)$.

Similarly, the total Chern class $c(E) = 1 + c_1(E) + \cdots$ satisfies $c(E \oplus F) = c(E)c(F)$, so we may think of it as a group homomorphism from $KU(X)$ to the abelian group U of all formal sums $1 + a_2 + a_4 + \cdots$ in $\hat{H}^*(X;\mathbb{Z})$. It follows that the Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ of a complex vector bundle define set maps $c_i\colon KU(X) \longrightarrow H^{2i}(X;\mathbb{Z})$. Again, since c vanishes on componentwise trivial bundles, each Chern class c_i factors through the projection $KU(X) \to \widetilde{KU}(X)$.

II.3.7.1 **Example 3.7.1.** For even spheres the Chern class $c_n\colon \widetilde{KU}(S^{2n}) \to H^{2n}(S^n;\mathbb{Z})$ is an isomorphism. We will return to this point in Ex. EII.3.6 3.6 and in §4.

II.3.7.2 **Example 3.7.2.** The map $\varinjlim [\mathbb{R}\mathbb{P}^\infty, BO_n] \to \widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$ of (II.3.1.0 3.1.0) cannot be onto. To see this, consider the element $\eta = 1 - [E_1]$ of $\widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$, where E_1 is the canonical line bundle. Since $w(-\eta) = w(E_1) = 1 + x$ we have $w(\eta) = (1+x)^{-1} = \sum_{i=0}^\infty x^i$, and $w_i(\eta) \neq 0$ for every $i \geq 0$. Axiom (SW1) implies that η cannot equal $[F] - \dim(F)$ for any bundle F.

Similarly, $\varinjlim [\mathbb{C}\mathbb{P}^\infty, BU_n] \to \widetilde{KU}(\mathbb{C}\mathbb{P}^\infty)$ cannot be onto; the argument is similar, again using the canonical line bundle: $c_i(1 - [E_1]) \neq 0$ for every $i \geq 0$.

EXERCISES

EII.3.1 **3.1.** Let X be a topological space. Show that there is an involution of $\mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to the complex conjugate bundle $[\bar{E}]$ of Ex. I.4.6. The corresponding involution c on $KU(X)$ can be nontrivial; use I.4.9.2 to show that c is multiplication by -1 on $\widetilde{KU}(S^2) \cong \mathbb{Z}$. (By Bott periodicity, this implies that c is multiplication by $(-1)^k$ on $\widetilde{KU}(S^{2k}) \cong \mathbb{Z}$.) Finally, show that the composite $KU(X) \to KO(X) \to KU(X)$ is the map $1 + c$ sending $[E]$ to $[E] + [\bar{E}]$.

EII.3.2 **3.2.** If $\amalg X_i$ is the disjoint union of spaces X_i, show that $K(\amalg X_i) \cong \prod K(X_i)$. Then construct a space X such that the map $\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X)$ of (3.1.0) is not onto.

EII.3.3 **3.3.** *External products.* Show that there is a bilinear map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ for every X_1 and X_2, sending $[E_1] \otimes [E_2]$ to $[\pi_1^*(E_1) \otimes \pi_2^*(E_2)]$, where $\pi_i \colon X_1 \times X_2 \to X_i$ is the projection. Then show that if $X_1 = X_2 = X$ the composition with the diagonal map $\Delta^* \colon K(X \times X) \to K(X)$ yields the usual product in the ring $K(X)$, sending $[E_1] \otimes [E_2]$ to $[E_1 \otimes E_2]$.

EII.3.4 **3.4.** Recall that the smash product $X \wedge Y$ of two based spaces is the quotient $X \times Y / X \vee Y$, where $X \vee Y$ is the union of $X \times \{*\}$ and $\{*\} \times Y$. Show that

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y).$$

EII.3.5 **3.5.** Show that $KU^{-2}(*) \otimes KU^{-n}(X) \to KU^{-n-2}(X)$ induces a "periodicity" isomorphism $\beta \colon KU^{-n}(X) \xrightarrow{\sim} KU^{-n-2}(X)$ for all n. *Hint:* Let β be a generator of $KU^{-2}(*) \cong \mathbb{Z}$, and use $S^2 \wedge S^n X \simeq S^{n+2} X$.

EII.3.6 **3.6.** Let X be a finite CW complex with only even-dimensional cells, such as $\mathbb{CP}^n$. Show that $KU(X)$ is a free abelian group on the set of cells of X, and that $KU(SX) = \mathbb{Z}$, so that $KU^{-1}(X) = 0$. Then use Example 3.7.1 to show that the total Chern class injects the group $\widetilde{KU}(X)$ into $\prod H^{2i}(X; \mathbb{Z})$. *Hint:* Use induction on $\dim(X)$ and the fact that X_{2n}/X_{2n-2} is a bouquet of $2n$-spheres.

EII.3.7 **3.7.** *Chern character for* $\mathbb{CP}^n$. Let E_1 be the canonical line bundle on $\mathbb{CP}^n$, and let x denote the class $[E_1] - 1$ in $KU(\mathbb{CP}^n)$. Use Chern classes and the previous exercise to show that $\{1, [E_1], [E_1 \otimes E_1], \dots, [E_1^{\otimes n}]\}$, and hence $\{1, x, x^2, \dots, x^n\}$, forms a basis of the free abelian group $KU(\mathbb{CP}^n)$. Then show that $x^{n+1} = 0$, so that the ring $KU(\mathbb{CP}^n)$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. We will see in Ex. 4.11 below that the Chern character ch maps the ring $KU(\mathbb{CP}^n)$ isomorphically onto the subring $\mathbb{Z}[t]/(t^{n+1})$ of $H^*(\mathbb{CP}^n; \mathbb{Q})$ generated by $t = e^{c_1(x)} - 1$.

EII.3.8 **3.8.** Consider the suspension $X = SY$ of a paracompact space Y. Use Ex. I.4.16 to show that $\varinjlim [X, BO_n] \cong \widetilde{KO}(X)$.

EII.3.9 **3.9.** If X is a finite CW complex, show by induction on the number of cells that both $KO(X)$ and $KU(X)$ are finitely generated abelian groups.

EII.3.10 **3.10.** Show that $KU(\mathbb{RP}^{2n}) = KU(\mathbb{RP}^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}/2^n$. *Hint:* Try the total Stiefel–Whitney class, using Proposition 3.3.

EII.3.11 **3.11.** Let X be a compact space with a nondegenerate basepoint. Show that $KO^{-n}(X) \cong [X, \Omega^n BO] \cong [X, \Omega^{n-1} O]$ and $KU^{-n}(X) \cong [X, \Omega^n BU] \cong [X, \Omega^{n-1} U]$ for all $n \geq 1$. In particular, $KU^{-1}(X) \cong [X, U]$ and $KO^{-1}(X) \cong [X, O]$.

EII.3.12 **3.12.** Let X be a compact space with a nondegenerate basepoint. Show that the homotopy groups of the topological groups $GL(\mathbb{R}^X) = \mathrm{Hom}(X, GL(\mathbb{R}))$ and $GL(\mathbb{C}^X) = \mathrm{Hom}(X, GL(\mathbb{C}))$ are (for $n > 0$):

$$\pi_{n-1} GL(\mathbb{R}^X) = KO^{-n}(X) \quad \text{and} \quad \pi_{n-1} GL(\mathbb{C}^X) = KU^{-n}(X).$$

EII.3.13 **3.13.** If $E \to X$ is a complex bundle, there is a quaternionic vector bundle $E_{\mathbb{H}} \to X$ with fibers $E_x \otimes_{\mathbb{C}} \mathbb{H}$, as in Ex. I.4.5; this induces the map $KU(X) \to KSp(X)$ mentioned in 3.4. Show that $E_{\mathbb{H}} \to X$, considered as a complex vector bundle, is isomorphic to the Whitney sum $E \oplus E$. Deduce that the composition $KU(X) \to KSp(X) \to KU(X)$ is multiplication by 2.

EII.3.14 **3.14.** Show that $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ as an $\mathbb{H}$-bimodule, on generators $1 \otimes 1 \pm j \otimes j$. This induces a natural isomorphism $V \otimes_{\mathbb{C}} \mathbb{H} \cong V \oplus V$ of vector spaces over $\mathbb{H}$. If $E \to X$ is a quaternionic vector bundle, with underlying complex bundle $uE \to X$, show that there is a natural isomorphism $(uE)_{\mathbb{H}} \cong E \oplus E$. Conclude that the composition $KSp(X) \to KU(X) \to KSp(X)$ is multiplication by 2.

EII.3.15 **3.15.** Let $\bar{E}$ be the complex conjugate bundle of a complex vector bundle $E \to X$; see Ex. I.4.6. Show that $\bar{E}_{\mathbb{H}} \cong E_{\mathbb{H}}$ as quaternionic vector bundles. This shows that $KU(X) \to KSp(X)$ commutes with the involution c of Ex. 3.1.

Using exercises 3.1 and 3.14, show that the composition $KSp(X) \to KO(X) \to KSp(X)$ is multiplication by 4.

4 Lambda and Adams Operations

II.4.1 **4.1.** A commutative ring K is called a λ-*ring* if we are given a family of set operations $\lambda^k\colon K \to K$ for $k \geq 0$ such that for all $x, y \in K$:

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K$;
- $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x)\lambda^{k-i}(y) = \lambda^k(x) + \lambda^{k-1}(x)\lambda^1(y) + \cdots + \lambda^k(y)$.

This last condition is equivalent to the assertion that there is a group homomorphism λ_t from the additive group of K to the multiplicative group $W(K) = 1 + tK[[t]]$ given by the formula $\lambda_t(x) = \sum \lambda^k(x)t^k$. A λ-*ideal* of K is an ideal I with $\lambda^n(I) \subseteq I$ for all $n \geq 1$.

Warning: Our notation of λ-ring follows Atiyah; Grothendieck and other authors call this a *pre-λ-ring*, reserving the term λ-ring for what we call a *special λ-ring*; see Definition 4.3.1 below.

II.4.1.1 **Example 4.1.1** (Binomial Rings)**.** The integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$ are λ-rings with $\lambda^k(n) = \binom{n}{k}$. If K is any $\mathbb{Q}$-algebra, we define $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $x \in K$ and $k \geq 1$; again the formula $\lambda^k(x) = \binom{x}{k}$ makes K into a λ-ring.

More generally, a *binomial ring* is a subring K of a $\mathbb{Q}$-algebra $K_{\mathbb{Q}}$ such that for all $x \in K$ and $k \geq 1$, $\binom{x}{k} \in K$. We make a binomial ring into a λ-ring by setting $\lambda^k(x) = \binom{x}{k}$. If K is a binomial ring then formally λ_t is given by the formula $\lambda_t(x) = (1+t)^x$. For example, if X is a topological space, then the ring $[X, \mathbb{Z}]$ is a λ-ring with $\lambda^k(f) = \binom{f}{k}$, the function sending x to $\binom{f(x)}{k}$.

The notion of λ-semiring is very useful in constructing λ-rings. Let M be a semiring (see §1); we know that the group completion $M^{-1}M$ of M is a ring. We call M a λ*–semiring* if it is equipped with operations $\lambda^k\colon M \to M$ such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ and $\lambda^k(x+y) = \sum \lambda^i(x)\lambda^{k-i}(y)$.

If M is a λ-semiring then the group completion $K = M^{-1}M$ is a λ-ring. To see this, note that sending $x \in M$ to the power series $\sum \lambda^k(x)t^k$ defines a monoid map $\lambda_t\colon M \to 1 + tK[[t]]$. By universality of K, this extends to a group homomorphism λ_t from K to $1 + tK[[t]]$, and the coefficients of $\lambda_t(x)$ define the operations $\lambda^k(x)$.

II.4.1.2 **Example 4.1.2** (Algebraic K_0)**.** Let R be a commutative ring and set $K = K_0(R)$. If P is a finitely generated projective R-module, consider the formula $\lambda^k(P) = [\wedge^k P]$. The decomposition $\wedge^k(P \oplus Q) \cong \sum (\wedge^i P) \otimes (\wedge^{k-i} Q)$ given in ch. I, §3 shows that $\mathbf{P}(R)$ is a λ-semiring. Hence $K_0(R)$ is a λ-ring.

Since $\mathrm{rank}(\wedge^k P) = \binom{\mathrm{rank}\,P}{k}$, it follows that the map rank: $K_0(R) \to H_0(R)$ of 2.3 is a morphism of λ-rings, and hence that $\widetilde{K}_0(R)$ is a λ-ideal of $K_0(R)$.

II.4.1.3 **Example 4.1.3** (Topological K^0)**.** Let X be a topological space and let $K(X)$ be either $KO(X)$ or $KU(X)$. If $E \to X$ is a vector bundle, let $\lambda^k(E)$ be the exterior power bundle $\wedge^k E$ of Ex. I.4.3. The decomposition of $\wedge^k(E \oplus F)$ given in Ex. I.4.3 shows that the monoid $\mathbf{VB}(X)$ is a λ-semiring. Hence $KO(X)$ and $KU(X)$ are λ-rings, and $KO(X) \to KU(X)$ is a morphism of λ-rings.

Since $\dim(\wedge^k E) = \binom{\dim E}{k}$, it follows that $KO(X) \to [X, \mathbb{Z}]$ and $KU(X) \to [X, \mathbb{Z}]$ are λ-ring morphisms, and that $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are λ-ideals.

II.4.1.4 **Example 4.1.4** (Representation Ring)**.** Let G be a finite group, and consider the complex representation ring $R(G)$, constructed in Example 1.6 as the group completion of $Rep_{\mathbb{C}}(G)$, the semiring of finite dimensional representations of G; as an abelian group $R(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of elements in G. The exterior powers $\Lambda^i(V)$ of a representation V are also G-modules, and the decomposition of $\Lambda^k(V \oplus W)$ as complex vector spaces used in 4.1.2 shows that $Rep_{\mathbb{C}}(G)$ is a λ-semiring. Hence $R(G)$ is a λ-ring. (It is true, but harder to show, that $R(G)$ is a special λ-ring; see Ex. 4.2.)

If $d = \dim_{\mathbb{C}}(V)$ then $\dim_{\mathbb{C}}(\Lambda^k V) = \binom{d}{k}$, so $\dim_{\mathbb{C}}$ is a λ-ring map from $R(G)$ to $\mathbb{Z}$. The kernel $\widetilde{R}(G)$ of this map is a λ-ideal of $R(G)$.

II.4.1.5 **Example 4.1.5.** Let X be a scheme, or more generally a locally ringed space (Ch. I, §5). We will define a ring $K_0(X)$ in §7 below, using the category $\mathbf{VB}(X)$. As an abelian group it is generated by the classes of vector bundles on X. We will see in Proposition 8.8 that the operations $\lambda^k[\mathcal{E}] = [\wedge^k \mathcal{E}]$ are well-defined on $K_0(X)$ and make it into a λ-ring. (The formula for $\lambda^k(x+y)$ will follow from Ex. I.5.4.)

Positive structures

4.2. Not every λ-ring is well-behaved. In order to avoid pathologies, we introduce a further condition, satisfied by the above examples: the λ-ring K must have a positive structure and satisfy the Splitting Principle.

II.4.2.1 **Definition 4.2.1.** By a *positive structure* on a λ-ring K we mean: (i) a λ-subring H^0 of K which is a binomial ring; (ii) a λ-ring surjection $\varepsilon\colon K \to H^0$ which is the identity on H^0 (ε is called the *augmentation*); and (iii) a subset $P \subset K$ (the *positive* elements), such that

(1) $\mathbb{N} = \{0, 1, 2, \cdots\}$ is contained in P.

(2) P is a λ-sub-semiring of K. That is, P is closed under addition, multiplication, and the operations λ^k.

(3) Every element of the kernel $\widetilde{K}$ of ε can be written as $p - q$ for some p, $q \in P$.

(4) If $p \in P$ then $\varepsilon(p) = n \in \mathbb{N}$. Moreover, $\lambda^i(p) = 0$ for $i > n$ and $\lambda^n(p)$ is a unit of K.

Condition (2) states that the group completion $P^{-1}P$ of P is a λ-subring of K; by (3) we have $P^{-1}P = \mathbb{Z} \oplus \widetilde{K}$. By (4), $\varepsilon(p) > 0$ for $p \neq 0$, so $P \cap (-P) = 0$; therefore $P^{-1}P$ is a partially ordered abelian group in the sense of §1. An element $\ell \in P$ with $\varepsilon(\ell) = 1$ is called a *line element*; by (4), $\lambda^1(\ell) = \ell$ and ℓ is a unit of K. That is, the line elements form a subgroup L of the units of K.

The λ-rings in examples 4.1.2–4.1.5 all have positive structures. The λ-ring $K_0(R)$ has a positive structure with

$$H^0 = H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}] \quad \text{and} \quad P = \{[P] : \mathrm{rank}(P) \text{ is constant}\};$$

the line elements are the classes of line bundles, so $L = \mathrm{Pic}(R)$. Similarly, the λ-rings $KO(X)$ and $KU(X)$ have a positive structure in which H^0 is $H^0(X,\mathbb{Z}) = [X,\mathbb{Z}]$ and P is $\{[E] : \dim(E) \text{ is constant}\}$, as long as we restrict to compact spaces or spaces with $\pi_0(X)$ finite, so that I.4.1.1 applies. Again, line elements are the classes of line bundles; for $KO(X)$ and $KU(X)$ we have $L = H^1(X;\mathbb{Z}/2)$ and $L = H^2(X;\mathbb{Z})$, respectively. For $R(G)$, the classes $[V]$ of representations V are the positive elements; H^0 is $\mathbb{Z}$, and L is the set of 1-dimensional representations of G. Finally, if X is a scheme (or locally ringed space) then in the positive structure on $K_0(X)$ we have $H^0 = H^0(X;\mathbb{Z})$ and P is $\{[\mathcal{E}] : \mathrm{rank}(\mathcal{E}) \text{ is constant}\}$; see I.5.1. The line bundles are again the line elements, so $L = \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ by I.5.10.1.

There is a natural group homomorphism "det" from K to L, which vanishes on H^0. If $p \in P$ we define $\det(p) = \lambda^n(p)$, where $\varepsilon(p) = n$. The formula for $\lambda^n(p+q)$ and the vanishing of $\lambda^i(p)$ for $i > \varepsilon(p)$ imply that $\det\colon P \to L$ is a monoid map, *i.e.,* that $\det(p+q) = \det(p)\det(q)$. Thus det extends to a map from $P^{-1}P$ to L. As $\det(n) = \binom{n}{n} = 1$ for every $n \geq 0$, $\det(\mathbb{Z}) = 1$. By (iii), defining $\det(H^0) = 1$ extends det to a map from K to L. When K is $K_0(R)$ the map det was introduced in §2. For $KO(X)$, det is the first Stiefel–Whitney class; for $KU(X)$, det is the first Chern class.

Having described what we mean by a positive structure on K, we can now state the Splitting Principle.

II.4.2.2 **Splitting Principle 4.2.2.** The *Splitting Principle* states that for every positive element p in K there is a extension $K \subset K'$ (of λ-rings with positive structure) such that p is a sum of line elements in K'.

The Splitting Principle for $KO(X)$ and $KU(X)$ holds by Ex. 4.12. Using algebraic geometry, we will show in 8.8 that the Splitting Principle holds for $K_0(R)$ as well as K_0 of a scheme. The Splitting Principle also holds for $R(G)$; see [9, 1.5]. The importance of the Splitting Principle lies in its relation to "special λ-rings," a notion we shall define after citing the following motivational result from [59, ch. I].

II.4.2.3 **Theorem 4.2.3.** *If K is a λ-ring with a positive structure, and $\mathbb{N}$ is cofinal in P, the Splitting Principle holds if and only if K is a special λ-ring.*

In order to define special λ-ring, we need the following technical example:

II.4.3 **Example 4.3** (Witt Vectors)**.** For every commutative ring R, the abelian group $W(R) = 1 + tR[[t]]$ has the structure of a commutative ring, natural in R; $W(R)$ is called the ring of (big) *Witt vectors* of R. The multiplicative identity of the ring $W(R)$ is $(1-t)$, and multiplication $*$ is completely determined by naturality, formal factorization of elements of $W(R)$ as $f(t) = \prod_{i=1}^{\infty}(1 - r_i t^i)$ and the formula:

$$(1 - rt) * f(t) = f(rt).$$

It is not hard to see that there are "universal" polynomials P_n in $2n$ variables so:

$$\left(\sum a_i t^i\right) * \left(\sum b_j t^j\right) = \sum c_n t^n, \text{ with } c_n = P_n(a_1, \dots, a_n; b_1, \dots, b_n).$$

If $\mathbb{Q} \subseteq R$ there is an isomorphism $\prod_{n=1}^{\infty} R \to W(R)$, $(r_1, \dots) \mapsto \prod \exp(1 - r_n t^n/n)$.

Grothendieck observed that there are operations λ^k on $W(R)$ making it into a λ-ring; they are defined by naturality, formal factorization and the formula

$$\lambda^k(1 - rt) = 0 \text{ for all } \; k \geq 2.$$

Another way to put it is that there are universal polynomials $P_{n,k}$ such that:

$$\lambda^k(\sum a_i t^i) = \sum b_n t^n, \text{ with } \; b_n = P_{n,k}(a_1, ..., a_{nk}).$$

II.4.3.1 **Definition 4.3.1.** A *special λ-ring* is a λ-ring K such that the group homomorphism $\lambda_t : K \to W(K)$ is a λ-ring homomorphism. Since $\lambda_t(x) = \sum \lambda^k(x) t^k$, a special λ-ring is a λ-ring K such that
- $\lambda^k(1) = 0$ for $k \neq 0, 1$
- $\lambda^k(xy)$ is $P_k(\lambda^1(x), ..., \lambda^k(x); \lambda^1(y), ..., \lambda^k(y))$, and
- $\lambda^n(\lambda^k(x)) = P_{n,k}(\lambda^1(x), ..., \lambda^{nk}(x))$.

II.4.3.2 **Example 4.3.2.** The formula $\lambda^n(s_1) = s_n$ defines a special λ-ring structure on the polynomial ring $U = \mathbb{Z}[s_1, ..., s_n, ...]$; see [AT 9, §2]. It is the free special λ-ring on the generator s_1, because if x is any element in any special λ-ring K then the map $U \to K$ sending s_n to $\lambda^n(x)$ is a λ-ring homomorphism. The λ-ring U cannot have a positive structure by Theorem II.4.6 4.6 below, since U has no nilpotent elements except 0.

Adams operations

For every augmented λ-ring K we can define the *Adams operations* $\psi^k : K \to K$ for $k \geq 0$ by setting $\psi^0(x) = \varepsilon(x)$, $\psi^1(x) = x$, $\psi^2(x) = x^2 - 2\lambda^2(x)$ and inductively

$$\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + \cdots + (-1)^k \lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1} k \lambda^k(x).$$

From this inductive definition we immediately deduce three facts:
- if ℓ is a line element then $\psi^k(\ell) = \ell^k$;
- if I is a λ-ideal with $I^2 = 0$ then $\psi^k(x) = (-1)^{k-1} k \lambda^k(x)$ for all $x \in I$;
- For every binomial ring H we have $\psi^k = 1$. Indeed, the formal identity $x \sum_{i=0}^{k-1} (-1)^i \binom{x}{i} = (-1)^{k+1} k \binom{x}{k}$ shows that $\psi^k(x) = x$ for all $x \in H$.

The operations ψ^k are named after J.F. Adams, who first introduced them in 1962 in his study of vector fields on spheres.

Here is a slicker, more formal presentation of the Adams operations. Define $\psi^k(x)$ to be the coefficient of t^k in the power series:

$$\psi_t(x) = \sum \psi^k(x) t^k = \varepsilon(x) - t \frac{d}{dt} \log \lambda_{-t}(x).$$

The proof that this agrees with the inductive definition of $\psi^k(x)$ is an exercise in formal algebra, which we relegate to Exercise EII.4.6 4.6 below.

II.4.4 **Proposition 4.4.** *Assume K satisfies the Splitting Principle. Then each ψ^k is a ring endomorphism of K, and $\psi^j\psi^k = \psi^{jk}$ for all $j, k \geq 0$.*

Proof. The logarithm in the definition of ψ_t implies that $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$, so each ψ^k is additive. The Splitting Principle and the formula $\psi^k(\ell) = \ell^k$ for line elements yield the formulas $\psi^k(pq) = \psi^k(p)\psi^k(q)$ and $\psi^j(\psi^k(p)) = \psi^{jk}(p)$ for positive p. The extension of these formulas to K is clear. $\square$

II.4.4.1 **Example 4.4.1.** Consider the λ-ring $KU(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ of 3.1.1. On $H^0 = \mathbb{Z}$, $\psi^k = 1$, but on $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$, ψ^k is multiplication by k^n. (See [7, 3.2.2].)

II.4.4.2 **Example 4.4.2.** Consider $KU(\mathbb{R}\mathbb{P}^{2n})$, which by Ex. 3.10 is $\mathbb{Z} \oplus \mathbb{Z}/2^n$. I claim that for all $x \in \widetilde{KU}(\mathbb{R}\mathbb{P}^{2n})$:

$$\psi^k(x) = \begin{cases} x \text{ if } k \text{ is odd} \\ 0 \text{ if } k \text{ is even.} \end{cases}$$

To see this, note that $\widetilde{KU}(\mathbb{R}\mathbb{P}^{2n}) \cong \mathbb{Z}/2^n$ is additively generated by $(\ell - 1)$, where ℓ is the nonzero element of $L = H^2(\mathbb{R}\mathbb{P}^{2n}; \mathbb{Z}) = \mathbb{Z}/2$. Since $\ell^2 = 1$, we see that $\psi^k(\ell - 1) = (\ell^k - 1)$ is 0 if k is even and $(\ell - 1)$ if k is odd. The assertion follows.

γ-operations

Associated to the operations λ^k are the operations $\gamma^k \colon K \to K$. To construct them, we assume that $\lambda^k(1) = 0$ for $k \neq 0, 1$. Note that if we set $s = t/(1-t)$ then $K[[t]] = K[[s]]$ and $t = s/(1+s)$. Therefore we can rewrite $\lambda_s(x) = \sum \lambda^i(x)s^i$ as a power series $\gamma_t(x) = \sum \gamma^k(x)t^k$ in t. By definition, $\gamma^k(x)$ is the coefficient of t^k in $\gamma_t(x)$. Since $\gamma_t(x) = \lambda_s(x)$ we have $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$. In particular $\gamma^0(x) = 1$, $\gamma^1(x) = x$ and $\gamma^k(x+y) = \sum \gamma^i(x)\gamma^{k-i}(y)$. That is, the γ-operations satisfy the axioms for a λ-ring structure on K. An elementary calculation, left to the reader, yields the useful identity:

II.4.5 **Formula 4.5.** $\gamma^k(x) = \lambda^k(x + k - 1)$. This implies that $\gamma^2(x) = \lambda^2(x) + x$ and

$$\gamma^k(x) = \lambda^k(x+k-1) = \lambda^k(x) + \binom{k-1}{1}\lambda^{k-1}(x) + \cdots + \binom{k-1}{k-2}\lambda^2(x) + x.$$

II.4.5.1 **Example 4.5.1.** If H is a binomial ring then for all $x \in H$ we have

$$\gamma^k(x) = \binom{x+k-1}{k} = (-1)^k\binom{-x}{k}.$$

II.4.5.2 **Example 4.5.2.** $\gamma^k(1) = 1$ for all k, because $\lambda_s(1) = 1 + s = 1/(1-t)$. More generally, if ℓ is a line element then $\gamma^k(\ell) = \ell$ for all $k \geq 1$.

II.4.5.3 **Lemma 4.5.3.** *If $p \in P$ is a positive element with $\varepsilon(p) = n$, then $\gamma^k(p - n) = 0$ for all $k > n$. In particular, if $\ell \in K$ is a line element then $\gamma^k(\ell - 1) = 0$ for every $k > 1$.*

Proof. If $k > n$ then $q = p + (k - n - 1)$ is a positive element with $\varepsilon(q) = k - 1$. Thus $\gamma^k(p - n) = \lambda^k(q) = 0$. □

If $x \in K$, the γ-*dimension* $\dim_\gamma(x)$ of x is defined to be the largest integer n for which $\gamma^n(x - \varepsilon(x)) \neq 0$, provided n exists. For example, $\dim_\gamma(h) = 0$ for every $h \in H^0$ and $\dim_\gamma(\ell) = 1$ for every line element ℓ (except $\ell = 1$ of course). By the above remarks if $p \in P$ and $n = \varepsilon(p)$ then $\dim_\gamma(p) = \dim_\gamma(p - n) \leq n$. The supremum of the $\dim_\gamma(x)$ for $x \in K$ is called the γ-*dimension* of K.

II.4.5.4 **Example 4.5.4.** If R is a commutative noetherian ring, the Bass-Serre Cancellation I.2.4 states that every element of $\widetilde{K}_0(R)$ is represented by $[P] - n$, where P is a finitely generated projective module of rank $< \dim(R)$. Hence $K_0(R)$ has γ-dimension at most $\dim(R)$.

Suppose that X is a CW complex with finite dimension d. The Real Cancellation Theorem I.4.3 allows us to use the same argument to deduce that $KO(X)$ has γ-dimension at most d; the Complex Cancellation Theorem I.4.4 shows that $KU(X)$ has γ-dimension at most $d/2$.

II.4.5.5 **Corollary 4.5.5.** *If K has a positive structure in which $\mathbb{N}$ is cofinal in P, then every element of $\widetilde{K}$ has finite γ-dimension.*

Proof. Recall that "$\mathbb{N}$ is cofinal in P" means that for every p there is a p' so that $p + p' = n$ for some $n \in \mathbb{N}$. Therefore every $x \in \widetilde{K}$ can be written as $x = p - m$ for some $p \in P$ with $m = \varepsilon(p)$. By Lemma 4.5.3, $\dim_\gamma(x) \leq m$. □

II.4.6 **Theorem 4.6.** *If every element of K has finite γ-dimension (e.g., K has a positive structure in which $\mathbb{N}$ is cofinal in P), then $\widetilde{K}$ is a nil ideal. That is, every element of $\widetilde{K}$ is nilpotent.*

Proof. Fix $x \in \widetilde{K}$, and set $m = \dim_\gamma(x)$, $n = \dim_\gamma(-x)$. Then both $\gamma_t(x) = 1 + xt + \gamma^2(x)t^2 + \cdots + \gamma^m(x)t^m$ and $\gamma_t(-x) = 1 - xt + \cdots + \gamma^n(-x)t^n$ are polynomials in t. Since $\gamma_t(x)\gamma_t(-x) = \gamma_t(0) = 1$, the polynomials $\gamma_t(x)$ and $\gamma_t(-x)$ are units in the polynomial ring $K[t]$. By I.3.12, the coefficients of these polynomials are nilpotent elements of K. □

II.4.6.1 **Corollary 4.6.1.** *The ideal $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$ for every commutative ring R.*

If X is compact then $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are the nilradicals of the rings $KO(X)$ and $KU(X)$, respectively.

II.4.6.2 **Example 4.6.2.** The conclusion of Theorem 4.6 fails for the representation ring $R(G)$ of a cyclic group of order 2. If σ denotes the 1-dimensional sign representation, then $L = \{1, \sigma\}$ and $\widetilde{R}(G) \cong \mathbb{Z}$ is generated by $(\sigma - 1)$. Since $(\sigma - 1)^2 = (\sigma^2 - 2\sigma + 1) = (-2)(\sigma - 1)$, we see that $(\sigma - 1)$ is not nilpotent, and in fact that $\widetilde{R}(G)^n = (2^{n-1})\widetilde{R}(G)$ for every $n \geq 1$. The hypothesis of Corollary 4.5.5 fails here because σ cannot be a summand of a trivial representation. In fact $\dim_\gamma(1 - \sigma) = \infty$, because $\gamma^n(1 - \sigma) = (1 - \sigma)^n = 2^{n-1}(1 - \sigma)$ for all $n \geq 1$.

The γ-Filtration

The *γ-filtration* on K is a descending sequence of ideals:

$$K = F^0_\gamma K \supset F^1_\gamma K \supset \cdots \supset F^n_\gamma K \supset \cdots .$$

It starts with $F^0_\gamma K = K$ and $F^1_\gamma K = \widetilde{K}$ (the kernel of ε). The first quotient F^0_γ/F^1_γ is clearly $H^0 = K/\widetilde{K}$. For $n \geq 2$, $F^n_\gamma K$ is defined to be the ideal of K generated by the products $\gamma^{k_1}(x_1)\cdots\gamma^{k_m}(x_m)$ with $x_i \in \widetilde{K}$ and $\sum k_i \geq n$. In particular, $F^n_\gamma K$ contains $\gamma^k(x)$ for all $x \in \widetilde{K}$ and $k \geq n$.

It follows immediately from the definition that $F^i_\gamma F^j_\gamma \subseteq F^{i+j}_\gamma$. For $j = 1$, this implies that the quotients $F^i_\gamma K/F^{i+1}_\gamma K$ are H^0-modules. We will prove that the quotient F^1_γ/F^2_γ is the group L of line elements in K:

II.4.7 **Theorem 4.7.** *If K satisfies the Splitting Principle, then the map $\ell \mapsto \ell - 1$ induces a group isomorphism, split by the map* det*:*

$$L \xrightarrow{\cong} F^1_\gamma K/F^2_\gamma K.$$

II.4.7.1 **Corollary 4.7.1.** *For every commutative ring R, the first two ideals in the γ-filtration of $K_0(R)$ are $F^1_\gamma = \widetilde{K}_0(R)$ and $F^2_\gamma = SK_0(R)$. (See 2.6.2.) In particular,*

$$F^0_\gamma/F^1_\gamma \cong H_0(R) \text{ and } \quad F^1_\gamma/F^2_\gamma \cong \operatorname{Pic}(R).$$

II.4.7.2 **Corollary 4.7.2.** *The first two quotients in the γ-filtration of $KO(X)$ are*

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \quad \text{and} \quad F^1_\gamma/F^2_\gamma \cong H^1(X;\mathbb{Z}/2).$$

The first few quotients in the γ-filtration of $KU(X)$ are

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \text{ and } \quad F^1_\gamma/F^2_\gamma \cong H^2(X;\mathbb{Z}).$$

For the proof of Theorem 4.7, we shall need the following consequence of the Splitting Principle. A proof of this principle may be found in Fulton-Lang [59, III.1].

Filtered Splitting Principle. Let K be a λ-ring satisfying the Splitting Principle, and let x be an element of $F^n_\gamma K$. Then there exists a λ-ring extension $K \subset K'$ such that $F^n_\gamma K = K \cap F^n_\gamma K'$, and x is an H-linear combination of products $(\ell_1 - 1)\cdots(\ell_m - 1)$, where the ℓ_i are line elements of K' and $m \geq n$.

Proof of Theorem 4.7. Since $(\ell_1 - 1)(\ell_2 - 1) \in F^2_\gamma K$, the map $\ell \mapsto \ell - 1$ is a homomorphism. If ℓ_1, ℓ_2, ℓ_3 are line elements of K,

$$\det((\ell_1 - 1)(\ell_2 - 1)\ell_3) = \det(\ell_1\ell_2\ell_3)\det(\ell_3)/det(\ell_1\ell_3)\det(\ell_2\ell_3) = 1.$$

By Ex. 4.3, the Filtered Splitting Principle implies that every element of $F^2_\gamma K$ can be written as a sum of terms $(\ell_1 - 1)(\ell_2 - 1)\ell_3$ in some extension K' of K. This shows that $\det(F^2_\gamma) = 1$, so det induces a map $\widetilde{K}/F^2_\gamma K \to L$. Now det is the inverse of the map $\ell \mapsto \ell - 1$ because for $p \in P$ the Splitting Principle shows that $p - \varepsilon(p) \equiv \det(p) - 1$ modulo $F^2_\gamma K$. □

II.4.8 **Proposition 4.8.** *If the γ-filtration on K is finite then $\widetilde{K}$ is a nilpotent ideal. If $\widetilde{K}$ is a nilpotent ideal which is finitely generated as an abelian group, then the γ-filtration on K is finite. That is, $F_\gamma^N K = 0$ for some N.*

Proof. The first assertion follows from the fact that $\widetilde{K}^n \subset F_\gamma^n K$ for all n. If $\widetilde{K}$ is additively generated by $\{x_1, ..., x_s\}$, then there is an upper bound on the k for which $\gamma^k(x_i) \neq 0$; using the sum formula there is an upper bound n on the k for which γ^k is nonzero on $\widetilde{K}$. If $\widetilde{K}^m = 0$ then clearly we must have $F_\gamma^{mn} K = 0$. □

II.4.8.1 **Example 4.8.1.** If X is a finite CW complex, both $KO(X)$ and $KU(X)$ are finitely generated abelian groups by Ex. 3.9. Therefore they have finite γ-filtrations.

II.4.8.2 **Example 4.8.2.** If R is a commutative noetherian ring of Krull dimension d, then $F_\gamma^{d+1} K_0(R) = 0$ by [59, V.3.10], even though $K_0(R)$ may not be a finitely generated abelian group.

II.4.8.3 **Example 4.8.3.** For the representation ring $R(G)$, G cyclic of order 2, we saw in Example 4.6.2 that $\widetilde{R}$ is not nilpotent. In fact $F_\gamma^n R(G) = \widetilde{R}^n = 2^{n-1}\widetilde{R} \neq 0$. An even worse example is the λ-ring $R_\mathbb{Q} = R(G) \otimes \mathbb{Q}$, because $F_\gamma^n R_\mathbb{Q} = \widetilde{R}_\mathbb{Q} \cong \mathbb{Q}$ for all $n \geq 1$.

II.4.8.4 **Remark 4.8.4.** Fix $x \in \widetilde{K}$. It follows from the nilpotence of the $\gamma^k(x)$ that there is an integer N such that $x^N = 0$, and for every $k_1, \dots, k_n$ with $\sum k_i \geq N$ we have

$$\gamma^{k_1}(x)\gamma^{k_2}(x) \cdots \gamma^{k_n}(x) = 0.$$

The best general bound for such an N is $N = mn = \dim_\gamma(x) \dim_\gamma(-x)$.

II.4.9 **Proposition 4.9.** *Let k, $n \geq 1$ be integers. If $x \in F_\gamma^n K$ then modulo $F_\gamma^{n+1} K$:*

$$\psi^k(x) \equiv k^n x; \qquad \textit{and} \qquad \lambda^k(x) \equiv (-1)^k k^{n-1} x.$$

Proof. If ℓ is a line element then modulo $(\ell - 1)^2$ we have

$$\psi^k(\ell - 1) = (\ell^{k-1} + ... + \ell + 1)(\ell - 1) \equiv k(\ell - 1).$$

Therefore if $\ell_1, \cdots, \ell_m$ are line elements and $m \geq n$ we have

$$\psi^k((\ell_1 - 1) \cdots (\ell_n - 1)) \equiv k^n (\ell_1 - 1) \cdots (\ell_n - 1) \text{ modulo } F_\gamma^{n+1} K.$$

The Filtered Splitting Principle implies that $\psi^k(x) \equiv k^n x$ modulo $F_\gamma^{n+1} K$ for every $x \in F_\gamma^n K$. For λ^k, we use the inductive definition of ψ^k to see that $k^n x = (-1)^{k-1} k \lambda^k(x)$ for every $x \in F_\gamma^n K$. The Filtered Splitting Principle allows us to consider the universal case $W = W_s$ of Exercise 4.4. Since there is no torsion in $F_\gamma^n W / F_\gamma^{n+1} W$, we can divide by k to obtain the formula $k^{n-1} x = (-1)^{k-1} \lambda^k(x)$. □

II.4.10 **Theorem 4.10** (Structure of $K \otimes \mathbb{Q}$). *Suppose that a λ-ring K has a positive structure in which every element has finite γ-dimension e.g., if $\mathbb{N}$ is cofinal in P. Then:*

(1) The eigenvalues of ψ^k on $K_\mathbb{Q} = K \otimes \mathbb{Q}$ are a subset of $\{1, k, k^2, k^3, ...\}$ for each k;

(2) The subspace $K_\mathbb{Q}^{(n)} = K_\mathbb{Q}^{(n,k)}$ of eigenvectors for $\psi^k = k^n$ is independent of k;

(3) $K_\mathbb{Q}^{(n)}$ is isomorphic to $F_\gamma^n K_\mathbb{Q}/F_\gamma^{n+1} K_\mathbb{Q} \cong (F_\gamma^n K/F_\gamma^{n+1} K) \otimes \mathbb{Q}$;

(4) $K_\mathbb{Q}^{(0)} \cong H^0 \otimes \mathbb{Q}$ and $K_\mathbb{Q}^{(1)} \cong L \otimes \mathbb{Q}$;

(5) $K \otimes \mathbb{Q}$ is isomorphic to the graded ring $K_\mathbb{Q}^{(0)} \oplus K_\mathbb{Q}^{(1)} \oplus \cdots \oplus K_\mathbb{Q}^{(n)} \oplus \cdots$.

Proof. For every positive p, consider the universal λ-ring $U_\mathbb{Q} = \mathbb{Q}[s_1, ...]$ of Example II.4.3.2 4.3.2, and the map $U_\mathbb{Q} \to K_\mathbb{Q}$ sending s_1 to p and s_k to $\lambda^k(p)$. If $\varepsilon(p) = n$ then s_i maps to zero for $i > n$ and each $s_i - \binom{n}{i}$ maps to a nilpotent element by Theorem II.4.6 4.6. The image A of this map is a λ-ring which is finite-dimensional over $\mathbb{Q}$, so A is an artinian ring. Clearly $F_\gamma^N A = 0$ for some large N. Consider the linear operation $\prod_{n=0}^N(\psi^k - k^n)$ on A; by Proposition II.4.9 4.9 it is trivial on each $F_\gamma^n/F_\gamma^{n+1}$, so it must be zero. Therefore the characteristic polynomial of ψ^k on A divides $\Pi(t - k^n)$, and has distinct integer eigenvalues. This proves (1) and that $K_\mathbb{Q}$ is the direct sum of the eigenspaces $K_\mathbb{Q}^{(n,k)}$ for ψ^k. As ψ^k preserves products, Proposition II.4.9 4.9 now implies (3) and (4). The rest is immediate from Theorem II.4.7 4.7. ☐

Chern class homomorphisms

II.4.11 **4.11.** The formalism in §3 for the Chern classes $c_i\colon KU(X) \to H^{2i}(X;\mathbb{Z})$ extends to the current setting. Suppose we are given a λ-ring K with a positive structure and a commutative graded ring $A = A^0 \oplus A^1 \oplus \cdots$. *Chern classes* on K with values in A are set maps $c_n\colon K \to A^n$ for $n \geq 0$ with $c_0(x) = 1$, satisfying the following axioms:

(CC0) The c_n send H^0 to zero (for $n \geq 1$): $c_n(h) = 0$ for every $h \in H^0$.

(CC1) *Dimension.* $c_n(p) = 0$ whenever p is positive and $n \geq \varepsilon(p)$.

(CC2) *Sum Formula.* For every x, y in K and every n:

$$c_n(x + y) = \sum_{i=0}^{n} c_i(x)c_{n-i}(y).$$

(CC3) *Normalization.* $c_1\colon L \to A^1$ is a group homomorphism. That is, for ℓ, ℓ':

$$c_1(\ell\ell') = c_1(\ell) + c_1(\ell').$$

The *total Chern class* of x is the element $c(x) = \sum c_i(x)$ of the completion $\hat{A} = \prod A^i$ of A. In terms of the total Chern class, (CC2) becomes the product formula

$$c(x+y) = c(x)c(y).$$

II.4.11.1 **Example 4.11.1.** The Stiefel–Whitney classes $w_i\colon KO(X) \to H^i(X;\mathbb{Z}/2)$ ($= A^i$) and the Chern classes $c_i\colon KU(X) \to H^{2i}(X;\mathbb{Z})$ ($= A^i$) are both Chern classes in this sense.

II.4.11.2 **Example 4.11.2.** Associated to the γ-filtration on K we have the associated graded ring $Gr_\gamma^\bullet K$ with $Gr_\gamma^i K = F_\gamma^i / F_\gamma^{i+1}$. For a positive element p in K, define $c_i(p)$ to be $\gamma^i(p - \varepsilon(p))$ modulo F_γ^{i+1}. The multiplicative formula for γ_t implies that $c_i(p+q) = c_i(p) + c_i(q)$, so that the c_i extend to classes $c_i\colon K \to Gr_\gamma^\bullet K$. The total Chern class $c\colon K \to Gr_\gamma^\bullet K$ is a group homomorphism with torsion kernel and cokernel, because by Theorem 4.10 and Ex. 4.10 the induced map $c_n\colon K_{\mathbb{Q}}^{(n)} \to Gr_\gamma^n K_{\mathbb{Q}} \cong K_{\mathbb{Q}}^{(n)}$ is multiplication by $(-1)^n(n-1)!$.

The Splitting Principle 4.2.2 implies the following additional Splitting Principle (see [59, I.3.1]).

Chern Splitting Principle. Given a finite set $\{p_i\}$ of positive elements of K, there is a λ-ring extension $K \subset K'$ in which each p_i splits as a sum of line elements, and a graded extension $A \subset A'$ such that the c_i extend to maps $c_i\colon K' \to (A')^i$ satisfying (CC1) and (CC2).

The existence of "Chern roots" is an important consequence of this Splitting Principle. Suppose that $p \in K$ is positive, and that in an extension K' of K we can write $p = \ell_1 + \cdots + \ell_n$, $n = \varepsilon(p)$. The *Chern roots of* p are the elements $a_i = c_1(\ell_i)$ in $(A')^1$; they determine the $c_k(p)$ in A^k. Indeed, because $c(p)$ is the product of the $c(\ell_i) = 1 + a_i$, we see that $c_k(p)$ is the k^{th} elementary symmetric polynomial $\sigma_k(a_1, ..., a_n)$ of the a_i in the larger ring A'. In particular, the first Chern class is $c_1(p) = \sum a_i$ and the "top" Chern class is $c_n(p) = \prod a_i$.

A famous theorem of Isaac Newton states that every symmetric polynomial in n variables $t_1, ..., t_n$ is in fact a polynomial in the symmetric polynomials $\sigma_k = \sigma_k(t_1, ..., t_n)$, $k = 1, 2, \cdots$. Therefore every symmetric polynomial in the Chern roots of p is also a polynomial in the Chern classes $c_k(p)$, and as such belongs to the subring A of A'. Here is an elementary application of these ideas.

II.4.11.3 **Proposition 4.11.3.** *Suppose that K satisfies the Splitting Principle 4.2.2. Then $c_n(\psi^k x) = k^n c_n(x)$ for all $x \in K$. That is, the following diagram commutes:*

$$\begin{array}{ccc} K & \xrightarrow{c_n} & A^n \\ \psi^k \downarrow & & \downarrow k^n \\ K & \xrightarrow{c_n} & A^n. \end{array}$$

II.4.11.4 **Corollary 4.11.4.** *If $\mathbb{Q} \subset A$ then c_n vanishes on $K_{\mathbb{Q}}^{(i)}$, $i \neq n$.*

Chern character

As an application of the notion of Chern roots, suppose given Chern classes $c_i\colon K \to A^i$, where for simplicity A is an algebra over $\mathbb{Q}$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch(p)$ to be the formal expansion

$$ch(p) = \sum_{i=0}^{n} \exp(a_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. The k^{th} term $\frac{1}{k!}\sum a_i^k$ is symmetric in the Chern roots, so it is a polynomial in the Chern classes $c_1(p), ..., c_k(p)$ and hence belongs to A^k. Therefore $ch(p)$ is a formal expansion of terms in A, *i.e.,* an element of $\hat{A} = \prod A^k$. For example, if ℓ is a line element of K then $ch(\ell)$ is just $\exp(c_1(\ell))$. From the definition, it is immediate that $ch(p+q) = ch(p)+ch(q)$, so ch extends to a map from $P^{-1}P$ to $\hat{A}$. Since $ch(1) = 1$, this is compatible with the given map $H^0 \to A^0$, and so it defines a map $ch\colon K \to \hat{A}$, called the *Chern character* on K. The first few terms in the expansion of the Chern character are

$$ch(x) = \varepsilon(x) + c_1(x) + \frac{1}{2}[c_1(x)^2 - c_2(x)] + \frac{1}{6}[c_1(x)^3 - 3c_1(x)c_2(x) + 3c_3(x)] + \cdots$$

An inductive formula for the term in $ch(x)$ is given in Exercise EII.4.14 4.14.

II.4.12 **Proposition 4.12.** *If $\mathbb{Q} \subset A$ then the Chern character is a ring homomorphism*

$$ch\colon K \to \hat{A}.$$

Proof. By the Splitting Principle, it suffices to verify that $ch(pq) = ch(p)ch(q)$ when p and q are sums of line elements. Suppose that $p = \sum \ell_i$ and $q = \sum m_j$ have Chern roots $a_i = c_1(\ell_i)$ and $b_j = c_1(m_j)$, respectively. Since $pq = \sum \ell_i m_j$, the Chern roots of pq are the $c_1(\ell_i m_j) = c_1(\ell_i) + c_1(m_j) = a_i + b_j$. Hence

$$ch(pq) = \sum ch(\ell_i m_j) = \sum \exp(a_i + b_j) = \sum \exp(a_i)\exp(b_j) = ch(p)ch(q). \quad \Box$$

II.4.12.1 **Corollary 4.12.1.** *Suppose that K has a positive structure in which every $x \in K$ has finite γ-dimension (e.g., $\mathbb{N}$ is cofinal in P). Then the Chern character lands in A, and the induced map from $K_{\mathbb{Q}} = \oplus K_{\mathbb{Q}}^{(n)}$ to A is a graded ring map. That is, the n^{th} term $ch_n\colon K_{\mathbb{Q}} \to A^n$ vanishes on $K_{\mathbb{Q}}^{(i)}$ for $i \neq n$.*

II.4.12.2 **Example 4.12.2.** The universal Chern character $ch\colon K_{\mathbb{Q}} \to K_{\mathbb{Q}}$ is the identity map. Indeed, by Ex. EII.4.10 4.10(b) and Ex. EII.4.14 4.14 we see that ch_n is the identity map on each $K_{\mathbb{Q}}^{(n)}$.

The following result was proven by M. Karoubi in Kar63 [99]. (See Exercise EII.4.11 4.11 for the proof when X is a finite CW complex.)

II.4.13 **Theorem 4.13.** *If X is a compact topological space and $\check{H}$ denotes Čech cohomology, then the Chern character is an isomorphism of graded rings.*

$$ch\colon KU(X) \otimes \mathbb{Q} \cong \bigoplus \check{H}^{2i}(X;\mathbb{Q})$$

II.4.13.1 **Example 4.13.1** (Spheres). For each even sphere, we know by Example II.3.7.1 3.7.1 that c_n maps $\widetilde{KU}(S^{2n})$ isomorphically onto $H^{2n}(S^{2n};\mathbb{Z}) = \mathbb{Z}$. The inductive formula for ch_n shows that in this case $ch(x) = \dim(x) + (-1)^n c_n(x)/(n-1)!$ for all $x \in KU(X)$. In this case it is easy to see directly that $ch\colon KU(S^{2n}) \otimes \mathbb{Q} \cong H^{2*}(S^{2n};\mathbb{Q})$

EXERCISES

EII.4.1 **4.1.** Show that in $K_0(R)$ or $K^0(X)$ we have

$$\lambda^k([P]-n) = \sum (-1)^i \binom{n+i-1}{i} [\wedge^{k-i}P].$$

EII.4.2 **4.2.** For every group G and every commutative ring A, let $R_A(G)$ denote the group $K_0(AG, A)$ of Ex. EII.2.14 2.14, *i.e.,* the group completion of the monoid $Rep(AG, A)$ of all AG-modules which are finitely generated projective as A-modules. Show that the $\wedge^k$ make $R_A(G)$ into a λ-ring with a positive structure given by $Rep(AG, A)$.

(a) If $A = \mathbb{C}$, show that $R_{\mathbb{C}}(G)$ satisfies the Splitting Principle and hence is a special λ-ring (by II.4.2.3 4.2.3); the line elements are the characters. Swan proved in Swan70 [194] that $R_A(G)$ satisfies the Splitting Principle for every A; another proof is in SGA6 [SGA6], VI(3.3).) This proves that $R_A(G)$ is a special λ-ring for every A.

(b) When $p = 0$ in A, show that $\psi^p = \Phi^*$ in $R_A(G)$, where $\Phi : A \to A$ is the Frobenius $\Phi(a) = a^p$. To do this, reduce to the case in which χ is a character and show that $\psi^k\chi(g) = \chi(g^p) = \chi(g)^p$.

EII.4.3 **4.3.** Suppose that a λ-ring K is generated as an H-algebra by line elements. Show that $F_\gamma^n = \widetilde{K}^n$ for all n, so the γ-filtration is the adic filtration defined by the ideal $\widetilde{K}$. Then show that if K is any λ-ring satisfying the Splitting Principle every element x of $F_\gamma^n K$ can be written in an extension K' of K as a product

$$x = (\ell_1 - 1)\cdots(\ell_m - 1)$$

of line elements with $m \geq n$. In particular, show that every $x \in F_\gamma^2$ can be written as a sum of terms $(\ell_i - 1)(\ell_j - 1)\ell$ in K'.

EII.4.4 **4.4.** *Universal special λ-ring.* Let W_s denote the Laurent polynomial ring $\mathbb{Z}[u_1, u_1^{-1}, ..., u_s, u_s^{-1}]$, and $\varepsilon\colon W_s \to \mathbb{Z}$ the augmentation defined by $\varepsilon(u_i) = 1$.

(a) Show that W_s is a λ-ring with a positive structure; the line elements are the monomials $u^\alpha = \prod u_i^{n_i}$. This implies that W_s is generated by the group $L \cong \mathbb{Z}^s$ of line elements, so by Exercise EII.4.3 4.3 the ideal $F_\gamma^n W_s$ is $\widetilde{W}^n$.

(b) Show that each $F_\gamma^n W / F_\gamma^{n+1} W$ is a torsionfree abelian group.

(c) If K is a special λ-ring show that any family $\{\ell_1, ..., \ell_s\}$ of line elements determines a λ-ring map $W_s \to K$ sending u_i to ℓ_i.

(d) (Splitting Principle for the free λ-ring) Let $U \to W_s$ be the λ-ring homomorphism sending s_1 to $\sum u_i$ (see 4.3.2). Show that U injects into $\varprojlim W_s$.

EII.4.5 **4.5.** A line element ℓ is called *ample* for K if for every $x \in \widetilde{K}$ there is an integer $N = N(x)$ such that for every $n \geq N$ there is a positive element p_n so that $\ell^n x = p_n - \varepsilon(p_n)$. (The terminology comes from Algebraic Geometry; see 8.8.4 below.) If K has an ample line element, show that every element of $\widetilde{K}$ is nilpotent.

EII.4.6 **4.6.** Verify that the inductive definition of ψ^k and the ψ_t definition of ψ^k agree.

EII.4.7 **4.7.** If p is prime, use the Splitting Principle to verify that $\psi^p(x) \equiv x^p$ modulo p for every $x \in K$.

EII.4.8 **4.8.** *Adams e-invariant.* Suppose given a map $f\colon S^{2m-1} \to S^{2n}$. The mapping cone $C(f)$ fits into a cofibration sequence $S^{2n} \overset{i}{\rightarrowtail} C(f) \overset{j}{\twoheadrightarrow} S^{2m}$. Associated to this is the exact sequence:

$$0 \to \widetilde{KU}(S^{2m}) \xrightarrow{j^*} \widetilde{KU}(C) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C)$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{2m}) \cong \mathbb{Z}$. Since j^* is a ring map, $y^2 = 0$.

(a) Show by applying ψ^k that $xy = 0$, and that if $m \neq 2n$ then $x^2 = 0$. (When $m = 2n$, x^2 defines the Hopf invariant of f; see the next exercise.)

(b) Show that $\psi^k(x) = k^n x + a_k y$ for appropriate integers a_k. Then show (for fixed x and y) that the rational number

$$e(f) = \frac{a_k}{k^m - k^n}$$

is independent of the choice of k.

(c) Show that a different choice of x only changes $e(f)$ by an integer, so that $e(f)$ is a well-defined element of $\mathbb{Q}/\mathbb{Z}$; $e(f)$ is called the *Adams e-invariant* of f.

(d) If f and f' are homotopic maps, it follows from the homotopy equivalence between $C(f)$ and $C(f')$ that $e(f) = e(f')$. By considering the mapping cone of $f \vee g$, show that the well-defined set map $e\colon \pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$ is a group homomorphism. J.F. Adams used this e-invariant to detect an important cyclic subgroup of $\pi_{2m-1}(S^{2n})$, namely the "image of J."

EII.4.9 **4.9.** *Hopf Invariant One.* Given a continuous map $f\colon S^{4n-1} \to S^{2n}$, define an integer $H(f)$ as follows. Let $C(f)$ be the mapping cone of f. As in the previous exercise, we have an exact sequence:

$$0 \to \widetilde{KU}(S^{4n}) \xrightarrow{j^*} \widetilde{KU}(C(f)) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C(f))$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{4n}) \cong \mathbb{Z}$. Since $i^*(x^2) = 0$, we can write $x^2 = Hy$ for some integer H; this integer $H = H(f)$ is called the *Hopf invariant* of f.

(a) Show that $H(f)$ is well-defined, up to $\pm$ sign.

(b) If $H(f)$ is odd, show that n is 1, 2, or 4. *Hint:* Use Ex. EII.4.7 4.7 to show that the integer a_2 of the previous exercise is odd. Considering $e(f)$, show that 2^n divides $p^n - 1$ for every odd p.

It turns out that the classical "Hopf maps" $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ all have Hopf invariant $H(f) = 1$. In contrast, for every even integer H there is a map $S^{4n-1} \to S^{2n}$ with Hopf invariant H.

EII.4.10 **4.10.** *Operations.* A natural *operation* τ on λ-rings is a map $\tau\colon K \to K$ defined for every λ-ring K such that $f\tau = \tau f$ for every λ-ring map $f\colon K \to K'$. The operations λ^k, γ^k, and ψ^k are all natural operations on λ-rings.

(a) If K satisfies the Splitting Principle II.4.2.2 4.2.2, generalize Proposition II.4.9 4.9 to show that every natural operation τ preserves the γ-filtration of K and that there are integers $\omega_n = \omega_n(\tau)$, independent of K, such that for every $x \in F^n_\gamma K$
$$\tau(x) \equiv \omega_n x \text{ modulo } F^{n+1}_\gamma K.$$

(b) Show that for $\tau = \gamma^k$ and $x \in F^n_\gamma$ we have:
$$\omega_n(\gamma^k) = \begin{cases} 0 \quad \text{if} \quad n < k \\ (-1)^{k-1}(k-1)! \text{ if} \quad n = k \\ \omega_n \neq 0 \quad \text{if} \quad n > k \end{cases}$$

(c) Show that $s_k \mapsto \lambda^k$ and $\tau \mapsto \tau(s_1)$ give λ-ring isomorphisms from the free λ-ring $U = \mathbb{Z}[s_1, s_2, \cdots]$ of II.4.3.2 4.3.2 to the ring of all natural operations on λ-rings. (See Atiyah [7, 3.1.7].)

EII.4.11 **4.11.** By Example II.4.13.1 4.13.1, the Chern character $ch\colon KU(S^n) \otimes \mathbb{Q} \to H^{2*}(S^n; \mathbb{Q})$ is an isomorphism for every sphere S^n. Use this to show that $ch\colon KU(X) \otimes \mathbb{Q} \to H^{2*}(X; \mathbb{Q})$ is an isomorphism for every finite CW complex X. *Hint:* If X_n is the n-skeleton of X then X_n/X_{n-1} is a bouquet of n-spheres.

EII.4.12 **4.12.** Let K be a λ-ring. Given a K-module M, construct the ring $K \oplus M$ in which $M^2 = 0$. Given a sequence of K-linear endomorphisms φ_k of M with $\varphi_1(x) = x$, show that the formulae $\lambda^k(x) = \varphi_k(x)$ extend the λ-ring structure on K to a λ-ring structure on $K \oplus M$. Then show that $K \oplus M$ has a positive structure if K does, and that $K \oplus M$ satisfies the Splitting Principle whenever K does. (The elements in $1 + M$ are to be the new line elements.)

EII.4.13 **4.13.** *Hirzebruch characters.* Suppose that A is an H^0-algebra and we fix a power series $\alpha(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ in $A^0[[t]]$. Suppose given Chern classes $c_i\colon K \to A^i$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch_\alpha(p)$ to be the formal expansion

$$ch_\alpha(p) = \sum_{i=0}^{n} \alpha(a_i) \sum_{k=0}^{\infty} \alpha_k \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. Show that $ch_\alpha(p)$ belongs to the formal completion $\hat{A}$ of A, and that it defines a group homomorphism $ch_\alpha\colon K \to \hat{A}$. This map is called the *Hirzebruch character* for α.

EII.4.14 **4.14.** Establish the following inductive formula for the n^{th} term ch_n in the Chern character:

$$ch_n - \frac{1}{n}\, c_1 ch_{n-1} + \cdots \pm \frac{1}{i!\binom{n}{i}}\, c_i ch_{n-i} + \cdots + \frac{(-1)^n}{(n-1)!}\, c_n = 0.$$

To do this, set $x = -t_i$ in the identity $\prod(x + a_i) = x^n + c_1 x^{n-1} + \cdots + c_n$.

5 K_0 of a Symmetric Monoidal Category

The idea of group completion in §1 can be applied to more categories than just the categories $\mathbf{P}(R)$ in §2 and $\mathbf{VB}(X)$ in §3. It applies to any category with a "direct sum", or more generally any natural product $\Box$ making the isomorphism classes of objects into an abelian monoid. This leads us to the notion of a symmetric monoidal category.

II.5.1 **Definition 5.1.** A *symmetric monoidal category* is a category S, equipped with a functor $\Box\colon S \times S \to S$, a distinguished object e and four basic natural isomorphisms:

$$e\Box s \cong s, \quad s\Box e \cong s, \quad s\Box(t\Box u) \cong (s\Box t)\Box u, \quad \text{and } s\Box t \cong t\Box s.$$

These basic isomorphisms must be "coherent" in the sense that two natural isomorphisms of products of $s_1, \dots, s_n$ built up from the four basic ones are the same whenever they have the same source and target. (We refer the reader to [Mac 116] for the technical details needed to make this definition of "coherent" precise.) Coherence permits us to write expressions without parentheses like $s_1\Box \cdots \Box s_n$ unambiguously (up to natural isomorphism).

II.5.1.1 **Example 5.1.1.** Any category with a direct sum $\oplus$ is symmetric monoidal; this includes additive categories like $\mathbf{P}(R)$ and $\mathbf{VB}(X)$ as we have mentioned. More generally, a category with finite coproducts is symmetric monoidal with $s\Box t = s \amalg t$. Dually, any category with finite products is symmetric monoidal with $s\Box t = s \times t$.

II.5.1.2 **Definition 5.1.2** ($K_0 S$)**.** Suppose that the isomorphism classes of objects of S form a *set*, which we call S^{iso}. If S is symmetric monoidal, this set S^{iso} is an abelian monoid with product $\Box$ and identity e. The group completion of this abelian monoid is called the *Grothendieck group* of S, and is written as $K_0^{\Box}(S)$, or simply as $K_0(S)$ if $\Box$ is understood.

From §1 we see that $K_0^{\Box}(S)$ may be presented with one generator $[s]$ for each isomorphism class of objects, with relations that $[s \Box t] = [s] + [t]$ for each s and t. From Proposition 1.1 we see that every element of $K_0^{\Box}(S)$ may be written as a difference $[s] - [t]$ for some objects s and t.

II.5.2 **Examples 5.2.** (1) The category $\mathbf{P}(R)$ of finitely generated projective modules over a ring R is symmetric monoidal under direct sum. Since the above definition is identical to that in §2, we see that we have $K_0(R) = K_0^{\oplus}(\mathbf{P}(R))$.

(2) Similarly, the category $\mathbf{VB}(X)$ of (real or complex) vector bundles over a topological space X is symmetric monoidal, with $\Box$ being the Whitney sum $\oplus$. From the definition we see that we also have $K(X) = K_0^{\oplus}(\mathbf{VB}(X))$, or more explicitly:

$$KO(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{R}}(X)), \qquad KU(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{C}}(X)).$$

(3) If R is a commutative ring, let $\mathbf{Pic}(R)$ denote the category of algebraic line bundles L over R and their isomorphisms (§I.3). This is a symmetric monoidal category with $\Box = \otimes_R$, and the isomorphism classes of objects already form a group, so $K_0\mathbf{Pic}(R) = \text{Pic}(R)$.

II.5.2.1 **Example 5.2.1** (Finite Sets)**.** Let $\mathbf{Sets}_{\text{fin}}$ denote the category of finite sets. The coproduct is the disjoint sum $\amalg$, and it is not hard to see that $K_0^{\amalg}(\mathbf{Sets}_{\text{fin}}) = \mathbb{Z}$.

Another monoidal operation on $\mathbf{Sets}_{\text{fin}}$ is the product ($\times$). However, since the empty set satisfies $\emptyset = \emptyset \times X$ for all X we have $K_0^{\times}(\mathbf{Sets}_{\text{fin}}) = 0$.

The category $\mathbf{Sets}_{\text{fin}}^{\times}$ of nonempty finite sets has for its isomorphism classes the set $\mathbb{N}_{>0} = \{1, 2, \dots\}$ of positive integers, and the product of finite sets corresponds to multiplication. Since the group completion of $(\mathbb{N}_{>0}, \times)$ is the multiplicative monoid $\mathbb{Q}_{>0}^{\times}$ of positive rational numbers, we have $K_0^{\times}(\mathbf{Sets}_{\text{fin}}^{\times}) \cong \mathbb{Q}_{>0}^{\times}$.

II.5.2.2 **Example 5.2.2** (Burnside Ring)**.** Suppose that G is a finite group, and let G-$\mathbf{Sets}_{\text{fin}}$ denote the category of finite G-sets. It is a symmetric monoidal category under disjoint union. We saw in Example 1.5 that $K_0(G\text{-}\mathbf{Sets}_{\text{fin}})$ is the Burnside Ring $A(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of subgroups of G.

II.5.2.3 **Example 5.2.3** (Representation ring)**.** The finite-dimensional complex representations of a finite group G form a category $\mathbf{Rep}_{\mathbb{C}}(G)$. It is symmetric monoidal under $\oplus$. We saw in Example 1.6 that $K_0\mathbf{Rep}_{\mathbb{C}}(G)$ is the representation ring $R(G)$ of G, which is a free abelian group on the classes $[V_1], \dots, [V_r]$ of the irreducible representations of G.

Cofinality

Let T be a full subcategory of a symmetric monoidal category S. If T contains e and is closed under finite products, then T is also symmetric monoidal. We say that T is *cofinal* in S if for every s in S there is an s' in S such that $s \Box s'$ is isomorphic to an element in T, *i.e.,* if the abelian monoid T^{iso} is cofinal in S^{iso} in the sense of §1. When this happens, we may restate Corollary II.1.3 1.3 as follows.

II.5.3 **Cofinality Theorem 5.3.** *Let T be cofinal in a symmetric monoidal category S. Then (assuming S^{iso} is a set):*

(1) $K_0(T)$ is a subgroup of $K_0(S)$;

(2) Every element of $K_0(S)$ is of the form $[s]-[t]$ for some s in S and t in T;

(3) If $[s]=[s']$ in $K_0(S)$ then $s\Box t \cong s'\Box t$ for some t in T.

II.5.4.1 **Example 5.4.1** (Free modules)**.** Let R be a ring. The category $\mathbf{Free}(R)$ of finitely generated free R-modules is cofinal (for $\Box = \oplus$) in the category $\mathbf{P}(R)$ of finitely generated projective modules. Hence $K_0\mathbf{Free}(R)$ is a subgroup of $K_0(R)$. In fact $K_0\mathbf{Free}(R)$ is a cyclic abelian group, and equals $\mathbb{Z}$ whenever R satisfies the Invariant Basis Property. Moreover, the subgroup $K_0\mathbf{Free}(R)$ of $K_0(R) = K_0\mathbf{P}(R)$ is the image of the map $\mathbb{Z} \to K_0(R)$ described in Lemma II.2.1 2.1.

$\mathbf{Free}(R)$ is also cofinal in the smaller category $\mathbf{P}^{\text{st.free}}(R)$ of finitely generated stably free modules. Since every stably free module P satisfies $P \oplus R^m \cong R^n$ for some m and n, the Cofinality theorem yields $K_0\mathbf{Free}(R) = K_0\mathbf{P}^{\text{st.free}}(R)$.

II.5.4.2 **Example 5.4.2.** Let R be a commutative ring. A finitely generated projective R-module is called *faithfully projective* if its rank is never zero. The tensor product of faithfully projective modules is again faithfully projective by Ex. I.EI.2.7 2.7. Hence the category $\mathbf{FP}(R)$ of faithfully projective R-modules is a symmetric monoidal category under the tensor product $\otimes_R$. For example, if R is a field, then the monoid $\mathbf{FP}^{\text{iso}}$ is the multiplicative monoid $(\mathbb{N}_{>0}, \times)$ of Example II.5.2.1 5.2.1, so in this case we have $K_0^{\otimes}\mathbf{FP}(R) \cong \mathbb{Q}^{\times}_{>0}$. We will describe the group $K_0^{\otimes}\mathbf{FP}(R)$ in the exercises below.

II.5.4.3 **Example 5.4.3** (Brauer groups)**.** Suppose first that F is a field, and let $\mathbf{Az}(F)$ denote the category of central simple F-algebras. This is a symmetric monoidal category with product $\otimes_F$, because if A and B are central simple then so is $A \otimes_F B$. The matrix rings $M_n(F)$ form a cofinal subcategory, with $M_m(F) \otimes_F M_n(F) \cong M_{mn}(F)$. From the previous example we see that the Grothendieck group of this subcategory is $\mathbb{Q}^{\times}_{>0}$. The classical *Brauer group* $\text{Br}(F)$ of the field F is the quotient of $K_0\mathbf{Az}(F)$ by this subgroup. That is, $\text{Br}(F)$ is generated by classes $[A]$ of central simple algebras with two families of relations: $[A \otimes_F B] = [A] + [B]$ and $[M_n(F)] = 0$.

More generally, suppose that R is a commutative ring. Recall (from Milne [127, IV]) that an R-algebra A is is called an *Azumaya algebra* if there is another R-algebra B such that $A \otimes_R B \cong M_n(R)$ for some n. The category $\mathbf{Az}(R)$

of Azumaya R-algebras is thus symmetric monoidal with product $\otimes_R$. If P is a faithfully projective R-module, $\mathrm{End}_R(P)$ is an Azumaya algebra. Since $\mathrm{End}_R(P \otimes_R P') \cong \mathrm{End}_R(P) \otimes_R \mathrm{End}_R(P')$, there is a monoidal functor End_R from $\mathbf{FP}(R)$ to $\mathbf{Az}(R)$, and a group homomorphism $K_0\mathbf{FP}(R) \to K_0\mathbf{Az}(R)$. The cokernel $\mathrm{Br}(R)$ of this map is called the *Brauer group* of R. That is, $\mathrm{Br}(R)$ is generated by classes $[A]$ of Azumaya algebras with two families of relations: $[A \otimes_R B] = [A] + [B]$ and $[\mathrm{End}_R(P)] = 0$.

G-bundles and equivariant K-theory

The following discussion is taken from Atiyah's very readable book [7]. Suppose that G is a finite group and that X is a topological space on which G acts continuously. A (complex) vector bundle E over X is called a *G-vector bundle* if G acts continuously on E, the map $E \to X$ commutes with the action of G, and for each $g \in G$ and $x \in X$ the map $E_x \to E_{gx}$ is a vector space homomorphism. The category $\mathbf{VB}_G(X)$ of G-vector bundles over X is symmetric monoidal under the usual Whitney sum, and we write $K_G^0(X)$ for the Grothendieck group $K_0^{\oplus}\mathbf{VB}_G(X)$. For example, if X is a point then $\mathbf{VB}_G(X) = \mathbf{Rep}_{\mathbb{C}}(G)$, so we have $K_G^0(\text{point}) = R(G)$. More generally, if x is a fixed point of X, then $E \mapsto E_x$ defines a monoidal functor from $\mathbf{VB}_G(X)$ to $\mathbf{Rep}_{\mathbb{C}}(G)$, and hence a group map $K_G^0(X) \to R(G)$.

If G acts trivially on X, every vector bundle E on X can be considered as a G-bundle with trivial action, and the tensor product $E \otimes V$ with a representation V of G is another G-bundle. The following result is proven on p. 38 of [7].

II.5.5 **Proposition 5.5** (Krull-Schmidt Theorem)**.** *Let $V_1, ..., V_r$ be a complete set of irreducible G-modules, and suppose that G acts trivially on X. Then for every G-bundle F over X there are unique vector bundles $E_i = \mathrm{Hom}_G(V_i, F)$ so that*

$$F \cong (E_1 \otimes V_1) \oplus \cdots \oplus (E_r \otimes V_r).$$

II.5.5.1 **Corollary 5.5.1.** *If G acts trivially on X then $K_G^0(X) \cong KU(X) \otimes_{\mathbb{Z}} R(G)$.*

The Witt ring $W(F)$ of a field

II.5.6 **5.6.** Symmetric bilinear forms over a field F provide another classical application of the K_0 construction. The following discussion is largely taken from Milnor and Husemoller's pretty book [133], and the reader is encouraged to look there for the connections with other branches of mathematics.

A *symmetric inner product space* (V, B) is a finite dimensional vector space V, equipped with a nondegenerate symmetric bilinear form $B : V \otimes V \to F$. The category $\mathbf{SBil}(F)$ of symmetric inner product spaces and form-preserving maps is symmetric monoidal, where the operation $\square$ is the orthogonal sum $(V, B) \oplus (V', B')$, defined as the vector space $V \oplus V'$, equipped with the bilinear form $\beta(v \oplus v', w \oplus w') = B(v, w) + B'(v', w')$.

A crucial role is played by the *hyperbolic plane* H, which is $V = F^2$ equipped with the bilinear form B represented by the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. An inner product space is called *hyperbolic* if it is isometric to an orthogonal sum of hyperbolic planes.

Let $(V,B)\otimes(V',B')$ denote the tensor product $V\otimes V'$, equipped with the bilinear form $\beta(v\otimes v', w\otimes w') = B(v,w)B'(v',w')$; this is also a symmetric inner product space, and the isometry classes of inner product spaces forms a semiring under $\oplus$ and $\otimes$ (see Ex. EII.5.10 5.10). Thus the Grothendieck group $GW(F) = K_0\mathbf{SBil}(F)$ is a commutative ring with unit $1 = \langle 1\rangle$; it is called the *Grothendieck-Witt ring* of F. The forgetful functor $\mathbf{SBil}(F)\to\mathbf{P}(F)$ sending (V,B) to V induces a ring augmentation $\varepsilon : GW(F)\to K_0(F)\cong\mathbb{Z}$. We write $\hat{I}$ for the augmentation ideal of $GW(F)$.

II.5.6.1 **Example 5.6.1.** For each $a\in F^\times$, we write $\langle a\rangle$ for the inner product space with $V = F$ and $B(v,w) = avw$. Clearly $\langle a\rangle\otimes\langle b\rangle\cong\langle ab\rangle$. Note that a change of basis $1\mapsto b$ of F induces an isometry $\langle a\rangle\cong\langle ab^2\rangle$ for every unit b, so the inner product space only determines a up to a square.

If $\mathrm{char}(F)\neq 2$, it is well known that every symmetric bilinear form is diagonalizable. Thus every symmetric inner product space is isometric to an orthogonal sum $\langle a_1\rangle\oplus\cdots\oplus\langle a_n\rangle$. For example, it is easy to see that $H\cong\langle 1\rangle\oplus\langle -1\rangle$. This also implies that $\hat{I}$ is additively generated by the elements $\langle a\rangle - 1$.

If $\mathrm{char}(F) = 2$, every symmetric inner product space is isomorphic to $\langle a_1\rangle\oplus\cdots\oplus\langle a_n\rangle\oplus N$, where N is hyperbolic; see M-SBF [133, I.3]. In this case $\hat{I}$ has the extra generator $H - 2$.

If $\mathrm{char}(F)\neq 2$, there is a Cancellation Theorem due to Witt: if X, Y, Z are inner product spaces, then $X\oplus Y\cong X\oplus Z$ implies that $Y\cong Z$. For a proof, we refer the reader to M-SBF [133]. We remark that cancellation fails if $\mathrm{char}(F) = 2$; see Ex. EII.5.11 5.11(d). The following definition is due to Knebusch.

II.5.6.2 **Definition 5.6.2.** Suppose that $\mathrm{char}(F)\neq 2$. The *Witt ring* $W(F)$ of F is defined to be the quotient of the ring $GW(F)$ by the subgroup $\{nH\}$ generated by the hyperbolic plane H. This subgroup is an ideal by Ex. EII.5.11 5.11, so $W(F)$ is also a commutative ring.

The augmentation $GW(F)\to\mathbb{Z}$ has $H\mapsto 2$, so it induces an augmentation $W(F)\xrightarrow{\varepsilon}\mathbb{Z}/2$. We write I for the augmentation ideal $\ker(\varepsilon)$ of $W(F)$.

When $\mathrm{char}(F) = 2$, $W(F)$ is defined similarly, as the quotient of $GW(F)$ by the subgroup of "split" spaces; see Ex. EII.5.11 5.11. In this case we have $2 = 0$ in the Witt ring $W(F)$, because the inner product space $\langle 1\rangle\oplus\langle 1\rangle$ is split (Ex. EII.5.11 5.11(d)).

When $\mathrm{char}(F)\neq 2$, the augmentation ideals of $GW(F)$ and $W(F)$ are isomorphic: $\hat{I}\cong I$. This is because $\varepsilon(nH) = 2n$, so that $\{nH\}\cap\hat{I} = 0$ in $GW(F)$.

Since $(V,B)+(V,-B) = 0$ in $W(F)$ by Ex. EII.5.11 5.11, every element of $W(F)$ is represented by an inner product space. In particular, I is additively generated by the classes $\langle a\rangle + \langle -1\rangle$, even if $\mathrm{char}(F) = 2$. The powers I^n of I form a decreasing chain of ideals $W(F)\supset I\supset I^2\supset\cdots$. We shall describe I/I^2 now, and return to this topic in chapter III, §7.

The discriminant of an inner product space (V,B) is a classical invariant with values in $F^\times/F^{\times 2}$, where $F^{\times 2}$ denotes $\{a^2 | a\in F^\times\}$. For each basis of V, there is a matrix M representing B, and the determinant of M is a unit of F. A change of basis replaces M by A^tMA, and $\det(A^tMA) = \det(M)\det(A)^2$, so $w_1(V,B) = \det(M)$ is a well defined element in $F^\times/F^{\times 2}$, called the *first Stiefel–Whitney class* of (V,B). Since $w_1(H) = -1$, we have to modify the definition slightly in order to get an invariant on the Witt ring.

II.5.6.3 **Definition 5.6.3.** If $\dim(V) = r$, the *discriminant* of (V,B) is defined to be the element $d(V,B) = (-1)^{r(r-1)/2}\det(M)$ of $F^\times/F^{\times 2}$.

For example, we have $d(H) = d(1) = 1$ but $d(2) = -1$. It is easy to verify that the discriminant of $(V,B)\oplus(V',B')$ is $(-1)^{rr'}d(V,B)d(V',B')$, where $r = \dim(V)$ and $r' = \dim(V')$. In particular, (V,B) and $(V,B)\oplus H$ have the same discriminant. It follows that the discriminant is a well-defined map from $W(F)$ to $F^\times/F^{\times 2}$, and its restriction to I is additive.

II.5.6.4 **Theorem 5.6.4.** *(Pfister) The discriminant induces an isomorphism between I/I^2 and $F^\times/F^{\times 2}$.*

Proof. Since the discriminant of $\langle a\rangle \oplus \langle -1\rangle$ is a, the map $d: I \to F^\times/F^{\times 2}$ is onto. This homomorphism annihilates I^2 because I^2 is additively generated by products of the form

$$\big(\langle a\rangle - 1\big)\big(\langle b\rangle - 1\big) = \langle ab\rangle + \langle -a\rangle + \langle -b\rangle + 1,$$

and these have discriminant 1. Setting these products equal to zero, the identity $\langle a\rangle + \langle -a\rangle = 0$ yields the congruence

$$\big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1 \qquad \mod I^2. \qquad (5.6.5)$$ II.5.6.5

Hence the formula $s(a) = \langle a\rangle - 1$ defines a surjective homomorphism $F^\times \xrightarrow{s} I/I^2$. Since $ds(a) = a$, it follows that s is an isomorphism with inverse induced by d. □

II.5.6.6 **Corollary 5.6.6.** *$W(F)$ contains $\mathbb{Z}/2$ as a subring (i.e., $2 = 0$) if and only if -1 is a square in F.*

II.5.6.7 **Classical Examples 5.6.7.** If F is an algebraically closed field, or more generally every element of F is a square, then $\langle a\rangle \cong \langle 1\rangle$ and $W(F) = \mathbb{Z}/2$.

If $F = \mathbb{R}$, every bilinear form is classified by its rank and signature. For example, $\langle 1\rangle$ has signature 1 but H has signature 0, with $H \otimes H \cong H \oplus H$. Thus $GW(\mathbb{R}) \cong \mathbb{Z}[H]/(H^2 - 2H)$ and the signature induces a ring isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$.

If $F = \mathbb{F}_q$ is a finite field with q odd, then $I/I^2 \cong \mathbb{Z}/2$, and an elementary argument due to Steinberg shows that the ideal I^2 is zero (Ex. 5.12). The structure of the ring $W(F)$ now follows from 5.6.6: if $q \equiv 3 \pmod 4$ then $W(F) = \mathbb{Z}/4$; if $q \equiv 1 \pmod 4$, $W(\mathbb{F}_q) = \mathbb{Z}/2[\eta]/(\eta^2)$, where $\eta = \langle a\rangle - 1$ for some $a \in F$.

If F is a finite field extension of the p-adic rationals, then $I^3 = 0$ and I^2 is cyclic of order 2. If p is odd and the residue field is $\mathbb{F}_q$, then $W(F)$ contains

$\mathbb{Z}/2$ as a subring if $q \equiv 1 \pmod 4$ and contains $\mathbb{Z}/4$ if $q \equiv 3 \pmod 4$. If $p = 2$ then $W(F)$ contains $\mathbb{Z}/2$ as a subring if and only if $\sqrt{-1} \in F$. Otherwise $W(F)$ contains $\mathbb{Z}/4$ or $\mathbb{Z}/8$, according to whether -1 is a sum of two squares, an issue which is somewhat subtle.

If $F = \mathbb{Q}$, the ring map $W(\mathbb{Q}) \to W(\mathbb{R}) = \mathbb{Z}$ is onto, with kernel N satisfying $N^3 = 0$. Since $I/I^2 = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, the kernel is infinite but under control.

Quadratic Forms

The theory of symmetric bilinear forms is closely related to the theory of quadratic forms, which we now sketch.

II.5.7 **Definition 5.7.** Let V be a vector space over a field F. A quadratic form on V is a function $q : V \to F$ such that $q(av) = a^2\, q(v)$ for every $a \in F$ and $v \in V$, and such that the formula $B_q(v, w) = q(v + w) - q(v) - q(w)$ defines a symmetric bilinear form B_q on V. We call (V, q) a *quadratic space* if B_q is nondegenerate, and call (V, B_q) the underlying symmetric inner product space. We write $\mathbf{Quad}(F)$ for the category of quadratic spaces and form-preserving maps.

The orthogonal sum $(V, q) \oplus (V', q')$ of two quadratic spaces is defined to be $V \oplus V'$ equipped with the quadratic form $v \oplus v' \mapsto q(v) + q'(v')$. This is a quadratic space, whose underlying symmetric inner product space is the orthogonal sum $(V, B_q) \oplus (V', B_{q'})$. Thus $\mathbf{Quad}(F)$ is a symmetric monoidal category, and the underlying space functor $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ sending (V, q) to (V, B_q) is monoidal.

Here is one source of quadratic spaces. Suppose that β is a (possibly non-symmetric) bilinear form on V. The function $q(v) = \beta(v, v)$ is visibly quadratic, with associated symmetric bilinear form $B_q(v, w) = \beta(v, w)+\beta(w, v)$. By choosing an ordered basis of V, it is easy to see that every quadratic form arises in this way. Note that when β is symmetric we have $B_q = 2\beta$; if $\mathrm{char}(F) \neq 2$ this shows that $\beta \mapsto \frac{1}{2}q$ defines a monoidal functor $\mathbf{SBil}(F) \to \mathbf{Quad}(F)$ inverse to the underlying functor, and proves the following result.

II.5.7.1 **Lemma 5.7.1.** *If* $\mathrm{char}(F) \neq 2$ *then the underlying space functor* $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ *is an equivalence of monoidal categories.*

A quadratic space (V, q) is said to be *split* if it contains a subspace N so that $q(N) = 0$ and $\dim(V) = 2\dim(N)$. For example, the quadratic forms $q(x, y) = xy + cy^2$ on $V = F^2$ are split.

II.5.7.2 **Definition 5.7.2.** The group $WQ(F)$ is defined to be the quotient of the group $K_0\mathbf{Quad}(F)$ by the subgroup of all split quadratic spaces.

It follows from Ex. 5.11 that the underlying space functor defines a homomorphism $WQ(F) \to W(F)$. By Lemma 5.7.1, this is an isomorphism when $\mathrm{char}(F) \neq 2$.

When $\mathrm{char}(F) = 2$, the underlying symmetric inner product space of a quadratic space (V, q) is always hyperbolic, and V is always even-dimensional;

see Ex. 5.13. In particular, $WQ(F) \to W(F)$ is the zero map when $\mathrm{char}(F) = 2$. By Ex. 5.13, $WQ(F)$ is a $W(F)$-module with $WQ(F)/I \cdot WQ(F)$ given by the Arf invariant. We will describe the rest of the filtration $I^n \cdot WQ(F)$ in III.7.10.4.

EXERCISES

EII.5.1 **5.1.** Let R be a ring and let $\mathbf{P}^\infty(R)$ denote the category of all countably generated projective R-modules. Show that $K_0^\oplus \mathbf{P}^\infty(R) = 0$.

EII.5.2 **5.2.** Suppose that the Krull-Schmidt Theorem holds in an additive category $\mathcal{C}$, *i.e.,* every object of $\mathcal{C}$ can be written as a finite direct sum of indecomposable objects, in a way that is unique up to permutation. Show that $K_0^\oplus(\mathcal{C})$ is the free abelian group on the set of isomorphism classes of indecomposable objects.

EII.5.3 **5.3.** Use Ex. 5.2 to prove Corollary 5.5.1.

EII.5.4 **5.4.** Let R be a commutative ring, and let $H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0})$ denote the free abelian group of all continuous maps $\mathrm{Spec}(R) \to \mathbb{Q}^\times_{>0}$. Show that $[P] \mapsto \mathrm{rank}(P)$ induces a split surjection from $K_0\mathbf{FP}(R)$ onto $H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0})$. In the next two exercises, we shall show that the kernel of this map is isomorphic to $\widetilde{K}_0(R) \otimes \mathbb{Q}$.

EII.5.5 **5.5.** Let R be a commutative ring, and let U_+ denote the subset of the ring $K_0(R) \otimes \mathbb{Q}$ consisting of all x such that $\mathrm{rank}(x)$ takes only positive values.

(a) Use the fact (Corollary 4.6.1) that the ideal $\widetilde{K}_0(R)$ is nilpotent to show that U_+ is an abelian group under multiplication, and that there is a split exact sequence

$$0 \to \widetilde{K}_0(R) \otimes \mathbb{Q} \xrightarrow{\exp} U_+ \xrightarrow{\mathrm{rank}} H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0}) \to 0.$$

(b) Show that $P \mapsto [P] \otimes 1$ is an additive function from $\mathbf{FP}(R)$ to the multiplicative group U_+, and that it induces a map $K_0\mathbf{FP}(R) \to U_+$.

EII.5.6 **5.6.** (Bass) Let R be a commutative ring. Show that the map $K_0\mathbf{FP}(R) \to U_+$ of the previous exercise is an isomorphism. *Hint:* The map is onto by Ex. 2.10. Conversely, if $[P] \otimes 1 = [Q] \otimes 1$ in U_+, show that $P \otimes R^n \cong Q \otimes R^n$ for some n.

EII.5.7 **5.7.** Suppose that a finite group G acts freely on X, and let X/G denote the orbit space. Show that $\mathbf{VB}_G(X)$ is equivalent to the category $\mathbf{VB}(X/G)$, and conclude that $K^0_G(X) \cong KU(X/G)$.

EII.5.8 **5.8.** Let R be a commutative ring. Show that the determinant of a projective module induces a monoidal functor $\det\colon \mathbf{P}(R) \to \mathbf{Pic}(R)$, and that the resulting map $K_0(\det)\colon K_0\mathbf{P}(R) \to K_0\mathbf{Pic}(R)$ is the determinant map $K_0(R) \to \mathrm{Pic}(R)$ of Proposition 2.6.

EII.5.9 **5.9.** Let G be a finite group. Given a finite G-set X and a $\mathbb{Z}[G]$-module M, the abelian group $X \times M$ carries a $\mathbb{Z}[G]$-module structure by $g(x,m) = (gx, gm)$. Show that $X \times -$ induces an additive functor from $\mathbf{P}(\mathbb{Z}[G])$ to itself (2.8). Then show that the pairing $(X, M) \mapsto X \times M$ makes $K_0(\mathbb{Z}[G])$ into a module over the Burnside ring $A(G)$.

EII.5.10 **5.10.** If $X = (V, B)$ and $X' = (V', B')$ are two inner product spaces, show that there is a nondegenerate bilinear form β on $V \otimes V'$ satisfying $\beta(v \otimes v', w \otimes w') = B(v,w)B'(v',w')$ for all $v, w \in V$ and $v', w' \in V'$. Writing $X \otimes X'$ for this inner product space, show that $X \otimes X' \cong X' \otimes X$ and $(X_1 \oplus X_2) \otimes X' \cong (X_1 \otimes X') \oplus (X_2 \otimes X')$. Then show that $X \otimes H \cong H \oplus \cdots \oplus H$.

EII.5.11 **5.11.** A symmetric inner product space $S = (V, B)$ is called *split* if it has a basis so that B is represented by a matrix $\left(\begin{smallmatrix} 0 & I \\ I & A \end{smallmatrix}\right)$. Note that the sum of split spaces is also split, and that the hyperbolic plane is split. We define $W(F)$ to be the quotient of $GW(F)$ by the subgroup of classes $[S]$ of split spaces.

(a) If $\mathrm{char}(F) \neq 2$, show that every split space S is hyperbolic. Conclude that this definition of $W(F)$ agrees with the definition given in 5.6.2.

(b) For any $a \in F^\times$, show that $\langle a \rangle \oplus \langle -a \rangle$ is split.

(c) If S is split, show that each $(V, B) \otimes S$ is split. In particular, $(V, B) \oplus (V, -B) = (V, B) \otimes \big(\langle 1 \rangle \oplus \langle -1 \rangle\big)$ is split. Conclude that $W(F)$ is also a ring when $\mathrm{char}(F) = 2$.

(d) If $\mathrm{char}(F) = 2$, show that the split space $S = \langle 1 \rangle \oplus \langle 1 \rangle$ is not hyperbolic, yet $\langle 1 \rangle \oplus S \cong \langle 1 \rangle \oplus H$. This shows that Witt Cancellation fails if $\mathrm{char}(F) = 2$. *Hint:* consider the associated quadratic forms. Then consider the basis $(1,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\langle 1 \rangle \oplus S$.

EII.5.12 **5.12.** If $a + b = 1$ in F, show that $\langle a \rangle \oplus \langle b \rangle \cong \langle ab \rangle \oplus \langle 1 \rangle$. Conclude that in both $GW(F)$ and $W(F)$ we have the Steinberg identity $\big(\langle a \rangle - 1\big)\big(\langle b \rangle - 1\big) = 0$.

EII.5.13 **5.13.** Suppose that $\mathrm{char}(F) = 2$ and that (V, q) is a quadratic form.

(a) Show that $B_q(v, v) = 0$ for every $v \in V$.

(b) Show that the underlying inner product space (V, B_q) is hyperbolic, hence split in the sense of Ex. 5.11. This shows that $\dim(V)$ is even, and that the map $WQ(F) \to W(F)$ is zero. *Hint:* Find two elements x, y in V so that $B_q(x, y) = 1$, and show that they span an orthogonal summand of V.

(c) If (W, β) is a symmetric inner product space, show that there is a unique quadratic form q' on $V' = V \otimes W$ satisfying $q'(v \oplus w) = q(v)\beta(w, w)$, such that the underlying bilinear form satisfies $B_{q'}(v \otimes w, v' \otimes w') = B_q(v, v')\beta(w, w')$. Show that this product makes $WQ(F)$ into a module over $W(F)$.

(d) (Arf invariant) Let $\wp : F \to F$ denote the additive map $\wp(a) = a^2 + a$. By (b), we may choose a basis $x_1, \dots, x_n$, $y_1 \dots, y_n$ of V so that each x_i, y_i span a hyperbolic plane. Show that the element $\Delta(V, q) = \sum q(x_i)q(y_i)$ of $F/\wp(F)$ is independent of the choice of basis, called the *Arf invariant* of the quadratic space (after C. Arf, who discovered it in 1941). Then show that Δ is an additive surjection. Using (c), H. Sah showed that the Arf invariant induces an isomorphism $WQ(F)/I \cdot WQ(F) \cong F/\wp(F)$.

(e) Consider the quadratic forms $q(a, b) = a^2 + ab + b^2$ and $q'(a, b) = ab$ on $V = F^2$. Show they are isometric if and only if F contains the field $\mathbb{F}_4$.

EII.5.14 **5.14.** (Kato) If $\mathrm{char}(F) = 2$, show that there is a ring homomorphism $W(F) \to F \otimes_{F^2} F$ sending $\langle a \rangle$ to $a^{-1} \otimes a$.

6 K_0 of an Abelian Category

Another important situation in which we can define Grothendieck groups is when we have a (skeletally) small abelian category. This is due to the natural notion of exact sequence in an abelian category. We begin by quickly reminding the reader what an abelian category is, defining K_0 and then making a set-theoretic remark.

It helps to read the definitions below with some examples in mind. The reader should remember that the prototype abelian category is the category **mod**-R of right modules over a ring R, the morphisms being R-module homomorphisms. The full subcategory with objects the free R-modules $\{0, R, R^2, \dots\}$ is additive, and so is the slightly larger full subcategory $\mathbf{P}(R)$ of finitely generated projective R-modules (this observation was already made in chapter I). For more information on abelian categories, see textbooks like [116] or [223].

II.6.1 **Definition 6.1.** (1) An *additive category* is a category containing a zero object '0' (an object which is both initial and terminal), having all products $A \times B$, and such that every set $\mathrm{Hom}(A, B)$ is given the structure of an abelian group in such a way that composition is bilinear. In an additive category the product $A \times B$ is also the coproduct $A \amalg B$ of A and B; we call it the *direct sum* and write it as $A \oplus B$ to remind ourselves of this fact.

(2) An *abelian category* $\mathcal{A}$ is an additive category in which (i) every morphism $f : B \to C$ has a kernel and a cokernel, and (ii) every monic arrow is a kernel, and every epi is a cokernel. (Recall that $f : B \to C$ is called *monic* if $fe_1 \neq fe_2$ for every $e_1 \neq e_2 : A \to B$; it is called *epi* if $g_1 f \neq g_2 f$ for every $g_1 \neq g_2 : C \to D$.)

(3) In an abelian category, we call a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ *exact* if $\ker(g)$ equals $\mathrm{im}(f) \equiv \ker\{B \to \mathrm{coker}(f)\}$. A longer sequence is *exact* if it is exact at all places. By the phrase *short exact sequence* in an abelian category $\mathcal{A}$ we mean an exact sequence of the form:

$$0 \to A' \to A \to A'' \to 0. \qquad (*)$$

II.6.1.1 **Definition 6.1.1** ($K_0\mathcal{A}$)**.** Let $\mathcal{A}$ be an abelian category. Its *Grothendieck group* $K_0(\mathcal{A})$ is the abelian group presented as having one generator $[A]$ for each object A of $\mathcal{A}$, with one relation $[A] = [A'] + [A'']$ for every short exact sequence $(*)$ in $\mathcal{A}$.

Here are some useful identities which hold in $K_0(\mathcal{A})$.
(a) $[0] = 0$ (take $A = A'$).
(b) if $A \cong A'$ then $[A] = [A']$ (take $A'' = 0$).
(c) $[A' \oplus A''] = [A'] + [A'']$ (take $A = A' \oplus A''$).
If two abelian categories are equivalent, their Grothendieck groups are naturally isomorphic, as (b) implies they have the same presentation. By (c), the group $K_0(\mathcal{A})$ is a quotient of the group $K_0^{\oplus}(\mathcal{A})$ defined in §5 by considering $\mathcal{A}$ as a symmetric monoidal category.

II.6.1.2 **Universal Property 6.1.2.** An *additive function* from $\mathcal{A}$ to an abelian group Γ is a function f from the objects of $\mathcal{A}$ to Γ such that $f(A) = f(A') + f(A'')$ for every short exact sequence $(*)$ in $\mathcal{A}$. By construction, the function $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\mathcal{A})$. This has the following universal property: any additive function f from $\mathcal{A}$ to Γ induces a unique group homomorphism $f\colon K_0(\mathcal{A}) \to \Gamma$, with $f([A]) = f(A)$ for every A.

For example, the direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$ of two abelian categories is also abelian. Using the universal property of K_0 it is clear that $K_0(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2)$. More generally, an arbitrary direct sum $\bigoplus \mathcal{A}_i$ of abelian categories is abelian, and we have $K_0(\bigoplus \mathcal{A}_i) \cong \bigoplus K_0(\mathcal{A}_i)$.

II.6.1.3 **Set-theoretic Considerations 6.1.3.** There is an obvious set-theoretic difficulty in defining $K_0\mathcal{A}$ when $\mathcal{A}$ is not small; recall that a category $\mathcal{A}$ is called *small* if the class of objects of $\mathcal{A}$ forms a set.

We will always implicitly assume that our abelian category $\mathcal{A}$ is *skeletally small*, *i.e.,* it is equivalent to a small abelian category $\mathcal{A}'$. In this case we define $K_0(\mathcal{A})$ to be $K_0(\mathcal{A}')$. Since any other small abelian category equivalent to $\mathcal{A}$ will also be equivalent to $\mathcal{A}'$, the definition of $K_0(\mathcal{A})$ is independent of this choice.

II.6.1.4 **Example 6.1.4** (All R-modules)**.** We cannot take the Grothendieck group of the abelian category **mod**-R because it is not skeletally small. To finesse this difficulty, fix an infinite cardinal number κ and let $\mathbf{mod}_\kappa$-R denote the full subcategory of **mod**-R consisting of all R-modules of cardinality $< \kappa$. As long as $\kappa \geq |R|$, $\mathbf{mod}_\kappa$-R is an abelian subcategory of **mod**-R having a set of isomorphism classes of objects. The Eilenberg Swindle I.2.8 applies to give $K_0(\mathbf{mod}_\kappa\text{-}R) = 0$. In effect, the formula $M \oplus M^\infty \cong M^\infty$ implies that $[M] = 0$ for every module M.

II.6.1.5 **6.1.5.** The natural type of functor $F\colon \mathcal{A} \to \mathcal{B}$ between two abelian categories is an *additive* functor; this is a functor for which all the maps $\mathrm{Hom}(A, A') \to \mathrm{Hom}(FA, FA')$ are group homomorphisms. However, not all additive functors induce homomorphisms $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

We say that an additive functor F is *exact* if it preserves exact sequences—that is, for every exact sequence (∗) in $\mathcal{A}$, the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact in $\mathcal{B}$. The presentation of K_0 implies that any exact functor F defines a group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by the formula $[A] \mapsto [F(A)]$.

Suppose given an inclusion $\mathcal{A} \subset \mathcal{B}$ of abelian categories, with $\mathcal{A}$ a full subcategory of $\mathcal{B}$. If the inclusion is an exact functor, we say that $\mathcal{A}$ is an *exact abelian subcategory* of $\mathcal{B}$. As with all exact functors, the inclusion induces a natural map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

II.6.2 **Definition 6.2** (G_0R)**.** If R is a (right) noetherian ring, let $\mathbf{M}(R)$ denote the subcategory of **mod-**R consisting of all finitely generated R-modules. The noetherian hypothesis implies that $\mathbf{M}(R)$ is an abelian category, and we write $G_0(R)$ for $K_0\mathbf{M}(R)$. (We will give a definition of $\mathbf{M}(R)$ and $G_0(R)$ for non-noetherian rings in Example II.7.1.4 7.1.4 below.)

The presentation of $K_0(R)$ in §2 shows that there is a natural map $K_0(R) \to G_0(R)$, called the *Cartan homomorphism* (send $[P]$ to $[P]$).

Associated to a ring homomorphism $f \colon R \to S$ are two possible maps on G_0: the contravariant transfer map and the covariant base change map.

When S is finitely generated as an R-module (*e.g.*, $S = R/I$), there is a "transfer" homomorphism $f_* \colon G_0(S) \to G_0(R)$. It is induced from the forgetful functor $f_* \colon \mathbf{M}(S) \to \mathbf{M}(R)$, which is exact.

Whenever S is flat as an R-module, there is a "base change" homomorphism $f^* \colon G_0(R) \to G_0(S)$. Indeed, the base change functor $f^* \colon \mathbf{M}(R) \to \mathbf{M}(S)$, $f^*(M) = M \otimes_R S$, is exact if and only if S is flat over R. We will extend the definition of f^* in §7 to the case in which S has a finite resolution by flat R-modules using Serre's Formula II.7.9.3 7.9.3: $f^*([M]) = \sum(-1)^i[\mathrm{Tor}_i^R(M, S)]$.

If F is a field then every exact sequence in $\mathbf{M}(F)$ splits, and it is easy to see that $G_0(F) \cong K_0(F) \cong \mathbb{Z}$. In particular, if R is an integral domain with field of fractions F, then there is a natural map $G_0(R) \to G_0(F) = \mathbb{Z}$, sending $[M]$ to the integer $\dim_F(M \otimes_R F)$.

II.6.2.1 **Example 6.2.1** (Abelian groups)**.** When $R = \mathbb{Z}$ the Cartan homomorphism is an isomorphism: $K_0(\mathbb{Z}) \cong G_0(\mathbb{Z}) \cong \mathbb{Z}$. To see this, first observe that the sequences

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

imply that $[\mathbb{Z}/n\mathbb{Z}] = [\mathbb{Z}] - [\mathbb{Z}] = 0$ in $G_0(\mathbb{Z})$ for every n. By the Fundamental Theorem of finitely generated Abelian Groups, every finitely generated abelian group M is a finite sum of copies of the groups $\mathbb{Z}$ and $\mathbb{Z}/n$, $n \geq 2$. Hence $G_0(\mathbb{Z})$ is generated by $[\mathbb{Z}]$. To see that $G_0(\mathbb{Z}) \cong \mathbb{Z}$, observe that since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module there is a homomorphism from $G_0(\mathbb{Z})$ to $G_0(\mathbb{Q}) \cong \mathbb{Z}$ sending $[M]$ to $r(M) = \dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$. In effect, r is an additive function; as such it induces a homomorphism $r \colon G_0(\mathbb{Z}) \to \mathbb{Z}$. As $r(\mathbb{Z}) = 1$, r is an isomorphism.

More generally, the Cartan homomorphism is an isomorphism whenever R is a principal ideal domain, and $K_0(R) \cong G_0(R) \cong \mathbb{Z}$. The proof is identical.

II.6.2.2 **Example 6.2.2** (p-groups)**.** Let $\mathbf{Ab}_p$ denote the abelian category of all finite abelian p-groups for some prime p. Then $K_0(\mathbf{Ab}_p) \cong \mathbb{Z}$ on generator $[\mathbb{Z}/p]$. To see this, we observe that the length $\ell(M)$ of a composition series for a finite p-group M is well-defined by the Jordan-Hölder Theorem. Moreover ℓ is an additive function, and defines a homomorphism $K_0(\mathbf{Ab}_p) \to \mathbb{Z}$ with $\ell(\mathbb{Z}/p) = 1$. To finish we need only observe that $\mathbb{Z}/p$ generates $K_0(\mathbf{Ab}_p)$; this follows by induction on the length of a p-group, once we observe that any $L \subset M$ yields $[M] = [L] + [M/L]$ in $K_0(\mathbf{Ab}_p)$.

II.6.2.3 **Example 6.2.3.** The category $\mathbf{Ab}_{\text{fin}}$ of all finite abelian groups is the direct sum of the categories $\mathbf{Ab}_p$ of Example 6.2.2. Therefore $K_0(\mathbf{Ab}_{\text{fin}}) = \bigoplus K_0(\mathbf{Ab}_p)$ is the free abelian group on the set $\{[\mathbb{Z}/p],\, p \text{ prime}\}$.

II.6.2.4 **Example 6.2.4.** The category $\mathbf{M}(\mathbb{Z}/p^n)$ of all finite $\mathbb{Z}/p^n$-modules is an exact abelian subcategory of $\mathbf{Ab}_p$, and the argument above applies verbatim to prove that the simple module $[\mathbb{Z}/p]$ generates the group $G_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$. In particular, the canonical maps from $G_0(\mathbb{Z}/p^n) = K_0\mathbf{M}(\mathbb{Z}/p^n)$ to $K_0(\mathbf{Ab}_p)$ are all isomorphisms.

Recall from Lemma 2.2 that $K_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$ on $[\mathbb{Z}/p^n]$. The Cartan homomorphism from $K_0 \cong \mathbb{Z}$ to $G_0 \cong \mathbb{Z}$ is not an isomorphism; it sends $[\mathbb{Z}/p^n]$ to $n[\mathbb{Z}/p]$.

II.6.2.5 **Definition 6.2.5** ($G_0(X)$)**.** Let X be a noetherian scheme. The category $\mathbf{M}(X)$ of all coherent $\mathcal{O}_X$-modules is an abelian category. (See [85, II.5.7].) We write $G_0(X)$ for $K_0\mathbf{M}(X)$. When $X = \operatorname{Spec}(R)$ this agrees with Definition 6.2: $G_0(X) \cong G_0(R)$, because of the equivalence of $\mathbf{M}(X)$ and $\mathbf{M}(R)$.

If $f\colon X \to Y$ is a morphism of schemes, there is a *base change functor* $f^*\colon \mathbf{M}(Y) \to \mathbf{M}(X)$ sending $\mathcal{F}$ to $f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$; see I.5.2. When f is flat, the base change f^* is exact and therefore the formula $f^*([\mathcal{F}]) = [f^*\mathcal{F}]$ defines a homomorphism $f^*\colon G_0(Y) \to G_0(X)$. Thus G_0 is contravariant for flat maps.

If $f\colon X \to Y$ is a finite morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and $f_*\colon \mathbf{M}(X) \to \mathbf{M}(Y)$ is an exact functor [EGA, I(1.7.8)]. In this case the formula $f_*([\mathcal{F}]) = [f_*\mathcal{F}]$ defines a "transfer" map $f_*\colon G_0(X) \to G_0(Y)$.

If $f\colon X \to Y$ is a proper morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and so are its higher direct images $R^if_*\mathcal{F}$. (This is Serre's "Theorem B"; see I.5.2 or [EGA, III(3.2.1)].) The functor $f_*\colon \mathbf{M}(X) \to \mathbf{M}(Y)$ is not usually exact (unless f is finite). Instead we have:

II.6.2.6 **Lemma 6.2.6.** *If $f\colon X \to Y$ is a proper morphism of noetherian schemes, there is a "transfer" homomorphism $f_*\colon G_0(X) \to G_0(Y)$. It is defined by the formula $f_*([\mathcal{F}]) = \sum(-1)^i[R^if_*\mathcal{F}]$. The transfer homomorphism makes G_0 functorial for proper maps.*

Proof. For each coherent $\mathcal{F}$ the $R^if_*\mathcal{F}$ vanish for large i, so the sum is finite. By 6.2.1 it suffices to show that the formula gives an additive function. But if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathbf{M}(X)$ there is a finite long exact sequence in $\mathbf{M}(Y)$:

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to R^1f_*\mathcal{F}' \to R^1f_*\mathcal{F} \to R^1f_*\mathcal{F}'' \to R^2f_*\mathcal{F}' \to \cdots$$

and the alternating sum of the terms is $f_*[\mathcal{F}'] - f_*[\mathcal{F}] + f_*[\mathcal{F}'']$. This alternating sum must be zero by Proposition 6.6 below, so f_* is additive as desired. (Functoriality is relegated to Ex. 6.15.) $\square$

The next lemma follows by inspection of the definition of the direct limit (or filtered colimit) $\mathcal{A} = \varinjlim \mathcal{A}_i$ of a filtered system of small categories; the objects and morphisms of $\mathcal{A}$ are the direct limits of the object and morphisms of the $\mathcal{A}_i$.

II.6.2.7 **Lemma 6.2.7** (Filtered colimits)**.** *Suppose that* $\{\mathcal{A}_i\}_{i\in I}$ *is a filtered family of small abelian categories and exact functors. Then the direct limit* $\mathcal{A} = \varinjlim \mathcal{A}_i$ *is also an abelian category, and*

$$K_0(\mathcal{A}) = \varinjlim K_0(\mathcal{A}_i).$$

II.6.2.8 **Example 6.2.8** (S-torsion modules)**.** Suppose that S is a multiplicatively closed set of elements in a noetherian ring R. Let $\mathbf{M}_S(R)$ be the subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that $Ms = 0$ for some $s \in S$. For example, if $S = \{p^n\}$ then $\mathbf{M}_S(\mathbb{Z}) = \mathbf{Ab}_p$ was discussed in Example 6.2.2. In general $\mathbf{M}_S(R)$ is not only the union of the $\mathbf{M}(R/RsR)$, but is also the union of the $\mathbf{M}(R/I)$ as I ranges over the ideals of R with $I \cap S \neq \phi$. By 6.2.7,

$$K_0\mathbf{M}_S(R) = \varinjlim_{I\cap S\neq\phi} G_0(R/I) = \varinjlim_{s\in S} G_0(R/RsR).$$

Devissage

The method behind the computation in Example 6.2.4 that $G_0(\mathbb{Z}/p^n) \cong K_0\mathbf{Ab}_p$ is called *devissage*, a French word referring to the "unscrewing" of the composition series. Here is a formal statement of the process, due to Alex Heller.

II.6.3 **Devissage Theorem 6.3.** *Let* $\mathcal{B} \subset \mathcal{A}$ *be small abelian categories. Suppose that (a)* $\mathcal{B}$ *is an exact abelian subcategory of* $\mathcal{A}$*, closed in* $\mathcal{A}$ *under subobjects and quotient objects; and (b) Every object* A *of* $\mathcal{A}$ *has a finite filtration* $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ *with all quotients* A_i/A_{i+1} *in* $\mathcal{B}$*.*
Then the inclusion functor $\mathcal{B} \subset \mathcal{A}$ *is exact and induces an isomorphism*

$$K_0(\mathcal{B}) \cong K_0(\mathcal{A}).$$

Proof. Let $i_*\colon K_0(\mathcal{B}) \to K_0(\mathcal{A})$ denote the canonical homomorphism. To see that i_* is onto, observe that every filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ yields $[A] = \sum [A_i/A_{i+1}]$ in $K_0(\mathcal{A})$. This follows by induction on n, using the observation that $[A_i] = [A_{i+1}] + [A_i/A_{i+1}]$. Since by (b) such a filtration exists with the A_i/A_{i+1} in $\mathcal{B}$, this shows that the canonical i_* is onto.

For each A in $\mathcal{A}$, fix a filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ with each A_i/A_{i+1} in $\mathcal{B}$, and define $f(A)$ to be the element $\sum [A_i/A_{i+1}]$ of $K_0(\mathcal{B})$. We claim that $f(A)$ is independent of the choice of filtration. Because any two filtrations have equivalent refinements (Ex. EII.6.2 6.2), we only need check refinements of our given filtration. By induction we need only check for one insertion, say changing $A_i \supset A_{i+1}$ to $A_i \supset A' \supset A_{i+1}$. Appealing to the exact sequence

$$0 \to A'/A_{i+1} \to A_i/A_{i+1} \to A_i/A' \to 0,$$

we see that $[A_i/A_{i+1}] = [A_i/A'] + [A'/A_{i+1}]$ in $K_0(\mathcal{B})$, as claimed.

Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, we may construct a filtration $\{A_i\}$ on A by combining our chosen filtration for A' with the inverse image in A of our chosen filtration for A''. For this filtration we have $\sum[A_i/A_{i+1}] = f(A') + f(A'')$. Therefore f is an additive function, and defines a map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. By inspection, f is the inverse of the canonical map i_*. □

II.6.3.1 **Corollary 6.3.1.** *Let I be a nilpotent ideal of a noetherian ring R. Then the inclusion* **mod**-$(R/I) \subset$ **mod**-R *induces an isomorphism*

$$G_0(R/I) \cong G_0(R).$$

Proof. To apply devissage, we need to observe that if M is a finitely generated R-module, the filtration $M \supseteq MI \supseteq MI^2 \supseteq \cdots \supseteq MI^n = 0$ is finite, and all the quotients MI^n/MI^{n+1} are finitely generated R/I-modules. □

Notice that this also proves the scheme version:

II.6.3.2 **Corollary 6.3.2.** *Let X be a noetherian scheme, and X_{red} the associated reduced scheme. Then $G_0(X) \cong G_0(X_{red})$.*

II.6.3.3 **Application 6.3.3** (R-modules with support)**.** Example II.6.2.2 6.2.2 can be generalized as follows. Given a central element s in a ring R, let $\mathbf{M}_s(R)$ denote the abelian subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that $Ms^n = 0$ for some n. That is, modules such that $M \supset Ms \supset Ms^2 \supset \cdots$ is a finite filtration. By devissage,

$$K_0\mathbf{M}_s(R) \cong G_0(R/sR).$$

More generally, suppose we are given an ideal I of R. Let $\mathbf{M}_I(R)$ be the (exact) abelian subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that the filtration $M \supset MI \supset MI^2 \supset \cdots$ is finite, *i.e.,* such that $MI^n = 0$ for some n. By devissage,

$$K_0\mathbf{M}_I(R) \cong K_0\mathbf{M}(R/I) = G_0(R/I).$$

II.6.3.4 **Example 6.3.4.** Let X be a noetherian scheme, and $i\colon Z \subset X$ the inclusion of a closed subscheme. Let $\mathbf{M}_Z(X)$ denote the abelian category of coherent $\mathcal{O}_X$-modules Z supported on Z, and $\mathcal{I}$ the ideal sheaf in $\mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I} \cong \mathcal{O}_Z$. Via the direct image $i_*\colon \mathbf{M}(Z) \subset \mathbf{M}(X)$, we can consider $\mathbf{M}(Z)$ as the subcategory of all modules M in $\mathbf{M}_Z(X)$ such that $\mathcal{I}M = 0$. Every M in $\mathbf{M}_Z(X)$ has a finite filtration $M \supset M\mathcal{I} \supset M\mathcal{I}^2 \supset \cdots$ with quotients in $\mathbf{M}(Z)$, so by devissage:

$$K_0\mathbf{M}_Z(X) \cong K_0\mathbf{M}(Z) = G_0(Z).$$

The Localization Theorem

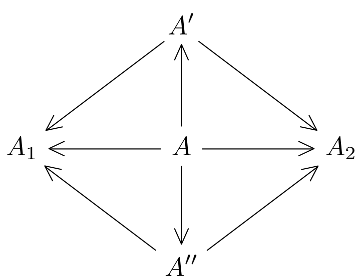
Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is an abelian subcategory $\mathcal{B}$ which is closed under subobjects, quotients and extensions. That is, if $0 \to B \to C \to D \to 0$ is exact in $\mathcal{A}$ then

$$C \in \mathcal{B} \Leftrightarrow B, D \in \mathcal{B}.$$

Now assume for simplicity that $\mathcal{A}$ is small. If $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, we can form a quotient abelian category $\mathcal{A}/\mathcal{B}$ as follows. Call a morphism f in $\mathcal{A}$ a $\mathcal{B}$-*iso* if $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{B}$. The objects of $\mathcal{A}/\mathcal{B}$ are the objects of $\mathcal{A}$, and morphisms $A_1 \to A_2$ are equivalence classes of diagrams in $\mathcal{A}$:

$$A_1 \xleftarrow{f} A' \xrightarrow{g} A_2, \quad f \text{ a } \mathcal{B}\text{-iso}.$$

Such a morphism is equivalent to $A_1 \leftarrow A'' \to A_2$ if and only if there is a commutative diagram:



where $A' \leftarrow A \to A''$ are $\mathcal{B}$-isos.

The composition with $A_2 \xleftarrow{f'} A''' \xrightarrow{h} A_3$ is $A_1 \xleftarrow{f} A' \longleftarrow A \longrightarrow A''' \xrightarrow{h} A_3$, where A is the pullback of A' and A''' over A_2. The proof that $\mathcal{A}/\mathcal{B}$ is abelian, and that the quotient functor $\operatorname{loc}\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact, may be found in [193, p. 44ff] or [61]. (See the appendix to this chapter.)

It is immediate from the construction of $\mathcal{A}/\mathcal{B}$ that $\operatorname{loc}(A) \cong 0$ if and only if A is an object of $\mathcal{B}$, and that for a morphism $f\colon A \to A'$ in $\mathcal{A}$, $\operatorname{loc}(f)$ is an isomorphism if and only if f is a $\mathcal{B}$-iso. In fact $\mathcal{A}/\mathcal{B}$ solves a universal problem (see *op. cit.*): if $T\colon \mathcal{A} \to \mathcal{C}$ is an exact functor such that $T(B) \cong 0$ for all B in $\mathcal{B}$, then there is a unique exact functor $T'\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ so that $T = T' \circ \operatorname{loc}$.

II.6.4 **Localization Theorem 6.4.** *(Heller) Let $\mathcal{A}$ be a small abelian category, and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then the following sequence is exact:*

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \xrightarrow{loc} K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

Proof. By the construction of $\mathcal{A}/\mathcal{B}$, $K_0(\mathcal{A})$ maps onto $K_0(\mathcal{A}/\mathcal{B})$ and the composition $K_0(\mathcal{B}) \to K_0(\mathcal{A}/\mathcal{B})$ is zero. Hence if Γ denotes the cokernel of the map $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ there is a natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$; to prove the theorem it suffices to give an inverse. For this it suffices to show that $\gamma(\operatorname{loc}(A)) = [A]$ defines an additive function from $\mathcal{A}/\mathcal{B}$ to Γ, because the induced map $\gamma\colon K_0(\mathcal{A}/\mathcal{B}) \to \Gamma$ will be inverse to the natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$.

Since loc: $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a bijection on objects, γ is well-defined. We claim that if $\mathrm{loc}(A_1) \cong \mathrm{loc}(A_2)$ in $\mathcal{A}/\mathcal{B}$ then $[A_1] = [A_2]$ in Γ. To see this, represent the isomorphism by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. As $\mathrm{loc}(g)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$, g is also a $\mathcal{B}$-iso. In $K_0(\mathcal{A})$ we have

$$[A] = [A_1] + [\ker(f)] - [\mathrm{coker}(f)] = [A_2] + [\ker(g)] - [\mathrm{coker}(g)].$$

Hence $[A] = [A_1] = [A_2]$ in Γ, as claimed.

To see that γ is an additive function, suppose given an exact sequence in $\mathcal{A}/\mathcal{B}$ of the form:

$$0 \to \mathrm{loc}(A_0) \xrightarrow{i} \mathrm{loc}(A_1) \xrightarrow{j} \mathrm{loc}(A_2) \to 0;$$

we have to show that $[A_1] = [A_0] + [A_2]$ in Γ. Represent j by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. Since $[A] = [A_1] + [\ker(f)] - [\mathrm{coker}(f)]$ in $K_0(\mathcal{A})$, $[A] = [A_1]$ in Γ. Applying the exact functor loc to

$$0 \to \ker(g) \to A \xrightarrow{g} A_2 \to \mathrm{coker}(g) \to 0,$$

we see that $\mathrm{coker}(g)$ is in $\mathcal{B}$ and that $\mathrm{loc}(\ker(g)) \cong \mathrm{loc}(A_0)$ in $\mathcal{A}/\mathcal{B}$. Hence $[\ker(g)] \equiv [A_0]$ in Γ, and in Γ we have

$$[A_1] = [A] = [A_2] + [\ker(g)] - [\mathrm{coker}(g)] \equiv [A_0] + [A_2]$$

proving that γ is additive, and finishing the proof of the Localization Theorem. □

II.6.4.1 **Application 6.4.1.** Let S be a central multiplicative set in a ring R, and let $\mathbf{mod}_S(R)$ denote the Serre subcategory of $\mathbf{mod}$-R consisting of S-torsion modules, *i.e.,* those R-modules M such that every $m \in M$ has $ms = 0$ for some $s \in S$. Then there is a natural equivalence between $\mathbf{mod}$-$(S^{-1}R)$ and the quotient category $\mathbf{mod}$-$R/\mathbf{mod}_S(R)$. If R is noetherian and $\mathbf{M}_S(R)$ denotes the Serre subcategory of $\mathbf{M}(R)$ consisting of finitely generated S-torsion modules, then $\mathbf{M}(S^{-1}R)$ is equivalent to $\mathbf{M}(R)/\mathbf{M}_S(R)$. The Localization exact sequence becomes:

$$K_0\mathbf{M}_S(R) \to G_0(R) \to G_0(S^{-1}R) \to 0.$$

In particular, if $S = \{s^n\}$ for some s then by Application II.6.3.3 6.3.3 we have an exact sequence

$$G_0(R/sR) \to G_0(R) \to G_0(R[\frac{1}{s}]) \to 0.$$

More generally, if I is an ideal of a noetherian ring R, we can consider the Serre subcategory $\mathbf{M}_I(R)$ of modules with some $MI^n = 0$ discussed in Application II.6.3.3 6.3.3. The quotient category $\mathbf{M}(R)/\mathbf{M}_I(R)$ is known to be isomorphic to the category $\mathbf{M}(U)$ of coherent $\mathcal{O}_U$-modules, where U is the open subset of $\mathrm{Spec}(R)$ defined by I. The composition of the isomorphism $K_0\mathbf{M}(R/I) \cong K_0\mathbf{M}_I(R)$ of II.6.3.3 6.3.3 with $K_0\mathbf{M}_I(R) \to K_0\mathbf{M}(R)$ is evidently the transfer map $i_*\colon G_0(R/I) \to G_0(R)$. Hence the Localization Sequence becomes the exact sequence

$$G_0(R/I) \xrightarrow{i_*} G_0(R) \to G_0(U) \to 0.$$

II.6.4.2 **Application 6.4.2.** Let X be a scheme, and $i\colon Z \subset X$ a closed subscheme with complement $j\colon U \subset X$. Let $\mathbf{mod}_Z(X)$ denote the Serre subcategory of $\mathcal{O}_X$-**mod** consisting of all $\mathcal{O}_X$-modules $\mathcal{F}$ with support in Z, *i.e.,* such that $\mathcal{F}|_U = 0$. Gabriel proved [61] that j^* induces an equivalence: $\mathcal{O}_U$-**mod** $\cong$ $\mathcal{O}_X$-**mod**/$\mathbf{mod}_Z(X)$.

Morover, if X is noetherian and $\mathbf{M}_Z(X)$ denotes the category of coherent sheaves supported in Z, then $\mathbf{M}(X)/\mathbf{M}_Z(X) \cong \mathbf{M}(U)$. The inclusion $i\colon Z \subset X$ induces an exact functor $i_*\colon \mathbf{M}(Z) \subset \mathbf{M}(X)$, and $G_0(Z) \cong K_0\mathbf{M}_Z(X)$ by Example 6.3.4. Therefore the Localization Sequence becomes:

$$G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0.$$

For example, if $X = \operatorname{Spec}(R)$ and $Z = \operatorname{Spec}(R/I)$, we recover the exact sequence in the previous application.

II.6.4.3 **Application 6.4.3** (Higher Divisor Class Groups)**.** Given a commutative noetherian ring R, let $D^i(R)$ denote the free abelian group on the set of prime ideals of height exactly i; this is generalizes the group of Weil divisors in Ch.I, §3. Let $\mathbf{M}^i(R)$ denote the category of finitely generated R-modules M whose associated prime ideals all have height $\geq i$. Each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$; see Ex. 6.9. Let $F^iG_0(R)$ denote the image of $K_0\mathbf{M}^i(R)$ in $G_0(R) = K_0\mathbf{M}(R)$. These subgroups form a filtration $\cdots \subset F^2 \subset F^1 \subset F^0 = G_0(R)$, called the *coniveau filtration* of $G_0(R)$.

It turns out that there is an equivalence $\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \bigoplus \mathbf{M}_{\mathfrak{p}}(R_{\mathfrak{p}})$, $ht(\mathfrak{p}) = i$. By Application 6.3.3 of devissage, $K_0\mathbf{M}_{\mathfrak{p}}(R_{\mathfrak{p}}) \cong G_0(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \cong \mathbb{Z}$, so there is an isomorphism $D^i(R) \xrightarrow{\cong} K_0\mathbf{M}^i/\mathbf{M}^{i+1}(R)$, $[\mathfrak{p}] \mapsto [R/\mathfrak{p}]$. By the Localization Theorem, we have an exact sequence

$$K_0\mathbf{M}^{i+1}(R) \to K_0\mathbf{M}^i(R) \to D^i(R) \to 0.$$

Thus $G_0(R)/F^1 \cong D^0(R)$, and each subquotient F^i/F^{i+1} is a quotient of $D^i(R)$.

For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(R)$ is defined to be the subgroup of $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R)$ generated by the classes $[R/\mathfrak{p}]$, $ht(\mathfrak{p}) \geq i$. This definition is due to L. Claborn and R. Fossum; the notation reflects a theorem (in V.9 below) that the kernel of $D^i(R) \to CH^i(R)$ is generated by rational equivalence. For example, we will see in Ex. 6.9 that if R is a Krull domain then $CH^1(R)$ is the usual divisor class group $Cl(R)$, and $G_0(R)/F^2 \cong \mathbb{Z} \oplus Cl(R)$.

Similarly, if X is a noetherian scheme, there is a coniveau filtration on $G_0(X)$. Let $\mathbf{M}^i(X)$ denote the subcategory of $\mathbf{M}(X)$ consisting of coherent modules whose support has codimension $\geq i$, and let $D^i(X)$ denote the free abelian group on the set of points of X having codimension i. Then each $\mathbf{M}^i(X)$ is a

Serre subcategory and $\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong \bigoplus \mathbf{M}_x(\mathcal{O}_{X,x})$, where x runs over all points of codimension i in X. Again by devissage, there is an isomorphism $K_0\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong D^i(X)$ and hence $G_0(X)/F^1 \cong D^0(X)$. For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(X)$ is defined to be the subgroup of $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(X)$ generated by the classes $[\mathcal{O}_Z]$, $\mathrm{codim}_X(Z) = i$. We will see later on (in V.9.4.1) that $CH^i(X)$ is the usual Chow group of codimension i cycles on X modulo rational equivalence, as defined in [58]. The verification that $CH^1(X) = Cl(X)$ is left to Ex. 6.10.

We now turn to a clasical application of the Localization Theorem: the Fundamental Theorem for G_0 of a noetherian ring R. Via the ring map $\pi\colon R[t] \to R$ sending t to zero, we have an inclusion $\mathbf{M}(R) \subset \mathbf{M}(R[t])$ and hence a transfer map $\pi_*\colon G_0(R) \to G_0(R[t])$. By 6.4.1 there is an exact localization sequence

$$G_0(R) \xrightarrow{\pi_*} G_0(R[t]) \xrightarrow{j^*} G_0(R[t,t^{-1}]) \to 0. \qquad (6.4.4)$$ II.6.4.4

Given an R-module M, the exact sequence of $R[t]$-modules

$$0 \to M[t] \xrightarrow{t} M[t] \to M \to 0$$

shows that in $G_0(R[t])$ we have

$$\pi_*[M] = [M] = [M[t]] - [M[t]] = 0.$$

Thus $\pi_* = 0$, because every generator $[M]$ of $G_0(R)$ becomes zero in $G_0(R[t])$. From the Localization sequence (6.4.4) it follows that j^* is an isomorphism. This proves the easy part of the following result.

II.6.5 **Fundamental Theorem for G_0-theory of Rings 6.5.** *For every noetherian ring R, the inclusions* $R \overset{i}{\hookrightarrow} R[t] \overset{j}{\hookrightarrow} R[t,t^{-1}]$ *induce isomorphisms*

$$G_0(R) \cong G_0(R[t]) \cong G_0(R[t,t^{-1}]).$$

Proof. The ring inclusions are flat, so they induce maps $i^*\colon G_0(R) \to G_0(R[t])$ and $j^*\colon G_0(R[t]) \to G_0(R[t,t^{-1}])$. We have already seen that j^* is an isomorphism; it remains to show that i^* is an isomorphism.

Because $R = R[t]/tR[t]$, there is a map $\pi^*\colon G_0(R[t]) \to G_0(R)$, given by Serre's formula: $\pi^*[M] = [M/Mt] - [\mathrm{ann}_M(t)]$, where $\mathrm{ann}_M(t) = \{x \in M : xt = 0\}$. (See Ex. 6.6 or 7.9.3 below.) Since $\pi^* i^*[M] = \pi^*[M[t]] = [M]$, the map i^* is an injection split by π^*.

We shall present Grothendieck's proof that $i^*\colon G_0(R) \to G_0(R[t])$ is onto, which assumes that R is a commutative ring. A proof in the non-commutative case (due to Serre) will be sketched in Ex. 6.13.

If $G_0(R) \neq G_0(R[t])$, we proceed by noetherian induction to a contradiction. Among all ideals J for which $G_0(R/J) \neq G_0(R/J[t])$, there is a maximal one. Replacing R by R/J, we may assume that $G_0(R/I) = G_0(R/I[t])$ for each $I \neq 0$ in R. Such a ring R must be reduced by Corollary 6.3.1. Let S be the set of non-zero divisors in R; by elementary ring theory $S^{-1}R$ is a finite product $\prod F_i$ of fields F_i, so $G_0(S^{-1}R) \cong \oplus G_0(F_i)$. Similarly $S^{-1}R[t] = \prod F_i[t]$ and $G_0(S^{-1}R[t]) \cong \oplus G_0(F_i[t])$. By Application 6.4.1 and Example 6.2.8 we have a diagram with exact rows:

$$\begin{array}{ccccccc}
\varinjlim G_0(R/sR) & \longrightarrow & G_0(R) & \longrightarrow & \oplus G_0(F_i) & \longrightarrow & 0 \\
\cong\downarrow i^* & & \downarrow i^* & & \downarrow & & \\
\varinjlim G_0(R/sR[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & \oplus G_0(F_i[t]) & \longrightarrow & 0.
\end{array}$$

Since the direct limits are taken over all $s \in S$, the left vertical arrow is an isomorphism by induction. Because each $F_i[t]$ is a principal ideal domain, 2.6.3 and Example 6.2.1 imply that the right vertical arrow is the sum of the isomorphisms

$$G_0(F_i) \cong K_0(F_i) \cong \mathbb{Z} \cong K_0(F_i[t]) \cong G_0(F_i[t]).$$

By the 5-lemma, the middle vertical arrow is onto, hence an isomorphism. □

We can generalize the Fundamental Theorem from rings to schemes by a slight modification of the proof. For every scheme X, let $X[t]$ and $X[t, t^{-1}]$ denote the schemes $X \times \mathrm{Spec}(\mathbb{Z}[t])$ and $X \times \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ respectively. Thus if $X = \mathrm{Spec}(R)$ we have $X[t] = \mathrm{Spec}(R[t])$ and $X[t, t^{-1}] = \mathrm{Spec}(R[t, t^{-1}])$. Now suppose that X is noetherian. Via the map $\pi\colon X \to X[t]$ defined by $t = 0$, we have an inclusion $\mathbf{M}(X) \subset \mathbf{M}(X[t])$ and hence a transfer map $\pi_*\colon G_0(X) \to G_0(X[t])$ as before. The argument we gave after (6.4.4) above goes through to show that $\pi_* = 0$ here too, because any generator $[\mathcal{F}]$ of $G_0(X)$ becomes zero in $G_0(X[t])$. By 6.4.2 we have an exact sequence

$$G_0(X) \xrightarrow{\pi_*} G_0(X[t]) \to G_0(X[t, t^{-1}]) \to 0$$

and therefore $G_0(X[t]) \cong G_0(X[t, t^{-1}])$.

II.6.5.1 **Fundamental Theorem for G_0-theory of Schemes 6.5.1.** *For every noetherian scheme X, the flat maps $X[t, t^{-1}] \overset{j}{\hookrightarrow} X[t] \xrightarrow{i} X$ induce isomorphisms:*

$$G_0(X) \cong G_0(X[t]) \cong G_0(X[t, t^{-1}]).$$

Proof. We have already seen that j^* is an isomorphism. By Ex. 6.7 there is a map $\pi^*\colon G_0(X[t]) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [\mathrm{ann}_{\mathcal{F}}(t)]$. Since $\pi^* i^*[\mathcal{F}] = (i\pi)^*[\mathcal{F}] = [\mathcal{F}]$, we again see that i^* is an injection, split by π^*.

It suffices to show that i^* is a surjection for all X. By noetherian induction, we may suppose that the result is true for all proper closed subschemes Z of X. In particular, if Z is the complement of an affine open subscheme $U = \mathrm{Spec}(R)$ of X, we have a commutative diagram whose rows are exact by Application 6.4.2.

$$\begin{array}{ccccccc}
G_0(Z) & \longrightarrow & G_0(X) & \longrightarrow & G_0(R) & \longrightarrow & 0 \\
\cong\downarrow i^* & & \downarrow i^* & & \cong\downarrow i^* & & \\
G_0(Z[t]) & \longrightarrow & G_0(X[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & 0
\end{array}$$

The outside vertical arrows are isomorphisms, by induction and Theorem II.6.5 6.5. By the 5-lemma, $G_0(X) \xrightarrow{i^*} G_0(X[t])$ is onto, and hence an isomorphism. □

Euler Characteristics

Suppose that $C_\bullet\colon\ 0 \to C_m \to \cdots \to C_n \to 0$ is a bounded chain complex of objects in an abelian category $\mathcal{A}$. We define the *Euler characteristic* $\chi(C_\bullet)$ of $C_\bullet$ to be the following element of $K_0(\mathcal{A})$:

$$\chi(C_\bullet) = \sum (-1)^i [C_i].$$

II.6.6 **Proposition 6.6.** *If $C_\bullet$ is a bounded complex of objects in $\mathcal{A}$, the element $\chi(C_\bullet)$ depends only upon the homology of $C_\bullet$:*

$$\chi(C_\bullet) = \sum (-1)^i [H_i(C_\bullet)].$$

In particular, if $C_\bullet$ is acyclic (exact as a sequence) then $\chi(C_\bullet) = 0$.

Proof. Write Z_i and B_{i-1} for the kernel and image of the map $C_i \to C_{i-1}$, respectively. Since $B_{i-1} = C_i/Z_i$ and $H_i(C_\bullet) = Z_i/B_i$, we compute in $K_0(\mathcal{A})$:

$$\begin{aligned} \sum (-1)^i [H_i(C_\bullet)] &= \sum (-1)^i [Z_i] - \sum (-1)^i [B_i] \\ &= \sum (-1)^i [Z_i] + \sum (-1)^i [B_{i-1}] \\ &= \sum (-1)^i [C_i] = \chi(C_\bullet). \end{aligned}$$ □

Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the abelian category of (possibly unbounded) chain complexes of objects in $\mathcal{A}$ having only finitely many nonzero homology groups. We call such complexes *homologically bounded*.

II.6.6.1 **Corollary 6.6.1.** *There is a natural surjection $\chi_H\colon K_0(\mathbf{Ch}^{hb}) \to K_0(\mathcal{A})$ sending $C_\bullet$ to $\sum (-1)^i [H_i(C_\bullet)]$. In particular, if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a exact sequence of homologically bounded complexes then:*

$$\chi_H(B_\bullet) = \chi_H(A_\bullet) + \chi_H(C_\bullet).$$

EXERCISES

EII.6.1 **6.1.** Let R be a ring and $\mathbf{mod}_{\mathrm{fl}}(R)$ the abelian category of R-modules with finite length. Show that $K_0\mathbf{mod}_{\mathrm{fl}}(R)$ is the free abelian group $\bigoplus_{\mathfrak{m}} \mathbb{Z}$, a basis being $\{[R/\mathfrak{m}], \mathfrak{m}$ a maximal right ideal of $R\}$. *Hint:* Use the Jordan-Hölder Theorem for modules of finite length.

EII.6.2 **6.2.** *Schreier Refinement Theorem.* Let $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_r = 0$ and $A = A'_0 \supseteq A'_1 \supseteq \cdots \supseteq A'_s = 0$ be two filtrations of an object A in an abelian category $\mathcal{A}$. Show that the subobjects $A_{i,j} = (A_i \cap A'_j) + A_{i+1}$, ordered lexicographically, form a filtration of A which refines the filtration $\{A_i\}$. By symmetry, there is also a filtration by the $A'_{j,i} = (A_i \cap A'_j) + A'_{j+1}$ which refines the filtration $\{A'_j\}$.

Prove *Zassenhaus' Lemma*, that $A_{i,j}/A_{i,j+1} \cong A'_{j,i}/A'_{j,i+1}$. This shows that the factors in the two refined filtrations are isomorphic up to a permutation; the slogan is that "any two filtrations have equivalent refinements."

EII.6.3 **6.3.** *Jordan-Hölder Theorem in $\mathcal{A}$.* An object A in an abelian category $\mathcal{A}$ is called *simple* if it has no proper subobjects. We say that an object A has *finite length* if it has a composition series $A = A_0 \supset \cdots \supset A_s = 0$ in which all the quotients A_i/A_{i+1} are simple. By Ex. 6.2, the Jordan-Hölder Theorem holds in $\mathcal{A}_{\mathrm{fl}}$: the simple factors in any composition series of A are unique up to permutation and isomorphism. Let $\mathcal{A}_{\mathrm{fl}}$ denote the subcategory of objects in $\mathcal{A}$ of finite length. Show that $\mathcal{A}_{\mathrm{fl}}$ is a Serre subcategory of $\mathcal{A}$, and that $K_0(\mathcal{A}_{\mathrm{fl}})$ is the free abelian group on the set of isomorphism classes of simple objects.

EII.6.4 **6.4.** Let $\mathcal{A}$ be a small abelian category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $A_1 \oplus C_1 \cong A_2 \oplus C_2$. *Hint:* First find sequences $0 \to D'_i \to D_i \to D''_i \to 0$ such that $A_1 \oplus D'_1 \oplus D''_1 \oplus D_2 \cong A_2 \oplus D'_2 \oplus D''_2 \oplus D_1$, and set $C_i = D'_i \oplus D''_i \oplus D_j$.

EII.6.5 **6.5.** *Resolution.* Suppose that R is a regular noetherian ring, *i.e.,* that every R-module has a finite projective resolution. Show that the Cartan homomorphism $K_0(R) \to G_0(R)$ is onto. (We will see in Theorem 7.8 that it is an isomorphism.)

EII.6.6 **6.6.** *Serre's Formula.* (Cf. 7.9.3) If s is a central element of a ring R, show that there is a map $\pi^*\colon G_0(R) \to G_0(R/sR)$ sending $[M]$ to $[M/Ms] - [\mathrm{ann}_M(s)]$, where $\mathrm{ann}_M(s) = \{x \in M : xs = 0\}$. Theorem 6.5 gives an example where π^* is onto, and if s is nilpotent the map is zero by devissage 6.3.1. *Hint:* Use the map $M \xrightarrow{s} M$.

EII.6.7 **6.7.** Let Y be a noetherian scheme over the ring $\mathbb{Z}[t]$, and let $X \overset{\pi}{\hookrightarrow} Y$ be the closed subscheme defined by $t = 0$. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, let $\mathrm{ann}_{\mathcal{F}}(t)$ denote the submodule of $\mathcal{F}$ annihilated by t. Show that there is a map $\pi^*\colon G_0(Y) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [\mathrm{ann}_{\mathcal{F}}(t)]$.

EII.6.8 **6.8.** (Heller-Reiner) Let R be a commutative domain with field of fractions F. If $S = R - \{0\}$, show that there is a well-defined map $\Delta : F^\times \to K_0\mathbf{M}_S(R)$ sending the fraction $r/s \in F^\times$ to $[R/rR] - [R/sR]$. Then use Ex. 6.4 to show that the localization sequence extends to the exact sequence

$$1 \to R^\times \to F^\times \xrightarrow{\Delta} K_0\mathbf{M}_S(R) \to G_0(R) \to \mathbb{Z} \to 0.$$

EII.6.9 **6.9.** *Weil Divisor Class groups.* Let R be a commutative noetherian ring.

(a) Show that each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$.

(b) Show that $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R) \cong CH^i(R) \oplus D^{i-1}(R)$. In particular, if R is a 1-dimensional domain then $G_0(R) = \mathbb{Z} \oplus CH^1(R)$.

(c) Show that each $F^iG_0(R)/F^{i+1}G_0(R)$ is a quotient of the group $CH^i(R)$, defined in Application 6.4.3.

(d) Suppose that R is a domain with field of fractions F. As in Ex. 6.8, show that there is an exact sequence generalizing Proposition I.3.6:

$$0 \to R^\times \to F^\times \xrightarrow{\Delta} D^1(R) \to CH^1(R) \to 0.$$

In particular, if R is a Krull domain, conclude that $CH^1(R) \cong Cl(R)$ and $G_0(R)/F^2 \cong \mathbb{Z} \oplus Cl(R)$.

(e) If $(R, \mathfrak{m})$ is a 1-dimensional local domain and $k_1, \dots, k_n$ are the residue fields of the normalization of R over $k = R/\mathfrak{m}$, show that $CH^1(R) \cong \mathbb{Z}/\gcd\{[k_i : k]\}$.

EII.6.10 **6.10.** Generalize the preceding exercise to a noetherian scheme X, as indicated in Application 6.4.3. *Hint:* F becomes the function field of X, and (d) becomes I.5.12.

EII.6.11 **6.11.** If S is a multiplicatively closed set of central elements in a noetherian ring R, show that

$$K_0\mathbf{M}_S(R) \cong K_0\mathbf{M}_S(R[t]) \cong K_0\mathbf{M}_S(R[t, t^{-1}]).$$

EII.6.12 **6.12.** *Graded modules.* When $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is a noetherian graded ring, let $\mathbf{M}_{gr}(S)$ denote the abelian category of finitely generated graded S-modules. Write σ for the shift automorphism $M \mapsto M(-1)$ of the category $\mathbf{M}_{gr}(S)$. Show that:

(a) $K_0\mathbf{M}_{gr}(S)$ is a module over the ring $\mathbb{Z}[\sigma, \sigma^{-1}]$.

(b) If S is flat over R, there is a map from the direct sum $G_0(R)[\sigma, \sigma^{-1}] = \bigoplus_{n\in\mathbb{Z}} G_0(R)\sigma^n$ to $K_0\mathbf{M}_{gr}(S)$ sending $[M]\sigma^n$ to $[\sigma^n(M \otimes S)]$.

(c) If $S = R$, the map in (b) is an isomorphism: $K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma, \sigma^{-1}]$.

(d) If $S = R[x_1, \cdots, x_m]$ with $x_1, \cdots, x_m$ in S_1, the map is surjective, *i.e.,* $K_0\mathbf{M}_{gr}(S)$ is generated by the classes $[\sigma^n M[x_1, ..., x_m]]$. We will see in Ex. 7.14 that the map in (b) is an isomorphism for $S = R[x_1, \cdots, x_m]$.

(e) Let $\mathcal{B}$ be the subcategory of $\mathbf{M}_{gr}(R[x, y])$ of modules on which y is nilpotent. Show that $\mathcal{B}$ is a Serre subcategory, and that

$$K_0\mathcal{B} \cong K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma, \sigma^{-1}].$$

EII.6.13 **6.13.** In this exercise we sketch Serre's proof of the Fundamental Theorem 6.5 when R is a non-commutative ring. We assume the results of the previous exercise. Show that the formula $j(M) = M/(y-1)M$ defines an exact functor $j\colon \mathbf{M}_{gr}(R[x,y]) \to \mathbf{M}(R[x])$, sending $\mathcal{B}$ to zero. In fact, j induces an equivalence

$$\mathbf{M}_{gr}(R[x,y])/\mathcal{B} \cong \mathbf{M}(R[x]).$$

Then use this equivalence to show that the map $i^*\colon G_0(R) \to G_0(R[x])$ is onto.

EII.6.14 **6.14.** *G_0 of projective space.* Let k be a field and set $S = k[x_0, \dots, x_m]$, with $X = \mathbb{P}^m_k$. Using the notation of Exercises 6.3 and 6.12, let $\mathbf{M}^b_{gr}(S)$ denote the Serre subcategory of $\mathbf{M}_{gr}(S)$ consisting of graded modules of finite length. It is well-known (see [85, II.5.15]) that every coherent $\mathcal{O}_X$-module is of the form $\widetilde{M}$ for some M in $\mathbf{M}_{gr}(S)$, *i.e.,* that the associated sheaf functor $\mathbf{M}_{gr}(S) \to \mathbf{M}(X)$ is onto, and that if M has finite length then $\widetilde{M} = 0$. In fact, there is an equivalence

$$\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(\mathbb{P}^m_k).$$

(See [85, Ex. II.5.9(c)].) Under this equivalence $\sigma^i(S)$ represents $\mathcal{O}_X(-i)$.

(a) Let F denote the graded S-module S^{m+1}, whose basis lies in degree 0. Use the Koszul exact sequence of I(5.4):

$$0 \to \sigma^{m+1}(\textstyle\bigwedge^{m+1} F) \to \cdots \to \sigma^2(\bigwedge^2 F) \to \sigma F \xrightarrow{x_0,\dots} S \to k \to 0$$

to show that in $K_0\mathbf{M}_{gr}(S)$ every finitely generated k-module M satisfies

$$[M] = \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \sigma^i\,[M \otimes_k S] = (1-\sigma)^{m+1}[M \otimes_k S].$$

(b) Show that in $G_0(\mathbb{P}^m_k)$ every $[\mathcal{O}_X(n)]$ is a linear combination of the classes $[\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)]$, and that

$$\textstyle\sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} [\mathcal{O}(-i)] = 0 \quad \text{in } G_0\,\mathbb{P}^m_k\,.$$

(c) We will see in Ex. 7.14 that the map in Ex. 6.12(b) is an isomorphism:

$$K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}].$$

Assume this calculation, and show that

$$G_0(\mathbb{P}^m_k) \cong \mathbb{Z}^m \quad \text{on generators } [\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)].$$

EII.6.15 **6.15.** *Naturality of f_*.* Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms between noetherian schemes. Show that $(gf)_* = g_*\,f_*$ as maps $G_0(X) \to G_0(Z)$.

EII.6.16 **6.16.** Let R be a noetherian ring, and $r \in R$. If r is a nonzerodivisor on modules M_j whose associated primes all have height i, and $0 = \sum \pm[M_j]$ in $K_0\mathbf{M}^i(R)$, show that $0 = \sum \pm[M_j/rM_j]$ in $K_0\mathbf{M}^{i+1}(R)$. *Hint:* By devissage, the formula holds in $K_0(R/I)$ for some product I of height i primes. Modify r to be a nonzerodivisor on R/I without changing the M_j/rM_j and use $f_* : G_0(R/I) \to G_0(R/(I + rR))$.

7 K_0 of an Exact Category

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, we may still talk about exact sequences: an *exact sequence* in $\mathcal{C}$ is a sequence of objects (and maps) in $\mathcal{C}$ which is exact as a sequence in $\mathcal{A}$. With hindsight, we know that it helps to require $\mathcal{C}$ to be closed under extensions. Thus we formulate the following definitions.

II.7.0 **Definition 7.0** (Exact categories)**.** An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a family of sequences in $\mathcal{C}$ of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0, \qquad (\dagger)$$

satisfying the following condition: there is an embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$ so that

(1) $\mathcal{E}$ is the class of all sequences $(\dagger)$ in $\mathcal{C}$ which are exact in $\mathcal{A}$;

(2) $\mathcal{C}$ is *closed under extensions* in $\mathcal{A}$ in the sense that if $(\dagger)$ is an exact sequence in $\mathcal{A}$ with $B, D \in \mathcal{C}$ then C is isomorphic to an object in $\mathcal{C}$.

The sequences in $\mathcal{E}$ are called the *short exact sequences* of $\mathcal{C}$. We will often abuse notation and just say that $\mathcal{C}$ is an exact category when the class $\mathcal{E}$ is clear. We call a map in $\mathcal{C}$ an *admissible monomorphism* (resp. an *admissible epimorphism*) if it occurs as the monomorphism i (resp. as the epi j) in some sequence $(\dagger)$ in $\mathcal{E}$.

The following hypothesis is commonly satisfied in applications, and is needed for Euler characteristics and the Resolution Theorem 7.6 below.

II.7.0.1 **7.0.1.** We say that $\mathcal{C}$ is *closed under kernels of surjections* in $\mathcal{A}$ provided that whenever a map $f\colon B \to C$ in $\mathcal{C}$ is a surjection in $\mathcal{A}$ then $\ker(f) \in \mathcal{C}$. The well-read reader will observe that Bass' definition of exact category in [15] is what we call an exact category closed under kernels of surjections.

An *exact functor* $F\colon \mathcal{B} \to \mathcal{C}$ between exact categories is an additive functor F carrying short exact sequences in $\mathcal{B}$ to exact sequences in $\mathcal{C}$. If $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, and the exact sequences in $\mathcal{B}$ are precisely the sequences $(\dagger)$ in $\mathcal{B}$ which are exact in $\mathcal{C}$, we call $\mathcal{B}$ an *exact subcategory* of $\mathcal{C}$. This is consistent with the notion of an exact abelian subcategory in 6.1.5.

II.7.1 **Definition 7.1** (K_0)**.** Let $\mathcal{C}$ be a small exact category. $K_0(\mathcal{C})$ is the abelian group having generators $[C]$, one for each object C of $\mathcal{C}$, and relations $[C] = [B] + [D]$ for every short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$.

As in 6.1.1, we have $[0] = 0$, $[B \oplus D] = [B] + [D]$ and $[B] = [C]$ if B and C are isomorphic. As before, we could actually define $K_0(\mathcal{C})$ when $\mathcal{C}$ is only skeletally small, but we shall not dwell on these set-theoretic intricacies. Clearly, $K_0(\mathcal{C})$ satisfies the universal property 6.1.2 for additive functions from $\mathcal{C}$ to abelian groups.

II.7.1.1 **Example 7.1.1.** The category $\mathbf{P}(R)$ of finitely generated projective R-modules is exact by virtue of its embedding in **mod**-R. As every exact sequence of projective modules splits, we have $K_0\mathbf{P}(R) = K_0(R)$.

Any additive category is a symmetric monoidal category under $\oplus$, and the above remarks show that $K_0(\mathcal{C})$ is a quotient of the group $K_0^{\oplus}(\mathcal{C})$ of §5. Since abelian categories are exact, Examples 6.2.1–6.2.4 show that these groups are not identical.

II.7.1.2 **Example 7.1.2** (Split exact categories)**.** A *split exact* category $\mathcal{C}$ is an exact category in which every short exact sequence in $\mathcal{E}$ is split (*i.e.,* isomorphic to $0 \to B \to B \oplus D \to D \to 0$). In this case we have $K_0(\mathcal{C}) = K_0^{\oplus}(\mathcal{C})$ by definition. For example, the category $\mathbf{P}(R)$ is split exact.

If X is a topological space, embedding $\mathbf{VB}(X)$ in the abelian category of families of vector spaces over X makes $\mathbf{VB}(X)$ into an exact category. When X is paracompact then, by the Subbundle Theorem I.4.1, $\mathbf{VB}(X)$ is a split exact category, so that $K^0(X) = K_0(\mathbf{VB}(X))$.

We will see in Exercise 7.7 that any additive category $\mathcal{C}$ may be made into a split exact category by equipping it with the class $\mathcal{E}_{split}$ of sequences isomorphic to $0 \to B \to B \oplus D \to D \to 0$

Warning. Every abelian category $\mathcal{A}$ has a natural exact category structure, but it also has the split exact structure. These will yield different K_0 groups in general, unless something like a Krull-Schmidt Theorem holds in $\mathcal{A}$. We will always use the natural exact structure unless otherwise indicated.

II.7.1.3 **Example 7.1.3** (K_0 of a scheme)**.** Let X be a scheme (or more generally a ringed space). The category $\mathbf{VB}(X)$ of algebraic vector bundles on X, introduced in (§I.5), is an exact category by virtue of its being an additive subcategory of the abelian category $\mathcal{O}_X$-**mod** of all $\mathcal{O}_X$-modules. If X is quasi-projective over a commutative ring, we write $K_0(X)$ for $K_0\mathbf{VB}(X)$. If X is noetherian, the inclusion $\mathbf{VB}(X) \subseteq \mathbf{M}(X)$ yields a *Cartan homomorphism* $K_0(X) \to G_0(X)$. We saw in I.5.3 that exact sequences in $\mathbf{VB}(X)$ do not always split, so $\mathbf{VB}(X)$ is not always a split exact category.

II.7.1.4 **Example 7.1.4** (G_0 of non-noetherian rings)**.** If R is a non-noetherian ring, the category $\mathbf{mod}_{\mathrm{fg}}(R)$ of all finitely generated R-modules will not be abelian, because $R \to R/I$ has no kernel inside this category. However, it is still an exact subcategory of **mod**-R, so once again we might try to consider the group $K_0\mathbf{mod}_{\mathrm{fg}}(R)$. However, it turns out that this definition does not have good properties (see Ex. 7.3 and 7.4).

Here is a more suitable definition, based upon [SGA6, I.2.9]. An R-module M is called *pseudo-coherent* if it has an infinite resolution $\cdots \to P_1 \to P_0 \to M \to 0$ by finitely generated projective R-modules. Pseudo-coherent modules are clearly finitely presented, and if R is right noetherian then every finitely generated module is pseudo-coherent. Let $\mathbf{M}(R)$ denote the category of all pseudo-coherent R-modules. The "Horseshoe Lemma" [223, 2.2.8] shows that $\mathbf{M}(R)$ is closed under extensions in **mod**-R, so it is an exact category. (It is also closed under kernels of surjections, and cokernels of injections in **mod-**R, as can be seen using the mapping cone.)

Now we define $G_0(R) = K_0\mathbf{M}(R)$. Note that if R is right noetherian then $\mathbf{M}(R)$ is the usual category of §6, and we have recovered the definition of $G_0(R)$ in 6.2.

II.7.1.5 **Example 7.1.5.** The opposite category $\mathcal{C}^{op}$ has an obvious notion of exact sequence: turn the arrows around in the exact sequences of $\mathcal{C}$. Formally, this arises from the inclusion of $\mathcal{C}^{op}$ in $\mathcal{A}^{op}$. Clearly $K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$.

II.7.1.6 **Example 7.1.6.** The direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ of two exact categories is also exact, the ambient abelian category being $\mathcal{A}_1 \oplus \mathcal{A}_2$. Clearly $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2) \cong K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$. More generally, the direct sum $\bigoplus \mathcal{C}_i$ of exact categories is an exact category (inside the abelian category $\oplus\mathcal{A}_i$), and as in 6.1.2 this yields $K_0(\oplus\mathcal{C}_i) \cong \oplus K_0(\mathcal{C}_i)$.

II.7.1.7 **Example 7.1.7** (Filtered colimits)**.** Suppose that $\{\mathcal{C}_i\}$ is a filtered family of exact subcategories of a fixed abelian category $\mathcal{A}$. Then $\mathcal{C} = \cup\mathcal{C}_i$ is also an exact subcategory of $\mathcal{A}$, and by inspection of the definition we see that

$$K_0(\bigcup \mathcal{C}_i) = \varinjlim K_0(\mathcal{C}_i).$$

The ambient $\mathcal{A}$ is unnecessary: if $\{\mathcal{C}_i\}$ is a filtered family of exact categories and exact functors, then $K_0(\varinjlim \mathcal{C}_i) = \varinjlim K_0(\mathcal{C}_i)$; see Ex. 7.9. As a case in point, if a ring R is the union of subrings R_α then $\mathbf{P}(R)$ is the direct limit of the $\mathbf{P}(R_\alpha)$, because every P in $\mathbf{P}(R)$ is finitely presented; we have $K_0(R) = \varinjlim K_0(R_\alpha)$, as in 2.1.6.

II.7.2 **Lemma 7.2** (Cofinality Lemma)**.** *Let $\mathcal{B}$ be an exact subcategory of $\mathcal{C}$ which is closed under extensions in $\mathcal{C}$, and which is cofinal in the sense that for every C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \oplus C'$ is in $\mathcal{B}$. Then $K_0\mathcal{B}$ is a subgroup of $K_0\mathcal{C}$.*

Proof. By 1.3 we know that $K_0^{\oplus}\mathcal{B}$ is a subgroup of $K_0^{\oplus}\mathcal{C}$. Given a short exact sequence $0 \to C_0 \to C_1 \to C_2 \to 0$ in $\mathcal{C}$, choose C_0' and C_2' in $\mathcal{C}$ so that $B_0 = C_0 \oplus C_0'$ and $B_2 = C_2 \oplus C_2'$ are in $\mathcal{B}$. Setting $B_1 = C_1 \oplus C_0' \oplus C_2'$, we have the short exact sequence $0 \to B_0 \to B_1 \to B_2 \to 0$ in $\mathcal{C}$. As $\mathcal{B}$ is closed under extensions in $\mathcal{C}$, $B_1 \in \mathcal{B}$. Therefore in $K_0^{\oplus}\mathcal{C}$:

$$[C_1] - [C_0] - [C_2] = [B_1] - [B_0] - [B_2].$$

Thus the kernel of $K_0^{\oplus}\mathcal{C} \to K_0\mathcal{C}$ equals the kernel of $K_0^{\oplus}\mathcal{B} \to K_0\mathcal{B}$, which implies that $K_0\mathcal{B} \to K_0\mathcal{C}$ is an injection. □

II.7.2.1 **Remark 7.2.1.** The proof shows that $K_0(\mathcal{C})/K_0(\mathcal{B}) \cong K_0^{\oplus}\mathcal{C}/K_0^{\oplus}\mathcal{B}$, and that every element of $K_0(\mathcal{C})$ has the form $[C] - [B]$ for some B in $\mathcal{B}$ and C in $\mathcal{C}$.

Idempotent completion.

II.7.3 **7.3.** A category $\mathcal{C}$ is called *idempotent complete* if every idempotent endomorphism e of an object C factors as $C \to B \to C$ with the composite $B \to C \to B$ being the identity. Given $\mathcal{C}$, we can form a new category $\widehat{\mathcal{C}}$ whose objects are pairs (C, e) with e an idempotent endomorphism of an object C of $\mathcal{C}$; a morphism from (C, e) to (C', e') is a map $f\colon C \to C'$ in $\mathcal{C}$ such that $f = e'fe$. The category $\widehat{\mathcal{C}}$ is idempotent complete, since an idempotent endomorphism f of (C, e) factors through the object (C, efe).

$\widehat{\mathcal{C}}$ is called the *idempotent completion* of $\mathcal{C}$. To see why, consider the natural embedding of $\mathcal{C}$ into $\widehat{\mathcal{C}}$ sending C to (C, id). It is easy to see that any functor from $\mathcal{C}$ to an idempotent complete category $\mathcal{D}$ must factor through a functor $\widehat{\mathcal{C}} \to \mathcal{D}$ that is unique up to natural equivalence. In particular, if $\mathcal{C}$ is idempotent complete then $\mathcal{C} \cong \widehat{\mathcal{C}}$.

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, then $\widehat{\mathcal{C}}$ is equivalent to a larger additive subcategory $\mathcal{C}'$ of $\mathcal{A}$ (see Ex. 7.6). Moreover, $\mathcal{C}$ is cofinal in $\widehat{\mathcal{C}}$, because (C, e) is a summand of C in $\mathcal{A}$. By the Cofinality Lemma 7.2, we see that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

II.7.3.1 **Example 7.3.1.** Consider the subcategory $\mathbf{Free}(R)$ of $\mathbf{M}(R)$ consisting of finitely generated free R-modules. The idempotent completion of $\mathbf{Free}(R)$ is the category $\mathbf{P}(R)$ of finitely generated projective modules. Thus the cyclic group $K_0\mathbf{Free}(R)$ is a subgroup of $K_0(R)$. If R satisfies the Invariant Basis Property (IBP), then $K_0\mathbf{Free}(R) \cong \mathbb{Z}$ and we have recovered the conclusion of Lemma 2.1.

II.7.3.2 **Example 7.3.2.** Let $R \to S$ be a ring homomorphism, and let $\mathcal{B}$ denote the full subcategory of $\mathbf{P}(S)$ on the modules of the form $P \otimes_R S$ for P in $\mathbf{P}(R)$. Since it contains all the free modules S^n, $\mathcal{B}$ is cofinal in $\mathbf{P}(S)$, so $K_0\mathcal{B}$ is a subgroup of $K_0(S)$. Indeed, $K_0\mathcal{B}$ is the image of the natural map $K_0(R) \to K_0(S)$.

Products

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be exact categories. A functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called *biexact* if $F(A, -)$ and $F(-, B)$ are exact functors for every A in $\mathcal{A}$ and B in $\mathcal{B}$, and $F(0, -) = F(-, 0) = 0$. (The last condition, not needed in this chapter, can always be arranged by replacing $\mathcal{C}$ by an equivalent category.) The following result is completely elementary.

II.7.4 **Lemma 7.4.** *A biexact functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$K_0\mathcal{A} \otimes K_0\mathcal{B} \to K_0\mathcal{C}.$$
$$[A] \otimes [B] \mapsto [F(A, B)]$$

II.7.4.1 **Application 7.4.1.** Let R be a commutative ring. The tensor product $\otimes_R$ defines a biexact functor $\mathbf{P}(R) \times \mathbf{P}(R) \to \mathbf{P}(R)$, as well as a biexact functor $\mathbf{P}(R) \times \mathbf{M}(R) \to \mathbf{M}(R)$. The former defines the product $[P][Q] = [P \otimes Q]$ in the commutative ring $K_0(R)$, as we saw in §2. The latter defines an action of $K_0(R)$ on $G_0(R)$, making $G_0(R)$ into a $K_0(R)$-module.

II.7.4.2 **Application 7.4.2.** Let X be a scheme (or more generally a locally ringed space) The tensor product of vector bundles defines a biexact functor $\mathbf{VB}(X) \times \mathbf{VB}(X) \to \mathbf{VB}(X)$ (see I.5.3). This defines a product on $K_0(X)$ satisfying $[\mathcal{E}][\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$. This product is clearly commutative and associative, so it makes $K_0(X)$ into a commutative ring. We will discuss this ring further in the next section.

If X is noetherian, recall from 6.2.5 that $G_0(X)$ denotes $K_0\mathbf{M}(X)$. Since the tensor product of a vector bundle and a coherent module is coherent, we have a functor $\mathbf{VB}(X) \times \mathbf{M}(X) \to \mathbf{M}(X)$. It is biexact (why?), so it defines an action of $K_0(X)$ on $G_0(X)$, making $G_0(X)$ into a $K_0(X)$-module.

II.7.4.3 **Application 7.4.3.** (Almkvist) If R is a ring, let $\mathbf{End}(R)$ denote the exact category whose objects (P, α) are pairs, where P is a finitely generated projective R-module and α is an endomorphism of P. A morphism $(P, \alpha) \to (Q, \beta)$ in $\mathbf{End}(R)$ is a morphism $f\colon P \to Q$ in $\mathbf{P}(R)$ such that $f\alpha = \beta f$, and exactness in $\mathbf{End}(R)$ is determined by exactness in $\mathbf{P}(R)$.

If R is commutative, the tensor product of modules gives a biexact functor

$$\begin{aligned} \otimes_R : \ \mathbf{End}(R) \times \mathbf{End}(R) &\to \mathbf{End}(R), \\ ((P, \alpha), (Q, \beta)) &\mapsto (P \otimes_R Q, \alpha \otimes_R \beta). \end{aligned}$$

As $\otimes_R$ is associative and symmetric up to isomorphism, the induced product makes $K_0\mathbf{End}(R)$ into a commutative ring with unit $[(R, 1)]$. The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $\alpha = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \to K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R, 1)] - [(R, 0)]$. Almkvist proved that $(P, \alpha) \mapsto \det(1 - \alpha t)$ defines an isomorphism of $\mathrm{End}_0(R)$ with the subgroup of the multiplicative group $W(R) = 1 + tR[[t]]$ consisting of all quotients $f(t)/g(t)$ of polynomials in $1 + tR[t]$ (see Ex. 7.18). Almkvist also proved that $\mathrm{End}_0(R)$ is a subring of $W(R)$ under the ring structure of 4.3.

If A is an R-algebra, $\otimes_R$ is also a pairing $\mathbf{End}(R) \times \mathbf{End}(A) \to \mathbf{End}(A)$, making $\mathrm{End}_0(A)$ into an $\mathrm{End}_0(R)$-module. We leave the routine details to the reader.

II.7.4.4 **Example 7.4.4.** If R is a ring, let $\mathbf{Nil}(R)$ denote the category whose objects (P, ν) are pairs, where P is a finitely generated projective R-module and ν is a nilpotent endomorphism of P. This is an exact subcategory of $\mathbf{End}(R)$. The forgetful functor $\mathbf{Nil}(R) \to \mathbf{P}(R)$ sending (P, ν) to P is exact, and is split by the exact functor $\mathbf{P}(R) \to \mathbf{Nil}(R)$ sending P to $(P, 0)$. Therefore $K_0(R) = K_0\mathbf{P}(R)$ is a direct summand of $K_0\mathbf{Nil}(R)$. We write $\mathrm{Nil}_0(R)$ for the kernel of $K_0\mathbf{Nil}(R) \to \mathbf{P}(R)$, so that there is a direct sum decomposition $K_0\mathbf{Nil}(R) = K_0(R) \oplus \mathrm{Nil}_0(R)$. Since $[P, \nu] = [P \oplus Q, \nu \oplus 0] - [Q, 0]$ in $K_0\mathbf{Nil}(R)$, we see that $\mathrm{Nil}_0(R)$ is generated by elements of the form $\big[(R^n, \nu)\big] - n\big[(R, 0)\big]$ for some n and some nilpotent matrix ν.

If A is an R-algebra, then the tensor product pairing on **End** restricts to a biexact functor $F\colon \mathbf{End}(R) \times \mathbf{Nil}(A) \to \mathbf{Nil}(A)$. The resulting bilinear map $K_0\mathbf{End}(R) \times K_0\mathbf{Nil}(A) \to K_0\mathbf{Nil}(A)$ is associative, and makes $\mathrm{Nil}_0(A)$ into a module over the ring $\mathrm{End}_0(R)$, and makes $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ an $\mathrm{End}_0(R)$-module map.

Any additive functor $T : \mathbf{P}(A) \to \mathbf{P}(B)$ induces an exact functor $\mathbf{Nil}(A) \to \mathbf{Nil}(B)$ and a homomorphism $\mathrm{Nil}_0(A) \to \mathrm{Nil}_0(B)$. If A and B are R-algebras and T is R-linear, $\mathrm{Nil}_0(A) \to \mathrm{Nil}_0(B)$ is an $\mathrm{End}_0(R)$-module homomorphism. (Exercise!)

II.7.4.5 **Example 7.4.5.** If R is a commutative regular ring, and $A = R[x]/(x^N)$, we will see in III.3.8.1 that $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ is an injection, identifying $\mathrm{Nil}_0(A)$ with the ideal $(1 + xtA[t])^\times$ of $\mathrm{End}_0(A)$, and identifying $[(A, x)]$ with $1 - xt$.

This isomorphism $\mathrm{Nil}_0(A) \cong (1+xtA[t])^\times$ is universal in the following sense. If B is an R-algebra and (P, ν) is in $\mathbf{Nil}(B)$, with $\nu^N = 0$, we may regard P as an A–B bimodule. By 2.8, this yields an R-linear functor $\mathbf{Nil}_0(A) \to \mathbf{Nil}_0(B)$ sending (A, x) to (P, ν). By 7.4.4, there is an $\mathrm{End}_0(R)$-module homomorphism $(1 + xtA[t])^\times \to \mathrm{Nil}_0(B)$ sending $1 - xt$ to $[(P, \nu)]$.

Euler characteristics can be useful in exact categories as well as in abelian categories, as the following analogue of Proposition 6.6 shows.

II.7.5 **Proposition 7.5.** *Suppose that $\mathcal{C}$ is closed under kernels of surjections in an abelian category $\mathcal{A}$. If $C_\bullet$ is a bounded chain complex in $\mathcal{C}$ whose homology $H_i(C_\bullet)$ is also in $\mathcal{C}$ then in $K_0(\mathcal{C})$:*

$$\chi(C_\bullet) = \sum(-1)^i[C_i] \quad \textit{equals} \quad \sum(-1)^i[H_i(C_\bullet)].$$

In particular, if $C_\bullet$ is any exact sequence in $\mathcal{C}$ then $\chi(C_\bullet) = 0$.

Proof. The proof we gave of 6.6 for abelian categories will go through, provided that the Z_i and B_i are objects of $\mathcal{C}$. Consider the exact sequences:

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$
$$0 \to B_i \to Z_i \to H_i(C_\bullet) \to 0.$$

Since $B_i = 0$ for $i \ll 0$, the following inductive argument shows that all the B_i and Z_i belong to $\mathcal{C}$. If $B_{i-1} \in \mathcal{C}$ then the first sequence shows that $Z_i \in \mathcal{C}$; since $H_i(C_\bullet)$ is in $\mathcal{C}$, the second sequence shows that $B_i \in \mathcal{C}$. □

II.7.5.1 **Corollary 7.5.1.** *Suppose $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$. If $f : C'_\bullet \to C_\bullet$ is a morphism of bounded complexes in $\mathcal{C}$, inducing an isomorphism on homology, then*

$$\chi(C'_\bullet) = \chi(C_\bullet).$$

Proof. Form cone(f), the mapping cone of f, which has $C_n \oplus C'_{n-1}$ in degree n. By inspection, $\chi(\mathrm{cone}(f)) = \chi(C_\bullet) - \chi(C'_\bullet)$. But cone($f$) is an exact complex because f is a homology isomorphism, so $\chi(\mathrm{cone}(f)) = 0$. □

The Resolution Theorem

We need a definition in order to state our next result. Suppose that $\mathcal{P}$ is an additive subcategory of an abelian category $\mathcal{A}$. A $\mathcal{P}$-*resolution* $P_\bullet \to C$ of an object C of $\mathcal{A}$ is an exact sequence in $\mathcal{A}$

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to C \to 0$$

in which all the P_i are in $\mathcal{P}$. The $\mathcal{P}$-*dimension* of C is the minimum n (if it exists) such that there is a resolution $P_\bullet \to C$ with $P_i = 0$ for $i > n$.

II.7.6 **Theorem 7.6** (Resolution Theorem). *Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive categories with $\mathcal{A}$ abelian ($\mathcal{A}$ gives the notion of exact sequence to $\mathcal{P}$ and $\mathcal{C}$). Assume that:*

(a) Every object C of $\mathcal{C}$ has finite $\mathcal{P}$-dimension; and

(b) $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$.

Then the inclusion $\mathcal{P} \subset \mathcal{C}$ induces an isomorphism $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$.

Proof. To see that $K_0(\mathcal{P})$ maps onto $K_0(\mathcal{C})$, observe that if $P_\bullet \to C$ is a finite $\mathcal{P}$-resolution, then the exact sequence

$$0 \to P_n \to \cdots \to P_0 \to C \to 0$$

has $\chi = 0$ by 7.5, so $[C] = \sum(-1)^i[P] = \chi(P_\bullet)$ in $K_0(\mathcal{C})$. To see that $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$, we will show that the formula $\chi(C) = \chi(P_\bullet)$ defines an additive function from $\mathcal{C}$ to $K_0(\mathcal{P})$. For this, we need the following lemma, due to Grothendieck.

II.7.6.1 **Lemma 7.6.1.** *Given a map $f\colon C' \to C$ in $\mathcal{C}$ and a finite $\mathcal{P}$-resolution $P_\bullet \to C$, there is a finite $\mathcal{P}$-resolution $P'_\bullet \to C'$ and a commutative diagram*

$$\begin{array}{ccccccccccccccc}
0 \to P'_m \to \cdots \to & P'_n & \to \cdots \to & P'_1 & \longrightarrow & P'_0 & \longrightarrow & C' & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & \\
0 \to & P_n & \to \cdots \to & P_1 & \longrightarrow & P_0 & \longrightarrow & C & \to 0
\end{array}$$

We will prove this lemma in a moment. First we shall use it to finish the proof of Theorem 7.6. Suppose given two finite $\mathcal{P}$-resolutions $P_\bullet \to C$ and $P'_\bullet \to C$ of an object C. Applying the lemma to the diagonal map $C \to C \oplus C$ and $P_\bullet \oplus P'_\bullet \to C \oplus C$, we get a $\mathcal{P}$-resolution $P''_\bullet \to C$ and a map $P''_\bullet \to P_\bullet \oplus P'_\bullet$ of complexes. Since the maps $P_\bullet \leftarrow P''_\bullet \to P'_\bullet$ are quasi-isomorphisms, Corollary 7.5.1 implies that $\chi(P_\bullet) = \chi(P''_\bullet) = \chi(P'_\bullet)$. Hence $\chi(C) = \chi(P_\bullet)$ is independent of the choice of $\mathcal{P}$-resolution.

Given a short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ and a $\mathcal{P}$-resolution $P_\bullet \to C$, the lemma provides a $\mathcal{P}$-resolution $P'_\bullet \to C'$ and a map $f\colon P'_\bullet \to P_\bullet$. Form the mapping cone complex cone(f), which has $P_n \oplus P'_{n-1}$ in degree n, and observe that $\chi(\text{cone}(f)) = \chi(P_\bullet) - \chi(P'_\bullet)$. The homology exact sequence

$$H_i(P') \to H_i(P) \to H_i(\text{cone}(f)) \to H_{i-1}(P') \to H_{i-1}(P)$$

shows that $H_i\text{cone}(f) = 0$ for $i \neq 0$, and $H_0(\text{cone}(f)) = C''$. Thus $\text{cone}(f) \to C''$ is a finite $\mathcal{P}$-resolution, and so

$$\chi(C'') = \chi(\text{cone}(f)) = \chi(P_\bullet) - \chi(P'_\bullet) = \chi(C) - \chi(C').$$

This proves that χ is an additive function, so it induces a map $\chi\colon K_0\mathcal{C} \to K_0(\mathcal{P})$. If P is in $\mathcal{P}$ then evidently $\chi(P) = [P]$, so χ is the inverse isomorphism to the map $K_0(\mathcal{P}) \to K_0(\mathcal{C})$. This finishes the proof of the Resolution Theorem 7.6. □

Proof of Lemma 7.6.1. We proceed by induction on the length n of $P_\bullet$. If $n = 0$, we may choose any $\mathcal{P}$-resolution of C'; the only nonzero map $P'_n \to P_n$ is $P'_0 \to C' \to C \cong P_0$. If $n \geq 1$, let Z denote the kernel (in $\mathcal{A}$) of $\varepsilon\colon P_0 \to C$ and let B denote the kernel (in $\mathcal{A}$) of $(\varepsilon, -f)\colon P_0 \oplus C' \to C$. As $\mathcal{C}$ is closed under kernels, both Z and B are in $\mathcal{C}$. Moreover, the sequence

$$0 \to Z \to B \to C' \to 0$$

is exact in $\mathcal{C}$ (because it is exact in $\mathcal{A}$). Choose a surjection $P'_0 \to B$ with P'_0 in $\mathcal{P}$, let f_0 be the composition $P'_0 \to B \to P_0$ and let Y denote the kernel of the surjection $P'_0 \to B \to C'$. By induction applied to the induced map $Y \to Z$, we can find a $\mathcal{P}$-resolution $P'_\bullet[+1]$ of Y and maps $f_i\colon P'_i \to P_i$ making the following diagram commute (the rows are not exact at Y and Z):

$$\begin{array}{ccccccccccc}
\cdots & P'_2 & \longrightarrow & P'_1 & \twoheadrightarrow & Y & \rightarrowtail & P'_0 & \longrightarrow & C' & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\
\cdots & P_2 & \longrightarrow & P_1 & \twoheadrightarrow & Z & \rightarrowtail & P_0 & \longrightarrow & C & \longrightarrow 0.
\end{array}$$

Splicing the rows by deleting Y and Z yields the desired $\mathcal{P}$-resolution of C'. □

II.7.7 **Definition 7.7** ($\mathbf{H}(R)$)**.** Given a ring R, let $\mathbf{H}(R)$ denote the category of all R-modules M having a finite resolution by finitely generated projective modules, and let $\mathbf{H}_n(R)$ denote the subcategory in which the resolutions have length $\leq n$. We write $pd_R(M)$ for the length of the smallest such resolution of M.

By the Horseshoe Lemma [223, 2.2.8], both $\mathbf{H}(R)$ and $\mathbf{H}_n(R)$ are exact subcategories of **mod**-R. The following Lemma shows that they are also closed under kernels of surjections in **mod**-R.

II.7.7.1 **Lemma 7.7.1.** *If $0 \to L \to M \xrightarrow{f} N \to 0$ is a short exact sequence of modules, with M in $\mathbf{H}_m(R)$ and N in $\mathbf{H}_n(R)$, then L is in $\mathbf{H}_\ell(R)$, where $\ell = \min\{m, n-1\}$.*

Proof. If $P_\bullet \to M$ and $Q_\bullet \to N$ are projective resolutions, and $P_\bullet \to Q_\bullet$ lifts f, then the kernel P'_0 of the surjection $P_0 \oplus Q_1 \to Q_0$ is finitely generated projective, and the truncated mapping cone $\cdots \to P_1 \oplus Q_2 \to P'_0$ is a resolution of L. □

II.7.7.2 **Corollary 7.7.2.** *$K_0(R) \cong K_0\mathbf{H}(R) \cong K_0\mathbf{H}_n(R)$ for all $n \geq 1$.*

Proof. Apply the Resolution Theorem to $\mathbf{P}(R) \subset \mathbf{H}(R)$. □

Here is a useful variant of the above construction. Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. We say a module M is *S-torsion* if $Ms = 0$ for some $s \in S$ (*cf.* Example II.6.2.8 6.2.8), and write $\mathbf{H}_S(R)$ for the exact subcategory $\mathbf{H}(R) \cap \mathbf{M}_S(R)$ of S-torsion modules M in $\mathbf{H}(R)$. Similarly, we write $\mathbf{H}_{n,S}(R)$ for the S-torsion modules in $\mathbf{H}_n(R)$. Note that $\mathbf{H}_{0,S}(R) = 0$, and that the modules R/sR belong to $\mathbf{H}_{1,S}(R)$.

II.7.7.3 **Corollary 7.7.3.** *$K_0\mathbf{H}_S(R) \cong K_0\mathbf{H}_{n,S}(R) \cong K_0\mathbf{H}_{1,S}(R)$ for all $n \geq 1$.*

Proof. We apply the Resolution Theorem with $\mathcal{P} = \mathbf{H}_{1,S}(R)$. By Lemma II.7.7.1 7.7.1, each $\mathbf{H}_{n,S}(R)$ is closed under kernels of surjections. Every N in $\mathbf{H}_{n,S}(R)$ is finitely generated, so if $Ns = 0$ there is an exact sequence

$$0 \to L \to (R/sR)^m \to N \to 0.$$

If $n \geq 2$ then L is in $\mathbf{H}_{n-1,S}(R)$ by Lemma II.7.7.1 7.7.1. By induction, L and hence N has a $\mathcal{P}$-resolution. □

II.7.7.4 **Corollary 7.7.4.** *If S is a multiplicatively closed set of central nonzerodivisors in a ring R, the sequence $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$ is exact.*

Proof. If $[P] - [R^n] \in K_0(R)$ vanishes in $K_0(S^{-1}R)$, $S^{-1}P$ is stably free (Cor. II.1.3 1.3). Hence there is an isomorphism $(S^{-1}R)^{m+n} \xrightarrow{\simeq} S^{-1}P \oplus (S^{-1}R)^m$. Clearing denominators yields a map $f\colon R^{m+n} \to P \oplus R^m$ whose kernel and cokernel are S-torsion. But $\ker(f) = 0$ because S consists of nonzerodivisors, and therefore $M = \text{coker}(f)$ is in $\mathbf{H}_{1,S}(R)$. But the map $K_0\mathbf{H}_S(R) \to K_0\mathbf{H}(R) = K_0(R)$ sends $[M]$ to $[M] = [P] - [R^n]$. □

Let R be a regular noetherian ring. Since every module has finite projective dimension, $\mathbf{H}(R)$ is the abelian category $\mathbf{M}(R)$ discussed in §6. Combining Corollary II.7.7.2 7.7.2 with the Fundamental Theorem for G_0 (II.6.5 6.5), we have:

II.7.8 **Theorem 7.8** (Fundamental Theorem for K_0 of regular rings)**.** *If R is a regular noetherian ring, then $K_0(R) \cong G_0(R)$. Moreover,*

$$K_0(R) \cong K_0(R[t]) \cong K_0(R[t, t^{-1}]).$$

If R is not regular, we can still use the localization sequence II.7.7.4 7.7.4 to get a partial result, which will be considerably strengthened by the Fundamental Theorem for K_0 in chapter III.

II.7.8.1 **Proposition 7.8.1.** *$K_0(R[t]) \to K_0(R[t, t^{-1}])$ is injective for every ring R.*

To prove this, we need the following lemma. Recall from Example II.7.4.4 7.4.4 that $\mathbf{Nil}(R)$ is the category of pairs (P, ν) with ν a nilpotent endomorphism of $P \in \mathbf{P}(R)$.

II.7.8.2 **Lemma 7.8.2.** *Let S be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then* $\mathbf{Nil}(R)$ *is equivalent to the category* $\mathbf{H}_{1,S}(R[t])$ *of all t-torsion $R[t]$-modules M in* $\mathbf{H}_1(R[t])$*.*

Proof. If (P,ν) is in $\mathbf{Nil}(R)$, let P_ν denote the $R[t]$-module P on which t acts as ν. It is a t-torsion module because $t^nP_\nu = \nu^nP = 0$ for large n. A projective resolution of P_ν is given by the "characteristic sequence" of ν:

$$0 \to P[t] \xrightarrow{t-\nu} P[t] \to P_\nu \to 0, \qquad (7.8.3)$$ II.7.8.3

Thus P_ν is an object of $\mathbf{H}_{1,S}(R[t])$. Conversely, each M in $\mathbf{H}_{1,S}(R[t])$ has a projective resolution $0 \to P \to Q \to M \to 0$ by finitely generated projective $R[t]$-modules, and M is killed by some power t^n of t. From the exact sequence

$$0 \to \mathrm{Tor}_1^{R[t]}\big(M, R[t]/(t^n)\big) \to P/t^nP \to Q/t^nQ \to M \to 0$$

and the identification of the first term with M we obtain the exact sequence $0 \to M \xrightarrow{t^n} P/t^nP \to P/t^nQ \to 0$. Since P/t^nP is a projective R-module and $pd_R(P/t^nQ) \le 1$, we see that M must be a projective R-module. Thus (M,t) is an object of $\mathbf{Nil}(R)$. □

Combining Lemma 7.8.2 with Corollary 7.7.3 yields:

II.7.8.4 **Corollary 7.8.4.** $K_0\mathbf{Nil}(R) \cong K_0\mathbf{H}_S(R[t])$.

Proof of Proposition 7.8.1. By Corollaries 7.7.4 and 7.8.4, we have an exact sequence

$$K_0\mathbf{Nil}(R) \to K_0(R[t]) \to K_0(R[t,t^{-1}]).$$

It suffices to show that the left map is zero. This map is induced by the forgetful functor $\mathbf{Nil}(R) \to \mathbf{H}(R[t])$ sending (P,ν) to P. Since the characteristic sequence (7.8.3) of ν shows that $[P] = 0$ in $K_0(R[t])$, we are done. □

Base change and Transfer Maps for Rings

II.7.9 **7.9.** Let $f\colon R \to S$ be a ring homomorphism. We have already seen that the base change $\otimes_R S\colon \mathbf{P}(R) \to \mathbf{P}(S)$ is an exact functor, inducing $f^*\colon K_0(R) \to K_0(S)$. If $S \in \mathbf{P}(R)$, we observed in 2.8.1 that the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ is exact, inducing the transfer map $f_*\colon K_0(S) \to K_0(R)$.

Using the Resolution Theorem, we can also define a transfer map f_* if $S \in \mathbf{H}(R)$. In this case every finitely generated projective S-module is in $\mathbf{H}(R)$, because if $P \oplus Q = S^n$ then $pd_R(P) \le pd_R(S^n) = pd_R(S) < \infty$. Hence there is an (exact) forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$, and we define the transfer map to be the induced map

$$f_*\colon K_0(S) = K_0\mathbf{P}(S) \to K_0\mathbf{H}(R) \cong K_0(R). \qquad (7.9.1)$$ II.7.9.1

A similar trick works to construct base change maps for the groups G_0. We saw in 6.2 that if S is flat as an R-module then $\otimes_R S$ is an exact functor

$\mathbf{M}(R) \to \mathbf{M}(S)$ and we obtained a map $f^*: G_0(R) \to G_0(S)$. More generally, suppose that S has finite flat dimension $\mathrm{fd}_R(S) = n$ as a left R-module, *i.e.,* that there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to S \to 0$$

of R-modules, with the F_i flat. Let $\mathcal{F}$ denote the full subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M with $\mathrm{Tor}_i^R(M,S) = 0$ for $i \neq 0$; $\mathcal{F}$ is an exact category concocted so that $\otimes_R S$ defines an exact functor from $\mathcal{F}$ to $\mathbf{M}(S)$. Not only does $\mathcal{F}$ contain $\mathbf{P}(R)$, but from homological algebra one knows that (if R is noetherian) every finitely generated R-module has a finite resolution by objects in $\mathcal{F}$; for any projective resolution $P_\bullet \to M$ the kernel of $P_n \to P_{n-1}$ (the n^{th} *syzygy*) of any projective resolution will be in $\mathcal{F}$. The long exact Tor sequence shows that $\mathcal{F}$ is closed under kernels, so the Resolution Theorem applies to yield $K_0(\mathcal{F}) \cong K_0\mathbf{M}(R) = G_0(R)$. Therefore if R is noetherian and $\mathrm{fd}_R(S) < \infty$ we can define the base change map $f^*: G_0(R) \to G_0(S)$ as the composite

$$G_0(R) \cong K_0(\mathcal{F}) \xrightarrow{\otimes} K_0\mathbf{M}(S) = G_0(S). \tag{7.9.2}$$ II.7.9.2

The following formula for f^* was used in §6 to show that $G_0(R) \cong G_0(R[x])$.

II.7.9.3 **Serre's Formula 7.9.3.** Let $f: R \to S$ be a map between noetherian rings with $\mathrm{fd}_R(S) < \infty$. Then the base change map $f^*: G_0(R) \to G_0(S)$ of (7.9.2) satisfies:

$$f^*([M]) = \sum (-1)^i \left[\mathrm{Tor}_i^R(M,S)\right].$$

Proof. Choose an $\mathcal{F}$-resolution $L_\bullet \to M$ (by R-modules L_i in $\mathcal{F}$):

$$0 \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0.$$

From homological algebra, we know that $\mathrm{Tor}_i^R(M,S)$ is the i^{th} homology of the chain complex $L_\bullet \otimes_R S$. By Prop. 7.5, the right-hand side of 7.9.3 equals

$$\chi(L_\bullet \otimes_R S) = \sum (-1)^i [L_i \otimes_R S] = f^*(\sum (-1)^i [L_i]) = f^*([M]). \quad \square$$

EXERCISES

EII.7.1 **7.1.** Suppose that $\mathbf{P}$ is an exact subcategory of an abelian category $\mathcal{A}$, closed under kernels of surjections in $\mathcal{A}$. Suppose further that every object of $\mathcal{A}$ is a quotient of an object of $\mathbf{P}$ (as in Corollary 7.7.2). Let $\mathbf{P}_n \subset \mathcal{A}$ be the full subcategory of objects having $\mathbf{P}$-dimension $\leq n$. Show that each $\mathbf{P}_n$ is an exact category closed under kernels of surjections, so that by the Resolution Theorem $K_0(\mathbf{P}) \cong K_0(\mathbf{P}_n)$. *Hint.* If $0 \to L \to P \to M \to 0$ is exact with $P \in \mathbf{P}$ and $M \in \mathbf{P}_1$, show that $L \in \mathbf{P}$.

EII.7.2 **7.2.** Let $\mathcal{A}$ be a small exact category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $A_1 \oplus C_1 \cong A_2 \oplus C_2$. (Cf. Ex. 6.4.)

EII.7.3 **7.3.** This exercise shows why the noetherian hypothesis was needed for G_0 in Corollary 6.3.1, and motivates the definition of $G_0(R)$ in 7.1.4. Let R be the ring $k \oplus I$, where I is an infinite-dimensional vector space over a field k, with multiplication given by $I^2 = 0$.

(a) (Swan) Show that $K_0\mathbf{mod}_{\text{fg}}(R) = 0$ but $K_0\mathbf{mod}_{\text{fg}}(R/I) = G_0(R/I) = \mathbb{Z}$.

(b) Show that every pseudo-coherent R-module (7.1.4) is isomorphic to R^n for some n. Conclude that $G_0(R) = \mathbb{Z}$.

EII.7.4 **7.4.** The groups $G_0(\mathbb{Z}[G])$ and $K_0\mathbf{mod}_{\text{fg}}(\mathbb{Z}[G])$ are very different for the free group G on two generators x and y. Let I be the two-sided ideal of $\mathbb{Z}[G]$ generated by y, so that $\mathbb{Z}[G]/I = \mathbb{Z}[x, x^{-1}]$. As a right module, $\mathbb{Z}[G]/I$ is not finitely presented.

(a) (Lück) Construct resolutions $0 \to \mathbb{Z}[G]^2 \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z}[G]/I \to \mathbb{Z}[G]/I \to \mathbb{Z} \to 0$. Conclude that $K_0\mathbf{mod}_{\text{fg}}(\mathbb{Z}[G]) = 0$.

(b) Gersten proved in [65] that $K_0(\mathbb{Z}[G]) = \mathbb{Z}$ by showing that every finitely presented $\mathbb{Z}[G]$-module is in $\mathbf{H}(\mathbb{Z}[G])$, i.e., has a finite resolution by finitely generated projective modules. Show that $G_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[G]) \cong \mathbb{Z}$.

EII.7.5 **7.5.** *Naturality of base change.* Let $R \xrightarrow{f} S \xrightarrow{g} T$ be maps between noetherian rings, with $\text{fd}_R(S)$ and $\text{fd}_S(T)$ finite. Show that $g^*f^* = (gf)^*$ as maps $G_0(R) \to G_0(T)$.

EII.7.6 **7.6.** *Idempotent completion.* Suppose that $(\mathcal{C}, \mathcal{E})$ is an exact category. Show that there is a natural way to make the idempotent completion $\widehat{\mathcal{C}}$ of $\mathcal{C}$ into an exact category, with $\mathcal{C}$ an exact subcategory. As noted in 7.3, this proves that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

EII.7.7 **7.7.** Let $\mathcal{C}$ be a small additive category, and $\mathcal{A} = \mathbf{Ab}^{\mathcal{C}^{op}}$ the (abelian) category of all additive contravariant functors from $\mathcal{C}$ to $\mathbf{Ab}$. The Yoneda embedding $h\colon \mathcal{C} \to \mathcal{A}$, defined by $h(C) = \text{Hom}_{\mathcal{C}}(-, C)$, embeds $\mathcal{C}$ as a full subcategory of $\mathcal{A}$. Show that every object of $\mathcal{C}$ is a projective object in $\mathcal{A}$. Then conclude that this embedding makes $\mathcal{C}$ into a split exact category (see 7.1.2).

EII.7.8 **7.8.** (Quillen) Let $\mathcal{C}$ be an exact category, with the family $\mathcal{E}$ of short exact sequences (and admissible monics i and admissible epis j)

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \qquad (\dagger)$$

as in Definition 7.0. Show that the following three conditions hold:

(1) Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$. If (†) is a sequence in $\mathcal{E}$ then i is a kernel for j (resp. j is a cokernel for i) in $\mathcal{C}$. The class $\mathcal{E}$ contains all of the sequences

$$0 \to B \xrightarrow{(1,0)} B \oplus D \xrightarrow{(0,1)} D \to 0.$$

(2) The class of admissible epimorphisms (resp. monomorphisms) is closed under composition. If (†) is in $\mathcal{E}$ and $B \to B''$, $D' \to D$ are maps in $\mathcal{C}$ then the base change sequence $0 \to B \to (C \times_D D') \to D' \to 0$ and the cobase change sequence $0 \to B'' \to (B'' \amalg_B C) \to D \to 0$ are in $\mathcal{E}$.

(3) If $C \to D$ is a map in $\mathcal{C}$ possessing a kernel, and there is a map $C' \to C$ in $\mathcal{C}$ so that $C' \to D$ is an admissible epimorphism, then $C \to D$ is an admissible epimorphism. Dually, if $B \to C$ has a cokernel and some $B \to C \to C''$ is admissible monomorphism, then so is $B \to C$.

Keller [102, App. A] has proven that (1) and (2) imply (3).

Quillen observed that a converse is true: let $\mathcal{C}$ be an additive category, equipped with a family $\mathcal{E}$ of sequences of the form (†). If conditions (1) and (2) hold, then $\mathcal{C}$ is an exact category in the sense of Definition 7.0. The ambient abelian category used in 7.0 is the category $\mathcal{L}$ of contravariant *left exact* functors: additive functors $F\colon \mathcal{C} \to \mathbf{Ab}$ which carry each (†) to a "left" exact sequence

$$0 \to F(D) \to F(C) \to F(B),$$

and the embedding $\mathcal{C} \subset \mathcal{L}$ is the Yoneda embedding.

We refer the reader to Appendix A of [200] for a detailed proof that $\mathcal{E}$ is the class of sequences in $\mathcal{C}$ which are exact in $\mathcal{L}$, as well as the following useful result: If $\mathcal{C}$ is idempotent complete then it is closed under kernels of surjections in $\mathcal{L}$.

EII.7.9 **7.9.** Let $\{\mathcal{C}_i\}$ be a filtered system of exact categories and exact functors. Use Ex. 7.8 to generalize Example 7.1.7, showing that $\mathcal{C} = \varinjlim \mathcal{C}_i$ is an exact category and that $K_0(\mathcal{C}) = \varinjlim K_0(\mathcal{C}_i)$.

EII.7.10 **7.10.** *Projection Formula for rings.* Suppose that R is a commutative ring, and A is an R-algebra which as an R-module is in $\mathbf{H}(R)$. By Ex. 2.1, $\otimes_R$ makes $K_0(A)$ into a $K_0(R)$-module. Generalize Ex. 2.2 to show that the transfer map $f_*\colon K_0(A) \to K_0(R)$ is a $K_0(R)$-module map, *i.e.,* that the *projection formula* holds:

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every} \quad x \in K_0(A),\ y \in K_0(R).$$

EII.7.11 **7.11.** For a localization $f : R \to S^{-1}R$ at a central set of nonzerodivisors, every $\alpha : S^{-1}P \to S^{-1}Q$ has the form $\alpha = \gamma/s$ for some $\gamma \in \mathrm{Hom}_R(P,Q)$ and $s \in S$. Show that $[(P, \gamma/s, Q)] \mapsto [Q/\gamma(P)] - [Q/sQ]$ defines an isomorphism $K_0(f) \to K_0\mathbf{H}_S(R)$ identifying the sequences (2.10.2) and 7.7.4.

EII.7.12 **7.12.** This exercise generalizes the Localization Theorem 6.4. Let $\mathcal{C}$ be an exact subcategory of an abelian category $\mathcal{A}$, closed under extensions and kernels of surjections, and suppose that $\mathcal{C}$ contains a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$. Let $\mathcal{C}/\mathcal{B}$ denote the full subcategory of $\mathcal{A}/\mathcal{B}$ on the objects of $\mathcal{C}$. Considering $\mathcal{B}$-isos $A \to C$ with C in $\mathcal{C}$, show that the following sequence is exact:

$$K_0(\mathcal{B}) \to K_0(\mathcal{C}) \xrightarrow{\text{loc}} K_0(\mathcal{C}/\mathcal{B}) \to 0.$$

EII.7.13 **7.13.** *δ-functors.* Let $T = \{T_i \colon \mathcal{C} \to \mathcal{A}, i \geq 0\}$ be a bounded homological δ-functor from an exact category $\mathcal{C}$ to an abelian category $\mathcal{A}$, *i.e.,* for every exact sequence (†) in $\mathcal{C}$ we have a finite long exact sequence in $\mathcal{A}$:

$$0 \to T_n(B) \to T_n(C) \to \cdots \to T_1(D) \xrightarrow{\delta} T_0(B) \to T_0(C) \to T_0(D) \to 0.$$

Let $\mathcal{F}$ denote the category of all C in $\mathcal{C}$ such that $T_i(C) = 0$ for all $i > 0$, and assume that every C in $\mathcal{C}$ is a quotient of some object of $\mathcal{F}$.

(a) Show that $K_0(\mathcal{F}) \cong K_0(\mathcal{C})$, and that T defines a map $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ sending $[C]$ to $\sum(-1)^i[T_iC]$. (Cf. Ex. 6.6 and 8.4 below.)

(b) Suppose that $f \colon X \to Y$ is a map of noetherian schemes, and that $\mathcal{O}_X$ has finite flat dimension over $f^{-1}\mathcal{O}_Y$. Show that there is a base change map $f^* \colon G_0(Y) \to G_0(X)$ satisfying $f^*g^* = (gf)^*$, generalizing (7.9.2) and Ex. 7.5.

EII.7.14 **7.14.** This exercise is a refined version of Ex. 6.12. Consider $S = R[x_0, \cdots, x_m]$ as a graded ring with $x_1, \cdots, x_n$ in S_1, and let $\mathbf{M}_{gr}(S)$ denote the exact category of finitely generated graded S-modules.

(a) Use Exercise 7.13 with $T_i = \mathrm{Tor}_i^S(-, R)$ to show that $K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}]$.

(b) Use (a) and Ex. 6.12(e) to obtain an exact sequence

$$G_0(R)[\sigma, \sigma^{-1}] \xrightarrow{i} G_0(R)[\sigma, \sigma^{-1}] \to G_0(R[x]) \to 0.$$

Then show that the map i sends α to $\alpha - \sigma\alpha$.

(c) Conclude that $G_0(R) \cong G_0(R[x])$.

EII.7.15 **7.15.** Let R be a noetherian ring. Show that the groups $K_0\mathbf{M}_i(R)$ of Application 6.4.3 are all $K_0(R)$-modules, and that the subgroups F^i in the coniveau filtration of $G_0(R)$ are $K_0(R)$-submodules. Conclude that if R is regular then the F^i are ideals in the ring $K_0(R)$.

EII.7.16 **7.16.** (Grayson) Show that the operations $\lambda^n(P, \alpha) = (\wedge^n P, \wedge^n \alpha)$ make $K_0\mathbf{End}(R)$ and $\mathrm{End}_0(R)$ into λ-rings. Then show that the ring map $\mathrm{End}_0(R) \to W(R)$ (of 7.4.3) is a λ-ring injection, where $W(R)$ is the ring of big Witt vectors of R (see Example 4.3). Conclude that $\mathrm{End}_0(R)$ is a special λ-ring (4.3.1).

The exact endofunctors $F_m : (P, \alpha) \mapsto (P, \alpha^m)$ and $V_m : (P, \alpha) \mapsto (P[t]/t^m - \alpha, t)$ on $\mathbf{End}(R)$ induce operators F_m and V_m on $\mathrm{End}_0(R)$. Show that they agree with the classical Frobenius and Verschiebung operators, respectively.

EII.7.17 **7.17.** This exercise is a refinement of 7.4.4. Let $F_n\mathbf{Nil}(R)$ denote the full subcategory of $\mathbf{Nil}(R)$ on the (P, ν) with $\nu^n = 0$. Show that $F_n\mathbf{Nil}(R)$ is an exact subcategory of $\mathbf{Nil}(R)$. If R is an algebra over a commutative ring k, show that the kernel $F_n \mathrm{Nil}_0(R)$ of $K_0F_n\mathbf{Nil}(R) \to K_0\mathbf{P}(R)$ is an $\mathrm{End}_0(k)$-module, and $F_n \mathrm{Nil}_0(R) \to \mathrm{Nil}_0(R)$ is a module map.

The exact endofunctor $F_m : (P, \nu) \mapsto (P, \nu^m)$ on $\mathbf{Nil}(R)$ is zero on $F_n\mathbf{Nil}(R)$. For $\alpha \in \mathrm{End}_0(k)$ and $(P, \nu) \in \mathrm{Nil}_0(R)$, show that $(V_m\alpha) \cdot (P, \nu) = V_m(\alpha \cdot F_m(P, \nu))$, and conclude that $V_m \mathrm{End}_0(k)$ acts trivially on the image of $F_m \mathrm{Nil}_0(A)$ in $\mathrm{Nil}_0(A)$.

EII.7.18 **7.18.** Let $\alpha_n = \alpha_n(a_1, ..., a_n)$ denote the $n \times n$ matrix over a commutative ring R:

$$\alpha_n(a_1, ..., a_n) = \begin{pmatrix} 0 & & & -a_n \\ 1 & 0 & & -a_{n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & -a_1 \end{pmatrix}.$$

(a) Show that $[(R^n, \alpha_n)] = 1 + a_1 t + \cdots + a_n t^n$ in $W(R)$. Conclude that the image of the map $\mathrm{End}_0(R) \to W(R)$ in 7.4.3 is indeed the subgroup of all quotients $f(t)/g(t)$ of polynomials in $1 + tR[t]$.

(b) Let A be an R-algebra. Recall that $(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n \nu)]$ in the $\mathrm{End}_0(R)$-module $\mathrm{Nil}_0(A)$ (see 7.4.4). Show that $(R^{n+1}, \alpha_{n+1}(a_1, \dots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)]$.

(c) Use 7.4.5 with $R = \mathbb{Z}[a_1, ..., a_n]$ to show that $(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)]$, $\beta = \alpha_n(a_1\nu, ..., a_n\nu^n)$. If $\nu^N = 0$, this is clearly independent of the a_i for $i \geq N$.

(d) Conclude that the $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(A)$ extends to a $W(R)$-module structure by the formula

$$\left(1 + \sum a_i t^i\right) * [(P, \nu)] = (R^n, \alpha_n(a_1, ..., a_n)) * [(P, \nu)], \quad n \gg 0.$$

EII.7.19 **7.19.** (Lam) If R is a commutative ring, and Λ is an R-algebra, we write $G_0^R(\Lambda)$ for $K_0\mathbf{Rep}_R(\Lambda)$, where $\mathbf{Rep}_R(\Lambda)$ denotes the full subcategory of $\mathbf{mod}$-Λ consisting of modules M which are finitely generated and projective as R-modules. If $\Lambda = R[G]$ is the group ring of a group G, the tensor product $M \otimes_R N$ of two $R[G]$-modules is again an $R[G]$-module where $g \in G$ acts by $(m \otimes n)g = mg \otimes ng$. Show that:

(a) $\otimes_R$ makes $G_0^R(R[G])$ an associative, commutative ring with identity $[R]$.

(b) $G_0^R(R[G])$ is an algebra over the ring $K_0(R)$, and $K_0(R[G])$ is a $G_0^R(R[G])$-module.

(c) If R is a regular ring and Λ is finitely generated projective as an R-module, $G_0^R(\Lambda) \cong G_0(\Lambda)$.

(d) If R is regular and G is finite, then $G_0(R[G])$ is a commutative $K_0(R)$-algebra, and that $K_0(R[G])$ is a module over $G_0(R[G])$.

EII.7.20 **7.20.** (Deligne) A *filtered object* in an abelian category $\mathcal{A}$ is an object A together with a finite filtration $\cdots \subseteq W_nA \subseteq W_{n+1}A \subseteq \cdots$; if A and B are filtered, a morphism $f : A \to B$ in $\mathcal{A}$ is filtered if $F(W_nA) \subseteq W_nB$ for all n. The category

$\mathcal{A}_{\rm filt}$ of filtered objects in $\mathcal{A}$ is additive but not abelian (because images and coimages can differ). Let $\mathcal{E}$ denote the family of all sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}_{\rm filt}$ such that each sequence $0 \to gr_n^W A \to gr_n^W B \to gr_n^W C \to 0$ is exact in $\mathcal{A}$.

(a) Show that $(\mathcal{A}_{\rm filt}, \mathcal{E})$ is an exact category. (See [22, 1.1.4].)

(b) Show that $K_0(\mathcal{A}_{\rm filt}) \cong \mathbb{Z} \times K_0(\mathcal{A})$.

EII.7.21 **7.21.** *Replete exact categories.* Let $\mathcal{C}$ be an additive category. A sequence $0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$ in $\mathcal{C}$ is called *replete* if i is the categorical kernel of j, and j is the categorical cokernel of i. Let $\mathcal{E}_{\rm rep}$ denote the class of all replete sequences, and show that $(\mathcal{C}, \mathcal{E}_{\rm rep})$ is an exact category.

EII.7.22 **7.22.** Fix a prime p, let $\mathbf{Ab}_p$ be the category of all finite abelian p-groups (6.2.2), and let $\mathcal{C}$ denote the full subcategory of all groups in $\mathbf{Ab}_p$ whose cyclic summands have even length (e.g., $\mathbb{Z}/p^{2i}$). Show that $\mathcal{C}$ is an additive category, but not an exact subcategory of $\mathbf{Ab}_p$ (7.0.1). Let $\mathcal{E}$ be the sequences in $\mathcal{C}$ which are exact in $\mathbf{Ab}_p$; is $(\mathcal{C}, \mathcal{E})$ an exact category?

EII.7.23 **7.23.** Give an example of a cofinal exact subcategory $\mathcal{B}$ of an exact category $\mathcal{C}$, such that the map $K_0\mathcal{B} \to K_0\mathcal{C}$ is not an injection (see 7.2).

EII.7.24 **7.24.** Suppose that $\mathcal{C}_i$ are exact categories. Show that the product category $\prod \mathcal{C}_i$ is an exact category. Need $K_0(\prod \mathcal{C}_i) \to \prod K_0(\mathcal{C}_i)$ be an isomorphism?

EII.7.25 **7.25.** (Claborn-Fossum). Set $R_n = \mathbb{C}[x_0, \cdots, x_n]/(\sum x_i^2 = 1)$. This is the complex coordinate ring of the n-sphere; it is a regular ring for every n, and $R_1 \cong \mathbb{C}[z, z^{-1}]$. In this exercise, we show that

$$\tilde{K}_0(R_n) \cong \widetilde{KU}(S^n) \cong \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \text{ is even, } (n \neq 0). \end{cases}$$

(a) Set $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$, so that $z\bar{z} = x_0^2 + x_1^2$. Show that

$$R_n[z^{-1}] \cong \mathbb{C}[z, z^{-1}, x_2, \dots, x_n] \quad \text{and} \quad R_n/zR_n \cong R_{n-2}[\bar{z}], \quad n \geq 2.$$

(b) Use (a) to show that $\tilde{K}_0(R_n) = 0$ for n odd, and that if n is even there is a surjection $\beta\colon K_0(R_{n-2}) \to \tilde{K}_0(R_n)$.

(c) If n is even, show that β sends $[R_{n-2}]$ to zero, and conclude that there is a surjection $\mathbb{Z} \to \tilde{K}_0(R_n)$.

Fossum produced a finitely generated projective R_{2n}-module P_n such that the map $\tilde{K}_0(R_{2n}) \to \widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ sends $[P_n]$ to the generator. (See [55].)

(d) Use the existence of P_n to finish the calculation of $K_0(R_n)$.

EII.7.26 **7.26.** (Keller) Recall that any exact category $\mathcal{C}$ embeds into the abelian category $\mathcal{L}$ of left exact functors $\mathcal{C} \to \mathbf{Ab}$, and is closed under extensions (see Ex. 7.8). The *countable envelope* $\mathcal{C}^e$ of $\mathcal{C}$ is the full subcategory of $\mathcal{L}$ consisting of all colimits of sequences $A_0 \rightarrowtail A_1 \rightarrowtail \cdots$ of admissible monics in $\mathcal{C}$. Show that countable direct sums exist in $\mathcal{C}^e$. Then use the Eilenberg Swindle (I.2.8) to show that $K_0(\mathcal{C}^e) = 0$.

8 K_0 of Schemes and Varieties

We have already introduced the Grothendieck group $K_0(X)$ of a scheme X in Example 7.1.3. By definition, it is $K_0\mathbf{VB}(X)$, where $\mathbf{VB}(X)$ denotes the (exact) category of vector bundles on X. The tensor product of vector bundles makes $K_0(X)$ into a commutative ring, as we saw in 7.4.2. This ring structure is natural in X: K_0 is a contravariant functor from schemes to commutative rings. Indeed, we saw in I.5.2 that a morphism of schemes $f\colon X \to Y$ induces an exact base change functor $f^*\colon \mathbf{VB}(Y) \to \mathbf{VB}(X)$, preserving tensor products, and such an exact functor induces a (ring) homomorphism $f^*\colon K_0(Y) \to K_0(X)$.

In this section we shall study $K_0(X)$ in more depth. Such a study requires that the reader has somewhat more familiarity with algebraic geometry than we assumed in the previous section, which is why this study has been isolated in its own section. We begin with two general invariants: the rank and determinant of a vector bundle.

The ring of continuous functions $X \to \mathbb{Z}$ is isomorphic to the global sections of the constant sheaf $\mathbb{Z}$, i.e., to the cohomology group $H^0(X;\mathbb{Z})$; see [85, I.1.0.3]. We saw in I.5.1 that the rank of a vector bundle $\mathcal{F}$ is a continuous function, so $\operatorname{rank}(\mathcal{F}) \in H^0(X;\mathbb{Z})$. Similarly, we saw in I.5.3 that the determinant of $\mathcal{F}$ is a line bundle on X, *i.e.,* $\det(\mathcal{F}) \in \operatorname{Pic}(X)$.

II.8.1 **Theorem 8.1.** *Let X be a scheme. Then $H^0(X;\mathbb{Z})$ is isomorphic to a subring of $K_0(X)$, and the rank of a vector bundle induces a split surjection of rings*

$$\operatorname{rank}\colon K_0(X) \to H^0(X;\mathbb{Z}).$$

Similarly, the determinant of a vector bundle induces a surjection of abelian groups

$$\det\colon K_0(X) \to \operatorname{Pic}(X).$$

Their sum $\operatorname{rank} \oplus \det\colon K_0(X) \to H^0(X;\mathbb{Z}) \oplus \operatorname{Pic}(X)$ *is a surjective ring map.*

The ring structure on $H^0(X;\mathbb{Z}) \oplus \operatorname{Pic}(X)$ is $(a_1, \mathcal{L}_1)\cdot(a_2,\mathcal{L}_2) = (a_1a_2, \mathcal{L}_1^{a_2} \otimes \mathcal{L}_2^{a_1})$.

Proof. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of vector bundles on X. At any point x of X we have an isomorphism of free $\mathcal{O}_x$-modules $\mathcal{F}_x \cong \mathcal{E}_x \oplus \mathcal{G}_x$, so $\operatorname{rank}_x(\mathcal{F}) = \operatorname{rank}_x(\mathcal{E}) + \operatorname{rank}_x(\mathcal{G})$. Hence each rank_x is an additive function on $\mathbf{VB}(X)$. As x varies rank becomes an additive function with values in $H^0(X;\mathbb{Z})$, so by 6.1.2 it induces a map $\operatorname{rank}\colon K_0(X) \to H^0(X;\mathbb{Z})$. This is a ring map, since the formula $\operatorname{rank}(\mathcal{E}\otimes\mathcal{F}) = \operatorname{rank}(\mathcal{E})\cdot\operatorname{rank}(\mathcal{F})$ may be checked at each point x. If $f\colon X \to \mathbb{N}$ is continuous, the componentwise free module $\mathcal{O}_X^f$ has rank f. It follows that rank is onto. Since the class of componentwise free $\mathcal{O}_X$-modules is closed under $\oplus$ and $\otimes$, the elements $[\mathcal{O}_X^f] - [\mathcal{O}_X^g]$ in $K_0(X)$ form a subring isomorphic to $H^0(X;\mathbb{Z})$.

Similarly, det is an additive function, because we have $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$ by Ex. I.5.4. Hence det induces a map $K_0(X) \to \operatorname{Pic}(X)$ by 6.1.2. If $\mathcal{L}$ is a line bundle on X, then the element $[\mathcal{L}] - [\mathcal{O}_X]$ of $K_0(X)$ has rank zero and determinant $\mathcal{L}$. Hence rank $\oplus$ det is onto; the proof that it is a ring map is given in Ex. 8.5. □

II.8.1.1 **Definition 8.1.1.** As in 2.3 and 2.6.1, the ideal $\widetilde{K}_0(X)$ of $K_0(X)$ is defined to be the kernel of the rank map, so that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \widetilde{K}_0(X)$ as an abelian group. In addition, we let $SK_0(X)$ denote the kernel of rank $\oplus$ det. By Theorem 8.1, these are both ideals of the ring $K_0(X)$. In fact, they form the beginning of the γ-filtration; see Theorem 4.7.

Regular Noetherian Schemes and the Cartan Map

Historically, the group $K_0(X)$ first arose when X is a smooth projective variety, in Grothendieck's proof of the Riemann-Roch Theorem 8.10 (see [30]). The following theorem was central to that proof.

Recall from §6 that $G_0(X)$ is the Grothendieck group of the category $\mathbf{M}(X)$ of coherent $\mathcal{O}_X$-modules. The inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$ induces a natural map $K_0(X) \to G_0(X)$, called the *Cartan homomorphism* (see 7.1.3).

II.8.2 **Theorem 8.2.** *If X is a separated regular noetherian scheme, then every coherent $\mathcal{O}_X$-module has a finite resolution by vector bundles, and the Cartan homomorphism is an isomorphism:*

$$K_0(X) \xrightarrow{\cong} G_0(X).$$

Proof. The first assertion is [SGA6, II, 2.2.3 and 2.2.7.1]. It implies that the Resolution Theorem 7.6 applies to the inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$. □

II.8.2.1 **Proposition 8.2.1** (Nonsingular Curves)**.** *Let X be a 1-dimensional separated regular noetherian scheme, such as a nonsingular curve. Then $SK_0(X) = 0$, and*

$$K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X).$$

Proof. Given Theorem 8.2, this does follow from Ex. 6.10 (see Example 8.2.2 below). However, we shall give a slightly different proof here.

Without loss of generality, we may assume that X is irreducible. If X is affine, this is just Corollary 2.6.3. Otherwise, choose any closed point P on X. By [85, Ex. IV.1.3] the complement $U = X - P$ is affine, say $U = \mathrm{Spec}(R)$. Under the isomorphism $\mathrm{Pic}(X) \cong Cl(X)$ of I.5.14, the line bundle $\mathcal{L}(P)$ corresponds to the class of the Weil divisor $[P]$. Hence the right-hand square commutes in the following diagram

$$\begin{array}{ccccccccc}
 & & G_0(P) & \xrightarrow{i_*} & \widetilde{K}_0(X) & \longrightarrow & \widetilde{K}_0(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{det} & & \cong \downarrow \mathrm{det} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\{\mathcal{L}(P)\}} & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & 0.
\end{array}$$

The top row is exact by 6.4.2 (and 8.2), and the bottom row is exact by I.5.14 and Ex. I.5.12. The right vertical map is an isomorphism by 2.6.2.

Now $G_0(P) \cong \mathbb{Z}$ on the class $[\mathcal{O}_P]$. From the exact sequence $0 \to \mathcal{L}(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0$ we see that $i_*[\mathcal{O}_P] = [\mathcal{O}_X] - [\mathcal{L}(-P)]$ in $K_0(X)$, and $\det(i_*[\mathcal{O}_P]) = \det \mathcal{L}(-P)^{-1}$ in $\mathrm{Pic}(X)$. Hence the isomorphism $G_0(P) \cong \mathbb{Z}$ is compatible with the above diagram. A diagram chase yields $\widetilde{K}_0(X) \cong \mathrm{Pic}(X)$. □

II.8.2.2 **Example 8.2.2** (Classes of subschemes)**.** Let X be a separated regular noetherian scheme. Given a subscheme Z of X, it is convenient to write $[Z]$ for the element $[\mathcal{O}_Z]$ in $K_0\mathbf{M}(X) = K_0(X)$. By Ex. 6.10(d) we see that $SK_0(X)$ is the subgroup of $K_0(X)$ generated by the classes $[Z]$ as Z runs through the irreducible subschemes of codimension ≥ 2. In particular, if $\dim(X) = 2$ then $SK_0(X)$ is generated by the classes $[P]$ of closed points (of codimension 2).

II.8.2.3 **Example 8.2.3** (Transfer for finite and proper maps)**.** Let $f\colon X \to Y$ be a finite morphism of separated noetherian schemes with Y regular. As pointed out in 6.2.5, the direct image f_* is an exact functor $\mathbf{M}(X) \to \mathbf{M}(Y)$. In this case we have a transfer map f_* on K_0 sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$: $K_0(X) \to G_0(X) \to G_0(Y) \cong K_0(Y)$.

If $f\colon X \to Y$ is a proper morphism of separated noetherian schemes with Y regular, we can use the transfer $G_0(X) \to G_0(Y)$ of Lemma 6.2.6 to get a functorial transfer map $f_*\colon K_0(X) \to K_0(Y)$, this time sending $[\mathcal{F}]$ to $\sum(-1)^i[R^i f_*\mathcal{F}]$.

II.8.2.4 **Example 8.2.4** (A non-separated scheme)**.** Here is an example of a regular but non-separated scheme X with $K_0\mathbf{VB}(X) \neq G_0(X)$. Let X be "affine n-space with a double origin" over a field k, where $n \geq 2$. This scheme is the union of two copies of $\mathbb{A}^n = \mathrm{Spec}(k[x_1, \dots, x_n])$ along $\mathbb{A}^n - \{0\}$. Using the localization sequence for either origin and the Fundamental Theorem 6.5, one can show that $G_0(X) = \mathbb{Z} \oplus \mathbb{Z}$. However the inclusion $\mathbb{A}^n \subset X$ is known to induce an equivalence $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^n)$ (see [EGA, IV(5.9)]), so by Theorem 7.8 we have $K_0\mathbf{VB}(X) \cong K_0(k[x_1, \dots, x_n]) \cong \mathbb{Z}$.

II.8.3 **Definition 8.3.** Let $\mathbf{H}(X)$ denote the category consisting of all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that, for each affine open subscheme $U = \mathrm{Spec}(R)$ of X, $\mathcal{F}|_U$ has a finite resolution by vector bundles. Since $\mathcal{F}|_U$ is defined by the finitely generated R-module $M = \mathcal{F}(U)$ this condition just means that M is in $\mathbf{H}(R)$.

If X is regular and separated, then we saw in Theorem 8.2 that $\mathbf{H}(X) = \mathbf{M}(X)$. If $X = \mathrm{Spec}(R)$, it is easy to see that $\mathbf{H}(X)$ is equivalent to $\mathbf{H}(R)$.

$\mathbf{H}(X)$ is an exact subcategory of $\mathcal{O}_X$-**mod**, closed under kernels of surjections, because each $\mathbf{H}(R)$ is closed under extensions and kernels of surjections in R-**mod**.

To say much more about the relation between $\mathbf{H}(X)$ and $K_0(X)$, we need to restrict our attention to quasi-compact schemes such that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle $\mathcal{E}_0$. This implies that every module $\mathcal{F} \in \mathbf{H}(X)$ has a finite resolution $0 \to \mathcal{E}_d \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ by vector bundles. Indeed, the kernel $\mathcal{F}'$ of a quotient map $\mathcal{E}_0 \to \mathcal{F}$ is always locally of lower projective dimension than $\mathcal{F}$, and X has a finite affine cover by $U_i = \mathrm{Spec}(R_i)$, it follows that the d^{th} syzygy is a vector bundle, where $d = \max\{pd_{R_i} M_i\}$, $M_i = \mathcal{F}(U_i)$.

For this condition to hold, it is easiest to assume that X is *quasi-projective* (over a commutative ring k), *i.e.,* a locally closed subscheme of some projective space $\mathbb{P}^n_k$. By [EGA, II, 4.5.5 and 4.5.10], this implies that every quasicoherent $\mathcal{O}_X$-module of finite type $\mathcal{F}$ is a quotient of some vector bundle $\mathcal{E}_0$ of the form $\mathcal{E}_0 = \bigoplus \mathcal{O}_X(n_i)$.

II.8.3.1 **Proposition 8.3.1.** *If X is quasi-projective (over a commutative ring), then* $K_0(X) \cong K_0\mathbf{H}(X)$.

Proof. Because $\mathbf{H}(X)$ is closed under kernels of surjections in $\mathcal{O}_X$-**mod**, and every object in $\mathbf{H}(X)$ has a finite resolution by vector bundles, the Resolution Theorem 7.6 applies to $\mathbf{VB}(X) \subset \mathbf{H}(X)$. □

II.8.3.2 **Technical remark 8.3.2.** Another assumption that guarantees that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle is that X be quasi-separated and quasi-compact with an ample family of line bundles. Such schemes are called *divisorial* in [SGA6, II.2.2.4]. For such schemes, the proof of 8.3.1 goes through to show that we again have $K_0\mathbf{VB}(X) \cong K_0\mathbf{H}(X)$.

II.8.3.3 **Example 8.3.3** (Restricting Bundles)**.** Given an open subscheme U of a quasi-projective scheme X, let $\mathcal{B}$ denote the full subcategory of $\mathbf{VB}(U)$ consisting of vector bundles $\mathcal{F}$ whose class in $K_0(U)$ is in the image of $j^*\colon K_0(X) \to K_0(U)$. We claim that the category $\mathcal{B}$ is cofinal in $\mathbf{VB}(U)$, so that $K_0\mathcal{B}$ is a subgroup of $K_0(U)$ by the Cofinality Lemma 7.2. To see this, note that each vector bundle $\mathcal{F}$ on U fits into an exact sequence $0 \to \mathcal{F}' \to \mathcal{E}_0 \to \mathcal{F} \to 0$, where $\mathcal{E}_0 = \bigoplus \mathcal{O}_U(n_i)$. But then $\mathcal{F} \oplus \mathcal{F}'$ is in $\mathcal{B}$, because in $K_0(U)$

$$[\mathcal{F} \oplus \mathcal{F}'] = [\mathcal{F}] + [\mathcal{F}'] = [\mathcal{E}_0] = \sum j^*[\mathcal{O}_X(n_i)].$$

Transfer Maps for Schemes

II.8.4 **8.4.** We can define a transfer map $f_*\colon K_0(X) \to K_0(Y)$ with $(gf)_* = g_*f_*$ associated to various morphisms $f\colon X \to Y$. If Y is regular, we have already done this in 8.2.3.

Suppose first that f is a finite map. In this case, the inverse image of any affine open $U = \operatorname{Spec}(R)$ of Y is an affine open $f^{-1}U = \operatorname{Spec}(S)$ of X, S is finitely generated as an R-module, and the direct image sheaf $f_*\mathcal{O}_X$ satisfies $f_*\mathcal{O}(U) = S$. Thus the direct image functor f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathcal{O}_Y$-modules (as pointed out in 6.2.5).

If f is finite and $f_*\mathcal{O}_X$ is a vector bundle then f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathbf{VB}(Y)$. Indeed, locally it sends each finitely generated projective S-module to a finitely generated projective R-module, as described in Example 2.8.1. Thus there is a canonical transfer map $f_*\colon K_0(X) \to K_0(Y)$ sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$.

If f is finite and $f_*\mathcal{O}_X$ is in $\mathbf{H}(X)$ then f_* sends $\mathbf{VB}(X)$ into $\mathbf{H}(X)$, because locally it is the forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$ of (7.9.1). Therefore f_* defines a homomorphism $K_0(X) \to K_0\mathbf{H}(Y)$. If Y is quasi-projective then composition with $K_0\mathbf{H}(Y) \cong K_0(Y)$ yields a "finite" transfer map $K_0(X) \to K_0(Y)$.

Now suppose that $f\colon X \to Y$ is a proper map between quasi-projective noetherian schemes. The transfer homomorphism $f_*\colon G_0(X) \to G_0(Y)$ was constructed in Lemma 6.2.6, with $f_*[\mathcal{F}] = \sum(-1)^i[R^if_*\mathcal{F}]$.

If in addition f has finite Tor-dimension, then we can also define a transfer map $f_*\colon K_0(X) \to K_0(Y)$, following [SGA6, IV.2.12.3]. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is called *f_*-acyclic* if $R^qf_*\mathcal{F} = 0$ for all $q > 0$. Let $\mathbf{P}(f)$ denote the category of all vector bundles $\mathcal{F}$ on X such that $\mathcal{F}(n)$ is f_*-acyclic for all $n \geq 0$. By the usual yoga of homological algebra, $\mathbf{P}(f)$ is an exact category, closed under cokernels of injections, and f_* is an exact functor from $\mathbf{P}(f)$ to $\mathbf{H}(Y)$. Hence the following lemma allows us to define the transfer map as

$$K_0(X) \xleftarrow{\cong} K_0\mathbf{P}(f) \xrightarrow{f_*} K_0\mathbf{H}(Y) \xleftarrow{\cong} K_0(Y) \qquad (8.4.1)$$ II.8.4.1

II.8.4.2 **Lemma 8.4.2.** *Every vector bundle $\mathcal{F}$ on a quasi-projective X has a finite resolution*

$$0 \to \mathcal{F} \to \mathcal{P}_0 \to \cdots \to \mathcal{P}_m \to 0$$

by vector bundles in $\mathbf{P}(f)$. Hence by the Resolution Theorem $K_0\mathbf{P}(f) \cong K_0(X)$.

Proof. For $n \geq 0$ the vector bundle $\mathcal{O}_X(n)$ is generated by global sections. Dualizing the resulting surjection $\mathcal{O}_X^r \to \mathcal{O}_X(n)$ and twisting n times yields a short exact sequence of vector bundles $0 \to \mathcal{O}_X \to \mathcal{O}_X(n)^r \to \mathcal{E} \to 0$. Hence for every vector bundle $\mathcal{F}$ on X we have a short exact sequence of vector bundles $0 \to \mathcal{F} \to \mathcal{F}(n)^r \to \mathcal{E}\otimes\mathcal{F} \to 0$. For all large n, the sheaf $\mathcal{F}(n)$ is f_*-acyclic (see [EGA, III.3.2.1] or [85, III.8.8]), and $\mathcal{F}(n)$ is in $\mathbf{P}(f)$. Repeating this process with $\mathcal{E}\otimes\mathcal{F}$ in place of $\mathcal{F}$, we obtain the desired resolution of $\mathcal{F}$. □

Like the transfer map for rings, the transfer map f_* is a $K_0(Y)$-module homomorphism. (This is the *projection formula*; see Ex. 7.10 and Ex. 8.3.)

Projective Space Bundles

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-compact scheme X, and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ denote the projective space bundle of Example I.5.8. (If $\mathcal{E}|_U$ is free over $U \subseteq X$ then $\mathbb{P}|_U$ is the usual projective space $\mathbb{P}^r_U$.) Via the structural map $\pi\colon \mathbb{P} \to X$, the base change map is a ring homomorphism $\pi^*\colon K_0(X) \to K_0(\mathbb{P})$, sending $[\mathcal{M}]$ to $[f^*\mathcal{M}]$, where $f^*\mathcal{M} = \mathcal{O}_{\mathbb{P}}\otimes_X\mathcal{M}$. In this section we give Quillen's proof [153, §8] of the following result, originally due to Berthelot [SGA6, VI.1.1].

II.8.5 **Projective Bundle Theorem 8.5.** *Let $\mathbb{P}$ be the projective space bundle of $\mathcal{E}$ over a quasi-compact scheme X. Then $K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis the twisting line bundles $\{1 = [\mathcal{O}_{\mathbb{P}}], [\mathcal{O}_{\mathbb{P}}(-1)], \dots, [\mathcal{O}_{\mathbb{P}}(-r)]\}$.*

II.8.6 **Corollary 8.6.** *As a ring, $K_0(\mathbb{P}^r_{\mathbb{Z}}) = \mathbb{Z}[z]/(z^{r+1})$, where $z = 1-[\mathcal{O}(-1)]$. (The relation $z^{r+1} = 0$ is Ex. 6.14(b); note that $z = [\mathbb{P}^{r-1}]$.)*

Hence $K_0(\mathbb{P}^r_X) \cong K_0(X)\otimes K_0(\mathbb{P}^r_{\mathbb{Z}}) = K_0(X)[z]/(z^{r+1})$.

8.7. To prove Theorem 8.5, we would like to apply the direct image functor π_* to a vector bundle $\mathcal{F}$ and get a vector bundle. This requires a vanishing condition. The proof of this result rests upon the following notion, which is originally due to Castelnuovo. It is named after David Mumford, who exploited it in [139].

II.8.7.1 **Definition 8.7.1.** A quasicoherent $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$ is called *Mumford-regular* if for all $q > 0$ the higher derived sheaves $R^q\pi_*\mathcal{F}(-q)$ vanish. Here $\mathcal{F}(n)$ is $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(n)$, as in Example I.5.3.1. We write **MR** for the additive category of all Mumford-regular vector bundles, and abbreviate $\otimes_X$ for $\otimes_{\mathcal{O}_X}$.

II.8.7.2 **Example 8.7.2.** If $\mathcal{N}$ is a quasicoherent $\mathcal{O}_X$-module then the standard cohomology calculations on projective spaces show that $\pi^*\mathcal{N} = \mathcal{O}_{\mathbb{P}} \otimes_X \mathcal{N}$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N} = \mathcal{N}$. More generally, if $n \geq 0$ then $\pi^*\mathcal{N}(n)$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N}(n) = Sym_n\mathcal{E} \otimes_X \mathcal{N}$. For $n < 0$ we have $\pi_*\pi^*\mathcal{N}(n) = 0$. In particular, $\mathcal{O}_{\mathbb{P}}(n) = \pi^*\mathcal{O}_X(n)$ is Mumford-regular for all $n \geq 0$.

If X is noetherian and $\mathcal{F}$ is coherent, then for $n \gg 0$ the twists $\mathcal{F}(n)$ are Mumford-regular, because the higher derived functors $R^q\pi_*\mathcal{F}(n)$ vanish for large n and also for $q > r$ (see [85, III.8.8]).

The following facts were discovered by Castelnuovo when $X = \mathrm{Spec}(\mathbb{C})$, and proven in [139, Lecture 14] as well as [153, §8]:

II.8.7.3 **Proposition 8.7.3.** *If $\mathcal{F}$ is Mumford-regular, then*

(1) The twists $\mathcal{F}(n)$ are Mumford-regular for all $n \geq 0$;

(2) Mumford-regular modules are π_*-acyclic, and in fact $R^q\pi_*\mathcal{F}(n) = 0$ for all $q > 0$ and $n \geq -q$;

(3) The canonical map $\varepsilon\colon \pi^*\pi_*(\mathcal{F}) \to \mathcal{F}$ is onto.

Remark. Suppose that X is affine. Since $\pi^*\pi_*(\mathcal{F}) = \mathcal{O}_{\mathbb{P}} \otimes_X \pi_*\mathcal{F}$, and $\pi_*\mathcal{F}$ is quasicoherent, item (3) states that Mumford-regular sheaves are generated by their global sections.

II.8.7.4 **Lemma 8.7.4.** *Mumford-regular modules form an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-***mod***, and π_* is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.*

Proof. Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of $\mathcal{O}_{\mathbb{P}}$-modules with both $\mathcal{F}'$ and $\mathcal{F}''$ Mumford-regular. From the long exact sequence

$$\cdots \to R^q\pi_*\mathcal{F}'(-q) \to R^q\pi_*\mathcal{F}(-q) \to R^q\pi_*\mathcal{F}''(-q) \to \cdots$$

we see that $\mathcal{F}$ is also Mumford-regular. Thus Mumford-regular modules are closed under extensions, *i.e.,* they form an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-**mod**. Since $\mathcal{F}'(1)$ is Mumford-regular, $R^1\pi_*\mathcal{F}' = 0$, and so we have an exact sequence

$$0 \to \pi_*\mathcal{F}' \to \pi_*\mathcal{F} \to \pi_*\mathcal{F}'' \to 0.$$

This proves that π_* is an exact functor. $\square$

The following results were proven by Quillen in [153, §8].

II.8.7.5 **Lemma 8.7.5.** *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}$.*

(1) $\mathcal{F}(n)$ is a Mumford-regular vector bundle on $\mathbb{P}$ for all large enough n;

(2) If $\mathcal{F}(n)$ is π_*-acyclic for all $n \geq 0$ then $\pi_*\mathcal{F}$ is a vector bundle on X.

(3) Hence by 8.7.3, if $\mathcal{F}$ is Mumford-regular then $\pi_*\mathcal{F}$ is a vector bundle on X.

(4) $\pi^*\mathcal{N} \otimes_{\mathbb{P}} \mathcal{F}(n)$ is Mumford-regular for all large enough n, and all quasicoherent $\mathcal{O}_X$-modules $\mathcal{N}$.

II.8.7.6 **Definition 8.7.6** (T_n)**.** Given a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$, we define a natural sequence of $\mathcal{O}_X$-modules $T_n = T_n\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}}$-modules $Z_n = Z_n\mathcal{F}$, starting with $T_0\mathcal{F} = \pi_*\mathcal{F}$ and $Z_{-1} = \mathcal{F}$. Let Z_0 be the kernel of the natural map $\varepsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ of Proposition 8.7.3. Inductively, we define $T_n\mathcal{F} = \pi_* Z_{n-1}(n)$ and define Z_n to be $\ker(\varepsilon)(-n)$, where ε is the canonical map from $\pi^*T_n = \pi^*\pi_*Z_{n-1}(n)$ to $Z_{n-1}(n)$.

Thus we have sequences (exact except possibly at $Z_{n-1}(n)$)

$$0 \to Z_n(n) \to \pi^*(T_n\mathcal{F}) \xrightarrow{\varepsilon} Z_{n-1}(n) \to 0 \qquad (8.7.7)$$ II.8.7.7

whose twists fit together into the sequence of the following theorem.

II.8.7.8 **Theorem 8.7.8** (Quillen's Resolution Theorem)**.** *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}(\mathcal{E})$, $\operatorname{rank}(\mathcal{E}) = r+1$. If $\mathcal{F}$ is Mumford-regular then $Z_r = 0$, and the sequences (8.7.7) are exact for $n \geq 0$, so there is an exact sequence*

$$0 \to (\pi^*T_r\mathcal{F})(-r) \xrightarrow{\varepsilon(-r)} \cdots \to (\pi^*T_i\mathcal{F})(-i) \xrightarrow{\varepsilon(-i)} \cdots \xrightarrow{\varepsilon(-1)} (\pi^*T_0\mathcal{F}) \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Moreover, each $\mathcal{F} \mapsto T_i\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.

Proof. We first prove by induction on $n \geq 0$ that (a) the module $Z_{n-1}(n)$ is Mumford-regular, (b) $\pi_*Z_n(n) = 0$ and (c) the canonical map $\varepsilon\colon \pi^*T_n \to Z_{n-1}(n)$ is onto, *i.e.,* that (8.7.7) is exact for n.

We are given that (a) holds for $n = 0$, so we suppose that (a) holds for n. This implies part (c) for n by Proposition 8.7.3. Inductively then, we are given that (8.7.7) is exact, so $\pi_*Z_n(n) = 0$ and the module $Z_n(n+1)$ is Mumford-regular by Ex. 8.6. That is, (b) holds for n and (a) holds for $n+1$. This finishes the first proof by induction.

Using (8.7.7), another induction on n shows that (d) each $\mathcal{F} \mapsto Z_{n-1}\mathcal{F}(n)$ is an exact functor from Mumford-regular modules to itself, and (e) each $\mathcal{F} \mapsto T_n\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules. Note that (d) implies (e) by Lemma 8.7.4, since $T_n = \pi_*Z_{n-1}(n)$.

Since the canonical resolution is obtained by splicing the exact sequences (8.7.7) together for $n = 0, ..., r$, all that remains is to prove that $Z_r = 0$, or equivalently, that $Z_r(r) = 0$. From (8.7.7) we get the exact sequence

$$R^{q-1}\pi_*Z_{n+q-1}(n) \to R^q\pi_*Z_{n+q}(n) \to R^q\pi_*(\pi^*T_{n+q}(-q))$$

which allows us to conclude, starting from (b) and 8.7.2, that $R^q\pi_*(Z_{n+q}) = 0$ for all $n, q \geq 0$. Since $R^q\pi_* = 0$ for all $q > r$, this shows that $Z_r(r)$ is Mumford-regular. Since $\pi^*\pi_*Z_r(r) = 0$ by (b), we see from Proposition 8.7.3(3) that $Z_r(r) = 0$ as well. □

II.8.7.9 **Corollary 8.7.9.** *If $\mathcal{F}$ is Mumford-regular, each $T_i\mathcal{F}$ is a vector bundle on X.*

Proof. For every $n \geq 0$, the n^{th} twist of Quillen's resolution 8.7.8 yields exact sequences of π_*-acyclic modules. Thus applying π_* yields an exact sequence of $\mathcal{O}_X$-modules, which by 8.7.2 is

$$0 \to T_n \to \mathcal{E} \otimes T_{n-1} \to \cdots \to Sym_{n-i}\mathcal{E} \otimes T_i \to \cdots \to \pi_*\mathcal{F}(n) \to 0.$$

The result follows from this sequence and induction on i, since $\pi_*\mathcal{F}(n)$ is a vector bundle by Lemma 8.7.5(3). □

Let $\mathbf{MR}(n)$ denote the n^{th} twist of $\mathbf{MR}$; it is the full subcategory of $\mathbf{VB}(\mathbb{P})$ consisting of vector bundles $\mathcal{F}$ such that $\mathcal{F}(-n)$ is Mumford-regular. Since twisting is an exact functor, each $\mathbf{MR}(n)$ is an exact category. By Lemma 8.7.3 we have

$$\mathbf{MR} = \mathbf{MR}(0) \subseteq \mathbf{MR}(-1) \subseteq \cdots \subseteq \mathbf{MR}(n) \subseteq \mathbf{MR}(n-1) \subseteq \cdots .$$

II.8.7.10 **Proposition 8.7.10.** *The inclusions $\mathbf{MR}(n) \subset \mathbf{VB}(\mathbb{P})$ induce isomorphisms $K_0\mathbf{MR} \cong K_0\mathbf{MR}(n) \cong K_0(\mathbb{P})$.*

Proof. The union of the $\mathbf{MR}(n)$ is $\mathbf{VB}(\mathbb{P})$ by Lemma 8.7.5(1). By Example 7.1.7 we have $K_0\mathbf{VB}(\mathbb{P}) = \varinjlim K_0\mathbf{MR}(n)$, so it suffices to show that each inclusion $\mathbf{MR}(n) \subset \mathbf{MR}(n-1)$ induces an isomorphism on K_0. For $i > 0$, let $u_i\colon \mathbf{MR}(n-1) \to \mathbf{MR}(n)$ be the exact functor $\mathcal{F} \mapsto \mathcal{F}(i) \otimes_X \wedge^i\mathcal{E}$. It induces a homomorphism $u_i\colon K_0\mathbf{MR}(n-1) \to K_0\mathbf{MR}(n)$. By Proposition 7.5 (Additivity), applied to the Koszul resolution

$$0 \to \mathcal{F} \to \mathcal{F}(1) \otimes_X \mathcal{E} \to \cdots \to \mathcal{F}(r+1) \otimes_X \wedge^{r+1}\mathcal{E} \to 0$$

we see that the map $\sum_{i>0}(-1)^{i-1}u_i$ is an inverse to the map $\iota_n\colon K_0\mathbf{MR}(n) \to K_0\mathbf{MR}(n-1)$ induced by the inclusion. Hence ι_n is an isomorphism, as desired. □

Proof of Projective Bundle Theorem 8.5. Each T_n is an exact functor from $\mathbf{MR}$ to $\mathbf{VB}(X)$ by Theorem 8.7.8 and 8.7.9. Hence we have a homomorphism

$$t\colon K_0\mathbf{MR} \to K_0(X)^{r+1}, \qquad [\mathcal{F}] \mapsto ([T_0\mathcal{F}], -[T_1\mathcal{F}], \ldots, (-1)^r[T_r\mathcal{F}]).$$

This fits into the diagram

$$K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{t} K_0(X)^{r+1} \xrightarrow{u} K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{v} K_0(X)^{r+1}$$

where $u(a_0, ..., a_r) = \pi^*a_0 + \pi^*a_1 \cdot [\mathcal{O}_{\mathbb{P}}(-1)] + \cdots + \pi^*a_r \cdot [\mathcal{O}_{\mathbb{P}}(-r)]$ and $v[\mathcal{F}] = ([\pi_*\mathcal{F}], [\pi_*\mathcal{F}(1)], \ldots, [\pi_*\mathcal{F}(r)])$. The composition ut sends $[\mathcal{F}]$ to the alternating sum of the $[(\pi^*T_i\mathcal{F})(-i)]$, which equals $[\mathcal{F}]$ by Quillen's Resolution Theorem 8.7.8. Hence u is a surjection.

Since the (i,j) component of vu sends $\mathcal{N}_j$ to $\pi_*(\pi^*\mathcal{N}_j(i-j)) = Sym_{i-j}\mathcal{E} \otimes_X \mathcal{N}_j$ by Example 8.7.2, it follows that the composition vu is given by a lower triangular matrix with ones on the diagonal. Therefore vu is an isomorphism, so u is injective. □

λ-operations in $K_0(X)$

The following result was promised in Example 4.1.5.

II.8.8 **Proposition 8.8.** *The operations $\lambda^k[\mathcal{F}] = [\wedge^k\mathcal{F}]$ are well-defined on $K_0(X)$, and make $K_0(X)$ into a λ-ring.*

Proof. It suffices to show that the formula $\lambda_t(\mathcal{F}) = \sum[\wedge^k\mathcal{F}]t^k$ defines an additive homomorphism from $\mathbf{VB}(X)$ to the multiplicative group $1 + tK_0(X)[[t]]$. Note that the constant term in $\lambda_t(\mathcal{F})$ is 1 because $\wedge^0\mathcal{F} = \mathcal{O}_X$. Suppose given an exact sequence of vector bundles $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. By Ex. I.5.4, each $\wedge^k\mathcal{F}$ has a finite filtration whose associated quotient modules are the $\wedge^i\mathcal{F}' \otimes \wedge^{k-i}\mathcal{F}''$, so in $K_0(X)$ we have

$$[\wedge^k\mathcal{F}] = \sum[\wedge^i\mathcal{F}' \otimes \wedge^{k-i}\mathcal{F}''] = \sum[\wedge^i\mathcal{F}'] \cdot [\wedge^{k-i}\mathcal{F}''].$$

Assembling these equations yields the formula $\lambda_t(\mathcal{F}) = \lambda_t(\mathcal{F}')\lambda_t(\mathcal{F}'')$ in the group $1+tK_0(X)[[t]]$, proving that λ_t is additive. Hence λ_t (and each coefficient λ^k) is well-defined on $K_0(X)$. □

II.8.8.1 **Splitting Principle 8.8.1.** (see 4.2.2) Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of a vector bundle $\mathcal{E}$ over a quasi-compact scheme X. Then $K_0(\mathbb{F}(\mathcal{E}))$ is a free module over the ring $K_0(X)$, and $f^*[\mathcal{E}]$ is a sum of line bundles $\sum[\mathcal{L}_i]$.

Proof. Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of $\mathcal{E}$; by Theorem I.5.9 the bundle $f^*\mathcal{E}$ has a filtration by sub-vector bundles whose successive quotients $\mathcal{L}_i$ are line bundles. Hence $f^*[\mathcal{E}] = \sum[\mathcal{L}_i]$ in $K_0(\mathbb{F}(\mathcal{E}))$. Moreover, we saw in I.5.8 that the flag bundle is obtained from X by a sequence of projective space bundle extensions, beginning with $\mathbb{P}(\mathcal{E})$. By the Projective Bundle Theorem 8.5, $K_0(\mathbb{F}(\mathcal{E}))$ is obtained from $K_0(X)$ by a sequence of finite free extensions. □

The λ-ring $K_0(X)$ has a positive structure in the sense of Definition 4.2.1. The "positive elements" are the classes $[\mathcal{F}]$ of vector bundles, and the augmentation $\varepsilon\colon K_0(X) \to H^0(X;\mathbb{Z})$ is given by Theorem 8.1. In this vocabulary, the "line elements" are the classes $[\mathcal{L}]$ of line bundles on X, and the subgroup L of units in $K_0(X)$ is just $\mathrm{Pic}(X)$. The following corollary now follows from Theorems 4.2.3 and 4.7.

II.8.8.2 **Corollary 8.8.2.** *$K_0(X)$ is a special λ-ring. Consequently, the first two ideals in the γ-filtration of $K_0(X)$ are $F^1_\gamma = \widetilde{K}_0(X)$ and $F^2_\gamma = SK_0(X)$. In particular,*

$$F^0_\gamma/F^1_\gamma \cong H^0(X;\mathbb{Z}) \qquad \textit{and} \qquad F^1_\gamma/F^2_\gamma \cong \mathrm{Pic}(X).$$

II.8.8.3 **Corollary 8.8.3.** *For every commutative ring R, $K_0(R)$ is a special λ-ring.*

II.8.8.4 **Proposition 8.8.4.** *If X is quasi-projective, or more generally if X has an ample line bundle $\mathcal{L}$ then every element of $\widetilde{K}_0(X)$ is nilpotent. Hence $\widetilde{K}_0(X)$ is a nil ideal of $K_0(X)$.*

Proof. By Ex. 4.5, it suffices to show that $\ell = [\mathcal{L}]$ is an ample line element. Given $x = [\mathcal{E}] - [\mathcal{F}]$ in $\widetilde{K}_0(X)$, the fact that $\mathcal{L}$ is ample implies that $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for all large n. Hence there are short exact sequences

$$0 \to \mathcal{G}_n \to \mathcal{O}_X^{r_n} \to \mathcal{F}(n) \to 0$$

and therefore in $K_0(X)$ we have the required equation:

$$\ell^n x = [\mathcal{E}(n)] - [\mathcal{O}_X^{r_n}] + [\mathcal{G}_n] = [\mathcal{E}(n) \oplus \mathcal{G}_n] - r_n. \qquad \square$$

II.8.8.5 **Remark 8.8.5.** If X is noetherian and quasiprojective of dimension d, then $\widetilde{K}_0(X)^{d+1} = 0$, because it lies inside F_γ^{d+1}, which vanishes by [SGA6, VI.6.6] or [59, V.3.10]. (See Example 4.8.2.)

If X is a nonsingular algebraic variety, the Chow groups $CH^i(X)$ are defined to be the quotient of $D^i(X)$, the free group on the integral codimension i subvarieties, by rational equivalence; see 6.4.3. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank n, form the projective space bundle $\mathbb{P}(\mathcal{E})$ and flag bundle $\mathbb{F}(\mathcal{E})$ of $\mathcal{E}$; see I.4.10. The Projective Bundle Theorem (see [58]) states that the Chow group $CH^*(\mathbb{P}(\mathcal{E}))$ is a free graded module over $CH^*(X)$ with basis $\{1, \xi, \dots, \xi^{n-1}\}$, where $\xi \in CH^1(\mathbb{P}(\mathcal{E}))$ is the class of a divisor corresponding to $\mathcal{O}(1)$. We define the Chern classes $c_i(\mathcal{E})$ in $CH^i(X)$ to be $(-1)^i$ times the coefficients of ξ^n relative to this basis, with $c_i(\mathcal{E}) = 0$ for $i > n$, with $c_0(\mathcal{E}) = 1$. Thus we have the equation in $CH^n(\mathbb{P}(\mathcal{E}))$:

$$\xi^n - c_1\xi^{n-1} + \cdots + (-1)^i c_i \xi^{n-i} + \cdots + (-1)^n c_n = 0.$$

If $\mathcal{E}$ is trivial then $\xi^n = 0$, and all the c_i vanish except c_0. If $\mathcal{E}$ has rank 1 then there is a divisor D with $\mathcal{E} = \mathcal{L}(D)$ and $\xi = [D]$ and $c_1(\mathcal{E}) = \xi = [D]$.

II.8.9 **Proposition 8.9.** *The classes $c_i(\mathcal{E})$ define Chern classes on $K_0(X)$ with values in $CH^*(X)$.*

Proof. (Grothendieck, 1957) We have already established axioms (CC0) and (CC1); the Normalization axiom (CC3) follows from the observation that

$$c_1(\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)) = [D_1] + [D_2] = c_1(\mathcal{L}(D_1)) + c_1(\mathcal{L}(D_2)).$$

We now invoke the Splitting Principle, that we may assume that $\mathcal{E}$ has a filtration with invertible sheaves $\mathcal{L}_j$ as quotients; this is because $CH^*(\mathbb{P}(\mathcal{E}))$ embeds into $CH^*(\mathbb{F}(\mathcal{E}))$, where such a filtration exists; see I.5.9. Since $\prod(\xi - c_1(\mathcal{L}_j)) = 0$ in $CH^n(\mathbb{F}(\mathcal{E}))$, expanding the product gives the coefficients $c_i(\mathcal{E})$; the coefficients of each ξ^k establish the Sum Formula (CC2). $\square$

II.8.9.1 **Corollary 8.9.1.** *If X is nonsingular, the Chern classes induce isomorphisms $c_i : K_0^{(i)}(X) \cong CH^i(X) \otimes \mathbb{Q}$, and the Chern character induces a ring isomorphism $ch : K_0(X) \otimes \mathbb{Q} \cong CH^*(X) \otimes \mathbb{Q}$.*

Proof. By 4.12 the Chern character $K_0(X) \to CH^*(X) \otimes \mathbb{Q}$ is a ring homomorphism. By 4.11.4, c_n vanishes on $K_0^{(i)}(X)$ for $i \neq n$, and by 4.12.1 it is a graded ring map, where $K_0(X) \otimes \mathbb{Q}$ is given the γ-filtration. Let $F^r K_0(X)$ denote the image of $K_0\mathbf{M}(X) \to K_0(X)$; it is well known (see [59, V.3] that $F^r_\gamma K_0(X) \subseteq F^r K_0(X)$. We will prove by induction on r that $F^r K_0(X) \cong \oplus_{i \geq r} CH^i(X)$. By 6.4.3, there is a canonical surjection $CH^i(X) \to F^i K_0(X)/F^{i+1}K_0(X)$ sending $[Z]$ to the class of $[\mathcal{O}_Z]$; removing a closed subvariety of Z, we can assume that Z is a complete intersection. In that case, $c_i(\mathcal{O}_Z) \cong (-1)^i(i-1)![Z]$ by Ex. 8.7. □

We cite the following result from Fulton [58]. For any smooth X, the *Todd class* $\mathrm{td}(X)$ is defined to be the Hirzebruch character (Ex. 4.13) of the tangent bundle of X for the power series $x/(1-e^{-x})$. If $a = a_0 + \cdots + a_d \in CH^*(X)$ with $a_i \in CH^i(X)$ and $d = \dim(X)$, we write $\deg(a)$ for the image of a_d under the degree map $CH^d(X) \to \mathbb{Z}$.

II.8.10 **Theorem 8.10** (Riemann-Roch Theorem)**.** *Let X be a nonsingular projective variety over a field k, and let $\mathcal{E}$ be a locally free sheaf of rank n on X. Then the Euler characteristic $\chi(\mathcal{E}) = \sum(-1)^i \dim H^i(X,\mathcal{E})$ equals $\deg(ch(\mathcal{E}) \cdot td(X))$.*

More generally, if $f : X \to Y$ is a smooth projective morphism, then the pushforward $f_* : K_0(X) \to K_0(Y)$ satisfies

$$ch(f_*x) = f_*(ch(x) \cdot td(T_f)),$$

where T_f is the relative tangent sheaf of f.

$$\begin{array}{ccc} K_0(X) & \xrightarrow{ch(\cdot td)} & CH^*(X) \otimes \mathbb{Q} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{ch} & CH^*(Y) \otimes \mathbb{Q} \end{array}$$

II.8.11 **Limits of schemes 8.11.** Here is the analogue for schemes of the fact that every commutative ring is the filtered union of its finitely generated (noetherian) subrings. By [EGA, IV.8.2.3], every quasi-compact separated scheme X is the inverse limit of a filtered inverse system $i \mapsto X_i$ of noetherian schemes, each finitely presented over $\mathbb{Z}$, with affine transition maps.

Let $i \mapsto X_i$ be any filtered inverse system of schemes such that the transition morphisms $X_i \to X_j$ are affine, and let X be the inverse limit scheme $\varprojlim X_i$. This scheme exists by [EGA, IV.8.2]. In fact, over an affine open subscheme $\mathrm{Spec}(R_j)$ of any X_j we have affine open subschemes $\mathrm{Spec}(R_i)$ of each X_i, and the corresponding affine open subscheme of X is $\mathrm{Spec}(\varinjlim R_i)$. By [EGA, IV.8.5] every vector bundle on X comes from a bundle on some X_j, and two bundles on X_j are isomorphic over X just in case they are isomorphic over some X_i. Thus the filtered system of groups $K_0(X_i)$ has the property that

$$K_0(X) = \varinjlim K_0(X_i).$$

EXERCISES

EII.8.1 **8.1.** Suppose that Z is a closed subscheme of a quasi-projective scheme X, with complement U. Let $\mathbf{H}_Z(X)$ denote the subcategory of $\mathbf{H}(X)$ consisting of modules supported on Z.

(a) Suppose that $U = \mathrm{Spec}(R)$ for some ring R, and that Z is locally defined by a nonzerodivisor. (The ideal $\mathcal{I}_Z$ is invertible; see I.5.12.) As in Cor. 7.7.4, show that there is an exact sequence: $K_0\mathbf{H}_Z(X) \to K_0(X) \to K_0(U)$.

(b) Suppose that Z is contained in an open subset V of X which is regular. Show that $\mathbf{H}_Z(X)$ is the abelian category $\mathbf{M}_Z(X)$ of 6.4.2, so that $K_0\mathbf{H}_Z(X) \cong G_0(Z)$. Then apply Ex. 7.12 to show that there is an exact sequence
$$G_0(Z) \to K_0(X) \to K_0(U) \to 0.$$

(c) (Deligne) Let R be a 2-dimensional noetherian domain which is not Cohen-Macaulay. If M is in $\mathbf{H}(R)$ then $\mathrm{pd}(M) = 1$, by the Auslander-Buchsbaum equality [223, 4.4.15]. Setting $X = \mathrm{Spec}(R)$ and $Z = \{\mathfrak{m}\}$, show that $\mathbf{H}_Z(X) = 0$. *Hint:* If M is the cokernel of $f : R^m \to R^m$, show that $\det(f)$ must lie in a height 1 prime $\mathfrak{p}$ and conclude that $M_\mathfrak{p} \neq 0$.

There are 2-dimensional normal domains where $K_0(R) \to G_0(R)$ is not into [217]; for these R the sequence $K_0\mathbf{H}_Z(X) \to K_0(X) \to K_0(U)$ is not exact.

EII.8.2 **8.2.** Let X be a curve over an algebraically closed field. By Ex. I.5.7, $K_0(X)$ is generated by classes of line bundles. Show that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X)$.

EII.8.3 **8.3.** *Projection Formula for schemes.* Suppose that $f\colon X \to Y$ is a proper map between quasi-projective schemes, both of which have finite Tor-dimension.

(a) Given $\mathcal{E}$ in $\mathbf{VB}(X)$, consider the subcategory $\mathbf{L}(f)$ of $\mathbf{M}(Y)$ consisting of coherent $\mathcal{O}_Y$-modules which are Tor-independent of both $f_*\mathcal{E}$ and $f_*\mathcal{O}_X$. Show that $G_0(Y) \cong K_0\mathbf{L}(f)$.

(b) Set $x = [E] \in K_0(X)$. By (8.4.1), we can regard f_*x as an element of $K_0(Y)$. Show that $f_*(x \cdot f^*y) = f_*(x) \cdot y$ for every $y \in G_0(Y)$.

(c) Using 7.4.2 and the ring map $f^*\colon K_0(Y) \to K_0(X)$, both $K_0(X)$ and $G_0(X)$ are $K_0(Y)$-modules. Show that the transfer maps $f_*\colon G_0(X) \to G_0(Y)$ of Lemma 6.2.6 and $f_*\colon K_0(X) \to K_0(Y)$ of (8.4.1) are $K_0(Y)$-module homomorphisms, *i.e.,* that the *projection formula* holds for every $y \in K_0(Y)$:
$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every } x \in K_0(X) \text{ or } x \in G_0(X).$$

EII.8.4 **8.4.** Suppose given a commutative square of quasi-projective schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with $X' = X \times_Y Y'$ and f proper. Assume that g has finite flat dimension, and that X and Y' are Tor-independent over Y, *i.e.*, for $q > 0$ and all $x \in X$, $y' \in Y'$ and $y \in Y$ with $y = f(x) = g(y')$ we have

$$\mathrm{Tor}_q^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

Show that $g^* f_* = f'_* g'^*$ as maps $G_0(X) \to G_0(Y')$.

EII.8.5 **8.5.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be vector bundles of ranks r_1 and r_2, respectively. Modify Ex. I.2.7 to show that $\det(\mathcal{F}_1 \otimes \mathcal{F}_2) \cong (\det \mathcal{F}_1)^{r_2} \otimes (\det \mathcal{F}_2)^{r_1}$. Conclude that $K_0(X) \to H^0(X; \mathbb{Z}) \oplus \mathrm{Pic}(X)$ is a ring map.

EII.8.6 **8.6.** Let $\pi\colon \mathbb{P} \to X$ be a projective space bundle as in 8.5, and let $\mathcal{F}$ be a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module. Let $\mathcal{N}$ denote the kernel of the canonical map $\varepsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$. Show that $\mathcal{N}(1)$ is Mumford-regular, and that $\pi_*\mathcal{N} = 0$.

EII.8.7 **8.7.** Suppose that Z is a codimension i subvariety of a nonsingular X, with conormal bundle $\mathcal{E}$; $\mathcal{E} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ is exact. Show that $c_i([\mathcal{O}_Z]) = (-1)^i(i-1)![Z]$ in $CH^i(X)$. *Hint:* Passing to a flag bundle of $\mathcal{E}$, show that $[\mathcal{O}_Z]$ is a product of terms $1 - L_j$, for line bundles $L_j = [\mathcal{O}_{D_j}]$ of divisors D_j. Now use the product formula for the total Chern class and $c_1(L_j) = [D_j]$. Since this is a formal computation, it suffices to compute $\ln c([\mathcal{O}_Z])$.

9 K_0 of a Waldhausen category

It is useful to be able to define the Grothendieck group $K_0(\mathcal{C})$ of a more general type of category than exact categories, by adding a notion of weak equivalence. A structure that generalizes well to higher K-theory is that of a category of cofibrations and weak equivalences, which we shall call a "Waldhausen category" for brevity. The definitions we shall use are due to Friedhelm Waldhausen, although the ideas for K_0 are due to Grothendieck and were used in SGA6 [SGA6].

We need to consider two families of distinguished morphisms in a category $\mathcal{C}$, the cofibrations and the weak equivalences. For this we use the following device. Suppose that we are given a family $\mathcal{F}$ of distinguished morphisms in a category $\mathcal{C}$. We assume that these distinguished morphisms are closed under composition, and contain every identity. It is convenient to regard these distinguished morphisms as the morphisms of a subcategory of $\mathcal{C}$, which by abuse of notation we also call $\mathcal{F}$.

II.9.1 **Definition 9.1.** Let $\mathcal{C}$ be a category equipped with a subcategory $co = co(\mathcal{C})$ of morphisms in a category $\mathcal{C}$, called "cofibrations" (and indicated with feathered arrows $\rightarrowtail$). The pair $(\mathcal{C}, co)$ is called a *category with cofibrations* if the following axioms are satisfied:
(W0) Every isomorphism in $\mathcal{C}$ is a cofibration;
(W1) There is a distinguished zero object '0' in $\mathcal{C}$, and the unique map $0 \rightarrowtail A$ in $\mathcal{C}$ is a cofibration for every A in $\mathcal{C}$;
(W2) If $A \rightarrowtail B$ is a cofibration, and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \cup_A C$ of these two maps exists in $\mathcal{C}$, and moreover the map $C \rightarrowtail B \cup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array}$$

These axioms imply that two constructions make sense in $\mathcal{C}$: (1) the coproduct $B \amalg C$ of any two objects exists in $\mathcal{C}$ (it is the pushout $B \cup_0 C$), and (2) every cofibration $i\colon A \rightarrowtail B$ in $\mathcal{C}$ has a cokernel B/A (this is the pushout $B \cup_A 0$ of i along $A \to 0$). We refer to $A \rightarrowtail B \twoheadrightarrow B/A$ as a *cofibration sequence* in $\mathcal{C}$.

For example, any abelian category is naturally a category with cofibrations: the cofibrations are the monomorphisms. More generally, we can regard any exact category as a category with cofibrations by letting the cofibrations be the admissible monics; axiom (W2) follows from Ex. 7.8(2). In an exact category, the cofibration sequences are exactly the admissible exact sequences.

II.9.1.1 **Definition 9.1.1.** A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (abbreviated '*w.e.*' and indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the following "Glueing axiom" must be satisfied:

(W3) *Glueing for weak equivalences.* For every commutative diagram of the form

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array}$$

(in which the vertical maps are weak equivalences and the two right horizontal maps are cofibrations), the induced map

$$B \cup_A C \to B' \cup_{A'} C'$$

is also a weak equivalence.

Although a Waldhausen category is really a triple $(\mathcal{C}, co, w)$, we will usually drop the (co, w) from the notation and just write $\mathcal{C}$. We say that $\mathcal{C}$ (or just $w\mathcal{C}$) is *saturated* if: whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence if and only if g is.

II.9.1.2 **Definition 9.1.2** $(K_0(\mathcal{C}))$**.** Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$ is the abelian group presented as having one generator $[C]$ for each object C of $\mathcal{C}$, subject to the relations

(1) $[C] = [C']$ if there is a weak equivalence $C \xrightarrow{\sim} C'$

(2) $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

Of course, in order for this to be set-theoretically meaningful, we must assume that the weak equivalence classes of objects form a set. We shall occasionally use the notation $K_0(w\mathcal{C})$ for $K_0(\mathcal{C})$ to emphasize the choice of $w\mathcal{C}$ as weak equivalences.

These relations imply that $[0] = 0$ and $[B \amalg C] = [B] + [C]$, as they did in §6 for abelian categories. Because pushouts preserve cokernels, we also have $[B \cup_A C] = [B] + [C] - [A]$. However, weak equivalences add a new feature: $[C] = 0$ whenever $0 \simeq C$.

II.9.1.3 **Example 9.1.3.** Any exact category $\mathcal{A}$ becomes a Waldhausen category, with cofibrations being admissible monics and weak equivalences being isomorphisms. By construction, the Waldhausen definition of $K_0(\mathcal{A})$ agrees with the exact category definition of $K_0(\mathcal{A})$ given in §7.

More generally, any category with cofibrations $(\mathcal{C}, co)$ may be considered as a Waldhausen category in which the category of weak equivalences is the category $\mathrm{iso}\,\mathcal{C}$ of all isomorphisms. In this case $K_0(\mathcal{C}) = K_0(\mathrm{iso}\,\mathcal{C})$ has only the relation (2). We could of course have developed this theory in §7 as an easy generalization of the preceding paragraph.

II.9.1.4 **Topological Example 9.1.4.** To show that we need not have additive categories, we give a topological example due to Waldhausen. Let $\mathcal{R} = \mathcal{R}(*)$ be the category of based CW complexes with countably many cells (we need a bound on the cardinality of the cells for set-theoretic reasons). Morphisms are cellular maps, and $\mathcal{R}_f = \mathcal{R}_f(*)$ is the subcategory of finite based CW complexes. Both are Waldhausen categories: "cofibration" is a cellular inclusion, and "weak equivalence" means weak homotopy equivalence (isomorphism on homotopy groups). The coproduct $B \vee C$ is obtained from the disjoint union of B and C by identifying their basepoints.

The Eilenberg Swindle shows that $K_0\mathcal{R} = 0$. In effect, the infinite coproduct C^∞ of copies of a fixed complex C exists in $\mathcal{R}$, and equals $C \vee C^\infty$. In contrast, the finite complexes have interesting K-theory:

II.9.1.5 **Proposition 9.1.5.** *$K_0\mathcal{R}_f \cong \mathbb{Z}$.*

Proof. The inclusion of S^{n-1} in the n-disk D^n has $D^n/S^{n-1} \cong S^n$, so $[S^{n-1}] + [S^n] = [D^n] = 0$. Hence $[S^n] = (-1)^n[S^0]$. If C is obtained from B by attaching an n-cell, $C/B \cong S^n$ and $[C] = [B] + [S^n]$. Hence $K_0\mathcal{R}_f$ is generated by $[S^0]$. Finally, the reduced Euler characteristic $\chi(C) = \sum(-1)^i \dim \tilde{H}^i(X;\mathbb{Q})$ defines a surjection from $K_0\mathcal{R}_f$ onto $\mathbb{Z}$, which must therefore be an isomorphism. □

II.9.1.6 **BiWaldhausen Categories 9.1.6.** In general, the opposite $\mathcal{C}^{op}$ need not be a Waldhausen category, because the quotients $B \twoheadrightarrow B/A$ need not be closed under composition: the family quot$(\mathcal{C})$ of these quotient maps need not be a subcategory of $\mathcal{C}^{op}$. We call $\mathcal{C}$ a *category with bifibrations* if $\mathcal{C}$ is a category with cofibrations, $\mathcal{C}^{op}$ is a category with cofibrations, $co(\mathcal{C}^{op}) = $ quot$(\mathcal{C})$, the canonical map $A \amalg B \to A \times B$ is always an isomorphism, and A is the kernel of each quotient map $B \twoheadrightarrow B/A$. We call $\mathcal{C}$ a *biWaldhausen category* if $\mathcal{C}$ is a category with bifibrations, having a subcategory $w(\mathcal{C})$ so that both $(\mathcal{C}, co, w)$ and $(\mathcal{C}^{op}, \text{quot}, w^{op})$ are Waldhausen categories. The notions of bifibrations and biWaldhausen category are self-dual, so we have:

II.9.1.7 **Lemma 9.1.7.** *$K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$ for every biWaldhausen category.*

Example 9.1.3 shows that exact categories are biWaldhausen categories. We will see in 9.2 below that chain complexes form another important family of biWaldhausen categories.

II.9.1.8 **Exact Functors 9.1.8.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ between Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and pushouts along a cofibration. The last condition means that the canonical map $FB \cup_{FA} FC \to F(B \cup_A C)$ is an isomorphism for every cofibration $A \rightarrowtail B$. Clearly, an exact functor induces a group homomorphism $K_0(F)\colon K_0\mathcal{C} \to K_0\mathcal{D}$.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in $\mathcal{A}$ are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

For example, suppose that $\mathcal{C}$ and $\mathcal{D}$ are exact categories (in the sense of §7), considered as Waldhausen categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is exact in the above sense if and only if F is additive and preserves short exact sequences, *i.e.,* F is an exact functor between exact categories in the sense of §7. The routine verification of this assertion is left to the reader.

Here is an elementary consequence of the definition of exact functor. Let $\mathcal{A}$ and $\mathcal{C}$ be Waldhausen categories and F, F', F'' three exact functors from $\mathcal{A}$ to $\mathcal{C}$. Suppose moreover that there are natural transformations $F' \Rightarrow F \Rightarrow F''$ so that for all A in $\mathcal{A}$

$$F'A \rightarrowtail FA \twoheadrightarrow F''A$$

is a cofibration sequence in $\mathcal{C}$. Then $[FA] = [F'A] + [F''A]$ in $K_0\mathcal{C}$, so as maps from $K_0\mathcal{A}$ to $K_0\mathcal{C}$ we have $K_0(F) = K_0(F') + K_0(F'')$.

Chain complexes

II.9.2 **9.2.** Historically, one of the most important families of Waldhausen categories are those arising from chain complexes. The definition of K_0 for a category of (co)chain complexes dates to the 1960's, being used in [SGA6] to study the Riemann-Roch Theorem. We will work with chain complexes here, although by reindexing we could equally well work with cochain complexes.

Given a small abelian category $\mathcal{A}$, let $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ denote the category of all chain complexes in $\mathcal{A}$, and let $\mathbf{Ch}^b$ denote the full subcategory of all bounded complexes. The following structure makes $\mathbf{Ch}$ into a Waldhausen category, with $\mathbf{Ch}^b(\mathcal{A})$ as a Waldhausen subcategory. We will show below that $K_0\mathbf{Ch}(\mathcal{A}) = 0$ but that $K_0\mathbf{Ch}^b(\mathcal{A}) \cong K_0(\mathcal{A})$.

A cofibration $C \to D$ is a chain map such that every map $C_n \rightarrowtail D_n$ is monic in $\mathcal{A}$. Thus a cofibration sequence is just a short exact sequence of chain complexes. A weak equivalence $C \xrightarrow{\sim} D$ is a quasi-isomorphism, *i.e.,* a chain map inducing isomorphisms on homology.

Here is a slightly more general construction, taken from [SGA6, IV(1.5.2)]. Suppose that $\mathcal{B}$ is an exact category, embedded in an abelian category $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{B})$, resp. $\mathbf{Ch}^b(\mathcal{B})$, denote the category of all (resp. all bounded) chain complexes in $\mathcal{B}$. A cofibration $A_\bullet \rightarrowtail B_\bullet$ in $\mathbf{Ch}(\mathcal{B})$ (resp. $\mathbf{Ch}^b\mathcal{B}$) is a map which is a degreewise admissible monomorphism, *i.e.,* such that each $C_n = B_n/A_n$ is in $\mathcal{B}$, yielding short exact sequences $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ in $\mathcal{B}$. To define the weak equivalences, we use the notion of homology in the ambient abelian category $\mathcal{A}$: let $w\mathbf{Ch}(\mathcal{B})$ denote the family of all chain maps in $\mathbf{Ch}(\mathcal{B})$ which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathcal{A})$. With this structure, both $\mathbf{Ch}(\mathcal{B})$ and $\mathbf{Ch}^b(\mathcal{B})$ become Waldhausen subcategories of $\mathbf{Ch}(\mathcal{A})$.

Subtraction in $K_0\mathbf{Ch}$ and $K_0\mathbf{Ch}^b$ is given by shifting indices on complexes. To see this, recall from [223, 1.2.8] that the n^{th} *translate* of C is defined to be the chain complex $C[n]$ which has C_{i+n} in degree i. (If we work with cochain complexes then C^{i-n} is in degree i.) Moreover, the *mapping cone complex* cone(f) of a chain complex map $f\colon B \to C$ fits into a short exact sequence of complexes:

$$0 \to C \to \text{cone}(f) \to B[-1] \to 0.$$

Therefore in K_0 we have $[C] + [B[-1]] = [\text{cone}(f)]$. In particular, if f is the identity map on C, the cone complex is exact and hence *w.e.* to 0. Thus we have $[C] + [C[-1]] = [\text{cone}(\text{id})] = 0$. We record this observation as follows.

II.9.2.1 **Lemma 9.2.1.** *Let $\mathcal{C}$ be any Waldhausen subcategory of $\mathbf{Ch}(\mathcal{A})$ closed under translates and the formation of mapping cones. Then $[C[n]] = (-1)^n[C]$ in $K_0(\mathcal{C})$. In particular, this is true in $K_0\mathbf{Ch}(\mathcal{B})$ and $K_0\mathbf{Ch}^b(\mathcal{B})$ for every exact subcategory $\mathcal{B}$ of $\mathcal{A}$.*

A chain complex C is called *bounded below* (resp. *bounded above*) if $C_n = 0$ for all $n \ll 0$ (resp. all $n \gg 0$). If C is bounded above, then each infinite direct sum $C_n \oplus C_{n+2} \oplus \cdots$ is finite, so the infinite direct sum of shifts

$$B = C \oplus C[2] \oplus C[4] \oplus \cdots \oplus C[2n] \oplus \cdots$$

is defined in $\mathbf{Ch}$. From the exact sequence $0 \to B[2] \to B \to C \to 0$, we see that in $K_0\mathbf{Ch}$ we have the Eilenberg swindle: $[C] = [B] - [B[2]] = [B] - [B] = 0$. A similar argument shows that $[C] = 0$ if C is bounded below. But every chain complex C fits into a short exact sequence

$$0 \to B \to C \to D \to 0$$

in which B is bounded above and D is bounded below. (For example, take $B_n = 0$ for $n > 0$ and $B_n = C_n$ otherwise.) Hence $[C] = [B] + [D] = 0$ in $K_0\mathbf{Ch}$. This shows that $K_0\mathbf{Ch} = 0$, as asserted.

If $\mathcal{B}$ is any exact category, the natural inclusion of $\mathcal{B}$ into $\mathbf{Ch}^b(\mathcal{B})$ as the chain complexes concentrated in degree zero is an exact functor. Hence it induces a homomorphism $K_0(\mathcal{B}) \to K_0\mathbf{Ch}^b(\mathcal{B})$.

II.9.2.2 **Theorem 9.2.2.** *([SGA6, I.6.4]) Let $\mathcal{A}$ be an abelian category. Then*

$$K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}),$$

and the class $[C]$ of a chain complex C in $K_0\mathcal{A}$ is the same as its Euler characteristic, namely $\chi(C) = \sum(-1)^i[C_i]$.

Similarly, if $\mathcal{B}$ is an exact category closed under kernels of surjections in an abelian category (in the sense of 7.0.1), then $K_0(\mathcal{B}) \cong K_0\mathbf{Ch}^b(\mathcal{B})$, and again we have $\chi(C) = \sum(-1)^i[C_i]$ in $K_0(\mathcal{B})$.

Proof. We give the proof for $\mathcal{A}$; the proof for $\mathcal{B}$ is the same, except one cites 7.5 in place of 6.6. As in Proposition 6.6 (or 7.5), the Euler characteristic $\chi(C)$ of a bounded complex is the element $\sum(-1)^i[C_i]$ of $K_0(\mathcal{A})$. We saw in 6.6 (and 7.5.1) that $\chi(B) = \chi(C)$ if $B \to C$ is a weak equivalence (quasi-isomorphism). If $B \rightarrowtail C \twoheadrightarrow D$ is a cofibration sequence in $\mathbf{Ch}^b$, then from the short exact sequences $0 \to B_n \to C_n \to D_n \to 0$ in $\mathcal{A}$ we obtain $\chi(C) = \chi(B) + \chi(C/B)$ by inspection (as in 7.5.1). Hence χ satisfies the relations needed to define a homomorphism χ from $K_0(\mathbf{Ch}^b)$ to $K_0(\mathcal{A})$. If C is concentrated in degree 0 then $\chi(C) = [C_0]$, so the composite map $K_0(\mathcal{A}) \to K_0(\mathbf{Ch}^b) \to K_0(\mathcal{A})$ is the identity.

It remains to show that $[C] = \chi(C)$ in $K_0\mathbf{Ch}^b$ for every complex

$$C\colon 0 \to C_m \to \cdots \to C_n \to 0.$$

If $m = n$, then $C = C_n[-n]$ is the object C_n of $\mathcal{A}$ concentrated in degree n; we have already observed that $[C] = (-1)^n[C_n[0]] = (-1)^n[C_n]$ in this case. If $m > n$, let B denote the subcomplex consisting of C_n in degree n, and zero elsewhere. Then $B \rightarrowtail C$ is a cofibration whose cokernel C/B has shorter length than C. By induction, we have the desired relation in $K_0\mathbf{Ch}^b$, finishing the proof:

$$[C] = [B] + [C/B] = \chi(B) + \chi(C/B) = \chi(C). \qquad \square$$

II.9.2.3 **Remark 9.2.3** (K_0 and derived categories)**.** Let $\mathcal{B}$ be an exact category. Theorem 9.2.2 states that the group $K_0\mathbf{Ch}^b(\mathcal{B})$ is independent of the choice of ambient abelian category $\mathcal{A}$, as long as $\mathcal{B}$ is closed under kernels of surjections in $\mathcal{A}$. This is the group $k(\mathcal{B})$ introduced in [SGA6], Expose IV(1.5.2). (The context of [SGA6] was triangulated categories, and the main observation in *op. cit.* is that this definition only depends upon the derived category $D^b_{\mathcal{B}}(\mathcal{A})$. See Ex. 9.5 below.)

We warn the reader that if $\mathcal{B}$ is not closed under kernels of surjections in $\mathcal{A}$, then $K_0\mathbf{Ch}^b(\mathcal{B})$ can differ from $K_0(\mathcal{B})$. (See Ex. 9.11).

If $\mathcal{A}$ is an abelian category, or even an exact category, the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ has another Waldhausen structure with the same weak equivalences: we redefine cofibration so that $B \to C$ is a cofibration if and only if each $B_i \to C_i$ is a *split* injection in $\mathcal{A}$. If split $\mathbf{Ch}^b$ denotes $\mathbf{Ch}^b$ with this new Waldhausen structure, then the inclusion split $\mathbf{Ch}^b \to \mathbf{Ch}^b$ is an exact functor, so it induces a surjection $K_0(\text{split}\,\mathbf{Ch}^b) \to K_0(\mathbf{Ch}^b)$.

II.9.2.4 **Lemma 9.2.4.** *If $\mathcal{A}$ is an exact category then*

$$K_0(\text{split}\,\mathbf{Ch}^b) \cong K_0(\mathbf{Ch}^b) \cong K_0(\mathcal{A}).$$

Proof. Lemma 9.2.1 and enough of the proof of 9.2.2 go through to prove that $[C[n]] = (-1)^n[C]$ and $[C] = \sum(-1)^n[C_n]$ in $K_0(\text{split}\,\mathbf{Ch}^b)$. Hence it suffices to show that $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\text{split}\,\mathbf{Ch}^b)$. If A is an object of $\mathcal{A}$, let $[A]$ denote the class in $K_0(\text{split}\,\mathbf{Ch}^b)$ of the complex which is A concentrated in degree zero. Any short exact sequence $E\colon 0 \to A \to B \to C \to 0$ in $\mathcal{A}$ may be regarded as an (exact) chain complex concentrated in degrees 0, 1 and 2 so:

$$[E] = [A] - [B] + [C]$$

in $K_0(\text{split}\,\mathbf{Ch}^b)$. But E is weakly equivalent to zero, so $[E] = 0$. Hence $A \mapsto [A]$ is an additive function, defining a map $K_0(\mathcal{A}) \to K_0(\text{split}\,\mathbf{Ch}^b)$. $\square$

II.9.3 **Example 9.3** (Extension Categories)**.** If $\mathcal{B}$ is a category with cofibrations, the cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{B}$ form the objects of a category $\mathcal{E} = \mathcal{E}(\mathcal{B})$. A morphism $E \to E'$ in $\mathcal{E}$ is a commutative diagram:

$$\begin{array}{ccccccc} E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E': & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

We make $\mathcal{E}$ into a category with cofibrations by declaring that a morphism $E \to E'$ in $\mathcal{E}$ is a cofibration if $A \to A'$, $C \to C'$ and $A' \cup_A B \to B'$ are cofibrations in $\mathcal{B}$. This is required by axiom (W2), and implies that the composite $B \rightarrowtail A' \cup_A B \rightarrowtail B'$ is a cofibration too. If $\mathcal{B}$ is a Waldhausen category then so is $\mathcal{E}(\mathcal{B})$: a weak equivalence in $\mathcal{E}$ is a morphism whose component maps $A \to A'$, $B \to B'$, $C \to C'$ are weak equivalences in $\mathcal{B}$.

Here is a useful variant. If $\mathcal{A}$ and $\mathcal{C}$ are Waldhausen subcategories of $\mathcal{B}$, the extension category $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\mathcal{C}$ by $\mathcal{A}$ is the Waldhausen subcategory of the extension category of $\mathcal{B}$ consisting of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ with A in $\mathcal{A}$ and C in $\mathcal{C}$. Clearly, $\mathcal{E}(\mathcal{B}) = \mathcal{E}(\mathcal{B}, \mathcal{B}, \mathcal{B})$.

There is an exact functor $\amalg\colon \mathcal{A} \times \mathcal{C} \to \mathcal{E}$, sending (A, C) to $A \rightarrowtail A \amalg C \twoheadrightarrow C$. Conversely, there are three exact functors (s, t and q) from $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, which send $A \rightarrowtail B \twoheadrightarrow C$ to A, B and C, respectively. By the above remarks, if $\mathcal{A} = \mathcal{B} = \mathcal{C}$ then $t_* = s_* + q_*$ as maps $K_0(\mathcal{E}) \to K_0(\mathcal{B})$.

II.9.3.1 **Proposition 9.3.1.** $K_0(\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})) \cong K_0(\mathcal{A}) \times K_0(\mathcal{C})$.

Proof. Since (s, q) is a left inverse to $\amalg$, $\amalg_*$ is a split injection from $K_0(\mathcal{A}) \times K_0(\mathcal{C})$ to $K_0(\mathcal{E})$. Thus it suffices to show that for every $E\colon A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ we have $[E] = [\amalg(A, 0)] + [\amalg(0, C)]$ in $K_0(\mathcal{E})$. This relation follows from the fundamental relation II.9.1.2 9.1.2(2) of K_0, given that

$$\begin{array}{ccccccc} \amalg(A,0): & & A & \overset{=}{\rightarrowtail} & A & \twoheadrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \downarrow \\ E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \end{array}$$

is a cofibration in $\mathcal{E}$ with cokernel $\amalg(0, C) : 0 \rightarrowtail C \twoheadrightarrow C$. □

II.9.3.2 **Example 9.3.2** (Higher Extension categories)**.** Here is a generalization of the extension category $\mathcal{E} = \mathcal{E}_2$ constructed above. Let $\mathcal{E}_n$ be the category whose objects are sequences of n cofibrations in a Waldhausen category $\mathcal{C}$:

$$A: \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n.$$

A morphism $A \to B$ in $\mathcal{E}_n$ is a natural transformation of sequences, and is a weak equivalence if each component $A_i \to B_i$ is a *w.e.* in $\mathcal{C}$. It is a cofibration when for each $0 \leq i < j < k \leq n$ the map of cofibration sequences

$$\begin{array}{ccccc} A_j/A_i & \rightarrowtail & A_k/A_i & \twoheadrightarrow & A_k/A_j \\ \downarrow & & \downarrow & & \downarrow \\ B_j/B_i & \rightarrowtail & B_k/B_i & \twoheadrightarrow & B_k/B_j \end{array}$$

is a cofibration in $\mathcal{E}$. The reader is encouraged in Ex. 9.4 to check that $\mathcal{E}_n$ is a Waldhausen category, and to compute $K_0(\mathcal{E}_n)$.

II.9.4 **Theorem 9.4** (Cofinality Theorem)**.** *Let $\mathcal{B}$ be a Waldhausen subcategory of $\mathcal{C}$ closed under extensions. If $\mathcal{B}$ is cofinal in $\mathcal{C}$ (in the sense that for all C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \amalg C'$ is in $\mathcal{B}$), then $K_0(\mathcal{B})$ is a subgroup of $K_0(\mathcal{C})$.*

Proof. Considering $\mathcal{B}$ and $\mathcal{C}$ as symmetric monoidal categories with product $\amalg$, we have $K_0^{\amalg}(\mathcal{B}) \subset K_0^{\amalg}(\mathcal{C})$ by 1.3. The proof of cofinality for exact categories (Lemma 7.2) goes through verbatim to prove that $K_0(\mathcal{B}) \subset K_0(\mathcal{C})$. □

II.9.4.1 **Remark 9.4.1.** The proof shows that $K_0(\mathcal{C})/K_0(\mathcal{B}) \cong K_0^{\amalg}\mathcal{C}/K_0^{\amalg}\mathcal{B}$, and that every element of $K_0(\mathcal{C})$ has the form $[C] - [B]$ for some B in $\mathcal{B}$ and C in $\mathcal{C}$.

Products

II.9.5 **9.5.** Our discussion in 7.4 about products in exact categories carries over to the Waldhausen setting. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Waldhausen categories, and suppose given a functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. The following result is completely elementary:

II.9.5.1 **Lemma 9.5.1.** *If each $F(A, -)\colon \mathcal{B} \to \mathcal{C}$ and $F(-, B)\colon \mathcal{A} \to \mathcal{C}$ is an exact functor, then $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$K_0\mathcal{A} \otimes K_0\mathcal{B} \to K_0\mathcal{C}$$
$$[A] \otimes [B] \mapsto [F(A, B)].$$

Note that the 3×3 diagram in $\mathcal{C}$ determined by $F(A \rightarrowtail A', B \rightarrowtail B')$ yields the following relation in $K_0(\mathcal{C})$.

$$[F(A', B')] = [F(A, B)] + [F(A'/A, B)] + [F(A, B'/B)] + [F(A'/A, B'/B)]$$

Higher K-theory will need this relation to follow from more symmetric considerations, viz. that $F(A \rightarrowtail A', B \rightarrowtail B')$ should represent a cofibration in the category $\mathcal{E}$ of all cofibration sequences in $\mathcal{C}$. With this in mind, we introduce the following definition.

II.9.5.2 **Definition 9.5.2.** A functor $F\colon \mathcal{A} \times \mathcal{B} \rightarrowtail \mathcal{C}$ between Waldhausen categories is called *biexact* if each $F(A, -)$ and $F(-, B)$ is exact, and the following condition is satisfied:

- For every pair of cofibrations ($A \rightarrowtail A'$ in $\mathcal{A}$, $B \rightarrowtail B'$ in $\mathcal{B}$) the following map must be a cofibration in $\mathcal{C}$:

$$F(A', B) \cup_{F(A,B)} F(A, B') \rightarrowtail F(A', B').$$

Our next result requires some notation. Suppose that a category with cofibrations $\mathcal{C}$ has two notions of weak equivalence, a weak one v and a stronger one w. (Every map in v belongs to w.) We write $v\mathcal{C}$ and $w\mathcal{C}$ for the two Waldhausen categories $(\mathcal{C}, co, v)$ and $(\mathcal{C}, co, w)$. The identity on $\mathcal{C}$ is an exact functor $v\mathcal{C} \to w\mathcal{C}$.

Let $\mathcal{C}^w$ denote the full subcategory of all w-acyclic objects in $\mathcal{C}$, *i.e.,* those C for which $0 \rightarrowtail C$ is in $w(\mathcal{C})$; $\mathcal{C}^w$ is a Waldhausen subcategory (9.1.8) of $v\mathcal{C}$, *i.e.,* of the category $\mathcal{C}$ with the v-notion of weak equivalence.

Recall from 9.1.1 that $w\mathcal{C}$ is called *saturated* if, whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence if and only if g is.

II.9.6 **Theorem 9.6** (Localization Theorem)**.** *Suppose that $\mathcal{C}$ is a category with cofibrations, endowed with two notions ($v \subset w$) of weak equivalence, with $w\mathcal{C}$ saturated, and that $\mathcal{C}^w$ is defined as above.*

Assume in addition that every map $f\colon C_1 \to C_2$ in $\mathcal{C}$ factors as the composition of a cofibration $C_1 \rightarrowtail C$ and an equivalence $C \xrightarrow{\sim} C_2$ in $v(\mathcal{C})$.

Then the exact inclusions $\mathcal{C}^w \to v\mathcal{C} \to w\mathcal{C}$ induce an exact sequence

$$K_0(\mathcal{C}^w) \to K_0(v\mathcal{C}) \to K_0(w\mathcal{C}) \to 0.$$

Proof. Our proof of this is similar to the proof of the Localization Theorem 6.4 for abelian categories. Clearly $K_0(v\mathcal{C})$ maps onto $K_0(w\mathcal{C})$ and $K_0(\mathcal{C}^w)$ maps to zero. Let L denote the cokernel of $K_0(\mathcal{C}^w) \to K_0(v\mathcal{C})$; we will prove the theorem by showing that $\lambda(C) = [C]$ induces a map $K_0(w\mathcal{C}) \to L$ inverse to the natural surjection $L \to K_0(w\mathcal{C})$. As $v\mathcal{C}$ and $w\mathcal{C}$ have the same notion of cofibration, it suffices to show that $[C_1] = [C_2]$ in L for every equivalence $f\colon C_1 \to C_2$ in $w\mathcal{C}$. Our hypothesis that f factors as $C_1 \rightarrowtail C \xrightarrow{\sim} C_2$ implies that in $K_0(v\mathcal{C})$ we have $[C_2] = [C] = [C_1] + [C/C_1]$. Since $w\mathcal{C}$ is saturated, it contains $C_1 \rightarrowtail C$. The following lemma implies that C/C_1 is in $\mathcal{C}^w$, so that $[C_2] = [C_1]$ in L. This is the relation we needed to have λ define a map $K_0(w\mathcal{C}) \to L$, proving the theorem. □

II.9.6.1 **Lemma 9.6.1.** *If $B \overset{\sim}{\rightarrowtail} C$ is both a cofibration and a weak equivalence in a Waldhausen category, then $0 \rightarrowtail C/B$ is also a weak equivalence.*

Proof. Apply the Glueing Axiom (W3) to the following diagram. □

$$\begin{array}{ccccc}
0 & \longleftarrow & B & = = = & B \\
\| & & \sim\| & & \sim\| \\
0 & \longleftarrow & B & \rightarrowtail & C.
\end{array}$$

Here is a simple application of the Localization Theorem. Let $(\mathcal{C}, co, v)$ be a Waldhausen category, and G an abelian group. Given a surjective homomorphism $\pi\colon K_0(\mathcal{C}) \to G$, we let $\mathcal{C}^\pi$ denote the Waldhausen subcategory of $\mathcal{C}$ consisting of all objects C such that $\pi([C]) = 0$.

II.9.6.2 **Proposition 9.6.2.** *Assume that every morphism in a Waldhausen category $\mathcal{C}$ factors as the composition of a cofibration and a weak equivalence. There is a short exact sequence*

$$0 \to K_0(\mathcal{C}^\pi) \to K_0(\mathcal{C}) \xrightarrow{\pi} G \to 0.$$

Proof. Define $w\mathcal{C}$ to be the family of all morphisms $A \to B$ in $\mathcal{C}$ with $\pi([A]) = \pi([B])$. This satisfies axiom (W3) because $[C \cup_A B] = [B] + [C] - [A]$, and the factorization hypothesis ensures that the Localization Theorem 9.6 applies to $v \subseteq w$. Since $\mathcal{C}^\pi$ is the category $\mathcal{C}^w$ of w-acyclic objects, this yields exactness at $K_0(\mathcal{C})$. Exactness at $K_0(\mathcal{C}^\pi)$ will follow from the Cofinality theorem 9.4, provided we show that $\mathcal{C}^\pi$ is cofinal. Given an object C, factor the map $C \to 0$ as a cofibration $C \rightarrowtail C''$ followed by a weak equivalence $C'' \xrightarrow{\sim} 0$. If C' denotes C''/C, we compute in G that

$$\pi([C \amalg C']) = \pi([C]) + \pi([C']) = \pi([C] + [C']) = \pi([C'']) = 0.$$

Hence $C \amalg C'$ is in $\mathcal{C}^\pi$, and $\mathcal{C}^\pi$ is cofinal in $\mathcal{C}$. □

II.9.7 **Theorem 9.7** (Approximation Theorem)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between two Waldhausen categories. Suppose also that F satisfies the following conditions:*

(a) A morphism f in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a weak equivalence in $\mathcal{B}$.

(b) Given any map $b\colon F(A) \to B$ in $\mathcal{B}$, there is a cofibration $a\colon A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $b'\colon F(A') \xrightarrow{\sim} B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.

(c) If b is a weak equivalence, we may choose a to be a weak equivalence in $\mathcal{A}$.

Then F induces an isomorphism $K_0\mathcal{A} \cong K_0\mathcal{B}$.

Proof. Applying (b) to $0 \rightarrowtail B$, we see that for every B in $\mathcal{B}$ there is a weak equivalence $F(A') \xrightarrow{\sim} B$. If $F(A) \xrightarrow{\sim} B$ is a weak equivalence, so is $A \xrightarrow{\sim} A'$ by (c). Therefore not only is $K_0\mathcal{A} \to K_0\mathcal{B}$ onto, but the set W of weak equivalence classes of objects of $\mathcal{A}$ is isomorphic to the set of $w.e.$ classes of objects in $\mathcal{B}$.

Now $K_0\mathcal{B}$ is obtained from the free abelian group $\mathbb{Z}[W]$ on the set W by modding out by the relations $[C] = [B] + [C/B]$ corresponding to the cofibrations $B \rightarrowtail C$ in $\mathcal{B}$. Given $F(A) \xrightarrow{\sim} B$, hypothesis (b) yields $A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $F(A') \xrightarrow{\sim} C$ in $\mathcal{B}$. Finally, the Glueing Axiom (W3) applied to

$$\begin{array}{ccccc} 0 & \longleftarrow & F(A) & \rightarrowtail & F(A') \\ \| & & \downarrow\wr & & \downarrow\wr \\ 0 & \longleftarrow & B & \rightarrowtail & C \end{array}$$

implies that the map $F(A'/A) \to C/B$ is a weak equivalence. Therefore the relation $[C] = [B] + [C/B]$ is equivalent to the relation $[A'] = [A] + [A'/A]$ in the free abelian group $\mathbb{Z}[W]$, and already holds in $K_0\mathcal{A}$. This yields $K_0\mathcal{A} \cong K_0\mathcal{B}$, as asserted. □

II.9.7.1 **Approximation for saturated categories 9.7.1.** If $\mathcal{B}$ is saturated (9.1.1), then condition (c) is redundant in the Approximation Theorem, because $F(a)$ is a weak equivalence by (b) and hence by (a) the map a is a *w.e.* in $\mathcal{A}$.

II.9.7.2 **Example 9.7.2.** Recall from Example 9.1.4 that the category $\mathcal{R}(*)$ of based CW complexes is a Waldhausen category. Let $\mathcal{R}_{hf}(*)$ denote the Waldhausen subcategory of all based CW-complexes weakly homotopic to a finite CW complex. The Approximation Theorem applies to the inclusion of $\mathcal{R}_f(*)$ into $\mathcal{R}_{hf}(*)$; this may be seen by using the Whitehead Theorem and elementary obstruction theory. Hence

$$K_0\mathcal{R}_{hf}(*) \cong K_0\mathcal{R}_f(*) \cong \mathbb{Z}.$$

II.9.7.3 **Example 9.7.3.** If $\mathcal{A}$ is an exact category, the Approximation Theorem applies to the inclusion split $\mathbf{Ch}^b \subset \mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Lemma 9.2.4, yielding a more elegant proof that $K_0(\text{split}\,\mathbf{Ch}^b) = K_0(\mathbf{Ch}^b)$. To see this, observe that any chain complex map $f\colon A \to B$ factors through the mapping cylinder complex $\text{cyl}(f)$ as the composite $A \rightarrowtail \text{cyl}(f) \xrightarrow{\sim} B$, and that split $\mathbf{Ch}^b$ is saturated (9.1.1).

II.9.7.4 **Homologically bounded complexes 9.7.4.** Fix an abelian category $\mathcal{A}$, and consider the Waldhausen category $\mathbf{Ch}(\mathcal{A})$ of all chain complexes over $\mathcal{A}$, as in 9.2. We call a complex $C_\bullet$ *homologically bounded* if it is exact almost everywhere, *i.e.,* if only finitely many of the $H_i(C)$ are nonzero. Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of the homologically bounded complexes, and let $\mathbf{Ch}^{hb}_-(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of all bounded above, homologically bounded chain complexes $0 \to C_n \to C_{n-1} \to \cdots$. These are all saturated biWaldhausen categories (see 9.1.1 and 9.1.6). We will prove that

$$K_0\mathbf{Ch}^{hb}(\mathcal{A}) \cong K_0\mathbf{Ch}^{hb}_-(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}) \cong K_0(\mathcal{A}),$$

the final isomorphism being Theorem 9.2.2. From this and Proposition 6.6 it follows that if C is homologically bounded then

$$[C] = \sum (-1)^i [H_i(\mathcal{A})] \quad \text{in} \quad K_0\mathcal{A}.$$

We first claim that the Approximation Theorem 9.7 applies to $\mathbf{Ch}^b \subset \mathbf{Ch}^{hb}_-$, yielding $K_0\mathbf{Ch}^b \cong K_0\mathbf{Ch}^{hb}_-$. If $C_\bullet$ is bounded above then each good truncation $\tau_{\ge n}C = (\cdots C_{n+1} \to Z_n \to 0)$ of C is a bounded subcomplex of C such that $H_i(\tau_{\ge n}C)$ is $H_i(C)$ for $i \ge n$, and 0 for $i < n$. (See [223, 1.2.7].) Therefore $\tau_{\ge n}C \xrightarrow{\sim} C$ is a quasi-isomorphism for small n ($n \ll 0$). If B is a bounded complex, any map $f\colon B \to C$ factors through $\tau_{\ge n}C$ for small n; let A denote the mapping cylinder of $B \to \tau_{\ge n}C$ (see [223, 1.5.8]). Then A is bounded and f factors as the cofibration $B \rightarrowtail A$ composed with the weak equivalence $A \xrightarrow{\sim} \tau_{\ge n}C \xrightarrow{\sim} C$. Thus we may apply the Approximation Theorem, as claimed.

The Approximation Theorem does not apply to $\mathbf{Ch}_-^{hb} \subset \mathbf{Ch}^{hb}$, but rather to $\mathbf{Ch}_+^{hb} \subset \mathbf{Ch}^{hb}$, where the "+" indicates bounded below chain complexes. The argument for this is the same as for $\mathbf{Ch}^b \subset \mathbf{Ch}_-^{hb}$. Since these are biWaldhausen categories, we can apply 9.1.7 to $\mathbf{Ch}_-^{hb}(\mathcal{A})^{op} = \mathbf{Ch}_+^{hb}(\mathcal{A}^{op})$ and $\mathbf{Ch}^{hb}(\mathcal{A})^{op} = \mathbf{Ch}^{hb}(\mathcal{A}^{op})$ to get

$$K_0\mathbf{Ch}_-^{hb}(\mathcal{A}) = K_0\mathbf{Ch}_+^{hb}(\mathcal{A}^{op}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A}^{op}) = K_0\mathbf{Ch}^{hb}(\mathcal{A}).$$

This completes our calculation that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A})$.

II.9.7.5 **Example 9.7.5** (K_0 and Perfect Complexes)**.** Let R be a ring. A chain complex $M_\bullet$ of R-modules is called *perfect* if there is a quasi-isomorphism $P_\bullet \xrightarrow{\sim} M_\bullet$, where $P_\bullet$ is a bounded complex of finitely generated projective R-modules, *i.e.,* $P_\bullet$ is a complex in $\mathbf{Ch}^b(\mathbf{P}(R))$. The perfect complexes form a Waldhausen subcategory $\mathbf{Ch}_{\text{perf}}(R)$ of $\mathbf{Ch}(\textbf{mod-}R)$. We claim that the Approximation Theorem applies to $\mathbf{Ch}^b(\mathbf{P}(R)) \subset \mathbf{Ch}_{\text{perf}}(R)$, so that

$$K_0\mathbf{Ch}_{\text{perf}}(R) \cong K_0\mathbf{Ch}^b\mathbf{P}(R) \cong K_0(R).$$

To see this, consider the intermediate Waldhausen category $\mathbf{Ch}^b_{\text{perf}}$ of bounded perfect complexes. The argument of Example 9.7.4 applies to show that $K_0\mathbf{Ch}^b_{\text{perf}} \cong K_0\mathbf{Ch}_{\text{perf}}(R)$, so it suffices to show that the Approximation Theorem applies to $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}^b_{\text{perf}}$. This is an elementary application of the projective lifting property, which we relegate to Exercise 9.2.

II.9.7.6 **Example 9.7.6** (G_0 and Pseudo-coherent Complexes)**.** Let R be a ring. A complex $M_\bullet$ of R-modules is called *pseudo-coherent* if there exists a quasi-isomorphism $P_\bullet \xrightarrow{\sim} M_\bullet$, where $P_\bullet$ is a bounded below complex $\cdots \to P_{n+1} \to P_n \to 0$ of finitely generated projective R-modules, *i.e.,* $P_\bullet$ is a complex in $\mathbf{Ch}_+(\mathbf{P}(R))$. For example, if R is noetherian we can consider any finitely generated module M as a pseudo-coherent complex concentrated in degree zero. Even if R is not noetherian, it follows from Example 7.1.4 that M is pseudo-coherent as an R-module if and only if it is pseudo-coherent as a chain complex. (See [SGA6, I.2.9].)

The pseudo-coherent complexes form a Waldhausen subcategory $\mathbf{Ch}_{\text{pcoh}}(R)$ of $\mathbf{Ch}(\textbf{mod-}R)$, and the subcategory $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$ of homologically bounded pseudo-coherent complexes is also Waldhausen. Moreover, the above remarks show that $\mathbf{M}(R)$ is a Waldhausen subcategory of both of them. We will see in Ex. 9.7 that the Approximation Theorem applies to the inclusions of $\mathbf{Ch}^b\mathbf{M}(R)$ and $\mathbf{Ch}_+^{hb}\mathbf{P}(R)$ in $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$, so that in particular we have

$$K_0\mathbf{Ch}^b\mathbf{M}(R) \cong K_0\mathbf{Ch}_+^{hb}\mathbf{P}(R) \cong K_0\mathbf{Ch}^{hb}_{\text{pcoh}}(R) \cong G_0(R).$$

Chain complexes with support

Suppose that S is a multiplicatively closed set of central elements in a ring R. Let $\mathbf{Ch}^b_S\mathbf{P}(R)$ denote the Waldhausen subcategory of $\mathcal{C} = \mathbf{Ch}^b\mathbf{P}(R)$ consisting of complexes E such that $S^{-1}E$ is exact, and write $K_0(R \text{ on } S)$ for $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$.

The category $\mathbf{Ch}^b_S\mathbf{P}(R)$ is the category $\mathcal{C}^w$ of the Localization Theorem 9.6, where w is the family of all morphisms $P \to Q$ in $\mathcal{C}$ such that $S^{-1}P \to S^{-1}Q$ is a quasi-isomorphism. By Theorem 9.2.2 we have $K_0(\mathcal{C}) = K_0(R)$. Hence there is an exact sequence

$$K_0(R \text{ on } S) \to K_0(R) \to K_0(w\mathcal{C}) \to 0.$$

II.9.8 **Theorem 9.8.** *The localization* $w\mathcal{C} \to \mathbf{Ch}^b\mathbf{P}(S^{-1}R)$ *induces an injection on* K_0*, so there is an exact sequence*

$$K_0(R \textit{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

Proof. Let $\mathcal{B}$ denote the category of $S^{-1}R$-modules of the form $S^{-1}P$ for P in $\mathbf{P}(R)$. By Example 7.3.2 and Theorem 9.2.2, $K_0\mathbf{Ch}^b(\mathcal{B}) = K_0(\mathcal{B})$ is a subgroup of $K_0(S^{-1}R)$. Therefore the result follows from the following Proposition. □

II.9.8.1 **Proposition 9.8.1.** *The Approximation Theorem 9.7 applies to* $w\mathcal{C} \to \mathbf{Ch}^b(\mathcal{B})$.

Proof. Let P be a complex in $\mathbf{Ch}^b\mathbf{P}(R)$ and $b\colon S^{-1}P \to B$ a map in $\mathcal{B}$. Because each B_n has the form $S^{-1}Q_n$ and each $B_n \to B_{n-1}$ is $s_n^{-1}d_n$ for some $s_n \in S$ and $d_n\colon Q_n \to Q_{n-1}$ such that $d_n d_{n-1} = 0$, B is isomorphic to the localization $S^{-1}Q$ of a bounded complex Q in $\mathbf{P}(R)$, and some sb is the localization of a map $f\colon P \to Q$ in $\mathbf{Ch}^b\mathbf{P}(R)$. Hence f factors as $P \rightarrowtail \mathrm{cyl}(f) \xrightarrow{\sim} Q$. Since b is the localization of f, followed by an isomorphism $S^{-1}Q \cong B$ in $\mathcal{B}$, it factors as desired. □

EXERCISES

EII.9.1 **9.1.** *Retracts of a space.* Fix a CW complex X and let $\mathcal{R}(X)$ be the category of CW complexes Y obtained from X by attaching cells, and having a retraction $Y \to X$. Let $\mathcal{R}_f(X)$ be the subcategory of those Y obtained by attaching only finitely many cells. Let $\mathcal{R}_{\mathrm{fd}}(X)$ be the subcategory of those Y which are finitely dominated, *i.e.,* are retracts up to homotopy of spaces in $\mathcal{R}_f(X)$. Show that $K_0\mathcal{R}_f(X) \cong \mathbb{Z}$ and $K_0\mathcal{R}_{\mathrm{fd}}(X) \cong K_0(\mathbb{Z}[\pi_1 X])$. *Hint*: The cellular chain complex of the universal covering space $\widetilde{Y}$ is a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

EII.9.2 **9.2.** Let R be a ring. Use the projective lifting property to show that the Approximation Theorem applies to the inclusion $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}^b_{\mathrm{perf}}$ of Example 9.7.5. Conclude that $K_0(R) = K_0\mathbf{Ch}_{\mathrm{perf}}(R)$.

If S is a multiplicatively closed set of central elements of R, show that the Approximation Theorem also applies to the inclusion of $\mathbf{Ch}^b_S\mathbf{P}(R)$ in $\mathbf{Ch}_{\mathrm{perf},S}(R)$, and conclude that $K_0(R \text{ on } S) \cong K_0\mathbf{Ch}_{\mathrm{perf},S}(R)$.

EII.9.3 **9.3.** Consider the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Theorem 9.2.2 as a Waldhausen category in which the weak equivalences are the isomorphisms, $\mathrm{iso}\,\mathbf{Ch}^b$, as in

Example 9.1.3. Let $\mathbf{Ch}^b_{\rm acyc}$ denote the subcategory of complexes whose differentials are all zero. Show that $\mathbf{Ch}^b_{\rm acyc}$ is equivalent to the category $\oplus_{n\in\mathbb{Z}}\mathcal{A}$, and that the inclusion in $\mathbf{Ch}^b$ induces an isomorphism

$$K_0(\mathrm{iso}\,\mathbf{Ch}^b) \cong \bigoplus\nolimits_{n\in\mathbb{Z}} K_0(\mathcal{A}).$$

EII.9.4 **9.4.** *Higher Extension categories.* Consider the category $\mathcal{E}_n$ constructed in Example 9.3.2, whose objects are sequences of n cofibrations in $\mathcal{C}$. Show that $\mathcal{E}_n$ is a category with cofibrations, that $\mathcal{E}_n$ is a Waldhausen category when $\mathcal{C}$ is, and in that case

$$K_0(\mathcal{E}_n) \cong \bigoplus\nolimits_{i=1}^{n} K_0(\mathcal{C}).$$

EII.9.5 **9.5.** ([SGA6, IV(1.6)]) Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$, or more generally any exact subcategory of $\mathcal{A}$ closed under extensions and kernels of surjections. Let $\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}^b(\mathcal{A})$ of bounded complexes C with $H_i(C)$ in $\mathcal{B}$ for all i. Show that

$$K_0\mathcal{B} \cong K_0\mathbf{Ch}^b(\mathcal{B}) \cong K_0\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A}).$$

EII.9.6 **9.6.** *Perfect injective complexes.* Let R be a ring and let $\mathbf{Ch}^+_{\rm inj}(R)$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ consisting of perfect bounded below cochain complexes of injective R-modules $0 \to I^m \to I^{m+1}\cdots$. (Recall from Example 9.7.5 that $I^\bullet$ is called *perfect* if it is quasi-isomorphic to a bounded complex $P^\bullet$ of finitely generated projective modules.) Show that

$$K_0\mathbf{Ch}^+_{\rm inj}(R) \cong K_0(R).$$

EII.9.7 **9.7.** *Pseudo-coherent complexes and $G_0(R)$.* Let R be a ring. Recall from Example 9.7.6 that $\mathbf{Ch}^{hb}_{\rm pcoh}(R)$ denotes the Waldhausen category of all homologically bounded pseudo-coherent chain complexes of R-modules. Show that:

(a) The category $\mathbf{M}(R)$ is a Waldhausen subcategory of $\mathbf{Ch}^{hb}_{\rm pcoh}(R)$.

(b) $K_0\mathbf{Ch}_{\rm pcoh}(R) = K_0\mathbf{Ch}_+\mathbf{P}(R) = 0$.

(c) The Approximation Theorem applies to the inclusions of both $\mathbf{Ch}^{hb}_+\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_+\mathbf{P}(R)$ in $\mathbf{Ch}^{hb}_{\rm pcoh}(R)$, and to $\mathbf{Ch}^b\mathbf{M}(R) \subset \mathbf{Ch}^{hb}_-\mathbf{M}(R)$. *Hint:* See 9.7.4.

This shows that $G_0(R) \cong K_0\mathbf{Ch}^{hb}_+\mathbf{P}(R) \cong K_0\mathbf{Ch}^{hb}_{\rm pcoh}(R)$.

EII.9.8 **9.8.** *Pseudo-coherent complexes and G_0^{der}.* Let X be a scheme. A cochain complex $E^\bullet$ of $\mathcal{O}_X$-modules is called *strictly pseudo-coherent* if it is a bounded above complex of vector bundles, and *pseudo-coherent* if it is locally quasi-isomorphic to a strictly pseudo-coherent complex, *i.e.,* if every point $x \in X$ has a neighborhood U, a strictly pseudo-coherent complex $P^\bullet$ on U and a quasi-isomorphism $P^\bullet \to E^\bullet|_U$. Let $\mathbf{Ch}^{hb}_{\rm pcoh}(X)$ denote the Waldhausen category of all pseudo-coherent complexes $E^\bullet$ which are homologically bounded, and set $G_0^{der}(X) = K_0\mathbf{Ch}^{hb}_{\rm pcoh}(X)$; this is the definition used in [SGA6], Exposé IV(2.2).

(a) If X is a noetherian scheme, show that every coherent $\mathcal{O}_X$-module is a pseudo-coherent complex concentrated in degree zero, so that we may consider $\mathbf{M}(X)$ as a Waldhausen subcategory of $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$. Then show that a complex $E^\bullet$ is pseudo-coherent if and only if it is homologically bounded above and all the homology sheaves of $E^\bullet$ are coherent $\mathcal{O}_X$-modules.

(b) If X is a noetherian scheme, show that $G_0(X) \cong G_0^{der}(X)$.

(c) If $X = \mathrm{Spec}(R)$ for a ring R, show that $G_0^{der}(X)$ is isomorphic to the group $K_0\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ of the previous exercise.

EII.9.9

9.9. Let Z be a closed subscheme of X. Let $\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$ denote the subcategory of complexes in $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ which are acyclic on $X - Z$, and define $G_0(X \text{ on } Z)$ to be $K_0\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$.

(a) If X is a noetherian scheme, show that $G_0(Z) \cong G_0(X \text{ on } Z)$.

(b) Show that there is an exact sequence

$$G_0(X \text{ on } Z) \to G_0^{der}(X) \to G_0^{der}(X - Z).$$

EII.9.10

9.10. *Perfect complexes and* K_0^{der}. Let X be a scheme. A complex $E^\bullet$ of $\mathcal{O}_X$-modules is called *strictly perfect* if it is a bounded complex of vector bundles, *i.e.,* a complex in $\mathbf{Ch}^b\mathbf{VB}(X)$. A complex is called *perfect* if it is locally quasi-isomorphic to a strictly perfect complex, *i.e.,* if every point $x \in X$ has a neighborhood U, a strictly perfect complex $P^\bullet$ on U and a quasi-isomorphic $P^\bullet \to E^\bullet|_U$. Write $\mathbf{Ch}_{\mathrm{perf}}(X)$ for the Waldhausen category of all perfect complexes, and $K_0^{der}(X)$ for $K_0\mathbf{Ch}_{\mathrm{perf}}(X)$; this is the definition used in [SGA6], Exposé IV(2.2).

(a) If $X = \mathrm{Spec}(R)$, show that $K_0(R) \cong K_0^{der}(X)$. *Hint:* show that the Approximation Theorem 9.7 applies to $\mathbf{Ch}_{\mathrm{perf}}(R) \subset \mathbf{Ch}_{\mathrm{perf}}(X)$.

(b) If X is noetherian, show that the category $\mathcal{C} = \mathbf{Ch}^{qc}_{\mathrm{perf}}$ of perfect complexes of quasi-coherent $\mathcal{O}_X$-modules also has $K_0(\mathcal{C}) = K_0^{der}(X)$.

(c) If X is a regular noetherian scheme, show that a homologically bounded complex is perfect if and only if it is pseudo-coherent, and conclude that $K_0^{der}(X) \cong G_0(X)$.

(d) Let X be the affine plane with a double origin over a field k, obtained by glueing two copies of $\mathbb{A}^2 = \mathrm{Spec}(k[x,y])$ together; X is a regular noetherian scheme. Show that $K_0\mathbf{VB}(X) = \mathbb{Z}$ but $K_0^{der}(X) = \mathbb{Z} \oplus \mathbb{Z}$. *Hint.* Use the fact that $\mathbb{A}^2 \to X$ induces an isomorphism $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^2)$ and the identification of $K_0^{der}(X)$ with $G_0(X)$ from part (c).

EII.9.11

9.11. Give an example of an exact subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ in which $K_0(\mathcal{B}) \neq K_0\mathbf{Ch}^b(\mathcal{B})$. Here $\mathbf{Ch}^b(\mathcal{B})$ is the Waldhausen category defined in 9.2. Note that $\mathcal{B}$ cannot be closed under kernels of surjections, by Theorem 9.2.2.

EII.9.12 **9.12.** *Finitely dominated complexes.* Let $\mathcal{C}$ be a small exact category, closed under extensions and kernels of surjections in an ambient abelian category $\mathcal{A}$ (Definition 7.0.1). A bounded below complex $C_\bullet$ of objects in $\mathcal{C}$ is called *finitely dominated* if there is a bounded complex $B_\bullet$ and two maps $C_\bullet \to B_\bullet \to C_\bullet$ whose composite $C_\bullet \to C_\bullet$ is chain homotopic to the identity. Let $\mathbf{Ch}_+^{\mathrm{fd}}(\mathcal{C})$ denote the category of finitely dominated chain complexes of objects in $\mathcal{C}$. (If $\mathcal{C}$ is abelian, this is the category $\mathbf{Ch}_+^{hb}(\mathcal{C})$ of Example 9.7.4.)

(a) Let e be an idempotent endomorphism of an object C, and let $\mathrm{tel}(e)$ denote the nonnegative complex

$$\cdots \xrightarrow{e} C \xrightarrow{1-e} C \xrightarrow{e} C \to 0.$$

Show that $\mathrm{tel}(e)$ is finitely dominated.

(b) Let $\hat{\mathcal{C}}$ denote the idempotent completion 7.3 of $\mathcal{C}$. Show that there is a map from $K_0(\hat{\mathcal{C}})$ to $K_0\mathbf{Ch}_+^{\mathrm{fd}}(\mathcal{C})$ sending $[(C,e)]$ to $[\mathrm{tel}(e)]$.

(c) Show that the map in (b) induces an isomorphism $K_0(\hat{\mathcal{C}}) \cong K_0\mathbf{Ch}_+^{\mathrm{fd}}(\mathcal{C})$.

EII.9.13 **9.13.** Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. Show that $K_0\mathbf{H}_S(R) \cong K_0(R \text{ on } S)$, and compare Cor. 7.7.4 to Theorem 9.8.

EII.9.14 **9.14.** (Grayson's Trick) Let $\mathcal{B}$ be a Waldhausen subcategory of $\mathcal{C}$ closed under extensions. Suppose that $\mathcal{B}$ is cofinal in $\mathcal{C}$, so that $K_0(\mathcal{B}) \subseteq K_0(\mathcal{C})$ by the Cofinality Theorem 9.4. Define an equivalence relation $\sim$ on objects of $\mathcal{C}$ by $C \sim C'$ if there are B, B' in $\mathcal{B}$ with $C \amalg B \cong C' \amalg B'$.
(a) Given a cofibration sequence $C' \rightarrowtail C \twoheadrightarrow C''$ in $\mathcal{C}$, use the proof of the Cofinality Theorem 7.2 to show that $C \sim C' \amalg C''$.
(b) Conclude that $C \sim C'$ if and only if $[C] - [C']$ is in $K_0(\mathcal{B}) \subseteq K_0(\mathcal{C})$. (See Remark 9.4.1.)
(c) For each sequence $C_1, \dots, C_n$ of objects in $\mathcal{C}$ such that $[C_1] = \cdots = [C_n]$ in $K_0(\mathcal{C})/K_0(\mathcal{B})$, show that there is a C' in $\mathcal{C}$ so that each $C_i \amalg C'$ is in $\mathcal{B}$.
(d) If $K_0(\mathcal{B}) = K_0(\mathcal{C})$, show that $\mathcal{B}$ is *strictly cofinal* in $\mathcal{C}$, meaning that for every C in $\mathcal{C}$ there is a B in $\mathcal{B}$ so that $C \amalg B$ is in $\mathcal{B}$.

EII.9.15 **9.15.** *Triangulated Categories.* If $\mathcal{C}$ is a triangulated category, the Grothendieck group $k(\mathcal{C})$ is the free abelian group on the objects, modulo the relation that $[A] - [B] + [C] = 0$ for every triangle $A \to B \to C \to A[1]$. (a) If $\mathcal{B}$ is an additive category, regarded as a split exact category (7.1.2), show that $K_0(\mathcal{B})$ is isomorphic to $k(\mathbf{K}^b\mathcal{B})$. (b) If $\mathcal{B}$ is an exact subcategory of an abelian category, closed under kernels, show that $K_0(\mathcal{B})$ is isomorphic to $k(\mathbf{K}^b\mathcal{B})$. *Hint.* See 9.2.2. (c) If $\mathcal{C}$ has a bounded t-structure with heart $\mathcal{A}$ [22], show that $K_0(\mathcal{A}) \cong k(\mathcal{C})$.

Appendix. Localizing by calculus of fractions

If $\mathcal{C}$ is a category and S is a collection of morphisms in $\mathcal{C}$, then the *localization of* $\mathcal{C}$ *with respect to* S is a category $\mathcal{C}_S$, together with a functor $\text{loc}\colon \mathcal{C} \to \mathcal{C}_S$ such that

(1) For every $s \in S$, loc(s) is an isomorphism

(2) If $F\colon \mathcal{C} \to \mathcal{D}$ is any functor sending S to isomorphisms in $\mathcal{D}$, then F factors uniquely through $\text{loc}\colon \mathcal{C} \to \mathcal{C}_S$.

Example. We may consider any ring R as an additive category $\mathcal{R}$ with one object. If S is a central multiplicative subset of R, there is a ring $S^{-1}R$ obtained by localizing R at S, and the corresponding category is $\mathcal{R}_S$. The useful fact that every element of the ring $S^{-1}R$ may be written in standard form $s^{-1}r = rs^{-1}$ generalizes to morphisms in a localization $\mathcal{C}_S$, provided that S is a "locally small multiplicative system" in the following sense.

II.A.1 **Definition A.1.** A collection S of morphisms in $\mathcal{C}$ is called a *multiplicative system* if it satisfies the following three self-dual axioms:

(FR1) S is closed under composition and contains the identity morphisms 1_X of all objects X of $\mathcal{C}$. That is, S forms a subcategory of $\mathcal{C}$ with the same objects.

(FR2) (Ore condition) (a) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathcal{C}$ there is a commutative diagram in $\mathcal{C}$ with $s \in S$:

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y. \end{array}$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f and s.") (b) The dual statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t and g") is also valid.

(FR3) (Cancellation) If $f, g\colon X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:

(a) $sf = sg$ for some $s\colon Y \to Z$ in S

(b) $ft = gt$ for some $t\colon W \to X$ in S.

We say that S is a *right multiplicative system* if it satisfies (FR1) and (FR2a), and if (FR3a) implies (FR3b) Left multiplicative systems are defined dually.

II.A.1.1 **Example A.1.1.** If S is a multiplicatively closed subset of a ring R, then S forms a multiplicative system if and only if S is a "2–sided denominator set." (One-sided denominator sets (left and right) correspond to left and right multiplicative systems.) The localization of rings at denominator sets was the original application of Øystein Ore.

II.A.1.2 **Example A.1.2.** (Gabriel) Let $\mathcal{B}$ be a Serre subcategory (see §6) of an abelian category $\mathcal{A}$, and let S be the collection of all $\mathcal{B}$-*isos*, *i.e.,* those maps f such that $\ker(f)$ and $\operatorname{coker}(f)$ is in $\mathcal{B}$. Then S is a multiplicative system in $\mathcal{A}$; the verification of axioms (FR2), (FR3) is a pleasant exercise in diagram chasing. In this case, $\mathcal{A}_S$ is the quotient abelian category $\mathcal{A}/\mathcal{B}$ discussed in the Localization Theorem 6.4.

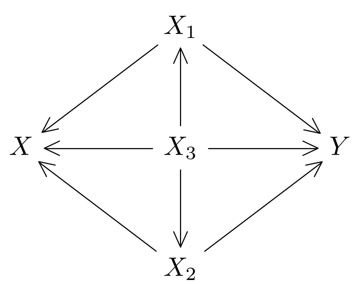
We would like to say that every morphism $X \to Z$ in $\mathcal{C}_S$ is of the form fs^{-1}. However, the issue of whether or this construction makes sense (in our universe) involves delicate set-theoretic questions. The following notion is designed to avoid these set-theoretic issues.

We say that S is *locally small* (on the left) if for each X in $\mathcal{C}$ there is a *set* S_X of morphisms $X' \xrightarrow{s} X$ in S such that every map $Y \to X$ in S factors as $Y \to X' \xrightarrow{s} X$ for some $s \in S_X$.

II.A.2 **Definition A.2** (Fractions)**.** A (left) *fraction* between X and Y is a chain in $\mathcal{C}$ of the form:

$$fs^{-1}: \quad X \xleftarrow{s} X_1 \xrightarrow{f} Y, \qquad s \in S.$$

Call fs^{-1} *equivalent* to $X \leftarrow X_2 \to Y$ just in case there is a chain $X \leftarrow X_3 \to Y$ fitting into a commutative diagram in $\mathcal{C}$:



It is easy to see that this is an equivalence relation. Write $\operatorname{Hom}_S(X, Y)$ for the equivalence classes of such fractions between X and Y. ($\operatorname{Hom}_S(X, Y)$ is a set when S is locally small.)

We cite the following theorem without proof from [223, 10.3.7], relegating its routine proof to Exercises A.1 and A.2.

II.A.3 **Gabriel-Zisman Theorem A.3.** *Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Then the localization $\mathcal{C}_S$ of $\mathcal{C}$ exists, and may be constructed by the following "calculus" of left fractions.*

$\mathcal{C}_S$ has the same objects as $\mathcal{C}$, but $\operatorname{Hom}_{\mathcal{C}_S}(X, Y)$ is the set of equivalence classes of chains $X \leftarrow X' \to Y$ with $X' \to X$ in S, and composition is given by the Ore condition. The functor loc: $\mathcal{C} \to \mathcal{C}_S$ sends $X \to Y$ to the chain $X \xleftarrow{=} X \to Y$, and if $s\colon X \to Y$ is in S its inverse is represented by $Y \leftarrow X \xrightarrow{=} X$.

II.A.3.1 **Corollary A.3.1.** *Two parallel arrows $f, g\colon X \to Y$ become identified in $\mathcal{C}_S$ if and only if the conditions of (FR3) hold.*

II.A.3.2 **Corollary A.3.2.** *Suppose that $\mathcal{C}$ has a zero object, and that S is a multiplicative system in $\mathcal{C}$. Assume that S is saturated in the sense that if s and st are in S then so is t. Then for every X in $\mathcal{C}$:*

$$loc(X) \cong 0 \Leftrightarrow \textit{The zero map } \; X \xrightarrow{0} X \; \textit{ is in } \; S.$$

Proof. Since loc(0) is a zero object in $\mathcal{C}_S$, $\mathrm{loc}(X) \cong 0$ if and only if the parallel maps $0, 1 \colon X \to X$ become identified in $\mathcal{C}_S$. □

Now let $\mathcal{A}$ be an abelian category, and $\mathbf{C}$ a full subcategory of the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$, closed under translation and the formation of mapping cones. Let $\mathbf{K}$ be the quotient category of $\mathbf{C}$, obtained by identifying chain homotopic maps in $\mathbf{C}$. Let Q denote the family of (chain homotopy equivalence classes of) quasi-isomorphisms in $\mathbf{C}$. The following result states that Q forms a multiplicative system in $\mathbf{K}$, so that we can form the localization $\mathbf{K}_Q$ of $\mathbf{K}$ with respect to Q by the calculus of fractions.

II.A.4 **Lemma A.4.** *The family Q of quasi-isomorphisms in the chain homotopy category $\mathbf{K}$ forms a multiplicative system.*

Proof. (FR1) is trivial. To prove (FR2), consider a diagram $X \xrightarrow{u} Y \xleftarrow{s} Z$ with $s \in Q$. Set $C = \mathrm{cone}(s)$, and observe that C is acyclic. If $f \colon Y \to C$ is the natural map, set $W = \mathrm{cone}(fu)$, so that the natural map $t \colon W \to X[-1]$ is a quasi-isomorphism. Now the natural projections from each $W_n = Z_{n-1} \oplus Y_n \oplus X_{n-1}$ to Z_{n-1} form a morphism $v \colon W \to Z[-1]$ of chain complexes making the following diagram commute:

$$\begin{array}{ccccccccc}
 & & X & \xrightarrow{fu} & C & \longrightarrow & W & \xrightarrow{t} & X[-1] \\
 & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle =} & & {\scriptstyle v}\downarrow & & \downarrow \\
Z & \xrightarrow{s} & Y & \xrightarrow{f} & C & \longrightarrow & Z[-1] & \xrightarrow{s[-1]} & Y[-1].
\end{array}$$

Applying $X \mapsto X[1]$ to the right square gives the first part of (FR2); the second part is dual and is proven similarly.

To prove (FR3), we suppose given a quasi-isomorphism $s \colon Y \to Y'$ and set $C = \mathrm{cone}(s)$; from the long exact sequence in homology we see that C is acyclic. Moreover, if v denotes the map $C[1] \to Y$ then there is an exact sequence:

$$\mathrm{Hom}_{\mathbf{K}}(X, C[1]) \xrightarrow{v} \mathrm{Hom}_{\mathbf{K}}(X, Y) \xrightarrow{s} \mathrm{Hom}_{\mathbf{K}}(X, Y')$$

(see WHomo [223, 10.2.8]). Given f and g, set $h = f - g$. If $sh = 0$ in $\mathbf{K}$, there is a map $w \colon X \to C[1]$ such that $h = vw$. Setting $X' = \mathrm{cone}(w)[1]$, the natural map $X' \xrightarrow{t} X$ must be a quasi-isomorphism because C is acyclic. Moreover, $wt = 0$, so we have $ht = vwt = 0$, *i.e.*, $ft = gt$. □

II.A.5 **Definition A.5.** Let $\mathbf{C} \subset \mathbf{Ch}(\mathcal{A})$ be a full subcategory closed under translation and the formation of mapping cones. The *derived category* of $\mathbf{C}$, $\mathbf{D}(\mathbf{C})$, is defined to be the localization $\mathbf{K}_Q$ of the chain homotopy category $\mathbf{K}$ at the multiplicative system Q of quasi-isomorphisms. The *derived category* of $\mathcal{A}$ is $\mathbf{D}(\mathcal{A}) = \mathbf{D}(\mathbf{Ch}(\mathcal{A}))$.

Another application of calculus of fractions is Verdier's formation of quotient triangulated categories by thick subcategories. We will use Rickard's definition of thickness, which is equivalent to Verdier's.

II.A.6 **Definition A.6.** Let $\mathbf{K}$ be any triangulated category (see [223, 10.2.1]). A full additive subcategory $\mathcal{E}$ of $\mathbf{K}$ is called *thick* if:

(1) In any distinguished triangle $A \to B \to C \to A[1]$, if two out of A, B, C are in $\mathcal{E}$ then so is the third.

(2) if $A \oplus B$ is in $\mathcal{E}$ then both A and B are in $\mathcal{E}$.

If $\mathcal{E}$ is a thick subcategory of $\mathbf{K}$, we can form a quotient triangulated category $\mathbf{K}/\mathcal{E}$, parallel to Gabriel's construction of a quotient abelian category in A.1.2. That is, $\mathbf{K}/\mathcal{E}$ is defined to be $S^{-1}\mathbf{K}$, where S is the family of maps whose cone is in $\mathcal{E}$. By Ex. A.6, S is a saturated multiplicative system of morphisms, so $S^{-1}\mathbf{K}$ can be constructed by the calculus of fractions (Theorem A.3).

To justify this definition, note that because S is saturated it follows from A.3.2 and A.6(2) that: (a) $X \cong 0$ in $\mathbf{K}/\mathcal{E}$ if and only if X is in $\mathcal{E}$, and (b) a morphism $f : X \to Y$ in $\mathbf{K}$ becomes an isomorphism in $\mathbf{K}/\mathcal{E}$ if and only if f is in S.

We conclude with a more recent application, due to M. Schlichting [164].

A.7 **Definition A.7.** Let $A \subset \mathcal{B}$ be exact categories, with $\mathcal{A}$ closed under extensions, admissible subobjects and admissible quotients in $\mathcal{B}$. We say that $\mathcal{A}$ is *right filtering* in $\mathcal{B}$ if every map from an object B of $\mathcal{B}$ to an object of $\mathcal{A}$ factors through an admissible epi $B \twoheadrightarrow A$ with A in $\mathcal{A}$.

A morphism of $\mathcal{B}$ is called a *weak isomorphism* if it is a finite composition of admissible monics with cokernel in $\mathcal{A}$ and admissible epis with kernel in $\mathcal{A}$. We write $\mathcal{B}/\mathcal{A}$ for the localization of $\mathcal{B}$ with respect to the weak isomorphisms.

II.A.7.1 **Proposition A.7.1.** *If $\mathcal{A}$ is right filtering in $\mathcal{B}$, then the class Σ of weak isomorphisms is a right multiplicative system. By the Gabriel-Zisman Theorem A.3, $\mathcal{B}/\mathcal{A}$ may be constructed using a calculus of right fractions.*

Proof. By construction, weak isomorphisms are closed under composition, so (FR1) holds. Given an admissible $t : Z \twoheadrightarrow Y$ in $\mathcal{B}$ with kernel in $\mathcal{A}$ and $g : X \to Y$, the base change $s : Z \times_Y X \twoheadrightarrow X$ is an admissible epi in Σ and the canonical map $Z \times_Y X \to Z \to Y$ equals gs. Given an admissible monic $t : Z \rightarrowtail Y$ with kernel A' in $\mathcal{A}$, the map $X \to A'$ factors throught an admissible epi $q : X \twoheadrightarrow A$ with A in $\mathcal{A}$ because $\mathcal{A} \subset \mathcal{B}$ is right filtering; the kernel $W \rightarrowtail X$ of q is in Σ and $W \to X \to Y$ factors through a universal map $W \to Z$. An arbitrary t in Σ is a finite composition of these two types, so by induction on the length of t, we see that Σ satisfies (FR2a).

Finally, suppose that $sf = sg$ for some weak isomorphism $s : Y \to Z$ and $f, g : X \to Y$. If s is an admissible monic, then $f = g$ already. If s is an admissible epi, $f - g$ factors through the kernel $A \rightarrowtail Y$ of s. Because $\mathcal{A}$ is right filtering in $\mathcal{B}$, there is an admissible exact sequence $W \overset{t}{\rightarrowtail} X \twoheadrightarrow A$ with A in $\mathcal{A}$, such that $f - g$ factors through A. Hence t is a weak equivalence and $ft = gt$. As before, induction shows that (a) implies (b) in axiom (FR3). □

EXERCISES

EII.A.1 **A.1.** Show that the construction of the Gabriel-Zisman Theorem II.A.3 A.3 makes $\mathcal{C}_S$ into a category by showing that composition is well-defined and associative.

EII.A.2 **A.2.** If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending S to isomorphisms, show that F factors uniquely through the Gabriel-Zisman category $\mathcal{C}_S$ of the previous exercise as $\mathcal{C} \to \mathcal{C}_S \to \mathcal{D}$. This proves the Gabriel-Zisman Theorem II.A.3 A.3, that $\mathcal{C}_S$ is indeed the localization of $\mathcal{C}$ with respect to S.

EII.A.3 **A.3.** Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and let S be a multiplicative system in $\mathcal{C}$ such that $S \cap \mathcal{B}$ is a multiplicative system in $\mathcal{B}$. Assume furthermore that one of the following two conditions holds:

(a) Whenever $s : C \to B$ is in S with B in $\mathcal{B}$, there is a morphism $f : B' \to C$ with B' in $\mathcal{B}$ such that $sf \in S$

(b) Condition (a) with the arrows reversed, for $s : B \to C$.

Show that the natural functor $\mathcal{B}_S \to \mathcal{C}_S$ is fully faithful, so that $\mathcal{B}_S$ can be identified with a full subcategory of $\mathcal{C}_S$.

EII.A.4 **A.4.** Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor between two abelian categories, and let S be the family of morphisms s in $\mathbf{Ch}(\mathcal{A})$ such that $F(s)$ is a quasi-isomorphism. Show that S is a multiplicative system in $\mathbf{Ch}\mathcal{A}$.

EII.A.5 **A.5.** Suppose that $\mathbf{C}$ is a subcategory of $\mathbf{Ch}(\mathcal{A})$ closed under translation and the formation of mapping cones, and let Σ be the family of all chain homotopy equivalences in $\mathbf{C}$. Show that the localization $\mathbf{C}_\Sigma$ is the quotient category $\mathbf{K}$ of $\mathbf{C}$ described before Lemma II.A.4 A.4. Conclude that the derived category $\mathbf{D}(\mathbf{C})$ is the localization of $\mathbf{C}$ at the family of all quasi-isomorphisms. *Hint:* If two maps $f_1, f_2 : X \to Y$ are chain homotopic then they factor through a common map $f : \text{cyl}(X) \to Y$ out of the mapping cylinder of X.

EII.A.6 **A.6.** Let $\mathcal{E}$ be a thick subcategory of a triangulated category $\mathbf{K}$, and S the morphisms whose cone is in $\mathcal{E}$, as in II.A.6 A.6. Show that S is a multiplicative system of morphisms. Then show that S is saturated in the sense of II.A.3.2 A.3.2.

Chapter III

K_1 and K_2 of a ring

Let R be an associative ring with unit. In this chapter, we introduce the classical definitions of the groups $K_1(R)$ and $K_2(R)$. These definitions use only linear algebra and elementary group theory, as applied to the groups $GL(R)$ and $E(R)$. We also define relative groups for K_1 and K_2, as well as the negative K-groups $K_{-n}(R)$ and the Milnor K-groups $K_n^M(R)$.

In the next chapter we will give another definition: $K_n(R) = \pi_n K(R)$ for all $n \geq 0$, where $K(R)$ is a certain topological space built using the category $\mathbf{P}(R)$ of finitely generated projective R-modules. We will then have to prove that these topologically defined groups agree with the definition of $K_0(R)$ in chapter II, as well as with the classical constructions of $K_1(R)$ and $K_2(R)$ in this chapter.

1 The Whitehead Group K_1 of a ring

Let R be an associative ring with unit. Identifying each $n \times n$ matrix g with the larger matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives an embedding of $GL_n(R)$ into $GL_{n+1}(R)$. The union of the resulting sequence

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \cdots$$

is called the *infinite general linear group* $GL(R)$.

Recall that the commutator subgroup $[G, G]$ of a group G is the subgroup generated by its commutators $[g, h] = ghg^{-1}h^{-1}$. It is always a normal subgroup of G, and has a universal property: the quotient $G/[G, G]$ is an abelian group, and every homomorphism from G to an abelian group factors through $G/[G, G]$.

III.1.1 **Definition 1.1.** $K_1(R)$ is the abelian group $GL(R)/[GL(R), GL(R)]$.

The universal property of $K_1(R)$ is this: every homomorphism from $GL(R)$ to an abelian group must factor through the natural quotient $GL(R) \to K_1(R)$. Depending upon our situation, we will sometimes think of $K_1(R)$ as an additive group, and sometimes as a multiplicative group.

A ring map $R \to S$ induces a natural map from $GL(R)$ to $GL(S)$, and hence from $K_1(R)$ to $K_1(S)$. That is, K_1 is a functor from rings to abelian groups.

III.1.1.1 **Example 1.1.1** (SK_1)**.** If R happens to be commutative, the determinant of a matrix provides a group homomorphism from $GL(R)$ onto the group $R^\times$ of units of R. It is traditional to write $SK_1(R)$ for the kernel of the induced surjection $\det\colon K_1(R) \to R^\times$. The *special linear group* $SL_n(R)$ is the subgroup of $GL_n(R)$ consisting of matrices with determinant 1, and $SL(R)$ is their union. Since the natural inclusion of the units $R^\times$ in $GL(R)$ as $GL_1(R)$ is split by the homomorphism $\det\colon GL(R) \to R^\times$, we see that $GL(R)$ is the semidirect product $SL(R) \rtimes R^\times$, and there is a direct sum decomposition: $K_1(R) = R^\times \oplus SK_1(R)$.

III.1.1.2 **Example 1.1.2.** If F is a field, then $K_1(F) = F^\times$. We will see this below (see Lemma III.1.2.2 1.2.2 and Example III.1.3.1 1.3.1 below), but it is more fun to deduce this from an 1899 theorem of L. E. J. Dickson, that $SL_n(F)$ is the commutator subgroup of both $GL_n(F)$ and $SL_n(F)$, with only two exceptions: $GL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2) \cong \Sigma_3$, which has order 6, and $GL_2(\mathbb{F}_3)$, which has center $\{\pm I\}$ and quotient $PGL_2(\mathbb{F}_3) = GL_2(\mathbb{F}_3)/\{\pm I\}$ isomorphic to Σ_4.

III.1.1.3 **Example 1.1.3.** If R is the product $R' \times R''$ of two rings, then $K_1(R) = K_1(R') \oplus K_1(R'')$. Indeed, $GL(R)$ is the product $GL(R') \times GL(R'')$, and the commutator subgroup decomposes accordingly.

III.1.1.4 **Example 1.1.4.** For all n, the Morita equivalence between R and $S = M_n(R)$ (see II.II.2.7.2 2.7.2) produces an isomorphism between $M_{mn}(R) = \mathrm{End}_R(R^m \otimes R^n)$ and $M_m(M_n(R)) = \mathrm{End}_S(S^m)$. It is easy to see that the resulting isomorphism of units $GL_{mn}(R) \cong GL_m(M_n(R))$ is compatible with stablization in m, giving an isomorphism $GL(R) \cong GL(M_n(R))$. Hence $K_1(R) \cong K_1(M_n(R))$.

We will show that the commutator subgroup of $GL(R)$ is the subgroup $E(R)$ generated by "elementary" matrices. These are defined as follows.

III.1.2 **Definition 1.2.** If $i \neq j$ are distinct positive integers and $r \in R$ then the *elementary matrix* $e_{ij}(r)$ is the matrix in $GL(R)$ which has 1 in every diagonal spot, has r in the (i,j)-spot, and is zero elsewhere.

$E_n(R)$ denotes the subgroup of $GL_n(R)$ generated by all elementary matrices $e_{ij}(r)$ with $1 \le i,j \le n$, and the union $E(R)$ of the $E_n(R)$ is the subgroup of $GL(R)$ generated by all elementary matrices.

III.1.2.1 **Example 1.2.1.** A *signed permutation matrix* is one which permutes the standard basis $\{e_i\}$ up to sign, *i.e.*, it permutes the set $\{\pm e_1, \dots, \pm e_n\}$. The following signed permutation matrix belongs to $E_2(R)$:

$$\bar{w}_{12} := e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By changing the subscripts, we see that the signed permutation matrices $\bar{w}_{ij}$ belong to $E_n(R)$ for $n \ge i,j$. Since the products $\bar{w}_{jk}\bar{w}_{ij}$ correspond to cyclic permutations of 3 basis elements, every matrix corresponding to an even permutation of basis elements belongs to $E_n(R)$. Moreover, if $g \in GL_n(R)$ then we see by Ex. I.EI.1.11 1.11 that $E_{2n}(R)$ contains the matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$.

III.1.2.2 **1.2.2.** If we interpret matrices as linear operators on column vectors, then $e_{ij}(r)$ is the elementary row operation of adding r times row j to row i, and $E_n(R)$ is the subset of all matrices in $GL_n(R)$ which may be reduced to the identity matrix using only these row operations. The quotient set $GL_n(R)/E_n(R)$ measures the obstruction to such a reduction.

If F is a field this obstruction is $F^\times$, and is measured by the determinant. That is, $E_n(F) = SL_n(F)$ for all $n \geq 1$. Indeed, standard linear algebra shows that every matrix of determinant 1 is a product of elementary matrices.

III.1.2.3 **Remark 1.2.3** (Surjections)**.** If I is an ideal of R, each homomorphism $E_n(R) \to E_n(R/I)$ is onto, because the generators $e_{ij}(r)$ of $E_n(R)$ map onto the generators $e_{ij}(\bar{r})$ of $E_n(R/I)$. In contrast, the maps $GL_n(R) \to GL_n(R/I)$ are usually not onto unless I is a radical ideal (Ex. I.1.12(iv)). Indeed, the obstruction is measured by the group $K_0(I) = K_0(R, I)$; see Proposition 2.3 below.

III.1.2.4 **Division rings 1.2.4.** The same linear algebra that we invoked for fields shows that if D is a division ring (a "skew field") then every invertible matrix may be reduced to a diagonal matrix $\text{diag}(r, 1, ..., 1)$, and that $E_n(D)$ is a normal subgroup of $GL_n(D)$. Thus each $GL_n(D)/E_n(D)$ is a quotient group of the nonabelian group $D^\times$. Dieudonné proved in 1943 that in fact $GL_n(D)/E_n(D) = D^\times/[D^\times, D^\times]$ for all $n > 1$ (except for $n = 2$ when $D = \mathbb{F}_2$). In particular, $K_1(D) = GL_n(D)/E_n(D)$ for all $n \geq 3$. A proof of this result is sketched in Exercise 1.2 below.

If D is a d-dimensional division algebra over its center F (which must be a field), then $d = n^2$ for some integer n, and n is called the *Schur index* of D. Indeed, there are (many) field extensions E of F such that $D \otimes_F E \cong M_n(E)$; such a field is called a *splitting field* for D. For example, any maximal subfield $E \subset D$ has $[E : F] = n$ and is a splitting field.

For any splitting field E, the inclusions $D \hookrightarrow M_n(E)$ and $M_r(D) \hookrightarrow M_{nr}(E)$ induce maps $D^\times \subset GL_n(E) \xrightarrow{\det} E^\times$ and $GL_r(D) \to GL_{nr}(E) \xrightarrow{\det} E^\times$ whose image lies in the subgroup $F^\times$ of $E^\times$. (Indeed, if E/F is Galois, the image is fixed by the Galois group $Gal(E/F)$ and hence lies in $F^\times$.) The induced maps $D^\times \to F^\times$ and $GL_r(D) \to F^\times$ are called the *reduced norms* N_{red} for D, and are independent of E. For example, if $D = \mathbb{H}$ is the quaternions then $F = \mathbb{R}$, and $N_{\text{red}}(t+ix+jy+kz) = t^2+x^2+y^2+z^2$. It is easy to check here that N_{red} induces $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times \subset \mathbb{R}^\times$.

Now if A is any central simple F-algebra then $A \cong M_r(D)$ for some D, and $M_m(A) \cong M_{mr}(D)$. The induced maps $N_{\text{red}} : GL_m(A) \cong GL_{mr}(D) \to F^\times$ are sometimes called the reduced norm for A, and the kernel of this map is written as $SL_m(A)$. We define $SK_1(A)$ to be the kernel of the induced map

$$N_{\text{red}} \colon K_1(A) \cong K_1(D) \to K_1(F) = F^\times.$$

In 1950 S. Wang showed that $SK_1(D) = 1$ if F is a number field. For every real embedding $\sigma : F \hookrightarrow \mathbb{R}$, $D \otimes_F \mathbb{R}$ is a matrix algebra over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; it is called *unramified* in case $\mathbb{H}$ occurs. The Hasse-Schilling-Maass norm theorem describes the image of the reduced norm, and hence $K_1(D)$:

$$K_1(D) \xrightarrow[N_{\text{red}}]{\simeq} \{x \in F^\times : \sigma(x) > 0 \text{ in } \mathbb{R} \text{ for all ramified } \sigma\}.$$

Wang also showed that $SK_1(D) = 1$ if the Schur index of D is squarefree. In 1976 V. Platanov produced the first examples of a D with $SK_1(D) \neq 1$, by constructing a map from $SK_1(D)$ to a subquotient of the Brauer group $\mathrm{Br}(F)$. We will see in 1.7.2 below that the group $SK_1(D)$ has exponent n.

III.1.2.5 **Remark 1.2.5.** There is no a priori reason to believe that the subgroups $E_n(R)$ are normal, except in special cases. For example, we shall show in Ex. 1.3 that if R has stable range $d+1$ then $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. Vaserstein proved in [205] that $K_1(R) = GL_n(R)/E_n(R)$ for all $n \geq d+2$.

If R is commutative, we can do better: $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq 3$. This theorem was proven by A. Suslin in [179]; we give Suslin's proof in Ex. 1.9. Suslin also gave examples of Dedekind domains for which $E_2(R)$ is not normal in $GL_2(R)$ in [178]. For noncommutative rings, the $E_n(R)$ are only known to be normal for large n, and only then when the ring R has finite stable range in the sense of Ex. I.1.5; see Ex. 1.3 below.

III.1.3 **Commutators 1.3.** Here are some easy-to-check formulas for multiplying elementary matrices. Fixing the indices, we have $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$, and $e_{ij}(-r)$ is the inverse of $e_{ij}(r)$. The commutator of two elementary matrices is easy to compute and simple to describe (unless $j = k$ and $i = \ell$):

$$[e_{ij}(r), e_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ e_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases} \qquad (1.3.1)$$ III.1.3.1

Recall that a group is called *perfect* if $G = [G, G]$. If a subgroup H of G is perfect, then $H \subseteq [G, G]$. The group $E(R)$ is perfect, as are most of its finite versions:

III.1.3.2 **Lemma 1.3.2.** *If $n \geq 3$ then $E_n(R)$ is a perfect group.*

Proof. If i, j, k are distinct then $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$. □

We know from Example 1.1.2 that $E_2(R)$ is not always perfect; in fact $E_2(\mathbb{F}_2)$ and $E_2(\mathbb{F}_3)$ are solvable groups.

Rather than become enmeshed in technical issues, it is useful to "stabilize" by increasing the size of the matrices we consider. One technical benefit of stability is given in Ex. 1.4. The following stability result was proven by J.H.C. Whitehead in the 1950 paper [227], and in some sense is the origin of K-theory.

III.1.3.3 **Whitehead's Lemma 1.3.5.** *$E(R)$ is the commutator subgroup of $GL(R)$. Hence $K_1(R) = GL(R)/E(R)$.*

Proof. The commutator subgroup contains $E(R)$ by Lemma 1.3.2. Conversely, every commutator in $GL_n(R)$ can be expressed as a product in $GL_{2n}(R)$:

$$[g,h] = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}. \qquad (1.3.4)$$ III.1.3.4

But we saw in Example 1.2.1 that each of these terms is in $E_{2n}(R)$. □

III.1.3.5 **Example 1.3.5.** If F is a field then $K_1(F) = F^\times$, because we have already seen that $E(F) = SL(F)$. Similarly, if R is a Euclidean domain such as $\mathbb{Z}$ or $F[t]$ then it is easy to show that $SK_1(R) = 1$ and hence $K_1(R) = R^\times$; see Ex. 1.5. In particular, $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ and $K_1(F[t]) = F^\times$.

To get a feeling for the non-commutative situation, suppose that D is a division ring. Diedonné's calculation of $GL_n(D)/E_n(D)$ (described in 1.2.4 and Ex. 1.2) gives an isomorphism $K_1(D) \cong D^\times/[D^\times, D^\times]$.

III.1.3.6 **Example 1.3.6.** If F is a finite field extension of $\mathbb{Q}$ (a number field) and R is an integrally closed subring of F, then Bass, Milnor and Serre proved in [19, 4.3] that $SK_1(R) = 0$, so that $K_1(R) \cong R^\times$. We mention that if R is finitely generated over $\mathbb{Z}$ then, by the Dirichlet Unit Theorem, $K_1(R) = R^\times$ is a finitely generated abelian group isomorphic to $\mu(F) \oplus \mathbb{Z}^{s-1}$, where $\mu(F)$ denotes the cyclic group of all roots of unity in F and s is the number of "places at infinity" for R.

III.1.3.7 **Example 1.3.7** (Vaserstein)**.** If $r, s \in R$ are such that $1 + rs$ is a unit, then so is $1 + sr$ because $(1 + sr)(1 - s(1 + rs)^{-1}r) = 1$. The subgroup $W(R)$ of $R^\times$ generated by the $(1 + rs)(1 + sr)^{-1}$ belongs to $E_2(R)$ by Ex. 1.1. For $R = M_2(\mathbb{F}_2)$, $W(R) = R^\times \cong \Sigma_3$ and $K_1(R) = 1$ but $R^\times \neq [R^\times, R^\times]$. If T is the subring of upper triangular matrices in $M_2(\mathbb{F}_2)$, its group of units is abelian ($T^\times \cong \mathbb{Z}/2$), but $K_1(T) = 1$ since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (1+rs)(1+sr)^{-1}$ for $r = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Vaserstein has shown [207] that $W(R) = [R^\times, R^\times]$ if $\Lambda = R/\mathrm{rad}(R)$ is a product of matrix rings, none of which is $M_2(\mathbb{F}_2)$, and at most one of the factors in Λ is $\mathbb{F}_2$. In particular, $W(R) = [R^\times, R^\times]$ for every local ring R. (See Ex. 1.1.)

III.1.4 **Lemma 1.4.** *If R is a semilocal ring then the natural inclusion of $R^\times = GL_1(R)$ into $GL(R)$ induces an isomorphism $K_1(R) \cong R^\times/W(R)$,*

If R is a commutative semilocal ring, then

$$SK_1(R) = 1 \quad \textit{and} \quad K_1(R) = R^\times.$$

Proof. By Example 1.1.1 (or 1.3.7 and Ex. 1.2 in the noncommutative case) it suffices to prove that $R^\times$ maps onto $K_1(R)$. This will follow by induction on n once we show that $GL_n(R) = E_n(R)GL_{n-1}(R)$. Let J denote the Jacobson radical of R, so that R/J is a finite product of matrix algebras over division rings. By Examples 1.1.3, 1.1.4 and 1.2.4, $(R/J)^\times$ maps onto $K_1(R/J)$; in fact by Exercise 1.3 we know that every $\bar{g} \in GL_n(R/J)$ is a product $\bar{e}\bar{g}_1$, where $\bar{e} \in E_n(R/J)$ and $\bar{g}_1 \in GL_1(R/J)$.

Given $g \in GL_n(R)$, its reduction $\bar{g}$ in $GL_n(R/J)$ may be decomposed as above: $\bar{g} = \bar{e}\bar{g}_1$. By Remark 1.2.3, we can lift $\bar{e}$ to an element $e \in E_n(R)$. The matrix $e^{-1}g$ is congruent to the diagonal matrix $\bar{g}_1$ modulo J, so its diagonal entries are all units and its off-diagonal entries lie in J. Using elementary row operations $e_{ij}(r)$ with $r \in J$, it is an easy matter to reduce $e^{-1}g$ to a diagonal matrix, say to $D = \mathrm{diag}(r_1, ..., r_n)$. By Ex. I.1.11, the matrix $\mathrm{diag}(1, ..., 1, r_n, r_n^{-1})$ is in $E_n(R)$. Multiplying D by this matrix yields a matrix in $GL_{n-1}(R)$, finishing the induction and the proof. □

Commutative Banach Algebras

Let R be a commutative Banach algebra over the real or complex numbers. For example, R could be the ring $\mathbb{R}^X$ of continuous real-valued functions of a compact space X. As subspaces of the metric space of $n \times n$ matrices over R, the groups $SL_n(R)$ and $GL_n(R)$ are topological groups.

III.1.5 **Proposition 1.5.** *$E_n(R)$ is the path component of the identity matrix in the special linear group $SL_n(R)$, $n \geq 2$. Hence we may identify the group $SK_1(R)$ with the group $\pi_0 SL(R)$ of path components of the topological space $SL(R)$.*

Proof. To see that $E_n(R)$ is path-connected, fix an element $g = \prod e_{i_\alpha j_\alpha}(r_\alpha)$. The formula $t \mapsto \prod e_{i_\alpha j_\alpha}(r_\alpha t)$, $0 \leq t \leq 1$ defines a path in $E_n(R)$ from the identity to g. To prove that $E_n(R)$ is open subset of $SL_n(R)$ (and hence a path-component), it suffices to prove that $E_n(R)$ contains U_{n-1}, the set of matrices $1+(r_{ij})$ in $SL_n(R)$ with $||r_{ij}|| < \frac{1}{n-1}$ for all i, j. We will actually show that each matrix in U_{n-1} can be expressed naturally as a product of n^2+5n-6 elementary matrices, each of which depends continuously upon the entries $r_{ij} \in R$.

Set $u = 1 + r_{11}$. Since $\frac{n-2}{n-1} < ||u||$, u has an inverse v with $||v|| < \frac{n-1}{n-2}$. Subtracting vr_{1j} times the first column from the j^{th} we obtain a matrix $1 + r'_{ij}$ whose first row is $(u, 0, ..., 0)$ and

$$||r'_{ij}|| < \frac{1}{n-1} + \frac{n-1}{n-2}\left(\frac{1}{n-1}\right)^2 = \frac{1}{n-2}.$$

We can continue to clear out entries in this way so that after $n(n-1)$ elementary operations we have reduced the matrix to diagonal form.

By Ex. I.1.10, any diagonal matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ is the product of 6 elementary matrices. By induction, it follows that any diagonal $n \times n$ matrix of determinant 1 can be written naturally as a product of $6(n-1)$ elementary matrices. □

Let V denote the path component of 1 in the topological group $R^\times$, *i.e.*, the kernel of $R^\times \to \pi_0 R^\times$. By Ex. 1.12, V is a quotient of the additive group R.

III.1.5.1 **Corollary 1.5.1.** *If R is a commutative Banach algebra, there is a natural surjection from $K_1(R)$ onto $\pi_0 GL(R) = \pi_0(R^\times) \times \pi_0 SL(R)$. The kernel of this map is the divisible subgroup V of $R^\times$.*

III.1.5.2 **Example 1.5.2.** If $R=\mathbb{R}$ then $K_1(\mathbb{R})=\mathbb{R}^\times$ maps onto $\pi_0 GL(\mathbb{R})=\{\pm 1\}$, and the kernel is the uniquely divisible multiplicative group $V=(0,\infty)$. If $R=\mathbb{C}$ then $V=\mathbb{C}^\times$, because $K_1(\mathbb{C})=\mathbb{C}^\times$ but $\pi_0 GL(\mathbb{C})=0$.

III.1.5.3 **Example 1.5.3.** Let X be a compact space with a nondegenerate basepoint. Then $SK_1(\mathbb{R}^X)$ is the group $\pi_0 SL(\mathbb{R}^X)=[X,SL(\mathbb{R})]=[X,SO]$ of homotopy classes of maps from X to the infinite special orthogonal group SO. By Ex. II.3.11 we have $\pi_0 GL(\mathbb{R}^X)=[X,O]=KO^{-1}(X)$, and there is a short exact sequence

$$0\to\mathbb{R}^X\xrightarrow{\exp}K_1(\mathbb{R}^X)\to KO^{-1}(X)\to 0.$$

Similarly, $SK_1(\mathbb{C}^X)$ is the group $\pi_0 SL(\mathbb{C}^X)=[X,SL(\mathbb{C})]=[X,SU]$ of homotopy classes of maps from X to the infinite special unitary group SU. Since $\pi_0 GL(\mathbb{C}^X)=[X,U]=KU^{-1}(X)$ by II.3.5.1 and Ex. II.3.11, there is a natural surjection from $K_1(\mathbb{C}^X)$ onto $KU^{-1}(X)$, and the kernel V is the divisible group of all contractible maps $X\to\mathbb{C}^\times$.

III.1.5.4 **Example 1.5.4.** When X is the circle S^1 we have $SK_1(\mathbb{R}^{S^1})=[S^1,SO]=\pi_1 SO=\mathbb{Z}/2$. On the other hand, we have $\pi_0 SL_2(\mathbb{R}^{S^1})=\pi_1 SL_2(\mathbb{R})=\pi_1 SO_2=\mathbb{Z}$, generated by the matrix $A=\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. Since $\pi_1 SO_2(\mathbb{R})\to\pi_1 SO$ is onto, the matrix A represents the nonzero element of $SK_1(\mathbb{R}^{S^1})$.

The ring $R=\mathbb{R}[x,y]/(x^2+y^2-1)$ may be embedded in the ring $\mathbb{R}^{S^1}$ by $x\mapsto\cos(\theta), y\mapsto\sin(\theta)$. Since the matrix $\left(\begin{smallmatrix}x & y\\ -y & x\end{smallmatrix}\right)$ maps to A, it represents a nontrivial element of $SK_1(R)$. In fact it is not difficult to show that $SK_1(R)\cong\mathbb{Z}/2$ using Mennicke symbols (Ex. 1.10).

K_1 and projective modules

Now let P be a finitely generated projective R-module. Choosing an isomorphism $P\oplus Q\cong R^n$ gives a group homomorphism from $\mathrm{Aut}(P)$ to $GL_n(R)$. (Send α to $\alpha\oplus 1_Q$.)

III.1.6 **Lemma 1.6.** *The homomorphism from* $\mathrm{Aut}(P)$ *to* $GL(R)=\bigcup GL_n(R)$ *is well-defined up to inner automorphism of* $GL(R)$*. Hence there is a well-defined homomorphism* $\mathrm{Aut}(P)\to K_1(R)$*.*

Proof. First suppose that Q and n are fixed. Two different isomorphisms between $P\oplus Q$ and R^n must differ by an automorphism of R^n, *i.e.*, by an element $g\in GL_n(R)$. Thus if $\alpha\in\mathrm{Aut}(P)$ maps to the matrices A and B, respectively, we must have $A=gBg^{-1}$. Next we observe that there is no harm in stabilizing, *i.e.*, replacing Q by $Q\oplus R^m$ and $P\oplus Q\cong R^n$ by $P\oplus(Q\oplus R^m)\cong R^{n+m}$. This is because $GL_n(R)\to GL(R)$ factors through $GL_{n+m}(R)$. Finally, suppose given a second isomorphism $P\oplus Q'\cong R^m$. Since $Q\oplus R^m\cong R^n\oplus Q'$, we may stabilize both Q and Q' to make them isomorphic, and invoke the above argument. □

III.1.6.1 **Corollary 1.6.1.** *If R and S are rings, there is a natural external product operation* $K_0(R)\otimes K_1(S)\to K_1(R\otimes S)$*.*

If R is commutative and S is an R-algebra, there is a natural product operation $K_0(R) \otimes K_1(S) \to K_1(S)$*, making* $K_1(S)$ *into a module over the ring* $K_0(R)$*.*

Proof. For each finitely generated projective R-module P and each m, Lemma III.1.6 1.6 provides a homomorphism $\mathrm{Aut}(P \otimes S^m) \to K_1(R \otimes S)$. For each $\beta \in GL_m(S)$, let $[P] \cdot \beta$ denote the image of the automorphism $1_P \otimes \beta$ of $P \otimes S^m$ under this map. Fixing β and m, the isomorphism $(P \oplus P') \otimes S^m \cong (P \otimes S^m) \oplus (P' \otimes S^m)$ yields the identity $[P \oplus P'] \cdot \beta = [P] \cdot \beta + [P'] \cdot \beta$ in $K_1(R \otimes S)$. Hence $P \mapsto [P] \cdot \beta$ is an additive function of $P \in \mathbf{P}(R)$, so (by definition) it factors through $K_0(R)$. Now fix P; the map $GL_m(S) \to K_1(R \otimes S)$ given by $\beta \mapsto [P] \cdot \beta$ is compatible with stabilization in m. Thus it factors through a map $GL(S) \to K_1(R \otimes S)$, and through a map $K_1(S) \to K_1(R \otimes S)$. This shows that the product is well-defined and bilinear.

When R is commutative, $K_0(R)$ is a ring by II, §2. If S is an R-algebra, there is a ring map $R \otimes S \to S$. Composing the external product with $K_1(R \otimes S) \to K_1(S)$ yields a natural product operation $K_0(R) \otimes K_1(S) \to K_1(S)$. The verification that $[P \otimes_R Q] \cdot \beta = [P] \cdot ([Q] \cdot \beta)$ is routine. □

Here is a homological interpretation of $K_1(R)$. Recall that the first homology $H_1(G; \mathbb{Z})$ of any group G is naturally isomorphic to $G/[G, G]$. (See WHomo [223, 6.1.11] for a proof.) For $G = GL(R)$ this yields

$$K_1(R) = H_1(GL(R); \mathbb{Z}) = \lim_{n \to \infty} H_1(GL_n(R); \mathbb{Z}). \qquad (1.6.2)$$ III.1.6.2

By Lemma III.1.6 1.6, we also have well-defined compositions

$$H_1(\mathrm{Aut}(P); \mathbb{Z}) \to H_1(GL_n(R); \mathbb{Z}) \to K_1(R),$$

which are independent of the choice of isomorphism $P \oplus Q \cong R^n$.

Here is another description of $K_1(R)$ in terms of the category $\mathbf{P}(R)$ of finitely generated projective R-modules. Consider the translation category $t\mathbf{P}$ of $\mathbf{P}(R)$: its objects are isomorphism classes of finitely generated projective modules, and the morphisms between P and P' are the isomorphism classes of Q such that $P \oplus Q \cong P'$. This is a filtering category WHomo [223, 2.6.13], and $P \mapsto H_1(\mathrm{Aut}(P); \mathbb{Z})$ is a well-defined functor from $t\mathbf{P}$ to abelian groups. Hence we can take the filtered direct limit of this functor. Since the free modules are cofinal in $t\mathbf{P}$, we see from (III.1.6.2 1.6.2) that we have

III.1.6.3 **Corollary 1.6.3.** *(Bass)* $K_1(R) \cong \varinjlim_{P \in t\mathbf{P}} H_1(\mathrm{Aut}(P); \mathbb{Z})$*.*

Recall from II.II.2.7 2.7 that if two rings R and S are *Morita equivalent* then the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$ are equivalent. By Corollary III.1.6.3 1.6.3 we have the following:

III.1.6.4 **Proposition 1.6.4** (Morita invariance of K_1)**.** *The group* $K_1(R)$ *depends only upon the category* $\mathbf{P}(R)$*. That is, if R and S are Morita equivalent rings then* $K_1(R) \cong K_1(S)$*. In particular, the isomorphism of* III.1.1.4 *1.1.4 arises in this way:*

$$K_1(R) \cong K_1(M_n(R)).$$

Transfer maps on K_1

Let $f\colon R \to S$ be a ring homomorphism. We will see later on that a *transfer homomorphism* $f_*\colon K_1(S) \to K_1(R)$ is defined whenever S has a finite R-module resolution by finitely generated projective R-modules. This construction requires a definition of K_1 for an exact category such as $\mathbf{H}(R)$, and is analogous to the transfer map in II(7.9.1) for K_0. Without this machinery, we can still construct the transfer map when S is finitely generated projective as an R-module, using the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$; this is the analogue of the method used for the K_0 transfer map in Example II.2.8.1.

III.1.7 **Lemma 1.7.** *Any additive functor $\mathbf{P}(S) \xrightarrow{T} \mathbf{P}(R)$ induces a natural homomorphism $K_1(T)\colon K_1(S) \to K_1(R)$, and $T_1 \oplus T_2$ induces the sum $K_1(T_1) + K_1(T_2)$.*

Proof. The functor T induces an evident functor $t\mathbf{P}(S) \to t\mathbf{P}(R)$. If P is a finitely generated projective S-module, T also induces a homomorphism $\mathrm{Aut}_S(P) \to \mathrm{Aut}_R(TP)$ and hence $H_1(\mathrm{Aut}_S(P);\mathbb{Z}) \to H_1(\mathrm{Aut}_R(TP);\mathbb{Z})$. As P varies, these assemble to give a natural transformation of functors from the translation category $t\mathbf{P}(S)$ to abelian groups. Since $K_1(S) = \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_S(P);\mathbb{Z})$ by Corollary 1.6.3, taking the direct limit over $t\mathbf{P}(S)$ yields the desired map

$$K_1(S) \to \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_R(P);\mathbb{Z}) \to \varinjlim_{Q\in\mathbf{P}(R)} H_1(\mathrm{Aut}_R(Q);\mathbb{Z}) = K_1(R). \quad \square$$

III.1.7.1 **Corollary 1.7.1.** *Suppose that S is finitely generated projective as an R-module. Then the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural transfer homomorphism $f_*\colon K_1(S) \to K_1(R)$. If R is commutative, the composite*

$$K_1(R) \xrightarrow{f^*} K_1(S) \xrightarrow{f_*} K_1(R)$$

is multiplication by $[S] \in K_0(R)$.

Proof. When T is the forgetful map, so that $K_1(S) \to K_1(R)$ is the transfer map f_*, we compute the composite f_*f^* by computing its effect upon an element $\alpha \in GL_n(R)$. The matrix $f^*(\alpha) = 1_S \otimes_R \alpha$ lies in $GL_n(S)$. To apply f_* we consider $1_S \otimes_R \alpha$ as an element of the group $\mathrm{Aut}_R(S^n) = \mathrm{Aut}_R(S \otimes_R R^n)$, which we then map into $GL(R)$. But this is just the product $[S]\cdot\alpha$ of 1.6.1. $\square$

When $j : F \to E$ is a finite field extension, it is easy to see from 1.1.2 that the transfer map $j_* : E^\times \to F^\times$ is the classical norm map. For this reason, the transfer map is sometimes called the *norm map.*

III.1.7.2 **Example 1.7.2.** Let D be a division algebra of dimension $d = n^2$ over its center F, and recall from 1.2.4 that $SK_1(D)$ is the kernel of the reduced norm N_{red}. We will show that $SK_1(D)$ has exponent n by showing that $i^*N_{\mathrm{red}} : K_1(D) \to K_1(D)$ is multiplication by n.

To see this, choose a maximal subfield E with inclusions $F \xrightarrow{j} E \xrightarrow{\sigma} D$. By the definition of $N_{\rm red}$, composing it with $j^*: F^\times \subset E^\times$ yields the transfer map $\sigma_*: K_1(D) \to K_1(E)$. Therefore, $i^* N_{\rm red} = \sigma^* j^* N_{\rm red} = \sigma^*\sigma_*$. Hence it suffices to show that $\sigma^*\sigma_*: K_1(D) \to K_1(D)$ is multiplication by n. By 1.7, $\sigma^*\sigma_*$ is induced by the additive self-map $T: M \mapsto M \otimes_D (D \otimes_E D)$ of $\mathbf{P}(D)$. Since $D \otimes_E D \cong D^n$ as a D-bimodule, $T(M) \cong M^n$ and the assertion follows from 1.7.

The transfer map $i_*: K_1(D) \to K_1(F)$ associated to $i: F \subset D$ is induced from the classical norm map $N_{D/F}: D^\times \subset GL_d(F) \to F^\times$. In fact, the norm map is n times the reduced norm $N_{\rm red}: D^\times \to F^\times$ of 1.2.4; see Ex. 1.16 below. Moreover, the composition $i^* i_*: K_1(D) \to K_1(D)$ is multiplication by d since it corresponds to the additive self-map $M \mapsto M \otimes_D (D \otimes_F D)$ of $\mathbf{P}(D)$, and $D \otimes_F D \cong D^d$ as a D-bimodule (see II.2.8.1).

III.1.7.3 **Corollary 1.7.3.** $K_1(R) = 0$ *for every flasque ring* R.

Proof. Recall from II.2.1.3 that a ring R is *flasque* if there is an additive self-functor T $(P \mapsto P \otimes_R M)$ on $\mathbf{P}(R)$ together with a natural transformation $\theta_P: P \oplus T(P) \cong T(P)$. By 1.7, the induced self-map on $K_1(R)$ satisfies $x + T(x) = T(x)$ (and hence $x = 0$) for all $x \in K_1(R)$. □

Here is an application of 1.7 that anticipates the higher K-theory groups with coefficients in chapter IV.

III.1.7.4 **Definition 1.7.4.** For each natural number m, we define $K_1(R;\mathbb{Z}/m)$ to be the relative group $K_0(\cdot m)$ of II.2.10, where $\cdot m$ is the endo-functor of $\mathbf{P}(R)$ sending P to $P^m = P \otimes_R R^m$. Since the P^m are cofinal, we see by Ex. II.2.15 and Ex. 1.14, that it fits into a universal coefficient sequence:

$$K_1(R) \xrightarrow{m} K_1(R) \to K_1(R;\mathbb{Z}/m) \to K_0(R) \xrightarrow{m} K_0(R).$$

III.1.8 **Example 1.8** (Whitehead group Wh_1)**.** If R is the group ring $\mathbb{Z}[G]$ of a group G, the (first) Whitehead group $Wh_1(G)$ is the quotient of $K_1(\mathbb{Z}[G])$ by the subgroup generated by ± 1 and the elements of G, considered as elements of GL_1. If G is abelian, then $\mathbb{Z}[G]$ is a commutative ring and $\pm G$ is a subgroup of $K_1(\mathbb{Z}[G])$, so by 1.3.4 we have $Wh_1(G) = (\mathbb{Z}[G]^\times / \pm G) \oplus SK_1(\mathbb{Z}[G])$. If G is finite then $Wh_1(G)$ is a finitely generated group whose rank is $r - q$, where r and q are the number of simple factors in $\mathbb{R}[G]$ and $\mathbb{Q}[G]$, respectively. This and other calculations related to $Wh_1(G)$ may be found in R. Oliver's excellent sourcebook [146].

The group $Wh_1(G)$ arose in Whitehead's 1950 study [227] of simple homotopy types. Two finite CW complexes have the same simple homotopy type if they are connected by a finite sequence of "elementary expansions and collapses." Given a homotopy equivalence $f\colon K \to L$ of complexes with fundamental group G, the *torsion* of f is an element $\tau(f) \in Wh_1(G)$. Whitehead proved that $\tau(f) = 0$ if and only if f is a simple homotopy equivalence, and that every element of $Wh_1(G)$ is the torsion of some f. An excellent source for the geometry behind this is [43].

III.1.9 **Example 1.9** (The s-Cobordism Theorem)**.** Here is another area of geometric topology in which Whitehead torsion has played a crucial role: piecewise-linear ("PL") topology. We say that a triple (W, M, M') of compact PL manifolds is an *h-cobordism* if the boundary of W is the disjoint union of M and M', and both inclusions $M \subset W$, $M' \subset W$ are homotopy equivalences. In this case we can define the torsion τ of $M \subset W$, as an element of $Wh_1(G)$, $G = \pi_1 M$. The *s-cobordism theorem* states that if M is fixed with $\dim(M) \geq 5$ then $(W, M, M') \cong (M \times [0,1], M \times 0, M \times 1)$ if and only if $\tau = 0$. Moreover, every element of $Wh_1(G)$ arises as the torsion of some h-cobordism (W, M, M').

Here is an application. Suppose given an h-cobordism (W, M, M'), and let N be the union of W, the cone on M and the cone on M'. Then N is PL homeomorphic to the suspension ΣM of M iff $(W, M, M') \cong (M \times [0,1], M \times 0, M \times 1)$ if and only if $\tau = 0$.

This gives a counterexample to the "Hauptvermutung" that two homeomorphic complexes would be PL homeomorphic. Indeed, if (W, M, M') is an h-cobordism with nonzero torsion, then N and ΣM cannot be PL homeomorphic, yet the theory of "engulfing" implies that they must be homeomorphic manifolds.

Another application, due to Smale, is the Generalized Poincaré Conjecture. Let N be an n-dimensional PL manifold of the homotopy type of the sphere S^n, $n \geq 5$. Then N is PL homeomorphic to S^n. To see this, let W be obtained by removing two small disjoint n-discs D_1, D_2 from N. The boundary of these discs is the boundary of W, and (W, S^{n-1}, S^{n-1}) is an h-cobordism. Its torsion must be zero since $\pi_1(S^{n-1}) = 0$ and $Wh_1(0) = 0$. Hence W is $S^{n-1} \times [0,1]$, and this implies that $N = W \cup D_1 \cup D_2$ is S^n.

EXERCISES

EIII.1.1 **1.1.** If $r, s, t \in R$ are such that $(1 + rs)t = 1$, show that $(1 + rs)(1 + sr)^{-1} \in E_2(R)$. *Hint:* Start by calculating $e_{12}(r + rsr)e_{21}(st + s)e_{12}(-r)e_{21}(-s)$.

If r is a unit of R, or if $r, s \in \mathrm{rad}(R)$, show that $(1+rs)(1+sr)^{-1} \in [R^\times, R^\times]$. Conclude that if R is a local ring then $W(R) = [R^\times, R^\times]$. *Hint:* If $r, s \in \mathrm{rad}(R)$, then $t = 1 + s - sr$ is a unit; compute $[t^{-1} + r, t]$ and $(1 + rs)(1 + r)$.

EIII.1.2 **1.2.** *Semilocal rings.* Let R be a noncommutative semilocal ring (Ex. II.2.6). Show that there exists a unique "determinant" map from $GL_n(R)$ onto the abelian group $R^\times/W(R)$ of Lemma 1.4 with the following properties: (i) $\det(e) = 1$ for every elementary matrix e, and (ii) If $\rho = \mathrm{diag}(r, 1, ..., 1)$ and $g \in GL_n(R)$ then $\det(\rho \cdot g) = r \cdot \det(g)$. Then show that det is a group homomorphism: $\det(gh) = \det(g)\det(h)$. Conclude that $K_1(R) \cong R^\times/W(R)$.

EIII.1.3 **1.3.** Suppose that a ring R has stable range $sr(R) = d + 1$ in the sense of Ex. I.1.5. (For example, R could be a d-dimensional commutative noetherian ring.) This condition describes the action of $E_{d+2}(R)$ on unimodular rows in R^{d+2}.

(a) Show that $GL_n(R) = GL_{d+1}(R)E_n(R)$ for all $n > d + 1$, and deduce that $GL_{d+1}(R)$ maps onto $K_1(R)$.

(b) Show that $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. *Hint:* Conjugate $e_{nj}(r)$ by $g \in GL_{d+2}(R)$.

EIII.1.4 **1.4.** Let R be the polynomial ring $F[x,y]$ over a field F. P.M. Cohn proved that the matrix $g = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$ is not in $E_2(R)$. Show that g is in $E_3(R) \cap GL_2(R)$.

EIII.1.5 **1.5.** Let R be a Euclidean domain, such as $\mathbb{Z}$ or the polynomial ring $F[t]$ over a field. Show that $E_n(R) = SL_n(R)$ for all n, and hence that $SK_1(R) = 0$.

EIII.1.6 **1.6.** Here is another interpretation of the group law for K_1. For each m, n, let $\oplus_{mn}$ denote the group homomorphism $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$ sending (α, β) to the block diagonal matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Show that in $K_1(R)$ we have $[\alpha \oplus_{mn} \beta] = [\alpha][\beta]$.

EIII.1.7 **1.7.** Let $E = \text{End}_R(R^\infty)$ be the ring of infinite row-finite matrices over R of Ex. I.1.7. Show that $K_1(E) = 0$. *Hint:* If $\alpha \in GL_n(E)$, form the block diagonal matrix $\alpha^\infty = \text{diag}(\alpha, \alpha, \dots)$ in $\text{Aut}(V) \cong GL(E)$, where V is an infinite sum of copies of $(R^\infty)^n$, and show that $\alpha \oplus \alpha^\infty$ is conjugate to α^∞.

EIII.1.8 **1.8.** In this exercise we show that the center of $E(R)$ is trivial. First show that any matrix in $GL_n(R)$ commuting with $E_n(R)$ must be a diagonal matrix $\text{diag}(r, ..., r)$ with r in the center of R. Conclude that no element in $E_{n-1}(R)$ is in the center of $E_n(R)$, and pass to the limit as $n \to \infty$.

EIII.1.9 **1.9.** In this exercise we suppose that R is a commutative ring, and give Suslin's proof that $E_n(R)$ is a normal subgroup of $GL_n(R)$ when $n \geq 3$. Let $v = \sum_{i=1}^n v_i e_i$ be a column vector, and let u, w be row vectors such that $u \cdot v = 1$ and $w \cdot v = 0$.

(a) Show that $w = \sum_{i<j} r_{ij}(v_j e_i - v_i e_j)$, where $r_{ij} = w_i u_j - w_j u_i$.

(b) Conclude that the matrix $I_n + (v \cdot w)$ is in $E_n(R)$ if $n \geq 3$.

(c) If $g \in GL_n(R)$ and $i < j$, let v be the i^{th} column of g and w the j^{th} row of g^{-1}, so that $w \cdot v = 0$. Show that $g e_{ij}(r) g^{-1} = I_n + (v \cdot rw)$ for all $r \in R$. By (b), this proves that $E_n(R)$ is normal.

EIII.1.10 **1.10.** *Mennicke symbols.* Let (r, s) be a unimodular row over a commutative ring R. We define the *Mennicke symbol* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ to be the class in $SK_1(R)$ of the matrix $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$, where $t, u \in R$ satisfy $ru - st = 1$. Show that this Mennicke symbol is independent of the choice of t and u, that $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s' \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} ss' \\ r \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} s+xr \\ r \end{smallmatrix}\right]$.

If R is noetherian of dimension 1, or more generally has $sr(R) \leq 2$, then we know by Ex. 1.3 that $GL_2(R)$ maps onto $K_1(R)$, and hence $SK_1(R)$ is generated by Mennicke symbols.

EIII.1.11 **1.11.** *Transfer.* Suppose that R is a Dedekind domain and $\mathfrak{p}$ is a prime ideal of R. Show that there is a map π_* from $K_1(R/\mathfrak{p}) = (R/\mathfrak{p})^\times$ to $SK_1(R)$ sending $\bar{s} \in (R/\mathfrak{p})^\times$ to the Mennicke symbol $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$, where $s \in R$ maps to $\bar{s}$ and $r \in R$ is an element of $\mathfrak{p} - \mathfrak{p}^2$ relatively prime to s. Another construction of the transfer map π_* will be given in chapter V.

EIII.1.12 **1.12.** If R is a commutative Banach algebra, let $\exp(R)$ denote the image of the exponential map $R \to R^\times$. Show that $\exp(R)$ is the path component of 1 in $R^\times$.

EIII.1.13 **1.13.** If H is a normal subgroup of a group G, then G acts upon H and hence its homology $H_*(H;\mathbb{Z})$ by conjugation. Since H always acts trivially upon its homology [223, 6.7.8], the group G/H acts upon $H_*(H;\mathbb{Z})$. Taking $H = E(R)$ and $G = GL(R)$, use Example 1.2.1 to show that $GL(R)$ and $K_1(R)$ act trivially upon the homology of $E(R)$.

EIII.1.14 **1.14.** (Swan) Let $T : \mathbf{P}(R) \to \mathbf{P}(S)$ be an additive functor, such as the base change f^* associated to a ring map $f : R \to S$. In II.2.10 we constructed a relative group $K_0(T)$. Since $K_0(T)$ is abelian, we can concatenate the K_1 map of Lemma 1.7 to (II.2.10.2) to get a sequence which is exact at $K_0(T)$ and (if T is cofinal) at $K_0(R)$:

$$K_1(R) \xrightarrow{T} K_1(S) \to K_0(T) \to K_0(R) \xrightarrow{T} K_0(S).$$

In this exercise, we show that the sequence is also exact at $K_1(S)$.

(a) We say that $(P,\alpha,Q) \sim (P',\alpha',Q')$ if there are $N, N' \in \mathbf{P}(R)$ and a commutator γ in $\mathrm{Aut}_S\, T(Q \oplus N)$ so that $(P \oplus N, \gamma(\alpha \oplus 1), Q \oplus N)$ is isomorphic to $(P' \oplus N', \alpha' \oplus 1, Q' \oplus N')$ in $\mathbf{P}(T)$. Show that $\sim$ is an equivalence relation.

(b) Show that the equivalence classes of $\sim$ form an abelian group under $\oplus$.

(c) If $(P,\alpha,Q) \sim (P',\alpha',Q')$, show that $[(P,\alpha,Q)] \cong [(P',\alpha',Q')]$ in $K_0(T)$.

(d) If $[(P,\alpha,Q)] \cong [(P',\alpha',Q')]$ in $K_0(T)$, show that $(P,\alpha,Q) \sim (P',\alpha',Q')$. *Hint:* Show that the relations for $K_0(T)$ hold in the group of (b). To do so, write $P \cong P' \oplus P''$ and $Q \cong Q' \oplus Q''$ in the exact sequence II(2.10.1) in $\mathbf{P}(T)$.

(e) Use (d) to show that if $\alpha \in \mathrm{Aut}_S\, T(R^n)$ and $[(R^n,\alpha,R^n)] = 0$ in $K_0(T)$ then (after increasing n) there is an isomorphism $(p,q) : (R^n,\alpha,R^n) \cong (R^n,\gamma,R^n)$ in $\mathbf{P}(T)$. Conclude that $[\alpha]$ is the image of $[q^{-1}p] \in K_1(R)$, proving exactness of the sequence at $K_1(S)$.

EIII.1.15 **1.15.** *Suspension rings.* Let R be any ring. Recall from Ex. I.1.8 that the cone ring $C(R)$ is the ring of row-and-column-finite matrices over R. The finite matrices in $C(R)$ form a 2-sided ideal $M(R)$, and the quotient $S(R) = C(R)/M(R)$ is called the *suspension ring* of R. Use Exercise 1.14 and 1.7.3, together with II.2.1.3 and II.2.7.2 to show that $K_1S(R) \cong K_0(R)$.

EIII.1.16 **1.16.** Let D be a division algebra of dimension $d = n^2$ over its center F. Show that the norm (or transfer) map $K_1(D) \to K_1(F)$ is n times the reduced norm N_{red} of III.1.2.4 1.2.4. *Hint:* Choose a maximal subfield E and show that the map $K_1(D) \to K_1(E)$ induced by the norm is induced by the additive map $M \mapsto M \otimes_D (D \otimes_F E)$ from $\mathbf{P}(D)$ to $\mathbf{P}(E)$. Then show that $D \otimes_F E \cong D^n$ as a D-E bimodule.

EIII.1.17 **1.17.** Let D be a division algebra, finite dimensional over its center F, and let E be any finite extension of F which is a splitting field of D, *i.e.,* $E \otimes_F D \cong M_n(E)$.

(a) Show that the following three maps $\theta_E : K_1(E) \to K_1(D)$ agree.
(i) $K_1(E) \cong K_1(M_n(E)) = K_1(E \otimes_F D) \xrightarrow{\text{transfer}} K_1(D)$;
(ii) $K_1(E) \to K_1(M_r(D)) \cong K_1(D)$, where $E \subset M_r(D)$;
(iii) $K_1(T)$, where $T : \mathbf{P}(E) \to \mathbf{P}(D)$ is $T(M) = M \otimes_E V$ for a simple $E \otimes_F D$-module V.

(b) If $j : E \to L$ is a finite field map over F, show that $\theta_E = \theta_L\, j_*$.

(c) If $\sigma \in \text{Aut}(E/F)$, then $\theta_E = \theta_E \sigma$.

EIII.1.18 **1.18.** If A is any finite-dimensional semisimple algebra over a field with center C, construct a reduced norm $A^\times \to C^\times$ and define $SL_n(A)$ to be the kernel of the reduced norm $GL_n(A) \to C^\times$. Show that the kernel $SK_1(A)$ of the induced map $K_1(A) \to C^\times$ is isomorphic to $SL_n(A)/E_n(A)$ for all $n \geq 3$.

2 Relative K_1

Let I be an ideal in a ring R. We write $GL(I)$ for the kernel of the natural map $GL(R) \to GL(R/I)$; the notation reflects the fact that $GL(I)$ is independent of R (see Ex. I.1.10). In addition, we define $E(R,I)$ to be the smallest normal subgroup of $E(R)$ containing the elementary matrices $e_{ij}(x)$ with $x \in I$. More generally, for each n we define $E_n(R,I)$ to be the normal subgroup of $E_n(R)$ generated by the matrices $e_{ij}(x)$ with $x \in I$ and $1 \le i \ne j \le n$. Clearly $E(R,I)$ is the union of the subgroups $E_n(R,I)$.

III.2.1 **Relative Whitehead Lemma 2.1.** *$E(R,I)$ is a normal subgroup of $GL(I)$, and contains the commutator subgroup of $GL(I)$.*

Proof. For any matrix $g = 1+\alpha \in GL_n(I)$, the identity

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g^{-1}\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g\alpha & 1 \end{pmatrix}.$$

shows that the matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$ is in $E_{2n}(R,I)$. (The product of the first 3 matrices is in $E_{2n}(R,I)$.) Hence if $h \in E_n(R,I)$ then the conjugate

$$\begin{pmatrix} ghg^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$$

is in $E(R,I)$. Finally, if $g,h \in GL_n(I)$ then $[g,h]$ is in $E_{2n}(R,I)$ by equation (1.3.4). □

III.2.2 **Definition 2.2.** The relative group $K_1(R,I)$ is defined to be the quotient $GL(I)/E(R,I)$. By the Relative Whitehead Lemma, it is an abelian group.

The inclusion of $GL(I)$ in $GL(R)$ induces a map $K_1(R,I) \to K_1(R)$. More generally, if $R \to S$ is a ring map sending I into an ideal I' of S, the natural maps $GL(I) \to GL(I')$ and $E(R) \to E(S)$ induce a map $K_1(R,I) \to K_1(S,I')$.

III.2.2.1 **Remark 2.2.1.** Suppose that $R \to S$ is a ring map sending an ideal I of R isomorphically onto an ideal of S. The induced map $K_1(R,I) \to K_1(S,I)$ must be a surjection, as both groups are quotients of $GL(I)$. However, Swan discovered that they need not be isomorphic; a simple example is given in Ex. 2.3 below.

Vaserstein proved in [206, 14.2] that $K_1(R,I)$ is independent of R if and only if $I = I^2$. One direction is easy (Ex. 2.10): if $I = I^2$ then the commutator subgroup of $GL(I)$ is perfect, and equal to $E(R,I)$. Thus $K_1(R,I) = GL(I)/[GL(I), GL(I)]$, a group which is independent of R (when $I = I^2$). (See Ex. 2.6 when R is commutative.)

III.2.3 **Proposition 2.3.** *There is an exact sequence*

$$K_1(R,I) \to K_1(R) \to K_1(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

Proof. By Ex. II.2.3 there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

Since the K_1 groups are quotients of the GL groups, and $E(R)$ maps onto $E(R/I)$, this gives exactness except at $K_1(R)$. Suppose $g \in GL(R)$ maps to zero under $GL(R) \to K_1(R) \to K_1(R/I)$. Then the reduction $\bar{g}$ of g mod I is in $E(R/I)$. Since $E(R)$ maps onto $E(R/I)$, there is a matrix e in $E(R)$ mapping to $\bar{g}$, *i.e.*, ge^{-1} is in the kernel $GL(I)$ of $GL(R) \to GL(R/I)$. Hence the class of ge^{-1} in $K_1(R,I)$ is defined, and maps to the class of g in $K_1(R)$. This proves exactness at the remaining spot. □

The relative group $SK_1(R,I)$

If R happens to be commutative, the determinant map $K_1(R) \to R^\times$ of Example 1.1.1 induces a relative determinant map det: $K_1(R,I) \to GL_1(I)$, since the determinant of a matrix in $GL(I)$ is congruent to 1 modulo I. It is traditional to write $SK_1(R,I)$ for the kernel of det, so the canonical map $GL_1(I) \to K_1(R,I)$ induces a direct sum decomposition $K_1(R,I) = GL_1(I) \oplus SK_1(R,I)$ compatible with the decomposition $K_1(R) = R^\times \oplus SK_1(R)$ of Example 1.1.1. Here are two important cases in which $SK_1(R,I)$ vanishes:

III.2.4 **Lemma 2.4.** *Let I be a radical ideal in R. Then:*

1. *$K_1(R,I)$ is a quotient of the multiplicative group $1 + I = GL_1(I)$.*
2. *If R is a commutative ring, then $SK_1(R,I) = 0$ and $K_1(R,I) = 1 + I$.*

Proof. As in the proof of Lemma 1.4, it suffices to show that $GL_n(I) = E_n(R,I)GL_{n-1}(I)$ for $n \geq 2$. If (x_{ij}) is a matrix in $GL_n(I)$ then x_{nn} is a unit of R, and for $i < n$ the entries x_{in}, x_{ni} are in I. Multiplying by the diagonal matrix $\mathrm{diag}(1,\dots,1,x_{nn},x_{nn}^{-1})$, we may assume that $x_{nn} = 1$. Now multiplying on the left by the matrices $e_{in}(-x_{in})$ and on the right by $e_{ni}(-x_{ni})$ reduces the matrix to one in $GL_{n-1}(I)$. □

The next theorem (and its variant) extends the calculation mentioned in Example 1.3.6 above. We cite them from [19, 4.3], mentioning only that the proof involves calculations with Mennicke symbols (see Ex. 1.10 and 2.5) for finitely generated R, *i.e.,* Dedekind rings of arithmetic type.

III.2.5 **Theorem 2.5** (Bass-Milnor-Serre)**.** *Let R be an integrally closed subring of a number field F, and I an ideal of R. Then*

(1) If F has any embedding into $\mathbb{R}$ then $SK_1(R,I) = 0$.

(2) If F is "totally imaginary" (has no embedding into $\mathbb{R}$), then $SK_1(R,I) \cong C_n$ is a finite cyclic group whose order n divides the order w_1 of the group of roots of unity in R. The exponent $\mathrm{ord}_p\, n$ of p in the integer n is the minimum over all prime ideals $\mathfrak{p}$ of R containing I of the integer

$$\inf\left\{\mathrm{ord}_p\, w_1, \sup\{0, \left[\frac{\mathrm{ord}_{\mathfrak{p}}(I)}{\mathrm{ord}_{\mathfrak{p}}(p)} - \frac{1}{p-1}\right]\right\}.$$

III.2.5.1 **Variant 2.5.1** (Bass-Milnor-Serre)**.** Let R be the coordinate ring of a smooth affine curve over a finite field. Then $SK_1(R) = 0$.

The Mayer-Vietoris Exact Sequence

Suppose we are given a ring map $f\colon R \to S$ and an ideal I of R mapped isomorphically into an ideal of S. Then we have a Milnor square, as in I.2.6:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I. \end{array}$$

III.2.6 **Theorem 2.6** (Mayer-Vietoris)**.** *Given a Milnor square as above, there is an exact sequence*

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \xrightarrow{\pm} K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Proof. By Theorem II.2.9 we have an exact sequence

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Since $K_0(R)$ is abelian, we may replace $GL(S/I)$ by $K_1(S/I)$ in this sequence. This gives the sequence of the theorem, and exactness at all the K_0 places. Also by II.2.9, the image of $\partial\colon K_1(S/I) \to K_0(R)$ is the double coset space

$$GL(S) \backslash GL(S/I) / GL(R/I).$$

Note that $E(S) \to E(S/I)$ is onto. Therefore the kernel of ∂ is the subgroup of $K_1(S/I)$ generated by the images of $GL(S)$ and $GL(R/I)$, and the sequence is exact at $K_1(S/I)$. To prove exactness at the final spot, suppose given $\bar{g} \in GL_n(R/I)$, $h \in GL_n(S)$ and an elementary matrix $\bar{e} \in E(S/I)$ such that $\bar{f}(\bar{g})\bar{e} \equiv h \pmod{I}$. Lifting $\bar{e}$ to an $e \in E_n(S)$ (by Remark 1.2.3) yields $\bar{f}(\bar{g}) \equiv he^{-1} \pmod{I}$. Since R is the pullback of S and R/I, there is a $g \in GL_n(R)$, equivalent to $\bar{g}$ modulo I, such that $f(g) = he^{-1}$. This establishes exactness at the final spot. $\square$

EXERCISES

EIII.2.1 **2.1.** Suppose we are given a Milnor square in which R and S are commutative rings. Using the Units-Pic sequence (I.3.10), conclude that there are exact sequences

$$SK_1(R,I) \to SK_1(R) \to SK_1(R/I) \xrightarrow{\partial} SK_0(I) \to SK_0(R) \to SK_0(R/I),$$

$$SK_1(R) \to SK_1(S) \oplus SK_1(R/I) \xrightarrow{\partial} SK_0(R) \to SK_0(S) \oplus SK_0(R/I) \to SK_0(S/I).$$

EIII.2.2 **2.2.** *Rim Squares.* Let C_p be a cyclic group of prime order p with generator t, and let $\zeta = e^{2\pi i/p}$. The ring $\mathbb{Z}[\zeta]$ is the integral closure of $\mathbb{Z}$ in the number field $\mathbb{Q}(\zeta)$. Let $f\colon \mathbb{Z}[C_p] \to \mathbb{Z}[\zeta]$ be the ring surjection sending t to ζ, and let I denote the kernel of the augmentation $\mathbb{Z}[C_p] \to \mathbb{Z}$.

(a) Show that I is isomorphic to the ideal of $\mathbb{Z}[\zeta]$ generated by $\zeta - 1$, so that we have a Milnor square with the rings $\mathbb{Z}[C_p]$, $\mathbb{Z}[\zeta]$, $\mathbb{Z}$ and $\mathbb{F}_p$.

(b) Show that for each $k = 1, ..., p-1$ the element $(\zeta^k - 1)/(\zeta - 1) = 1 + \cdots + \zeta^{k-1}$ is a unit of $\mathbb{Z}[\zeta]$, mapping onto $k \in \mathbb{F}_p^\times$.

These elements are often called *cyclotomic units*, and generate a subgroup of finite index in $\mathbb{Z}[\zeta]^\times$. If $p \ge 3$, the Dirichlet Unit Theorem says that the units of $\mathbb{Z}[\zeta]$ split as the direct sum of the finite group $\{\pm\zeta^k\}$ of order $2p$ $(p \ne 2)$ and a free abelian group of rank $(p-3)/2$.

(c) Conclude that if $p > 3$ then both $K_1(\mathbb{Z}[C_p])$ and $Wh_1(C_p)$ are nonzero. In fact, $SK_1(\mathbb{Z}[C_p]) = 0$.

EIII.2.3 **2.3.** *Failure of Excision for K_1.* Here is Swan's simple example to show that $K_1(R, I)$ depends upon R. Let F be a field and let R be the algebra of all upper triangular matrices $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in $M_2(F)$. Let I be the ideal of all such matrices with $x = z = 0$, and let R_0 be the commutative subalgebra $F \oplus I$. Show that $K_1(R_0, I) \cong F$ but that $K_1(R, I) = 0$. *Hint:* Calculate $e_{21}(r)e_{12}(y)e_{21}(-r)$.

EIII.2.4 **2.4.** (Vaserstein) If I is an ideal of R, and $x \in I$ and $r \in R$ are such that $(1+rx)$ is a unit, modify Ex. 1.1 to show that $(1+rx)(1+xr)^{-1}$ is in $E_2(R, I)$.

If I is a radical ideal and $W = W(R, I)$ denotes the subgroup of units generated by the $(1+rx)(1+xr)^{-1}$, use Lemma 2.4 to conclude that $(1+I)/W$ maps onto $K_1(R, I)$. Vaserstein proved in [205] that $K_1(R, I) \cong (1+I)/W$ for every radical ideal.

EIII.2.5 **2.5.** *Mennicke symbols.* If I is an ideal of a commutative ring R, $r \in (1+I)$ and $s \in I$, we define the *Mennicke symbol* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ to be the class in $SK_1(R, I)$ of the matrix $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$, where $t \in I$ and $u \in (1+I)$ satisfy $ru - st = 1$. Show that this Mennicke symbol is independent of the choice of t and u, with $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s \\ r' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s \\ rr' \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s' \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} ss' \\ r \end{smallmatrix}\right]$. (*Hint:* Use Ex. 1.10.) Finally, show that if $t \in I$ then

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} s + rt \\ r \end{bmatrix} = \begin{bmatrix} s \\ r + st \end{bmatrix}.$$

EIII.2.6 **2.6.** *The obstruction to excision.* Let $R \to S$ be a map of commutative rings, sending an ideal I of R isomorphically onto an ideal of S. Given $x \in I$ and $s \in S$, let $\psi(x, s)$ denote the Mennicke symbol $\left[\begin{smallmatrix} x \\ 1-sx \end{smallmatrix}\right]$ in $SK_1(R, I)$.

(a) Verify that $\psi(x, s)$ vanishes in $SK_1(S, I)$.

(b) Prove that ψ is bilinear, and that $\psi(x, s) = 1$ if either $x \in I^2$ or $s \in R$. Thus ψ induces a map from $(I/I^2) \otimes (S/R)$ to $SK_1(R, I)$.

(c) Prove that the Leibniz rule holds: $\psi(x, ss') = \psi(sx, s')\psi(s'x, s)$.

For every map $R \to S$, the S-module $\Omega^1_{S/R}$ of *relative Kähler differentials* is presented with generators ds, $s \in S$, subject to the following relations: $d(s + s') = ds + ds'$, $d(ss') = s\,ds' + s'\,ds$, and if $r \in R$ then $dr = 0$. (See [223].)

(d) (Vorst) Show that $\Omega^1_{S/R} \otimes_S I/I^2$ is the quotient of $(S/R) \otimes (I/I^2)$ by the subgroup generated by the elements $s \otimes s'x + s' \otimes sx - ss' \otimes x$. Then conclude that ψ induces a map $\Omega^1_{S/R} \otimes_S I/I^2 \to SK_1(R, I)$.

Swan proved in [195] that the resulting sequence is exact:

$$\Omega^1_{S/R} \otimes_S I/I^2 \xrightarrow{\psi} SK_1(R, I) \to SK_1(S, I) \to 1.$$

EIII.2.7 **2.7.** Suppose that the ring map $R \to R/I$ is split by a map $R/I \to R$. Show that $K_1(R) \cong K_1(R/I) \oplus K_1(R, I)$. The corresponding decomposition of $K_0(R)$ follows from the ideal sequence 2.3, or from the definition of $K_0(I)$, since $R \cong R/I \oplus I$; see Ex. II.2.4.

EIII.2.8 **2.8.** Suppose that $p^r = 0$ in R for some prime p. Show that $K_1(R, pR)$ is a p-group. Conclude that the kernel of the surjection $K_1(R) \to K_1(R/pR)$ is also a p-group.

EIII.2.9 **2.9.** If I is a nilpotent ideal in a $\mathbb{Q}$-algebra R, or even a complete radical ideal, show that $K_1(R, I) \cong I/[R, I]$, where $[R, I]$ is the subgroup spanned by all elements $[r, x] = rx - xr$, $r \in R$ and $x \in I$. In particular, this proves that $K_1(R, I)$ is uniquely divisible. *Hint:* If $[R, I] = 0$, $\ln : 1 + I \to I$ is a bijection. If not, use Ex. 2.4 and the Campbell-Hausdorff formula.

EIII.2.10 **2.10.** Suppose that I is an ideal satisfying $I = I^2$. Show that $[GL(I), GL(I)]$ is a perfect group. Conclude that $E(R, I) = [GL(I), GL(I)]$ and hence that $K_1(R, I)$ is independent of R. *Hint:* Use the commutator formulas (1.3.1).

3 The Fundamental Theorems for K_1 and K_0

The Fundamental Theorem for K_1 is a calculation of $K_1(R[t,t^{-1}])$, and describes one of the many relationships between K_1 and K_0. The core of this calculation depends upon the construction of an exact sequence (see 3.2 below and II.7.8.1):

$$K_1(R[t]) \to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{H}_{\{t^n\}}(R[t]) \to 0$$

We will construct a localization sequence connecting K_1 and K_0 in somewhat greater generality first. Recall from chapter II, Theorem 9.8 that for any multiplicatively closed set S of central elements in a ring R there is an exact sequence $K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$, where $K_0(R \text{ on } S)$ denotes K_0 of the Waldhausen category $\mathbf{Ch}^b_S\mathbf{P}(R)$. If S consists of nonzerodivisors, $K_0(R \text{ on } S)$ also equals $K_0\mathbf{H}_S(R)$ by Ex. II.9.13; see Corollary II.7.7.4.

Our first goal is to extend this sequence to the left using K_1, and we begin by constructing the boundary map ∂.

Let α be an endomorphism of R^n. We say that α is an *S-isomorphism* if $S^{-1}\ker(\alpha) = S^{-1}\text{coker}(\alpha) = 0$, or equivalently, $\alpha/1 \in GL_n(S^{-1}R)$. Write $\text{cone}(\alpha)$ for the *mapping cone* of α, which is the chain complex $R^n \xrightarrow{-\alpha} R^n$ concentrated in degrees 0 and 1; see [223, 1.5.1]. It is clear that α is an S-isomorphism if and only if $\text{cone}(\alpha) \in \mathbf{Ch}^b_S\mathbf{P}(R)$.

III.3.1 **Lemma 3.1.** *Let S be a multiplicatively closed set of central elements in a ring R. Then there is a group homomorphism*

$$K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S)$$

sending each S-isomorphism α to the class $[\text{cone}(\alpha)]$ of the mapping cone of α. In particular, each $s \in S$ is an endomorphism of R so $\partial(s)$ is the class of the chain complex $\text{cone}(s) : R \xrightarrow{-s} R$.

Before proving this lemma, we give one important special case. When S consists of nonzerodivisors, every S-isomorphism α must be an injection, and $\text{coker}(\alpha)$ is a module of projective dimension one, *i.e.*, an object of $\mathbf{H}_S(R)$. Moreover, under the isomorphism $K_0\mathbf{Ch}^b_S\mathbf{P}(R) \cong K_0\mathbf{H}_S(R)$ of Ex. II.9.13, the class of $\text{cone}(\alpha)$ in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ corresponds to the element $[\text{coker}(\alpha)]$ of $K_0\mathbf{H}_S(R)$. Thus we immediately have:

III.3.1.1 **Corollary 3.1.1.** *If S consists of nonzerodivisors then there is a homomorphism $K_1(S^{-1}R) \xrightarrow{\partial} K_0\mathbf{H}_S(R)$ sending each S-isomorphism α to $[\text{coker}(\alpha)]$, and sending $s \in S$ to $[R/sR]$.*

Proof of 3.1. If $\beta \in \mathrm{End}(R^m)$ is also an S-isomorphism, then the diagram

$$\begin{array}{ccccc}
0 & \rightarrowtail & R^n & \overset{=}{\twoheadrightarrow} & R^n \\
\downarrow & & \downarrow{\scriptstyle (1,\beta)} & & \downarrow{\scriptstyle \beta} \\
R^n & \overset{(1\,0)}{\rightarrowtail} & R^n \oplus R^n & \overset{(0\,1)}{\twoheadrightarrow} & R^n \\
\downarrow{\scriptstyle \alpha\beta} & & \downarrow{\scriptstyle \binom{\alpha\beta}{-\alpha}} & & \downarrow \\
R^n & \overset{=}{\rightarrowtail} & R^n & \twoheadrightarrow & 0
\end{array}$$

is a short exact sequence in $\mathbf{Ch}^b_S\mathbf{P}(R)$, where we regard the columns as chain complexes. Since the middle column of the diagram is quasi-isomorphic to its subcomplex $0 \to 0 \oplus R^n \xrightarrow{-\alpha} R^n$, we get the relation

$$\big[\mathrm{cone}(\alpha)\big] - \big[\mathrm{cone}(\alpha\beta)\big] = \big[\mathrm{cone}(\beta)[-1]\big] = -\big[\mathrm{cone}(\beta)\big],$$

or

$$\big[\mathrm{cone}(\alpha\beta)\big] = \big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(\beta)\big] \qquad (3.1.2)$$ III.3.1.2

in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. In particular, if β is the diagonal matrix $\mathrm{diag}(t, ..., t)$ then $\mathrm{cone}(\beta)$ is the direct sum of n copies of $\mathrm{cone}(t)$, so we have

$$\big[\mathrm{cone}(\alpha t)\big] = \big[\mathrm{cone}(\alpha)\big] + n\big[\mathrm{cone}(t)\big]. \qquad (3.1.3)$$ III.3.1.3

Every $g \in GL_n(S^{-1}R)$ can be represented as α/s for some S-isomorphism α and some $s \in S$, and we define $\partial(g) = \partial(\alpha/s)$ to be the element $\big[\mathrm{cone}(\alpha)\big] - n\big[\mathrm{cone}(s)\big]$ of $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. By (3.1.3) we have $\partial(\alpha/s) = \partial(\alpha t/st)$, which implies that $\partial(g)$ is independent of the choice of α and s. By (3.1.2) this implies that ∂ is a well-defined homomorphism from each $GL_n(S^{-1}R)$ to $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. Finally, the maps ∂ are compatible with the inclusions $GL_n \subset GL_{n+1}$, because

$$\begin{aligned}\partial\begin{pmatrix}\alpha/s & 0\\ 0 & 1\end{pmatrix} = \partial\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}/s\right) &= \left[\mathrm{cone}\begin{pmatrix}\alpha & 0\\ 0 & s\end{pmatrix}\right] - (n+1)\big[\mathrm{cone}(s)\big]\\ &= \big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(s)\big] - (n+1)\big[\mathrm{cone}(s)\big] = \partial(\alpha/s).\end{aligned}$$

Hence ∂ extends to $GL(S^{-1}R)$, and hence must factor through the universal map to $K_1(S^{-1}R)$. □

III.3.1.4 **Example 3.1.4** (Key Example)**.** For the Fundamental Theorem, we shall need the following special case of this construction. Let T be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then the map ∂ goes from $K_1(R[t, t^{-1}])$ to $K_0\mathbf{H}_T(R[t])$. If ν is a nilpotent endomorphism of R^n then $t - \nu$ is a T-isomorphism, because its inverse is the polynomial $t^{-1}(1 + \nu t^{-1} + \nu^2 t^{-2} + \dots)$. If (R^n, ν) denotes the $R[t]$-module R^n on which t acts as ν,

$$\partial(t - \nu) = \big[R[t]^n/(t-\nu)\big] = \big[(R^n, \nu)\big],$$

$$\partial(1 - \nu t^{-1}) = \partial(t - \nu) - \partial(t \cdot \mathrm{id}_n) = \big[(R^n, \nu)\big] - n\big[(R, 0)\big].$$

We can also compose ∂ with the product $K_0(R)\otimes K_1(\mathbb{Z}[t,\frac{1}{t}]) \to K_1(R[t,\frac{1}{t}])$ of Corollary 1.6.1. Given a finitely generated projective R-module P, the product $[P]\cdot t$ is the image of $t\cdot \mathrm{id}_{P[t,t^{-1}]}$ under the map $\mathrm{Aut}(P[t,t^{-1}]) \to K_1(R[t,t^{-1}])$ of Lemma 1.6. To compute $\partial([P]\cdot t)$, choose Q such that $P\oplus Q\cong R^n$. Since the cokernel of $t\cdot\mathrm{id}_{P[t]}\colon P[t]\to P[t]$ is the $R[t]$-module $(P,0)$, we have an exact sequence of $R[t]$-modules:

$$0 \to R[t]^n \xrightarrow{t\cdot \mathrm{id}_{P[t]}\oplus 1\cdot\mathrm{id}_{Q[t]}} R[t]^n \to (P,0)\to 0.$$

Therefore we have the formula $\partial([P]\cdot t) = \big[(P,0)\big]$.

III.3.1.5 **Lemma 3.1.5.** *$K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ is generated by the classes $[Q_\bullet]$ of chain complexes concentrated in degrees 0 and 1, i.e., by complexes $Q_\bullet$ of the form $Q_1\to Q_0$.*

The kernel of $K_0\mathbf{Ch}^b_S\mathbf{P}(R) \to K_0(R)$ is generated by the complexes $R^n \xrightarrow{\alpha} R^n$ associated to S-isomorphisms, i.e., by the classes $\partial(\alpha)=\big[\mathrm{cone}(\alpha)\big]$.

Proof. By the Shifting Lemma II.9.2.1, K_0 is generated by bounded complexes of the form $0\to P_n\to\cdots\to P_0\to 0$. If $n\ge 2$, choose a free R-module $F=R^N$ mapping onto $H_0(P_\bullet)$. By assumption, we have $sH_0(P_\bullet)=0$ for some $s\in S$. By the projective lifting property, there are maps f_0, f_1 making the diagram

$$\begin{array}{ccccccc}
F & \xrightarrow{\;s\;} & F & \longrightarrow & F/sF & \longrightarrow & 0\\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow\!\!\downarrow & & \\
P_1 & \longrightarrow & P_0 & \longrightarrow & H_0(P) & \longrightarrow & 0
\end{array}$$

commute. Thus if $Q_\bullet$ denotes the complex $F\xrightarrow{s}F$ we have a chain map $Q_\bullet \xrightarrow{f} P_\bullet$ inducing a surjection on H_0. The mapping cone of f fits into a cofibration sequence $P_\bullet \rightarrowtail \mathrm{cone}(f) \twoheadrightarrow Q_\bullet[-1]$ in $\mathbf{Ch}^b_S\mathbf{P}(R)$, so we have $[P_\bullet] = [Q_\bullet]+[\mathrm{cone}(f)]$ in $K_0(R \text{ on } S)$. Moreover, $H_0(\mathrm{cone}(f))=0$, so there is a decomposition $P_1\oplus F\cong P_0\oplus P_1'$ so that the mapping cone is the direct sum of an exact complex $P_0\xrightarrow{\cong}P_0$ and a complex $P'_\bullet$ of the form

$$0\to P_n\to\cdots\to P_3\to P_2\oplus F\to P_1'\to 0.$$

Since $P'_\bullet$ has length $n-1$, induction on n implies that $[\mathrm{cone}(f)]=[P'_\bullet]$ is a sum of terms of the form $[Q_1\to Q_0]$.

Hence every element of K_0 has the form $x=[P_1\xrightarrow{\alpha}P_0]-[Q_1\xrightarrow{\beta}Q_0]$. Choose $s\in S$ so that $s\beta^{-1}$ is represented by an S-isomorphism $Q_0\xrightarrow{\gamma}Q_1$; adding γ to both terms of x, as well as the appropriate zero term $Q'\xrightarrow{=}Q'$, we may assume that $Q_1=Q_0=R^n$, i.e., that the second term of x is the mapping cone of some S-isomorphism $\beta\in\mathrm{End}(R^n)$. With this reduction, the map to $K_0(R)$ sends x to $[P_1]-[P_0]$. If this vanishes, then P_1 and P_0 are stably isomorphic. Adding the appropriate $P'\xrightarrow{=}P'$ makes $P_1=P_0=R^m$ for some m, and writes x in the form

$$x=\mathrm{cone}(\alpha)-\mathrm{cone}(\beta)=\partial(\alpha)-\partial(\beta). \quad \square$$

III.3.2 **Theorem 3.2.** *Let S be a multiplicatively closed set of central elements in a ring R. Then the map ∂ of Lemma 3.1 fits into an exact sequence*

$$K_1(R) \to K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

Proof. We have proven exactness at $K_0(R)$ in Theorem II.9.8, and the composition of any two consecutive maps is zero by inspection. Exactness at $K_0(R \text{ on } S)$ was proven in Lemma 3.1.5. Hence it suffices to establish exactness at $K_1(S^{-1}R)$.

For reasons of exposition, we shall give the proof when S consists of nonzerodivisors, relegating the general proof (which is similar but more technical) to Exercise 3.5. The point of this simplification is that we can work with the exact category $\mathbf{H}_S(R)$. In particular, for every S-isomorphism α the class of the module $\operatorname{coker}(\alpha)$ is simpler to manipulate than the class of the mapping cone.

Recall from the proof of Lemma 3.1 that every element of $GL_n(S^{-1}R)$ can be represented as α/s for some S-isomorphism $\alpha \in \operatorname{End}(R^n)$ and some $s \in S$, and that $\partial(\alpha/s)$ is defined to be $\big[\operatorname{coker}(\alpha)\big] - [R^n/sR^n]$. If $\partial(\alpha/s) = 0$, then from Ex. II.7.2 there are short exact sequences in $\mathbf{H}_S(R)$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $\operatorname{coker}(\alpha) \oplus C_1 \cong (R^n/sR^n) \oplus C_2$. By Ex. 3.4 we may add terms to C', C'' to assume that $C' = \operatorname{coker}(\alpha')$ and $C'' = \operatorname{coker}(\alpha'')$ for appropriate S-isomorphisms of some R^m. By the Horseshoe Lemma ([223, 2.2.8]) we can construct two exact sequences of projective resolutions (for $i = 1, 2$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^m & \rightarrowtail & R^{2m} & \twoheadrightarrow & R^m & \longrightarrow & 0 \\
 & & \alpha' \downarrow & & \alpha_i \downarrow & & \alpha'' \downarrow & & \\
0 & \longrightarrow & R^m & \rightarrowtail & R^{2m} & \twoheadrightarrow & R^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \rightarrowtail & C_i & \twoheadrightarrow & C'' & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Inverting S makes each α_i an isomorphism conjugate to $\begin{pmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{pmatrix}$. Thus in $K_1(S^{-1}R)$ we have $[\alpha_1] = [\alpha'] + [\alpha''] = [\alpha_2]$. On the other hand, the two endomorphisms $\alpha \oplus \alpha_1$ and $s \cdot \mathrm{id}_n \oplus \alpha_2$ of R^{2m+n} have isomorphic cokernels by construction. Lemma 3.2.1 below implies that in $K_1(S^{-1}R)$ we have

$$[\alpha/s] = [\alpha \oplus \alpha_1] - [s \cdot \mathrm{id}_n \oplus \alpha_2] = g \qquad \text{for some } g \in GL(R).$$

This completes the proof of Theorem 3.2. □

III.3.2.1 **Lemma 3.2.1.** *Suppose that S consists of nonzerodivisors. If $\alpha, \beta \in \operatorname{End}_R(R^n)$ are S-isomorphisms with $R^n/\alpha R^n$ isomorphic to $R^n/\beta R^n$, then there is a $g \in GL_{4n}(R)$ such that $[\alpha] = [g][\beta]$ in $K_1(S^{-1}R)$.*

Proof. Put $M = \operatorname{coker}(\alpha) \oplus \operatorname{coker}(\beta)$, and let $\gamma\colon R^n/\alpha R^n \cong R^n/\beta R^n$ be an automorphism. By Ex. 3.3(b) with $Q = R^{2n}$ we can lift the automorphism $\left(\begin{smallmatrix} 0 & \gamma^{-1} \\ \gamma & 0 \end{smallmatrix}\right)$ of M to an automorphism γ_0 of R^{4n}. If π_1 and π_2 denote the projections $R^{4n} \xrightarrow{(pr,0,0,0)} \operatorname{coker}(\alpha)$, and $R^{4n} \xrightarrow{(0,pr,0,0)} \operatorname{coker}(\beta)$, respectively, then we have $\gamma\pi_1 = \pi_2\gamma_0$. This yields a commutative diagram

$$\begin{array}{ccccccccc}
]0 & \longrightarrow & R^{4n} & \xrightarrow{(\alpha,1,1,1)} & R^{4n} & \xrightarrow{\pi_1} & R^n/\alpha R^n & \longrightarrow & 0 \\
 & & \gamma_1 \big\downarrow & & \cong \big\downarrow \gamma_0 & & \cong \big\downarrow \gamma & & \\
0 & \longrightarrow & R^{4n} & \xrightarrow{(1,\beta,1,1)} & R^{4n} & \xrightarrow{\pi_2} & R^n/\beta R^n & \longrightarrow & 0
\end{array}$$

in which γ_1 is the induced map. Since γ and γ_0 are isomorphisms, so is γ_1. Because $\gamma_0(\alpha, 1, 1, 1) = (1, \beta, 1, 1)\gamma_1$ in $GL_{4n}(S^{-1}R)$, we have $[\gamma_0] + [\alpha] = [\beta] + [\gamma_1]$, or $[\alpha] = [\gamma_1\gamma_0^{-1}][\beta]$ in $K_1(S^{-1}R)$. □

NK_1 and the group Nil_0

III.3.3 **Definition 3.3** (NF). If F is any functor from rings to abelian groups, we write $NF(R)$ for the cokernel of the natural map $F(R) \to F(R[t])$; NF is also a functor on rings. Moreover, the ring map $R[t] \xrightarrow{t=1} R$ provides a splitting $F(R[t]) \to F(R)$ of the natural map, so we have a natural decomposition $F(R[t]) \cong F(R) \oplus NF(R)$.

In particular, when F is K_n ($n = 0, 1$) we have functors NK_n and a decomposition $K_n(R[t]) \cong K_n(R) \oplus NK_n(R)$. Since the ring maps $R[t] \xrightarrow{t=r} R$ are split surjections for every $r \in R$, we see by Proposition 2.3 and Ex. 2.7 that for every r we also have

$$NK_0(R) \cong K_0(R[t], (t-r)) \quad \text{and} \quad NK_1(R) \cong K_1(R[t], (t-r)).$$

We will sometimes speak about NF for functors F defined on any category of rings closed under polynomial extensions and containing the map "$t = 1$," such as k-algebras or commutative rings. For example, the functors NU and $N\operatorname{Pic}$ were discussed briefly for commutative rings in chapter I, Ex. 3.17 and 3.19.

III.3.3.1 **Example 3.3.1.** (Chase) Suppose that A is an algebra over $\mathbb{Z}/p$. Then the group $NK_1(A)$ is a p-group. To see this, first observe that it is true for the algebras $A_n = \mathbb{Z}/p[x]/(x^n)$ by 2.4, since $(1 + tf(x,t))^p = 1 + t^p f(x^p, t^p)$. Then observe that by Higman's trick (3.5.1 below) every element of $NK_1(A)$ is the image of $1 - xt \in NK_1(A_n)$ (for some n) under a map $A_n \to M_n(A)$, $x \mapsto \nu$.

By Ex. 2.8, $NK_1(A)$ is also a p-group for every $\mathbb{Z}/p^r$-algebra A.

III.3.3.2 **Example 3.3.2.** If A is an algebra over a field k of characteristic zero, then $NK_1(A)$ is a uniquely divisible abelian group. In fact, $NK_1(A)$ has the structure of a k-vector space; see Ex. 3.7.

III.3.4 **Definition 3.4** (F-regular rings)**.** We say that a ring R is *F-regular* if $F(R) = F(R[t_1, \dots, t_n])$ for all n. Since $NF(R[t]) = NF(R) \oplus N^2F(R)$, we see by induction on p that R is F-regular if and only if $N^pF(R) = 0$ for all $p \geq 1$.

For example, Traverso's theorem (I.3.11) states that a commutative ring R is Pic-regular if and only if R_{red} is seminormal. We also saw in I.3.12 that commutative rings are U-regular (U=units) if and only if R is reduced.

We saw in II.6.5 that any regular ring is K_0-regular. We will see in Theorem 3.8 below that regular rings are also K_1-regular, and we will see in chapter V that they are K_m-regular for every m. Rosenberg has also shown that commutative C^*-algebras are K_m-regular for all m; see [162].

III.3.4.1 **Lemma 3.4.1.** *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Then the kernel of $F(R) \to F(R_0)$ embeds in $NF(R)$ and even in the kernel of $NF(R) \to NF(R_0)$.*

In particular, if $NF(R) = 0$ then $F(R) \cong F(R_0)$.

Proof. Let f denote the ring map $R \to R[t]$ defined by $f(r_n) = r_n t^n$ for every $r_n \in R_n$. Since the composition of f and "$t = 1$" is the identity on R, $F(f)$ is an injection. Let Q denote the kernel of $F(R) \to F(R_0)$, so that $F(R) = F(R_0) \oplus Q$. Since the composition of f and "$t = 0$" is the projection $R \to R_0 \to R$, Q embeds into the kernel $NF(R)$ of the evaluation $F(R[t]) \to F(R)$ at $t = 0$. Similarly, since the composition of f and $R[t] \to R_0[t]$ is projection $R \to R_0 \to R_0[t]$, Q embeds into the kernel of $NF(R) \to NF(R_0)$. □

A typical application of this result is that if R is a graded seminormal algebra with R_0 a field then $\text{Pic}(R) = 0$.

III.3.4.2 **Application 3.4.2.** It follows that $NF(R)$ is a summand of $N^2F(R)$ and hence $N^pF(R)$ for all $p > 0$. Indeed, the first part is the application of 3.4.1 to $R[s]$, and the second part is obtained by induction, replacing F by $N^{p-2}F$.

If $s \in S$ is central, the algebra map $S[x] \to S[x]$, $x \to sx$, induces an operation $[s] : NK_0(S) \to NK_0(S)$. (It is the multiplication by $1 - st \in W(S)$ in Ex. 3.7.) Write S_s for $S[1/s]$.

III.3.4.3 **Theorem 3.4.3.** *(Vorst) $NK_0(S_s)$ is the "localization" of $NK_0(S)$ along $[s]$:*

$$NK_0(S_s) \cong \varinjlim(NK_0(S) \xrightarrow{[s]} NK_0(S) \xrightarrow{[s]} \cdots).$$

In particular, if S is K_0-regular, so is S_s.

Proof. Write I for the ideal (x) of $S_s[x]$ and set $R = S + I$. Then $NK_0(S_s) = K_0(I)$ by Ex. II.2.3. But $R = \varinjlim(S[x] \to S[x] \to \cdots)$ and I is the direct limit of $xS[x] \to xS[x] \to \cdots$, so $K_0(I) = \varinjlim(K_0(xS[x]) \to \cdots)$ as claimed. □

III.3.4.4 **Corollary 3.4.4.** *If A is K_0-regular then so is $A[s, s^{-1}]$.*

III.3.5 **3.5.** We are going to describe the group $NK_1(R)$ in terms of nilpotent matrices. For this, we need the following trick, which was published by Graham Higman in 1940. For clarity, if $I = fA$ is an ideal in A we write $GL(A, f)$ for $GL(I)$.

III.3.5.1 **Higman's Trick 3.5.1.** For every $g \in GL(R[t], t)$ there is a nilpotent matrix ν over R such that $[g] = [1 - \nu t]$ in $K_1(R[t])$.

Similarly, for every $g \in GL(R[t, t^{-1}], t-1)$ there is a nilpotent matrix ν over R such that $[g] = [1 - \nu(t-1)]$ in $K_1(R[t, t^{-1}], t-1)$.

Proof. Every invertible $p \times p$ matrix over $R[t]$ can be written as a polynomial $g = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_n t^n$ with the γ_i in $M_p(R)$. If g is congruent to the identity modulo t, then $\gamma_0 = 1$. If $n \geq 2$ and we write $g = 1 - ht + \gamma_n t^n$, then modulo $E_{2p}(R[t], t)$ we have

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} g & \gamma_n t^{n-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1-ht & \gamma_n t^{n-1} \\ -t & 1 \end{pmatrix} = 1 - \begin{pmatrix} h & -\gamma_n t^{n-2} \\ 1 & 0 \end{pmatrix} t.$$

By induction on n, $[g]$ is represented by a matrix of the form $1 - \nu t$. The matrix ν is nilpotent by Ex. EIII.3.1 3.1.

Over $R[t, t^{-1}]$ we can use a similar argument. After multiplying by a power of t, we may write g as a polynomial in t. Such a polynomial may be rewritten as a polynomial $\sum \gamma_i x^i$ in $x = (t-1)$. If g is congruent to the identity modulo $(t-1)$ then again we have $\gamma_0 = 1$. By Higman's trick (applied to x), we may reduce g to a matrix of the form $1 - \nu x$, and again ν must be nilpotent by Ex. EIII.3.1 3.1. ☐

We will also need the category $\mathbf{Nil}(R)$ of II.II.7.4.4 7.4.4. Recall that the objects of this category are pairs (P, ν), where P is a finitely generated projective R-module and ν is a nilpotent endomorphism of P. Let T denote the multiplicative set $\{t^n\}$ in $R[t]$. From II.II.7.8.4 7.8.4 we have

$$K_0(R[t] \text{ on } T) \cong K_0\mathbf{Nil}(R) \cong K_0(R) \oplus \mathrm{Nil}_0(R),$$

where $\mathrm{Nil}_0(R)$ is the subgroup generated by elements of the form $\big[(R^n, \nu)\big] - n\big[(R, 0)\big]$ for some n and some nilpotent matrix ν.

III.3.5.2 **Lemma 3.5.2.** *For every ring R, the product with $t \in K_1(\mathbb{Z}[t, t^{-1}])$ induces a split injection $K_0(R) \xrightarrow{\cdot t} K_1(R[t, t^{-1}])$.*

Proof. Since the forgetful map $K_0\mathbf{Nil}(R) \to K_0(R)$ sends $\big[(P, \nu)\big]$ to $[P]$, the calculation in Example III.3.1.4 3.1.4 shows that the composition

$$K_0(R) \xrightarrow{\cdot t} K_1(R[t, t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to K_0(R)$$

is the identity map. Hence the first map is a split injection. ☐

Momentarily changing variables from t to s, we now define an additive function τ from $\mathbf{Nil}(R)$ to $K_1(R[s])$. Given an object (P,ν), let $\tau(P,\nu)$ be the image of the automorphism $1-\nu s$ of $P[s]$ under the natural map $\mathrm{Aut}(P[s]) \to K_1(R[s])$ of Lemma 1.6. Given a short exact sequence

$$0 \to (P',\nu') \to (P,\nu) \to (P'',\nu'') \to 0$$

in $\mathbf{Nil}(R)$, a choice of a splitting $P \cong P' \oplus P''$ allows us to write

$$(1-\nu s) = \begin{pmatrix} 1-\nu' s & \gamma s \\ 0 & 1-\nu'' s \end{pmatrix} = \begin{pmatrix} 1-\nu' s & 0 \\ 0 & 1-\nu'' s \end{pmatrix} \begin{pmatrix} 1 & \gamma' s \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{Aut}(P[s])$. Hence in $K_1(R[s])$ we have $[1-\nu s] = [1-\nu' s][1-\nu'' s]$. Therefore τ is an additive function, and induces a homomorphism $\tau\colon K_0\mathbf{Nil}(R) \to K_1(R[s])$. Since $\tau(P,0)=1$ for all P and $1-\nu s$ is congruent to 1 modulo s, we see that τ is actually a map from $\mathrm{Nil}_0(R)$ to $K_1(R[s],s)$.

III.3.5.3 **Proposition 3.5.3.** *$\mathrm{Nil}_0(R) \cong NK_1(R)$, and $K_0\mathbf{Nil}(R) \cong K_0(R) \oplus NK_1(R)$.*

Proof. For convenience, we identify s with t^{-1}, so that $R[s,s^{-1}] = R[t,t^{-1}]$. Applying Lemma 3.1 to $R[t]$ and $T = \{1,t,t^2,\dots\}$, form the composition

$$\begin{aligned} K_1(R[s],s) \to K_1(R[s]) \to & K_1(R[s,s^{-1}]) \\ = & K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R \text{ on } T) \to \mathrm{Nil}_0(R). \end{aligned} \tag{3.5.4}$$ III.3.5.4

Let us call this composition δ. We claim that τ is the inverse of δ. By Higman's Trick, every element of $K_1(R[s],s)$ is represented by a matrix $1-\nu s$ with ν nilpotent. In Example 3.1.4 we saw that $\delta(1-\nu s) = \big[(R^n,\nu)\big] - n\big[(R,0)\big]$. By the construction of τ we have the desired equations: $\tau\delta(1-\nu s) = \tau\big[(R^n,\nu)\big] = (1-\nu s)$ and

$$\delta\tau\Big(\big[(R^n,\nu)\big] - n\big[(R,0)\big]\Big) = \delta(1-\nu s) = \big[(R^n,\nu)\big] - n\big[(R,0)\big]. \quad \square$$

III.3.5.5 **Corollary 3.5.5.** *$K_1(R[s]) \to K_1(R[s,s^{-1}])$ is an injection for every ring R.*

Proof. By Ex. 2.7, we have $K_1(R[s]) \cong K_1(R) \oplus K_1(R[s],s)$. Since $K_1(R)$ is a summand of $K_1(R[s,s^{-1}])$, the isomorphism $\delta\colon K_1(R[s],s) \cong \mathrm{Nil}_0(R)$ of (3.5.4) factors through $K_1(R[s],s) \to K_1(R[s,s^{-1}])/K_1(R)$. This quotient map must then be an injection. The result follows. $\square$

The Fundamental Theorems for K_1 and K_0

III.3.6 **Fundamental Theorem for K_1 3.6.** *For every ring R, there is a split surjection $K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R)$, with inverse $[P] \mapsto [P]\cdot t$. This map fits into a naturally split exact sequence:*

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R) \to 0.$$

Consequently, we have a natural direct sum decomposition:

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus NK_1(R) \oplus NK_1(R).$$

Proof. We merely assemble the pieces of the proof from 3.5. The first assertion is just Lemma 3.5.2. The natural maps from $K_1(R)$ into $K_1(R[t])$, $K_1(R[t^{-1}])$ and $K_1(R[t,t^{-1}])$ are injections, split by "$t=1$" (as in 3.5), so the obviously exact sequence

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R)\oplus K_1(R) \xrightarrow{\pm} K_1(R) \to 0 \qquad (3.6.1)$$ III.3.6.1

is a summand of the sequence we want to prove exact. From Proposition II.7.8.1, Theorem 3.2 and Corollary 3.5.5, we have an exact sequence

$$0 \to K_1(R[t]) \to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to 0. \qquad (3.6.2)$$ III.3.6.2

Since $K_0\mathbf{Nil}(R) \cong K_0(R)\oplus \mathrm{Nil}_0(R)$, the map ∂ in (3.6.2) is split by the maps of 3.5.2 and 3.5.3. The sequence in the Fundamental Theorem for K_1 is obtained by rearranging the terms in sequences (3.6.1) and (3.6.2). □

In order to formulate the corresponding Fundamental Theorem for K_0, we define $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t])\oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])$. We will reprove the following result more formally in the next section.

III.3.7 **Fundamental Theorem for K_0 3.7.** *For every ring R, there is a naturally split exact sequence:*

$$0 \to K_0(R) \xrightarrow{\Delta} K_0(R[t])\oplus K_0(R[t^{-1}]) \xrightarrow{\pm} K_0(R[t,t^{-1}]) \xrightarrow{\partial} K_{-1}(R) \to 0.$$

Consequently, we have a natural direct sum decomposition:

$$K_0(R[t,t^{-1}]) \cong K_0(R)\oplus K_{-1}(R)\oplus NK_0(R)\oplus NK_0(R).$$

Proof. Let s be a second indeterminate. The Fundamental Theorem for K_1, applied to the variable t, gives a natural decomposition

$$K_1(R[s,t,t^{-1}]) \cong K_1(R[s])\oplus NK_1(R[s])\oplus NK_1(R[s])\oplus K_0(R[s]),$$

and similar decompositions for the other terms in the map

$$K_1(R[s,t,t^{-1}])\oplus K_1(R[s^{-1},t,t^{-1}]) \to K_1(R[s,s^{-1},t,t^{-1}]).$$

Therefore the cokernel of this map also has a natural splitting. But the cokernel is $K_0(R[t,t^{-1}])$, as we see by applying the Fundamental Theorem for K_1 to the variable s. □

III.3.8 **Theorem 3.8.** *If R is a regular ring, $K_1(R[t])\cong K_1(R)$ and there is a natural isomorphism $K_1(R[t,t^{-1}])\cong K_1(R)\oplus K_0(R)$.*

Proof. Consider the category $\mathbf{M}_t(R[t])$ of finitely generated t-torsion $R[t]$-modules; by devissage II.6.3.3, $K_0\mathbf{M}_t(R[t]) \cong K_0(R)$. Since R is regular, every such module has a finite resolution by finitely generated projective $R[t]$-modules, *i.e.,* $\mathbf{M}_t(R[t])$ is the same as the category $\mathbf{H}_t(R[t])$ of II.7.7. By II.7.8.4,

$$K_0\mathbf{Nil}(R) \cong K_0\mathbf{H}_t(R[t]) \cong K_0(R).$$

Hence $\mathrm{Nil}_0(R) = 0$. By 3.5.3, $NK_1(R) = 0$ and $K_1(R[t]) \cong K_1(R)$. The description of $K_1(R[t,t^{-1}])$ now comes from the Fundamental Theorem 3.6. □

III.3.8.1 **Example 3.8.1.** If R is a commutative regular ring, and $A = R[x]/(x^N)$, it follows from 2.4 and 3.8 that $SK_1(A[t]) = SK_1(A)$ and hence (by 3.5.3 and I.3.12) $\mathrm{Nil}_0(A) \cong NK_1(A) \cong (1+tA[t])^\times = (1+xtA[t])^\times$. This isomorphism sends $[(P,\nu)] \in \mathrm{Nil}_0(A)$ to $\det(1-\nu t) \in (1+tA[t])^\times$. By inspection, this is the restriction of the canonical $\mathrm{End}_0(A)$-module map $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ of II.7.4.4, followed by the inclusion $\mathrm{End}_0(A) \subset W(A)$ of II.7.4.3. It follows that $\mathrm{Nil}_0(A)$ is an ideal of the ring $\mathrm{End}_0(A)$.

EXERCISES

EIII.3.1 **3.1.** Let A be a ring and $a \in A$, show that the following are equivalent:
(i) a is nilpotent;
(ii) $1 - at$ is a unit of $A[t]$;
(iii) $1 - a(t-1)$ is a unit of $A[t,t^{-1}]$.

EIII.3.2 **3.2.** Let $\alpha, \beta\colon P \to Q$ be two maps between finitely generated projective R-modules. If S is a central multiplicatively closed set in R and $S^{-1}\alpha, S^{-1}\beta$ are isomorphisms, then $g = \beta^{-1}\alpha$ is an automorphism of $S^{-1}P$. Show that $\partial(g) = [\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$. In particular, if S consists of nonzerodivisors, show that $\partial(g) = [\mathrm{coker}(\alpha)] - [\mathrm{coker}(\beta)]$.

EIII.3.3 **3.3.** (Bass) Prove that every module M in $\mathbf{H}(R)$ has a projective resolution $P. \to M$ such that every automorphism α of M lifts to an *automorphism* of the chain complex $P.$. To do so, proceed as follows.

(a) Fix a surjection $\pi\colon Q \to M$, and use Ex. I.1.11 to lift the automorphism $\alpha \oplus \alpha^{-1}$ of $M \oplus M$ to an automorphism β of $Q \oplus Q$.

(b) Defining $e\colon Q \oplus Q \to M$ to be $e(x,y) = \pi(x)$, show that every automorphism of M can be lifted to an automorphism of $Q \oplus Q$.

(c) Set $P_0 = Q \oplus Q$, and repeat the construction on $Z_0 = \ker(e)$ to get a finite resolution $P.$ of M with the desired property.

EIII.3.4 **3.4.** Suppose that S consists of nonzerodivisors, and that M is a module in $\mathbf{H}_S(R)$.

(a) Prove that there is a module M' and an S-isomorphism $\alpha \in \mathrm{End}(R^m)$ so that $\mathrm{coker}(\alpha) = M \oplus M'$. *Hint:* Modify the proof of Lemma 3.1.5, where M is the cokernel of a map $P_1 \xrightarrow{\beta} P_0$.

(b) Given S-isomorphisms $\alpha', \alpha'' \in \mathrm{End}(R^m)$ and a short exact sequence of S-torsion modules $0 \to \mathrm{coker}(\alpha') \to M \to \mathrm{coker}(\alpha'') \to 0$, show that there is an S-isomorphism $\alpha \in R^{2m}$ with $M \cong \mathrm{coker}(\alpha)$.

EIII.3.5 **3.5.** Modify the proofs of the previous two exercises to prove Theorem 3.2 when S contains zerodivisors.

EIII.3.6 **3.6.** *Noncommutative localization.* By definition, a multiplicatively closed set S in a ring R is called a *right denominator set* if it satisfies the following two conditions: (i) For any $s \in S$ and $r \in R$ there exists an $s' \in S$ and $r' \in R$ such that $sr' = rs'$; (ii) if $sr = 0$ for any $r \in R$, $s \in S$ then $rs' = 0$ for some $s' \in S$. This is the most general condition under which a (right) ring of fractions $S^{-1}R$ exists, in which every element of $S^{-1}R$ has the form $r/s = rs^{-1}$, and if $r/1 = 0$ then some $rs = 0$ in R.

Prove Theorem 3.2 when S is a right denominator set consisting of nonzerodivisors. To do this, proceed as follows.

(a) Show that for any finite set of elements x_i in $S^{-1}R$ there is an $s \in S$ and $r_i \in R$ so that $x_i = r_i/s$ for all i.

(b) Reprove II.7.7.3 and II.9.8 for denominator sets, using (a); this yields exactness at $K_0(R)$.

(c) Modify the proof of Lemma 3.1 and 3.1.5 to construct the map ∂ and prove exactness at $K_0\mathbf{H}_S(R)$.

(d) Modify the proof of Theorem 3.2 to prove exactness at $K_1(S^{-1}R)$.

EIII.3.7 **3.7.** Let A be an algebra over a commutative ring R. Recall from Ex. II.7.18 that $NK_1(A) = \mathrm{Nil}_0(A)$ is a module over the ring $W(R) = 1 + tR[[t]]$ of Witt vectors II.4.3. In this exercise we develop a little of the structure of $W(R)$, which yields information about the structure of $NK_1(A)$ and hence (by Theorem 3.7) the structure of $NK_0(R)$.

(a) If $1/p \in R$ for some prime integer p, show that $W(R)$ is an algebra over $\mathbb{Z}[1/p]$. Conclude that $NK_1(A)$ and $NK_0(A)$ are uniquely p-divisible abelian groups. *Hint:* use the fact that the coefficients in the power series expansion for $r(t) = (1+t)^{1/p}$ only involve powers of p.

(b) If $\mathbb{Q} \subseteq R$, consider the exponential map $\prod_{i=1}^{\infty} R \to W(R)$, sending $(r_1, \ldots)$ to $\prod_{i=1}^{\infty} \exp(-r_i t^i/i)$. This is an isomorphism of abelian groups, whose inverse (the "ghost map") is given by the coefficients of $f \mapsto -t\ d/dt(\ln f)$. Show that this is a ring isomorphism. Conclude that $NK_1(A)$ and $NK_0(A)$ have the structure of R-modules.

(c) If $n \in \mathbb{Z}$ is nonzero, Stienstra showed that $NK_1(A)[1/n] \cong NK_1(A[1/n])$. Use this to show that if G is a finite group of order n then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n.

EIII.3.8 **3.8.** If I is a nilpotent ideal in a $\mathbb{Q}$-algebra A, show that $NK_1(A,I) \to K_1(A,I)$ is onto. Thus Ex. 3.7 gives another proof that $K_1(A,I)$ is divisible (Ex. 2.9).

EIII.3.9 **3.9.** If $s \in S$ is central, show that $NK_1(S_s)$ is a localization of $NK_1(S)$ in the sense of 3.4.3. Conclude that if S is K_1-regular then S is K_0-regular. *Hint:* Use the sequence of Ex. 2.6.

EIII.3.10 **3.10.** (Karoubi) Let S be a multiplicatively closed set of central nonzerodivisors in a ring A. We say that a ring homomorphism $f : A \to B$ is an *analytic isomorphism* along S if $f(S)$ consists of central nonzerodivisors in B, and if $A/sA \cong B/sB$ for every $s \in S$. (This implies that the s-adic completions of A and B are isomorphic, whence the name.)

If f is an analytic isomorphism along S, show that $M \mapsto M \otimes_A B$ defines an equivalence of categories $\mathbf{H}_S(A) \cong \mathbf{H}_S(B)$. (One proof is given in V.7.5 below.) Using Theorem 3.2 and Ex. II.9.13, this shows that we have an exact sequence

$$K_1(S^{-1}A) \oplus K_1(B) \to K_1(S^{-1}B) \to K_0(A) \to K_0(S^{-1}A) \oplus K_0(B) \to K_0(S^{-1}B).$$

Hint: For M in $\mathbf{H}^1_S(A)$, show that $\mathrm{Tor}_1^A(M,B) = 0$, so that the functor $\mathbf{H}^1_S(A) \to \mathbf{H}^1_S(B)$ is exact. Then use Lemma II.7.7.1 to show that $\mathbf{H}_S(A)$ is the category of modules having a finite resolution by modules in $\mathbf{H}^1_S(A)$, and similarly for $\mathbf{H}_S(B)$.

4 Negative K-theory

In the last section, we defined $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])$. Of course we can keep going, and define all the negative K-groups by induction on n:

III.4.1 **Definition 4.1.** For $n > 0$, we inductively define $K_{-n}(R)$ to be the cokernel of the map

$$K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t,t^{-1}]).$$

Clearly, each K_{-n} is a functor from rings to abelian groups. It follows from Theorem II.7.8 that if R is regular noetherian then $K_n(R) = 0$ for all $n < 0$.

To describe the properties of these negative K-groups, it is convenient to cast the Fundamental Theorems above in terms of Bass' notion of *contracted functors*. With this in mind, we make the following definitions.

III.4.1.1 **Definition 4.1.1** (Contracted functors)**.** Let F be a functor from rings to abelian groups. For each R, we define $LF(R)$ to be the cokernel of the map $F(R[t]) \oplus F(R[t^{-1}]) \to F(R[t,t^{-1}])$. We write $Seq(F,R)$ for the following sequence, where $\Delta(a) = (a,a)$ and $\pm(b,c) = b - c$:

$$0 \to F(R) \xrightarrow{\Delta} F(R[t]) \oplus F(R[t^{-1}]) \xrightarrow{\pm} F(R[t,t^{-1}]) \to LF(R) \to 0.$$

We say that F is *acyclic* if $Seq(F,R)$ is exact for all R. We say that F is a *contracted functor* if F is acyclic and in addition there is a splitting $h = h_{t,R}$ of the defining map $F(R[t,t^{-1}]) \to LF(R)$, a splitting which is natural in both t and R. The notation F_{-1} is sometimes used for LF.

By iterating this definition, we can speak about the functors NLF, L^2F, etc. For example, Definition 4.1 states that $K_{-n} = L^n(K_0)$.

As with the definition of NF (3.3), it will occasionally be useful to define LF etc. on a more restricted class of rings, such as commutative algebras. Suppose that $\mathcal{R}$ is a category of rings such that if R is in $\mathcal{R}$ then so are $R[t]$, $R[t,t^{-1}]$ and the maps $R \to R[t] \rightrightarrows R[t,t^{-1}]$. Then the definitions of LF, L^nF and $Seq(F,R)$ in 4.1.1 make sense for any functor F from $\mathcal{R}$ to any abelian category.

III.4.1.2 **Example 4.1.2** (Fundamental Theorem for K_{-n})**.** We can restate the Fundamental Theorems for K_1 and K_0 as the assertions that these are contracted functors. It follows from Proposition 4.2 below that each K_{-n} is a contracted functor; by Definition 4.1, this means that there is a naturally split exact sequence:

$$0 \to K_{-n}(R) \xrightarrow{\Delta} K_{-n}(R[t]) \oplus K_{-n}(R[t^{-1}]) \xrightarrow{\pm} K_{-n}(R[t,t^{-1}]) \xrightarrow{\partial} K_{-n-1}(R) \to 0.$$

III.4.1.3 **Example 4.1.3** (Units)**.** Let $U(R) = R^\times$ denote the group of units in a commutative ring R. By Ex. I.3.17, U is a contracted functor with contraction $LU(R) = [\mathrm{Spec}(R), \mathbb{Z}]$; the splitting map $LU(R) \to U(R[t,t^{-1}])$ sends a continuous function $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ to the unit t^f of $R[t,t^{-1}]$. From Ex. 4.2 below we see that the functors L^2U and NLU are zero. Thus we can write a simple formula for the units of any extension $R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. If R is reduced, so that $NU(R)$ vanishes (Ex. I.3.17), then we just have

$$U(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) = U(R) \times \coprod_{i=1}^{n} [\mathrm{Spec}(R), \mathbb{Z}] \cdot t_i.$$

III.4.1.4 **Example 4.1.4** (Pic)**.** Recall from chapter I, §3 that the Picard group $\mathrm{Pic}(R)$ of a commutative ring is a functor, and that $N\,\mathrm{Pic}(R) = 0$ exactly when R_{red} is seminormal. By Ex. I.3.18 the sequence $\mathit{Seq}(\mathrm{Pic}, R)$ is exact. In fact Pic is a contracted functor with $NL\,\mathrm{Pic} = L^2\,\mathrm{Pic} = 0$; see [221]. The group $L\,\mathrm{Pic}(R)$ is the étale cohomology group $H^1_{\mathrm{et}}(\mathrm{Spec}(R), \mathbb{Z})$.

A *morphism of contracted functors* is a natural transformation $\eta\colon F \Rightarrow F'$ between two contracted functors such that the following square commutes for all R.

$$\begin{array}{ccc} LF(R) & \xrightarrow{h} & F(R[t,t^{-1}]) \\ {\scriptstyle (L\eta)_R}\downarrow & & \downarrow{\scriptstyle \eta_{R[t,t^{-1}]}} \\ LF'(R) & \xrightarrow{h'} & F'(R[t,t^{-1}]) \end{array}$$

III.4.2 **Proposition 4.2.** *Let $\eta\colon F \Rightarrow F'$ be a morphism of contracted functors. Then both $\ker(\eta)$ and $\mathrm{coker}(\eta)$ are also contracted functors.*

In particular, if F is contracted, then NF and LF are also contracted functors. Moreover, there is a natural isomorphism of contracted functors $NLF \cong LNF$.

Proof. If $C \xrightarrow{\phi} D$ is a morphism between split exact sequences, which have compatible spittings, then the sequences $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are always split exact, with splittings induced from the splittings of C and D. Applying this remark to $\mathit{Seq}(F,R) \to \mathit{Seq}(F',R)$ shows that $\ker(\eta)$ and $\mathrm{coker}(\eta)$ are contracted functors: both $\mathit{Seq}(\ker(\eta), R)$ and $\mathit{Seq}(\mathrm{coker}(\eta), R)$ are split exact. It also shows that

$$0 \to \ker(\eta)(R) \to F(R) \xrightarrow{\eta_R} F'(R) \to \mathrm{coker}(\eta)(R) \to 0$$

is an exact sequence of contracted functors.

Since $NF(R)$ is the cokernel of the morphism $F(R) \to F'(R) = F(R[t])$ and $LF(R)$ is the cokernel of the morphism $\pm$ in $\mathit{Seq}(F,R)$, both NF and LF are contracted functors. Finally, the natural isomorphism $NLF(R) \cong LNF(R)$ arises from inspecting one corner of the large commutative diagram represented by

$$0 \to \mathit{Seq}(F, R[s], s) \to \mathit{Seq}(F, R[s]) \to \mathit{Seq}(F,R) \to 0. \quad \square$$

III.4.2.1 **Example 4.2.1** (SK_1)**.** If R is a commutative ring, it follows from Examples 1.1.1 and 4.1.3 that $\det\colon K_1(R) \to U(R)$ is a morphism of contracted functors. Hence SK_1 is a contracted functor. The contracted map $L\det$ is the map rank: $K_0(R) \to H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$ of II.2.3; it follows that $L(SK_1)(R) = \widetilde{K}_0(R)$. From Ex. 4.2 we also have $L^2(SK_1)(R) = L\widetilde{K}_0(R) = K_{-1}(R)$.

We can give an elegant formula for $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$, using the following notation. If $p(N, L) = \sum m_{ij} N^i L^j$ is any formal polynomial in N and L with integer coefficients $m_{ij} > 0$, and F is a functor from rings to abelian groups, we set $p(N, L)F$ equal to the direct sum (over i and j) of m_{ij} copies of each group $N^i L^j F(R)$.

III.4.2.2 **Corollary 4.2.2.** *$F(R[t_1, \dots, t_n]) \cong (1 + N)^n F(R)$ for every F. If F is a contracted functor, then $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) \cong (1 + 2N + L)^n F(R)$.*

Proof. The case $n = 1$ follows from the definitions; the general case follows by induction. □

For example, if $L^2 F = 0$ and R is F-regular, then $(1+2N+L)^n F(R)$ stands for $F(R) \oplus n\, LF(R)$. In particular, the formula for units in Example 4.1.3 is just the case $F = U$ of 4.2.2.

III.4.2.3 **Example 4.2.3.** Since $L^j K_0 = K_{-j}$, $K_0(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$ is the direct sum of many pieces $N^i K_{-j}(R)$, including $K_{-n}(R)$ and $\binom{n}{j}$ copies of $K_{-j}(R)$.

From 3.4.4 we see that if R is K_n-regular for some $n \le 0$ then R is also K_{n-1}-regular. In particular, if R is K_0-regular then R is also K_n-regular for all $n < 0$.

III.4.2.4 **Conjecture 4.2.4.** *Let R be a commutative noetherian ring of Krull dimension d. It is conjectured that $K_{-j}(R)$ vanishes for all $j > d$, and that R is K_{-d}-regular; see [217]. This is so for $d = 0, 1$ by Exercises 4.3 and 4.4, and Example 4.3.1 below shows that the bound is best possible. It was recently shown to be true for $\mathbb{Q}$-algebras in [46].*

The Mayer-Vietoris sequence

Suppose that $f\colon R \to S$ is a ring map, and I is an ideal of R mapped isomorphically into an ideal of S. By Theorem 2.6 there is an exact "Mayer-Vietoris" sequence:

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \xrightarrow{\pm} K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Applying the contraction operation L to this sequence gives a sequence relating K_0 to K_{-1}, whose first three terms are identical to the last three terms of the displayed sequence. Splicing these together yields a longer sequence. Repeatedly applying L and splicing sequences leads to the following result.

III.4.3 **Theorem 4.3** (Mayer-Vietoris)**.** *Suppose we are given a ring map $f\colon R \to S$ and an ideal I of R mapped isomorphically into an ideal of S. Then the Mayer-*

Vietoris sequence of Theorem 2.6 continues as a long exact Mayer-Vietoris sequence of negative K-groups.

$$\xrightarrow{\Delta} \begin{bmatrix} K_0(S)\oplus \\ K_0(R/I) \end{bmatrix} \xrightarrow{\pm} K_0(S/I) \xrightarrow{\partial} K_{-1}(R) \xrightarrow{\Delta} \begin{bmatrix} K_{-1}(S)\oplus \\ K_{-1}(R/I) \end{bmatrix} \xrightarrow{\pm} K_{-1}(S/I) \xrightarrow{\partial} \cdots$$

$$\cdots \to K_{-n+1}(S/I) \xrightarrow{\partial} K_{-n}(R) \xrightarrow{\Delta} \begin{bmatrix} K_{-n}(S)\oplus \\ K_{-n}(R/I) \end{bmatrix} \xrightarrow{\pm} K_{-n}(S/I) \xrightarrow{\partial} \cdots$$

III.4.3.1 **Example 4.3.1.** (B. Dayton) Fix a regular ring R, and let $\Delta^n(R)$ denote the coordinate ring $R[t_0,\dots,t_n]/(f)$, $f = t_0\cdots t_n(1-\sum t_i)$ of the n-dimensional tetrahedron over R. Using $I = (1-\sum t_i)\Delta^n(R)$ and $\Delta^n(R)/I \cong R[t_1,\dots,t_n]$ via $t_0 \mapsto 1-(t_1+\cdots+t_n)$, we have a Milnor square

$$\begin{array}{ccc} \Delta^n(R) & \longrightarrow & A_n \\ \downarrow & & \downarrow \\ R[t_1,\dots,t_n] & \longrightarrow & \Delta^{n-1}(R) \end{array}$$

where $A_n = R[t_0,\dots,t_n]/(t_0\cdots t_n)$. By Ex. 4.8, the negative K-groups of A_n vanish and $K_i(A_n) = K_i(R)$ for $i = 0,1$. Thus $K_0(\Delta^n(R)) \cong K_0(R) \oplus K_1(\Delta^{n-1}(R))/K_1(R)$ for $n>0$, and $K_{-j}(\Delta^n(R)) \cong K_{1-j}(\Delta^{n-1}(R))$ for $j>0$. These groups vanish for $j>n$, with $K_{-n}(\Delta^n(R)) \cong K_0(R)$. In particular, if F is a field then $\Delta^n(F)$ is an n-dimensional noetherian ring with $K_{-n}(\Delta^n(F)) \cong \mathbb{Z}$; see Conjecture 4.2.4.

When we have introduced higher K-theory, we will see that in fact $K_0(\Delta^n(R)) \cong K_n(R)$ and $K_1(\Delta^n(R)) \cong K_{n+1}(R)$. (See IV, Ex. 12.1.) This is just one way in which higher K-theory appears in classical K-theory.

Theories of Negative K-theory

Here is an alternative approach to defining negative K-theory, due to Karoubi and Villamayor [100].

III.4.4 **Definition 4.4.** A *theory of negative K-theory* for (nonunital) rings consists of a sequence of functions K_n ($n \le 0$) from nonunital rings to abelian groups, together with natural boundary maps $\partial : K_n(R/I) \to K_{n-1}(I)$ for every 2-sided ideal $I \subset R$, satisfying the following axioms.

(1) $K_0(R)$ is the Grothendieck group of chapter II;

(2) $K_n(I) \to K_n(R) \to K_n(R/I) \xrightarrow{\partial} K_{n-1}(I) \to K_{n-1}(R)$ is exact for every ideal $I \subset R$;

(3) If Λ is a flasque ring (II.2.1.3), then $K_n(\Lambda) = 0$ for all $n \le 0$;

(4) The inclusion $R \subset M(R) = \cup M_m(R)$ induces an isomorphism $K_n(R) \cong K_n(M(R))$ for each $n \le 0$.

III.4.4.1 **Example 4.4.1.** Bass' negative K-groups (4.1) form a theory of negative K-theory for rings. This follows from the contraction of 2.3 (see Ex. 4.5), Ex. 4.9 and the contraction of Morita Invariance 1.6.4.

III.4.4.2 **Example 4.4.2.** Embedding $M(R)$ as an ideal in a flasque ring Λ, axiom (2) shows that $K_{-1}R \cong K_0(\Lambda/M(R))$. This was the approach used by Karoubi and Villamayor in [100] to inductively define a theory of negative K-theory; see Ex. 4.10.

III.4.4.3 **Example 4.4.3.** If A is a hensel local ring then $K_{-1}(A) = 0$. This was proven by Drinfeld in [49], using a *Calkin category* model for negative K-theory.

III.4.5 **Theorem 4.5.** *Every theory of negative K-theory for rings is canonically isomorphic to the negative K-theory of this section.*

Proof. Suppose that $\{K'_n\}$ is another theory of negative K-theory for rings. We will show that there are natural isomorphisms $h_n(A) : K_n(A) \to K'_n(A)$ commuting with the boundary operators. By induction, we may assume that h_n is given. Since $C(R)$ is flasque (II.2.1.3), and $S(R) = C(R)/M(R)$, the axioms yield isomorphisms $\partial : K_nS(R) \cong K_{n-1}(R)$ and $\partial' : K'_nS(R) \cong K'_{n-1}(R)$. We define $h_{n-1}(R) : K_{n-1}(R) \to K'_{n-1}(R)$ to be $\partial' \circ h_n(SR) \circ \partial^{-1}$.

It remains to check that the h_n commute with the boundary maps associated to an ideal $I \subset R$. Since $M(I)$ is an ideal in $C(I)$, $C(R)$ and $M(R)$, the axioms yield $K_nS(I) \cong K_nC(R)/M(I)$ and similarly for K'_n. The naturality of ∂ and ∂' relative to $M(R/I) = M(R)/M(I) \to C(R)/M(I)$ yield the diagram

$$\begin{array}{ccccccccc}
K_n(R/I) & \xrightarrow{\cong} & K_nM(R/I) & \longrightarrow & K_nC(R)/M(I) & \xleftarrow{\cong} & K_nS(I) & \xrightarrow{\partial} & K_{n-1}(I) \\
\cong\downarrow h_n(R/I) & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow h_{n-1}(I) \\
K'_n(R/I) & \xrightarrow{\cong} & K'_nM(R/I) & \longrightarrow & K'_nC(R)/M(I) & \xleftarrow{\cong} & K'_nS(I) & \xrightarrow{\partial'} & K'_{n-1}(I).
\end{array}$$

Since the horizontal composites are the given maps $\partial : K_n(R/I) \to K_{n-1}(I)$ and $\partial' : K'_n(R/I) \to K'_{n-1}(I)$, we have the desired relation: $\partial' h_n(R/I) \cong h_{n-1}(I)\,\partial$. □

EXERCISES

EIII.4.1 **4.1.** Suppose that $0 \to F' \to F \to F'' \to 0$ is an exact sequence of functors, with F' and F'' contracted. Show that F is acyclic, but need not be contracted.

EIII.4.2 **4.2.** For a commutative ring R, let $H_0(R)$ denote the group $[\mathrm{Spec}(R), \mathbb{Z}]$ of all continuous functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Show that $NH_0 = LH_0 = 0$, *i.e.*, that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

EIII.4.3 **4.3.** Let R be an Artinian ring. Show that R is K_0-regular, and that $K_{-n}(R) = 0$ for all $n > 0$.

EIII.4.4 **4.4.** (Bass-Murthy) Let R be a 1-dimensional commutative noetherian ring with finite normalization $\widetilde{R}$ and conductor ideal I. Show that R is K_{-1}-regular, and that $K_{-n}(R) = 0$ for all $n \geq 2$. If $h_0(R)$ denotes the rank of the free abelian group $H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$, show that $K_{-1}(R) \cong L\,\mathrm{Pic}(R) \cong \mathbb{Z}^r$, where $r = h_0(R) - h_0(\widetilde{R}) + h_0(\widetilde{R}/I) - h_0(R/I)$.

Now suppose that R is any 1-dimensional commutative noetherian ring. Even if its normalization is not finitely generated over R, show that R is K_{-1}-regular, and that $K_{-n}(R) = 0$ for all $n \geq 2$.

EIII.4.5 **4.5.** (Carter) Let $f : R \to R'$ be a ring homomorphism. In II.2.10 we defined a group $K_0(f)$ and showed in Ex. 1.14 that it fits into an exact sequence

$$K_1(R) \xrightarrow{f^*} K_1(R') \to K_0(f) \to K_0(R) \xrightarrow{f^*} K_0(R').$$

Show that $A \mapsto K_0(f \otimes A)$ defines a functor on commutative rings A, and define $K_{-n}(f)$ to be $L^n K_0(f \otimes -)$. Show that each $K_{-n}(f)$ is an acyclic functor, and that the above sequence continues into negative K-theory as:

$$\cdots \to K_0(R) \to K_0(R') \xrightarrow{\partial} K_{-1}(f) \to K_{-1}(R) \to$$
$$K_{-1}(R') \xrightarrow{\partial} K_{-2}(f) \to K_{-2}(R) \to \cdots$$

With the help of higher K-theory to define $K_1(f)$ and to construct the product "$\cdot t$", it will follow that $K_0(f)$ and hence every $K_{-n}(f)$ is a contracted functor.

EIII.4.6 **4.6.** Let $T : \mathbf{P}(R) \to \mathbf{P}(R')$ be any cofinal additive functor. Show that the functor $K_0(T)$ of II.2.10 and its contractions $K_{-n}(T)$ are acyclic, and that they extend the sequence of Ex. 1.14 into a long exact sequence, as in the previous exercise.

When T is the endofunctor $\cdot m$ of 1.7.4, we write $K_0(R; \mathbb{Z}/m)$ for $LK_0(\cdot m)$ and $K_{-n}(R; \mathbb{Z}/m)$ for $L^{n+1}K_0(\cdot m)$. Show that the sequence of 1.7.4 extends to a long exact sequence

$$K_0(R) \xrightarrow{m} K_0(R) \to K_0(R; \mathbb{Z}/m) \to K_{-1}(R) \xrightarrow{m} K_{-1}(R) \to K_{-1}(R; \mathbb{Z}/m) \cdots .$$

EIII.4.7 **4.7.** Let G be a finite group of order n, and let $\widetilde{R}$ be a "maximal order" in $\mathbb{Q}[G]$. It is well known that $\widetilde{R}$ is a regular ring containing $\mathbb{Z}[G]$, and that $I = n\widetilde{R}$ is an ideal of $\mathbb{Z}[G]$; see [15, p. 560]. Show that $K_{-n}\mathbb{Z}[G] = 0$ for $n \geq 2$, and that K_{-1} has the following resolution by free abelian groups:

$$0 \to \mathbb{Z} \to H_0(\widetilde{R}) \oplus H_0(\mathbb{Z}/n[G]) \to H_0(\widetilde{R}/n\widetilde{R}) \to K_{-1}(\mathbb{Z}[G]) \to 0.$$

D. Carter has shown in [39] that $K_{-1}\mathbb{Z}[G] \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^s$, where s equals the number of simple components $M_{n_i}(D_i)$ of the semisimple ring $\mathbb{Q}[G]$ such that the Schur index of D is even (see 1.2.4), but the Schur index of D_p is odd at each prime p dividing n. In particular, if G is abelian then $K_{-1}\mathbb{Z}[G]$ is torsionfree (see [15, p. 695]).

EIII.4.8 **4.8.** *Coordinate hyperplanes.* Let R be a regular ring. By induction on n and Ex. 2.6, show that the graded rings $A_n = R[t_0, \dots, t_n]/(t_0 \cdots t_n)$ are K_i-regular for all $i \le 1$. Conclude that $K_1(A_n) = K_1(R)$, $K_0(A_n) = K_0(R)$ and $K_i(A_n) = 0$ for all $i < 0$.

Show that the rings $\Delta^n(R)$ of Example 4.3.1 are also K_1-regular.

EIII.4.9 **4.9.** Let Λ be a flasque ring. Show that $\Lambda[t, t^{-1}]$ is also flasque, and conclude that $K_n(\Lambda) = 0$ for all $n \le 0$.

EIII.4.10 **4.10.** (Karoubi) Recall from Ex. 1.15 that the suspension ring $S(R)$ satisfies $\partial : K_1(S(R)) \cong K_0(R)$. For each $n \ge 0$, set $K_0S^n(R) = K_0(S^n(R))$. Show that the functors $\{K'_n = K_0S^{-n}\}$ form a theory of negative K-theory for rings, and conclude that $K_n(R) \cong K_0(S^n(R))$.

EIII.4.11 **4.11.** (Karoubi) Let $f : A \to B$ be an analytic isomorphism along S in the sense of Ex. 3.10. Using Ex. 4.5, show that there is an exact sequence for all $n \le 0$, continuing the sequence of Ex. 3.10:

$$\cdots \to K_{n+1}(S^{-1}A) \oplus K_{n+1}(B) \to K_{n+1}(S^{-1}B) \to$$
$$K_n(A) \to K_n(S^{-1}A) \oplus K_n(B) \to K_n(S^{-1}B) \to \cdots .$$

EIII.4.12 **4.12.** (Reid) Let $f = y^2 - x^3 + x^2$ in $k[x, y]$ and set $B = k[x, y]/(f)$. Using Theorem 4.3, show that $K_{-1}(B) \cong K_{-1}(B_{(x,y)}) \cong \mathbb{Z}$. Let A be the subring $k + \mathfrak{m}$ of $k[x, y]$, where $\mathfrak{m} = fk[x, y]$; show that $K_{-2}(A) \cong K_{-2}(A_{\mathfrak{m}}) \cong \mathbb{Z}$. Writing the integrally closed ring A as the union of finitely generated normal subrings $k[f, xf, yf, ...]$, conclude that there is a 2-dimensional normal ring A_0, finitely generated over k, with $K_{-2}(A_0) \ne 0$.

EIII.4.13 **4.13.** (Reid) We saw in Example 4.4.3 that $K_{-1}(A) = 0$ for every hensel local ring. In this exercise we construct a complete local 2-dimensional ring with $K_{-2}(\hat{A}) \ne 0$. Let A be the ring of Exercise 4.12, and $\hat{A}$ its completion at the maximal ideal $\mathfrak{m}$. Let $\hat{A_f}$ denote the completion of A at the ideal Af. Using Ex. 4.10, show that $K_{-2}(A) \cong K_{-2}(\hat{A_f}) \cong K_{-2}(\hat{A})$, and hence that $K_{-2}(\hat{A}) \ne 0$.

5 K_2 of a ring

The group K_2 of a ring was defined by J. Milnor in 1967, following a 1962 paper by R. Steinberg on Universal Central Extensions of Chevalley groups. Milnor's 1971 book [131] is still the best source for the fundamental theorems about it. In this section we will give an introduction to the subject, but we will not prove the harder theorems.

Following Steinberg, we define a group in terms of generators and relations designed to imitate the behavior of the elementary matrices, as described in (1.3.1). To avoid technical complications, we shall avoid any definition of $St_2(R)$.

III.5.1 **Definition 5.1.** For $n \geq 3$ the *Steinberg group* $St_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations"

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s) \tag{5.1.2}$$ III.5.1.1

$$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases} \tag{5.1.3}$$ III.5.1.2

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n \colon St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\varinjlim St_n(R)$, and observe that by stabilizing the ϕ_n induce a surjection $\phi \colon St(R) \to E(R)$.

III.5.2 **Definition 5.2.** The group $K_2(R)$ is the kernel of $\phi \colon St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.2.1 below that $K_2(R)$ is an abelian group. Moreover, it is clear that St and K_2 are both covariant functors from rings to groups, just as GL and K_1 are.

III.5.2.1 **Theorem 5.2.1.** *(Steinberg) $K_2(R)$ is an abelian group. In fact it is precisely the center of $St(R)$.*

Proof. If $x \in St(R)$ commutes with every element of $St(R)$, then $\phi(x)$ must commute with all of $E(R)$. But the center of $E(R)$ is trivial (by Ex. 1.8) so $\phi(x) = 1$, *i.e.,* $x \in K_2(R)$. Thus the center of $St(R)$ is contained in $K_2(R)$.

Conversely, suppose that $y \in St(R)$ satisfies $\phi(y) = 1$. Then in $E(R)$ we have

$$\phi([y, p]) = \phi(y)\phi(p)\phi(y)^{-1}\phi(p)^{-1} = \phi(p)\phi(p)^{-1} = 1$$

for every $p \in St(R)$. Choose an integer n large enough that y can be expressed as a word in the symbols $x_{ij}(r)$ with $i, j < n$. For each element $p = x_{kn}(s)$ with $k < n$ and $s \in R$, the Steinberg relations imply that the commutator $[y, p]$ is an element of the subgroup P_n of $St(R)$ generated by the symbols $x_{in}(r)$ with $i < n$. On the other hand, we know by Ex. 5.2 that ϕ maps P_n injectively into $E(R)$. Since $\phi([y, p]) = 1$ this implies that $[y, p] = 1$. Hence y commutes with every generator $x_{kn}(s)$ with $k < n$.

By symmetry, this proves that y also commutes with every generator $x_{nk}(s)$ with $k < n$. Hence y commutes with all of $St_n(R)$, since it commutes with every $x_{kl}(s) = [x_{kn}(s), x_{nl}(1)]$ with $k, l < n$. Since n can be arbitrarily large, this proves that y is in the center of $St(R)$. □

III.5.2.2 **Example 5.2.2.** The group $K_2(\mathbb{Z})$ is cyclic of order 2. This calculation uses the Euclidean algorithm to rewrite elements of $St(\mathbb{Z})$, and is given in §10 of [131]. In fact, Milnor proves that the symbol $\{-1, -1\} = \{x_{12}(1)x_{21}(-1)x_{12}(1)\}^4$ is the only nontrivial element of $\ker(\phi_n)$ for all $n \geq 3$. It is easy to see that $\{-1, -1\}$ is in the kernel of each ϕ_n, because the 2×2 matrix $e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has order 4 in $GL_n(\mathbb{Z})$. We will see in Example 6.2.1 below that $\{-1, -1\}$ is still nonzero in $K_2(\mathbb{R})$.

Tate has used the same Euclidean algorithm type techniques to show that $K_2(\mathbb{Z}[\sqrt{-7}])$ and $K_2(\mathbb{Z}[\sqrt{-15}])$ are also cyclic of order 2, generated by the symbol $\{-1, -1\}$, while $K_2(R) = 1$ for the imaginary quadratic rings $R = \mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-3}]$, $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-11}]$. See the appendix to [21] for details.

III.5.2.3 **Example 5.2.3.** For every field F we have $K_2(F[t]) = K_2(F)$. This was originally proven by R. K. Dennis and J. Sylvester using the same Euclidean algorithm type techniques as in the previous example. We shall not describe the details, because we shall see in chapter V that $K_2(R[t]) = K_2(R)$ for every regular ring.

Universal Central Extensions

5.3. The Steinberg group $St(R)$ can be described in terms of universal central extensions, and the best exposition of this is in Milnor's book [131, §5]. Properly speaking, this is a subject in pure group theory; see [192, 2.9]. However, since extensions of a group G are classified by the cohomology group $H^2(G; \mathbb{Z})$, the theory of universal central extensions is also a part of homological algebra; see [223, §6.9]. Here are the relevant definitions.

Let G be a group and A an abelian group. A *central extension* of G by A is a short exact sequence of groups $1 \to A \to X \xrightarrow{\pi} G \to 1$ such that A is in the center of X. We say that a central extension is *split* if it is isomorphic to an extension of the form $1 \to A \to A \times G \xrightarrow{pr} G \to 1$, where $pr(a, g) = g$.

If A or π is clear from the context, we may omit it from the notation. For example, $1 \to K_2(R) \to St(R) \to E(R) \to 1$ is a central extension by Steinberg's Theorem 5.2.1, but we usually just say that $St(R)$ is a central extension of $E(R)$.

Two extensions X and Y of G by A are said to be *equivalent* if there is an isomorphism $f\colon X \to Y$ which is the identity on A and which induces the identity map on G. It is well-known that the equivalence classes of central extensions of G by a fixed group A are in 1–1 correspondence with the elements of the cohomology group $H^2(G;A)$; see [223, §6.6].

More generally, by a *homomorphism over* G from $X \xrightarrow{\pi} G$ to another central extension $1 \to B \to Y \xrightarrow{\tau} G \to 1$ we mean a group map $f\colon X \to Y$ such that $\pi = \tau f$.

III.5.3.1 **Definition 5.3.1.** A *universal central extension* of G is a central extension $X \xrightarrow{\pi} G$ such that for every other central extension $Y \xrightarrow{\tau} G$ there is a unique homomorphism f over G from X to Y. Clearly a universal central extension is unique up to isomorphism over G, provided it exists.

III.5.3.2 **Lemma 5.3.2.** *If G has a universal central extension $X \xrightarrow{\pi} G$, then both G and X must be perfect groups.*

Proof. Otherwise $B = X/[X,X]$ is nontrivial, and there would be two homomorphisms over G from X to the central extension $1 \to B \to B \times G \to G \to 1$, namely the maps $(0,\pi)$ and (pr,π), where $pr : X \to B$ is the natural projection. □

III.5.3.3 **Lemma 5.3.3.** *If X and Y are central extensions of G, and X is a perfect group, there is at most one homomorphism over G from X to Y.*

Proof. If f and f_1 are two such homomorphisms, then for any x and x' in X we can write $f_1(x) = f(x)c$, $f_1(x') = f(x')c'$ for elements c and c' in the center of Y. Therefore $f_1(xx'x^{-1}(x')^{-1}) = f(xx'x^{-1}(x')^{-1})$. Since the commutators $[x,x'] = xx'x^{-1}(x')^{-1}$ generate X we must have $f_1 = f$. □

III.5.3.4 **Example 5.3.4.** Every presentation of G gives rise to two natural central extensions as follows. A presentation corresponds to the choice of a free group F mapping onto G, and a description of the kernel $R \subset F$. Since $[R,F]$ is a normal subgroup of F, we may form the following central extensions:

$$1 \to R/[R,F] \to F/[R,F] \to G \to 1,$$

$$1 \to (R \cap [F,F])/[R,F] \to [F,F]/[R,F] \to [G,G] \to 1. \qquad (5.3.5)$$ III.5.3.5

The group $(R \cap [F,F])/[R,F]$ in (5.3.5) is the homology group $H_2(G;\mathbb{Z})$; this identity was discovered in 1941 by Hopf [223, 6.8.8]. If $G = [G,G]$, then both are extensions of G, and (5.3.5) is the universal central extension by the following theorem.

III.5.4 **Recognition Theorem 5.4.** *Every perfect group G has a universal central extension, namely the extension (5.3.5):*

$$1 \to H_2(G;\mathbb{Z}) \to [F,F]/[R,F] \to G \to 1.$$

Let X be any central extension of G, the following are equivalent: (1) X is a universal central extension; (2) X is perfect, and every central extension of X splits; (3) $H_1(X;\mathbb{Z}) = H_2(X;\mathbb{Z}) = 0$.

Proof. Given any central extension X of G, the map $F \to G$ lifts to a map $h\colon F \to X$ because F is free. Since $h(R)$ is in the center of X, $h([R,F]) = 1$. Thus h induces a map from $[F,F]/[R,F]$ to X over G. This map is unique by Lemma 5.3.3. This proves that (5.3.5) is a universal central extension, and proves the equivalence of (1) and (3). The implication (1)⇒(2) is Lemma 5.3.2 and Ex. 5.7, and (2) ⇒ (1) is immediate. □

III.5.5 **Theorem 5.5.** *(Kervaire, Steinberg) The Steinberg group $St(R)$ is the universal central extension of $E(R)$. Hence*

$$K_2(R) \cong H_2(E(R);\mathbb{Z}).$$

This theorem follows immediately from the Recognition Theorem 5.4, and the following splitting result:

III.5.5.1 **Proposition 5.5.1.** *If $n \geq 5$, every central extension $Y \xrightarrow{\pi} St_n(R)$ is split. Hence $St_n(R)$ is the universal central extension of $E_n(R)$.*

Proof. We first show that if $j \neq k$ and $l \neq i$ then every two elements $y, z \in Y$ with $\pi(y) = x_{ij}(r)$ and $\pi(z) = x_{kl}(s)$ must commute in Y. Pick t distinct from i, j, k, l and choose $y', y'' \in Y$ with $\pi(y') = x_{it}(1)$ and $\pi(y'') = x_{tj}(r)$. The Steinberg relations imply that both $[y', z]$ and $[y'', z]$ are in the center of Y, and since $\pi(y) = \pi([y', y''])$ this implies that z commutes with $[y', y'']$ and y.

We now choose distinct indices i, j, k, l and elements $u, v, w \in Y$ with

$$\pi(u) = x_{ij}(1), \quad \pi(v) = x_{jk}(s) \quad \text{and } \pi(w) = x_{kl}(r).$$

If G denotes the subgroup of Y generated by u, v, w then its commutator subgroup $[G,G]$ is generated by elements mapping under π to $x_{ik}(s)$, $x_{jl}(sr)$ or $x_{il}(sr)$. From the first paragraph of this proof it follows that $[u,w] = 1$ and that $[G,G]$ is abelian. By Ex. 5.3 we have $[[u,v],w] = [u,[v,w]]$. Therefore if $\pi(y) = x_{ik}(s)$ and $\pi(z) = x_{jl}(sr)$ we have $[y,w] = [u,z]$. Taking $s = 1$, this identity proves that the element

$$y_{il}(r) = [u,z], \qquad \text{where } \pi(u) = x_{ij}(1),\ \pi(z) = x_{jl}(r)$$

doesn't depend upon the choice of j, nor upon the lifts u and z of $x_{ij}(1)$ and $x_{jl}(r)$.

We claim that the elements $y_{ij}(r)$ satisfy the Steinberg relations, so that there is a group homomorphism $St_n(R) \to Y$ sending $x_{ij}(r)$ to $y_{ij}(r)$. Such a homomorphism will provide the desired splitting of the extension π. The first paragraph of this proof implies that if $j \neq k$ and $l \neq i$ then $y_{ij}(r)$ and $y_{kl}(s)$ commute. The identity $[y,w] = [u,z]$ above may be rewritten as

$$[y_{ik}(r), y_{kl}(s)] = y_{il}(rs) \quad \text{for } i,k,l \text{ distinct.}$$

The final relation $y_{ij}(r)y_{ij}(s) = y_{ij}(r+s)$ is a routine calculation with commutators left to the reader. □

III.5.5.2 **Remark 5.5.2** (Stability for K_2)**.** The kernel of $St_n(R) \to E_n(R)$ is written as $K_2(n,R)$, and there are natural maps $K_2(n,R) \to K_2(R)$. If R is noetherian of dimension d, or more generally has $sr(R) = d+1$, then the following stability result holds: $K_2(n,R) \cong K_2(R)$ for all $n \geq d+3$. This result evolved in the mid-1970's as a sequence of results by Dennis, Vaserstein, van der Kallen and Suslin-Tulenbaev. We refer the reader to section 19C20 of *Math Reviews* for more details.

Transfer maps on K_2

Here is a description of $K_2(R)$ in terms of the translation category $t\mathbf{P}(R)$ of finitely generated projective R-modules, analogous to the description given for K_1 in Corollary 1.6.3.

III.5.6 **Proposition 5.6.** *(Bass)* $K_2(R) \cong \varinjlim_{P \in t\mathbf{P}} H_2([\mathrm{Aut}(P), \mathrm{Aut}(P)]; \mathbb{Z})$.

Proof. If G is a group, then G acts by conjugation upon $[G,G]$ and hence upon the homology $H = H_2([G,G];\mathbb{Z})$. Taking coinvariants, we obtain the functor H_2' from groups to abelian groups defined by $H_2'(G) = H_0(G;H)$. By construction, G acts trivially upon $H_2'(G)$ and commutes with direct limits of groups.

Note that if G acts trivially upon $H = H_2([G,G];\mathbb{Z})$ then $H_2'(G) = H$. For example, $GL(R)$ acts trivially upon the homology of $E(R) = [GL(R), GL(R)]$ by Ex. 1.13. By Theorem 5.5 this implies that $H_2'(GL(R)) = H_2(E(R);\mathbb{Z}) = K_2(R)$.

Since morphisms in the translation category $t\mathbf{P}(R)$ are well-defined up to isomorphism, it follows that $P \mapsto H_2'(\mathrm{Aut}(P))$ is a well-defined functor from $t\mathbf{P}(R)$ to abelian groups. Hence we can take the filtered colimit of this functor, as we did in 1.6.3. Since the free modules are cofinal in $t\mathbf{P}(R)$, the result follows from the identification of the colimit as

$$\lim_{n\to\infty} H_2'(GL_n(R)) \cong H_2'(GL(R)) = K_2(R). \quad \square$$

III.5.6.1 **Corollary 5.6.1** (Morita Invariance of K_2)**.** *The group $K_2(R)$ is determined by the category $\mathbf{P}(R)$. Thus, if R and S are Morita equivalent rings (see II.2.7) then $K_2(R) \cong K_2(S)$. In particular, there are isomorphisms on K_2:*

$$K_2(R) \cong K_2(M_n(R)).$$

III.5.6.2 **Corollary 5.6.2.** *Any additive functor $T : \mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural homomorphism $K_2(T)\colon K_2(S) \to K_2(R)$, and $T_1 \oplus T_2$ induces the sum $K_2(T_1) + K_2(T_2)$.*

Proof. The proof of 1.7 goes through, replacing $H_1(\mathrm{Aut}\,P)$ by $H_2'(\mathrm{Aut}\,P)$. $\square$

III.5.6.3 **Corollary 5.6.3** (Transfer maps)**.** *Let $f\colon R \to S$ be a ring homomorphism such that S is finitely generated projective as an R-module. Then the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural transfer homomorphism $f_*\colon K_2(S) \to K_2(R)$.*

If R is commutative, so that $K_2(R)$ is a $K_0(R)$-module by Ex. 5.4, the composition $f_*f^*\colon K_2(R) \to K_2(S) \to K_2(R)$ is multiplication by $[S] \in K_0(R)$. In particular, if S is free of rank n, then f_*f^* is multiplication by n.

Proof. The composite f_*f^* is obtained from the self-map $T(P) = P \otimes_R S$ of $\mathbf{P}(S)$. It induces the self-map $\otimes_R S$ on $t\mathbf{P}(R)$ giving rise to multiplication by $[S]$ on $K_2(R)$ in Ex. 5.5. ☐

We will see in chapter V that we can also define a transfer map $K_2(S) \to K_2(R)$ when S is a finite R-algebra of finite projective dimension over R.

III.5.6.4 **Example 5.6.4.** Let D be a division algebra of dimension $d = n^2$ over its center F. As in Example 1.7.2, the transfer $i_* : K_2(D) \to K_2(F)$ has a kernel of exponent n^2, since i^*i_* is induced by the functor $T(M) = M \otimes_D (D \otimes_F D) \cong M^d$ and hence is multiplication by n^2.

If E is a splitting field for D, the construction of Ex. 1.17 yields a natural map $\theta_E : K_2(E) \to K_2(D)$. If n is squarefree, Merkurjev and Suslin constructed a reduced norm $N_{\text{red}} : K_2(D) \to K_2(F)$ such that $N_{\text{red}}\theta_E = N_{E/F}$; see [125]. If $K_2(F) \to K_2(E)$ is injective, it is induced by the norm map $K_2(D) \to K_2(E)$, as in 1.2.4.

Relative K_2 and relative Steinberg groups

Given an ideal I in a ring R, we may construct the augmented ring $R \oplus I$, with multiplication $(r, x)(s, y) = (rs, ry + xs + xy)$. This ring is equipped with two natural maps $pr, add\colon R \oplus I \to R$, defined by $pr(r, x) = r$ and $add(r, x) = r + x$. This "double" ring was used to define the relative group $K_0(I)$ in Ex. II.2.3.

Let $St'(R, I)$ denote the normal subgroup of $St(R \oplus I)$ generated by all $x_{ij}(0, v)$ with $v \in I$. Clearly there is a map from $St'(R, I)$ to the subgroup $E(R \oplus I, 0 \oplus I)$ of $GL(R \oplus I)$ (see Lemma 2.1), and an exact sequence

$$1 \to St'(R, I) \to St(R \oplus I) \xrightarrow{pr} St(R) \to 1.$$

The following definition is taken from [103] and [115], and modifies [131].

III.5.7 **Definition 5.7.** The *relative Steinberg group* $St(R, I)$ is defined to be the quotient of $St'(R, I)$ by the normal subgroup generated by all "cross-commutators" $[x_{ij}(0, u), x_{kl}(v, -v)]$ with $u, v \in I$.

The homomorphism $St(R \oplus I) \xrightarrow{\text{add}} St(R)$ sends these cross-commutators to 1, so it induces a homomorphism $St(R, I) \xrightarrow{\text{add}} St(R)$ whose image is the normal subgroup generated by the $x_{ij}(v)$, $v \in I$. By the definition of $E(R, I)$, the surjection $St(R) \to E(R)$ maps $St(R, I)$ onto $E(R, I)$. We define $K_2(R, I)$ to be the kernel of the map $St(R, I) \to E(R, I)$.

III.5.7.1 **Theorem 5.7.1.** *If I is an ideal of a ring R, then the exact sequence of Proposition 2.3 extends to an exact sequence*

$$K_2(R,I) \to K_2(R) \to K_2(R/I) \to K_1(R,I) \to K_1(R) \to K_1(R/I) \to K_0(I) \cdots$$

Proof. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K_2(R,I) & \to & St(R,I) & \to & GL(I) & \to & K_1(R,I) \\
\downarrow & & \downarrow{\scriptstyle \text{add}} & & \downarrow{\scriptstyle \text{into}} & & \downarrow \\
K_2(R) & \to & St(R) & \to & GL(R) & \to & K_1(R) \\
\downarrow & & \downarrow{\scriptstyle \text{onto}} & & \downarrow & & \downarrow \\
K_2(R/I) & \to & St(R/I) & \to & GL(R/I) & \to & K_1(R/I)
\end{array}$$

The exact sequence now follows from the Snake Lemma and Ex. 5.1. □

If I and J are ideals in a ring R with $I \cap J = 0$, we may also consider I as an ideal of R/J. As in §1, these rings form a Milnor square:

$$\begin{array}{ccc}
R & \longrightarrow & R/J \\
\downarrow & & \downarrow \\
R/I & \longrightarrow & R/(I+J).
\end{array}$$

III.5.8 **Theorem 5.8** (Mayer-Vietoris)**.** *If I and J are ideals of R with $I \cap J = 0$, then the Mayer-Vietoris sequence of Theorem 2.6 can be extended to K_2:*

$$\begin{array}{l}
K_2(R) \xrightarrow{\Delta} K_2(R/I) \oplus K_2(R/J) \xrightarrow{\pm} K_2(R/I+J) \xrightarrow{\partial} \\
K_1(R) \xrightarrow{\Delta} K_1(R/I) \oplus K_1(R/J) \xrightarrow{\pm} K_1(R/I+J) \xrightarrow{\partial} K_0(R) \longrightarrow \cdots
\end{array}$$

Proof. Set $S = R/J$. By Ex. 5.10, we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
K_2(R,I) & \to & K_2(R) & \to & K_2(R/I) & \to & K_1(R,I) & \to & K_1(R) & \to & K_1(R/I) \\
\twoheaddownarrow{\scriptstyle \text{onto}} & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
K_2(S,I) & \to & K_2(S) & \to & K_2(S/I) & \to & K_1(S,I) & \to & K_1(S) & \to & K_1(S/I)
\end{array}$$

By chasing this diagram, we obtain the exact Mayer-Vietoris sequence. □

Commutative Banach Algebras

Let R be a commutative Banach algebra over the real or complex numbers. Just as $SK_1(R) = \pi_0 SL(R)$ and $K_1(R)$ surjects onto $\pi_0 GL(R)$ (by 1.5 and 1.5.1), there is a relation between $K_2(R)$ and $\pi_1 GL(R)$.

III.5.9 **Proposition 5.9.** *Let R be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1 SL(R) = \pi_1 E(R)$.*

Proof. ([131, p. 59]) By Proposition 1.5, we know that $E_n(R)$ is the path component of the identity in the topological group $SL_n(R)$, so $\pi_1 SL(R) = \pi_1 E(R)$. Using the exponential map $M_n(R) \to GL_n(R)$, we see that $E_n(R)$ is locally contractible, so it has a universal covering space $\widetilde{E}_n$. The group map $\widetilde{E}_n \to E_n(R)$ is a central extension with kernel $\pi_1 E_n(R)$. Taking the direct limit as $n \to \infty$, we get a central extension $1 \to \pi_1 E(R) \to \widetilde{E} \to E(R) \to 1$. By universality, there is a unique homomorphism $\tilde{\phi}\colon St(R) \to \widetilde{E}$ over $E(R)$, and hence a unique map $K_2(R) \to \pi_1 E(R)$. Thus it suffices to show that $\tilde{\phi}$ is onto.

The map $\tilde{\phi}$ may be constructed explicitly as follows. Let $\tilde{e}_{ij}(r) \in \widetilde{E}$ be the endpoint of the path which starts at 1 and lifts the path $t \mapsto e_{ij}(tr)$ in $E(R)$. We claim that the map $\tilde{\phi}$ sends $x_{ij}(r)$ to $\tilde{e}_{ij}(r)$. To see this, it suffices to show that the Steinberg relations (5.1) are satisfied. But the paths $\tilde{e}_{ij}(tr)\tilde{e}_{ij}(ts)$ and $[\tilde{e}_{ij}(tr), \tilde{e}_{kl}(s)]$ cover the two paths $e_{ij}(tr)e_{ij}(s)$ and $[e_{ij}(tr), e_{kl}(s)]$ in $E(R)$. Evaluating at $t = 1$ yields the Steinberg relations.

By Proposition 1.5 there is a neighborhood U_n of 1 in $SL_n(R)$ in which we may express every matrix g as a product of elementary matrices $e_{ij}(r)$, where r depends continuously upon g. Replacing each $e_{ij}(r)$ with $\tilde{e}_{ij}(r)$ defines a continuous lifting of U_n to $\widetilde{E}_n$. Therefore the image of each map $\tilde{\phi}\colon St_n(R) \to \widetilde{E}_n$ contains a neighborhood $\widetilde{U}_n$ of 1. Since any open subset of a connected group (such as $\widetilde{E}_n$) generates the entire group, this proves that each $\tilde{\phi}_n$ is surjective. Passing to the limit as $n \to \infty$, we see that $\tilde{\phi}\colon St(R) \to \widetilde{E}$ is also surjective. □

III.5.9.1 **Example 5.9.1.** If $R = \mathbb{R}$ then $\pi_1 SL(\mathbb{R}) \cong \pi_1 SO$ is cyclic of order 2. It follows that $K_2(\mathbb{R})$ has at least one nontrivial element. In fact, the symbol $\{-1, -1\}$ of Example 5.2.2 maps to the nonzero element of $\pi_1 SO$. We will see in 6.8.3 below that the kernel of $K_2(\mathbb{R}) \to \pi_1 SO$ is a uniquely divisible abelian group with uncountably many elements.

III.5.9.2 **Example 5.9.2.** Let X be a compact space with a nondegenerate basepoint. By Ex. II.3.11, we have $KO^{-2}(X) \cong [X, \Omega SO] = \pi_1 SL(\mathbb{R}^X)$, so $K_2(\mathbb{R}^X)$ maps onto the group $KO^{-2}(X)$.

Similarly, since $\Omega U \simeq \mathbb{Z} \times \Omega SU$, we see by Ex. II.3.11 that $KU^{-2}(X) \cong [X, \Omega U] = [X, \mathbb{Z}] \times [X, \Omega SU]$. Since $\pi_1 SL(\mathbb{C}^X) = \pi_1(SU^X) = [X, \Omega SU]$ and $[X, \mathbb{Z}]$ is a subgroup of $\mathbb{C}^X$, we can combine Proposition 5.9 with Example 1.5.3 to obtain the exact sequence

$$K_2(\mathbb{C}^X) \to KU^{-2}(X) \to \mathbb{C}^X \xrightarrow{\exp} K_1(\mathbb{C}^X) \to KU^{-1}(X) \to 0.$$

Steinberg symbols

If two matrices $A, B \in E(R)$ commute, we can construct an element in $K_2(R)$ by lifting their commutator to $St(R)$. To do this, choose $a, b \in St(R)$ with $\phi(a) = A$, $\phi(b) = B$ and define

$$A \bigstar B = [a, b] \in K_2(R).$$

This definition is independent of the choice of a and b because any other lift will equal ac, bc' for central elements c, c', and $[ac, bc'] = [a, b]$.

If $P \in GL(R)$ then $(PAP^{-1}) \bigstar (PBP^{-1}) = A \bigstar B$. To see this, suppose that $A, B, P \in GL_n(R)$ and let $g \in St_{2n}(R)$ be a lift of the block diagonal matrix $D = \mathrm{diag}(P, P^{-1})$. Since gag^{-1} and gbg^{-1} lift PAP^{-1} and PBP^{-1} and $[a, b]$ is central we have the desired relation: $[gag^{-1}, gbg^{-1}] = g[a, b]g^{-1} = [a, b]$.

The $\bigstar$ symbol is also skew-symmetric and bilinear: $(A \bigstar B)(B \bigstar A) = 1$ and $(A_1 A_2) \bigstar B = (A_1 \bigstar B)(A_2 \bigstar B)$. These relations are immediate from the commutator identies $[a, b][b, a] = 1$ and $[a_1 a_2, b] = [a_1, [a_2, b]][a_2, b][a_1, b]$.

III.5.10 **Definition 5.10.** If r, s are commuting units in a ring R, we define the Steinberg symbol $\{r, s\} \in K_2(R)$ to be

$$\{r, s\} = \begin{pmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{pmatrix} \bigstar \begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{pmatrix} = \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix} \bigstar \begin{pmatrix} s & & \\ & s^{-1} & \\ & & 1 \end{pmatrix}.$$

Because the $\bigstar$ symbols are skew-symmetric and bilinear, so are the Steinberg symbols: $\{r, s\}\{s, r\} = 1$ and $\{r_1 r_2, s\} = \{r_1, s\}\{r_2, s\}$.

III.5.10.1 **Example 5.10.1.** For any unit r of R we set $w_{ij}(r) = x_{ij}(r) x_{ji}(-r^{-1}) x_{ij}(r)$ and $h_{ij}(r) = w_{ij}(r) w_{ij}(-1)$. In $GL(R)$, $\phi w_{ij}(r)$ is the monomial matrix with r and $-r^{-1}$ in the (i, j) and (j, i) places, while $\phi h_{ij}(r)$ is the diagonal matrix with r and r^{-1} in the i^{th} and j^{th} diagonal spots. By definition we then have:

$$\{r, s\} = [h_{12}(r), h_{13}(s)] = [h_{ij}(r), h_{ik}(s)].$$

III.5.10.2 **Lemma 5.10.2.** *If both r and $1 - r$ are units of R, then in $K_2(R)$ we have:*

$$\{r, 1 - r\} = 1 \qquad \textit{and} \qquad \{r, -r\} = 1.$$

Proof. By Ex. EIII.5.8 5.8, $w_{12}(-1) = w_{21}(1) = x_{21}(1) x_{12}(-1) x_{21}(1)$, $w_{12}(r) x_{21}(1) = x_{12}(-r^2) w_{12}(r)$ and $x_{21}(1) w_{12}(s) = w_{12}(s) x_{12}(-s^2)$. If $s = 1 - r$ we can successively use the identities $r - r^2 = rs$, $r + s = 1$, $s - s^2 = rs$ and $\frac{1}{r} + \frac{1}{s} = \frac{1}{rs}$ to obtain:

$$\begin{aligned} w_{12}(r) w_{12}(-1) w_{12}(s) &= x_{12}(-r^2) w_{12}(r) x_{12}(-1)\ w_{12}(s) x_{12}(-s^2) \\ &= x_{12}(rs) x_{21}(-r^{-1}) x_{12}(0) x_{21}(-s^{-1}) x_{12}(rs) \\ &= x_{12}(rs) x_{21}(\frac{-1}{rs}) x_{12}(rs) \\ &= w_{12}(rs). \end{aligned}$$

Multiplying by $w_{12}(-1)$ yields $h_{12}(r)h_{12}(s) = h_{12}(rs)$ when $r + s = 1$. By Ex. 5.9, this yields the first equation $\{r, s\} = 1$. Since $-r = (1 - r)/(1 - r^{-1})$, the first equation implies

$$\{r, -r\} = \{r, 1 - r\}\{r, 1 - r^{-1}\}^{-1} = \{r^{-1}, 1 - r^{-1}\} = 1, \qquad (5.10.3)$$ III.5.10.3

which is the second equation. □

III.5.10.4 **Remark 5.10.4.** The equation $\{r, -r\} = 1$ holds more generally for every unit r, even if $1 - r$ is not a unit. This follows from the fact that $K_2(\mathbb{Z}[r, \frac{1}{r}])$ injects into $K_2(\mathbb{Z}[r, \frac{1}{r}, \frac{1}{1-r}])$, a fact we shall establish in chapter V, 6.1.3. For a direct proof, see [131, 9.8].

The following useful result was proven for fields and division rings in §9 of [131]. It was extended to commutative semilocal rings by Dennis and Stein [48], and we cite it here for completeness.

III.5.10.5 **Theorem 5.10.5.** *If R is a field, division ring, local ring, or even a semilocal ring, then $K_2(R)$ is generated by the Steinberg symbols $\{r, s\}$.*

III.5.11 **Definition 5.11** (Dennis-Stein symbols)**.** If $r, s \in R$ commute and $1 - rs$ is a unit then the element

$$\langle r, s\rangle = x_{ji}\big(-s(1 - rs)^{-1}\big)x_{ij}(-r)x_{ji}(s)x_{ij}\big((1 - rs)^{-1}r\big)h_{ij}(1 - rs)^{-1}$$

of $St(R)$ belongs to $K_2(R)$, because $\phi\langle r, s\rangle = 1$. By Ex. 5.11, it is independent of the choice of $i \neq j$, and if r is a unit of R then $\langle r, s\rangle = \{r, 1 - rs\}$. If I is an ideal of R and $s \in I$ then we can even consider $\langle r, s\rangle$ as an element of $K_2(R, I)$; see 5.7. These elements are called *Dennis-Stein symbols* because they were first studied in [48], where the following identities were established.

(D1) $\langle r, s\rangle\langle s, r\rangle = 1$

(D2) $\langle r, s\rangle\langle r, t\rangle = \langle r, s + t - rst\rangle$

(D3) $\langle r, st\rangle = \langle rs, t\rangle\langle tr, s\rangle$ (this holds in $K_2(R, I)$ if any of r, s, or t are in I.)

We warn the reader that the meaning of the symbol $\langle r, s\rangle$ changed circa 1980. We use the modern definition of this symbol, which equals $\langle -r, s\rangle^{-1}$ in the old literature, including that of *loc. cit.* By (D3) of our definition, $\langle r, 1\rangle = 0$ for all r.

The following result is essentially due to Maazen, Stienstra and van der Kallen. However, their work preceded the correct definition of $K_2(R, I)$ so the correct historical reference is [103].

III.5.11.1 **Theorem 5.11.1.** *(a) Let R be a commutative local ring, or a field. Then $K_2(R)$ may be presented as the abelian group generated by the symbols $\langle r, s\rangle$ with $r, s \in R$ such that $1 - rs$ is a unit, subject only to the relations (D1), (D2) and (D3).*

(b) Let I be a radical ideal, contained in a commutative ring R. Then $K_2(R, I)$ may be presented as the abelian group generated by the symbols $\langle r, s\rangle$ with either $r \in R$ and $s \in I$, or else $r \in I$ and $s \in R$. These generators are subject only to the relations (D1), (D2), and the relation (D3) whenever r, s, or t is in I.

The product $K_1(R) \otimes K_1(R) \to K_2(R)$

Let R be a commutative ring, and suppose given two invertible matrices $g \in GL_m(R)$, $h \in GL_n(R)$. Identifying the tensor product $R^m \otimes R^n$ with R^{m+n}, then $g \otimes 1_n$ and $1_m \otimes h$ are commuting automorphisms of $R^m \otimes R^n$. Hence there is a ring homomorphism from $A = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ to $E = \mathrm{End}_R(R^m \otimes R^n) \cong M_{m+n}(R)$ sending x and y to $g \otimes 1_n$ and $1_m \otimes h$. Recall that by Morita Invariance III.5.6 5.6.1 the natural map $K_2(R) \to K_2(E)$ is an isomorphism.

III.5.12 **Definition 5.12.** The element $\{g, h\}$ of $K_2(R)$ is defined to be the image of the Steinberg symbol $\{x, y\}$ under the homomorphism $K_2(A) \to K_2(E) \cong K_2(R)$.

Note that if $m = n = 1$ this agrees with the definition of the usual Steinberg symbol in III.5.10 5.10, because $R = E$.

III.5.12.1 **Lemma 5.12.1.** *The symbol $\{g, h\}$ is independent of the choice of m and n, and is skew-symmetric. Moreover, for each $\alpha \in GL_m(R)$ we have $\{g, h\} = \{\alpha g \alpha^{-1}, h\}$.*

Proof. If we embed $GL_m(R)$ and $GL_n(R)$ in $GL_{m'}(R)$ and $GL_{n'}(R)$, respectively, then we embed E into the larger ring $E' = \mathrm{End}_R(R^{m'} \otimes R^{n'})$, which is also Morita equivalent to R. Since the natural maps $K_2(R) \to K_2(E) \to K_2(E')$ are isomorphisms, and $K_2(A) \to K_2(E) \to K_2(E') \cong K_2(R)$ defines the symbol with respect to the larger embedding, the symbol is independent of m and n.

Any linear automorphism of R^{m+n} induces an inner automorphism of E. Since the composition of $R \to E$ with such an automorphism is still $R \to E$, the symbol $\{g, h\}$ is unchanged by such an operation. Applying this to $\alpha \otimes 1_n$, the map $A \to E \to E$ sends x and y to $\alpha g \alpha^{-1} \otimes 1_n$ and $1_m \otimes h$, so $\{g, h\}$ must equal $\{\alpha g \alpha^{-1}, h\}$.

As another application, note that if $m = n$ the inner automorphism of E induced by $R^m \otimes R^n \cong R^n \otimes R^m$ sends $\{h, g\}$ to the image of $\{y, x\}$ under $K_2(A) \to K_2(E)$. This proves skew-symmetry, since $\{y, x\} = \{x, y\}^{-1}$. □

III.5.12.2 **Theorem 5.12.2.** *For every commutative ring R, there is a skew-symmetric bilinear pairing $K_1(R) \otimes K_1(R) \to K_2(R)$ induced by the symbol $\{g, h\}$.*

Proof. We first show that the symbol is bimultiplicative when g and g' commute in $GL_m(R)$. Mapping $A[z, z^{-1}]$ into E by $z \mapsto g' \otimes 1_n$ allows us to deduce $\{gg', h\} = \{g, h\}\{g', h\}$ from the corresponding property of Steinberg symbols. If g and g' do not commute, the following trick establishes bimultiplicativity:

$$\{gg', h\} = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}, h \right\} = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, h \right\} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}, h \right\} = \{g, h\}\{g', h\}.$$

If either g or h is a commutator, this implies that the symbol $\{g,h\}$ vanishes in the abelian group $K_2(R)$. Since the symbol $\{g,h\}$ is compatible with stabilization, it describes a function $K_1(R) \times K_1(R) \to K_2(R)$ which is multiplicative in each entry: $\{gg',h\} = \{g,h\}\{g',h\}$. If we write K_1 and K_2 additively the function is additive in each entry, i.e., bilinear. □

EXERCISES

EIII.5.1 **5.1.** *Relative Steinberg groups.* Let I be an ideal in a ring R. Show that there is an exact sequence $St(R,I) \xrightarrow{\text{add}} St(R) \to St(R/I) \to 1$.

EIII.5.2 **5.2.** Consider the function $\rho_n\colon R^{n-1} \to St_n(R)$ sending $(r_1,...,r_{n-1})$ to the product $x_{1n}(r_1)x_{2n}(r_2)\cdots x_{n-1,n}(r_{n-1})$. The Steinberg relations show that this is a group homomorphism.

Show that ρ is an injection by showing that the composite $\phi\rho\colon R^{n-1} \to St_n(R) \to GL_n(R)$ is an injection. Then show that the elements $x_{ij}(r)$ with $i,j < n$ normalize the subgroup $P_n = \rho(R^n)$ of $St_n(R)$, *i.e.*, that $x_{ij}(r)P_nx_{ij}(-r) = P_n$.

Use this and induction to show that the subgroup T_n of $St_n(R)$ generated by the $x_{ij}(r)$ with $i < j$ maps isomorphically onto the subgroup of lower triangular matrices in $GL_n(R)$.

EIII.5.3 **5.3.** Let G be a group whose commutator group $[G,G]$ is abelian. Prove that the Jacobi identity holds for every $u,v,w \in G$:

$$[u,[v,w]][v,[w,u]][w,[u,v]] = 1.$$

If in addition $[u,w] = 1$ this implies that $[[u,v],w] = [u,[v,w]]$.

EIII.5.4 **5.4.** *Product with* K_0. Construct a product operation $K_0(R)\otimes K_2(A) \to K_2(A)$, assuming that R is commutative and A is an associative R-algebra. To do this, fix a finitely generated projective R-module P. Each isomorphism $P \oplus Q = R^n$ gives rise to a homomorphism $h^P : GL_m(A) \to GL_{mn}(A) \subset GL(A)$ sending α to $\alpha\otimes 1$ and $E_m(A)$ to $E(A)$. Show that h^P is well-defined up to conjugation by an element of $E(A)$. Since conjugation acts trivially on homology, this implies that the induced map ${h^P}_*\colon H_2(E_m(A);\mathbb{Z}) \to H_2(E(A);\mathbb{Z}) = K_2(A)$ is well-defined. Then show that ${h^{P\oplus Q}}_* = {h^P}_* \oplus {h^Q}_*$ and pass to the limit as $m \to \infty$ to obtain the required endomorphism $[P]\cdot$ of $K_2(A)$.

EIII.5.5 **5.5.** If R is commutative and $P \in \mathbf{P}(R)$, show that $Q \mapsto Q \otimes_R P$ defines a functor from the translation category $t\mathbf{P}(A)$ to itself for every R-algebra A, and that the resulting endomorphism of $K_2(A) = \varinjlim H_2([\mathrm{Aut}(Q),\mathrm{Aut}(Q)])$ is the map h^P_* of the previous exercise. Use this description to show that the product makes $K_2(A)$ into a module over the ring $K_0(R)$.

EIII.5.6 **5.6.** *Projection Formula.* Suppose that $f\colon R \to S$ is a finite map of commutative rings, with $S \in \mathbf{P}(R)$. Show that for all $r \in K_i(R)$ and $s \in K_j(S)$ with $i+j=2$ we have $f_*(f^*(r)\cdot s) = r \cdot f_*(s)$ in $K_2(R)$. The case $i=0$ states that the transfer $f_*\colon K_2(S) \to K_2(R)$ is $K_0(R)$-linear, while the case $i=1$ yields the useful formula $f_*\{r,s\} = \{r, Ns\}$ for Steinberg symbols in $K_2(R)$, where $r \in R^\times$, $s \in S^\times$ and $Ns = f_*(s) \in R^\times$ is the norm of s.

EIII.5.7 **5.7.** If $Y \xrightarrow{\rho} X$ and $X \xrightarrow{\pi} G$ are central extensions, show that the "composition" $Y \xrightarrow{\pi\rho} G$ is also a central extension. If X is a universal central extension of G, conclude that every central extension $Y \xrightarrow{\rho} X$ splits.

EIII.5.8 **5.8.** Show that the following identities hold in $St(R)$ (for i, j and k distinct).
(a) $w_{ij}(r)w_{ij}(-r) = 1$;
(b) $w_{ik}(r)x_{ij}(s)w_{ik}(-r) = x_{kj}(-r^{-1}s)$;
(c) $w_{ij}(r)x_{ij}(s)w_{ij}(-r) = x_{ji}(-r^{-1}sr^{-1})$;
(d) $w_{ij}(r)x_{ji}(s)w_{ij}(-r) = x_{ij}(-rsr)$;
(e) $w_{ij}(r)w_{ji}(r^{-1}) = 1$;

EIII.5.9 **5.9.** Use the previous exercise to show that $\{r,s\} = h_{ij}(rs)h_{ij}(s)^{-1}h_{ij}(r)^{-1}$. *Hint:* Conjugate $h_{ij}(s)$ by $w_{ik}(r)w_{ik}(-1)$.

EIII.5.10 **5.10.** *Excision.* If I and J are ideals in a ring R with $I \cap J = 0$, we may also consider I as an ideal of R/J. Show that $St(R,I)$ surjects onto $St(R/J,I)$, while the subgroups $E(R,I)$ and $E(R/J,I)$ of $GL(I)$ are equal. Use the 5-lemma to conclude that $K_1(R,I) \cong K_1(R/J,I)$ and that $K_2(R,I) \to K_2(R/J,I)$ is onto.

In fact, the sequence $I/I^2 \otimes_{R\otimes R} J/J^2 \to K_2(R,I) \to K_2(R/J,I) \to 0$ is exact, where the first map sends $x \otimes y$ to $\langle x,y\rangle$; see Swan71 [195].

EIII.5.11 **5.11.** *Dennis-Stein symbols.* Let $\langle r,s\rangle_{ij}$ denote the element of $St(R)$ given in Definition III.5.11 5.11. Show that this element is in $K_2(R)$. Then use Ex. EIII.5.8 5.8 to show that if $w = w_{ik}(1)w_{j\ell}(1)w_{k\ell}^2(1)$ (so that $\phi(w)$ is the permutation matrix sending i,j to k,ℓ) then $w\,\langle r,s\rangle_{ij}\,w^{-1} = \langle r,s\rangle_{k\ell}$. This shows that the Dennis-Stein symbol is independent of the choice of indices i,j.

EIII.5.12 **5.12.** Let A be an abelian group and F a field. Show that, for all $n \geq 5$, homomorphisms $K_2(F) \xrightarrow{c} A$ are in 1–1 correspondence with central extensions of $SL_n(F)$ having kernel A.

EIII.5.13 **5.13.** If p is an odd prime, use Theorem III.5.11.1 5.11.1 to show that $K_2(\mathbb{Z}/p^n) = 1$. If $n \geq 2$, show that $K_2(\mathbb{Z}/2^n) \cong K_2(\mathbb{Z}/4) \cong \{\pm 1\}$ on $\{-1,-1\} = \langle -1,-2\rangle = \langle 2,2\rangle$.

Using the Mayer-Vietoris sequence III.5.8 5.8, conclude that $K_1(R) = R^\times = \{\pm 1\}$ for the ring $R = \mathbb{Z}[x]/(x^2 - p^{2n})$. Note that $R/(x \pm p^n)R = \mathbb{Z}$.

EIII.5.14 **5.14.** Let R be a commutative ring, and let Ω_R^1 denote the module of Kähler differentials of R over $\mathbb{Z}$, as in Ex. EIII.2.6 2.6.

(a) If I is a radical ideal of R, show that there is a surjection from $K_2(R,I)$ onto $I \otimes_R \Omega^1_{R/I}$, sending $\langle x,r\rangle$ to $x \otimes dr$ ($r \in R$, $x \in I$).

(b) If $I^2 = 0$, show that the kernel of the map in (a) is generated by the Dennis-Stein symbols $\langle x, y\rangle$ with $x, y \in I$.

(c) (Van der Kallen) The *dual numbers* over R is the ring $R[\varepsilon]$ with $\varepsilon^2 = 0$. If $\frac{1}{2} \in R$, show that the map $K_2(R[\varepsilon], \varepsilon) \to \Omega^1_R$ of part (a) is an isomorphism.

(d) Let k be a field. Show that the group $K_2(k[[t]], t)$ is ℓ-divisible for every ℓ invertible in k. If $\mathrm{char}(K) = p > 0$, show that this group is *not* p-divisible.

EIII.5.15 **5.15.** Assume the fact that $\mathbb{Z}/2$ is K_2-regular (see III.3.4 3.4 and chapter V). Show that:

(a) $K_2(\mathbb{Z}/4[x])$ is an elementary abelian 2-group with basis $\langle 2, 2\rangle$, $\langle 2x^n, x\rangle$, and $\langle 2x^{2n+1}, 2\rangle$, $n \geq 0$. *Hint:* Split the map $K_2(A, 2) \to \Omega^1_{A/2} \cong \mathbb{Z}/2[x]$ of Ex. EIII.5.14 5.14 and use $0 = \langle 2f, 1\rangle = \langle 2(f + f^2), 2\rangle$.

(b) The group $K_2(\mathbb{Z}/4[x, y])$ is an elementary abelian 2-group with basis $\langle 2, 2\rangle$, $\langle 2x^m y^n, x\rangle$, $\langle 2x^m y^n, y\rangle$ $(m, n \geq 0)$ and $\langle 2x^m y^n, x\rangle$ (one of m, n odd).

(c) Consider the maps $\partial_1 : NK_2(\mathbb{Z}/4) \to K_2(\mathbb{Z}/4)$ and $\partial_2 : N^2K_2(\mathbb{Z}/4) \to NK_2(\mathbb{Z}/4)$ induced by the map $\mathbb{Z}/4[x] \to \mathbb{Z}/4$ sending x to 1, and the map $\mathbb{Z}/4[x, y] \to \mathbb{Z}/4[x]$ sending y to $1 - x$, respectively. Show that the following sequence is exact:

$$N^2K_2(\mathbb{Z}/4) \xrightarrow{\partial_2} NK_2(\mathbb{Z}/4) \xrightarrow{\partial_1} K_2(\mathbb{Z}/4) \to 0.$$

6 K_2 of fields

The following theorem was proven by Hideya Matsumoto in 1969. We refer the reader to Milnor [131, §12] for a self-contained proof.

III.6.1 **Matsumoto's Theorem 6.1.** *If F is a field then $K_2(F)$ is the abelian group generated by the set of Steinberg symbols $\{x, y\}$ with $x, y \in F^\times$, subject only to the relations:*

(Bilinearity) $\{xx', y\} = \{x, y\}\{x', y\}$ *and* $\{x, yy'\} = \{x, y\}\{x, y'\}$*;*

(Steinberg Identity) $\{x, 1 - x\} = 1$ *for all* $x \neq 0, 1$*.*

In other words, $K_2(F)$ is the quotient of $F^\times \otimes F^\times$ by the subgroup generated by the elements $x \otimes (1 - x)$. Note that the calculation (5.10.3) implies that $\{x, -x\} = 1$ for all x, and this implies that the Steinberg symbols are skew-symmetric: $\{x, y\}\{y, x\} = \{x, -xy\}\{y, -xy\} = \{xy, -xy\} = 1$.

III.6.1.1 **Corollary 6.1.1.** $K_2(\mathbb{F}_q) = 1$ *for every finite field* $\mathbb{F}_q$*.*

Proof. If x generates the cyclic group $\mathbb{F}_q^\times$, we must show that the generator $x \otimes x$ of the cyclic group $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ vanishes in K_2. If q is even, then $\{x, x\} = \{x, -x\} = 1$, so we may suppose that q is odd. Since $\{x, x\}^2 = 1$ by skew-symmetry, we have $\{x, x\} = \{x, x\}^{mn} = \{x^m, x^n\}$ for every odd m and n. Since odd powers of x are the same as non-squares, it suffices to find a non-square u such that $1 - u$ is also a non-square. But such a u exists because $u \mapsto (1 - u)$ is an involution on the set $\mathbb{F}_q - \{0, 1\}$, and this set consists of $(q - 1)/2$ non-squares but only $(q - 3)/2$ squares. □

III.6.1.2 **Example 6.1.2.** Let $F(t)$ be a rational function field in one variable t over F. Then $K_2(F)$ is a direct summand of $K_2F(t)$.

To see this, we construct a map $\lambda\colon K_2F(t) \to K_2(F)$ inverse to the natural map $K_2(F) \to K_2F(t)$. To this end, we define the *leading coefficient* of the rational function $f(t) = (a_0t^n + \cdots + a_n)/(b_0t^m + \cdots + b_m)$ to be $\mathrm{lead}(f) = a_0/b_0$ and set $\lambda(\{f, g\}) = \{\mathrm{lead}(f), \mathrm{lead}(g)\}$. To see that this defines a homomorphism $K_2F(t) \to K_2(F)$, we check the presentation in Matsumoto's Theorem. Bilinearity is clear from $\mathrm{lead}(f_1f_2) = \mathrm{lead}(f_1)\,\mathrm{lead}(f_2)$, and $\{\mathrm{lead}(f), \mathrm{lead}(1-f)\} = 1$ holds in $K_2(F)$ because $\mathrm{lead}(1-f)$ is either 1, $1-\mathrm{lead}(f)$ or $-\mathrm{lead}(f)$, according to whether $m > n$, $m = n$ or $m < n$.

Because K_2 commutes with filtered colimits, it follows that $K_2(F)$ injects into $K_2F(T)$ for *every* purely transcendental extension $F(T)$ of F.

III.6.1.3 **Lemma 6.1.3.** *For every field extension $F \subset E$, the kernel of $K_2(F) \to K_2(E)$ is a torsion subgroup.*

Proof. E is an algebraic extension of some purely transcendental extension $F(X)$ of F, and $K_2(F)$ injects into $K_2F(X)$ by Example 6.1.2. Thus we may assume that E is algebraic over F. Since E is the filtered union of finite extensions, we may even assume that E/F is a finite field extension. But in this case the result holds because (by 5.6.3) the composite $K_2(F) \to K_2(E) \to K_2(F)$ is multiplication by the integer $[E : F]$. □

The next result is useful for manipulations with symbols.

III.6.1.4 **Lemma 6.1.4.** *(Bass-Tate) If $E = F(u)$ is a field extension of F, then every symbol of the form $\{b_1u - a_1, b_2u - a_2\}$ ($a_i, b_i \in F$) is a product of symbols $\{c_i, d_i\}$ and $\{c_i, u - d_i\}$ with $c_i, d_i \in F$.*

Proof. Bilinearity allows us to assume that $b_1 = b_2 = 1$. Set $x = u - a_1$, $y = u - a_2$ and $a = a_2 - a_1$, so $x = a + y$. Then $1 = \frac{a}{x} + \frac{y}{x}$ yields the relation $1 = \{\frac{a}{x}, \frac{y}{x}\}$. Using $\{x, x\} = \{-1, x\}$, this expands to the desired expression: $\{x, y\} = \{a, y\}\{-1, x\}\{a^{-1}, x\}$. □

Together with the Projection Formula (Ex. 5.6), this yields:

III.6.1.5 **Corollary 6.1.5.** *If $E = F(u)$ is a quadratic field extension of F, then $K_2(E)$ is generated by elements coming from $K_2(F)$, together with elements of the form $\{c, u - d\}$. Thus the transfer map $N_{E/F}\colon K_2(E) \to K_2(F)$ is completely determined by the formulas*

$$N_{E/F}\{c, d\} = \{c, d\}^2, \quad N_{E/F}\{c, u - d\} = \{c, N(u - d)\} \quad (c, d \in F).$$

III.6.1.6 **Example 6.1.6.** Since $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$, every element of $K_2(\mathbb{C})$ is a product of symbols $\{r, s\}$ and $\{r, e^{i\theta}\}$ with $r, s, \theta \in \mathbb{R}$. Moreover, $N\{r, e^{i\theta}\} = 1$ in $K_2(\mathbb{R})$. Under the automorphism of $K_2(\mathbb{C})$ induced by complex conjugation, the symbols of the first kind are fixed and the symbols of the second kind are sent to their inverses. We will see in Theorem 6.4 below that $K_2(\mathbb{C})$ is uniquely divisible, *i.e.*, a vector space over $\mathbb{Q}$, and the decomposition of $K_2(\mathbb{C})$ into eigenspaces for ± 1 corresponds to symbols of the first and second kind.

III.6.1.7 **Example 6.1.7.** Let F be an algebraically closed field. By Lemma 6.1.4, $K_2F(t)$ is generated by linear symbols of the form $\{a, b\}$ and $\{t - a, b\}$. It will follow from 6.5.2 below that every element u of $K_2F(t)$ uniquely determines finitely many elements $a_i \in F$, $b_i \in F^\times$ so that $u = \lambda(u) \prod\{t - a_i, b_i\}$, where $\lambda(u) \in K_2(F)$ was described in Example 6.1.2.

Steinberg symbols

III.6.2 **Definition 6.2.** A *Steinberg symbol* on a field F with values in a multiplicative abelian group A is a bilinear map $c\colon F^\times \otimes F^\times \to A$ satisfying $c(r, 1-r) = 1$. By Matsumoto's Theorem, these are in 1–1 correspondence with homomorphisms $K_2(F) \xrightarrow{c} A$.

III.6.2.1 **Example 6.2.1.** There is a Steinberg symbol $(x, y)_\infty$ on the field $\mathbb{R}$ with values in the group $\{\pm 1\}$. Define $(x, y)_\infty$ to be: -1 if both x and y are negative, and $+1$ otherwise. The Steinberg identity $(x, 1 - x)_\infty = +1$ holds because x and $1 - x$ cannot be negative at the same time. The resulting map $K_2(\mathbb{R}) \to \{\pm 1\}$ is onto because $(-1, -1)_\infty = -1$. This shows that the symbol $\{-1, -1\}$ in $K_2(\mathbb{Z})$ is nontrivial, as promised in 5.2.2, and even shows that $K_2(\mathbb{Z})$ is a direct summand in $K_2(\mathbb{R})$.

For our next two examples, recall that a *local field* is a field F which is complete under a discrete valuation v, and whose residue field k_v is finite. Classically, every local field is either a finite extension of the p-adic rationals $\hat{\mathbb{Q}}_p$ or of $\mathbb{F}_p((t))$.

III.6.2.2 **Example 6.2.2** (Hilbert symbols)**.** Let F be a local field containing $\frac{1}{2}$. The Hilbert (quadratic residue) symbol on F is defined by setting $c_F(r,s) \in \{\pm 1\}$ equal to $+1$ or -1, depending on whether or not the equation $rx^2 + sy^2 = 1$ has a solution in F. Bilinearity is classical when F is local; see [147, p. 164]. The Steinberg identity is trivial, because $x = y = 1$ is always a solution when $r + s = 1$.

Of course, the definition of $c_F(r,s)$ makes sense for any field of characteristic $\neq 2$, but it will not always be a Steinberg symbol because it can fail to be bilinear in r. It is a Steinberg symbol when $F = \mathbb{R}$, because the Hilbert symbol $c_{\mathbb{R}}(r,s)$ is the same as the symbol $(r,s)_\infty$ of the previous example.

III.6.2.3 **Example 6.2.3** (norm residue symbols)**.** The roots of unity in a local field F form a finite cyclic group μ, equal to the group μ_m of all m^{th} roots of unity for some integer m with $\frac{1}{m} \in F$. The classical m^{th} *power norm residue symbol* is a map $K_2(F) \to \mu_m$ defined as follows (see [167] for more details).

Because $F^{\times m}$ has finite index in $F^\times$, there is a finite "Kummer" extension K containing the m^{th} roots of every element of F. The Galois group $G_F = Gal(K/F)$ is canonically isomorphic to $\mathrm{Hom}(F^\times, \mu_m)$, with the automorphism g of K corresponding to the homomorphism $\zeta\colon F^\times \to \mu_m$ sending $a \in F^\times$ to $\zeta(a) = g(x)/x$, where $x^m = a$. In addition, the cokernel of the norm map $K^\times \xrightarrow{N} F^\times$ is isomorphic to G_F by the "norm residue" isomorphism of local class field theory. The composite $F^\times \longrightarrow F^\times/NK^\times \cong G_F \cong \mathrm{Hom}(F^\times, \mu_m)$, written as $x \mapsto (x, -)_F$, is adjoint to a nondegenerate bilinear map $(\ ,\ )_F\colon F^\times \otimes F^\times \to \mu_m$.

The Steinberg identity $(a, 1-a)_F = 1$ is proven by noting that $(1-a)$ is a norm from the intermediate field $E = F(x)$, $x^m = a$. Since $G_E \subset G_F$ corresponds to the norm map $E^\times/N_{K/E}K^\times \hookrightarrow F^\times/N_{K/F}K^\times$, the element g of $G_F = Gal(K/F)$ corresponding to the map $\zeta(a) = (a, 1-a)_F$ from $F^\times$ to μ_m must belong to G_E, *i.e.*, ζ must extend to a map $E^\times \to \mu_m$. But then $(a, 1-a)_F = \zeta(a) = \zeta(x)^m = 1$.

The name "norm residue" comes from the fact that for each x, the map $y \mapsto \{x, y\}$ is trivial if and only if $x \in NK^\times$. Since a primitive m^{th} root of unity ζ is not a norm from K, it follows that there is an $x \in F$ such that $(\zeta, x)_F \neq 1$. Therefore the norm residue symbol is a split surjection with inverse $\zeta^i \mapsto \{\zeta^i, x\}$.

The role of the norm residue symbol is explained by the following structural result, whose proof we cite from the literature.

III.6.2.4 **Moore's Theorem 6.2.4.** *If F is a local field, then $K_2(F)$ is the direct sum of a uniquely divisible abelian group U and a finite cyclic group, isomorphic under the norm residue symbol to the group $\mu = \mu_m$ of roots of unity in F.*

Proof. We have seen that the norm residue symbol is a split surjection. A proof that its kernel U is divisible, due to C. Moore, is given in the Appendix to [131]. The fact that U is torsionfree (hence uniquely divisible) was proven by Tate [198] when $\mathrm{char}(F) = p$, and by Merkurjev [124] when $\mathrm{char}(F) = 0$. □

III.6.2.5 **Example 6.2.5** (2-adic rationals)**.** The group $K_2(\hat{\mathbb{Q}}_2)$ is the direct sum of the cyclic group of order 2 generated by $\{-1,-1\}$ and a uniquely divisible group. Since $x^2 + y^2 = -1$ has no solution in $F = \hat{\mathbb{Q}}_2$ we see from definition (6.2.2) that the Hilbert symbol $c_F(-1,-1) = -1$.

Tame symbols

Every discrete valuation v on a field F provides a Steinberg symbol. Recall that v is a homomorphism $F^\times \to \mathbb{Z}$ such that $v(r+s) \geq \min\{v(r), v(s)\}$. By convention, $v(0) = \infty$, so that the ring R of all r with $v(r) \geq 0$ is a discrete valuation ring (DVR). The units $R^\times$ form the set $v^{-1}(0)$, and the maximal ideal of R is generated by any $\pi \in R$ with $v(\pi) = 1$. The residue field k_v is defined to be $R/(\pi)$. If $u \in R$, we write $\bar{u}$ for the image of u under $R \to k_v$.

III.6.3 **Lemma 6.3.** *For every discrete valuation v on F there is a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$, defined by*

$$\partial_v(\{r,s\}) = (-1)^{v(r)v(s)}\overline{\left(\frac{s^{v(r)}}{r^{v(s)}}\right)}.$$

This symbol is called the tame symbol of the valuation v. The tame symbol is onto, because if $u \in R^\times$ then $v(u) = 0$ and $\partial_v(\pi, u) = \bar{u}$.

Proof. Writing $r = u_1\pi^{v_1}$ and $s = u_2\pi^{v_2}$ with $u_1, u_2 \in R^\times$, we must show that $\partial_v(r,s) = (-1)^{v_1v_2}\frac{\bar{u}_2^{v_1}}{\bar{u}_1^{v_2}}$ is a Steinberg symbol. By inspection, $\partial_v(r,s)$ is an element of $k_v^\times$, and ∂_v is bilinear. To see that $\partial_v(r,s) = 1$ when $r + s = 1$ we consider several cases. If $v_1 > 0$ then r is in the maximal ideal, so $s = 1 - r$ is a unit and $\partial_v(r,s) = \bar{s}^{v_1} = 1$. The proof when $v_2 > 0$ is the same, and the case $v_1 = v_2 = 0$ is trivial. If $v_1 < 0$ then $v(\frac{1}{r}) > 0$ and $\frac{1-r}{r} = -1 + \frac{1}{r}$ is congruent to $-1 \pmod{\pi}$. Since $v(r) = v(1-r)$, we have

$$\partial_v(r, 1-r) = (-1)^{v_1}\left(\frac{1-r}{r}\right)^{v_1} = (-1)^{v_1}(-1)^{v_1} = 1. \quad \square$$

III.6.3.1 **Remark 6.3.1** (Ramification)**.** Suppose that E is a finite extension of F, and that w is a valuation on E over the valuation v on F. Then there is an integer e, called the *ramification index*, such that $w(r) = e \cdot v(r)$ for every $r \in F$. The natural map $K_2(F) \to K_2(E)$ is compatible with the tame symbols in the sense that for every $r_1, r_2 \in F^\times$ we have $\partial_w(r_1, r_2) = \partial_v(r_1, r_2)^e$ in $k_w^\times$.

$$\begin{array}{ccc} K_2(F) & \xrightarrow{\partial_v} & k_v^\times \\ \downarrow & & e \downarrow x \mapsto x^e \\ K_2(E) & \xrightarrow{\partial_w} & k_w^\times \end{array}$$

Let S denote the integral closure of R in E. Then S has finitely many prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_n$ lying over $\mathfrak{p}$, with corresponding valuations $w_1, ..., w_n$ on E. We say that S is *unramified* over R if the ramification indices $e_1, ..., e_n$ are all 1; in this case the diagonal inclusion $\Delta\colon k_v^\times \hookrightarrow \prod_i k_{w_i}^\times$ is compatible with the tame symbols in the sense that $\Delta\partial_v(r_1, r_2)$ is the product of the $\partial_{w_i}(r_1, r_2)$.

III.6.3.2 **Corollary 6.3.2.** *If F contains the rational function field $\mathbb{Q}(t)$ or $\mathbb{F}_p(t_1, t_2)$, then $K_2(F)$ has the same cardinality as F. In particular, if F is uncountable then so is $K_2(F)$.*

Proof. By hypothesis, F contains a transcendental element t. Choose a subset $X = \{x_\alpha\}$ of F so that $X \cup \{t\}$ is a transcendence basis for F over its ground field F_0, and set $k = F_0(X)$. Then the subfield $k(t)$ of F has a t-adic valuation with residue class field k. Hence $K_2(k(t))$ contains a subgroup $\{t, k^\times\}$ mapped isomorphically under the tame symbol to $k^\times$. By Lemma 6.1.3, the kernel of $k^\times \to K_2(k(t)) \to K_2(F)$ is contained in the torsion subgroup $\mu(k)$ of roots of unity in k. Thus the cardinality of $K_2(F)$ is bounded below by the cardinality of $k^\times/\mu(k)$. Since F is an algebraic extension of $k(t)$, and k contains either $\mathbb{Q}$ or $\mathbb{F}_p(t_2)$, we have the inequality $|F| = |k| = |k^\times/\mu(k)| \le |K_2(F)|$. The other inequality $|K_2(F)| \le |F|$ is immediate from Matsumoto's Theorem, since F is infinite. □

III.6.4 **Theorem 6.4.** *(Bass-Tate) When F is an algebraically closed field, $K_2(F)$ is a uniquely divisible abelian group.*

Theorem 6.4 is an immediate consequence of proposition 6.4.1 below. To see this, recall that an abelian group is uniquely divisible when it is uniquely p-divisible for each prime p; a group is said to be *uniquely p-divisible* if it is p-divisible and has no p-torsion.

III.6.4.1 **Proposition 6.4.1.** *(Bass-Tate) Let p be a prime number such that each polynomial $t^p - a$ ($a \in F$) splits in $F[t]$ into linear factors. Then $K_2(F)$ is uniquely p-divisible.*

Proof. The hypothesis implies that $F^\times$ is p-divisible. Since the tensor product of p-divisible abelian groups is always uniquely p-divisible, $F^\times \otimes F^\times$ is uniquely p-divisible. Let R denote the kernel of the natural surjection $F^\times \otimes F^\times \to K_2(F)$. By inspection (or by the Snake Lemma), $K_2(F)$ is p-divisible and the p-torsion subgroup of $K_2(F)$ is isomorphic to R/pR.

Therefore it suffices to prove that R is p-divisible. Now R is generated by the elements $\psi(a) = (a) \otimes (1-a)$ of $F^\times \otimes F^\times$ ($a \in F - \{0, 1\}$), so it suffices to show that each $\psi(a)$ is in pR. By hypothesis, there are $b_i \in F$ such that $t^p - a = \prod(t - b_i)$ in $F[t]$, so $1 - a = \prod(1 - b_i)$ and $b_i^p = a$ for each i. But then we compute in $F^\times \otimes F^\times$:

$$\psi(a) = (a) \otimes (1-a) = \sum (a) \otimes (1 - b_i) = \sum (b_i)^p \otimes (1 - b_i) = p \sum \psi(b_i). \quad \square$$

III.6.4.2 **Corollary 6.4.2.** *If F is a perfect field of characteristic p, then $K_2(F)$ is uniquely p-divisible.*

The Localization Sequence for K_2

The following result will be proven in chapter V, 6.6.1, but we find it useful to quote this result now. If $\mathfrak{p}$ is a nonzero prime ideal of a Dedekind domain R, the local ring $R_\mathfrak{p}$ is a discrete valuation ring, and hence determines a tame symbol.

III.6.5 **Localization Theorem 6.5.** *Let R be a Dedekind domain, with field of fractions F. Then the tame symbols $K_2(F) \xrightarrow{\partial_\mathfrak{p}} (R/\mathfrak{p})^\times$ associated to the prime ideals of R fit into a long exact sequence*

$$\coprod_{\mathfrak{p}} K_2(R/\mathfrak{p}) \to K_2(R) \to K_2(F) \xrightarrow{\partial=\coprod \partial_\mathfrak{p}} \coprod_{\mathfrak{p}} (R/\mathfrak{p})^\times \to SK_1(R) \to 1$$

where the coproducts are over all nonzero prime ideals $\mathfrak{p}$ of R, and the maps from $(R/\mathfrak{p})^\times = K_1(R/\mathfrak{p})$ to $SK_1(R)$ are the transfer maps of Ex. 1.11. The transfer maps $K_2(R/\mathfrak{p}) \to K_2(R)$ will be defined in chapter V.

III.6.5.1 **Application 6.5.1** ($K_2\mathbb{Q}$)**.** If $R = \mathbb{Z}$ then, since $K_2(\mathbb{Z}/p) = 1$ and $SK_1(\mathbb{Z}) = 1$, we have an exact sequence $1 \to K_2(\mathbb{Z}) \to K_2(\mathbb{Q}) \xrightarrow{\partial} \coprod \mathbb{F}_p^\times \to 1$. As noted in Example 6.2.1, this sequence is split by the symbol $(r,s)_\infty$, so we have $K_2(\mathbb{Q}) \cong K_2(\mathbb{Z}) \oplus \coprod \mathbb{F}_p^\times$.

III.6.5.2 **Application 6.5.2** (Function fields)**.** If R is the polynomial ring $F[t]$ for some field F, we know that $K_2(F[t]) = K_2(F)$ (see 5.2.3). Moreover, the natural map $K_2(F) \to K_2F(t)$ is split by the leading coefficient symbol λ of Example 6.1.2. Therefore we have a split exact sequence

$$1 \to K_2(F) \to K_2F(t) \xrightarrow{\partial} \coprod_{\mathfrak{p}} (F[t]/\mathfrak{p})^\times \to 1.$$

III.6.5.3 **Weil Reciprocity Formula 6.5.3.** Just as in the case $R = \mathbb{Z}$, there is a valuation on $F(t)$ not arising from a prime ideal of $F[t]$. In this case, it is the valuation $v_\infty(f) = -deg(f)$ associated with the point at infinity, *i.e.*, with parameter t^{-1}. Since the symbol $(f,g)_\infty$ vanishes on $K_2(F)$, it must be expressable in terms of the tame symbols $\partial_\mathfrak{p}(f,g) = (f,g)_\mathfrak{p}$. The appropriate reciprocity formula first appeared in Weil's 1940 paper on the Riemann Hypothesis for curves:

$$(f,g)_\infty \cdot \prod_{\mathfrak{p}} N_\mathfrak{p}(f,g)_\mathfrak{p} = 1 \qquad \text{in } F^\times.$$

In Weil's formula, '$N_\mathfrak{p}$' denotes the usual norm map $(F[t]/\mathfrak{p})^\times \to F^\times$. To establish this reciprocity formula, we observe that $K_2F(t)/K_2F = \coprod (F[t]/\mathfrak{p})^\times$ injects into $K_2\bar{F}(t)/K_2\bar{F}$, where $\bar{F}$ is the algebraic closure of F. Thus we may assume that F is algebraically closed. By Example 6.1.7, $K_2F(t)$ is generated by linear symbols of the form $\{a, t-b\}$. But $(a, t-b)_\infty = a$ and $\partial_{t-b}(a, t-b) = a^{-1}$, so the formula is clear.

Our next structural result was discovered by Merkurjev and Suslin in 1981, and published in their landmark paper [125]; see [66, 8.4]. Recall that an automorphism σ of a field E induces an automorphism of $K_2(E)$ sending $\{x,y\}$ to $\{\sigma x, \sigma y\}$.

III.6.6 **Theorem 6.6** (Hilbert's Theorem 90 for K_2)**.** *Let E/F be a cyclic Galois field extension of prime degree p, and let σ be a generator of $Gal(E/F)$. Then the following sequence is exact, where N denotes the transfer map on K_2:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F).$$

Merkurjev and Suslin gave this result the suggestive name "Hilbert's Theorem 90 for K_2," because of its formal similarity to the following result, which is universally called "Hilbert's Theorem 90 (for units)" because it was the 90^{th} theorem in Hibert's classical 1897 survey of algebraic number theory, *Theorie der Algebraische Zahlkörper.*

III.6.6.1 **Theorem 6.6.1** (Hilbert's Theorem 90 for units)**.** *Let E/F be a cyclic Galois field extension, and let σ be a generator of $Gal(E/F)$. If $1-\sigma$ denotes the map $a \mapsto a/\sigma(a)$, then the following sequence is exact:*

$$1 \to F^\times \to E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times.$$

We omit the proof of Hilbert's Theorem 90 for K_2 (and for K_n^M; see III.7.8.4 7.8.4 below), since the proof does not involve K-theory, contenting ourselves with two special cases: when $p = \text{char}(F)$ (III.7.8.3 7.8.3) and the following special case.

III.6.6.2 **Proposition 6.6.2.** *Let F be a field containing a primitive n^{th} root of unity ζ, and let E be a cyclic field extension of degree n, with σ a generator of $Gal(E/F)$.*

Suppose in addition that the norm map $E^\times \xrightarrow{N} F^\times$ is onto, and that F has no extension fields of degree $< n$. Then the following sequence is exact:

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F) \to 1.$$

Proof. Since $N\zeta = 1$, Hilbert's Theorem 90 gives an $r \in E$ with $\sigma(r) = \zeta r$. Setting $c = N(r) \in F$, it is well-known and easy to see that $E = F(r)$, $r^n = c$.

Again by Hilbert's Theorem 90 for units and our assumption about norms, $E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times \to 1$ is an exact sequence of abelian groups. Applying the right exact functor $\otimes F^\times$ retains exactness. Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
E^\times \otimes F^\times & \xrightarrow{(1-\sigma)\otimes 1} & E^\times \otimes F^\times & \xrightarrow{N\otimes 1} & F^\times \otimes F^\times & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
K_2(E) & \xrightarrow{1-\sigma} & K_2(E) & \longrightarrow & C & \longrightarrow & 1
\end{array}$$

in which C denotes the cokernel of $1-\sigma$.

Now every element of E is a polynomial $f(r)$ in r of degree $< n$, and $f(t)$ is a product of linear terms $b_i t - a_i$ by our assumption. By Lemma III.6.1.4 6.1.4, every element of $K_2(E)$ is a product of symbols of the form $\{a, b\}$ and $\{a, r-b\}$. Therefore the vertical maps $F^\times \otimes E^\times \to K_2(E)$ are onto in the above diagram. Hence γ is onto.

If $a \in F^\times$ and $x \in E^\times$ then the projection formula (Ex. 5.6) yields

$$N(1-\sigma)\{a,x\} = N\{a, x/(\sigma x)\} = \{a, Nx/N(\sigma x)\} = 1.$$

Hence the transfer map $K_2(E) \to K_2(F)$ factors through C. A diagram chase shows that it suffices to show that γ is a Steinberg symbol, so that it factors through $K_2(F)$. For this we must show that for all $y \in E$ we have $\gamma(Ny \otimes (1 - Ny)) = 1$, i.e., that $\{y, 1-Ny\} \in (1-\sigma)K_2(E)$.

Fix $y \in E$ and set $z = N_{E/F}(y) \in F$. Factor $t^n - z = \prod f_i$ in $F[t]$, with the f_i irreducible, and let F_i denote the field $F(x_i)$, where $f_i(x_i) = 0$ and $x_i^n = z$. Setting $t = 1$, $1 - z = \prod f_i(1) = \prod N_{F_i/F}(1-x_i)$. Setting $E_i = E \otimes_F F_i$, so that $N_{F_i/F}(1-x_i) = N_{E_i/E}(1-x_i)$ and $\sigma(x_i) = x_i$, the projection formula (Ex. 5.6) gives

$$\{y, 1-z\} = \prod N_{E_i/E}\{y, 1-x_i\} = \prod N_{E_i/E}\{y/x_i, 1-x_i\}.$$

Thus it suffices to show that each $N_{E_i/E}\{y/x_i, 1-x_i\}$ is in $(1-\sigma)K_2(E)$. Now E_i/F_i is a cyclic extension whose norm $N = N_{E_i/F_i}$ satisfies $N(y/x_i) = N(y)/x_i^n = 1$. By Hilbert's Theorem 90 for units, $y/x_i = v_i/\sigma v_i$ for some $v_i \in E_i$. We now compute:

$$N_{E_i/E}\{y/x_i, 1-x_i\} = N_{E_i/E}\{v_i/\sigma v_i, 1-x_i\} = (1-\sigma)N_{E_i/E}\{v_i, 1-x_i\}. \qquad \square$$

Here are three pretty applications of Hilbert's Theorem 90 for K_2. When F is a perfect field, the first of these has already been proven in Proposition 6.4.1.

III.6.7 **Theorem 6.7.** *If char$(F) = p \neq 0$, then the group $K_2(F)$ has no p-torsion.*

Proof. Let x be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism σ satisfying $\sigma(x) = x + 1$. By 6.5.2, $K_2(F)$ is a subgroup of both $K_2F(x)$ and $K_2F(y)$, and the projection formula shows that the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to u^p.

Now fix $u \in K_2(F)$ satisfying $u^p = 1$; we shall prove that $u = 1$. By Hilbert's Theorem 90 for K_2, $u = (1-\sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Every prime ideal $\mathfrak{p}$ of $F[x]$ is unramified over $\mathfrak{p}_y = \mathfrak{p} \cap F[y]$, because $F[x]/\mathfrak{p}$ is either equal to, or an Artin-Schrier extension of, $F[y]/\mathfrak{p}_y$. By 6.3.1 and 6.5.2, we have a commutative diagram in which the vertical maps ∂ are surjective:

$$\begin{array}{ccccc}
K_2F(y) & \xrightarrow{\;i^*\;} & K_2F(x) & \xrightarrow{\;1-\sigma\;} & K_2F(x) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
\coprod_{\mathfrak{p}_y}(F[y]/\mathfrak{p}_y)^\times & \xrightarrow{\;\Delta\;} & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times & \xrightarrow{\;1-\sigma\;} & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times
\end{array}$$

We claim that the bottom row is exact. By decomposing the row into subsequences invariant under σ, we see that there are two cases to consider. If a prime

$\mathfrak{p}$ is not fixed by σ, then the fields $F[x]/\sigma^i\mathfrak{p}$ are all isomorphic to $E = F[y]/\mathfrak{p}_y$, and for $a_i \in E^\times$ we have

$$(1-\sigma)(a_0, a_1, \dots, a_{p-1}) = (a_0 a_{p-1}^{-1}, a_1 a_0^{-1}, \dots, a_{p-1} a_{p-2}^{-1})$$

in $\prod_{i=0}^{p-1}(F[x]/\sigma^i\mathfrak{p})^\times$. This vanishes if and only if the a_i agree, in which case $(a_0, \dots, a_{p-1})$ is the image of $a \in E^\times$. On the other hand, if σ fixes $\mathfrak{p}$ then $F[x]/\mathfrak{p}$ is a cyclic Galois extension of $E = F[y]/\mathfrak{p}_y$. Therefore if $a \in F[x]/\mathfrak{p}$ and $(1-\sigma)a = a/(\sigma a)$ equals 1, then $a = \sigma(a)$, *i.e.*, $a \in E$. This establishes the claim.

A diagram chase shows that since $1 = \partial u = \partial(1-\sigma)v$, there is a v_0 in $K_2F(y)$ with $\partial(v) = \partial(i^* v_0)$. Since $i^* = \sigma i^*$, we have $(1-\sigma)i^*v_0 = 1$. Replacing v by $v(i^*v_0)^{-1}$, we may assume that $\partial(v) = 1$, *i.e.*, that v is in the subgroup $K_2(F)$ of $K_2F(x)$. Therefore we have $u = v(\sigma v)^{-1} = 1$. As u was any element of $K_2(F)$ satisfying $u^p = 1$, $K_2(F)$ has no p-torsion. □

III.6.7.1 **Example 6.7.1.** If $F = \mathbb{F}_q(t)$, $q = p^r$, we have $K_2(F) = \coprod(\mathbb{F}_q[t]/\mathfrak{p})^\times$. Since the units of each finite field $\mathbb{F}_q[t]/\mathfrak{p}$ form a cyclic group, and its order can be arbitrarily large (yet prime to p), $K_2\mathbb{F}_q(t)$ is a very large torsion group.

III.6.8 **Theorem 6.8.** *If F contains a primitive n^{th} root of unity ζ, then every element of $K_2(F)$ of exponent n has the form $\{\zeta, x\}$ for some $x \in F^\times$.*

Proof. We first suppose that n is a prime number p. Let x be an indeterminate and $y = x^p$; the Galois group of the field extension $F(x)/F(y)$ is generated by an automorphism σ satisfying $\sigma(x) = \zeta x$. By Application 6.5.2, $K_2(F)$ is a subgroup of $K_2F(x)$, and by the projection formula the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to u^p.

Fix $u \in K_2(F)$ satisfying $u^p = 1$. By Hilbert's Theorem 90 for K_2, if $u^p = 1$ then $u = (1-\sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Now the extension $F[y] \subset F[x]$ is unramified at every prime ideal except $\mathfrak{p} = (x)$. As in the proof of Theorem 6.7, we have a commutative diagram whose bottom row is exact:

$$\begin{array}{ccccc}
K_2F(y) & \xrightarrow{\ i^*\ } & K_2F(x) & \xrightarrow{\ 1-\sigma\ } & K_2F(x) \\
\downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial'} \\
\coprod_{\mathfrak{p}_y \neq (y)} (F[y]/\mathfrak{p}_y)^\times & \xrightarrow{\ \Delta\ } & \coprod_{\mathfrak{p} \neq (x)} (F[x]/\mathfrak{p})^\times & \xrightarrow{\ 1-\sigma\ } & \coprod_{\mathfrak{p} \neq (x)} (F[x]/\mathfrak{p})^\times
\end{array}$$

As before, we may modify v by an element from $K_2F(y)$ to arrange that $\partial_{\mathfrak{p}}(v) = 1$ for all $\mathfrak{p} \neq (x)$. For $\mathfrak{p} = (x)$, let $a \in F = F[x]/(x)$ be such that $\partial_{(x)}(v) = a$ and set $v' = v\{a, x\}$. Then $\partial_{(x)}(v') = 1$ and $\partial_{\mathfrak{p}}(v') = \partial_{\mathfrak{p}}(v) = 1$ for every other $\mathfrak{p}$. It follows from 6.5.2 that v' is in $K_2(F)$. Therefore $(1-\sigma)v' = 1$; since $v = v'\{a, x\}^{-1}$ this implies that u has the asserted form:

$$u = (1-\sigma)\{a, x\}^{-1} = \{a, x\}^{-1}\{a, \zeta x\} = \{a, \zeta\}.$$

Now we proceed inductively, supposing that $n = mp$ and that the theorem has been proven for m (and p). If $u \in K_2(F)$ has exponent n then u^p has exponent m, so there is an $x \in F^\times$ so that $u^p = \{\zeta^p, x\}$. The element $u\{\zeta, x\}^{-1}$ has exponent p, so it equals $\{\zeta^m, y\} = \{\zeta, y^m\}$ for some $y \in F^\times$. Hence $u = \{\zeta, xy^m\}$, as required. □

III.6.8.1 **Remark 6.8.1.** Suslin also proved the following result in [186]. Let F be a field containing a primitive p^{th} root of unity ζ, and let $F_0 \subset F$ be the subfield of constants. If $x \in F_0^\times$ and $\{\zeta, x\} = 1$ in $K_2(F)$ then $\{\zeta, x\} = 1$ in $K_2(F_0)$. If $\{\zeta, y\} = 1$ in $K_2(F)$ for some $y \in F^\times$ then $y = xz^p$ for some $x \in F_0^\times$ and $z \in F^\times$.

III.6.8.2 **Application 6.8.2.** We can use Theorem 6.8 to give another proof of Theorem 6.4, that when F is an algebraically closed field, the group $K_2(F)$ is uniquely divisible. Fix a prime p. For each $a \in F^\times$ there is an α with $\alpha^p = a$. Hence $\{a, b\} = \{\alpha, b\}^p$, so $K_2(F)$ is p-divisible. If $p \neq \mathrm{char}(F)$ then there is no p-torsion because $\{\zeta, a\} = \{\zeta, \alpha\}^p = 1$. Finally, if $\mathrm{char}(F) = p$, there is no p-torsion either by Theorem 6.7.

III.6.8.3 **Application 6.8.3** ($K_2\mathbb{R}$)**.** Theorem 6.8 states that $\{-1, -1\}$ is the only element of order 2 in $K_2\mathbb{R}$. Indeed, if r is a positive real number then:

$$\{-1, r\} = \{-1, \sqrt{r}\}^2 = 1, \quad \text{and} \quad \{-1, -r\} = \{-1, -1\}\{-1, r\} = \{-1, -1\}.$$

Note that $\{-1, -1\}$ is in the image of $K_2(\mathbb{Z})$, which is a summand by either Example 6.2.1 or Example 5.9.1. Recall from Example 6.1.6 that the image of $K_2\mathbb{R}$ in the uniquely divisible group $K_2\mathbb{C}$ is the eigenspace $(K_2\mathbb{C})^+$, and that the composition $K_2\mathbb{R} \to K_2\mathbb{C} \xrightarrow{N} K_2\mathbb{R}$ is multiplication by 2, so its kernel is $K_2(\mathbb{Z})$. It follows that

$$K_2\mathbb{R} \cong K_2(\mathbb{Z}) \oplus (K_2\mathbb{C})^+.$$

K_2 and the Brauer group

Let F be a field. Recall from II.5.4.3 that the Brauer group $\mathrm{Br}(F)$ is generated by the classes of central simple algebras with two relations: $[A \otimes_F B] = [A] \cdot [B]$ and $[M_n(F)] = 0$. Here is one classical construction of elements in the Brauer group; it is a special case of the construction of crossed product algebras.

III.6.9 **Example 6.9** (Cyclic algebras)**.** Let ζ be a primitive n^{th} root of unity in F, and $\alpha, \beta \in F^\times$. The *cyclic algebra* $A = A_\zeta(\alpha, \beta)$ is defined to be the associative algebra with unit, which is generated by two elements x, y subject to the relations $x^n = \alpha \cdot 1$, $y^n = \beta \cdot 1$ and $yx = \zeta xy$. Thus A has dimension n^2 over F, a basis being the monomials $x^i y^j$ with $0 \le i, j < n$. The identity $(x + y)^n = (\alpha + \beta) \cdot 1$ is also easy to check.

When $n = 2$ (so $\zeta = -1$), cyclic algebras are called *quaternion algebras*. The name comes from the fact that the usual quaternions $\mathbb{H}$ are the cyclic algebra $A(-1, -1)$ over $\mathbb{R}$. Quaternion algebras arise in the Hasse invariant of quadratic forms.

It is classical, and not hard to prove, that A is a central simple algebra over F; see [10, §8.5]. Moreover, the n-fold tensor product $A \otimes_F A \otimes_F \cdots \otimes_F A$ is a matrix algebra; see [10, Theorem 8.12]. Thus we can consider $[A] \in \mathrm{Br}(F)$ as an element of exponent n. We shall write ${}_n\mathrm{Br}(F)$ for the subgroup of $\mathrm{Br}(F)$ consisting of all elements x with $x^n = 1$, so that $[A] \in {}_n\mathrm{Br}(F)$

For example, the following lemma shows that $A_\zeta(1, \beta)$ must be a matrix ring because $x^n = 1$. Thus $[A_\zeta(1, \beta)] = 1$ in $\mathrm{Br}(F)$.

III.6.9.1 **Lemma 6.9.1.** *Let A be a central simple algebra of dimension n^2 over a field F containing a primitive n^{th} root of unity ζ. If A contains an element $u \notin F$ such that $u^n = 1$, then $A \cong M_n(F)$.*

Proof. The subalgebra $F[u]$ of A spanned by u is isomorphic to the commutative algebra $F[t]/(t^n - 1)$. Since $t^n - 1 = \prod(t - \zeta^i)$, the Chinese Remainder Theorem yields $F[u] \cong F \times F \times \cdots \times F$. Hence $F[u]$ contains n idempotents e_i with $e_i e_j = 0$ for $i \neq j$. Therefore A splits as the direct sum $e_1 A \oplus \cdots \oplus e_n A$ of right ideals. By the Artin-Wedderburn theorem, if $A = M_d(D)$ then A can be the direct sum of at most d right ideals. Hence $d = n$, and A must be isomorphic to $M_n(F)$. ☐

III.6.9.2 **Proposition 6.9.2** (n^{th} power norm residue symbol)**.** *If F contains ζ, a primitive n^{th} root of unity, there is a homomorphism $K_2(F) \to \mathrm{Br}(F)$ sending $\{\alpha, \beta\}$ to the class of the cyclic algebra $A_\zeta(\alpha, \beta)$.*

Since the image is a subgroup of exponent n, we shall think of the power norm residue symbol as a map $K_2(F)/nK_2(F) \to {}_n\mathrm{Br}(F)$.

This homomorphism is sometimes also called the *Galois symbol*.

Proof. From Ex. 6.12 we see that in $\mathrm{Br}(F)$ we have $[A_\zeta(\alpha, \beta)] \cdot [A_\zeta(\alpha, \gamma)] = [A_\zeta(\alpha, \beta\gamma)]$. Thus the map $F^\times \times F^\times \to \mathrm{Br}(F)$ sending (α, β) to $[A_\zeta(\alpha, \beta)]$ is bilinear. To see that it is a Steinberg symbol we must check that $A = A_\zeta(\alpha, 1-\alpha)$ is isomorphic to the matrix algebra $M_n(F)$. Since the element $x+y$ of A satisfies $(x+y)^n = 1$, Lemma 6.9.1 implies that A must be isomorphic to $M_n(F)$. ☐

III.6.9.3 **Remark 6.9.3.** Merkurjev and Suslin proved in [125] that $K_2(F)/mK_2(F)$ is isomorphic to the subgroup ${}_m\mathrm{Br}(F)$ of elements of order m in $\mathrm{Br}(F)$ when $\mu_m \subset F$. By Matsumoto's Theorem, this implies that the m-torsion in the Brauer group is generated by cyclic algebras. The general description of $K_2(F)/m$, due to Merkurjev-Suslin, is given in 6.10.4; see VI.3.1.1.

The Galois symbol

We can generalize the power norm residue symbol to fields not containing enough roots of unity by introducing Galois cohomology. Here are the essential facts we shall need; see [223] or [127].

III.6.10 **Sketch of Galois Cohomology 6.10.** Let F_{sep} denote the separable closure of a field F, and let $G = G_F$ denote the Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$. The family of subgroups $G_E = \mathrm{Gal}(F_{\mathrm{sep}}/E)$, as E runs over all finite extensions of F, forms a basis for a topology of G. A G-module M is called *discrete* if the multiplication $G \times M \to M$ is continuous.

For example, the abelian group $\mathbb{G}_m = F_{\text{sep}}^{\times}$ of units of F_{sep} is a discrete module, as is the subgroup μ_m of all m^{th} roots of unity. We can also make the tensor product of two discrete modules into a discrete module, with G acting diagonally. For example, the tensor product $\mu_m^{\otimes 2} = \mu_m \otimes \mu_m$ is also a G-discrete module. Note that the three G-modules $\mathbb{Z}/m$, μ_m and $\mu_m^{\otimes 2}$ have the same underlying abelian group, but are isomorphic G_F-modules only when $\mu_m \subset F$.

The G-invariant subgroup M^G of a discrete G-module M is a left exact functor on the category of discrete G_F-modules. The *Galois cohomology groups* $H^i_{\text{et}}(F;M)$ are defined to be its right derived functors. In particular, $H^0_{\text{et}}(F;M)$ is just M^G.

If E is a finite separable field extension of F then $G_E \subset G_F$. Thus there is a forgetful functor from G_F-modules to G_E-modules, inducing maps $H^i_{\text{et}}(F;M) \to H^i_{\text{et}}(E;M)$. In the other direction, the induced module functor from G_E-modules to G_F-modules gives rise to cohomological transfer maps $tr_{E/F}\colon H^i_{\text{et}}(E;M) \to H^i_{\text{et}}(F;M)$; see [223, 6.3.9 and 6.11.11].

III.6.10.1 **Example 6.10.1** (Kummer Theory)**.** The cohomology of the module $\mathbb{G}_m$ is of fundamental importance. Of course $H^0_{\text{et}}(F, \mathbb{G}_m) = F^{\times}$. By Hilbert's Theorem 90 for units, and a little homological algebra [223, 6.11.16], we also have $H^1_{\text{et}}(F;\mathbb{G}_m) = 0$ and $H^2_{\text{et}}(F;\mathbb{G}_m) \cong \text{Br}(F)$.

If m is prime to char(F), the exact sequence of discrete modules

$$1 \to \mu_m \to \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \to 1$$

is refered to as the *Kummer sequence*. Writing $\mu_m(F)$ for the group μ_m^G of all m^{th} roots of unity in F, the corresponding cohomology sequence is also called the *Kummer sequence*.

$$1 \to \mu_m(F) \to F^{\times} \xrightarrow{m} F^{\times} \to H^1_{\text{et}}(F;\mu_m) \to 1$$
$$1 \to H^2_{\text{et}}(F;\mu_m) \to \text{Br}(F) \xrightarrow{m} \text{Br}(F)$$

This yields isomorphisms $H^1_{\text{et}}(F;\mu_m) \cong F^{\times}/F^{\times m}$ and $H^2_{\text{et}}(F;\mu_m) \cong {}_m\text{Br}(F)$. If $\mu_m \subset F^{\times}$, this yields a natural isomorphism $H^2_{\text{et}}(F;\mu_m^{\otimes 2}) \cong {}_m\text{Br}(F) \otimes \mu_m(F)$.

There are also natural cup products in cohomology, such as the product

$$F^{\times} \otimes F^{\times} \to H^1_{\text{et}}(F;\mu_m) \otimes H^1_{\text{et}}(F;\mu_m) \xrightarrow{\cup} H^2_{\text{et}}(F;\mu_m^{\otimes 2}) \qquad (6.10.2)$$ III.6.10.2

which satisfies the following *projection formula:* if E/F is a finite separable extension, $a \in F^{\times}$ and $b \in E^{\times}$, then $tr_{E/F}(a \cup b) = a \cup N_{E/F}(b)$.

III.6.10.3 **Proposition 6.10.3** (Galois symbol)**.** *The bilinear pairing* (6.10.2) *induces a Steinberg symbol* $K_2(F)/mK_2(F) \to H^2_{et}(F;\mu_m^{\otimes 2})$ *for every* m *prime to* char(F).

Proof. It suffices to show that $a \cup (1-a)$ vanishes for every $a \in F - \{0,1\}$. Fixing a, factor the separable polynomial $t^m - a = \prod f_i$ in $F[t]$ with the f_i

irreducible, and let F_i denote the field $F(x_i)$ with $f_i(x_i) = 0$. Setting $t = 1$, $1 - a = \prod_i N_{F_i/F}(1 - x_i)$. Writing H^2_{et} additively, we have

$$a \cup (1-a) = \sum_i a \cup N_{F_i/F}(1-x_i) = \sum_i tr_{F_i/F}\big(a \cup (1-x_i)\big)$$
$$= m \sum_i tr_{F_i/F}\big(x_i \cup (1-x_i)\big).$$

Since the group $H^2_{\text{et}}(F; \mu_m^{\otimes 2})$ has exponent m, all these elements vanish, as desired. □

III.6.10.4 **Remark 6.10.4.** Suppose that F contains a primitive m^{th} root of unity ζ. If we identify $\mathbb{Z}/m$ with μ_m via $1 \mapsto \zeta$, we have a natural isomorphism

$$_m\text{Br}(F) \cong {}_m\text{Br}(F) \otimes \mathbb{Z}/m \cong {}_m\text{Br}(F) \otimes \mu_m \cong H^2_{\text{et}}(F; \mu_m^{\otimes 2}).$$

Tate showed in [198] that this isomorphism identifies the Galois symbol of Proposition 6.10.3 with the m^{th} power norm residue symbol of Proposition 6.9.2. The Merkurjev-Suslin isomorphism of [125] cited above in Remark 6.9.3 is a special case of the more general assertion that this symbol induces an isomorphism: $K_2(F)/mK_2(F) \cong H^2_{\text{et}}(F; \mu_m^{\otimes 2})$ for all fields F of characteristic prime to m. See Chapter VI, 4.1.1.

EXERCISES

EIII.6.1 **6.1.** Given a discrete valuation on a field F, with residue field k and parameter π, show that there is a surjection $\lambda: K_2(F) \to K_2(k)$ given by the formula $\lambda\{u\pi^i, v\pi^j\} = \{\bar{u}, \bar{v}\}$. Example 6.1.2 is a special case of this, in which $\pi = t^{-1}$.

EIII.6.2 **6.2.** (Bass-Tate) If $E = F(u)$ is a field extension of F, and $e_1, e_2 \in E$ are monic polynomials in u of some fixed degree $d > 0$, show that $\{e_1, e_2\}$ is a product of symbols $\{e_1, e_2'\}$ and $\{e, e_2''\}$ with e, e_2', e_2'' polynomials of degree $< d$. This generalizes Lemma 6.1.4.

EIII.6.3 **6.3.** (Bass-Tate) Let k be a field and set $F = k((t))$.

(a) Show that $K_2(F) \cong K_2(k) \times k^\times \times K_2(k[[t]], t)$.

(b) Show that the group $K_2(k[[t]], t)$ is torsionfree; by Ex. 5.14, it is uniquely divisible if $\text{char}(k) = 0$. *Hint:* Use Theorem 6.7 and the proof of 6.4.1.

EIII.6.4 **6.4.** If F is a number field with r_1 distinct embeddings $F \hookrightarrow \mathbb{R}$, show that the r_1 symbols $(\ ,\ )_\infty$ on F define a surjection $K_2(F) \to \{\pm 1\}^{r_1}$.

EIII.6.5 **6.5.** If $\bar{F}$ denotes the algebraic closure of a field F, show that $K_2(\bar{\mathbb{Q}}) = K_2(\bar{\mathbb{F}}_p) = 1$.

EIII.6.6 **6.6.** *2-adic symbol on* $\mathbb{Q}$. Any nonzero rational number r can be written uniquely as $r = (-1)^i 2^j 5^k u$, where $i, k \in \{0, 1\}$ and u is a quotient of integers congruent to 1 (mod 8). If $s = (-1)^{i'} 2^{j'} 5^{k'} u'$, set $(r, s)_2 = (-1)^{ii'+jj'+kk'}$. Show that this is a Steinberg symbol on $\mathbb{Q}$, with values in $\{\pm 1\}$.

EIII.6.7 **6.7.** Let $((r, s))_p$ denote the Hilbert symbol on $\hat{\mathbb{Q}}_p$ (6.2.2), and $(r, s)_p$ the tame symbol $K_2(\hat{\mathbb{Q}}_p) \to \mathbb{F}_p^\times$. Assume that p is odd, so that there is a unique surjection $\varepsilon\colon \mathbb{F}_p^\times \to \{\pm 1\}$. Show that $((r, s))_p = \varepsilon\left((r, s)_p\right)$ for all $r, s \in \hat{\mathbb{Q}}_p^\times$.

EIII.6.8 **6.8.** *Quadratic Reciprocity.* If $r, s \in \mathbb{Q}^\times$, and $(r, s)_2$ is the 2-adic symbol of Ex. 6.6, show that

$$(r, s)_\infty \ (r, s)_2 \ \prod_{p \neq 2} ((r, s))_p = +1.$$

Hint: From 6.5.1 and Ex. 6.7, the 2-adic symbol of Ex. 6.6 must satisfy some relation of the form

$$(r, s)_2 = {(r, s)_\infty}^{\varepsilon_\infty} \prod_{p \neq 2} {((r, s))_p}^{\varepsilon_p},$$

where the exponents ε_p are either 0 or 1. Since $(-1, -1)_2 = (-1, -1)_\infty$ we have $\varepsilon_\infty = 1$. If p is a prime not congruent to 1 (mod 8), consider $\{2, p\}$ and $\{-1, p\}$. If p is a prime congruent to 1 (mod 8), Gauss proved that there is a prime $q < \sqrt{p}$ such that p is not a quadratic residue modulo q. Then $((p, q))_q = -1$, even though $(p, q)_\infty = (p, q)_2 = 1$. Since we may suppose inductively that ε_q equals 1, this implies that $\varepsilon_p \neq 0$.

EIII.6.9 **6.9.** (Suslin) Suppose that a field F is algebraically closed in a larger field E. Use Lemma 6.1.3 and Remark 6.8.1 to show that $K_2(F)$ injects into $K_2(E)$.

EIII.6.10 **6.10.** Let F be a field, and let $\Omega_F^1 = \Omega_{F/\mathbb{Z}}^1$ denote the vector space of absolute Kähler differentials (see Ex. 2.6). The n^{th} exterior power of Ω_F^1 is written as Ω_F^n. Show that there is a homomorphism $K_2(F) \to \Omega_F^2$ sending $\{x, y\}$ to $\frac{dx}{x} \wedge \frac{dy}{y}$. This map is not onto, because the image is in the kernel of the de Rham differential $d : \Omega_F^2 \to \Omega_F^3$.

EIII.6.11 **6.11.** If F is a field of transcendence degree κ over the ground field, Ω_F^1 is a vector space of dimension κ. Now suppose that κ is an infinite cardinal number, so that Ω_F^n is also a vector space of dimension κ for all $n > 1$. Show that the image of the map $K_2(F) \to \Omega_F^2$ in the previous exercise is an abelian group of rank κ.

In particular, if F is a local field then the uniquely divisible summand U of $K_2(F)$ in Moore's Theorem (6.2.4) is uncountable.

EIII.6.12 **6.12.** Show that $A_\zeta(\alpha, \beta) \otimes A_\zeta(\alpha, \gamma) \cong M_n(A)$, where $A = A_\zeta(\alpha, \beta\gamma)$. This isomorphism is used to construct the Galois symbol in 6.10.3.

Hint: Let x', y' generate $A_\zeta(\alpha, \beta)$ and x'', y'' generate $A_\zeta(\alpha, \gamma)$, and show that $x', y = y'y''$ generate A. Then show that $u = (x')^{-1}x'' + y''$ has $u^n = 1$. (For another proof, see [10, Ex. 8.5.2].)

7 Milnor K-theory of fields

Fix a field F, and consider the tensor algebra of the group $F^\times$,

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus (F^\times \otimes F^\times \otimes F^\times) \oplus \cdots .$$

To keep notation straight, we write $l(x)$ for the element of degree one in $T(F^\times)$ corresponding to $x \in F^\times$.

III.7.1 **Definition 7.1.** The graded ring $K_*^M(F)$ is defined to be the quotient of $T(F^\times)$ by the ideal generated by the homogeneous elements $l(x)\otimes l(1-x)$ with $x \neq 0, 1$. The *Milnor K-group* $K_n^M(F)$ is defined to be the subgroup of elements of degree n. We shall write $\{x_1, \dots, x_n\}$ for the image of $l(x_1) \otimes \cdots \otimes l(x_n)$ in $K_n^M(F)$.

That is, $K_n^M(F)$ is presented as the group generated by symbols $\{x_1, \dots, x_n\}$ subject to two defining relations: $\{x_1, \dots, x_n\}$ is multiplicative in each x_i, and equals zero if $x_i + x_{i+1} = 1$ for some i.

The name comes from the fact that the ideas in this section first arose in Milnor's 1970 paper [130]. Clearly we have $K_0^M(F) = \mathbb{Z}$, and $K_1^M = F^\times$ (with the group operation written additively). By Matsumoto's Theorem 6.1 we also have $K_2^M(F) = K_2(F)$, the elements $\{x, y\}$ being the usual Steinberg symbols, except that the group operation in $K_2^M(F)$ is written additively.

Since $\{x_i, x_{i+1}\} + \{x_{i+1}, x_i\} = 0$ in $K_2^M(F)$, we see that interchanging two entries in $\{x_1, \dots, x_n\}$ yields the inverse. It follows that these symbols are alternating: for any permutation π with sign $(-1)^\pi$ we have

$$\{x_{\pi(1)}, \dots, x_{\pi(n)}\} = (-1)^\pi \{x_1, \dots, x_n\}.$$

III.7.2 **Examples 7.2.** (a) If $\mathbb{F}_q$ is a finite field, then $K_n^M(\mathbb{F}_q) = 0$ for all $n \geq 2$, because $K_2^M(\mathbb{F}_q) = 0$ by Cor. 6.1.1. If F has transcendence degree 1 over a finite field (a global field of finite characteristic), Bass and Tate proved in [21] that $K_n^M(F) = 0$ for all $n \geq 3$.

(b) If F is algebraically closed then $K_n^M(F)$ is uniquely divisible. Divisibility is clear because $F^\times$ is divisible. The proof that there is no p-torsion is the same as the proof for $n = 2$ given in Theorem 6.4, and is relegated to Ex. 7.3.

(c) When $F = \mathbb{R}$ we can define a symbol $K_n^M(\mathbb{R}) \to \{\pm 1\}$ by the following formula: $(x_1, \dots, x_n)_\infty$ equals -1 if all the x_i are negative, and equals $+1$ otherwise. When $n = 2$ this is the symbol defined in Example 6.2.1.

To construct it, extend $\mathbb{Z} \to \mathbb{Z}/2$ to a ring homomorphism $T(\mathbb{R}^\times) \to (\mathbb{Z}/2)[t]$ by sending $l(x)$ to t if $x < 0$ and to 0 if $x > 0$. This sends the elements $l(x) \otimes l(1-x)$ to zero (as in 6.2.1), so it induces a graded ring homomorphism $K_*^M(\mathbb{R}) \to (\mathbb{Z}/2)[t]$. The symbol above is just the degree n part of this map.

By induction on n, it follows that $K_n^M(\mathbb{R})$ is the direct sum of a cyclic group of order 2 generated by $\{-1, \dots, -1\}$, and a divisible subgroup. In particular, this shows that $K_*^M(\mathbb{R})/2K_*^M(\mathbb{R})$ is the polynomial ring $(\mathbb{Z}/2)[\epsilon]$ on $\epsilon = l(-1)$. Using the norm map we shall see later that the divisible subgroup of $K_n^M(\mathbb{R})$ is in fact uniquely divisible. This gives a complete description of each $K_n^M(\mathbb{R})$ as an abelian group.

(d) When F is a number field, let r_1 be the number of embeddings of F into $\mathbb{R}$. Then we have a map from $K_n^M(F)$ to the torsion subgroup $(\mathbb{Z}/2)^{r_1}$ of $K_n^M(\mathbb{R})^{r_1}$. Bass and Tate proved in [21] that this map is an isomorphism for all $n \geq 3$: $K_n^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Tame symbols

Recall from Lemma 6.3 and Ex. 6.1 that every discrete valuation v on F induces a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$ and a map $K_2(F) \xrightarrow{\lambda} K_2(k_v)$. These symbols extend to all of Milnor K-theory; the ∂_v are called *higher tame symbols*, and the λ are called *specialization maps*.

III.7.3 **Theorem 7.3** (Specialization maps and higher tame symbols). *For every discrete valuation v on F, there are two surjections*

$$K_n^M(F) \xrightarrow{\partial_v} K_{n-1}^M(k_v) \quad \text{and} \quad K_n^M(F) \xrightarrow{\lambda} K_n^M(k_v)$$

satisfying the following conditions. Let $R = \{r \in F : v(r) \geq 0\}$ be the valuation ring, and π a parameter for v. If $u_i \in R^\times$, and $\bar{u}_i$ denotes the image of u_i in $k_v = R/(\pi)$ then

$$\lambda\{u_1\pi^{i_1}, \dots, u_n\pi^{i_n}\} = \{\bar{u}_1, \dots, \bar{u}_n\}, \qquad \partial_v\{\pi, u_2, \dots, u_n\} = \{\bar{u}_2, \dots, \bar{u}_n\}.$$

In particular, $\partial_v\colon K_2^M(F) \to k_v^\times$ is the tame symbol of Lemma 6.3, and $\lambda\colon K_2(F) \to K_2(k)$ is the specialization map of Example 6.1.2 and Ex. 6.1.

Proof. (Serre) Let L denote the graded $K_*^M(k_v)$-algebra generated by an indeterminate Π in L_1, with the relation $\{\Pi, \Pi\} = \{-1, \Pi\}$. We claim that the group homomorphism

$$d\colon F^\times \to L_1 = l(k_v^\times) \oplus \mathbb{Z}\cdot\Pi, \qquad d(u\pi^i) = l(\bar{u}) + i\Pi$$

satisfies the relation: for $r \neq 0, 1$, $d(r)d(1-r) = 0$ in L_2. If so, the presentation of $K_*^M(F)$ shows that d extends to a graded ring homomorphism $d\colon K_*^M(F) \to L$. Since L_n is the direct sum of $K_n^M(k_v)$ and $K_{n-1}^M(k_v)$, we get two maps: $\lambda\colon K_n^M(F) \to K_n^M(k_v)$ and $\partial_v\colon K_n^M(F) \to K_{n-1}^M(k_v)$. The verification of the relations is routine, and left to the reader.

If $1 \neq r \in R^\times$, then either $1 - r \in R^\times$ and $d(r)d(1-r) = \{\bar{r}, 1-\bar{r}\} = 0$, or else $v(1-r) = i > 0$ and $d(r) = l(1) + 0\cdot\Pi = 0$ so $d(r)d(1-r) = 0\cdot d(1-r) = 0$. If $v(r) > 0$ then $1 - r \in R^\times$ and the previous argument implies that $d(1-r)d(r) = 0$. If $r \notin R$ then $1/r \in R$, and we see from (5.10.3) and the above that $d(r)d(1-r) = d(1/r)d(-1/r)$. Therefore it suffices to show that $d(r)d(-r) = 0$ for every $r \in R$. If $r = \pi$ this is the given relation upon L, and if $r \in R^\times$ then $d(r)d(-r) = \{r, -r\} = 0$ by (5.10.3). Since the product in L is anticommutative, the general case $r = u\pi^i$ follows from this. □

III.7.3.1 **Corollary 7.3.1** (Rigidity). *Suppose that F is complete with respect to the valuation v, with residue field $k = k_v$. For every integer q prime to* char(k), *the maps $\lambda \oplus \partial_v\colon K_n^M(F)/q \to K_n^M(k)/q \oplus K_{n-1}^M(k)/q$ are isomorphisms for all n.*

Proof. Since the valuation ring R is complete, Hensel's Lemma implies that the group $1 + \pi R$ is q-divisible. It follows that $l(1 + \pi R) \cdot K_{n-1}^M(F)$ is also q-divisible. But by Ex. 7.2 this is the kernel of the map $d\colon K_n^M(F) \to L_n \cong K_n^M(k_v) \oplus K_{n-1}^M(k_v)$. □

III.7.3.2 **Example 7.3.2** (Leading Coefficients)**.** As in Example 6.1.2, $K_n^M(F)$ is a direct summand of $K_n^M F(t)$. To see this, we consider the valuation $v_\infty(f) = -deg(f)$ on $F(t)$ of Example 6.5.3. Since t^{-1} is a parameter, each polynomial $f = ut^{-i}$ has $\text{lead}(f) = \bar{u}$. The map $\lambda\colon K_n^M F(t) \to K_n^M(F)$, given by $\lambda\{f_1, \dots, f_n\} = \{\text{lead}(f_1), \dots, \text{lead}(f_n)\}$, is clearly inverse to the natural map $K_n^M(F) \to K_n^M F(t)$.

Except for v_∞, every discrete valuation v on $F(t)$ which is trivial on F is the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ associated to a prime ideal $\mathfrak{p}$ of $F[t]$. In this case k_v is the field $F[t]/\mathfrak{p}$, and we write $\partial_\mathfrak{p}$ for ∂_v.

III.7.4 **Theorem 7.4.** *(Milnor) There is a split exact sequence for each n, natural in the field F, and split by the map λ:*

$$0 \to K_n^M(F) \to K_n^M F(t) \xrightarrow{\partial = \coprod \partial_\mathfrak{p}} \coprod_\mathfrak{p} K_{n-1}^M(F[t]/\mathfrak{p}) \to 0.$$

Proof. Let L_d denote the subgroup of $K_n^M F(t)$ generated by those symbols $\{f_1, \dots, f_r\}$ such that all the polynomials f_i have degree $\leq d$. By Example 7.3.2, L_0 is a summand isomorphic to $K_n^M(F)$. Since $K_n^M F(t)$ is the union of the subgroups L_d, the theorem will follow from Lemma 7.4.2 below, using induction on d. □

Let π be an irreducible polynomial of degree d and set $k = k_\pi = F[t]/(\pi)$. Then each element $\bar{a}$ of k is represented by a unique polynomial $a \in F[t]$ of degree $< d$.

III.7.4.1 **Lemma 7.4.1.** *There is a unique homomorphism $h = h_\pi\colon K_{n-1}^M(k) \to L_d/L_{d-1}$ carrying $\{\bar{a}_2, \dots, \bar{a}_n\}$ to the class of $\{\pi, a_2, \dots, a_n\}$ modulo L_{d-1}.*

Proof. The formula gives a well-defined set map h from $k^\times \times \cdots \times k^\times$ to L_d/L_{d-1}. To see that it is linear in $\bar{a}_2$, suppose that $\bar{a}_2 = \bar{a}_2' \bar{a}_2''$. If $a_2 \neq a_2' a_2''$ then there is a nonzero polynomial f of degree $< d$ with $a_2 = a_2' a_2'' + f\pi$. Since $f\pi/a_2 = 1 - a_2' a_2''/a_2$ we have $\{f\pi/a_2, a_2' a_2''/a_2\} = 0$. Multiplying by $\{a_3, \dots, a_n\}$ gives

$$\{\pi, a_2' a_2''/a_2, a_3, \dots, a_n\} \equiv 0 \qquad \text{modulo } L_{d-1}.$$

Similarly, h is linear in $a_3, \dots, a_n$. To see that the multilinear map h factors through $K_{n-1}^M(k)$, we observe that if $\bar{a}_i + \bar{a}_{i+1} = 1$ in k then $a_i + a_{i+1} = 1$ in F. □

III.7.4.2 **Lemma 7.4.2.** *The homomorphisms $\partial_{(\pi)}$ and h_π induce an isomorphism between L_d/L_{d-1} and the direct sum $\oplus_\pi K_{n-1}^M(k_\pi)$ as π ranges over all monic irreducible polynomials of degree d in $F[t]$.*

Proof. Since π cannot divide any polynomial of degree $< d$, the maps $\partial_{(\pi)}$ vanish on L_{d-1} and induce maps $\bar{\partial}_{(\pi)}: L_d/L_{d-1} \to K^M_{n-1}(k_\pi)$. By inspection, the composition of $\oplus h_\pi$ with the direct sum of the $\bar{\partial}_{(\pi)}$ is the identity on $\oplus_\pi K^M_{n-1}(k_\pi)$. Thus it suffices to show that $\oplus h_\pi$ maps onto L_d/L_{d-1}. By Ex. 6.2, L_d is generated by L_{d-1} and symbols $\{\pi, a_2, \dots, a_n\}$ where π has degree d and the a_i have degree $< d$. But each such symbol is h_π of an element of $K^M_{n-1}(k_\pi)$, so $\oplus h_\pi$ is onto. □

The Transfer Map

Let v_∞ be the valuation on $F(t)$ with parameter t^{-1}. The formulas in Theorem 7.3 defining ∂_∞ show that it vanishes on $K^M_*(F)$. By Theorem 7.4, there are unique homomorphisms $N_\mathfrak{p}: K^M_n(F[t]/\mathfrak{p}) \to K^M_n(F)$ so that $-\partial_\infty = \sum_\mathfrak{p} N_\mathfrak{p}\partial_\mathfrak{p}$.

III.7.5 **Definition 7.5.** Let E be a finite field extension of F generated by an element a. Then the *transfer map*, or *norm map* $N = N_{a/F}: K^M_*(E) \to K^M_*(F)$, is the unique map $N_\mathfrak{p}$ defined above, associated to the kernel $\mathfrak{p}$ of the map $F[t] \to E$ sending t to a.

We can calculate the norm of an element $x \in K^M_n(E)$ as $N_\mathfrak{p}(x) = -\partial_{v_\infty}(y)$, where $y \in K^M_{n+1}F(t)$ is such that $\partial_\mathfrak{p}(y) = x$ and $\partial_{\mathfrak{p}'}(y) = 0$ for all $\mathfrak{p}' \neq \mathfrak{p}$.

If $n = 0$, the transfer map $N: \mathbb{Z} \to \mathbb{Z}$ is multiplication by the degree $[E : F]$ of the field extension, while if $n = 1$ the map $N: E^\times \to F^\times$ is the usual norm map; see Ex. 7.5. We will show in 7.6 below that N is independent of the choice of $a \in E$ for all n. First we make two elementary observations.

If we let N_∞ denote the identity map on $K^M_n(F)$, and sum over the set of all discrete valuations on $F(t)$ which are trivial on F, the definition of the N_v yields the:

III.7.5.1 **Weil Reciprocity Formula 7.5.1.** $\sum_v N_v\partial_v(x) = 0$ for all $x \in K^M_n F(t)$.

III.7.5.2 **Projection Formula 7.5.2.** Let $E = F(a)$. Then for $x \in K^M_*(F)$ and $y \in K^M_*(E)$ the map $N = N_{a/F}$ satisfies $N\{x, y\} = \{x, N(y)\}$.

Proof. The inclusions of F in $F(t)$ and $F[t]/\mathfrak{p}$ allow us to view $K^M_*F(t)$ and $K^M_*(F[t]/\mathfrak{p})$ as graded modules over the ring $K^M_*(F)$. It follows from Theorem 7.4 that each $\partial_\mathfrak{p}$ is a graded module homomorphism of degree -1. This remark also applies to v_∞ and ∂_∞, because $F(t) = F(t^{-1})$. Therefore each $N_\mathfrak{p}$ is a graded module homomorphism of degree 0. □

Taking $y = 1$ in $K^M_0(E) = \mathbb{Z}$, so $N(y) = [E : F]$ by Ex. 7.5, this yields

III.7.5.3 **Corollary 7.5.3.** *If the extension E/F has degree d, then the composition $K^M_*(F) \to K^M_*(E) \xrightarrow{N} K^M_*(F)$ is multiplication by d. In particular, the kernel of $K^M_*(F) \to K^M_*(E)$ is annihilated by d.*

III.7.6 **Definition 7.6.** Let $E = F(a_1, \ldots, a_r)$ be a finite field extension of F. The *transfer map* $N_{E/F}\colon K_*^M(E) \to K_*^M(F)$ is defined to be the composition of the transfer maps defined in 7.5:

$$K_n^M(E) \xrightarrow{N_{a_r}} K_n^M(F(a_1, \ldots, a_{r-1})) \xrightarrow{N_{a_{r-1}}} \cdots\ K_n^M(F(a_1)) \xrightarrow{N_{a_1}} K_n^M(F).$$

The transfer map is well-defined by the following result of K. Kato.

III.7.6.1 **Theorem 7.6.1.** *(Kato) The transfer map $N_{E/F}$ is independent of the choice of elements $a_1, \ldots, a_r$ such that $E = F(a_1, \ldots, a_r)$. In particular, if $F \subset F' \subset E$ then $N_{E/F} = N_{F'/F} N_{E/F'}$.*

The key trick used in the proof of this theorem is to fix a prime p and pass from F to a union F' of finite extensions of F of degree prime to p such that the degree of every finite extension of F' is a power of p. By Corollary 7.5.3 the kernel of $K_n^M(F) \to K_n^M(F')$ has no p-torsion.

III.7.6.2 **Lemma 7.6.2.** *(Kato) If E is a normal extension of F, and $[E : F]$ is a prime number p, then the map $N_{E/F} = N_{a/F}\colon K_*^M(E) \to K_*^M(F)$ does not depend upon the choice of a such that $E = F(a)$.*

Proof. If also $E = F(b)$, then from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ we see that $\delta(x) = N_{a/F}(x) - N_{b/F}(x)$ is annihilated by p. If $\delta(x) \neq 0$ for some $x \in K_n^M(E)$ then, again by Corollary 7.5.3, $\delta(x)$ must be nonzero in $K_n^M(F')$, where F' is a maximl prime-to-p extension, a union of finite extensions of F of degree prime to p. Again by Ex. 7.7, we see that we may replace F by F' and x by its image in $K_n^M(EF')$. Since the degree of every finite extension of F' is a power of p, the assertion for F' follows from Ex. 7.6, since the Projection Formula 7.5.2 yields $N_{a/F'}\{y, x_2, \ldots, x_n\} = \{N(y), x_2, \ldots, x_n\}$. □

III.7.6.3 **Corollary 7.6.3.** *If in addition F is a complete discrete valuation field with residue field k_v, and the residue field of E is k_w, the following diagram commutes.*

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{\partial_w} & K_{n-1}^M(k_w) \\ \downarrow N & & \downarrow N \\ K_n^M(F) & \xrightarrow{\partial_v} & K_{n-1}^M(k_v) \end{array}$$

Proof. Ex. 7.6 implies that for each $u \in K_n^M(E)$ there is a finite field extension F' of F such that $[F' : F]$ is prime to p and the image of u in $K_n(EF')$ is generated by elements of the form $u' = \{y, x_2, \ldots, x_n\}$ ($y \in EF'$, $x_i \in F'$). By Ex. 7.7 and Ex. 7.8 it suffices to prove that $N_{k_w/k_v}\partial_w(u) = \partial_v(N_{EF'/F'}u)$ for every element u of this form. But this is an easy computation. □

III.7.6.4 **Proposition 7.6.4.** *(Kato) Let E and $F' = F(a)$ be extensions of F with E/F normal of prime degree p. If $E' = E(a)$ denotes the composite field, the following diagram commutes.*

$$\begin{array}{ccc} K_*^M(E') & \xrightarrow{N_{a/E}} & K_*^M(E) \\ \downarrow N & & \downarrow N \\ K_*^M(F') & \xrightarrow{N_{a/F}} & K_*^M(F). \end{array}$$

Proof. The vertical norm maps are well-defined by Lemma 7.6.2. Let $\pi \in F[t]$ and $\pi' \in E[t]$ be the minimal polynomials of a over F and E, respectively. Given $x \in K_n^M(E')$, we have $N_{a/E}(x) = -\partial_\infty(y)$, where $y \in K_{n+1}^M E(t)$ satisfies $\partial_{\pi'}(y) = x$ and $\partial_w(y) = 0$ if $w \neq w_{\pi'}$. If v is a valuation on $F(t)$, Ex. 7.9 gives:

$$\partial_v(N_{E(t)/F(t)}y) = \sum_{w|v} N_{E(w)/F(v)}(\partial_w y) = \begin{cases} N_{E'/F'}(x) & \text{if } v = v_\pi \\ N_{E/F}(\partial_\infty y) & \text{if } v = v_\infty \\ 0 & \text{else} \end{cases}$$

in $K_*^M(F')$. Two applications of Definition 7.5 give the desired calculation:

$$N_{a/F}(N_{E'/F'}x) = -\partial_\infty(N_{E(t)/F(t)}y) = -N_{E/F}(\partial_\infty y) = N_{E/F}(N_{a/F}x). \quad \square$$

Proof of Theorem 7.6.1. As in the proof of Lemma 7.6.2, we see from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ that the indeterminacy is annihilated by $[E : F]$. Using the key trick of passing to a larger field, we may assume that the degree of every finite extension of F is a power of a fixed prime p.

Let us call a tower of intermediate fields $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ *maximal* if $[F_i : F_{i-1}] = p$ for all i. By Lemma 7.6.2, the transfer maps $N\colon K_*^M(F_i) \to K_*^M(F_{i-1})$ are independent of the choice of a such that $F_i = F_{i-1}(a)$. If $F \subset F_1 \subset E$ and $F \subset F' \subset E$ are maximal towers, Proposition 7.6.4 states that $N_{F'/F}N_{E/F'} = N_{F_1/F}N_{E/F_1}$, because if $F' \neq F_1$ then $E = F'F_1$. It follows by induction on $[E : F]$ that if $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ is a maximal tower then the composition of the norm maps

$$K_n^M(E) \xrightarrow{N} K_n^M(F_{r-1}) \xrightarrow{N} \cdots K_n^M(F_1) \xrightarrow{N} K_n^M(F)$$

is independent of the choice of maximal tower.

Comparing any tower to a maximal tower, we see that it suffices to prove that if $F \subset F_1 \subset F'$ is a maximal tower and $F' = F(a)$ then $N_{a/F} = N_{F_1/F}N_{F'/F_1}$. But this is just Proposition 7.6.4 with $E = F_1$ and $E' = F'$. $\square$

The dlog *symbol and* $\nu(n)_F$

For any field F, we write Ω_F^n for the n^{th} exterior power of the vector space $\Omega_F^1 = \Omega_{F/\mathbb{Z}}^1$ of Kähler differentials (Ex. 2.6, 6.10). The direct sum over n forms a graded-commutative ring Ω_F^*, and the map dlog : $F^\times \to \Omega_F^1$ sending a to $\frac{da}{a}$ extends to a graded ring map from the tensor algebra $T(F^\times)$ to Ω_F^*. By Ex. 6.10, $l(a) \otimes\ l(1-a)$ maps to zero, so it factors through the quotient ring $K_*^M(F)$ of $T(F^\times)$. We record this observation for later reference.

III.7.7 **Lemma 7.7.** *If F is any field, there is a graded ring homomorphism*

$$\mathrm{dlog}: K_*^M(F) \to \Omega_F^*, \qquad \mathrm{dlog}\{a_1, \dots, a_n\} = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

Now let F be a field of characteristic $p \neq 0$, so that $d(a^p) = p\,da = 0$. In fact, if $\{x_i\}$ is a p-basis of F over F^p then the symbols dx_i form a basis of the F-vector space Ω_F^1. Note that the set $d\Omega_F^{n-1}$ of all symbols $da_1 \wedge \cdots \wedge da_n$ forms an F^p-vector subspace of Ω_F^n.

III.7.7.1 **Definition 7.7.1.** If $\mathrm{char}(F) = p \neq 0$, let $\nu(n)_F$ denote the kernel of the Artin-Schrier operator $\wp : \Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$, which is defined by

$$\wp\left(x \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}\right) = (x^p - x)\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

(In the literature, $\wp + 1$ is the inverse of the "Cartier" operator.)

Clearly $\wp(\mathrm{dlog}\{a_1, \dots, a_n\}) = 0$, so the image of the dlog map lies in $\nu(n)_F$. The following theorem, which implies that these symbols span $\nu(n)_F$, was proven by Kato [101] for $p = 2$; for general p it was proven by Bloch, Kato and Gabber [26, 2.1].

III.7.7.2 **Theorem 7.7.2.** *(Bloch-Kato-Gabber) Let F be a field of characteristic $p \neq 0$. Then the* dlog *map induces an isomorphism for every $n \geq 0$:*

$$K_n^M(F)/pK_n^M(F) \cong \nu(n)_F.$$

Using this result, Bloch and Kato also proved that the p-torsion subgroup of $K_n^M(F)$ is divisible [26, 2.8]. Using this divisibility, Izhboldin found the following generalization of Theorem 6.7; see [95].

III.7.8 **Izhboldin's Theorem 7.8.** *If $\mathrm{char}(F) = p$, the groups $K_n^M(F)$ have no p-torsion.*

Proof. We proceed by induction on n, the case $n = 2$ being Theorem 6.7. As in the proof of Theorem 6.7, let x be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism σ satisfying $\sigma(x) = x + 1$. By Theorem 7.4, we can regard $K_n^M(F)$ as a subgroup of both $K_n^M F(x)$ and $K_n^M F(y)$.

For all field extensions E of $F(y)$ linearly disjoint from $F(x)$, i.e., with no root of $t^p - t - y$, write $E(x)$ for the field $E \otimes_{F(y)} F(x)$. Let $I(E)$ denote the set of all p-torsion elements in $K_n^M E(x)$ of the form $v - \sigma(v)$, $v \in K_n^M E(x)$, and let $P(E)$ denote the p-torsion subgroup of the kernel of the norm map $N_{x/E} : K_n^M E(x) \to K_n^M(E)$. Since $N\sigma(v) = N(v)$, $I(E) \subseteq P(E)$. Both $I(E)$ and $P(E)$ vary naturally with E, and are equal by Proposition 7.8.2 below.

Fix $u \in K_n^M(F)$ with $pu = 0$. The projection formula 7.5.2 shows that the norm map $K_n^M E(x) \to K_n^M F(y)$ sends u to $pu = 0$. Hence $u \in P(F(y))$. By Proposition 7.8.2, $u \in I(F(y))$, i.e., there is a $v \in K_n^M F(x)$ so that $u = v - \sigma(v)$ in $K_n^M F(x)$. Now apply the leading coefficient symbol λ of 7.3.2; since $\lambda(\sigma v) = \lambda(v)$ we have: $u = \lambda(u) = \lambda(v) - \lambda(\sigma v) = 0$. This proves Izhboldin's theorem. $\square$

Before proceeding to Proposition 7.8.2, we need some facts about the group $I(E)$ defined in the proof of 7.8. We first claim that the transcendental extension $E \subset E(t)$ induces an isomorphism $I(E) \cong I(E(t))$. Indeed, since $E(x,t)$ is purely transcendental over $E(x)$, Theorem 7.4 and induction on n imply that $K_n^M E(x) \to K_n^M E(x,t)$ is an isomorphism on p-torsion subgroups, and the claim follows because the leading coefficient symbol 7.3.2 commutes with σ.

III.7.8.1 **Lemma 7.8.1.** *The group $I(E)$ is p-divisible.*

Proof. Pick $v \in K_n^M E(x)$ so that $u = v - \sigma(v)$ is in $I(E)$. Now we invoke the Bloch-Kato result, mentioned above, that the p-torsion subgroup of $K_n^M(L)$ is divisible for every field L of characteristic p. By Theorem 7.7.2, this implies that u vanishes in $K_n^M E(x)/p \cong \nu(n)_{E(x)}$. By Ex. 7.12 and Theorem 7.7.2, the class of v mod p comes from an element $w \in K_n^M(E)$, i.e., $v - w = pv'$ for some $v' \in K_n^M E(x)$. Then $u = v - \sigma(v) = pv' - p\sigma(v')$, it follows that $u' = v' - \sigma(v')$ is an element of $I(E)$ with $u = pu'$. $\square$

We next claim that if E/E' is a purely inseparable field extension then $I(E') \to I(E)$ is onto. For this we may assume that $E^p \subseteq E' \subset E$. The composition of the Frobenius map $E \to E^p$ with this inclusion induces the endomorphism of $K_n^M(E)$ sending $\{a_1, \dots, a_n\}$ to $\{a_1^p, \dots, a_n^p\} = p^n\{a_1, \dots, a_n\}$. Hence this claim follows from Lemma 7.8.1.

III.7.8.2 **Proposition 7.8.2.** *For all E containing $F(y)$, linearly disjoint from $F(x)$, $P(E) = I(E)$.*

Proof. We shall show that the obstruction $V(E) = P(E)/I(E)$ vanishes. This group has exponent p, because if $u \in P(E)$ then

$$\begin{aligned} pu &= pu - N_{EL/E}u = (p - 1 - \sigma - \cdots - \sigma^{p-1})u \\ &= \big((1-\sigma) + (1-\sigma^2) + \cdots + (1-\sigma^{p-1})\big)u \end{aligned}$$

is in $(1-\sigma)K_n^M E(x)$ and hence in $I(E)$. It follows that $V(E)$ injects into $I(E')$ whenever E'/E is an extension of degree prime to p.

Now we use the "Severi-Brauer" trick; this trick will be used again in chapter V, 1.6, in connection with Severi-Brauer varieties. For each $b \in E$ we let E_b denote the field $E(t_1, \dots, t_{p-1}, \beta)$ with $t_1, \dots, t_{p-1}$ purely transcendental over E and $\beta^p - \beta - y + \sum b^i t_i^p = 0$. It is known that b is in the image of the norm map $E_b(x)^\times \to E_b^\times$; see [96]. Since $E \cdot (E_b)^p$ is purely transcendental over E (on $\beta, t_2^p, ..., t_{p-1}^p$), it follows that $I(E) \to I(E_b)$ is onto. Since $E_b(x)$ is purely transcendental over $E(x)$ (why?), $I(E(x)) = I(E_b(x))$ and $K_n^M E(x)$ embeds in $K_n^M E_b(x)$ by Theorem 7.4. Hence $K_n^M(E(x))/I(E)$ embeds in $K_n^M E_b(x)/I(E_b)$. Since $V(E) \subset K_n^M E(x)/I(E)$ by definition, we see that $V(E)$ also embeds into $V(E_b)$.

Now if we take the composite of all the fields E_b, $b \in E$, and then form its maximal algebraic extension E' of degree prime to p, it follows that $V(E)$ embeds into $V(E')$. Repeating this construction a countable number of times yields an extension field E'' of E such that $V(E)$ embeds into $V(E'')$ and every element of E'' is a norm from $E''(x)$. Hence it suffices to prove that $V(E'') = 0$. The proof in this special case is completely parallel to the proof of Proposition III.6.6.2, and we leave the details to Ex. 7.13. □

This completes the proof of Izhboldin's Theorem 7.8.

III.7.8.3 **Corollary 7.8.3** (Hilbert's Theorem 90 for K_*^M)**.** *Let $j : F \subset L$ be a degree p field extension, with* char$(F) = p$*, and let σ be a generator of $G = Gal(L/F)$. Then $K_n^M(F) \cong K_n^M(L)^G$, and the following sequence is exact for all $n > 0$:*

$$0 \to K_n^M(F) \xrightarrow{j^*} K_n^M(L) \xrightarrow{1-\sigma} K_n^M(L) \xrightarrow{N} K_n^M(F).$$

Proof. Since $K_n^M(F)$ has no p-torsion, Corollary 7.5.3 implies that j^* is an injection. To prove exactness at the next spot, suppose that $v \in K_n^M(L)$ has $\sigma(v) = v$. By Ex. 7.12 and Theorem 7.7.2, the class of $v \bmod p$ comes from an element $w \in K_n^M(F)$, i.e., $v - j^*(w) = pv'$ for some $v' \in K_n^M(L)$. Hence $p\sigma(v') = \sigma(pv') = pv'$. Since $K_n^M(L)$ has no p-torsion, $\sigma(v') = v'$. But then pv' equals $j^*N(v') = \sum \sigma^i(v')$, and hence $v = j^*(w) + j^*(Nv')$. In particular, this proves that $K_n^M(F) \cong K_n^M(L)^G$.

To prove exactness at the final spot, note that G acts on $K_n^M(L)$, and that $\ker(N)/\mathrm{im}(1-\sigma)$ is isomorphic to the cohomology group $H^1(G, K_n^M(L))$; see [223, 6.2.2]. Now consider the exact sequence of $Gal(L/F)$-modules

$$0 \to K_n^M(L) \xrightarrow{p} K_n^M(L) \xrightarrow{(7.7.2)} \nu(n)_L \to 0.$$

Using Ex. 7.12, the long exact sequence for group cohomology begins

$$0 \to K_n^M(F) \xrightarrow{p} K_n^M(F) \to \nu(n)_F \to H^1(G, K_n^M(L)) \xrightarrow{p} H^1(G, K_n^M(L)).$$

But $K_n^M(F)$ maps onto $\nu(n)_F$ by Theorem 7.7.2, and the group $H^1(G, A)$ has exponent p for all G-modules A [223, 6.5.8]. It follows that $H^1(G, K_n^M(L)) = 0$, so $\ker(N) = \mathrm{im}(1-\sigma)$, as desired. □

III.7.8.4 **Remark 7.8.4.** Hilbert's Theorem 90 for K_n^M, which extends Theorem 6.6, states that for any Galois extension $F \subset E$ of degree p, with σ generating Gal(E/F), the following sequence is exact:

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N_{E/F}} K_n^M(F).$$

This is a consequence of the Norm Residue Theorem (Chapter VI, 4.1) and is due to Voevodsky; we refer the reader to [82, 3.2] for a proof.

Relation to the Witt ring

Let F be a field of characteristic $\neq 2$. Recall from 5.6 of chapter II that the *Witt ring* $W(F)$ is the quotient of the Grothendieck group $K_0\mathbf{SBil}(F)$ of symmetric inner product spaces over F by the subgroup $\{nH\}$ generated by the hyperbolic form $\langle 1\rangle \oplus \langle -1\rangle$. The dimension of the underlying vector space induces an augmentation $K_0\mathbf{SBil}(F) \to \mathbb{Z}$, sending $\{nH\}$ isomorphically onto $2\mathbb{Z}$, so it induces an augmentation $\varepsilon\colon W(F) \to \mathbb{Z}/2$.

We shall be interested in the augmentation ideals $I = \ker(\varepsilon)$ of $W(F)$ and $\hat{I}$ of $K_0\mathbf{SBil}(F)$. Since $H \cap \hat{I} = 0$, we have $\hat{I} \cong I$. Now I is generated by the classes $\langle a\rangle - 1$, $a \in F - \{0, 1\}$. The powers I^n of I form a decreasing chain of ideals $W(F) \supset I \supset I^2 \supset \cdots$.

For convenience, we shall write $K_n^M(F)/2$ for $K_n^M(F)/2K_n^M(F)$.

III.7.9 **Theorem 7.9.** *(Milnor) There is a unique surjective homomorphism*

$$s_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$$

sending each product $\{a_1, \dots, a_n\}$ in $K_n^M(F)$ to the product $\prod_{i=1}^n \big(\langle a_i\rangle - 1\big)$ modulo I^{n+1}. The homomorphisms s_1 and s_2 are isomorphisms.

Proof. Because $\big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1$ modulo I^2 (II.5.6.5), the map $l(a_1) \times \cdots \times l(a_n) \mapsto \prod\big(\langle a_i\rangle - 1\big)$ is a multilinear map from $F^\times$ to I^n/I^{n+1}. Moreover, if $a_i + a_{i+1} = 1$ for any i, we know from Ex. II.5.12 that $\big(\langle a_i\rangle - 1\big)\big(\langle a_{i+1}\rangle - 1\big) = 0$. By the presentation of $K_*^M(F)$, this gives rise to a group homomorphism from $K_n^M(F)$ to I^n/I^{n+1}. It annihilates $2K_*^M(F)$ because $\langle a^2\rangle = 1$:

$$2s_n\{a_1, \dots, a_n\} = s_n\{a_1^2, a_2 \dots\} = \big(\langle a_1^2\rangle - 1\big)\prod_{i=2}^n \big(\langle a_i\rangle - 1\big) = 0.$$

It is surjective because I is generated by the $\big(\langle a\rangle - 1\big)$. When $n = 1$ the map is Pfister's isomorphism $F^\times/F^{\times 2} \cong I/I^2$ of II.5.6.4. We will see that s_2 is an isomorphism in Corollary 7.10.3 below, using the Hasse invariant w_2. □

III.7.9.1 **Example 7.9.1.** For the real numbers $\mathbb{R}$, we have $W(\mathbb{R}) = \mathbb{Z}$ and $I = 2\mathbb{Z}$ on $s_1(-1) = \langle -1\rangle - 1 = 2\langle -1\rangle$. On the other hand, we saw in Example 7.2(c) that $K_n^M(\mathbb{R})/2 \cong \mathbb{Z}/2$ on $\{-1, \dots, -1\}$. In this case each s_n is the isomorphism $\mathbb{Z}/2 \cong 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$.

At the other extreme, if F is algebraically closed then $W(F) = \mathbb{Z}/2$. Since $K_n^M(F)$ is divisible, $K_n^M(F)/2 = 0$ for all $n \geq 1$. Here s_n is the isomorphism $0 = 0$.

III.7.9.2 **Remark 7.9.2.** In 1970, Milnor asked if the surjection $s_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$ is an isomorphism for all n and F, $\mathrm{char}(F) \neq 2$ (on p. 332 of [130]). Milnor proved this was so for local and global fields. This result was proven for all fields and all n by Orlov, Vishik and Voevodsky in [148].

III.7.10 **Stiefel-Whitney invariant 7.10.** The total *Stiefel–Whitney invariant* $w(M)$ of the symmetric inner product space $M = \langle a_1\rangle \oplus \cdots \oplus \langle a_n\rangle$ is the element of $\prod_{i=0}^\infty K_i^M(F)/2$ defined by the formula

$$w(M) = \prod_{i=1}^n \big(1 + l(a_i)\big) = 1 + l(a_1 \cdots a_n) + \cdots + \{a_1, \dots, a_n\}$$

The lemma below shows that $w(M)$ is independent of the representation of M as a direct sum of 1-dimensional forms. We write $w(M) = 1 + w_1(M) + w_2(M) + \cdots$, where the i^{th} *Stiefel–Whitney invariant* $w_i(M) \in K_i^M(F)/2$ equals the i^{th} elementary symmetric function of $l(a_1), \dots, l(a_n)$.

For example, $w_1(M) = a_1 \cdots a_n \in F^\times/F^{\times 2}$ is just the classical "discriminant" of M defined in II.5.6.3, while the second elementary symmetric function $w_2(M) = \sum_{i<j}\{a_i, a_j\}$ lies in $K_2(F)/2$ and is called the *Hasse invariant* of M; see [133].

For $M = \langle a\rangle \oplus \langle b\rangle$ we have $w_1(M) = ab$ and $w_2(M) = \{a, b\}$, with $w_i(M) = 0$ for $i \geq 3$. In particular, the hyperbolic plane H has $w_i(H) = 0$ for all $i \geq 2$.

III.7.10.1 **Lemma 7.10.1.** *$w(M)$ is a well-defined unit in the ring $\prod_{i=1}^\infty K_i^M(F)/2$. It satisfies the Whitney sum formula*

$$w(M \oplus N) = w(M)w(N),$$

so w extends to a function on $K_0\mathbf{SBil}(F)$. Hence each Stiefel–Whitney invariant w_i extends to a function $K_0\mathbf{SBil}(F) \xrightarrow{w_i} K_i^M(F)/2$.

Proof. To show that $w(M)$ is well defined, it suffices to consider the rank two case. Suppose that $\langle a\rangle \oplus \langle b\rangle \cong \langle \alpha\rangle \oplus \langle \beta\rangle$. Then the equation $ax^2 + by^2 = \alpha$ must have a solution x, y in F. The case $y = 0$ (or $x = 0$) is straightforward, since $\langle \alpha\rangle = \langle ax^2\rangle = \langle a\rangle$, so we may assume that x and y are nonzero. Since the discriminant w_1 is an invariant, we have $ab = \alpha\beta u^2$ for some $u \in F$, and all we must show is that $\{a, b\} = \{\alpha, \beta\}$ in $K_2(F)/2$. The equation $1 = ax^2/\alpha + by^2/\alpha$ yields the equation

$$0 = \{ax^2/\alpha, by^2/\alpha\} \equiv \{a, b\} + \{\alpha, \alpha\} - \{a, \alpha\} - \{b, \alpha\} \equiv \{a, b\} - \{\alpha, ab/\alpha\}$$

in $K_2(F)/2K_2(F)$. Substituting $ab = \alpha\beta u^2$, this implies that $\{a, b\} \equiv \{\alpha, \beta\}$ modulo $2K_2(F)$, as desired. □

III.7.10.2 **Example 7.10.2.** Since $I \cong \hat{I}$, we may consider the w_i as functions on $I \subseteq W(F)$. However, care must be taken as $w_2(M)$ need not equal $w_2(M \oplus H)$. For example, $w_2(M \oplus H) = w_2(M) + \{w_1(M), -1\}$. In particular, $w_2(H \oplus H) = \{-1, -1\}$ can be nontrivial. The *Hasse–Witt invariant* of an element $x \in I \subset W(F)$ is defined to be $h(x) = w_2(V, B)$, where (V, B) is an inner product space representing x so that $\dim(V) \equiv 0 \bmod 8$.

III.7.10.3 **Corollary 7.10.3.** *The Hasse invariant $w_2\colon \hat{I} \to K_2(F)/2$ induces an isomorphism from $\hat{I}^2/\hat{I}^3 \cong I^2/I^3$ to $K_2^M(F)/2$, inverse to the map s_2 of Theorem 7.9.*

Proof. By Ex. 7.11, w_2 vanishes on the ideal $\hat{I}^3 \cong I^3$, and hence defines a function from $\hat{I}^2/\hat{I}^3$ to $K_2(F)/2$. Since the total Stiefel–Whitney invariant of $s_2\{a, b\} = (\langle a\rangle - 1)(\langle b\rangle - 1)$ is $1 + \{a, b\}$, this function provides an inverse to the function s_2 of Theorem 7.9. □

If $\mathrm{char}(F) = 2$, there is an elegant formula for the filtration quotients of the Witt ring $W(F)$ and the $W(F)$-module $WQ(F)$ (see §II.5) due to K. Kato [101]. Recall from 7.7.2 that $K_n^M(F)/2 \cong \nu(n)_F$, where $\nu(n)_F$ is the kernel of the operator $\wp$. The case $n = 0$ of Kato's result was described in Ex. II.5.13(d).

III.7.10.4 **Theorem 7.10.4.** *(Kato [101]) Let F be a field of characteristic 2. Then the map s_n of Theorem 7.9 induces an isomorphism $K_n^M(F)/2 \cong \nu(n)_F \cong I^n/I^{n+1}$, and there is a short exact sequence*

$$0 \to I^n/I^{n+1} \to \Omega_F^n \xrightarrow{\wp} \Omega_F^n/d\Omega_F^{n-1} \to I^n\ WQ(F)/I^{n+1}\ WQ(F) \to 0.$$

The Norm Residue symbol

For the next result, we need some facts about Galois cohomology, expanding slightly upon the facts mentioned in 6.10. Assuming that n is prime to $\mathrm{char}(F)$, there are natural cohomology cup products $H^i_{\mathrm{et}}(F;M) \otimes H^j_{\mathrm{et}}(F;N) \xrightarrow{\cup} H^{i+j}_{\mathrm{et}}(F;M\otimes N)$ which are associative in M and N. This makes the direct sum $H^*_{\mathrm{et}}(F;M^{\otimes *}) = \oplus_{i=0}^{\infty} H^i_{\mathrm{et}}(F;M^{\otimes i})$ into a graded-commutative ring for every $\mathbb{Z}/n$-module M over the Galois group $Gal(F_{\mathrm{sep}}/F)$. (By convention, $M^{\otimes 0}$ is $\mathbb{Z}/n$.) In particular, both $H^*_{\mathrm{et}}(F;\mathbb{Z}/n)$ and $H^*_{\mathrm{et}}(F;\mu_n^{\otimes *})$ are rings, and are isomorphic only when F contains a primitive n^{th} root of unity.

III.7.11 **Theorem 7.11** (Norm Residue Symbols)**.** *(Bass-Tate) Fix a field F and an integer n prime to* $\mathrm{char}(F)$*.*

(1) If F contains a primitive n^{th} root of unity, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H^1_{et}(F;\mathbb{Z}/n)$ extends uniquely to a graded ring homomorphism

$$h_F\colon K_*^M(F)/n \to H^*(F;\mathbb{Z}/n).$$

(2) More generally, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H^1(F;\mu_n)$ extends uniquely to a graded ring homomorphism

$$h_F\colon K_*^M(F)/n \to H^*_{et}(F;\mu_n^{\otimes *}) = \oplus_{i=0}^{\infty} H^i_{et}(F;\mu_n^{\otimes i}).$$

The individual maps $K_i^M(F) \to H^i_{et}(F;\mu_n^{\otimes i})$ are called the norm residue symbols, and also higher Galois symbols.

Proof. The first assertion is just a special case of the second assertion. As in (6.10.2), the Kummer isomorphism induces a map from the tensor algebra $T(F^\times)$ to $H^*_{\mathrm{et}}(F;\mu_n^{\otimes *})$, which in degree i is the iterated cup product

$$F^\times \otimes \cdots F^\times = (F^\times)^{\otimes n} \cong \left(H^1_{\mathrm{et}}(F;\mu_n)\right)^{\otimes i} \xrightarrow{\cup} H^i_{\mathrm{et}}(F;\mu_n^{\otimes i}).$$

By Proposition 6.10.3, the Steinberg identity is satisfied in $H^2_{\mathrm{et}}(F,\mu_n^{\otimes 2})$. Hence the presentation of $K_*^M(F)$ yields a ring homomorphism from $K_*^M(F)$ to $H^*_{\mathrm{et}}(F;\mu_n^{\otimes *})$. □

III.7.11.1 **Remark 7.11.1.** In his seminal paper [130], Milnor studied the norm residue symbol for $K_n^M(F)/2$ and stated (on p.340) that, "I do not know any examples for which the homomorphism h_F fails to be bijective." Voevodsky proved that h_F is an isomorphism for $n = 2^\nu$ in his 2003 paper [211]. The proof that the h_F is an isomorphism for all n prime to $\mathrm{char}(F)$ was proven a few years later; see VI.4.1.1.

EXERCISES

EIII.7.1 **7.1.** Let v be a discrete valuation on a field F. Show that the maps $\lambda\colon K_n^M(F) \to K_n^M(k_v)$ and $\partial_v\colon K_n^M(F) \to K_{n-1}^M(k_v)$ of Theorem 7.3 are independent of the choice of parameter π, and that they vanish on $l(u)\cdot K_{n-1}^M(F)$ whenever $u \in (1+\pi R)$. Show that the map λ also vanishes on $l(\pi)\cdot K_{n-1}^M(F)$.

EIII.7.2 **7.2.** Continuing Exercise 7.1, show that the kernel of the map $d\colon K_n^M(F) \to L_n$ of Theorem 7.3 is exactly $l(1+\pi R)\cdot K_{n-1}^M(F)$. Conclude that the kernel of the map λ is exactly $l(1+\pi R)\cdot K_{n-1}^M(F) + l(\pi)\cdot K_{n-1}^M(F)$.

EIII.7.3 **7.3.** (Bass-Tate) Generalize Theorem 6.4 to show that for all $n \geq 2$:

(a) If F is an algebraically closed field, then $K_n^M(F)$ is uniquely divisible.

(b) If F is a perfect field of characteristic p then $K_n^M(F)$ is uniquely p-divisible.

EIII.7.4 **7.4.** Let F be a local field with valuation v and finite residue field k. Show that $K_n^M(F)$ is divisible for all $n \geq 3$. *Hint:* By Moore's Theorem 6.2.4, $K_n^M(F)$ is ℓ-divisible unless F has a ℓ^{th} root of unity. Moreover, for every $x \notin F^{\times\ell}$ there is a $y \notin F^{\times\ell}$ so that $\{x,y\}$ generates $K_2(F)/\ell$. Given a,b,c with $\{b,c\} \notin \ell K_2(F)$, find $a',b' \notin F^{\times\ell}$ so that $\{b',c\} \equiv 0$ and $\{a',b'\} \equiv \{a,b\}$ modulo $\ell K_2(F)$, and observe that $\{a,b,c\} \equiv \{a',b',c\} \equiv 0$.

In fact, I. Sivitskii has shown that $K_n^M(F)$ is uniquely divisible for $n \geq 3$ when F is a local field. See [169]. We will give a proof of this in VI.7.1.

EIII.7.5 **7.5.** Let $E = F(a)$ be a finite extension of F, and consider the transfer map $N = N_{a/F}\colon K_n^M(E) \to K_n^M(F)$ in Definition 7.5. Use Weil's Formula (7.5.1) to show that when $n = 0$ the transfer map $N\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by $[E:F]$, and that when $n = 1$ the transfer map $N\colon E^\times \to F^\times$ is the usual norm map.

EIII.7.6 **7.6.** Suppose that the degree of every finite extension of a field F is a power of some fixed prime p. If E is an extension of degree p and $n > 0$, use Ex. 6.2 to show that $K_n^M(E)$ is generated by elements of the form $\{y, x_2, \dots, x_n\}$, where $y \in E^\times$ and the x_i are in $F^\times$.

EIII.7.7 **7.7.** *Ramification and the transfer.* Let F' and $E = F(a)$ be finite field extensions of F, and suppose that the irreducible polynomial $\pi \in F[t]$ of a has a decomposition $\pi = \prod \pi_i^{e_i}$ in $F'[t]$. Let E_i denote $F'(a_i)$, where each a_i has minimal polynomial π_i. Show that the following diagram commutes.

$$\begin{array}{ccc}
K_n^M(E) & \xrightarrow{e_1,\dots,e_r} & \oplus K_n^M(E_i) \\
{\scriptstyle N_a/F}\downarrow & & \downarrow{\scriptstyle \sum N_{a_i/F'}} \\
K_n^M(F) & \longrightarrow & K_n^M(F')
\end{array}$$

EIII.7.8 **7.8.** *Ramification and ∂_v.* Suppose that E is a finite extension of F, and that w is a valuation on E over the valuation v on F, with ramification index e. (See 6.3.1.) Use the formulas for ∂_v and ∂_w in Theorem 7.3 to show that for every $x \in K_n^M(F)$ we have $\partial_w(x) = e\cdot\partial_v(x)$ in $K_{n-1}^M(k_w)$.

EIII.7.9 **7.9.** If E/F is a normal extension of prime degree p, and v is a valuation on $F(t)$ trivial on F, show that $\partial_v N_{E(t)/F(t)} = \sum_w N_{E(w)/F(v)} \partial_w$, where the sum is over all the valuations w of $E(t)$ over v. *Hint:* If $F(t)_v$ and $E(t)_w$ denote the completions of $F(t)$ and $E(t)$ at v and w, respectively, use Ex. 7.7 and Lemma 7.6.3 to show that the following diagram commutes.

$$\begin{array}{ccccc} K^M_{n+1}E(t) & \longrightarrow & \bigoplus_w K^M_{n+1}E(t)_w & \xrightarrow{\partial} & \bigoplus_w K^M_n E(w) \\ \downarrow N_{E(t)/F(t)} & & \downarrow \sum_w N_{E(t)_w/F(t)_v} & & \downarrow \sum_w N_{e(w)/F(v)} \\ K^M_n F(t) & \longrightarrow & K^M_n F(t)_w & \xrightarrow{\partial} & K^M_n F(v) \end{array}$$

EIII.7.10 **7.10.** If v is a valuation on F, and $x \in K^M_i(F)$, $y \in K^M_j(F)$, show that

$$\partial_v(xy) = \lambda(x)\partial_v(y) + (-1)^j \partial_v(x)\rho(y)$$

where $\rho\colon K^M_*(F) \to K^M_*(k_v)$ is a ring homomorphism characterized by the formula $\rho(l(u\pi^i)) = l((-1)^i \bar{u})$.

EIII.7.11 **7.11.** Let $t = 2^{n-1}$ and set $z = \prod_{i=1}^n (\langle a_i \rangle - 1)$; this is a generator of the ideal $\hat{I}^n$ in $K_0\mathbf{SBil}(F)$. Writing s for $\{a_1, \dots, a_n, -1, -1, \dots, -1\} \in K^M_t(F)/2$, show that the Stiefel–Whitney invariant $w(z)$ is equal to: $1 + s$ if n is odd, and to $(1+s)^{-1}$ if n is even. This shows that the invariants w_i vanish on the ideal $\hat{I}^n$ if $i < t = 2^{n-1}$, and that w_t induces a homomorphism from $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$ to $K^M_t(F)/2$. For example, this implies that w_1 vanishes on $\hat{I}^2$, while w_2 and w_3 vanish on $\hat{I}^3$.

EIII.7.12 **7.12.** (Izhboldin) Let L/F be a field extension of degree $p = \mathrm{char}(F)$, with Galois group G. Show that Ω^n_F is isomorphic to $(\Omega^n_L)^G$, and that $\Omega^n_F/d\Omega^{n-1}_F$ is isomorphic to $(\Omega^n_L/d\Omega^{n-1}_L)^G$. Conclude that $\nu(n)_F \cong \nu(n)^G_L$.

EIII.7.13 **7.13.** In this exercise we complete the proof of Proposition 7.8.2, and establish a special case of 7.8.3. Suppose that $E(x)$ is a degree p field extension of E, $\mathrm{char}(E) = p$, and that σ is a generator of $Gal(E(x)/E)$. Suppose in addition that the norm map $E(x)^\times \to E^\times$ is onto, and that E has no extensions of degree $< p$. Modify the proof of proposition 6.6.2 to show that the following sequence is exact:

$$K^M_n E(x) \xrightarrow{1-\sigma} K^M_n E(x) \xrightarrow{N} K^M_n E \to 0.$$

EIII.7.14 **7.14.** Suppose that F is a field of infinite transcendence degree κ over its ground field. Show that the image of the dlog symbol of 7.7 lies in the kernel of $\Omega^n_F \xrightarrow{d} \Omega^{n+1}_F$. Using Ex. 6.11, show that $K^M_n(F)$ has cardinality κ for all $n > 0$.

If F is a local field, this and Ex. 7.4 implies that $K^M_n(F)$ is an uncountable, uniquely divisible abelian group.

Chapter IV

Definitions of higher K-theory

The higher algebraic K-groups of a ring R are defined to be the homotopy groups $K_n(R) = \pi_n K(R)$ of a certain topological space $K(R)$, which we shall construct in this chapter. Of course, the space $K(R)$ is rigged so that if $n = 0, 1, 2$ then $\pi_n K(R)$ agrees with the groups $K_n(R)$ constructed in chapters II and III.

We shall also define the higher K-theory of a category $\mathcal{A}$ in each of the three settings where $K_0(\mathcal{A})$ was defined in chapter II: when $\mathcal{A}$ is a symmetric monoidal category (§4), an exact category (§6) and a Waldhausen category (§8). In each case we build a "K-theory space" $K\mathcal{A}$ and define the group $K_n\mathcal{A}$ to be its homotopy groups: $K_n\mathcal{A} = \pi_n K\mathcal{A}$. Of course the group $\pi_0 K\mathcal{A}$ will agree with the corresponding group $K_0\mathcal{A}$ defined in chapter II.

We will show these definitions of $K_n\mathcal{A}$ coincide whenever they coincide for K_0. For example, the group $K_0(R)$ of a ring R was defined in §II.2 as K_0 of the category $\mathbf{P}(R)$ of finitely generated projective R-modules, but to define $K_0\mathbf{P}(R)$ we could also regard the category $\mathbf{P}(R)$ as being either a symmetric monoidal category (II.5.2), an exact category (II.7.1) or a Waldhausen category (II.9.1.3). We will show that the various constructions give homotopy equivalent spaces $K\mathbf{P}(R)$, and hence the same homotopy groups. Thus the groups $K_n(R) = \pi_n K\mathbf{P}(R)$ will be independent of the construction used.

Many readers will not be interested in the topological details, so we have designed this chapter to allow "surfing." Since the most non-technical way to construct $K(R)$ is to use the "+"–construction, we will do this in §1 below. The second (short) section defines K-theory with finite coefficients, as the homotopy groups of $K(R)$ with finite coefficients. These have proved to be remarkably useful in describing the structure of the groups $K_n(R)$, especially as related to étale cohomology. This is illustrated by the results in chapter VI.

In §3, we summarize the basic facts about the geometric realization BC of a category C, and the basic connection between category theory and homotopy theory needed for the rest of the constructions. Indeed, the K-theory space

$K\mathcal{A}$ is constructed in each setting using the geometric realization BC of some category C, concocted out of $\mathcal{A}$. For this, we assume only that the reader has a slight familiarity with cell complexes, or *CW complexes*, which are spaces obtained by successive attachment of cells, with the weak topology.

Sections 4–9 give the construction of the K-theory spaces. Thus in §4 we have group completion constructions for a symmetric monoidal category S, such as the $S^{-1}S$ construction, and the connection with the +–construction. It is used in §5 to construct λ-operations on $K(R)$. Quillen's Q–construction for abelian and exact categories is given in §6; in §7 we prove the "$+ = Q$" theorem, that the Q–construction and group completion constructions agree for split exact categories (II.7.1.2). The $wS_\bullet$ construction for Waldhausen categories is in §8, along with its connection to the Q–construction. In §9 we give an alternative construction for exact categories, due to Gillet and Grayson.

Section 10 gives a construction of the non-connective spectrum for algebraic K-theory of a ring, whose negative homotopy groups are the negative K-groups of Bass developed in Section III.4. Sections 11 and 12 are devoted to Karoubi-Villamayor K-theory and the homotopy-invariant version KH of K-theory. We will return to this topic in chapter V.

1 The BGL^+ definition for rings

Let R be an associative ring with unit. Recall from chapter III that the *infinite general linear group* $GL(R)$ is the union of the groups $GL_n(R)$, and that its commutator subgroup is the perfect group $E(R)$ generated by the elementary matrices $e_{ij}(r)$. Moreover the group $K_1(R)$ is defined to be the quotient $GL(R)/E(R)$.

In 1969, Quillen proposed defining the higher K-theory of a ring R to be the homotopy groups of a certain topological space, which he called $BGL(R)^+$. Before describing the elementary properties of Quillen's construction, and the related subject of acyclic maps, we present Quillen's description of $BGL(R)^+$ and define the groups $K_n(R)$ for $n \geq 1$.

For any group G, we can naturally construct a connected topological space BG whose fundamental group is G, but whose higher homotopy groups are zero. Details of this construction are in §3 below (see 3.1.3). Moreover, the homology of the topological space BG (with coefficients in a G-module M) coincides with the algebraic homology of the group G (with coefficients in M); the homology of a space X with coefficients in a $\pi_1(X)$-module is defined in [228, VI.1–4]. For $G = GL(R)$ we obtain the space $BGL(R)$, which is central to the following definition.

IV.1.1 **Definition 1.1.** The notation $BGL(R)^+$ will denote any CW complex X which has a distinguished map $BGL(R) \to BGL(R)^+$ such that

(1) $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$;

(2) $H_*(BGL(R); M) \xrightarrow{\cong} H_*(BGL(R)^+; M)$ for every $K_1(R)$-module M.

We will sometimes say that X is a *model* for $BGL(R)^+$.

For $n \geq 1$, $K_n(R)$ is defined to be the homotopy group $\pi_n BGL(R)^+$.

By Theorem 1.5 below, any two models are homotopy equivalent, *i.e.,* the space $BGL(R)^+$ is uniquely defined up to homotopy. Hence the homotopy groups $K_n(R)$ of $BGL(R)^+$ are well-defined up to a canonical isomorphism.

By construction, $K_1(R)$ agrees with the group $K_1(R) = GL(R)/E(R)$ defined in Chapter III. We will see in 1.7.1 below that $K_2(R) = \pi_2 BGL^+(R)$ agrees with the group $K_2(R)$ defined in Chapter III.

Several distinct models for $BGL(R)^+$ are described in 1.9 below. We will construct even more models for $BGL(R)^+$ in the rest of this chapter: the space $\mathbf{P}^{-1}\mathbf{P}(R)$ of §3, the space $\Omega BQ\mathbf{P}(R)$ of §5 and the space $\Omega(\mathrm{iso}\, S_\bullet S)$ arising from the Waldhausen construction in §8.

IV.1.1.1 **Definition 1.1.1.** Write $K(R)$ for the product $K_0(R) \times BGL(R)^+$. That is, $K(R)$ is the disjoint union of copies of the connected space $BGL(R)^+$, one for each element of $K_0(R)$. By construction, $K_0(R) = \pi_0 K(R)$. Moreover, it is clear that $\pi_n K(R) = \pi_n BGL(R)^+ = K_n(R)$ for $n \geq 1$.

IV.1.1.2 **Functoriality 1.1.2.** Each K_n is a functor from rings to abelian groups, while the topological spaces $BGL(R)^+$ and $K(R)$ are functors from rings to the homotopy category of topological spaces. However, without more information about the models used, the topological maps $BGL(R)^+ \to BGL(R')^+$ are only well-defined up to homotopy.

To see this, note that any ring map $R \to R'$ induces a natural group map $GL(R) \to GL(R')$, and hence a natural map $BGL(R) \to BGL(R')$. This induces a map $BGL(R)^+ \to BGL(R')^+$, unique up to homotopy, by Theorem 1.5 below. Thus the group maps $K_n(R) \to K_n(R')$ are well defined. Since the identity of R induces the identity on $BGL(R)^+$, only composition remains to be considered. Given a second map $R' \to R''$, the composition $BGL(R) \to BGL(R') \to BGL(R'')$ is induced by $R \to R''$ because BGL is natural. By uniqueness in Theorem 1.5, the composition $BGL(R)^+ \to BGL(R')^+ \to BGL(R'')^+$ must be homotopy equivalent to any a priori map $BGL(R)^+ \to BGL(R'')^+$.

It is possible to modify the components of $K(R) = K_0(R) \times BGL(R)^+$ up to homotopy equivalence in order to form a homotopy-commutative H-space in a functorial way, using other constructions (see 4.11.1). Because the map $K_1(R/I) \to K_0(R, I)$ is nontrivial (see III.2.3), $K(R)$ is *not* the product of the H-space $BGL(R)^+$ and the discrete group $K_0(R)$ in a natural way.

IV.1.1.3 **Transfer maps 1.1.3.** If $R \to S$ is a ring map such that $S \cong R^d$ as an R-module, the isomorphisms $S^m \cong R^{md}$ induce a group map $GL(S) \to GL(R)$ and hence a map $BGL(S)^+ \to BGL(R)^+$, again unique up to homotopy. On homotopy groups, the maps $K_n(S) \to K_n(R)$ are called *transfer maps*. We will see another construction of these maps in 6.3.2 below.

We shall be interested in the homotopy fiber of the map $BGL(R) \to BGL(R)^+$.

IV.1.2 **Homotopy Fiber 1.2.** The maps $\pi_*E \to \pi_*B$ induced by a continuous map $E \xrightarrow{f} B$ can always be made to fit into a long exact sequence, in a natural way. The *homotopy fiber* $F(f)$ of a f, relative to a basepoint $*_B$ of B, is the space of pairs (e, γ), where $e \in E$ and $\gamma : [0,1] \to B$ is a path in B starting at the basepoint $\gamma(0) = *_B$, and ending at $\gamma(1) = f(e)$. A sequence of based spaces $F \to E \xrightarrow{f} B$ with $F \to B$ constant is called a *homotopy fibration sequence* if the evident map $F \to F(f)$ (using $\gamma(t) = *_B$) is a homotopy equivalence.

The key property of the homotopy fiber is that (given a basepoint $*_E$ with $f(*_E) = *_B$) there is a long exact sequence of homotopy groups/pointed sets

$$\cdots \pi_{n+1}B \xrightarrow{\partial} \pi_n F(f) \to \pi_n E \to \pi_n B \xrightarrow{\partial} \pi_{n-1}F(f) \to \cdots$$
$$\cdots \xrightarrow{\partial} \pi_1 F(f) \to \pi_1 E \to \pi_1 B \xrightarrow{\partial} \pi_0 F(f) \to \pi_0 E \to \pi_0 B.$$

When $E \to B$ is an H-map of H-spaces, $F(f)$ is also an H-space, and the maps ending the sequence are product-preserving.

Acyclic Spaces and Acyclic Maps

The definition of $BGL(R)^+$ fits into the general framework of acyclic maps, which we now discuss. Our discussion of acyclicity is taken from [87] and [23].

IV.1.3 **Definition 1.3.** (Acyclic spaces) We call a topological space E *acyclic* if it has the homology of a point, that is, if $\widetilde{H}_*(E; \mathbb{Z}) = 0$.

IV.1.3.1 **Lemma 1.3.1.** *Let E be an acyclic space. Then E is connected, its fundamental group $G = \pi_1(E)$ is a perfect group, and $H_2(G; \mathbb{Z}) = 0$.*

Proof. The acyclic space E must be connected, as $H_0(E) = \mathbb{Z}$. Because $G/[G,G] = H_1(E;\mathbb{Z}) = 0$, we have $G = [G,G]$, *i.e.,* G is a perfect group. To calculate $H_2(G)$, observe that the universal covering space $\tilde{E}$ has $H_1(\tilde{E};\mathbb{Z}) = 0$. Moreover, the homotopy fiber (1.2) of the canonical map $E \to BG$ is homotopy equivalent to $\tilde{E}$; to see this, consider the long exact sequence of homotopy groups 1.2. The Serre Spectral Sequence for this homotopy fibration is $E^2_{pq} = H_p(G; H_q(\tilde{E};\mathbb{Z})) \Rightarrow H_{p+q}(E;\mathbb{Z})$ and the conclusion that $H_2(G;\mathbb{Z}) = 0$ follows from the associated exact sequence of low degree terms:

$$H_2(E;\mathbb{Z}) \to H_2(G;\mathbb{Z}) \xrightarrow{d^2} H_1(\tilde{E};\mathbb{Z})^G \to H_1(E;\mathbb{Z}) \to H_1(G;\mathbb{Z}). \qquad \square$$

IV.1.3.2 **Example 1.3.2.** (Volodin Spaces) The Volodin space $X(R)$ is an acyclic subspace of $BGL(R)$, constructed as follows. For each n, let $T_n(R)$ denote the subgroup of $GL_n(R)$ consisting of upper triangular matrices with 1's on the diagonal. As n varies, the union of these groups forms a subgroup $T(R)$ of $GL(R)$. Similarly we may regard the permutation groups Σ_n as subgroups of

$GL_n(R)$ by their representation as permutation matrices, and their union (the infinite permutation group Σ_∞) is a subgroup of $GL(R)$. For each $\sigma \in \Sigma_n$, let $T_n^\sigma(R)$ denote the subgroup of $GL_n(R)$ obtained by conjugating $T_n(R)$ by σ. For example, if $\sigma = (n \dots 1)$ then $T_n^\sigma(R)$ is the subgroup of lower triangular matrices.

Since the classifying spaces $BT_n(R)$ and their conjugates $BT_n(R)^\sigma$ are subspaces of $BGL_n(R)$, and hence of $BGL(R)$, we may form their union over all n and σ: $X(R) = \bigcup_{n,\sigma} BT_n(R)^\sigma$. The space $X(R)$ is acyclic (see Su81 [180]). Since $X(R)$ was first described by Volodin in 1971, it is usually called the *Volodin space* of R.

The image of the map $\pi_1 X(R) \to GL(R) = \pi_1 BGL(R)$ is the group $E(R)$. To see this, note that $\pi_1(X)$ is generated by the images of the $\pi_1 BT_n(R)^\sigma$, the image of the composition $\pi_1 BT_n^\sigma(R) \to \pi_1(X) \to \pi_1 BGL(R) = GL(R)$ is the subgroup $T_n^\sigma(R)$ of $E(R)$, and every generator $e_{ij}(r)$ of $E(R)$ is contained in some $T_n^\sigma(R)$.

IV.1.4 **Definition 1.4.** (Acyclic maps) Let X and Y be based connected CW complexes. A map $f\colon X \to Y$ is called *acyclic* if the homotopy fiber $F(f)$ of f is acyclic (has the homology of a point). This implies that $F(f)$ is connected and $\pi_1 F(f)$ is a perfect group.

From the exact sequence $\pi_1 F(f) \to \pi_1(X) \to \pi_1(Y) \to \pi_0 F(f)$ of homotopy groups/pointed sets, we see that if $X \to Y$ is acyclic then the map $\pi_1(X) \to \pi_1(Y)$ is onto, and its kernel P is a perfect normal subgroup of $\pi_1(X)$.

IV.1.4.1 **Definition 1.4.1.** Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based connected CW complex. An acyclic map $f\colon X \to Y$ is called a $+$–*construction* on X (relative to P) if P is the kernel of $\pi_1(X) \to \pi_1(Y)$.

IV.1.4.2 **Example 1.4.2.** If X is acyclic, the map $X \to$ point is acyclic. By Ex. EIV.1.2 1.2, it is a $+$–construction.

When Quillen introduced the notion of acyclic maps in 1969, he observed that both Y and the map f are determined up to homotopy by the subgroup P. This is the content of the following theorem; its proof uses topological obstruction theory. Part (1) is proven in Ex. EIV.1.4 1.4; an explicit proof may be found in §5 of Berrick [23].

IV.1.5 **Theorem 1.5.** *(Quillen) Let P be a perfect normal subgroup of $\pi_1(X)$. Then*

(1) There is a $+$–construction $f\colon X \to Y$ relative to P

(2) Let $f\colon X \to Y$ be a $+$–construction relative to P, and $g\colon X \to Z$ a map such that P vanishes in $\pi_1(Z)$. Then there is a map $h\colon Y \to Z$, unique up to homotopy, such that $g = hf$.

(3) In particular, if g is another $+$–construction relative to P, then the map h in (2) is a homotopy equivalence: $h\colon Y \xrightarrow{\sim} Z$.

IV.1.5.1 **Remark 1.5.1.** Every group G has a unique largest perfect subgroup P, called the *perfect radical* of G, and it is a normal subgroup of G; see Ex. 1.5. If no mention is made to the contrary, the notation X^+ will always denote the +–construction relative to the perfect radical of $\pi_1(X)$.

The first construction along these lines was announced by Quillen in 1969, so we have adopted Quillen's term "+–construction" as well as his notation. A good description of his approach may be found in [87] or [23].

IV.1.6 **Lemma 1.6.** *Let X and Y be connected CW complexes. A map $f\colon X \to Y$ is acyclic if and only if $H_*(X,M) \cong H_*(Y,M)$ for every $\pi_1(Y)$-module M.*

Proof. Suppose first that f is acyclic, with homotopy fiber $F(f)$. Since the map $\pi_1 F(f) \to \pi_1 Y$ is trivial, $\pi_1 F(f)$ acts trivially upon M. By the Universal Coefficient Theorem, $H_q(F(f);M) = 0$ for $q \neq 0$ and $H_0(F(f);M) = M$. Therefore $E^2_{pq} = 0$ for $q \neq 0$ in the Serre Spectral Sequence for f:

$$E^2_{pq} = H_p(Y; H_q(F(f);M)) \Rightarrow H_{p+q}(X;M).$$

The spectral sequence collapses to yield $H_p(X;M) \overset{\cong}{\to} H_p(Y;M)$ for all p.

Conversely, we suppose first that $\pi_1 Y = 0$ and $H_*(X;\mathbb{Z}) \cong H_*(Y;\mathbb{Z})$. By the Comparison Theorem for the Serre Spectral Sequences for $F(f) \to X \xrightarrow{f} Y$ and $* \to Y \xrightarrow{=} Y$, we have $\tilde{H}_*(F(f);\mathbb{Z}) = 0$. Hence $F(f)$ and f are acyclic.

The general case reduces to this by the following trick. Let $\tilde{Y}$ denote the universal covering space of Y, and $\tilde{X} = X \times_Y \tilde{Y}$ the corresponding covering space of X. Then there are natural isomorphisms $H_*(\tilde{Y};\mathbb{Z}) \cong H_*(Y;M)$ and $H_*(\tilde{X};\mathbb{Z}) \cong H_*(X;M)$, where $M = \mathbb{Z}[\pi_1(Y)]$. The assumption that $H_*(X;M) \cong H_*(Y;M)$ implies that the map $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ induces isomorphisms on integral homology. But $\pi_1(\tilde{Y}) = 0$ so, by the special case above, the homotopy fiber $F(\tilde{f})$ of $\tilde{f}$ is an acyclic space. But by path lifting we have $F(\tilde{f}) \cong F(f)$, so $F(f)$ is acyclic. Thus f is an acyclic map. ☐

Recall from III.5.4 that every perfect group P has a universal central extension $E \to P$, and that the kernel of this extension is the abelian group $H_2(P;\mathbb{Z})$.

IV.1.7 **Proposition 1.7.** *Let P be a perfect normal subgroup of a group G, with corresponding +–construction $f\colon BG \to BG^+$. If $F(f)$ is the homotopy fiber of f then $\pi_1 F(f)$ is the universal central extension of P, and $\pi_2(BG^+) \cong H_2(P;\mathbb{Z})$.*

Proof. We have an exact sequence

$$\pi_2(BG) \to \pi_2(BG^+) \to \pi_1 F(f) \to G \to G/P \to 1.$$

But $\pi_2(BG) = 0$, and $\pi_2(BG^+)$ is in the center of $\pi_1 F(f)$ by [228, IV.3.5]. Thus $\pi_1 F(f)$ is a central extension of P with kernel $\pi_2(BG^+)$. But $F(f)$ is acyclic, so $\pi_1 F(f)$ is perfect and $H_2(F;\mathbb{Z}) = 0$ by 1.3.1. By the Recognition Theorem III.5.4, $\pi_1 F(f)$ is the universal central extension of P. ☐

Recall from Theorem III.5.5 that the Steinberg group $St(R)$ is the universal central extension of the perfect group $E(R)$. Thus we have:

IV.1.7.1 **Corollary 1.7.1.** *The group $K_2(R) = \pi_2 BGL(R)^+$ is isomorphic to the group $K_2(R) \cong H_2(E(R);\mathbb{Z})$ of Chapter III.*

In fact, we will see in Ex. 1.8 and 1.9 that $K_n(R) \cong \pi_n(BE(R)^+)$ for all $n \geq 2$, and $K_n(R) \cong \pi_n(BSt(R)^+)$ for all $n \geq 3$, with $K_3(R) \cong H_3(St(R);\mathbb{Z})$.

IV.1.7.2 **Corollary 1.7.2.** *The fundamental group $\pi_1 X(R)$ of the Volodin space (1.3.2) is the Steinberg group $St(R)$.*

Construction Techniques

One problem with the +construction approach is the fact that $BGL(R)^+$ is not a uniquely defined space. It is not hard to see that $BGL(R)^+$ is an H-space (see Ex. 1.11). Quillen proved that that it is also an infinite loop space, and extends to an Ω-spectrum $\mathbf{K}(R)$. We omit the proof here, because it will follow from the $+ = Q$ theorem in Section 7.

Here is one of the most useful recognition criteria, due to Quillen. The proof is is an application of obstruction theory, which we omit (but see [64, 1.5].)

IV.1.8 **Theorem 1.8.** *The map $i : BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces. That is, for each map $f : BGL(R) \to H$, where H is an H-space, there is a map $g : BGL(R)^+ \to H$ so that $f = g\, i$, and such that the induced map $\pi_i(BGL(R)^+) \to \pi_i(H)$ is independent of g.*

IV.1.8.1 **Remark 1.8.1.** If $f_* : H_*(BGL(R),\mathbb{Z}) \xrightarrow{\cong} H_*(H,\mathbb{Z})$ is an isomorphism, then f is acyclic and g is a homotopy equivalence: $BGL(R)^+ \simeq H$. This gives another characterization of $BGL(R)^+$. The proof is indicated in Exercise 1.3.

IV.1.9 **Constructions 1.9.** Here are some ways that $BGL(R)^+$ may be constructed:

(i) Using point-set topology, *e.g.*, by attaching 2-cells and 3-cells to $BGL(R)$. If we perform this construction over $\mathbb{Z}$ and let $BGL(R)^+$ be the pushout of $BGL(\mathbb{Z})^+$ and $BGL(R)$ along $BGL(\mathbb{Z})$, this gives a construction which is functorial in R. This method is described Ex. 1.4, and in the books [23] and [163].

(ii) Using the Bousfield-Kan integral completion functor $\mathbb{Z}_\infty$: we set $BGL(R)^+ = \mathbb{Z}_\infty BGL(R)$. This approach, which is also functorial in R, is used in [50] and [64].

(iii) "Group completing" the H-space $\coprod_{n=0}^\infty BGL_n(R)$ yields an infinite loop space whose basepoint component is $BGL(R)^+$. This method will be discussed more in Section 3, and is due to G. Segal [165].

(iv) By taking BGL of a free simplicial ring $F_\bullet$ with an augmentation $F_0 \to R$ such that $F_\bullet \to R$ is a homotopy equivalence, as in [193]. Swan showed that the simplicial group $GL(F_\bullet)$ and the simplicial space $BGL(F_\bullet)$ are independent (up to simplicial homotopy) of the choice of resolution $F_\bullet \to R$, and that $\pi_1 BGL(F_\bullet) = \pi_0 GL(F_\bullet) = E(R)$. The *Swan K-theory space* $\Omega K^{Sw}(R)$ is defined to be the homotopy fiber of $BGL(F_\bullet) \to BGL(R)$, and we set $K_i^{Sw}(R) = \pi_{i-1}\Omega K^{Sw}(R)$ for $i \geq 1$ so that $K_1^{Sw}(R) = K_1(R)$ by construction. The space $\Omega K^{Sw}(R)$ is a model for the loop space $\Omega BGL(R)^+$.

As an application, if F is a free ring (without unit), we may take $F_\bullet$ to be the constant simplicial ring, so $\Omega K^{Sw}(F)$ is contractible, and $K_i^{Sw}(F) = 0$ for all i. Gersten proved in [65] (see V.6.5) that $BGL(F)^+$ is contractible; this was used by Don Anderson [2] to prove that the canonical map from $GL(R) = \Omega BGL(R)$ to $\Omega K^{Sw}(R)$ induces a homotopy equivalence $\Omega BGL(R)^+ \to \Omega K^{Sw}(R)$.

(v) *Volodin's construction.* Let $X(R)$ denote the acyclic Volodin space of Example 1.3.2. By Ex. 1.6, the quotient group $BGL(R)/X(R)$ is a model for $BGL(R)^+$.

An excellent survey of these constructions may be found in [64], except for details on Volodin's construction, which are in [180].

Products

If A and B are rings, any natural isomorphism $\varphi_{pq} : A^p \otimes B^q \cong (A \otimes B)^{pq}$ of $A \otimes B$-modules allows us to define a "tensor product" homomorphism $GL_p(A) \times GL_q(B) \to GL_{pq}(A \otimes B)$. This in turn induces continuous maps $\varphi_{p,q} : BGL_p(A)^+ \times BGL_q(B)^+ \to BGL_{pq}(A \otimes B)^+ \to BGL(A \otimes B)^+$. A different choice of φ yields a tensor product homomorphism conjugate to the original, and a new map $\varphi_{p,q}$ freely homotopic to the original. It follows that $\varphi_{p,q}$ is compatible up to homotopy with stabilization in p and q.

Since the target is an H-space (Ex. 1.11), we can define new maps $\gamma_{p,q}(a,b) = \varphi_{p,q}(a,b) - \varphi_{p,q}(a,*) - \varphi_{p,q}(*,b)$, where $*$ denotes the basepoint. Since $\gamma_{p,q}(a,*) = \gamma_{p,q}(*,b) = *$, and $\gamma_{p,q}$ is compatible with stabilization in p, q, it induces a map, well defined up to weak homotopy equivalence

$$\gamma : BGL(A)^+ \wedge BGL(B)^+ \to BGL(A \otimes B)^+.$$

Combining γ with the reduced join $\pi_p(X) \otimes \pi_q(Y) \to \pi_{p+q}(X \wedge Y)$ [228, p. 480] allows us to define a product map :

$$K_p(A) \otimes K_q(B) \to K_{p+q}(A \otimes B).$$

Loday proved the following result in [114].

IV.1.10 **Theorem 1.10.** *(Loday) The product map is natural in A and B, bilinear and associative. If A is commutative, the induced product*

$$K_p(A) \otimes K_q(A) \to K_{p+q}(A \otimes A) \to K_{p+q}(A)$$

is graded-commutative. Moreover, the special case $K_1(A) \otimes K_1(B) \to K_2(A \otimes B)$ coincides with the product defined in III.5.12.

IV.1.10.1 **Example 1.10.1.** *(Steinberg symbols)* If $r_1, ..., r_n$ are units of a commutative ring R, the product of the $r_i \in K_1(R)$ is an element $\{r_1, ..., r_n\}$ of $K_n(R)$. These elements are called Steinberg symbols, since the products $\{r_1, r_2\} \in K_2(R)$ agree with the Steinberg symbols of III.5.10. If F is a field, the universal property (III.7.1) of Milnor K-theory implies that there is a ring homomorphism $K_*^M(F) \to K_*(F)$. We will see in Ex. 1.12 that it need not be an injection.

IV.1.10.2 **Example 1.10.2.** Associated to the unit x of $\Lambda = \mathbb{Z}[x, x^{-1}]$ we choose a map $S^1 \to BGL(\Lambda)^+$, representing $[x] \in \pi_1 BGL(\Lambda)^+$. The pairing γ induces a map $BGL(R)^+ \wedge S^1$ to $BGL(R[x, x^{-1}])^+$. By adjunction, this yields a map $BGL(R)^+ \to \Omega BGL(R[x, x^{-1}])^+$. A spectrum version of this map is given in Ex. 4.14.

IV.1.10.3 **Example 1.10.3.** *(The K-theory Assembly Map)* If G is any group, the inclusion $G \subset \mathbb{Z}[G]^\times = GL(\mathbb{Z}[G])$ induces a map $BG \to BGL(\mathbb{Z}[G])^+$. If R is any ring, the product map $BGL(R)^+ \wedge BGL(\mathbb{Z}[G])^+ \to BGL(R[G])^+$ induces a map from $BGL(R)^+ \wedge (BG_+)$ to $BGL(R[G])^+$, where BG_+ denotes the disjoint union of BG and a basepoint. By Ex. 1.14, there is also a map from $K(R) \wedge (BG_+)$ to $K(R[G])$.

Now for any infinite loop space (or spectrum) $\mathbf{E}$, and any pointed space X, the homotopy groups of the space $\mathbf{E} \wedge X$ give the generalized homology of X with coefficients in $\mathbf{E}$, $H_n(BG; \mathbf{E})$. For $\mathbf{E} = K(R)$, $H_n(BG; \mathbf{K}(R))$ is the generalized homology of BG with coefficients in $K(R)$.

The map $H_n(BG; \mathbf{K}(R)) = \pi_n\big(BGL(R)^+ \wedge BG_+\big) \to K_n(R[G])$ which we have just constructed is called the K-theory *assembly map*, and it plays a critical role in the K-theory of group rings. It is due to Quinn and Loday [114], who observed that for $n = 0$ it is just the map $K_0(R) \to K_0(R[G])$, while for $n = 1$ it is the map $K_1(R) \oplus G/[G, G] \to K_1(R[G])$.

The *higher Whitehead Group* $Wh_n(G)$ is defined to be π_{n-1} of the homotopy fiber of the map $K(\mathbb{Z}) \wedge (BG_+) \to K(\mathbb{Z}[G])$. The above calculations show that $Wh_0(G)$ is Wall's finiteness obstruction (II.2.4), and the classical Whitehead group $Wh_1(G) = K_1(\mathbb{Z}[G])/\{\pm G\}$ of III.1.9.

If G is a torsionfree group, the *Isomorphism Conjecture* for G states that the assembly map $H_n(BG; \mathbf{K}(R)) \to K_n(R[G])$ should be an isomorphism for any regular ring R. There is a more general Isomorphism Conjecture for infinite groups with torsion, due to Farrell-Jones [54]; it replaces $H_n(BG; \mathbf{K}(R))$ by the equivariant homology of $E_{vc}G$, an equivariant version of the universal covering space EG of BG relative to the class of virtually cyclic subgroups of G.

Relative K-groups

IV.1.11 **Relative K-groups 1.11.** Given a ring homomorphism $f : R \to R'$, let $K(f)$ be the homotopy fiber of $K(R) \to K(R')$, and set $K_n(f) = \pi_n K(f)$. This construction is designed so that these relative groups fit into a long exact sequence:

$$\cdots K_{n+1}(R') \xrightarrow{\partial} K_n(f) \to K_n(R) \to K_n(R') \xrightarrow{\partial} \cdots$$
$$K_1(f) \to K_1(R) \to K_1(R') \xrightarrow{\partial} K_0(f) \to K_0(R) \to K_0(R').$$

Using the functorial homotopy-commutative H-space structure on $K(R)$ (see 1.1.2), it follows that each $K_n(f)$, including $K_0(f)$, is an abelian group.

When $R' = R/I$ for some ideal I, we write $K(R, I)$ for $K(R \to R/I)$. It is easy to see (Ex. 1.15) that $K_0(R, I)$ and $K_1(R, I)$ agree with the relative groups defined in Ex. II.2.3 and III.2.2, and that the ending of this sequence is the exact sequence of III.2.3 and III.5.7.1. Keune and Loday have shown that $K_2(R,I)$ agrees with the relative group defined in III.5.7.

IV.1.11.1 **Absolute Excision 1.11.1.** A non-unital ring I is said to *satisfy absolute excision* for K_n if $K_n(\mathbb{Z}\oplus I, I) \xrightarrow{\cong} K_n(R,I)$ is an isomorphism for every unital ring R containing I as an ideal; $\mathbb{Z}\oplus I$ is the canonical augmented ring (see Ex. I.1.10). By II, Ex. 2.3, every I satisfies absolute excision for K_0. By III, Remark 2.2.1, I satisfies absolute excision for K_1 if and only if $I = I^2$.

Suslin proved in [188] that I satisfies absolute excision for K_n if and only if the groups $\mathrm{Tor}_i^{\mathbb{Z}\oplus I}(\mathbb{Z},\mathbb{Z})$ vanish for $i = 1,\dots,n$. (Since $\mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}) = I/I^2$, this recovers the result for K_1.) In homological algebra, a non-unital ring I is called *H-unital* if every $\mathrm{Tor}_i(\mathbb{Z},\mathbb{Z})$ vanishes; Suslin's result says that I satisfies absolute excision for all K_n if and only if I is H-unital.

Together with a result of Suslin and Wodzicki [191], this implies that I satisfies absolute excision for $K_n\otimes\mathbb{Q}$ if and only if $I\otimes\mathbb{Q}$ satisfies absolute excision for K_n.

Suppose now that $I = I^2$. In this case the commutator subgroup of $GL(I)$ is perfect (III, Ex. 2.10). By Theorem 1.5 there is a $+$–construction $BGL(I)^+$ and a map from $BGL(I)^+$ to the basepoint component of $K(R,I)$. When I is H-unital, this is a homotopy equivalence; $\pi_n BGL(I)^+ \cong K_n(R,I)$ for all $n \ge 1$. This concrete version of absolute excision was proven by Suslin and Wodzicki in [191, 1.7].

IV.1.11.2 **Suspension Rings 1.11.2.** Let $C(R)$ be the *cone ring* of row-and-column finite matrices over a fixed ring R (Ex. I.1.8); by II.2.1.3, $C(R)$ is flasque, so $K(C(R))$ is contractible by Ex. 1.17. The *suspension ring* $S(R)$ of III, Ex. 1.15 is $C(R)/M(R)$, where $M(R)$ is the ideal of finite matrices over R. Since $M(R)\cong M(M(R))$ and $GL(R) = GL_1(M(R))$, we have $GL(R)\cong GL(M(R))$ and hence $BGL(R)^+\cong BGL(M(R))^+$. Since $M(R)$ is H-unital (see 1.11.1 and Ex. 1.20), it satisfies absolute excision and we have a fibration sequence

$$K_0(R)\times BGL(R)^+ \to BGL(C(R))^+ \to BGL(S(R))^+.$$

Since the middle term is contractible, this proves that $K_0(R)\times BGL(R)^+ \simeq \Omega BGL(S(R))^+$ so that $K_{n+1}S(R)\cong K_n(R)$ for all $n\ge 1$. ($K_0S(R)\cong K_{-1}(R)$ by III, Ex. 4.10.) This result was first proven by Gersten and Wagoner.

K-theory of finite fields

Next, we describe Quillen's construction for the K-theory of finite fields, arising from his work on the Adams Conjecture [152]. Adams had shown that the Adams operations ψ^k on topological K-theory (II.4.4) are represented by maps $\psi^k : BU\to BU$ in the sense that for each X the Adams operations on $\widetilde{KU}(X)$ are the maps:

$$\widetilde{KU}(X) = [X, BU] \xrightarrow{[X,\psi^k]} = \widetilde{KU}(X).$$

Fix a finite field $\mathbb{F}_q$ with $q = p^\nu$ elements. For each n, the Brauer lifting of the trivial and standard n-dimensional representations of $GL_n(\mathbb{F}_q)$ are n-dimensional complex representations, given by homomorphisms $1_n, \mathrm{id}_n : GL_n(\mathbb{F}_q) \to U$. Since BU is an H-space, we can form the difference $\rho_n = B(\mathrm{id}_n) - B(1_n)$ as a map $BGL_n(\mathbb{F}_q) \to BU$. Quillen observed that ρ_n and ρ_{n+1} are compatible up to homotopy with the inclusion of $BGL_n(\mathbb{F}_q)$ in $BGL_{n+1}(\mathbb{F}_q)$. (See 5.3.1 below.) Hence there is a map $\rho : BGL(\mathbb{F}_q) \to BU$, well defined up to homotopy. By Theorem 1.8, ρ induces a map from $BGL(\mathbb{F}_q)^+$ to BU, and hence maps $\rho_* : K_n(\mathbb{F}_q) \to \pi_n(BU) = \widetilde{KU}(S^n)$.

We will define operations λ^k and ψ^k on $K_*(\mathbb{F}_q)$ in 5.3.1 and Ex. 5.2 below, and show (5.5.2) that ψ^p is induced by the Frobenius on $\mathbb{F}_q$, so that ψ^q is the identity map on $K_n(\mathbb{F}_q)$. We will also see in 5.7.1 and 5.8 below that ρ_* commutes with the operations λ^k and ψ^k on $K_n(\mathbb{F}_q)$ and $\widetilde{KU}(S^n)$.

IV.1.12 **Theorem 1.12.** *(Quillen) The map $BGL(\mathbb{F}_q)^+ \to BU$ identifies $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of $\psi^q - 1$. That is, the following is a homotopy fibration.*

$$BGL(\mathbb{F}_q)^+ \xrightarrow{\rho} BU \xrightarrow{\psi^q - 1} BU$$

On homotopy groups, II.4.4.1 shows that ψ^q is multiplication by q^i on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$. Using the homotopy sequence 1.2 and Theorem 1.12, we immediately deduce:

IV.1.13 **Corollary 1.13.** *For every finite field $\mathbb{F}_q$, and $n \geq 1$, we have*

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1, \\ 0 & n \text{ even.} \end{cases}$$

Moreover, if $\mathbb{F}_q \subset \mathbb{F}_{q'}$ then $K_n(\mathbb{F}_q) \to K_n(\mathbb{F}_{q'})$ is an injection, identifying $K_n(\mathbb{F}_q)$ with $K_n(\mathbb{F}_{q'})^G$, where $G = \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$; the transfer map $K_n(\mathbb{F}_{q'}) \to K_n(\mathbb{F}_q)$ is onto (see 1.1.3).

IV.1.13.1 **Remark 1.13.1.** Clearly all products in the ring $K_*(\mathbb{F}_q)$ are trivial. We will see in section 2 that it is also possible to put a ring structure on the homotopy groups with mod-ℓ coefficients, $K_n(\mathbb{F}_q; \mathbb{Z}/\ell) = \pi_n(BGL(\mathbb{F}_q); \mathbb{Z}/\ell)$.

If $\ell \mid (q-1)$, the long exact sequence for homotopy with mod-ℓ coefficients (2.1.1) shows that $K_n(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ for all $n \geq 0$. The choice of a primitive unit $\zeta \in \mathbb{F}_q^\times$ and a primitive ℓth root of unity ω gives generators ζ for $K_1(\mathbb{F}_q; \mathbb{Z}/\ell)$ and the *Bott element* β for $K_2(\mathbb{F}_q; \mathbb{Z}/\ell)$, respectively. (The Bockstein sends β to $\omega \in K_1(\mathbb{F}_q)$.) Browder has shown [34] that $K_*(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta, \zeta]/(\zeta^2)$ as a graded ring, and that the natural isomorphism from the even part $\oplus_n K_{2n}(\mathbb{F}_q; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$ to $\oplus \pi_{2n}(BU; \mathbb{Z}/\ell)$ is a ring isomorphism.

If $p \neq \ell$, the algebraic closure $\bar{\mathbb{F}}_p$ is the union of the $\mathbb{F}_q$ where $q = p^\nu$ and $\ell \mid (q-1)$. Hence the ring $K_*(\bar{\mathbb{F}}_p; \mathbb{Z}/\ell)$ is the direct limit of the $K_*(\mathbb{F}_q; \mathbb{Z}/\ell)$. As each ζ vanishes and the Bott elements map to each other, we have:

$$K_*(\bar{\mathbb{F}}_p; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta] \cong \pi_*(BU; \mathbb{Z}/\ell).$$

IV.1.13.2 **Remark 1.13.2.** Browder has also shown in [34, 2.4] that the Bott element β in $K_2(\mathbb{C};\mathbb{Z}/m)$ maps to a generator of $\pi_2(BU;\mathbb{Z}/m) = \mathbb{Z}/m$ under the change-of-topology map. Hence the map $K_*(\mathbb{C};\mathbb{Z}/m) \to \pi_*(BU;\mathbb{Z}/m)$ is also onto. We will see in VI.1.4.1 that it is an isomorphism.

Homological Stability

Homological stability, the study of how the homology of a group like $GL_n(R)$ depends upon n, plays an important role in algebraic K-theory. The following theorem was proven by Suslin in [181], using Volodin's construction of $K(R)$. Recall from Ex. I.1.5 that the stable range of R, $sr(R)$, is defined in terms of unimodular rows; if R is commutative and noetherian it is at most $\dim(R) + 1$.

IV.1.14 **Theorem 1.14.** *Let R be a ring with stable range $sr(R)$. For $r \geq \max\{2n+1, n+sr(R)\}$ the maps $\pi_n BGL_r(R)^+ \to \pi_n BGL_{r+1}(R)^+$ are isomorphisms.*

Now assume that $r > sr(R) + 1$, so that $E_r(R)$ is a perfect normal subgroup of $GL_r(R)$ by Ex. III.1.3. The universal covering space of $BGL_r(R)^+$ is then homotopy equivalent to $BE_r(R)^+$ for (by Ex. 1.8). Applying the Hurewicz theorem (and the Comparison Theorem) to these spaces implies:

IV.1.14.1 **Corollary 1.14.1.** *In the range $r \geq \max\{2n+1, n+sr(R)\}$, the following maps are isomorphisms:*

$$H_n(BGL_r(R)) \to H_n(BGL_{r+1}(R)) \to H_n(BGL(R)^+);$$
$$H_n(BE_r(R)) \to H_n(BE_{r+1}(R)) \to H_n(BE(R)^+).$$

For example, suppose that R is an Artinian ring, so that $sr(R) = 1$ by Ex. I.1.5. Then $\pi_n BGL_r(R)^+ \cong K_n(R)$ and $H_n(BGL_r(R)) \cong H_n(BGL(R)^+)$ for all $r > 2n$. The following result, due to Suslin [183], improves this bound for fields.

IV.1.15 **Proposition 1.15.** *(Suslin) If F is an infinite field, then $H_n(GL_r(F)) \to H_n(GL(F))$ is an isomorphism for all $r \geq n$. In addition, there is a canonical isomorphism $H_n(GL_n(F))/im\, H_n(GL_{n-1}(F)) \cong K_n^M(F)$.*

IV.1.16 **Proposition 1.16.** *(Kuku) If R is a finite ring, then $K_n(R)$ is a finite abelian group for all $n > 0$.*

Proof. The case $n = 1$ follows from III.1.4 (or Ex. III.1.2); $K_1(R)$ is a quotient of $R^\times$. Since $K_n(R) = \pi_n BE(R)^+$ for $n > 1$ by Ex. 1.8, it suffices to show that the homology groups $H_n(E(R);\mathbb{Z})$ are finite for $n > 0$. But each $E_r(R)$ is a finite group, so the groups $H_n(BE_r(R);\mathbb{Z})$ are indeed finite for $n > 0$. ∎

Rank of K_n over number fields

It is a well known theorem of Cartan and Serre that the "rational" homotopy groups $\pi_n(X)\otimes\mathbb{Q}$ of an H-space X inject into the rational homology groups $H_n(X;\mathbb{Q})$, and that $\pi_*(X)\otimes\mathbb{Q}$ forms the primitive elements in the coalgebra structure on $H_*(X;\mathbb{Q})$. (See [134, p. 263].) For $X = BGL(R)^+$, which is an H-space by Ex. 1.11, this means that the groups $K_n(R)\otimes\mathbb{Q} = \pi_n(BGL(R)^+)\otimes\mathbb{Q}$ inject into the groups $H_*(GL(R);\mathbb{Q}) = H_*(BGL(R);\mathbb{Q}) = H_*(BGL(R)^+;\mathbb{Q})$ as the primitive elements. For $X = BSL(R)^+$, this means that the groups $K_n(R)\otimes\mathbb{Q}$ inject into $H_*(SL(R);\mathbb{Q})$ as the primitive elements for $n\geq 2$.

Now suppose that A is a finite dimensional semisimple algebra over $\mathbb{Q}$, such as a number field, and that R is a subring of A which is finitely generated over $\mathbb{Z}$ and has $R\otimes\mathbb{Q} = A$ (R is an *order*). In this case, Borel determined the ring $H^*(SL_m(R);\mathbb{Q})$ and hence the dual coalgebra $H_*(SL_m(R);\mathbb{Q})$ and hence its primitive part, $K_*(R)\otimes\mathbb{Q}$. (See the review MR0387496 of Borel's paper [29] by Garland.) The answer only depends upon the semisimple $\mathbb{R}$-algebra $A\otimes_{\mathbb{Q}}\mathbb{R}$.

More concretely, let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras (over $\mathbb{Q}$) of $SL_m(A\otimes\mathbb{R})$ and one of its maximal compact subgroups K. Borel first proved in [28, Thm. 1] that

$$H^q(SL_m(R);\mathbb{R})\cong H^q(SL_m(A);\mathbb{R})\cong H^q(\mathfrak{g},\mathfrak{k};\mathbb{R})\quad\text{for } m\gg q.$$

By the above remarks, this proves:

IV.1.17 **Theorem 1.17.** *(Borel) Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra. Then for every order R in A we have $K_n(R)\otimes\mathbb{Q}\cong K_n(A)\otimes\mathbb{Q}$ for all $n\geq 2$.*

Borel also calculated the ranks of these groups. Since A is a finite product of simple algebras A_i, and $K_n(A)$ is the product of the $K_n(A_i)$ by Ex. 1.7, we may assume that A is simple, *i.e.,* a matrix algebra over a division algebra. The center of A is then a number field F. It is traditional to write r_1 and r_2 for the number of factors of $\mathbb{R}$ and $\mathbb{C}$ in the $\mathbb{R}$-algebra $F\otimes_{\mathbb{Q}}\mathbb{R}$, so that $F\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$. Borel proved in [28, Thm. 2] [29, 12.2] that $H^*(SL(A),\mathbb{R})$ is a tensor product of r_1 exterior algebras having generators x_i in degrees $4i+1$ ($i\geq 1$) and r_2 exterior algebras having generators x_j in degrees $2j+1$ ($j\geq 1$). Taking primitive parts, this proves the following result:

IV.1.18 **Theorem 1.18.** *(Borel) Let F be a number field, and A a central simple F-algebra. Then for $n\geq 2$ we have $K_n(A)\otimes\mathbb{Q}\cong K_n(F)\otimes\mathbb{Q}$ and*

$$\operatorname{rank} K_n(A)\otimes\mathbb{Q} = \begin{cases} r_2, & n\equiv 3 \pmod 4\\ r_1+r_2, & n\equiv 1 \pmod 4\\ 0, & \textit{else.}\end{cases}$$

By Theorem 1.17, this also gives the rank of $K_n(R)$ for every order R.

In particular, these groups are torsion for every even $n\geq 2$.

IV.1.18.1 **Regulator Maps 1.18.1.** Borel's construction provides a specific map from $K_n(R)$ to the real vector space P_n of primitives in $H_n(SL(R), \mathbb{R})$; Borel observed that the image is a lattice Λ. There is another canonical lattice Λ' in P_n: the image of $\pi_n(X)$ for the symmetric space X contained in $K\backslash GL_m(\mathbb{C})$. The *higher regulator* of R is defined to be the positive real number R_n such that the volume of P_n/Λ is R_n times the volume of P_n/Λ'. Borel also proved that R_{2i+1} was a positive rational number times $\sqrt{D}\ \pi^{-m(i+1)}\zeta_F(i+1)$, where D is the discriminant of $F/\mathbb{Q}$ and ζ_F is the zeta function of F.

IV,1.18.2 **Example 1.18.2** (Group Rings)**.** The group ring $\mathbb{Z}[G]$ of a finite group G is an order in the semisimple algebra $\mathbb{Q}[G]$. Therefore Theorem 1.18 gives the rank of the groups $K_n(\mathbb{Z}[G])$ for $n \geq 2$. The rank of $K_1(\mathbb{Z}[G])$ was given in III.1.8, and does not follow this pattern. For example, if C_p is a cyclic group of prime order $p \geq 3$ then $r_1 + r_2 = (p+1)/2$ yet $K_1(\mathbb{Z}[C_p])$ has rank $(p-3)/2$.

$K_3(R)$ and $H_3(E(R), \mathbb{Z})$

The following material is due to Suslin [187]. Given an element α of $\pi_n(X)$ and an element β of $\pi_m(S^n)$, the composition product $\alpha \circ \beta$ is the element of $\pi_m(X)$ represented by $S^m \xrightarrow{\beta} S^n \xrightarrow{\alpha} X$. We will apply this to the Hopf element $\eta \in \pi_3(S^2)$, using the following observation.

If Y_n is the wedge of n copies of S^2, the Hilton-Milnor Theorem [228, XI(8.1)] says that

$$\Omega\Sigma Y_n \simeq \prod_{i=1}^{n} \Omega S^3 \times \prod_{i \neq j} \Omega S^5 \times Y_n',$$

where Y_n' is 5-connected and Σ is suspension. Note that $\pi_3(\Omega\Sigma S^2) = \pi_4(S^3) = \mathbb{Z}/2$, on the image of $\eta \in \pi_3(S^2)$. Hence $\pi_3(\Omega\Sigma Y_n) \cong (\mathbb{Z}/2)^n$. If Y_I is a wedge of copies of S^2 indexed by an infinite set I then (taking the filtered colimit over finite subsets of I) it follows that $\pi_3(\Omega\Sigma Y_I) \cong \oplus_I \mathbb{Z}/2$, generated by the factors $S^2 \to \Omega S^2 \to Y_I$.

IV.1.19 **Lemma 1.19.** *If X is a simply connected loop space, the composition product with η and the Hurewicz map $h : \pi_3(X) \to H_3(X, \mathbb{Z})$ fit into an exact sequence*

$$\pi_2(X) \xrightarrow{\circ\eta} \pi_3(X) \xrightarrow{h} H_3(X, \mathbb{Z}) \to 0.$$

Proof. Let I be a set of generators of $\pi_2(X)$; the maps $f(i) : S^2 \to X$ induce a map $f : Y \to X$, where $Y = \vee_I S^2$. The map f factors as $Y \to \Omega\Sigma Y \xrightarrow{\Omega f^*} X$, where $X = \Omega X'$ and $f^* : \Sigma Y \to X'$ is the adjoint of f. Since $\pi_2(Y) \to \pi_2(X)$ is onto, the sequence $\pi_3(\Omega\Sigma Y) \to \pi_3(X) \to H_3(X) \to 0$ is exact by Exercise 1.25.

As above, $\pi_3(\Omega\Sigma Y) \cong \oplus_I \pi_3(\Omega\Sigma S^2)$, and the i^{th} factor is the image of $\pi_3(S^2)$, generated by η. The map $\pi_3(\Omega\Sigma Y) \to \pi_3(X)$ sends the generator of the i^{th} factor to the composition product $f(i) \circ \eta : S^3 \xrightarrow{\eta} S^2 \to \Omega\Sigma S^2 \longrightarrow X$. Since $\pi_2(X)$ is generated by the $f(i)$, the result follows. □

IV.1.19.1 **Remark 1.19.1.** (Suslin) Lemma 1.19 holds for any simply connected H-space X. To see this, note that the Hilton-Milnor Theorem for $Y_n = \vee S^2$ states that the space ΩY_n is homotopy equivalent to $\prod_i \Omega S^2 \times \prod_{i \neq j} \Omega S^3 \times Y''$ where Y'' is 3-connected. Thus $\pi_3(Y_n) = \pi_2(\Omega Y_n)$ is the sum of $\mathbb{Z}^n = \oplus \pi_3(S^2)$ and $\oplus_{i \neq j} \pi_3(S^3)$, where the second summand is generated by the Whitehead products $[\iota_i, \iota_j]$ of the generators of $\pi_2(Y)$. These Whitehead products map to $[f(i), f(j)]$, which vanish in $\pi_3(X)$ when X is any H-space by [228, X(7.8)]. With this modification, the proof of Lemma 1.19 goes through.

IV.1.20 **Corollary 1.20.** *For any ring R the product with $[-1] \in K_1(\mathbb{Z})$ fits into an exact sequence*

$$K_2(R) \xrightarrow{[-1]} K_3(R) \xrightarrow{h} H_3(E(R), \mathbb{Z}) \to 0.$$

Proof. By Ex. 1.12(a), the map $\pi_3(S^2) \to K_1(\mathbb{Z})$ sends η to $[-1]$. Since $X = BGL(R^+)$ is an H-space, the composition product $\pi_2(X) \xrightarrow{\eta} \pi_3(X)$ is multiplication by the image of η in $\pi_1(X) = K_1(R)$, namely $[-1]$; see Ex. 1.12(e). The result follows from Lemma 1.19 and the observation in Exercise 1.8 that $\pi_n BE(R)^+ \to K_n(R)$ is an isomorphism for $n \geq 2$. □

EXERCISES

EIV.1.1 **1.1.** (Kervaire) Let X be a homology n-sphere, *i.e.,* a space with $H_*(X) = H_*(S^n)$. Show that there is a homotopy equivalence $S^n \to X^+$. *Hint:* Show that $\pi_1(X)$ is perfect if $n \neq 1$, so X^+ is simply connected, and use the Hurewicz theorem.

The binary icosohedral group $\Gamma = SL_2(\mathbb{F}_5)$ embeds in $O_3(\mathbb{R})$ as the symmetry group of both the dodecahedron and icosahedron. Show that the quotient $X = S^3/\Gamma$ is a homology 3-sphere, and conclude that the canonical map $S^3 \to X^+$ is a homotopy equivalence. (The fact that it is a homology sphere was discovered by Poincaré in 1904, and X is sometimes called the *Poincaré sphere.*)

EIV.1.2 **1.2.** (a) If F is an acyclic space, show that F^+ is contractible. (b) If $X \xrightarrow{f} Y$ is acyclic and $f_* : \pi_1(X) \cong \pi_1(Y)$, show that f is a homotopy equivalence.

EIV.1.3 **1.3.** Prove the assertions in Remark 1.8.1 using the following standard result: Let X and Y be H-spaces having the homotopy type of a CW complex. If $f : X \to Y$ is a map which induces an isomorphism $H_*(X, \mathbb{Z}) \cong H_*(Y, \mathbb{Z})$, show that f is a homotopy equivalence. *Hint:* Since $\pi_1(Y)$ acts trivially on the homotopy fiber F by [228, IV.3.6], the relative Hurewicz theorem [228, IV.7.2] shows that $\pi_*(F) = 0$.

EIV.1.4 **1.4.** Here is a point-set construction of X^+ relative to a perfect normal subgroup P. Form Y by attaching one 2-cell e_p for each element of P, so that $\pi_1(Y) = \pi_1(X)/P$. Show that $H_2(Y; \mathbb{Z})$ is the direct sum of $H_2(X; \mathbb{Z})$ and the free abelian group on the set $\{[e_p] : p \in P\}$. Next, prove that each homology class $[e_p]$ is represented by a map $h_p : S^2 \to Y$, and form Z by attaching 3-cells to Y (one for each $p \in P$) using the h_p. Finally, prove that Z is a model for X^+.

EIV.1.5 **1.5.** *Perfect Radicals.* Show that the subgroup generated by the union of perfect subgroups of any group G is itself a perfect subgroup. Conclude that G has a largest perfect subgroup P, called the *perfect radical* of G, and that it is a normal subgroup of G.

EIV.1.6 **1.6.** Let cone(i) denote the mapping cone of a map $F \xrightarrow{i} X$. If F is an acyclic space, show that the map $X \to \text{cone}(i)$ is acyclic. If F is a subcomplex of X then cone(i) is homotopy equivalent to the quotient space X/F, so $X \to X/F$ is also acyclic. Conclude that if $X(R)$ is the Volodin space of Example 1.3.2 then $BGL(R)/X(R)$ is a model for $BGL(R)^+$. *Hint:* Consider long exact sequences in homology.

EIV.1.7 **1.7.** Show that $BGL(R_1 \times R_2)^+ \simeq BGL(R_1)^+ \times BGL(R_2)^+$ and hence $K_n(R_1 \times R_2) \cong K_n(R_1) \times K_n(R_2)$ for every pair of rings R_1, R_2 and every n. *Hint:* Use 3.1(6) below to see that $BGL(R_1 \times R_2) \cong BGL(R_1) \times BGL(R_2)$.

EIV.1.8 **1.8.** Let P be a perfect normal subgroup of G, and let $BG \to BG^+$ be a +–construction relative to P. Show that BP^+ is homotopy equivalent to the universal covering space of BG^+. Hence $\pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 2$. *Hint:* BP is homotopy equivalent to a covering space of BG.

For $G = GL(R)$ and $P = E(R)$, this shows that $BE(R)^+$ is homotopy equivalent to the universal covering space of $BGL(R)^+$. Thus $K_n(R) \cong \pi_n BE(R)^+$ for $n \geq 2$.

(a) If R is a commutative ring, show that $SL(R) \hookrightarrow GL(R)$ induces isomorphisms $\pi_n BSL(R)^+ \cong K_n(R)$ for $n \geq 2$, and $\pi_1 BSL(R)^+ \cong SK_1(R)$. Conclude that the map $BSL(R)^+ \times B(R^\times) \to BGL(R)^+$ is a homotopy equivalence.

(b) If A is a finite semisimple algebra over a field, the subgroups $SL_n(A)$ of $GL_n(A)$ were defined in III.1.2.4. Show that $SL(A) \hookrightarrow GL(A)$ induces isomorphisms $\pi_n BSL(A)^+ \cong K_n(A)$ for $n \geq 2$, and $\pi_1 BSL(A)^+ \cong SK_1(A)$.

EIV.1.9 **1.9.** Suppose that $A \to S \to P$ is a universal central extension (III.5.3.1). In particular, S and P are perfect groups. Show that there is a homotopy fibration $BA \to BS^+ \to BP^+$. Conclude that $\pi_n(BS^+) = 0$ for $n \leq 2$, and that $\pi_n(BS^+) \cong \pi_n(BP^+) \cong \pi_n(BG^+)$ for all $n \geq 3$. In particular, $\pi_3(BP^+) \cong H_3(S;\mathbb{Z})$.

Since the Steinberg group $St(R)$ is the universal central extension of $E(R)$, this shows that $K_n(R) \cong \pi_n St(R)^+$ for all $n \geq 3$, and that $K_3(R) \cong H_3(St(R);\mathbb{Z})$.

EIV.1.10 **1.10.** For $n \geq 3$, let P_n denote the normal closure of the perfect group $E_n(R)$ in $GL_n(R)$, and let $BGL_n(R)^+$ denote the +–construction on $BGL_n(R)$ relative to P_n. Corresponding to the inclusions $GL_n \subset GL_{n+1}$ we can choose a sequence of maps $BGL_n(R)^+ \to BGL_{n+1}(R)^+$. Show that $\varinjlim BGL_n(R)^+$ is $BGL(R)^+$.

EIV.1.11 **1.11.** For each m and n, the group map $\Box : GL_m(R) \times GL_n(R) \to GL_{m+n}(R) \subset GL(R)$ induces a map $BGL_m(R) \times BGL_n(R) \to BGL(R) \to BGL(R)^+$. Show that these maps induce an H-space structure on $BGL(R)^+$.

EIV.1.12 **1.12.** In this exercise, we develop some properties of $B\Sigma_\infty^+$, where Σ_∞ denotes the union of the symmetric groups Σ_n. We will see in 4.9.3 that $\pi_n(B\Sigma_\infty^+)$ is the stable homotopy group π_n^s. The permutation representations $\Sigma_n \to GL_n(\mathbb{Z})$ (1.3.2) induce a map $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ and hence homomorphisms $\pi_n^s \to K_n(\mathbb{Z})$.

(a) Show that $\eta \in \pi_1^s \cong \mathbb{Z}/2$ maps to $[-1] \in K_1(\mathbb{Z})$.

(b) Show that the subgroups $\Sigma_m \times \Sigma_n$ of Σ_{m+n} induce an H-space structure on $B\Sigma_\infty^+$ such that $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ is an H-map. (See Ex. 1.11.)

(c) Modify the construction of Loday's product (1.10) to show that product permutations $\Sigma_m \times \Sigma_n \to \Sigma_{mn}$ induce a map $B\Sigma_\infty^+ \wedge B\Sigma_\infty^+ \to B\Sigma_\infty^+$ compatible with the corresponding map for $BGL(\mathbb{Z})^+$. The resulting product $\pi_m^s \otimes \pi_n^s \to \pi_{m+n}^s$ makes the stable homotopy groups into a graded-commutative ring, and makes $\pi_*^s \to K_*(\mathbb{Z})$ into a ring homomorphism.

(d) Show that the Steinberg symbol $\{-1,-1,-1,-1\}$ of 1.10.1 vanishes in $K_4(\mathbb{Z})$ and $K_4(\mathbb{Q})$. Since this symbol is nonzero in $K_4^M(\mathbb{Q})$ by III.7.2(c,d), this shows that the Milnor K-groups of a field need not inject into its Quillen K-groups. *Hint:* $\eta^3 \neq 0$ in π_3^s but $\eta^4 = 0$ in π_4^s.

(e) If $\beta \in \pi_{n+t}(S^n)$ and $\alpha \in K_n(R)$, show that the composition product $\alpha \circ \beta$ in $K_{n+t}(R)$ agrees with the product of α with $[\beta] \in \pi_t^s$.

EIV.1.13 **1.13.** Let A_∞ denote the union of the alternating groups A_n; it is a subgroup of Σ_∞ of index 2. A_∞ is a perfect group, since the A_n are perfect for $n \geq 5$.

(a) Show that $B\Sigma_\infty^+ \simeq BA_\infty^+ \times B(\mathbb{Z}/2)$, so $\pi_n BA_\infty^+ \cong \pi_n^s$ for all $n \geq 2$.

(b) Use Lemma 1.19 and $\pi_3^s \cong \mathbb{Z}/24$ to conclude that $H_3(A_\infty, \mathbb{Z}) \cong \mathbb{Z}/12$.

(c) Use the Künneth formula and (a) to show that $H_3(\Sigma_\infty, \mathbb{Z}) \cong H_3(\mathcal{A}_\infty, \mathbb{Z}) \oplus (\mathbb{Z}/2)^2$. This calculation was first done by Nakaoka [141].

EIV.1.14 **1.14.** Extend the product map γ of Theorem 1.10 to a map $K(A) \wedge K(B) \to K(A \otimes B)$, so that the induced maps $K_0(A) \times K_n(B) \to K_n(A \otimes B)$ agree with the products defined in III.1.6.1 and Ex. III.5.4.

EIV.1.15 **1.15.** Let I be an ideal in R. Show that the group $\pi_0 K(R, I)$ of 1.11 is isomorphic to the group $K_0(I)$ of Ex. II.2.3, and that the maps $K_1(R/I) \to K_0(I) \to K_0(R)$ in *loc. cit.* agree with the maps of 1.11. *Hint:* $\pi_0 K(R \oplus I, 0 \oplus I)$ must be $K_0(I)$.

Use Ex. III.2.7 to show that $\pi_1 K(R \to R/I)$ is isomorphic to the group $K_1(R, I)$ of III.2.2, and that the maps $K_2(R/I) \to K_1(R, I) \to K_1(R)$ in III.5.7.1 agree with those of 1.11.

EIV.1.16 **1.16.** If $f : R \to S$ is a ring homomorphism, show that the relative group $K_0(f)$ of 1.11 agrees with the relative group $K_0(f)$ of II.2.10.

EIV.1.17 **1.17.** (Wagoner) We say $GL(R)$ is *flabby* if there is a homomorphism $\tau : GL(R) \to GL(R)$ so that for each n the restriction $\tau_n : GL_n(R) \to GL(R)$ of τ is conjugate to the map $(1, \tau_n) : g \mapsto \begin{pmatrix} g & 0 \\ 0 & \tau_n(g) \end{pmatrix}$. In particular, τ_n and $(1, \tau_n)$ induce the same map $H_*(BGL_n(R)) \to H_*(BGL(R))$ by [223, 6.7.8].

(a) Assuming that $GL(R)$ is flabby, show that $BGL(R)$ is acyclic. By Ex. 1.2, this implies that $BGL(R)^+$ is contractible, *i.e.*, that $K_n(R) = 0$ for $n > 0$. *Hint:* The H-space structure (Ex. 1.11) makes $H_*(BGL(R))$ into a ring.

(b) Show that $GL(R)$ is flabby for every flasque ring R (see II.2.1.3). This shows that flasque rings have $K_n(R) = 0$ for all n. *Hint:* Modify Ex. II.2.15(a).

EIV.1.18 **1.18.** Suppose that I is a nilpotent ideal and that $p^\nu I = 0$ for some ν. Show that $H_*(GL(R); M) \cong H_*(GL(R/I); M)$ for every uniquely p-divisible module M. Conclude that the relative groups $K_*(R, I)$ are p-groups.

EIV.1.19 **1.19.** Suppose that I is a nilpotent ideal in a ring R, and that I is uniquely divisible as an abelian group. Show that $H_*(GL(R); M) \cong H_*(GL(R/I); M)$ for every torsion module M. Conclude that the relative groups $K_*(R, I)$ are uniquely divisible abelian groups.

EIV.1.20 **1.20.** Show that every ring with unit is H-unital (see 1.11.1). Then show that a non-unital ring I is H-unital if every finite subset of I is contained in a unital subring. (This shows that the ring $M(R)$ of finite matrices over R is H-unital.)

Finally, show that I is H-unital if for every finite subset $\{a_j\}$ of I there is an $e \in I$ such that $ea_j = a_j$. (An example of such an I is the non-unital ring of functions with compact support on $\mathbb{C}^n$.)

EIV.1.21 **1.21.** *Morita invariance.* For each $n > 0$, we saw in III.1.1.4 that $GL(R) \cong GL(M_n(R))$ via isomorphisms $M_m(R) \cong M_m(M_n(R))$. Deduce that there is a homotopy equivalence $BGL(R)^+ \simeq BGL(M_n(R))^+$ and hence isomorphisms $K_*(R) \cong K_*(M_n(R))$. (The cases $* = 0, 1, 2$ were given in II.2.7, III.1.6.4 and III.5.6.1.) We will give a more general proof in 6.3.5 below.

Compare this to the approach of 1.11.2, using $M(R)$.

EIV.1.22 **1.22.** *Loday symbols.* Let $r_1, ..., r_n$ be elements of R so that $r_n r_1 = 0$ and each $r_i r_{i+1} = 0$. Show that the elementary matrices $e_{n,1}(r_n)$ and $e_{i,i+1}(r_i)$ commute and define a ring homomorphism $B = \mathbb{Z}[x_1, 1/x_1, \dots, x_n, 1/x_n] \to M_n(R)$. Using Ex. 1.21, we define the *Loday symbol* $\langle\langle r_1, ..., r_n \rangle\rangle$ in $K_n(R)$ to be the image of $\{x_1, ..., x_n\}$ under $K_n(B) \to K_n(M_n(R)) \cong K_n(R)$.

EIV.1.23 **1.23.** Let $F \to E \xrightarrow{p} B$ and $F' \to E' \xrightarrow{p'} B'$ be homotopy fibrations (1.2), and suppose given pairings $e : E \wedge X \to E'$, $b : B \wedge X \to B'$ so that $p'e = b(p \wedge 1)$.

$$\begin{array}{ccccccc}
\Omega B \wedge X & \xrightarrow{\partial \wedge 1} & F \wedge X & \longrightarrow & E \wedge X & \xrightarrow{p \wedge 1} & B \wedge X \\
\downarrow{\scriptstyle \Omega b} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle b} \\
\Omega B' & \xrightarrow{\partial'} & F' & \longrightarrow & E' & \xrightarrow{p'} & B'.
\end{array}$$

Show there is a pairing $F \wedge X \xrightarrow{f} F'$ compatible with e, such that for $\beta \in \pi_*(B)$ and $\gamma \in \pi_*(X)$ the reduced join [228, p. 480] satisfies $\partial(\beta \wedge_b \gamma) = \partial(\beta) \wedge_f \gamma$.

EIV.1.24 **1.24.** Let $f : A \to B$ be a ring homomorphism and let $K(f)$ (resp., $K(f_C)$ be the relative groups (1.11), *i.e.*, the homotopy fiber of $K(A) \to K(B)$ (resp., $K(A \otimes C) \to K(B \otimes C)$). Use Ex. 1.23 to show that there is an induced pairing $K_*(f) \otimes K_*(C) \to K_*(f_C)$ such that for $\beta \in K_*(B)$ and $\gamma \in K_*(C)$ we have $\partial(\beta \wedge \gamma) = \partial(\beta) \wedge \gamma$ in $K_*(f_C)$.

When f is an R-algebra homomorphism, show that $K_*(f)$ is a right $K_*(R)$-module and that the maps in the relative sequence $K_{n+1}(B) \to K_n(f) \to K_n(A) \to K_n(B)$ of 1.11 are $K_*(R)$-module homomorphisms.

EIV.1.25 **1.25.** Suppose given a homotopy fibration sequence $F \to Y \to X$ with X, Y and F simply connected. Compare the long exact homotopy sequence (see 1.2) with the exact sequence of low degree terms in the Leray-Serre Spectral sequence (see [223, 5.3.3]) to show that there is an exact sequence $\pi_3(Y) \to \pi_3(X) \to H_3(X) \to 0$.

EIV.1.26 **1.26.** The Galois group $G = \mathrm{Gal}(\mathbb{F}_{q^i}/\mathbb{F}_q)$ acts on the group μ of units of $\mathbb{F}_{q^i}$, and also on the i-fold tensor product $\mu^{\otimes i} = \mu \otimes \cdots \otimes \mu$. By functoriality 1.1.2, G acts on $K_*(\mathbb{F}_q)$. Show that $K_{2i-1}(\mathbb{F}_q)$ is isomorphic to $\mu^{\otimes i}$ as a G-module.

EIV.1.27 **1.27.** *Monomial matrices.* Let F be a field and consider the subgroup M of $GL(F)$ consisting of matrices with only one nonzero entry in each row and column.

(a) Show that M is the wreath product $F^\times \wr \Sigma_\infty$, and contains $F^\times \wr A_\infty$ as a subgroup of index 2.

(b) Show that $[M, M]$ is the kernel of $\det : F^\times \wr A_\infty \to F^\times$, so $H_1(M) \cong F^\times \times \Sigma_2$.

(c) Show that $[M, M]$ is perfect, and $BM^+ \simeq B[M, M]^+ \times B(F^\times) \times B\Sigma_2$.

2 K-theory with finite coefficients

In addition to the usual K-groups $K_i(R)$, or the K-groups $K_i(\mathcal{C})$ of a category $\mathcal{C}$, it is often useful to study K-groups with coefficients "mod ℓ" $K_i(R;\mathbb{Z}/\ell)$ (or $K_i(\mathcal{C};\mathbb{Z}/\ell)$,) where ℓ is a positive integer. In this section we quickly recount the basic construction from mod ℓ homotopy theory. Basic properties of mod ℓ homotopy theory may be found in Neisendorfer [142].

Recall [142] that if $m \geq 2$ the mod ℓ *Moore space* $P^m(\mathbb{Z}/\ell)$ is the space formed from the sphere S^{m-1} by attaching an m-cell via a degree ℓ map. It is characterized as having only one nonzero reduced integral homology group, namely $\tilde{H}^m(P) = \mathbb{Z}/\ell$. The suspension of $P^m(\mathbb{Z}/\ell)$ is the Moore space $P^{m+1}(\mathbb{Z}/\ell)$, and as m varies these fit together to form a suspension spectrum $P^\infty(\mathbb{Z}/\ell)$, called the *Moore spectrum*.

IV.2.1 **Definition 2.1.** If $m \geq 2$, the mod ℓ homotopy "group" $\pi_m(X;\mathbb{Z}/\ell)$ of a based topological space X is defined to be the pointed set $[P^m(\mathbb{Z}/\ell), X]$ of based homotopy classes of maps from the Moore space $P^m(\mathbb{Z}/\ell)$ to X.

For a general space X, $\pi_2(X;\mathbb{Z}/\ell)$ isn't even a group, but the $\pi_m(X;\mathbb{Z}/\ell)$ are always groups for $m \geq 3$ and abelian groups for $m \geq 4$ [142]. If X is an H-space, such as a loop space, then these bounds improve by one. If $X = \Omega Y$ then we can define $\pi_1(X;\mathbb{Z}/\ell)$ as $\pi_2(Y;\mathbb{Z}/\ell)$; this is independent of the choice of Y by Ex. 2.1. More generally, if $X = \Omega^k Y_k$ for $k \gg 0$ and $P^m = P^m(\mathbb{Z}/\ell)$ then the formula

$$\pi_m(X;\mathbb{Z}/\ell) = [P^m, X] = [P^m, \Omega^k Y_k] \cong [P^{m+k}, Y_k] = \pi_{m+k}(Y_k;\mathbb{Z}/\ell)$$

shows that we can ignore these restrictions on m, and that $\pi_m(X;\mathbb{Z}/\ell)$ is an abelian group for all $m \geq 0$ (or even negative m, as long as $k > 2 + |m|$).

In particular, if X is an infinite loop space then abelian groups $\pi_m(X;\mathbb{Z}/\ell)$ are defined for all $m \in \mathbb{Z}$, using the explicit sequence of deloopings of X provided by the given structure on X.

IV.2.1.1 **Remark 2.1.1.** If $F \to E \to B$ is a Serre fibration there is a long exact sequence of groups/pointed sets (which is natural in the fibration):

$$\cdots \to \pi_{m+1}(B;\mathbb{Z}/\ell) \to \pi_m(F;\mathbb{Z}/\ell) \to \pi_m(E;\mathbb{Z}/\ell) \to \\ \pi_m(B;\mathbb{Z}/\ell) \to \pi_{m-1}(F;\mathbb{Z}/\ell) \to \cdots \to \pi_2(B;\mathbb{Z}/\ell).$$

This is just a special case of the fact that $\cdots \to [P,F] \to [P,E] \to [P,B]$ is exact for any CW complex P; see [228, III.6.18*].

If $m \geq 2$, the cofibration sequence $S^{m-1} \xrightarrow{\ell} S^{m-1} \longrightarrow P^m(\mathbb{Z}/\ell)$ defining $P^m(\mathbb{Z}/\ell)$ induces an exact sequence of homotopy groups/pointed sets

$$\pi_m(X) \xrightarrow{\ell} \pi_m(X) \to \pi_m(X;\mathbb{Z}/\ell) \xrightarrow{\partial} \pi_{m-1}(X) \xrightarrow{\ell} \pi_{m-1}(X).$$

If ℓ is odd, or divisible by 4, F. Peterson showed that there is even a non-canonical splitting $\pi_m(X;\mathbb{Z}/\ell) \to \pi_m(X)/\ell$ (see [34, 1.8]).

It is convenient to adopt the notation that if A is an abelian group then ${}_\ell A$ denotes the subgroup of all elements a of A such that $\ell \cdot a = 0$. This allows us to restate the above exact sequence in a concise fashion.

IV.2.2 **Universal Coefficient Sequence 2.2.** For all $m \geq 3$ there is a natural short exact sequence

$$0 \to (\pi_m X) \otimes \mathbb{Z}/\ell \to \pi_m(X; \mathbb{Z}/\ell) \xrightarrow{\partial} {}_\ell(\pi_{m-1} X) \to 0.$$

This sequence is split exact (but not naturally) when $\ell \not\equiv 2 \bmod 4$.

For π_2, the sequence (2.2) of pointed sets is also exact in a suitable sense; see [142, p. 3]. However this point is irrelevant for loop spaces, so we ignore it.

IV.2.2.1 **Example 2.2.1.** When $\ell = 2$, the sequence need not split. For example, it is known that $\pi_{m+2}(S^m; \mathbb{Z}/2) = \mathbb{Z}/4$ for $m \geq 3$, and that $\pi_2(BO; \mathbb{Z}/2) = \pi_3(O; \mathbb{Z}/2) = \mathbb{Z}/4$; see [3].

Here is another way to define mod ℓ homotopy groups, and hence $K_*(R; \mathbb{Z}/\ell)$.

IV.2.3 **Proposition 2.3.** *Suppose that X is a loop space, and let F denote the homotopy fiber of the map $X \to X$ which is multiplication by ℓ. Then $\pi_m(X; \mathbb{Z}/\ell) \cong \pi_{m-1}(F)$ for all $m \geq 2$.*

Proof. (Neisendorfer) Let Maps(A, X) be the space of pointed maps. If $S = S^k$ is the k-sphere then the homotopy groups of Maps(S^k, X) are the homotopy groups of X (reindexed by k), while if $P = P^k(\mathbb{Z}/\ell)$ is a mod ℓ Moore space, the homotopy groups of Maps(P, X) are the mod ℓ homotopy groups of X (reindexed by k).

Applying Maps$(-, X)$ to a cofibration sequence yields a fibration sequence, and applying Maps$(A, -)$ to a fibration sequence yields a fibration sequence; this may be formally deduced from the axioms (SM0) and (SM7) for any model structure, which hold for spaces (see [90]). Applying Maps$(-, X)$ to $S^k \to S^k \to P^{k+1}(\mathbb{Z}/\ell)$ shows that Maps(P, X) is the homotopy fiber of Maps$(S^k, X) \to$ Maps(S^k, X). Applying Maps$(S^k, -)$ to $F \to X \to X$ shows that Maps(S^k, F) is also the homotopy fiber, and is therefore homotopy equivalent to Maps(P, X). Taking the homotopy groups yields the result. □

IV. 2.3.1 **Example 2.3.1** (Spectra)**.** For fixed ℓ, the Moore spectrum $P^\infty(\mathbb{Z}/\ell)$ is equivalent to the (spectrum) cofiber of multiplication by ℓ on the sphere spectrum. If $\mathbf{E}$ is a spectrum, then (by S-duality) the homotopy groups $\pi_*(\mathbf{E}; \mathbb{Z}/\ell) = \varinjlim \pi_{*+r}(\mathbf{E}_r; \mathbb{Z}/\ell)$ are the same as the homotopy groups of the spectrum $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell)$.

Now suppose that $\mathcal{C}$ is either a symmetric monoidal category, or an exact category, or a Waldhausen category. We will construct a K-theory space $K(\mathcal{C})$ below (in 4.3, 6.3 and 8.5); in each case $K(\mathcal{C})$ is an infinite loop space.

IV.2.4 **Definition 2.4.** The mod ℓ K-groups of R are defined to be the abelian group:

$$K_m(R;\mathbb{Z}/\ell) = \pi_m(K(R);\mathbb{Z}/\ell), \quad m \in \mathbb{Z}.$$

Similarly, if the K-theory space $K(\mathcal{C})$ of a category $\mathcal{C}$ is defined then the mod ℓ K-groups of $\mathcal{C}$ are defined to be $K_m(\mathcal{C};\mathbb{Z}/\ell) = \pi_m(K(\mathcal{C});\mathbb{Z}/\ell)$.

By 2.1.1, if $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ is a sequence such that $K(\mathcal{C}_1) \to K(\mathcal{C}_2) \to K(\mathcal{C}_3)$ is a fibration, then there is a long exact sequence of abelian groups

$$\cdots \to K_{n+1}(\mathcal{C}_3;\mathbb{Z}/\ell) \to K_n(\mathcal{C}_1;\mathbb{Z}/\ell) \to K_n(\mathcal{C}_2;\mathbb{Z}/\ell) \to K_n(\mathcal{C}_3;\mathbb{Z}/\ell) \cdots$$

If $m \geq 2$ this definition states that $K_m(R;\mathbb{Z}/\ell) = [P^m(\mathbb{Z}/\ell), K(R)]$. Because $K(R) \simeq \Omega Y$, we can define $K_1(R;\mathbb{Z}/\ell)$ in a way that is independent of the choice of Y (Ex. 2.1); it agrees with the definition in III.1.7.4 (see Ex. 2.2). However, the groups $K_0(R;\mathbb{Z}/\ell)$ and $K_m(R;\mathbb{Z}/\ell)$ for $m < 0$ depend not only upon the loop space $K(R)$, but also upon the choice of the deloopings of $K(R)$ in the underlying K-theory spectrum $\mathbf{K}(R)$. In fact, the literature is not consistent about $K_m(R;\mathbb{Z}/\ell)$ when $m < 2$, even for $K_1(R;\mathbb{Z}/\ell)$. Similar remarks apply to the definition of $K_m(\mathcal{C};\mathbb{Z}/\ell)$.

By Universal Coefficients 2.2, the mod ℓ K-groups are related to the usual K-groups:

IV.2.5 **Universal Coefficient Theorem 2.5.** *There is a short exact sequence*

$$0 \to K_m(R) \otimes \mathbb{Z}/\ell \to K_m(R;\mathbb{Z}/\ell) \to {}_{\ell}K_{m-1}(R) \to 0$$

for every $m \in \mathbb{Z}$, $\mathcal{C}$, and ℓ. It is split exact unless $\ell \equiv 2 \bmod 4$. Ex. 2.3 shows that the splitting is not natural in R.

Similarly, if the K-theory of a category $\mathcal{C}$ is defined then we have an exact sequence

$$0 \to K_m(\mathcal{C}) \otimes \mathbb{Z}/\ell \to K_m(\mathcal{C};\mathbb{Z}/\ell) \to {}_{\ell}K_{m-1}(\mathcal{C}) \to 0$$

IV.2.5.1 **Example 2.5.1 ($\ell = 2$).** Since the isomorphism $\Omega^\infty\Sigma^\infty \to \mathbb{Z} \times BO$ factors through $K(\mathbb{Z})$ and $K(\mathbb{R})$, the universal coefficient theorem and 2.2.1 show that

$$K_2(\mathbb{Z};\mathbb{Z}/2) \cong K_2(\mathbb{R};\mathbb{Z}/2) \cong \pi_2(BO;\mathbb{Z}/2) = \mathbb{Z}/4.$$

It turns out [3] that for $\ell = 2$ the sequence for $K_m(R;\mathbb{Z}/2)$ is split whenever multiplication by $[-1] \in K_1(\mathbb{Z})$ is the zero map from $K_{m-1}(R)$ to $K_m(R)$. For example, this is the case for the finite fields $\mathbb{F}_q$, an observation made in [34].

IV.2.5.2 **Example 2.5.2.** (Bott elements) Suppose that R contains a primitive ℓ^{th} root of unity ζ. The Universal Coefficient Theorem 2.5 provides an element $\beta \in K_2(R;\mathbb{Z}/\ell)$, mapping to $\zeta \in {}_{\ell}K_1(R)$. This element is called the *Bott element*, and it plays an important role in the product structure of the ring $K_*(R;\mathbb{Z}/\ell)$. For finite fields, this role was mentioned briefly in Remark 1.13.1.

IV.2.5.3 **Remark 2.5.3.** A priori, β depends not only upon ζ but also upon the choice of the splitting in 2.5. One way to choose β is to observe that the inclusion of μ_ℓ in $GL_1(R)$ induces a map $B\mu_\ell \to BGL(R) \to BGL(R)^+$ and therefore a set function $\mu_\ell \to K_2(R;\mathbb{Z}/\ell)$. A posteriori, it turns out that this is a group homomorphism unless $\ell \equiv 2 \pmod 4$.

IV.2.6 **Example 2.6.** Let k be the algebraic closure of the field $\mathbb{F}_p$. Quillen's computation of $K_*(\mathbb{F}_q)$ in 1.13 shows that $K_n(k) = 0$ for m even ($m \geq 2$), and that $K_m(k) = \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ for m odd ($m \geq 1$). It follows that if ℓ is prime to p then:

$$K_m(k;\mathbb{Z}/\ell) = \begin{cases} \mathbb{Z}/\ell & \text{if } m \text{ is even, } m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $K_*(k;\mathbb{Z}/\ell)$ is the polynomial ring $\mathbb{Z}/\ell[\beta]$ on the Bott element $\beta \in K_2(k;\mathbb{Z}/\ell)$, under the K-theory product of 2.8 below. See 1.13.1 (and Chapter VI, 1.3.1) for more details.

The next result shows that we may always assume that ℓ is a power of a prime.

IV.2.7 **Proposition 2.7.** *If $\ell = q_1q_2$ with q_1 and q_2 relatively prime, then $\pi_m(X;\mathbb{Z}/\ell)$ is naturally isomorphic to $\pi_m(X;\mathbb{Z}/q_1) \times \pi_m(X;\mathbb{Z}/q_1)$.*

Proof. Set $P_1 = P^m(\mathbb{Z}/q_1)$ and $P_2 = P^m(\mathbb{Z}/q_2)$ and $P = P_1 \vee P_2$. Since P has only one nonzero reduced integral homology group, namely $\tilde{H}^m(P) = \mathbb{Z}/q_1 \times \mathbb{Z}/q_2 \cong \mathbb{Z}/\ell$, the natural map $P \to P^m(\mathbb{Z}/\ell)$ must be a homotopy equivalence. But then $\pi_m(X;\mathbb{Z}/\ell)$ is naturally isomorphic to

$$[P^m(\mathbb{Z}/q_1) \vee P^m(\mathbb{Z}/q_2), X] \cong [P^m(\mathbb{Z}/q_1), X] \times [P^m(\mathbb{Z}/q_2), X],$$

which is the required group $\pi_m(X;\mathbb{Z}/q_1) \times \pi_m(X;\mathbb{Z}/q_1)$. □

Products

If $\ell \geq 3$ is prime, there is a homotopy equivalence

$$P^m(\mathbb{Z}/\ell^\nu) \wedge P^n(\mathbb{Z}/\ell^\nu) \simeq P^{m+n}(\mathbb{Z}/\ell^\nu) \vee P^{m+n-1}(\mathbb{Z}/\ell^\nu).$$

The projections onto the first factor give a spectrum "product" map $P^\infty(\mathbb{Z}/\ell^\nu) \wedge P^\infty(\mathbb{Z}/\ell^\nu) \to P^\infty(\mathbb{Z}/\ell^\nu)$ which is homotopy associative and commutative unless $\ell^\nu = 3$. (The same thing is true when $\ell = 2$, except the product map does not exist if $2^\nu = 2$, it is not homotopy associative if $2^\nu = 4$ and it is not homotopy commutative when $2^\nu = 4, 8$.) These facts are due to Araki and Toda, and follow by S-duality from [142, 8.5–6]. So from now on, we shall exclude the pathological cases $\ell^\nu = 2, 3, 4, 8$.

If $\mathbf{E}$ is a homotopy associative and commutative ring spectrum, then so is the spectrum $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$, unless $\ell^\nu = 2, 3, 4, 8$. Applying this to $\mathbf{E} = \mathbf{K}(R)$ yields the following result.

IV.2.8 **Theorem 2.8.** *Let R be a commutative ring, and suppose $\ell^\nu \neq 2, 3, 4, 8$. Then $\mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu)$ is a homotopy associative and commutative ring spectrum. In particular, $K_*(R;\mathbb{Z}/\ell^\nu)$ is a graded-commutative ring.*

IV.2.8.1 **Scholium 2.8.1.** Browder [34] has observed that if $\pi_m(\mathbf{E}) = 0$ for all even $m > 0$ (and $m < 0$) then $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$ is a homotopy associative and commutative ring spectrum even for $\ell^\nu = 2, 3, 4, 8$. This applies in particular to $\mathbf{E} = \mathbf{K}(\mathbb{F}_q)$, as remarked in 1.13.1 and 2.6 above.

IV.2.8.2 **Corollary 2.8.2.** *If $\ell \geq 3$ and R contains a primitive ℓ^{th} root of unity ζ, and $\beta \in K_2(R;\mathbb{Z}/\ell)$ is the Bott element (2.5.2), there is a graded ring homomorphism $\mathbb{Z}/\ell[\zeta, \beta] \to K_*(R;\mathbb{Z}/\ell)$.*

If $\zeta \notin R$, there are elements $\beta' \in K_{2\ell-2}(R;\mathbb{Z}/\ell)$ and $\zeta' \in K_{2\ell-3}(R;\mathbb{Z}/\ell)$ whose images in $K_{2\ell-2}(R[\zeta];\mathbb{Z}/\ell)$ and $K_{2\ell-3}(R[\zeta];\mathbb{Z}/\ell)$ are $\beta^{\ell-1}$ and $\beta^{\ell-2}\zeta$, respectively.

Proof. The first assertion is immediate from 2.8 and 2.5.2. For the second assertion we may assume that $R = \mathbb{Z}$. Then the Galois group G of $\mathbb{Z}[\zeta]$ over $\mathbb{Z}$ is cyclic of order $\ell-1$, and we define β' to be the image of $-\beta^{\ell-1} \in K_{2\ell-2}(\mathbb{Z}[\zeta];\mathbb{Z}/\ell)$ under the transfer map i_* (1.1.3). Since i^*i_* is $\sum_{g\in G} g^*$ by Ex. 6.13,

$$i^*\beta' = -\sum g^*\beta^{\ell-1} = -(\ell-1)\beta^{\ell-1} = \beta^{\ell-1}.$$

Similarly, $\zeta' = i_*(-\zeta\beta^{\ell-2})$ has $i^*\zeta' = \zeta\beta^{\ell-2}$. □

IV.2.9 **The ℓ-adic completion 2.9.** Fix a prime ℓ. The *ℓ-adic completion* of a spectrum $\mathbf{E}$, $\hat{\mathbf{E}}_\ell$, is the homotopy limit (over ν) of the spectra $\mathbf{E} \wedge P^\infty(\mathbb{Z}/\ell^\nu)$. We let $\pi_n(\mathbf{E};\mathbb{Z}_\ell)$ denote the homotopy groups of this spectrum; if $\mathbf{E} = \mathbf{K}(R)$ we write $K_n(R;\mathbb{Z}_\ell)$ for the homotopy groups $\pi_n(\mathbf{K}(R);\mathbb{Z}_\ell)$ of the ℓ-adic completion $\hat{\mathbf{K}}(R)_\ell$. There is an extension

$$0 \to \varprojlim{}^1 \pi_{n+1}(\mathbf{E};\mathbb{Z}/\ell^\nu) \to \pi_n(\mathbf{E};\mathbb{Z}_\ell) \to \varprojlim \pi_n(\mathbf{E};\mathbb{Z}/\ell^\nu) \to 0.$$

If the homotopy groups $\pi_{n+1}(\mathbf{E};\mathbb{Z}/\ell^\nu)$ are finite, the $\varprojlim^1$ term vanishes and, by Universal Coefficients (2.5), $\pi_n(\mathbf{E};\mathbb{Z}_\ell)$ is an extension of the Tate module of $\pi_{n-1}(\mathbf{E})$ by the ℓ-adic completion of $\pi_n(\mathbf{E})$. (The (ℓ-primary) *Tate module* of an abelian group A is the inverse limit of the groups $\mathrm{Hom}(\mathbb{Z}/\ell^\nu, A)$.) For example, the Tate module of $K_1(\mathbb{C}) = \mathbb{C}^\times$ is $\mathbb{Z}_\ell$, so $K_2(\mathbb{C};\mathbb{Z}_\ell) = \pi_2(\mathbf{K}(\mathbb{C});\mathbb{Z}_\ell)$ is $\mathbb{Z}_\ell$.

If $\mathbf{E}$ is a homotopy associative and commutative ring spectrum then so is the homotopy limit $\hat{\mathbf{E}}_p$. Thus $\pi_*(\mathbf{E};\mathbb{Z}_\ell)$, and in particular $K_*(R;\mathbb{Z}_\ell)$, is also a graded-commutative ring.

We conclude with Gabber's Rigidity Theorem [60]. If I is an ideal in a commutative ring R, we say that (R, I) is a *Hensel pair* if for every finite commutative R-algebra C the map $C \to C/IC$ induces a bijection on idempotents. A *Hensel local ring* is a commutative local ring R such that $(R, \mathfrak{m})$ is a Hensel pair. These conditions imply that I is a radical ideal (Ex. I.2.1), and (R, I) is a Hensel pair whenever I is complete by Ex. I.2.2(i).

IV.2.10 **Theorem 2.10** (Rigidity Theorem). *Let (R, I) be a Hensel pair with $1/\ell \in R$. Then for all $n \geq 1$, $K_n(R;\mathbb{Z}/\ell) \xrightarrow{\cong} K_n(R/I;\mathbb{Z}/\ell)$ and $\widetilde{H}_*(GL(I),\mathbb{Z}/\ell) = 0$*

Gabber proves that $\widetilde{H}_*(GL(I),\mathbb{Z}/\ell) = 0$, and observes that this is equivalent to $K_n(R;\mathbb{Z}/\ell) \to K_n(R/I;\mathbb{Z}/\ell)$ being onto.

IV.2.10.1 **Example 2.10.1.** If $1/\ell \in R$ then $K_n(R[[x]];\mathbb{Z}/\ell) \cong K_n(R;\mathbb{Z}/\ell)$ for all $n \geq 0$.

IV.2.10.2 **Example 2.10.2.** A restriction like $n \geq 0$ is necessary. Les Reid [157] has given an example of a 2-dimensional hensel local $\mathbb{Q}$-algebra with $K_{-2}(R) = \mathbb{Z}$, and Drinfeld [49] has shown that $K_{-1}(I) = 0$.

EXERCISES

EIV.2.1 **2.1.** Suppose that X is a loop space. Show that $\pi_1(X;\mathbb{Z}/\ell)$ is independent of the choice of Y such that $X \simeq \Omega Y$. This shows that $K_1(R;\mathbb{Z}/\ell)$ and even $K_1(\mathcal{C};\mathbb{Z}/\ell)$ are well defined.

EIV.2.2 **2.2.** Show that the group $K_1(R;\mathbb{Z}/\ell)$ defined in 2.4 is isomorphic to the group defined in III.1.7.4. Using the Fundamental Theorem III.3.7 (and III.4.1.2), show that $K_0(R;\mathbb{Z}/\ell)$ and even the groups $K_n(R;\mathbb{Z}/\ell)$ for $n < 0$ which are defined in 2.4 are isomorphic to the corresponding groups defined in Ex. III.4.6.

EIV.2.3 **2.3.** Let R be a Dedekind domain with fraction field F. Show that the kernel of the map $K_1(R;\mathbb{Z}/\ell) \to K_1(F;\mathbb{Z}/\ell)$ is $SK_1(R)/\ell$. Hence it induces a natural map

$$_\ell\operatorname{Pic}(R) \xrightarrow{\rho} F^\times/F^{\times\ell}R^\times.$$

Note that $F^\times/R^\times$ is a free abelian group by I.3.6, so the target is a free $\mathbb{Z}/\ell$-module for every integer ℓ. Finally, use I.3.6 and I.3.8.1 to give an elementary description of ρ.

In particular, If R is the ring of integers in a number field F, the Bass-Milnor-Serre Theorem III.2.5 shows that the extension $K_1(R;\mathbb{Z}/\ell)$ of $_\ell\mathrm{Pic}(R)$ by $R^\times/R^{\times\ell}$ injects into $F^\times/F^{\times\ell}$, and that $_\ell K_0(R)$ is not a natural summand of $K_1(R;\mathbb{Z}/\ell)$. (If $1/\ell \in R$, the étale Chern class $K_1(R;\mathbb{Z}/\ell) \to H^1_{\text{et}}(\operatorname{Spec}(R),\mu_\ell)$ of V.11.10 is an isomorphism.)

EIV.2.4 **2.4.** If $n \geq 2$, there is a Hurewicz map $\pi_n(X;\mathbb{Z}/\ell) \to H_n(X;\mathbb{Z}/\ell)$ sending the class of a map $f : P^n \to X$ to $f_*[e]$, where $[e] \in H_n(P^n;\mathbb{Z}/\ell) \cong H_n(S^n;\mathbb{Z}/\ell)$ is the canonical generator. Its restriction to $\pi_n(X)/\ell$ is the reduction modulo ℓ of the usual Hurewicz homomorphism $\pi_n(X) \to H_n(X;\mathbb{Z})$.

(a) If $n \geq 3$, show that the Hurewicz map is a homomorphism. (If $n = 2$ and ℓ is odd, it is also a homomorphism.) *Hint:* Since P^n is a suspension, there is a comultiplication map $P^n \to P^n \vee P^n$.

If $n = 2$ and ℓ is even, the Hurewicz map h may not be a homomorphism, even if X is an infinite loop space. The precise formula is: $h(a + b) = h(a) + h(b) + (\ell/2)\{\partial a, \partial b\}$. (See [222].)

(b) In Example IV.2.5.1 2.5.1, show that the Hurewicz map from $K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$ to $H_2(SL(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2$ is nonzero on β and $2\beta = \{-1,-1\}$, but zero on 3β.

(c) If $n = 2$, show that the Hurewicz map is compatible with the action of $\pi_2(X)$ on $\pi_2(X;\mathbb{Z}/\ell)$ and on $H_2(X;\mathbb{Z}/\ell)$.

EIV.2.5 **2.5.** Show that $K_*(R;\mathbb{Z}/\ell^\nu)$ is a graded module over $K_*(R)$, associated to the evident pairing $\mathbf{K}(R) \wedge \mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu) \to \mathbf{K}(R) \wedge P^\infty(\mathbb{Z}/\ell^\nu)$.

EIV.2.6 **2.6.** Fix a prime ℓ and let $\mathbb{Z}/\ell^\infty$ denote the union of the groups $\mathbb{Z}/\ell^\nu$, which is a divisible torsion group. Show that there is a space $P^m(\mathbb{Z}/\ell^\infty) = \varinjlim P^m(\mathbb{Z}/\ell^\nu)$ such that $\pi_m(X;\mathbb{Z}/\ell^\infty) = [P^m(\mathbb{Z}/\ell^\infty), X]$ is the direct limit of the $\pi_m(X;\mathbb{Z}/\ell^\nu)$. Then show that there is a universal coefficient sequence for $m \geq 3$:

$$0 \to (\pi_m X) \otimes \mathbb{Z}/\ell^\infty \to \pi_m(X;\mathbb{Z}/\ell^\infty) \xrightarrow{\partial} (\pi_{m-1}X)_{\ell\text{-tors}} \to 0.$$

3 Geometric realization of a small category

Recall (II.6.1.3) that a *small* category is a category whose objects form a set. If C is a small category, its *geometric realization* BC is a CW complex constructed naturally out of C. By definition, BC is the geometric realization $|NC|$ of the nerve NC of C; see 3.1.4 below. However, it is characterized in a simple way.

IV.3.1 **Characterization 3.1.** The realization BC of a small category C is the CW complex uniquely characterized up to homeomorphism by the following properties. Let $\mathbf{n}$ denote the category with n objects $\{0, 1, \cdots, n-1\}$, with exactly one morphism $i \to j$ for each $i \le j$; $\mathbf{n}$ is an ordered set, regarded as a category.

(1) *Naturality.* A functor $F : C \to D$ induces a cellular map $BF : BC \to BD$, $BF \circ BG = B(FG)$ and $B(\mathrm{id}_C)$ is the identity map on BC.

(2) $B\mathbf{n}$ is the standard $(n-1)$–simplex Δ^{n-1}. The functor $\phi : \mathbf{i} \to \mathbf{n}$ induces the simplicial map $\Delta^{i-1} \to \Delta^{n-1}$ sending vertex j to vertex $\phi(j)$.

(3) BC is the colimit $\mathrm{colim}_\Phi\, B\mathbf{i}$, where Φ is the category whose objects are functors $\mathbf{n} \to C$, and whose morphisms are factorizations $\mathbf{i} \to \mathbf{n} \to C$. The corresponding map $B\mathbf{i} \to B\mathbf{n}$ is given by (2).

The following useful properties are consequences of this characterization:

(4) If C is a subcategory of D, BC is a subcomplex of BD;

(5) If C is the coproduct of categories C_α, $BC = \coprod BC_\alpha$;

(6) $B(C \times D)$ is homeomorphic to $(BC) \times (BD)$, where the product is given the compactly generated topology;

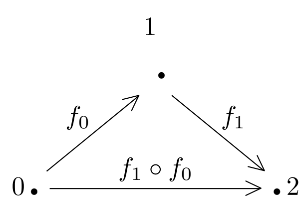
Here are some useful special cases of (2) for small n:

$B\mathbf{0} = \emptyset$ is the empty set, because $\mathbf{0}$ is the empty category.

$B\mathbf{1} = \{0\}$ is a one-point space, since $\mathbf{1}$ is the one object-one morphism category.

$B\mathbf{2} = [0, 1]$ is the unit interval, whose picture is: $0\bullet \longrightarrow \bullet 1$.

$B\mathbf{3}$ is the 2-simplex; the picture of this identification is:



The small categories form the objects of a category CAT, whose morphisms are functors. By (1), we see that geometric realization is a functor from CAT to the category of CW complexes and cellular maps.

IV.3.1.1 **Recipe 3.1.1.** The above characterization of the CW complex BC gives it the following explicit cellular decomposition. The 0-cells (vertices) are the objects of C. The 1-cells (edges) are the morphisms in C, excluding all identity morphisms, and they are attached to their source and target. For each pair (f, g) of composable maps in C, attach a 2-simplex, using the above picture of $B\mathbf{3}$ as the model. (Ignore pairs (f, g) where either f or g is an identity.) Inductively, given an n-tuple of composable maps in C (none an identity map), $c_0 \to c_1 \to \cdots \to c_n$, attach an n-simplex, using $B(\mathbf{n+1})$ as the model. By (3), BC is the union of these spaces, equipped with the weak topology.

Notice that this recipe implies a canonical cellular homeomorphism between BC and the realization BC^{op} of the opposite category C^{op}. In effect, the recipe doesn't notice which way the arrows run.

IV.3.1.2 **Example 3.1.2.** Let C_2 be the category with one object and two morphisms, 1 and σ, with $\sigma^2 = 1$. The recipe tells us that BC_2 has exactly one n-cell for each n, attached to the $(n-1)$-cell by a map of degree 2 (corresponding to the first and last faces of the n-simplex). Therefore the n-skeleton of BC_2 is the projective n-space $\mathbb{RP}^n$, and their union BC_2 is the infinite projective space $\mathbb{RP}^\infty$.

IV.3.1.3 **Example 3.1.3.** Any group G (or monoid) may be regarded as a category with one object. The realization BG of this category is the space studied in Section 1. The recipe 3.1.1 shows that BG has only one vertex, and one 1-cell for every nontrivial element of G.

Although the above recipe gives an explicit description of the cell decomposition of BC, it is a bit vague about the attaching maps. To be more precise, we shall assume that the reader has a slight familiarity with the basic notions in the theory of simplicial sets, as found for example in [223] or [118]. A simplicial set X is a contravariant functor $\mathbf{\Delta} \to \mathbf{Sets}$, where $\mathbf{\Delta}$ denotes the subcategory of ordered sets on the objects $\{\mathbf{0}, \mathbf{1}, ..., \mathbf{n}, ...\}$. Alternatively, it is a sequence of sets $X_0, X_1, \dots$, together with "face" maps $\partial_i : X_n \to X_{n-1}$ and "degeneracy maps" $\sigma_i : X_n \to X_{n+1}$ $(0 \le i \le n)$, subject to certain identities for the compositions of these maps.

We may break down the recipe for BC into two steps: we first construct a simplicial set NC, called the nerve of the category C, and then set $BC = |NC|$.

IV.3.1.4 **Definition 3.1.4** (The nerve of C)**.** The *nerve* NC of a small category C is the simplicial set defined by the following data. Its n-simplices are functors $c\colon \mathbf{n+1} \to C$, *i.e.,* diagrams in C of the form

$$c_0 \to c_1 \to \cdots \to c_n.$$

The i^{th} face $\partial_i(c)$ of this simplex is obtained by deleting c_i in the evident way; to get the i^{th} degeneracy $\sigma_i(c)$, one replaces c_i by $c_i \xrightarrow{=} c_i$.

The geometric realization $|X_\bullet|$ of a simplicial set $X_\bullet$ is defined to be the CW complex obtained by following the recipe 3.1.1 above, attaching an n-cell for each nondegenerate n-simplex x, identifying the boundary faces of the simplex with the $(n-1)$-simplices indexed by the $\partial_i x$. See [223, 8.1.6] or [118, §14] for more details.

BC is defined as the geometric realization $|NC|$ of the nerve of C. From this prescription, it is clear that BC is given by recipe 3.1.1 above.

By abuse of notation, we will say that a category is contractible, or connected, or has any other topological property if its geometric realization has that property. Similarly, we will say that a functor $F\colon C \to D$ is a homotopy equivalence if BF is a homotopy equivalence $BC \simeq BD$.

IV.3.2 **Homotopy-theoretic properties 3.2.** A natural transformation $\eta\colon F_0 \Rightarrow F_1$ between two functors $F_i\colon C \to D$ gives a homotopy $BC \times [0,1] \to BD$ between the maps BF_0 and BF_1. This follows from (4) and (6) of 3.1, because η may be viewed as a functor from $C \times \mathbf{2}$ to D whose restriction to $C \times \{i\}$ is F_i.

As a consequence, any adjoint pair of functors $L\colon C \to D$, $R\colon D \to C$ induces a homotopy equivalence between BC and BD, because there are natural transformations $LR \Rightarrow id_D$ and $id_C \Rightarrow RL$.

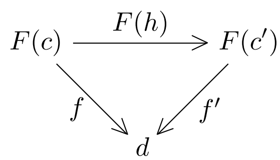
IV.3.2.1 **Example 3.2.1** (Smallness). Any equivalence $C_0 \xrightarrow{f} C$ between small categories induces a homotopy equivalence $BC_0 \xrightarrow{\sim} BC$, because F has an adjoint.

In practice, we will often work with a category C, such as $\mathbf{P}(R)$ or $\mathbf{M}(R)$, which is not actually a small category, but which is *skeletally small* (II.6.1.3). This means that C is equivalent to a small category, say to C_0. In this case, we can use BC_0 instead of the mythical BC, because any other choice for C_0 will have a homotopy equivalent geometric realization. We shall usually overlook this fine set-theoretic point in practice, just as we did in defining K_0 in Chapter II.

IV.3.2.2 **Example 3.2.2** (Initial objects). Any category with an initial object is contractible, because then the natural functor $C \to \mathbf{1}$ has a left adjoint. Similarly, any category with a terminal object is contractible.

For example, suppose given an object d of a category C. The *comma category* C/d of *objects over* d has as its objects the morphisms $f\colon c \to d$ in C with target d. A morphism in the comma category from f to $f'\colon c' \to d$ is a morphism $h\colon c \to c'$ so that $f = f'h$. The comma category C/d is contractible because it has a terminal object, namely the identity map $\mathrm{id}_d\colon d \xrightarrow{=} d$. The dual comma category $d\backslash C$ with objects $d \to c$ is similar, and left to the reader.

IV.3.2.3 **Definition 3.2.3** (Comma categories). Suppose given a functor $F\colon C \to D$ and an object d of D. The comma category F/d has as its objects all pairs (c, f) with c an object in C and f a morphism in D from $F(c)$ to d. By abuse of notation, we shall write such objects as $F(c) \xrightarrow{f} d$. A morphism in F/d from this object to $F(c') \xrightarrow{f'} d$ is a morphism $h\colon c \to c'$ in C so that the following diagram commutes in D.



There is a canonical forgetful functor $j\colon F/d \to C$, $j(c,f) = c$, and there is a natural transformation $\eta_{(c,f)} = f$ from the composite $F \circ j\colon F/d \to D$ to the constant functor with image d. So $B(F \circ j)$ is a contractible map. It follows that there is a natural continuous map from $B(F/d)$ to the homotopy fiber of $BC \to BD$.

There is a dual comma category $d\backslash F$, whose objects are written as $d \to F(c)$, and morphisms are morphisms $h\colon c \to c'$ in C. It also has a forgetful functor to C, and a map from $B(d\backslash F)$ to the homotopy fiber of $BC \to BD$. In fact, $d\backslash F = (d/F^{op})^{op}$.

In the same spirit, we can define comma categories F/D (resp., $D\backslash F$); an object is just an object of F/d (resp., of $d\backslash F$) for some d in D. A morphism in F/D from $(c, F(c) \to d)$ to $(c', F(c') \to d')$ is a pair of morphisms $c \to c'$, $d \to d'$ so that the two maps $F(c) \to d'$ agree; there is an evident forgetful functor $F/D \to C \times D$. A morphism in $D\backslash F$ from $(c, d \to F(c))$ to $(c', d' \to F(c'))$ is a pair of morphisms $c \to c'$, $d' \to d$ so that the two maps $d' \to F(c')$ agree; there is an evident forgetful functor $D\backslash F \to D^{op} \times C$.

The set π_0 of components of a category

The set $\pi_0(X)$ of connected components of any CW complex X can be described as the set of vertices modulo the incidence relation of edges. For BC this takes the following form. Let $\text{obj}(C)$ denote the set of objects of C, and write $\pi_0(C)$ for $\pi_0(BC)$.

IV.3.3 **Lemma 3.3.** *Let $\sim$ be the equivalence relation on obj(C) which is generated by the relation that $c \sim c'$ if there is a morphism in C between c and c'. Then*

$$\pi_0(C) = obj(C)/\sim .$$

IV.3.3.1 **Translation categories 3.3.1.** Suppose that G is a group, or even a monoid, acting on a set X. The *translation category* $G\!\int\!X$ is defined as the category whose objects are the elements of X, with $\text{Hom}(x,x') = \{g \in G | g \cdot x = x'\}$. By Lemma 3.3, $\pi_0(G\!\int\!X)$ is the orbit space X/G. The components of $G\!\int\!X$ are described in Ex. 3.2.

Thinking of a G-set X as a functor $G \to CAT$, the translation category becomes a special case of the following construction, due to Grothendieck.

IV.3.3.2 **Example 3.3.2.** Let I be a small category. Given a functor $X\colon I \to$ **Sets**, let $I\!\int\!X$ denote the category of pairs (i,x) with i an object of I and $x \in X(i)$, in which a morphism $(i,x) \to (i',x')$ is a morphism $f\colon i \to i'$ in I with $X(f)(x) = x'$. By Lemma 3.3 we have $\pi_0(I\int X) = \operatorname{colim}_{i\in I}\ X(i)$.

More generally, given a functor $X\colon I \to CAT$, let $I\int X$ denote the category of pairs (i,x) with i an object of I and x an object of $X(i)$, in which a morphism $(f,\phi)\colon (i,x) \to (i',x')$ is given by a morphism $f\colon i \to i'$ in I and a morphism $\phi\colon X(f)(x) \to x'$ in $X(i')$. Using Lemma 3.3, it is not hard to show that $\pi_0(I\int X) = \operatorname{colim}_{i\in I}\ \pi_0 X(i)$.

For example, if $F : C \to D$ is a functor then $d \mapsto F/d$ is a functor on D, and $D\int(F/-)$ is F/D, while $d \mapsto d\backslash F$ is a functor on D^{op}, and $D\int(-\backslash F)$ is $D^{op}\backslash F$.

The fundamental group π_1 of a category

Suppose that T is a set of morphisms in a category C. The *graph of* T is the 1-dimensional subcomplex of BC consisting of the edges corresponding to T and their incident vertices. We say that T is a *tree* in C if its graph is contractible (*i.e.,* a tree in the sense of graph theory). If C is connected then a tree T is maximal (a *maximal tree*) just in case every object of C is either the source or target of a morphism in T. By Zorn's Lemma, maximal trees exist when $C \neq \emptyset$.

Classically, the fundamental group $\pi_1(\Gamma)$ of the 1-skeleton Γ of BC is a free group on symbols $[f]$, one for every non-identity morphism f in C not in T. (The loop is the composite of f with the unique paths in the tree between the basepoint and the source and target of f.) The following well known formula for the fundamental group of BC is a straight-forward application of Van Kampen's theorem.

IV.3.4 **Lemma 3.4.** *Suppose that T is a maximal tree in a small connected category C. Then the group $\pi_1(BC)$ has the following presentation: it is generated by symbols $[f]$, one for every morphism in C, modulo the relations that*

(1) $[t] = 1$ for every $t \in T$, and $[id_c] = 1$ for the identity morphism id_c of each object c.

(2) $[f]\cdot[g] = [f \circ g]$ for every pair (f,g) of composable morphisms in C.

This presentation does not depend upon the choice of the object c_0 of C chosen as the basepoint. Geometrically, the class of $f : c_1 \to c_2$ is represented by the unique path in T from c_0 to c_1, followed by the edge f, followed by the unique path in T from c_2 back to c_0.

IV.3.4.1 **Application 3.4.1** (Groups)**.** Let G be a group, considered as a category with one object. Since BG has only one vertex, BG is connected. By Lemma 3.4 (with T empty) we see that $\pi_1(BG) = G$. In fact, $\pi_i(BG) = 0$ for all $i \geq 2$. (See Ex. 3.2.) BG is often called the *classifying space* of the group G, for reasons discussed in Examples 3.9.2 and 3.9.3 below.

IV.3.4.2 **Application 3.4.2** (Monoids)**.** If M is a monoid then BM has only one vertex. This time, Lemma 3.4 shows that the group $\pi = \pi_1(BM)$ is the group completion (Ex. II.1.1) of the monoid M.

For our purposes, one important thing about BG is that its homology is the same as the ordinary Eilenberg-Mac Lane homology of the group G (see [WHomo 223, 6.10.5 or 8.2.3]). In fact, if M is any G-module then we may consider M as a local coefficient system on BG (see IV.3.5.1 3.5.1). The cellular chain complex used to form the homology of BG with coefficients in M is the same as the canonical chain complex used to compute the homology of G, so we have $H_*(BG;M) = H_*(G;M)$. As a special case, we have $H_1(BG;\mathbb{Z}) = H_1(G;\mathbb{Z}) = G/[G,G]$, where $[G,G]$ denotes the commutator subgroup of G, *i.e.,* the subgroup of G generated by all commutators $[g,h] = ghg^{-1}h^{-1}$ $(g,h \in G)$.

IV.3.5 **The homology of C and BC 3.5.** The i^{th} homology of a CW complex X such as BC is given by the homology of the *cellular chain complex* $C_*(X)$. By definition, $C_n(X)$ is the free abelian group on the n-cells of X. If e is an $n+1$-cell and f is an n-cell, then the coefficient of $[f]$ in the boundary of $[e]$ is the degree of the map $S^n \xrightarrow{\varepsilon} X^{(n)} \xrightarrow{f} S^n$, where ε is the attaching map of e and the second map is the projection from $X^{(n)}$ (the n-skeleton of X) onto S^n given by the n-cell f.

For example, $H_*(BC;\mathbb{Z})$ is the homology of the unreduced cellular chain complex $C_*(BC)$, which in degree n is the free abelian group on the set of all n-tuples $(f_1,...,f_n)$ of composable morphisms in C, composable in the order $c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n$. The boundary map in this complex sends the generator $(f_1,...,f_n)$ to the alternating sum obtained by succesively deleting the c_i in the evident way:

$$(f_2,...,f_n) - (f_2f_1, f_3,...,f_n) + \cdots \pm (..., f_{i+1}f_i, ...) \mp \cdots \pm (..., f_nf_{n-1}) \mp (..., f_{n-1}).$$

More generally, for each functor $M\colon C \to \mathbf{Ab}$ we let $H_i(C;M)$ denote the i^{th} homology of the chain complex

$$\cdots \to \coprod_{c_0\to\cdots\to c_n} M(c_0) \to \cdots \to \coprod_{c_0\to c_1} M(c_0) \to \coprod_{c_0} M(c_0).$$

The final boundary map sends the copy of $M(c_0)$ indexed by $c_0 \xrightarrow{f} c_1$ to $M(c_0) \oplus M(c_1)$ by $x \mapsto (-x, fx)$. The cokernel of this map is the usual description for the colimit of the functor M, so $H_0(C;M) = \operatorname{colim}_{c\in C} M(c)$.

IV.3.5.1 **Definition 3.5.1** (Local coefficients)**.** A functor $C \to \mathbf{Sets}$ is said to be *morphism-inverting* if it carries all morphisms of C into isomorphisms. By Ex. EIV.3.1 3.1, morphism-inverting functors are in 1–1 correspondence with covering spaces of BC. Therefore the morphism-inverting functors $M\colon C \to \mathbf{Ab}$ are in 1–1 correspondence with local coefficient systems on the topological space BC. In this case, the groups $H_i(C;M)$ are canonically isomorphic to $H_i(BC;M)$, the topologist's homology groups of BC with local coefficients M. The isomorphism is given in [Wh 228, VI.4.8].

Bisimplicial Sets

A *bisimplicial set* X is a contravariant functor $\mathbf{\Delta}\times\mathbf{\Delta}\to\mathbf{Sets}$, where $\mathbf{\Delta}$ is the subcategory of ordered sets on the objects $\{\mathbf{0},\mathbf{1},...,\mathbf{n},...\}$. Alternatively, it is a doubly indexed family $X_{p,q}$ of sets, together with "horizontal" face and degeneracy maps ($\partial_i^h : X_{p,q}\to X_{p-1,q}$ and $\sigma_i^h : X_{p,q}\to X_{p+1,q}$) and "vertical" face and degeneracy maps ($\partial_i^v : X_{p,q}\to X_{p,q-1}$ and $\sigma_i^v : X_{p,q}\to X_{p,q+1}$), satisfying the horizontal and vertical simplicial identities and such that horizontal maps commute with vertical maps. In particular, each $X_{p,\bullet}$ and $X_{\bullet,q}$ is a simplicial set.

IV.3.6 **Definition 3.6.** The *geometric realization* BX of a bisimplicial set X is obtained by taking one copy of the product $\Delta^p\times\Delta^q$ for each element of $X_{p,q}$, inductively identifying its horizontal and vertical faces with the appropriate $\Delta^{p-1}\times\Delta^q$ or $\Delta^p\times\Delta^{q-1}$, and collapsing horizontal and vertical degeneracies. This construction is sometimes described as a coend: $BX=\int_{p,q}X_{p,q}\times\Delta^p\times\Delta^q$.

There is a diagonalization functor diag from bisimplicial sets to simplicial sets ($\operatorname{diag}(X)_p=X_{p,p}$), and it is well known (see [33, B.1]) that BX is homeomorphic to $B\operatorname{diag}(X)$. The following theorem is also well known; see [214, p. 164–5] or [153, p. 98] for example.

IV.3.6.1 **Theorem 3.6.1.** *Let $f:X\to Y$ be a map of bisimplicial sets.*
(i) If each simplicial map $X_{p,\bullet}\to Y_{p,\bullet}$ is a homotopy equivalence, so is $BX\to BY$.
(ii) If Y is the nerve of a category I (constant in the second simplicial coordinate), and $f^{-1}(i,\bullet)\to f^{-1}(j,\bullet)$ is a homotopy equivalence for every $i\to j$ in I, then each $B(f^{-1}(i,\bullet))\to BX\to B(I)$ is a homotopy fibration sequence.

IV.3.6.2 **Example 3.6.2.** (Quillen) If $F:C\to D$ is a functor, the canonical functor $D\backslash F\to C$ is a homotopy equivalence, where $D\backslash F$ is the comma category of Example 3.2.3. To see this, let X denote the bisimplicial set such that $X_{p.q}$ is the set of all pairs of sequences

$$(d_q\to\cdots\to d_0\to F(c_0),\ c_0\to\cdots\to c_p);$$

the horizontal and vertical faces come from the nerves of C and D. Consider the projection of X onto the nerve of C. Since NC_p is the discrete set of all sequences $c_0\to\cdots\to c_p$, the inverse image of this sequence is isomorphic to the nerve of $D\backslash F(c_0)$, and $D\backslash F(c_0)$ is contractible since it has a terminal object. Theorem 3.6.1 applies to yield $BX\xrightarrow{\sim}BC$. The simplicial set $\operatorname{diag}(X)$ is the nerve of $D\backslash F$, and the composition $B(D\backslash F)\xrightarrow{\sim}BX\xrightarrow{\sim}BC$ is the canonical map, whence the result.

Homotopy Fibers of Functors

If $F:C\to D$ is a functor, it is useful to study the realization map $BF\colon BC\to BD$ in terms of homotopy groups, and for this we want a category-theoretic interpretation of the homotopy fiber (1.2). The naïve approximations to the homotopy fiber are the realization of the comma categories F/d and its dual

$d\backslash F$. Indeed, we saw in 3.2.3 that there are continuous maps from both $B(F/d)$ and $B(d\backslash F)$ to the homotopy fiber.

Here is the fundamental theorem used to prove that two categories are homotopy equivalent. It was proven by Quillen in [153]. Note that it has a dual formulation, replacing $d\backslash F$ by F/d, because $BD \simeq BD^{op}$.

IV.3.7 **Theorem 3.7** (Quillen's Theorem A)**.** *Let $F\colon C \to D$ be a functor such that $d\backslash F$ is contractible for every d in D. Then $BF\colon BC \xrightarrow{\simeq} BD$ is a homotopy equivalence.*

Proof. Consider the comma category $D\backslash F$ of 3.2.3, which is equipped with functors $C \longleftarrow D\backslash F \longrightarrow D^{op}$ such that $BC \longleftarrow B(D\backslash F)$ is a homotopy equivalence (by 3.6.2). The functor $D\backslash F \to D\backslash D$, sending $(d \to F(c), c)$ to $(d \to F(c), F(c))$, fits into a commutative diagram of categories

$$\begin{array}{ccccc} C & \xleftarrow{\simeq} & D\backslash F & \longrightarrow & D^{op} \\ {\scriptstyle F}\downarrow & & \downarrow & & \| \\ D & \xleftarrow{\simeq} & D\backslash D & \longrightarrow & D^{op} \end{array}$$

Therefore it suffices to show that $B(D\backslash F) \to BD^{op}$ is a homotopy equivalence. This map factors as $B(D\backslash F) \simeq BX \xrightarrow{\pi} BD^{op}$, where X is the bisimplicial set of Example 3.6.2 and π is the projection. Consider the simplicial map $\pi_{\bullet,q}$ from $X_{\bullet,q}$ to the qth component of the nerve of D^{op}, which is the discrete set of all sequences $d_q \to \cdots \to d_0$ in D. For each such sequence, the inverse image in $X_{*,q}$ is the nerve of $d_0\backslash F$, which is assumed to be contractible. By Theorem 3.6.1, $B(D\backslash F) \simeq BX \to BD$ is a homotopy equivalence, as required. □

IV.3.7.1 **Example 3.7.1.** If $F\colon C \to D$ has a left adjoint L, then $d\backslash F$ is isomorphic to the comma category $L(d)\backslash C$, which is contractible by Example 3.2.2. In this case, Quillen's Theorem A recovers the observation in 3.2 that C and D are homotopy equivalent.

IV.3.7.2 **Example 3.7.2.** Consider the inclusion of monoids $i : \mathbb{N} \hookrightarrow \mathbb{Z}$ as a functor between categories with one object $*$. Then $*\backslash i$ is isomorphic to the translation category $\mathbb{N}\int\mathbb{Z}$, which is contractible (why?). Quillen's Theorem A shows that $B\mathbb{N} \simeq B\mathbb{Z} \simeq S^1$.

The *inverse image* $F^{-1}(d)$ of an object d is the subcategory of C consisting of all objects c with $F(c) = d$, and all morphisms h in C mapping to the identity of d. It is isomorphic to the full subcategory of F/d consisting of pairs $(c, F(c) \xrightarrow{=} d)$, and also to the full subcategory of pairs $(d \xrightarrow{=} F(c), c)$ of $d\backslash F$. It will usually not be homotopy equivalent to either F/d or $d\backslash F$.

One way to ensure that $F^{-1}(d)$ is homotopic to a comma category is to assume that F is either pre-fibered or pre-cofibered in the following sense.

IV.3.7.3 **Fibered and Cofibered functors 3.7.3.** (Cf. [SGA1, Exp. VI]) We say that a functor $F\colon C \to D$ is *pre-fibered* if for every d in D the inclusion $F^{-1}(d) \hookrightarrow d\backslash F$ has a right adjoint. This implies that $BF^{-1}(d) \simeq B(d\backslash F)$, and the *base change* functor $f^*\colon F^{-1}(d') \to F^{-1}(d)$ associated to a morphism $f\colon d \to d'$ in D is defined as the composite $F^{-1}(d') \hookrightarrow (d\backslash F) \to F^{-1}(d)$. F is called *fibered* if it is pre-fibered and $g^*f^* = (fg)^*$ for every pair of composable maps f, g, so that F^{-1} gives a contravariant functor from D to CAT.

Dually, we say that F is *pre-cofibered* if for every d the inclusion $F^{-1}(d) \hookrightarrow F/d$ has a left adjoint. In this case we have $BF^{-1}(d) \simeq B(F/d)$. The *cobase change* functor $f_*\colon F^{-1}(d) \to F^{-1}(d')$ associated to a morphism $f\colon d \to d'$ in D is defined as the composite $F^{-1}(d) \hookrightarrow (F/d') \to F^{-1}(d')$. F is called *cofibered* if it is pre-cofibered and $(fg)_* = f_*g_*$ for every pair of composable maps f, g, so that F^{-1} gives a covariant functor from D to CAT.

These notions allow us to state a variation on Quillen's Theorem A.

IV.3.7.4 **Corollary 3.7.4.** *Suppose that $F : C \to D$ is either pre-fibered or pre-cofibered, and that $F^{-1}(d)$ is contractible for each d in D. Then BF is a homotopy equivalence $BC \simeq BD$.*

IV.3.7.5 **Example 3.7.5.** Cofibered functors over D are in 1–1 correspondence with functors $D \to CAT$. We have already mentioned one direction: if $F\colon C \to D$ is cofibered, F^{-1} is a functor from D to CAT. Conversely, for each functor $X\colon D \to CAT$, the category $D\int X$ of Example 3.3.2 is cofibered over D by the forgetful functor $(d, x) \mapsto d$. It is easy to check that these are inverses: C is equivalent to $D\int F^{-1}$.

Here is the fundamental theorem used to construct homotopy fibration sequences of categories. It was originally proven in [153]. Note that it has a dual formulation, in which $d\backslash F$ is replaced by F/d; see Ex. 3.6.

IV.3.8 **Theorem 3.8** (Quillen's Theorem B)**.** *Let $F\colon C \to D$ be a functor such that for every morphism $d \to d'$ in D the induced functor $d'\backslash F \to d\backslash F$ is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$d\backslash F \xrightarrow{j} C \xrightarrow{F} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i B(d\backslash F) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

Proof. We consider the projection functor $X \xrightarrow{p} N(D^{op})$ of 3.6.2. Since $p^{-1}(d)$ is the nerve of $d\backslash F$, we may apply Theorem 3.6.1 to conclude that $B(d\backslash F) \to BX \to BD^{op}$ is a homotopy fibration sequence. Since $B(d\backslash F) \to B\,\mathrm{diag}(X) = B(D\backslash F) \xrightarrow{\simeq} BC$ is induced from $j : d\backslash F \to C$, the theorem follows from the diagram

$$\begin{array}{ccccc} d\backslash F & \longrightarrow & D\backslash F & \longrightarrow & D^{op} \\ \downarrow & & {\scriptstyle F}\downarrow & & \| \\ * \simeq d\backslash D & \longrightarrow & D\backslash D & \xrightarrow{\simeq} & D^{op}. \end{array} \qquad \square$$

IV.3.8.1 **Corollary 3.8.1.** *Suppose that F is pre-fibered, and for every $f : d \to d'$ in D the base change f^* is a homotopy equivalence. Then for each d in D the geometric realization of the sequence*

$$F^{-1}(d) \xrightarrow{j} C \xrightarrow{f} D$$

is a homotopy fibration sequence. Thus there is a long exact sequence

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \xrightarrow{j} \pi_i(BC) \xrightarrow{f} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

IV.3.9 **Topological categories 3.9.** If $C = C^{\text{top}}$ is a *topological* category (*i.e.,* the object and morphism sets form topological spaces), then the nerve of C^{top} is a simplicial topological space. Using the appropriate geometric realization of simplicial spaces, we can form the topological space $BC^{\text{top}} = |NC^{\text{top}}|$. It has the same underlying set as our previous realization BC^{δ} (the δ standing for "discrete," *i.e.,* no topology), but the topology of BC^{top} is more intricate. Since the identity may be viewed as a continuous functor $C^{\delta} \to C^{\text{top}}$ between topological categories, it induces a continuous map $BC^{\delta} \to BC^{\text{top}}$.

For example, any topological group $G = G^{\text{top}}$ is a topological category, so we need to distinguish between the two connected spaces BG^{δ} and BG^{top}. It is traditional to write BG for BG^{top}, reserving the notation BG^{δ} for the less structured space. As noted above, BG^{δ} has only one nonzero homotopy group: $\pi_1(BG^{\delta}) = G^{\delta}$. In contrast, the loop space $\Omega(BG^{\text{top}})$ is G^{top}, so $\pi_i BG^{\text{top}} = \pi_{i-1} G^{\text{top}}$ for $i > 0$.

IV.3.9.1 **Example 3.9.1.** Let $G = \mathbb{R}$ be the topological group of real numbers under addition. Then $B\mathbb{R}^{\text{top}}$ is contractible because $\mathbb{R}^{\text{top}}$ is, but $B\mathbb{R}^{\delta}$ is not contractible because $\pi_1(B\mathbb{R}^{\delta}) = \mathbb{R}$.

IV.3.9.2 **Example 3.9.2** (BU)**.** The unitary groups U_n are topological groups, and we see from I.4.10.1 that BU_n is homotopy equivalent to the infinite complex Grassmannian manifold G_n, which classifies n-dimensional complex vector bundles by Theorem I.4.10. The unitary group U_n is a deformation retract of the complex general linear group $GL_n(\mathbb{C})^{\text{top}}$. Thus BU_n and $BGL_n(\mathbb{C})^{\text{top}}$ are homotopy equivalent spaces. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BU \simeq BGL(\mathbb{C})^{\text{top}}$.

By Theorem II.3.2, $KU(X) \cong [X, \mathbb{Z} \times BU]$ and $\widetilde{KU}(X) \cong [X, BU]$ for every compact space X. By Ex. II.3.11 we also have $KU^{-n}(X) \cong [X, \Omega^n(\mathbb{Z} \times BU)]$ for all $n \geq 0$. In particular, for the one-point space $*$ the groups $KU^{-n}(*) = \pi_n(\mathbb{Z} \times BU)$ are periodic of order 2: $\mathbb{Z}$ if n is even, 0 if not. This follows from the observation in II.3.2 that the homotopy groups of BU are periodic — except for $\pi_0(BU)$, which is zero as BU is connected.

A refinement of Bott periodicity states that $\Omega U \simeq \mathbb{Z} \times BU$. Since $\Omega(BU) \simeq U$, we have $\Omega^2(\mathbb{Z} \times BU) \simeq \Omega^2 BU \simeq \mathbb{Z} \times BU$ and $\Omega^2 U \simeq U$. This yields the periodicity formula: $KU^{-n}(X) = KU^{-n-2}(X)$.

IV.3.9.3 **Example 3.9.3.** (BO) The orthogonal group O_n is a deformation retract of the real general linear group $GL_n(\mathbb{R})^{\text{top}}$. Thus the spaces BO_n and $BGL_n(\mathbb{R})^{\text{top}}$ are homotopy equivalent, and we see from I.4.10.1 that they are also homotopy equivalent to the infinite real Grassmannian manifold G_n. In particular, they classify n-dimensional real vector bundles by Theorem I.4.10. Taking the limit as $n \to \infty$, we have a homotopy equivalence $BO \simeq BGL(\mathbb{R})^{\text{top}}$.

Bott periodicity states that the homotopy groups of BO are periodic of order 8 — except for $\pi_0(BO) = 0$, and that the homotopy groups of $\mathbb{Z} \times BO$ are actually periodic of order 8. These homotopy groups are tabulated in II.3.1.1. A refinement of Bott periodicity states that $\Omega^7 O \simeq \mathbb{Z} \times BO$. Since $\Omega(BO) \simeq O$, we have $\Omega^8(\mathbb{Z} \times BO) \simeq \Omega^8(BO) \simeq \mathbb{Z} \times BO$ and $\Omega^8 O \simeq O$.

By Definition II.3.5 and Ex. II.3.11, the (real) topological K-theory of a compact space X is given by the formula $KO^{-n}(X) = [X, \Omega^n(\mathbb{Z} \times BO)], \quad n \geq 0$. This yields the periodicity formula: $KO^{-n}(X) = KO^{-n-8}(X)$.

Bicategories

One construction that has proven useful in constructing spectra is the geometric realization of a bicategory. Just as we could have regarded a small category $\mathcal{A}$ as a special type of simplicial set, via its nerve 3.1.4 ($\mathcal{A}_0$ and $\mathcal{A}_1$ are the objects and morphisms, all the other sets $\mathcal{A}_n$ are pullbacks and $\partial_1 : \mathcal{A}_2 = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_1$ defines composition), we can do the same with small bicategories.

IV.3.10 **Definition 3.10.** A small *bicategory* $\mathcal{C}$ is a bisimplicial set such that every row $\mathcal{C}_{\bullet,q}$ and column $\mathcal{C}_{p,\bullet}$ is the nerve of a category. We refer to elements of $\mathcal{C}_{0,0}$, $\mathcal{C}_{1,0}$, $\mathcal{C}_{0,1}$ and $\mathcal{C}_{1,1}$ as the objects, *horizontal* and *vertical morphisms*, and *bimorphisms*. A *bifunctor* between bicategories is a morphism of the underlying bisimplicial sets.

IV.3.10.1 **Example 3.10.1.** If $\mathcal{A}$ and $\mathcal{B}$ are categories, we can form the product bicategory $\mathcal{A} \otimes \mathcal{B}$. Its objects (resp., bimorphisms) are ordered pairs of objects (resp., morphisms) from $\mathcal{A}$ and $\mathcal{B}$. Its (p,q)-morphisms are pairs of functors $\mathbf{p+1} \to \mathcal{A}$, $\mathbf{q+1} \to \mathcal{B}$.

It is easy to see that $\text{diag}(\mathcal{A} \otimes \mathcal{B})$ is the product category $\mathcal{A} \times \mathcal{B}$, and that $B(\mathcal{A} \otimes \mathcal{B})$ is $B\mathcal{A} \times B\mathcal{B}$. In particular $B((\mathbf{p+1}) \times (\mathbf{q+1})) = \Delta^p \times \Delta^q$.

Bicategory terminology arose (in the 1960's) from the following paradigm.

IV.3.10.2 **Example 3.10.2.** For any category $\mathcal{B}$, bi$\mathcal{B}$ is the bicategory whose degree (p,q) part consists of commutative diagrams arising from functors $\mathbf{p+1} \times \mathbf{q+1} \to \mathcal{B}$. In particular, bimorphisms are commutative squares in $\mathcal{B}$; the horizontal and vertical edges of such a square are its associated horizontal and vertical morphisms. If $\mathcal{A}$ is a subcategory, $\mathcal{AB}$ is the sub-bicategory of bi$\mathcal{B}$ whose vertical maps are in $\mathcal{A}$.

We may also regard the small category $\mathcal{B}$ as a bicategory which is constant in the vertical direction ($\mathcal{B}_{p,q} = N\mathcal{B}_p$); this does not affect the homotopy type $B\mathcal{B}$ since $\operatorname{diag}\mathcal{B}$ recovers the category $\mathcal{B}$. The natural inclusion into $\operatorname{bi}\mathcal{B}$ is a homotopy equivalence by Ex. 3.13. It follows that any bifunctor $\mathcal{A}\otimes\mathcal{B}\to \operatorname{bi}\mathcal{C}$ induces a continuous map

$$B\mathcal{A}\times B\mathcal{B}\to B\operatorname{bi}\mathcal{C}\simeq B\mathcal{C}.$$

EXERCISES

EIV.3.1 **3.1.** *Covering spaces.* If $X: I\to \mathbf{Sets}$ is a morphism-inverting functor (3.5.1), use the recipe 3.1.1 to show that the forgetful functor $I\int X\to I$ of Example 3.3.2 makes $B(I\int X)$ into a covering space of BI with fiber $X(i)$ over each vertex i of BI.

Conversely, if $E\xrightarrow{\pi} BI$ is a covering space, show that $X(i)=\pi^{-1}(i)$ defines a morphism-inverting functor on I, where i is considered as a 0-cell of BI. Conclude that these constructions give a 1–1 correspondence between covering spaces of BI and morphism-inverting functors. (See [153, p. 90].)

EIV.3.2 **3.2.** *Translation categories.* Suppose that a group G acts on a set X, and form the translation category $G\int X$. Show that $B(G\int X)$ is homotopy equivalent to the disjoint union of the classifying spaces BG_x of the stabilizer subgroups G_x, one space for each orbit in X. For example, if X is the coset space G/H then $B(G\int X)\simeq BH$.

In particular, if $X=G$ is given the G-set structure $g\cdot g' = gg'$, this shows that $B(G\int G)$ is contractible, *i.e.,* the universal covering space of BG. Use this to calculate the homotopy groups of BG, as described in Example 3.4.1.

EIV.3.3 **3.3.** Let H be a subgroup of G, and $\iota\colon H\hookrightarrow G$ the inclusion as a subcategory.

(a) Show that $\iota/*$ is the category $H\int G$ of Ex. 3.1. Conclude that the homotopy fiber of $BH\to BG$ is the discrete set G/H, while $B\iota^{-1}(*)$ is a point.

(b) Use Ex. 3.2 to give another proof of (a).

EIV.3.4 **3.4.** If C is a filtering category [223, 2.6.13], show that BC is contractible. *Hint:* It suffices to show that all homotopy groups are trivial (see [228, V.3.5]). Any map from a sphere into a CW complex lands in a finite subcomplex, and every finite subcomplex of BC lands in the realization BD of a finite subcategory D of C; D lies in another subcategory D' of C which has a terminal object.

EIV.3.5 **3.5.** *Mapping telescopes.* If $\cup\mathbf{n}$ denotes the union of the categories $\mathbf{n}$ of (3.1), then a functor $\cup\mathbf{n}\xrightarrow{C} CAT$ is just a sequence $C_0\to C_1\to C_2\to\cdots$ of categories. Show that the geometric realization of the category $L=(\cup\mathbf{n})\int C$ of Example 3.3.2 is homotopy equivalent to BC, where C is the colimit of the C_n. In particular, this shows that $BL\simeq\lim_{n\to\infty}BC_n$. *Hint:* $C_n\simeq\mathbf{n}\int C$.

EIV.3.6 **3.6.** Suppose that $F\colon C \to D$ is pre-cofibered (Definition 3.7.3).

(a) Show that $F^{op} : C^{op} \to D^{op}$ is pre-fibered. If F is cofibered, F^{op} is fibered.

(b) Derive the dual formulation of Quillen's Theorem B, using F/d and F^{op}.

(c) If each cobase change functor f_* is a homotopy equivalence, show that the geometric realization of $F^{-1}(d) \longrightarrow C \xrightarrow{F} D$ is a homotopy fibration sequence for each d in D, and there is a long exact sequence:

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i BF^{-1}(d) \longrightarrow \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \cdots .$$

EIV.3.7 **3.7.** Let $F\colon C \to D$ be a cofibered functor (3.7.3). Construct a first quadrant double complex E^0 in which E^0_{pq} is the free abelian group on the pairs $(d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_q)$ of sequences of composable maps in C and D. By filtering the double complex by columns, show that the homology of the total complex Tot E^0 is $H_q(\text{Tot } E^0) \cong H_q(C;\mathbb{Z})$. Then show that the row filtration yields a spectral sequence converging to $H_*(C;\mathbb{Z})$ with $E^2_{pq} = H_p(D; H_qF^{-1})$, the homology of D with coefficients in the functor $d \mapsto H_q(F^{-1}(d);\mathbb{Z})$ described in 3.5.

EIV.3.8 **3.8.** A *lax functor* $\mathbf{M}\colon I \to CAT$ consists of functions assigning: (1) a category $\mathbf{M}(i)$ to each object i; (2) a functor $f_*\colon \mathbf{M}(i) \to \mathbf{M}(j)$ to every map $i \xrightarrow{f} j$ in I; (3) a natural transformation $(\mathrm{id}_i)_* \Rightarrow \mathrm{id}_{\mathbf{M}(i)}$ for each i; (4) a natural transformation $(fg)_* \Rightarrow f_*g_*$ for every pair of composable maps in I. This data is required to be "coherent" in the sense that the two transformations $(fgh)_* \Rightarrow f_*g_*h_*$ agree, and so do the various transformations $f_* \Rightarrow f_*$. For example, a functor is a lax functor in which (3) and (4) are identities.

Show that the definitions of objects and morphisms in Example 3.3.2 define a category $I\int\mathbf{M}$, where the map ϕ'' in the composition $(f'f, \phi'')$ of (f,ϕ) and (f',ϕ') is $(f'f)_*(x) \to f'_*f_*(x) \to f'_*(x') \to x''$. Show that the projection functor $\pi\colon I\int M \to I$ is pre-cofibered.

EIV.3.9 **3.9.** *Subdivision.* If $\mathcal{C}$ is a category, its *Segal subdivision* $Sub(\mathcal{C})$ is the category whose objects are the morphisms in $\mathcal{C}$; a morphism from $i : A \to B$ to $i' : A' \to B'$ is a pair of maps $(A' \longrightarrow A, B \longrightarrow B')$ so that i' is $A' \longrightarrow A \xrightarrow{i} B \longrightarrow B'$.

(a) Draw the Segal subdivisions of the interval **2** and the 2-simplex **3**.

(b) Show that the source and target functors $\mathcal{C}^{op} \longleftarrow Sub(\mathcal{C}) \longrightarrow \mathcal{C}$ are homotopy equivalences. *Hint:* Use Quillen's Theorem A and 3.2.2.

EIV.3.10 **3.10.** Given a simplicial set X, its *Segal subdivision* $Sub(X)$ is the sequence of sets $X_1, X_3, X_5, \dots$, made into a simplicial set by declaring the face maps $\partial'_i : X_{2n+1} \to X_{2n-1}$ to be $\partial_i\partial_{2n+1-i}$ and $\sigma'_i : X_{2n+1} \to X_{2n+3}$ to be $\sigma_i\sigma_{2n+1-i}$ $(0 \le i \le n)$.

If X is the nerve of a category $\mathcal{C}$, show that $Sub(X)$ is the nerve of the Segal subdivision category $Sub(\mathcal{C})$ of Ex. 3.9.

EIV.3.11 **3.11.** (Waldhausen) Let $f : X \to Y$ be a map of simplicial sets. For $y \in Y_n$, define the simplicial set $f/(n, y)$ to be the pullback of X and the n-simplex Δ^n along $f : X \to Y$ and the map $y : \Delta^n \to Y$. Thus an m-simplex consists of a map $\alpha : m \to n$ in $\mathbf{\Delta}$ and an $x \in X_m$ such that $f(x) = \alpha^*(y)$. Prove that:
(a) If each $f/(n, y)$ is contractible, then f is a homotopy equivalence;
(b) If for every $m \xrightarrow{\alpha} n$ in $\mathbf{\Delta}$ and every $y \in Y_n$ the map $f/(m, \alpha^* y) \to f/(n, y)$ is a homotopy equivalence, then each $|f/(n, y)| \to X \to Y$ is homotopy fibration sequence.
Hint: Any simplicial set X determines a category $\Delta^{op} \int X$ cofibered over Δ^{op}, by 3.3.2 and 3.7.5. Now apply Theorems A and B.

EIV.3.12 **3.12.** If $\mathcal{C}$ is a category, its *arrow category* $\mathcal{C}/\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a, b) : f \to f'$ in $\mathcal{C}/\mathcal{C}$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array}$$

If $f : A \to B$, the source $s(f) = A$ and target $t(f) = B$ of f define functors $\mathcal{C}/\mathcal{C} \to \mathcal{C}$. Show that s is a fibered functor, and that t is a cofibered functor. Then show that both s and t are homotopy equivalences.

EIV.3.13 **3.13.** *Swallowing Lemma.* If $\mathcal{A}$ is a subcategory of $\mathcal{B}$, show that the bicategory inclusion $\mathcal{B} \subset \mathcal{AB}$ of Example 3.10.2 induces a homotopy equivalence $B\mathcal{B} \simeq B(\mathcal{AB})$. When $\mathcal{A} = \mathcal{B}$ this proves that $B\mathcal{B} \simeq B(\text{bi}\mathcal{B})$. *Hint:* Show that $\mathcal{B} \simeq N_p(\mathcal{A})\mathcal{B}$ for all p.

EIV.3.14 **3.14.** *Diagonal Category.* (Waldhausen [214]) Show that the functor from small categories to small bicategories sending $\mathcal{B}$ to bi$\mathcal{B}$ (3.10.2) has a left adjoint, sending $\mathcal{C}$ to its *diagonal category*, and that the diagonal category of the bicategory $\mathcal{A} \otimes \mathcal{B}$ is the product category $\mathcal{A} \times \mathcal{B}$. *Hint:* both the horizontal and vertical morphisms of a bicategory $\mathcal{C}$ yield morphisms, and every bimorphism yields an equivalence relation for the composition of horizontal and vertical morphisms.

4 Symmetric Monoidal Categories

The geometric realization BS of a symmetric monoidal category is an H-space with a homotopy-commutative, homotopy-associative product. To see this, recall from Definition II.5.1 that a symmetric monoidal category is a category S with a functor $\Box\colon S \times S \to S$ which has a unit object "e" and is associative and is commutative, all up to coherent natural isomorphism. By 3.1(6) the geometric realization of $\Box$ is the "product" map $(BS) \times (BS) \cong B(S \times S) \to BS$. The natural isomorphisms $s\Box e \cong s \cong e\Box s$ imply that the vertex e is an identity up to homotopy, *i.e.,* that BS is an H-space. The other axioms imply that the product on BS is homotopy commutative and homotopy associative.

In many cases e is an initial object of S, and therefore the H-space BS is contractible by Example 3.2.2. For example, any additive category $\mathcal{A}$ is a symmetric monoidal category (with $\Box = \oplus$), and $e = 0$ is an initial object, so $B\mathcal{A}$ is contractible. Similarly, the category $\mathbf{Sets}_{\text{fin}}$ of finite sets is symmetric monoidal ($\Box$ being disjoint union) by I.5.2, and $e = \emptyset$ is initial, so $B\mathbf{Sets}_{\text{fin}}$ is contractible.

Here is an easy way to modify S in order to get an interesting H-space.

IV.4.1 **Definition 4.1.** Let $\operatorname{iso} S$ denote the subcategory of isomorphisms in S. It has the same objects as S, but its morphisms are the isomorphisms in S. Because $\operatorname{iso} S$ is also symmetric monoidal, $B(\operatorname{iso} S)$ is an H-space.

By Lemma 3.3, the abelian monoid $\pi_0(\operatorname{iso} S)$ is just the set of isomorphism classes of objects in S — the monoid S^{iso} considered in §II.5. In fact, $\operatorname{iso} S$ is equivalent to the disjoint union $\coprod \operatorname{Aut}_S(s)$ of the 1–object categories $\operatorname{Aut}_S(s)$, and $B(\operatorname{iso} S)$ is homotopy equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(s)$, $s \in S^{\text{iso}}$.

IV.4.1.1 **Example 4.1.1.** $B(\operatorname{iso} S)$ is often an interesting H-space.

(a) In the category $\mathbf{Sets}_{\text{fin}}$ of finite (pointed) sets, the group of automorphisms of any n-element set is isomorphic to the permutation group Σ_n. Thus the subcategory $\operatorname{iso}\mathbf{Sets}_{\text{fin}}$ is equivalent to $\coprod \Sigma_n$, the disjoint union of the one-object categories Σ_n. Thus the classifying space $B(\operatorname{iso}\mathbf{Sets}_{\text{fin}})$ is homotopy equivalent to the disjoint union of the classifying spaces $B\Sigma_n$, $n \geq 0$.

(b) The additive category $\mathbf{P}(R)$ of finitely generated projective R-modules has 0 as an initial object, so $B\mathbf{P}(R)$ is a contractible space. However, its subcategory $\mathbf{P} = \operatorname{iso}\mathbf{P}(R)$ of isomorphisms is more interesting. The topological space $B\mathbf{P}$ is equivalent to the disjoint union of the classifying spaces $B\operatorname{Aut}(P)$ as P runs over the set of isomorphism classes of finitely generated projective R-modules.

(c) Fix a ring R, and let $\mathbf{F}(R)$ be the category $\coprod GL_n(R)$ whose objects are the based free R-modules $\{0, R, R^2, \cdots, R^n, \cdots\}$ (these objects are distinct because the bases have different orders; see Section I.1). There are no maps in $\mathbf{F}(R)$ between R^m and R^n if $m \neq n$, and the self-maps of R^n form the group $GL_n(R)$. This is a symmetric monoidal category: $R^m\Box R^n = R^{m+n}$ by concatenation of bases; if a and b are morphisms, $a\Box b$ is the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

The space $B\mathbf{F}(R)$ is equivalent to the disjoint union of the classifying spaces $BGL_n(R)$.

If R satisfies the Invariant Basis Property (I.1.1), then $\mathbf{F}(R)$ is equivalent to a full subcategory of iso $\mathbf{P}(R)$. In this case, we saw in II.5.4.1 that $\mathbf{F}(R)$ is cofinal in iso $\mathbf{P}(R)$.

(d) Fix a commutative ring R, and let $S = \mathbf{Pic}(R)$ be the category of invertible R-modules and their isomorphisms. This is a symmetric monoidal category in which $\square$ is tensor product and e is R; see II.5.2(5). In this case, $S = \text{iso}\, S$ and S^{iso} is the Picard group $\text{Pic}(R)$ discussed in Section §I.3. By Lemma I.3.3, $\text{Aut}(L) = R^\times$ for every L. Thus $\mathbf{Pic}(R)$ is equivalent to a disjoint union of copies of $R^\times$, and $B(\mathbf{Pic})$ is homotopy equivalent to the product $\text{Pic}(R) \times B(R^\times)$.

(e) If F is a field, we saw in II.5.7 that the categories $\mathbf{SBil}(F)$ and $\mathbf{Quad}(F) = \mathbf{Quad}^+(F)$ of symmetric inner product spaces and quadratic spaces are symmetric monoidal categories. More generally, let A be any ring with involution, and $\epsilon = \pm 1$. Then the category $\mathbf{Quad}^\epsilon(A)$ of nonsingular ϵ-quadratic A-modules is a symmetric monoidal category with $\square = \oplus$ and $e = 0$. See [17, II] for more details.

(f) If G is a group, consider the category G-$\mathbf{Sets}_{\text{fin}}$ of free G-sets X having a finite number of orbits. This is symmetric monoidal under disjoint union (cf. II.5.2.2). If X has n orbits, then $\text{Aut}(X)$ is the wreath product $G \wr \Sigma_n$. As in (a), $B(G\text{-}\mathbf{Sets}_{\text{fin}})$ is equivalent to the disjoint union of the classifying spaces $B(G \wr \Sigma_n)$.

There is a monoidal functor G-$\mathbf{Sets}_{\text{fin}} \to \mathbf{P}(\mathbb{Z}[G])$ which sends X to the free abelian group on the set X.

The $S^{-1}S$ Construction

In [74], Quillen gave a construction of a category $S^{-1}S$ such that $K(S) = B(S^{-1}S)$ is a "group completion" of BS (see 4.4 below), provided that every map in S is an isomorphism and every translation $s\square\colon \text{Aut}_S(t) \to \text{Aut}_S(s\square t)$ is an injection. The motivation for this construction comes from the construction of the universal abelian group completion of an abelian monoid given in Chapter II, §1.

IV.4.2 **Definition 4.2.** ($S^{-1}S$) The objects of $S^{-1}S$ are pairs (m, n) of objects of S. A morphism in $S^{-1}S$ is an equivalence class of composites

$$(m_1, m_2) \xrightarrow{s\square} (s\square m_1, s\square m_2) \xrightarrow{(f,g)} (n_1, n_2).$$

This composite is equivalent to

$$(m_1, m_2) \xrightarrow{t\square} (t\square m_1, t\square m_2) \xrightarrow{(f',g')} (n_1, n_2)$$

exactly when there is an isomorphism $\alpha\colon s \xrightarrow{\simeq} t$ in S so that composition with $\alpha\square m_i$ sends f' and g' to f and g.

A (strict) monoidal functor $S \to T$ induces a functor $S^{-1}S \to T^{-1}T$.

IV.4.2.1 **Explanation 4.2.1.** There are two basic types of morphisms in $S^{-1}S$. The first type is a pair of maps $(f_1, f_2)\colon (m_1, m_2) \to (n_1, n_2)$ with $f_i\colon m_i \to n_i$ in S, arising from the inclusion of $S \times S$ in $S^{-1}S$. The second type is a formal map $s\Box\colon (m, n) \to (s\Box m, s\Box n)$.

We shall say that *translations are faithful* in S if every translation $\mathrm{Aut}(s) \to \mathrm{Aut}(s\Box t)$ in S is an injection. In this case every map in $S^{-1}S$ determines s, f and g up to unique isomorphism.

IV.4.2.2 **Remark 4.2.2.** $S^{-1}S$ is a symmetric monoidal category, with the product $(m, n)\Box(m', n') = (m\Box m', n\Box n')$, and the functor $S \to S^{-1}S$ sending m to (m, e) is monoidal. Hence the natural map $BS \to B(S^{-1}S)$ is an H-space map, and $\pi_0(S) \to \pi_0(S^{-1}S)$ is a map of abelian monoids.

In fact $\pi_0(S^{-1}S)$ is an abelian group, the inverse of (m, n) being (n, m), because of the existence of a morphism η in $S^{-1}S$ from (e, e) to $(m, n)\Box(n, m) = (m\Box n, n\Box m)$. Warning: η is not a natural transformation! See Ex. 4.3.

IV.4.3 **Definition 4.3.** Let S be a symmetric monoidal category in which every morphism is an isomorphism. Its K-groups are the homotopy groups of $B(S^{-1}S)$:

$$K_n^{\Box}(S) = \pi_n(BS^{-1}S).$$

It is sometimes convenient to write $K^{\Box}(S)$ for the geometric realization $B(S^{-1}S)$, and call it the *K-theory space* of S, so that $K_n^{\Box}(S) = \pi_n K^{\Box}(S)$. By 4.2, a (strict) monoidal functor $S \to T$ induces a map $K^{\Box}(S) \to K^{\Box}(T)$ and hence homomorphisms $K_n^{\Box}(S) \to K_n^{\Box}(T)$.

In order to connect this definition up with the definition of $K_0^{\Box}(S)$ given in Section II.5, we recall from 4.2.2 that the functor $S \to S^{-1}S$ induces a map of abelian monoids from $\pi_0(S) = S^{\mathrm{iso}}$ to $\pi_0(S^{-1}S)$.

IV.4.3.1 **Lemma 4.3.1.** *The abelian group $K_0^{\Box}(S) = \pi_0(S^{-1}S)$ is the group completion of the abelian monoid $\pi_0(S) = S^{iso}$. Thus Definition 4.3 agrees with the definition of $K_0^{\Box}(S)$ given in II.5.1.2.*

Proof. Let A denote the group completion of $\pi_0(S)$, and consider the function $\alpha(m, n) = [m] - [n]$ from the objects of $S^{-1}S$ to A. If $s \in S$ and $f_i\colon m_i \to n_i$ are morphisms in S then in A we have $\alpha(m, n) = \alpha(s\Box m, s\Box n)$ and $\alpha(m_1, m_2) = [m_1] - [m_2] = [n_1] - [n_2] = \alpha(n_1, n_2)$. By Lemma 3.3, α induces a set map $\pi_0(S^{-1}S) \to A$. By construction, α is an inverse to the universal homomorphism $A \to \pi_0(S^{-1}S)$. $\Box$

Group Completions

Group completion constructions for K-theory were developed in the early 1970's by topologists studying infinite loop spaces. These constructions all apply to symmetric monoidal categories.

Any discussion of group completions depends upon the following well-known facts (see [228, III.7]). Let X be a homotopy commutative, homotopy associative

H-space. Its set of components $\pi_0 X$ is an abelian monoid, and $H_0(X;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[\pi_0(X)]$. Moreover, the integral homology $H_*(X;\mathbb{Z})$ is an associative graded-commutative ring with unit.

We say that a homotopy associative H-space X is *group-like* if it has a homotopy inverse; see [228, III.4]. Of course this implies that $\pi_0(X)$ is a group. When X is a CW complex, the converse holds: if the monoid $\pi_0(X)$ is a group, then X is group-like. (If $\pi_0(X) = 0$ this is [228, X.2.2]; if $\pi_0(X)$ is a group, the proof in *loc. cit.* still goes through as the shear map $\pi_0(X)^2 \to \pi_0(X)^2$ is an isomorphism.)

For example, if $S = \operatorname{iso} S$ then $\pi_0(BS)$ is the abelian monoid S^{iso} of isomorphism classes, and $H_0(BS;\mathbb{Z})$ is the monoid ring $\mathbb{Z}[S^{\text{iso}}]$. In this case, the above remarks show that BS is grouplike if and only S^{iso} is an abelian group under the operation $\square$.

IV.4.4 **Definition 4.4** (Group Completion)**.** Let X be a homotopy commutative, homotopy associative H-space. A *group completion* of X is an H-space Y, together with an H-space map $X \to Y$, such that $\pi_0(Y)$ is the group completion of the abelian monoid $\pi_0(X)$ (in the sense of Section I.1), and the homology ring $H_*(Y;k)$ is isomorphic to the localization $\pi_0(X)^{-1}H_*(X;k)$ of $H_*(X;k)$ by the natural map, for all commutative rings k.

If X is a CW complex (such as $X = BS$), we shall assume that Y is also a CW complex. This hypothesis implies that the group completion Y is group-like.

IV.4.4.1 **Lemma 4.4.1.** *If X is a group-like H-space then X its own group completion, and any other group completion $f : X \to Y$ is a homotopy equivalence.*

Proof. Since f is a homology isomorphism, it is an isomorphism on π_0 and π_1. Therefore the map of basepoint components is a +–construction relative to the subgroup 1 of $\pi_1(X)$, and Theorem 1.5 implies that $X \simeq Y$. $\square$

IV.4.4.2 **Example 4.4.2** (Picard groups)**.** Let R be a commutative ring, and consider the symmetric monoidal category $S = \mathbf{Pic}(R)$ of Example II.5.2(5). Because $\pi_0(S)$ is already a group, S and $S^{-1}S$ are homotopy equivalent (by Lemma 4.4.1). Therefore we get

$$K_0\mathbf{Pic}(R) = \operatorname{Pic}(R), \quad K_1\mathbf{Pic}(R) = U(R) \quad \text{and } K_n\mathbf{Pic}(R) = 0 \quad \text{for } n \geq 2.$$

The determinant functor from $\mathbf{P} = \operatorname{iso}\mathbf{P}(R)$ to $\mathbf{Pic}(R)$ constructed in Section I.3 gives a map from $K(R) = K(\mathbf{P})$ to $K\mathbf{Pic}(R)$. Upon taking homotopy groups, this yields the familiar maps $\det\colon K_0(R) \to \operatorname{Pic}(R)$ of II.2.6 and $\det\colon K_1(R) \to R^\times$ of III.1.1.1.

A *phantom map* $\phi : X \to Y$ is a map such that, for every finite CW complex A, every composite $A \to X \to Y$ is null homotopic, *i.e.*, $\phi_* : [A,X] \to [A,Y]$ is the zero map. If $f : X \to Y$ is a group completion then so is $f + \phi : X \to Y$ for every phantom map ϕ. Thus the group completion is not unique up to homotopy equivalence whenever phantom maps exist.

The following result, taken from [40, 1.2], shows that phantom maps are essentially the only obstruction to uniqueness of group completions. We say that two maps $X \to Y$ are *weakly homotopic* if they induce the same map on homotopy classes $[A, X] \to [A, Y]$; if Y is an H-space, this means that their difference is a phantom map.

IV.4.4.3 **Theorem 4.4.3.** *Let X be an H-space such that $\pi_0(X)$ is either countable or contains a countable cofinal submonoid. If $f' : X \to X'$ and $f'' : X \to X''$ are two group completions, then there is a homotopy equivalence $g : X' \to X''$, unique up to weak homotopy, such that gf' and f'' are weak homotopy equivalent. (The map g is also a weak H-map.)*

The fact that gf' and f'' are weak homotopy equivalent implies that g is a homology isomorphism, and hence is a homotopy equivalence by 4.4.1.

IV.4.5 **4.5.** One can show directly that $\mathbb{Z} \times BGL(R)^+$ is a group completion of BS when $S = \coprod GL_n(R)$; see Ex. 4.9. We will see in Theorem 4.8 below that the K-theory space $B(S^{-1}S)$ is another group completion of BS, and then give an explicit homotopy equivalence between $B(S^{-1}S)$ and $\mathbb{Z} \times BGL(R)^+$ in 4.9. Here are some other methods of group completion:

IV.4.5.1 **Example 4.5.1** (Segal's ΩB Method)**.** If X is a topological *monoid*, such as $\coprod BGL_n(R)$ or $\coprod B\Sigma_n$, then we can form BX, the geometric realization of the (one-object) topological category X (see 3.9). In this case, ΩBX is an infinite loop space and the natural map $X \to \Omega BX$ is a group completion. For example, if X is the one-object monoid $\mathbb{N}$ then $B\mathbb{N} \simeq S^1$, and $\Omega B\mathbb{N} \simeq \Omega S^1 \simeq \mathbb{Z}$. That is, $\pi_0(\Omega B\mathbb{N})$ is $\mathbb{Z}$, and every component of $\Omega B\mathbb{N}$ is contractible. See [1] for more details.

IV.4.5.2 **Example 4.5.2** (Machine Methods)**.** (See [1].) If X isn't quite a monoid, but the homotopy associativity of its product is nice enough, then there are constructions called "infinite loop space machines" which can construct a group completion Y of X, and give Y the structure of an infinite loop space. All machines produce the same infinite loop space Y (up to homotopy); see [121]. Some typical machines are described in [165], and [119].

The realization $X = BS$ of a symmetric monoidal category S is nice enough to be used by infinite loop space machines. These machines produce an infinite loop space $K(S)$ and a map $BS \to K(S)$ which is a group completion. Most infinite loop machines will also produce explicit deloopings of $K(S)$ in the form of an Ω-spectrum $\mathbf{K}(S)$, the *K-theory spectrum* of S, which is connective in the sense that $\pi_n\mathbf{K}(S) = 0$ for $n < 0$. The production of $\mathbf{K}(S)$ is natural enough that monoidal functors between symmetric monoidal categories induce maps of the corresponding spectra.

Pairings and Products

A *pairing* of symmetric monoidal categories is a functor $\otimes : S_1 \times S_2 \to S$ such that $s \otimes 0 = 0 \otimes s = 0$, and there is a coherent natural bi-distributivity law

$$(a + a') \otimes (b + b') \cong (a \otimes b) \square (a \otimes b') \square (a' \otimes b) \square (a' \otimes b').$$

If $S_1 = S_2 = S$, we will just call this a pairing on S. Instead of making this technical notion precise, we refer the reader to May [120, §2] and content ourselves with two examples from 4.1.1: the product of finite sets is a pairing on $\mathbf{Sets}_{\text{fin}}$, and the tensor product of based free modules is a pairing $\mathbf{F}(A)\times\mathbf{F}(B) \to \mathbf{F}(A\otimes B)$. The free module functor from $\mathbf{Sets}_{\text{fin}}$ to $\mathbf{F}(A)$ preserves these pairings. The following theorem was proven by Peter May in [120, 1.6 and 2.1].

IV.4.6 **Theorem 4.6.** *A pairing $S_1 \times S_2 \to S$ of symmetric monoidal categories determines a natural pairing $K(S_1) \wedge K(S_2) \to K(S)$ of infinite loop spaces in 4.5.2, as well as a pairing of Ω-spectra $\mathbf{K}(S_1) \wedge \mathbf{K}(S_2) \to \mathbf{K}(S)$. This in turn induces bilinear products $K_p(S_1) \otimes K_q(S_2) \to K_{p+q}(S)$. There is also a commutative diagram*

$$\begin{array}{ccccc} BS_1 \times BS_2 & \longrightarrow & BS_1 \wedge BS_2 & \xrightarrow{B\otimes} & BS \\ \downarrow & & \downarrow & & \downarrow \\ K(S_1) \times K(S_2) & \longrightarrow & K(S_1) \wedge K(S_2) & \xrightarrow{B\otimes} & K(S). \end{array}$$

From Theorem 4.6 and the constructions in 1.10 and 4.9, repectively Ex. 1.12 and 4.9.3, we immediately deduce:

IV.4.6.1 **Corollary 4.6.1.** *When S is $\mathbf{Sets}_{fin}$ or $\mathbf{F}(R)$, the product defined by Loday (in 1.10) agrees with the product in Theorem 4.6.*

IV.4.6.2 **Remark 4.6.2.** If there is a pairing $S \times S \to S$ which is associative up to natural isomorphism, then $\mathbf{K}(S)$ can be given the structure of a ring spectrum. This is the case when S is $\mathbf{Sets}_{\text{fin}}$ as well as (for commutative R) $\mathbf{F}(R)$ and iso $\mathbf{P}(R)$.

Actions on other categories

To show that $B(S^{-1}S)$ is a group completion of BS, we need to fit the definition of $S^{-1}S$ into a more general framework.

IV.4.7 **Definition 4.7.** A monoidal category S is said to *act upon* a category X by a functor $\Box\colon S \times X \to X$ if there are natural isomorphisms $s\Box(t\Box x) \cong (s\Box t)\Box x$ and $e\Box x \cong x$ for $s, t \in S$ and $x \in X$, satisfying coherence conditions for the products $s\Box t\Box u\Box x$ and $s\Box e\Box x$ analogous to the coherence conditions defining S.

For example, S acts on itself by $\Box$. If X is a discrete category, S acts on X exactly when the monoid $\pi_0(S)$ acts on the underlying set of objects in X.

Here is the analogue of the translation category construction (3.3.1) associated to a monoid acting on a set.

IV.4.7.1 **Definition 4.7.1.** If S acts upon X, the category $\langle S, X\rangle$ has the same objects as X. A morphism from x to y in $\langle S, X\rangle$ is an equivalence class of pairs $(s, s\Box x \xrightarrow{\phi} y)$, where $s \in S$ and ϕ is a morphism in X. Two pairs (s,ϕ) and (s',ϕ') are equivalent in case there is an isomorphism $s \cong s'$ identifying ϕ' with $s'\Box x \cong s\Box x \xrightarrow{\phi} y$.

We shall write $S^{-1}X$ for $\langle S, S\times X\rangle$, where S acts on both factors of $S\times X$. Note that when $X=S$ this definition recovers the definition of $S^{-1}S$ given in 4.2 above. If S is symmetric monoidal, then the formula $s\Box(t\Box x)=(s\Box t,x)$ defines an action of S on $S^{-1}X$.

For example, if every arrow in S is an isomorphism, then e is an initial object of $\langle S,S\rangle$ and therefore the space $S^{-1}\mathbf{1}\simeq B\langle S,S\rangle$ is contractible.

We say that S acts *invertibly* upon X if each translation functor $s\Box\colon X\to X$ is a homotopy equivalence. For example, S acts invertibly on $S^{-1}X$ (if S is symmetric) by the formula $s\Box(t,x)=(s\Box t,x)$, the homotopy inverse of the translation $(t,x)\mapsto(s\Box t,x)$ being the translation $(t,x)\mapsto(t,s\Box x)$, because of the natural transformation $(t,x)\mapsto(s\Box t,s\Box x)$.

Now π_0S is a multiplicatively closed subset of the ring $H_0(S)=\mathbb{Z}[\pi_0S]$, so it acts on $H_*(X)$ and acts invertibly upon $H_*(S^{-1}X)$. Thus the functor $X\to S^{-1}X$ sending x to $(0,x)$ induces a map

$$(\pi_0S)^{-1}H_q(X)\to H_q(S^{-1}X). \tag{4.7.2}$$ IV.4.7.2

IV.4.8 **Theorem 4.8.** *(Quillen) If every map in S is an isomorphism and translations are faithful in S, then (4.7.2) is an isomorphism for all X and q.*

In particular, $B(S^{-1}S)$ is a group completion of the H-space BS.

Proof. (See [74, p. 221].) By Ex. 4.5, the projection functor $\rho\colon S^{-1}X\to\langle S,S\rangle$ is cofibered with fiber X. By Ex. 3.7 there is an associated spectral sequence $E^2_{pq}=H_p(\langle S,S\rangle;H_q(X))\Rightarrow H_{p+q}(S^{-1}X)$. Localizing this at the multiplicatively closed subset π_0S of $H_0(S)$ is exact, and π_0S already acts invertibly on $H_*(S^{-1}X)$ by Ex. 3.7, so there is also a spectral sequence $E^2_{pq}=H_p(\langle S,S\rangle;M_q)\Rightarrow H_{p+q}(S^{-1}X)$, where $M_q=(\pi_0S)^{-1}H_q(X)$. But the functors M_q are morphism-inverting on $\langle S,S\rangle$ (3.5.1), so by Ex. 3.1 and the contractibility of $\langle S,S\rangle$, the group $H_p(\langle S,S\rangle;M_q)$ is zero for $p\neq 0$, and equals M_q for $p=0$. Thus the spectral sequence degenerates to the claimed isomorphism (4.7.2).

The final assertion is immediate from this and Definition 4.4, given Remark 4.2.2 and Lemma 4.3.1. $\square$

Bass gave a classical definition of $K_1(S)$ and $K_2(S)$ in [17, p. 197]; we gave them implicitly in III.1.6.3 and III.5.6. We can now state these classical definitions, and show that they coincide with the K-groups defined in this section.

IV.4.8.1 **Corollary 4.8.1.** *If $S=iso\,S$ and translations are faithful in S, then:*

$$K_1(S)=\varinjlim_{s\in S}H_1(\mathrm{Aut}(s);\mathbb{Z}),$$

$$K_2(S)=\varinjlim_{s\in S}H_2([\mathrm{Aut}(s),\mathrm{Aut}(s)];\mathbb{Z}).$$

Proof. ([219]) The localization of $H_q(BS)=\oplus_{s\in S}H_q(\mathrm{Aut}(s))$ at $\pi_0(X)=S^{\mathrm{iso}}$ is the direct limit of the groups $H_q(\mathrm{Aut}(s))$, taken over the translation category of all $s\in S$. Since $\pi_1(X)=H_1(X;\mathbb{Z})$ for every H-space X, this gives the formula for $K_1(S)=\pi_1B(S^{-1}S)$.

For K_2 we observe that any monoidal category S is the filtered colimit of its monoidal subcategories having countably many objects. Since $K_2(S)$ and Bass' H_2 definition commute with filtered colimits, we may assume that S has countably many objects. In this case the proof is relegated to Exercise 4.10. □

Relation to the +–construction

Let $S = \mathbf{F}(R) = \coprod GL_n(R)$ be the monoidal category of based free R-modules, as in Example 4.1.1(c). In this section, we shall establish the following result, identifying the +–construction on $BGL(R)$ with the basepoint component of $K(S) = B(S^{-1}S)$.

IV.4.9 **Theorem 4.9.** *When S is $\coprod GL_n(R)$, $K(S) = B(S^{-1}S)$ is the group completion of $BS = \coprod BGL_n(R)$, and*

$$B(S^{-1}S) \simeq \mathbb{Z} \times BGL(R)^+.$$

As Theorems 4.8 and 1.8 suggest, we first need to find an acyclic map from $BGL(R)$ to the connected basepoint component of $B(S^{-1}S)$. This is done by the following "mapping telescope" construction (illustrated in Figure 4.9.1).

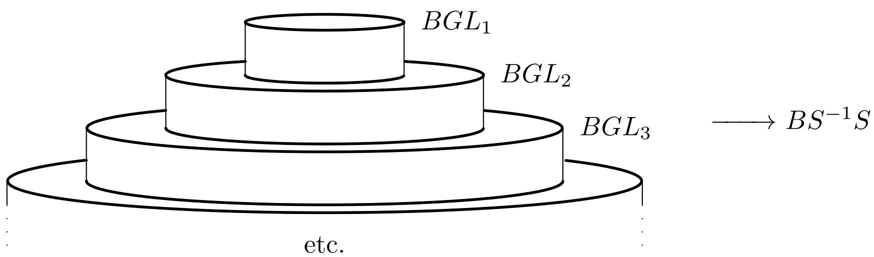


Figure 4.9.1: The mapping telescope of $BGL(R)$ and $B(S^{-1}S)$. IV.4.9.1

Any group map η from $GL_n(R)$ to $\mathrm{Aut}_{S^{-1}S}(R^n, R^n)$ gives a map from $BGL_n(R)$ to $B(S^{-1}S)$. For the specific maps $\eta = \eta_n$ defined by $\eta_n(g) = (g, 1)$, the diagram

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^n, R^n) \\ {\scriptstyle \square R}\downarrow & & \downarrow{\scriptstyle \square(R,R)} \\ GL_{n+1}(R) & \xrightarrow{\eta} & \mathrm{Aut}(R^{n+1}, R^{n+1}) \end{array}$$

commutes, *i.e.,* there is a natural transformation from η to $\eta(\square R)$. The resulting homotopy of maps $\eta \simeq \eta(\square R) : BGL_n(R) \to B(S^{-1}S)$ gives the map from the "mapping telescope" construction of $BGL(R)$ to $B(S^{-1}S)$; see Ex. 3.5. In fact, this map lands in the connected component Y_S of the identity in $B(S^{-1}S)$. Since $B(S^{-1}S)$ is an H-space, so is the connected component Y_S of the identity.

Proof of Theorem 4.9. (Quillen) We shall show that the map $BGL(R) \to Y_S$ is an isomorphism on homology with coefficients $\mathbb{Z}$. By the remark following Theorem 1.8, this will induce a homotopy equivalence $BGL(R)^+ \to Y_S$.

Let $e \in \pi_0 BS$ be the class of R. By Theorem 4.8, $H_*B(S^{-1}S)$ is the localization of the ring $H_*(BS)$ at $\pi_0(S) = \{e^n\}$. But this localization is the colimit of the maps $H_*(BS) \to H_*(BS)$ coming from the translation $\oplus R : S \to S$. Hence $H_*B(S^{-1}S) \cong H_*(Y_S) \otimes \mathbb{Z}[e, e^{-1}]$, where Y_S denotes the basepoint component of $B(S^{-1}S)$, and $H_*(Y_S) \cong \operatorname{colim} H_*(BGL_n(R)) = H_*(BGL(R))$. This means that the map $BGL(R) \to Y_S$ is a homology isomorphism, as required. $\square$

IV.4.9.2 **Example 4.9.2.** (Segal) Consider the symmetric monoidal category $S = \coprod \Sigma_n$, equivalent to the category $\mathbf{Sets}_{\mathrm{fin}}$ of Example 4.1.1(a). The infinite symmetric group Σ_∞ is the union of the symmetric groups Σ_n along the inclusions $\Box 1$ from Σ_n to Σ_{n+1}, and these inclusions assemble to give a map from the mapping telescope construction of $B\Sigma_\infty$ to $B(S^{-1}S)$, just as they did for $GL(R)$ (see Figure 4.9.1). Moreover the proof of Theorem 4.9 formally goes through to prove that $B(S^{-1}S) \simeq K(\mathbf{Sets}_{\mathrm{fin}})$ is homotopy equivalent to $\mathbb{Z} \times B\Sigma_\infty^+$. This is the equivalence of parts (a) and (b) in the following result. We refer the reader to [14] and [1, §4.2] for the equivalence of parts (b) and (c).

IV.4.9.3 **The Barratt-Priddy-Quillen-Segal Theorem 4.9.3.** *The three infinite loop spaces below are the same:*

(a) The group completion $K(\mathbf{Sets}_{fin})$ of $B\mathbf{Sets}_{fin}$;

(b) $\mathbb{Z} \times B\Sigma_\infty^+$, where Σ_∞ is the union of the symmetric groups Σ_n; and

(c) The infinite loop space $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n$.

Hence the groups $K_n(\mathbf{Sets}_{fin})$ are the stable homotopy groups of spheres, π_n^s.

More generally, suppose that S has a countable sequence of objects $s_1, \dots$ such that $s_{n+1} = s_n \Box a_n$ for some $a_n \in S$, and satisfying the cofinality condition that for every $s \in S$ there is an s' and an n so that $s \Box s' \cong s_n$. In this case we can form the group $\operatorname{Aut}(S) = \operatorname{colim}_{n\to\infty} \operatorname{Aut}_S(s_n)$.

IV.4.10 **Theorem 4.10.** *Let $S = iso\,S$ be a symmetric monoidal category whose translations are faithful, and suppose the above condition is satisfied, so that the group $Aut(S)$ exists. Then the commutator subgroup E of $Aut(S)$ is a perfect normal subgroup, $K_1(S) = \operatorname{Aut}(S)/E$, and the $+$–construction on $B\operatorname{Aut}(S)$ is the connected component of the identity in the group completion $K(S)$. Thus*

$$K(S) \simeq K_0(S) \times B\operatorname{Aut}(S)^+.$$

Proof. ([219]) The assertions about E are essentially on p. 355 of Bass [15]. On the other hand, the mapping telescope construction mentioned above gives an acyclic map from $B\operatorname{Aut}(S)$ to the basepoint component of $B(S^{-1}S)$, and such a map is by definition a $+$–construction. $\square$

IV.4.10.1 **Example 4.10.1.** Consider the subcategory $\coprod G\wr\Sigma_n$ of the category G-$\mathbf{Sets}_{\rm fin}$ of free G sets introduced in IV.4.1.1 4.1.1(f). The group $\mathrm{Aut}(S)$ is the (small) infinite wreath product $G\wr\Sigma_\infty = \cup G\wr\Sigma_n$, so we have $K(G\text{-}\mathbf{Sets}_{\rm fin}) \simeq \mathbb{Z}\times B(G\wr\Sigma_\infty)^+$. On the other hand, the Barratt-Priddy theorem BP71 [14] identifies this with the infinite loop space $\Omega^\infty S^\infty(BG_+)$ associated to the disjoint union BG_+ of BG and a point.

The monoidal functor G-$\mathbf{Sets}_{\rm fin}\to \mathbf{P}(\mathbb{Z}[G])$ of IV.4.1.1 4.1.1(f) induces a group homomorphism $G\wr\Sigma_\infty \to GL(\mathbb{Z}[G])$ and hence maps $B(G\wr\Sigma_\infty)^+\to BGL(\mathbb{Z}[G])^+$ and $\Omega^\infty S^\infty(BG_+)\simeq K(G\text{-}\mathbf{Sets}_{\rm fin})\to K(\mathbb{Z}[G])$.

This map is a version of the "assembly map" (IV.1.10.3 1.10.3) in the following sense. If R is any ring, there is a product map $K(R)\wedge K(\mathbb{Z}[G])\to K(R[G])$; see IV.1.10 1.10 and IV.4.6 4.6. This yields a map from $K(R)\wedge\Omega^\infty S^\infty(BG_+)$ to $K(R[G])$. Now the space $K(R)\wedge BG$ is included as a direct factor in $K(R)\wedge\Omega^\infty S^\infty(BG_+)$ (split by the "Snaith splitting"). Since the homotopy groups of the first space give the generalized homology of BG with coefficients in $K(R)$, $H_n(BG;K(R))$, we get homomorphisms $H_n(BG;K(R))\to K(R[G])$. It is not known if $K(R)\wedge BG$ has a complementary factor which maps trivially.

Cofinality

A monoidal functor $f:S\to T$ is called *cofinal* if for every t in T there is a t' and an s in S so that $t\Box t'\cong f(s)$; cf. II.II.5.3 5.3. For example, the functor $\mathbf{F}(R)\to\mathbf{P}(R)$ of Example IV.4.1.1 4.1.1(c) is cofinal, because every projective module is a summand of a free one. For $\mathbf{Pic}(R)$, the one-object subcategory $R^\times$ is cofinal.

IV.4.11 **Cofinality Theorem 4.11.** *Suppose that $f:S\to T$ is cofinal. Then*

(a) If T acts on X then S acts on X via f, and $S^{-1}X\simeq T^{-1}X$.

(b) If $\mathrm{Aut}_S(s)\cong\mathrm{Aut}_T(fs)$ for all $s\in S$ then the basepoint components of $K(S)$ and $K(T)$ are homotopy equivalent. Thus $K_n(S)\cong K_n(T)$ for all $n\ge 1$.

Proof. By cofinality, S acts invertibly on X if and only if T acts invertibly on X. Hence Ex. EIV.4.6 4.6 yields

$$S^{-1}X\xrightarrow{\simeq}T^{-1}(S^{-1}X)\cong S^{-1}(T^{-1}X)\xleftarrow{\simeq}T^{-1}X.$$

An alternate proof of part (a) is sketched in Ex. EIV.4.8 4.8.

For part (b), let Y_S and Y_T denote the connected components of $B(S^{-1}S)$ and $B(T^{-1}T)$. Writing the subscript $s\in S$ to indicate a colimit over the translation category IV.3.3.1 3.3.1 of $\pi_0(S)$, and similarly for the subscript $t\in T$, Theorem IV.4.8 4.8 yields:

$$\begin{aligned}H_*(Y_S)&=\mathrm{colim}_{s\in S}\,H_*(B\,\mathrm{Aut}(s))=\mathrm{colim}_{s\in S}\,H_*(B\,\mathrm{Aut}(fs))\\&\cong\mathrm{colim}_{t\in T}\,H_*(B\,\mathrm{Aut}(t))=H_*(Y_T).\end{aligned}$$

Hence the connected H-spaces Y_S and Y_T have the same homology, and this implies that they are homotopy equivalent. □

Note that $K_0(\mathbf{F}(R)) = \mathbb{Z}$ is not the same as $K_0(\mathbf{P}(R)) = K_0(R)$ in general, although $K_n(\mathbf{F}) \cong K_n(\mathbf{P})$ for $n \geq 1$ by the Cofinality Theorem IV.4.11 4.11(b). By Theorem IV.4.9 4.9 this establishes the following important result.

IV.4.11.1 **Corollary 4.11.1.** *Let $S = iso\,\mathbf{P}(R)$ be the category of finitely generated projective R-modules and their isomorphisms. Then*

$$B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+.$$

IV.4.11.2 **Remark 4.11.2.** Consider the 0-connected cover $\mathbf{K}(R)\langle 0\rangle$ of $\mathbf{K}(R)$, the spectrum constructed by an infinite loop space machine from iso $\mathbf{P}(R)$, as in IV.4.5.2 4.5.2. By IV.4.8 4.8 and IV.4.11.1 4.11.1, $BGL(R)^+$ is the 0^{th} space of the spectrum $\mathbf{K}(R)\langle 0\rangle$. In particular, it provides a canonical way to view $BGL(R)^+$ as an infinite loop space.

IV.4.12 **4.12.** Let's conclude with a look back at the other motivating examples in IV.4.1.1 4.1.1. In each of these examples, every morphism is an isomorphism and the translations are faithful, so the classifying space of $S^{-1}S$ is a group completion of BS.

IV.4.12.1 **Example 4.12.1** (Stable homotopy groups)**.** The "free R-module" on a finite set determines a functor from $\mathbf{Sets}_{\text{fin}}$ to $\mathbf{P}(R)$, or from the subcategory $\coprod \Sigma_n$ of $\mathbf{Sets}_{\text{fin}}$ to $\coprod GL_n(R)$. This functor identifies the symmetric group Σ_n with the permutation matrices in $GL_n(R)$. Applying group completions, Theorem IV.4.9 4.9 and IV.4.9.3 4.9.3 show that this gives a map from $\Omega^\infty S^\infty$ to $K(R)$, hence maps $\pi_n^s \to K_n(R)$.

IV.4.12.2 **Example 4.12.2** (L-theory)**.** Let $S = \mathbf{Quad}^\epsilon(A)$ denote the category of nonsingular ϵ-quadratic A-modules, where $\epsilon = \pm 1$ and A is any ring with involution B72; Bak [17, 11]. The K-groups of this category are the L-groups ${}_\epsilon L_n(A)$ of Karoubi and others. For this category, the sequence of hyperbolic spaces H^n is cofinal (by Ex. II.EII.5.11 5.11), and the automorphism group of H^n is the *orthogonal group* ${}_\epsilon O_n$. The infinite orthogonal group ${}_\epsilon O = {}_\epsilon O(A)$, which is the direct limit of the groups ${}_\epsilon O_n$, is the group $\mathrm{Aut}(S)$ in this case. By Theorem IV.4.10 4.10, we have

$$K(\mathbf{Quad}^\epsilon(A)) \simeq {}_\epsilon L_0(A) \times B_\epsilon O^+.$$

When $A = \mathbb{R}$, the classical orthogonal group O is ${}_{+1}O$. When $A = \mathbb{C}$ and the involution is complex conjugation, the classical unitary group U is ${}_{+1}O(\mathbb{C})$. For more bells and whistles, and classical details, we refer the reader to Bak [11].

IV.4.12.3 **Example 4.12.3** (Topological K-theory)**.** When R is a topological ring (such as $\mathbb{R}$ or $\mathbb{C}$), we can think of $\mathbf{P}(R)$ as a *topological* symmetric monoidal category. Infinite loop space machines (IV.4.5.2 4.5.2) also accept topological symmetric monoidal categories, and we write $K(R^{\text{top}})$ for $K(\mathbf{P}(R)^{\text{top}})$. The change-of-topology functor $\mathbf{P}(R) \to \mathbf{P}(R)^{\text{top}}$ induces natural infinite loop space maps from $K(R)$ to $K(R^{\text{top}})$. The naturality of these maps allows us to utilize infinite loop space machinery. As an example of the usefulness, we remark that

$$K(\mathbb{R}^{\text{top}}) \simeq \mathbb{Z} \times BO \qquad \text{and} \qquad K(\mathbb{C}^{\text{top}}) \simeq \mathbb{Z} \times BU.$$

EXERCISES

EIV.4.1 **4.1.** Let $\mathbb{N}$ be the additive monoid $\{0, 1, ...\}$, considered as a symmetric monoidal category with one object. Show that $\langle \mathbb{N}, \mathbb{N} \rangle$ is the union $\cup \mathbf{n}$ of the ordered categories $\mathbf{n}$, and that $\mathbb{N}^{-1}\mathbb{N}$ is a poset, each component being isomorphic to $\cup \mathbf{n}$.

EIV.4.2 **4.2.** Show that a sequence $X_0 \to X_1 \to \cdots$ of categories determines an action of $\mathbb{N}$ on the disjoint union $X = \coprod X_n$, and that $\langle \mathbb{N}, X \rangle$ is the mapping telescope category $\cup \mathbf{n} \int X$ of Ex. 3.5.

EIV.4.3 **4.3.** (Thomason) Let S be symmetric monoidal, and let $\iota\colon S^{-1}S \to S^{-1}S$ be the functor sending (m, n) to (n, m) and (f_1, f_2) to (f_2, f_1). Show that there is no natural transformation $0 \Rightarrow \mathrm{id} \Box \iota$. *Hint:* The obvious candidate is given in 4.2.2.

Thomason has shown that $B\iota$ is the homotopy inverse for the H-space structure on $B(S^{-1}S)$, but for subtle reasons.

EIV.4.4 **4.4.** If S is a symmetrical monoidal category, so is its opposite category S^{op}. Show that the group completions $K(S)$ and $K(S^{op})$ are homotopy equivalent.

EIV.4.5 **4.5.** (Quillen) Suppose that $S = \mathrm{iso}\, S$, and that translations in S are faithful (4.2.1). Show that the projection $S^{-1}X \xrightarrow{\rho} \langle S, S \rangle$ is cofibered, where $\rho(s, x) = s$.

EIV.4.6 **4.6.** Let $S = \mathrm{iso}\, S$ be a monoidal category whose translations are faithful (4.2.1). Suppose that S acts invertibly upon a category X. Show that the functors $X \to S^{-1}X$ $(x \mapsto (s, x))$ are homotopy equivalences for every s in S. If S acts upon a category Y, then S always acts invertibly upon $S^{-1}Y$, so this shows that $S^{-1}Y \simeq S^{-1}(S^{-1}Y)$. *Hint:* Use Exercises 3.6 and 4.5, and the contractibility of $\langle S, S \rangle$.

EIV.4.7 **4.7.** Suppose that every map in X is monic, and that each translation $\mathrm{Aut}_S(s) \xrightarrow{\Box x} \mathrm{Aut}_X(s \Box x)$ is an injection. Show that the sequence

$$S^{-1}S \xrightarrow{\Box x} S^{-1}X \xrightarrow{\pi} \langle S, X \rangle$$

is a homotopy fibration for each x in X, where π is projection onto the second factor. In particular, if $\langle S, X \rangle$ is contractible, this proves that $S^{-1}S \xrightarrow{\Box x} S^{-1}X$ is a homotopy equivalence. *Hint:* Show that π and $S^{-1}\pi\colon S^{-1}(S^{-1}X) \to \langle S, X \rangle$ are cofibered, and use the previous exercise.

EIV.4.8 **4.8.** Use Exercises 4.5 and 4.6 to give another proof of the Cofinality theorem 4.11(a).

EIV.4.9 **4.9.** Fix a ring R and set $S = \coprod GL_n(R)$. The maps $BGL_n(R) \to BGL(R) \to \{n\} \times BGL(R)^+$ assemble to give a map from BS to $\mathbb{Z} \times BGL(R)^+$. Use Ex. 1.11 to show that it is an H-space map. Then show directly that this makes $\mathbb{Z} \times BGL(R)^+$ into a group completion of BS.

EIV.4.10 **4.10.** Let S be a symmetric monoidal category with countably many objects, so that the group $\mathrm{Aut}(S)$ exists and its commutator subgroup E is perfect, as in 4.10. Let F denote the homotopy fiber of the H-space map $B\,\mathrm{Aut}(S)^+ \to B(K_1S)$.

(a) Show that $\pi_1(F) = 0$ and $H_2(F;\mathbb{Z}) \cong \pi_2(F) \cong K_2(S)$.

(b) Show that the natural map $BE \to F$ induces $H_*(BE) \cong H_*(F)$, so that $F = BE^+$. *Hint:* ([219]) Show that K_1S acts trivially upon the homology of BE and F, and apply the comparison theorem for spectral sequences.

(c) Conclude that $K_2(S) \cong H_2(E) \cong \varinjlim_{s\in S} H_2([\mathrm{Aut}(s), \mathrm{Aut}(s)];\mathbb{Z})$.

EIV.4.11 **4.11.** If $f : X \to Y$ is a functor, we say that an action of S on X is *fiberwise* if $S \times X \xrightarrow{\square} X \xrightarrow{f} Y$ equals the projection $S \times X \to X$ followed by f.

(a) Show that a fiberwise action on X restricts to an action of S on each fiber category $X_y = f^{-1}(y)$, and that f induces a functor $S^{-1}X \to Y$ whose fibers are the categories $S^{-1}(X_y)$.

(b) If f is a fibered functor (3.7.3), we say that a fiberwise action is *cartesian* if the base change maps commute with the action of S on the fibers. Show that in this case $S^{-1}X \to Y$ is a fibered functor.

EIV.4.12 **4.12.** Let G be a group, and G-$\mathbf{Sets}_{\mathrm{fin}}$ as in 4.1.1(f).

(a) Using 4.10.1, show that $K_1(G\text{-}\mathbf{Sets}_{\mathrm{fin}}) \cong G/[G,G] \times \{\pm 1\}$.

(b) Using Exercise II.5.9, show that the groups $K_n(\mathbb{Z}[G])$ are modules over the Burnside ring $A(G) = K_0G\text{-}\mathbf{Sets}_{\mathrm{fin}}$.

(c) If G is abelian, show that the product of G-sets defines a pairing in the sense of Theorem 4.6. Conclude that $K_*G\text{-}\mathbf{Sets}_{\mathrm{fin}}$ is a ring. Using the free module functor, show that $K_*G\text{-}\mathbf{Sets}_{\mathrm{fin}} \to K_*(\mathbb{Z}[G])$ is a ring homomorphism.

EIV.4.13 **4.13.** (a) Show that the idempotent completion $\hat{S}$ (II.7.3) of a symmetric monoidal category S is also symmetric monoidal, and that $S \to \hat{S}$ is a cofinal monoidal functor. Conclude that the basepoint components of $K(S)$ and $K(\hat{S})$ are homotopy equivalent.
(b) Show that a pairing $S_1 \times S_2 \to S$ induces a pairing $\hat{S}_1 \times \hat{S}_2 \to \hat{S}$ and hence (by 4.6) a pairing of spectra $\mathbf{K}(\hat{S}_1) \wedge \mathbf{K}(\hat{S}_2) \to \mathbf{K}(\hat{S})$.
(c) By (b), there is a pairing $\mathbf{K}(A) \wedge \mathbf{K}(B) \to \mathbf{K}(A \otimes B)$ for every pair of rings A, B. Using 4.6.1, deduce that the induced product agrees with the extension of Loday's product 1.10 described in Ex. 1.14.

EIV.4.14 **4.14.** Construct a morphism of spectra $S^1 \to \mathbf{K}(\mathbb{Z}[x, x^{-1}])$ which, as in 1.10.2, represents $[x] \in K_1(\mathbb{Z}[x, x^{-1}])$. Using the previous exercise, show that it induces a product map $\cup x : \mathbf{K}(R) \to \Omega\mathbf{K}(R[x, x^{-1}])$, natural in the ring R.

5 λ-operations in higher K-theory

IV.5

Let A be a commutative ring. In Section §II.4 we introduced the operations $\lambda^k : K_0(A) \to K_0(A)$ and showed that they endow $K_0(A)$ with the structure of a special λ-ring (II.4.3.1). The purpose of this section is to extend this structure to operations $\lambda^k : K_n(A) \to K_n(A)$ for all n. Although many constructions of λ-operations have been proposed in more exotic settings, we shall restrict our attention in this section to operations defined using the $+$–construction.

We shall begin with a general construction, which produces the operations $\wedge^k$ as a special case. Fix an arbitrary group G. If $\rho : G \to \mathrm{Aut}(P)$ is any representation of G in a finitely generated projective A-module P, any isomorphism $P \oplus Q \cong A^N$ gives a map $q(\rho) : BG \to B\,\mathrm{Aut}(P) \to BGL_N(A) \to BGL(A)^+$. A different embedding of P in A^N will give a map which is homotopic to the first, because the two maps only differ by conjugation and $BGL(A)^+$ is an H-space. (The action of $\pi_1(H)$ on $[X, H]$ is trivial for any H-space H and any space X; see [228, III.4.18]). Hence the map $q(\rho)$ is well-defined up to homotopy.

IV.5.1 **Example 5.1.** Recall from I.3 that the k^{th} exterior power $\wedge^k(P)$ of a finitely generated projective A-module P is also a projective module, of rank $\binom{\mathrm{rank}\,P}{k}$. Because $\wedge^k$ is a functor, it determines a group map $\wedge^k_P : \mathrm{Aut}(P) \to \mathrm{Aut}(\wedge^k P)$, i.e., a representation, for each P. We write Λ^k_P for $q(\wedge^k)$. Note that $\Lambda^0_P = *$.

Now any connected H-space, such as $BGL(A)^+$, has a multiplicative inverse (up to homotopy). Given a map $f : X \to H$, this allows us to construct maps $-f$, and to take formal $\mathbb{Z}$-linear combinations of maps.

IV.5.2 **Definition 5.2.** If P has rank n, we define $\lambda^k_P : B\,\mathrm{Aut}(P) \to BGL(A)^+$ to be the map

$$\lambda^k_P = \sum_{i=0}^{k-1} (-1)^i \binom{n+i-1}{i} \Lambda^{k-i}_P.$$

One can show directly that the maps λ^k_P are compatible with the inclusions of P in $P \oplus Q$, up to homotopy of course, giving the desired operations $\lambda^k : BGL(A)^+ \to BGL(A)^+$ (see Ex. 5.1). However, it is more useful to encode this bookkeeping in the Representation Ring $R_A(G)$, an approach which is due to Quillen.

Recall from II, Ex. 4.2, that the representation ring $R_A(G)$ is the Grothendieck group of the representations of G in finitely generated projective A-modules. We saw in *loc. cit.* that $R_A(G)$ is a special λ-ring.

IV.5.3 **Proposition 5.3.** *If* $0 \to (P', \rho') \to (P, \rho) \to (P'', \rho'') \to 0$ *is a short exact sequence of representations of* G*, then* $q(\rho) = q(\rho') + q(\rho'')$ *in* $[BG, BGL(A)^+]$.

Hence there is a natural map $q : R_A(G) \to [BG, BGL(A)^+]$.

Proof. It is clear from the H-space structure on $BGL(A)^+$ that $q(\rho \oplus \rho') = q(\rho) + q(\rho')$. By the above remarks, we may suppose that P' and P'' are free modules, of ranks m and n respectively. By universality, it suffices to consider the case in which $G = G_{m,n}$ is the automorphism group of the sequence, i.e., the upper triangular group $\begin{pmatrix} \mathrm{Aut}(P') & \mathrm{Hom}(P'', P') \\ 0 & \mathrm{Aut}(P'') \end{pmatrix}$. Quillen proved in [156] that in the limit, the inclusions $i : \mathrm{Aut}(P') \times \mathrm{Aut}(P'') \hookrightarrow G_{m,n}$ induce a homology isomorphism

$$\varinjlim H_*(G_{m,n}) \cong H_*(GL(A) \times GL(A)).$$

It follows that for any connected H-space H we have $[\varinjlim BG_{m,n}, H] \cong [BGL(A) \times BGL(A), H]$. Taking $H = BGL(A)^+$ yields the result. □

IV.5.3.1 **Example 5.3.1.** If ρ is a representation on a rank n module P, the elements $[\rho] - n$ and $\lambda^k([\rho] - n)$ of $R_A(G)$ determine maps $BG \to BGL(A)^+$. When $G = \mathrm{Aut}(P)$ and $\rho = \mathrm{id}_P$ is the tautological representation, it follows from the formula of Ex. II.4.2 that $\lambda^k([\mathrm{id}_P] - n)$ is the map Λ^k_P of 5.1.

We can now define the operations λ^k on $[BGL(A)^+, BGL(A)^+]$. As n varies, the representations id_n of $GL_n(A)$ are related by the relation $i_n^* \mathrm{id}_{n+1} = \mathrm{id}_n \oplus 1$, where $i_n : GL_n(A) \hookrightarrow GL_{n+1}(A)$ is the inclusion. Hence the virtual characters $\rho_n = \mathrm{id}_n - n \cdot 1$ satisfy $\rho_n = i_n^* \rho_{n+1}$. Since $i^* : R_A(GL_n A) \to R_A(GL_{n+1} A)$ is a homomorphism of λ-rings, we also have $\lambda^k \rho_n = i^*(\lambda^k \rho_{n+1})$. Hence we get a compatible family of homotopy classes $\lambda^k_n \in [BGL_n(A), BGL(A)^+]$.

Because each $BGL_n(A) \to BGL_{n+1}(A)$ is a closed cofibration, it is possible to inductively construct maps $\lambda^k_n : BGL_n(A) \to BGL(A)^+$ which are strictly compatible, so that by passing to the limit they determine a continuous map $\lambda^k_\infty : BGL(A) \to BGL(A)^+$ and even

$$\lambda^k : BGL(A)^+ \to BGL(A)^+.$$

The construction in Example 5.3.1 clearly applies to any compatible family of elements in the rings $R_A(GL_n(A))$. Indeed, we have a map

$$\varprojlim R_A(GL_n(A)) \to \varprojlim [BGL_n(A), BGL(A)^+] = [BGL(A)^+, BGL(A)^+]. \tag{5.3.2}$$ IV.5.3.2

For example, the operations ψ^k and γ^k may be defined in this way; see Ex. 5.2.

IV.5.4 **Definition 5.4.** If X is any based space, and $f : X \to BGL(A)^+$ any map, we define $\lambda^k f : X \to BGL(A)^+$ to be the composition of f and λ^k. This defines operations on $[X, BGL(A)^+]$ which we also refer to as λ^k. When $X = S^n$, we get operations $\lambda^k : K_n(A) \to K_n(A)$.

IV.5.4.1 **Example 5.4.1.** When $n = 1$ and $a \in A^\times$ is regarded as an element of $K_1(A)$, the formulas $\lambda^k(a) = a$ and $\psi^k(a) = a^k$ are immediate from the formula 5.2 for λ^k_A.

The abelian group $[X, BGL(A)^+]$ inherits an associative multiplication from the product on $BGL(A)^+$ described in 1.10: one uses the composition $X \xrightarrow{\Delta} X \wedge X \to BGL(A)^+ \wedge BGL(A)^+$. If $X = S^n$ for $n > 0$ (or if X is any suspension), this is the zero product because then the map $X \to X \wedge X$ is homotopic to 0.

Now recall from §II.4 that a λ-ring must satisfy $\lambda^0(x) = 1$, which requires an identity. In contrast, our λ^0 is zero. To fix this, we extend the operations to $K_0(A) \times [X, BGL(A)^+]$ by

$$\lambda^k(a, x) = (\lambda^k(a), \lambda^k(x) + a \cdot \lambda^{k-1}(x) + \cdots + \lambda^i(a)\lambda^{k-i}(x) + \cdots + \lambda^{k-1}(a)x.$$

Thus $\lambda^0(a, x) = (\lambda^0(a), \lambda^0(x)) = (1, 0)$, as required.

IV.5.5 **Theorem 5.5.** *For any based space X, the λ^k make $K_0(A) \times [X, BGL(A)^+]$ into a special λ-ring*

Proof. It suffices to consider the universal case $X = BGL(A)^+$. Since $\pi_1(X) = K_1(A)$, we have a map $R_A(K_1 A) \to [X, X]$. Via the transformation q of 5.3, we are reduced to checking identities in the rings $R_A(GL_n(A))$ by (5.3.2). For example, the formula $\lambda^k(x + y) = \sum \lambda^i(x)\lambda^{k-i}(y)$ comes from the identity

$$\lambda^k \circ \oplus = \sum \lambda^i \otimes \lambda^{k-i}$$

in $R_A(GL_m(A) \times GL_n(A))$. Similarly, the formal identities for $\lambda^k(xy)$ and $\lambda^n(\lambda^k x)$, listed in II.4.3.1 and which need to hold in special λ-rings, already hold in $R_A(G)$ and so hold in our setting via the map q. □

IV.5.5.1 **Corollary 5.5.1.** *If $n > 0$ then $\lambda^k : K_n(A) \to K_n(A)$ is additive, and we have $\psi^k(x) = (-1)^{k-1} k \lambda^k$.*

Proof. Since the products are zero, this is immediate from the formulas in II.4.1 and II.4.4 for $\lambda^k(x + y)$ and $\psi^k(x)$. □

If A is an algebra over a field of characteristic p, the Frobenius endomorphism Φ of A is defined by $\Phi(a) = a^p$. We say that A is *perfect* if Φ is an automorphism, *i.e.,* if A is reduced and for every $a \in A$ there is a $b \in A$ with $a = b^p$.

IV.5.5.2 **Corollary 5.5.2.** *If A is an algebra over a field of characteristic p, ψ^p is the Frobenius Φ^* on $K_n(A)$, $n>0$, and more generally on $[X, BGL(A)^+]$ for all X.*

Proof. This follows from the fact (Ex. II.4.2) that $\psi^p = \Phi^*$ on the representation ring $R_A(G)$, together with the observation that $q(\Phi^*) : K_n(A) \to K_n(A)$ is induced by $\Phi : A \to A$ by naturality in A. □

IV.5.6 **Proposition 5.6.** *If A is a perfect algebra over a field of characteristic p, then $K_n(A)$ is uniquely p-divisible for all $n > 0$.*

Proof. Since $n > 0$, we see from 5.5.1 that $\psi^p(x) = (-1)^{p-1} p \lambda^p(x)$ for $x \in K_n(A)$. Since $\psi^p = \Phi^*$ is an automorphism, so is multiplication by p. □

For any based space X there is a space FX homotopy equivalent to $B(\pi_1 X)$ and a natural map $X \to FX$ with $\pi_1(X) \xrightarrow{\cong} \pi_1(FX)$; if X is a simplicial space, FX is just the 2-coskeleton of X. Composing this map with the q of 5.3 gives a natural transformation $R_A(\pi_1 X) \to [X, BGL(A)^+]$ of functors from based spaces to groups.

IV.5.7 **Proposition 5.7.** *The natural transformation* $R_A(\pi_1 X) \xrightarrow{q} [X, BGL(A)^+]$ *is universal for maps to representable functors. That is, for any connected* H*-space* H *and any natural transformation* $\eta_X : R_A(\pi_1 X) \to [X, H]$ *there is a map* $f : BGL(A)^+ \to H$*, unique up to homotopy, such that* η_X *is the composite*

$$R_A(\pi_1 X) \xrightarrow{q} [X, BGL(A)^+] \xrightarrow{f} [X, H].$$

Like Theorem 1.8, this is proven by obstruction theory. Essentially, one considers the system of spaces $X = BGL_n(A)$ and the maps $BGL_n(A) \to H$ defined by $\eta_X(\mathrm{id}_n)$. See [89, 2.4] for details.

IV.5.7.1 **Example 5.7.1.** The above construction of operations works in the topological setting, allowing us to construct λ-operations on $[X, BU]$ extending the operations in II.4.1.3. It follows that $[X, BGL(\mathbb{C})^+] \to [X, BU]$ commutes with the operations λ^k and ψ^k for every X.

IV.5.8 **Example 5.8** (Finite fields)**.** Let $\mathbb{F}_q$ be a finite field, and $\mathbb{F}_q^\times \to \mathbb{C}^\times$ a homomorphism. It induces a homomorphism $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$ called the *Brauer lifting*. The composition of Brauer lifting with $R_{\mathbb{C}}(\pi_1 X) \to [X, BGL(\mathbb{C})^+]$ induces the map $BGL(\mathbb{F}_q)^+ \to BGL(\mathbb{C})^+ \to BU$ discussed in 1.12 above. Now an elementary calculation with characters (which we omit) shows that the Brauer lifting is actually a homomorphism of λ-rings. It follows from 5.7 and 5.7.1 that $[X, BGL(\mathbb{F}_q)^+] \to [X, BGL(\mathbb{C})^+] \to [X, BU]$ are homomorphisms of λ-rings. This was used in Theorem 1.12 to calculate $K_n(\mathbb{F}_q)$.

Compatibility with products

IV.5.9 **Theorem 5.9.** *The Adams operations* ψ^k *are compatible with the product on* K*-theory, in the sense that* $\psi^k(x \cdot y) = \psi^k(x) \cdot \psi^k(y)$ *for* $x \in K_m(A)$ *and* $y \in K_n(A)$.

Proof. It suffices to show that the following diagram commutes up to weak homotopy:

$$\begin{array}{ccc} BGL(A)^+ \wedge BGL(A)^+ & \longrightarrow & BGL(A)^+ \\ {\scriptstyle \psi^k \wedge \psi^k}\downarrow & & \downarrow{\scriptstyle \psi^k} \\ BGL(A)^+ \wedge BGL(A)^+ & \longrightarrow & BGL(A)^+. \end{array}$$

Via Proposition 5.7, this follows from the fact that the $R_A(GL_m(A) \times GL_n(A))$ are λ-rings for all m and n. $\square$

IV.5.9.1 **Example 5.9.1.** If F is a field then $\psi^k = k^2$ on $K_2(F)$. This is because $K_2(F)$ is generated by Steinberg symbols $\{a,b\}$ (III.6.1), and Example 5.4.1 implies that $\psi^k\{a,b\} = \{a^k, b^k\} = k^2\{a,b\}$. The same argument shows that $\psi^k = k^n$ on the image of $K_n^M(F) \to K_n(F)$; see §III.7.

IV.5.9.2 **Example 5.9.2.** For finite fields, we have $\psi^k(x) = k^i\,x$ for $x \in K_{2i-1}(\mathbb{F}_q)$. This follows from Example 5.8 and the fact (II.4.4.1) that $\psi^k = k^i$ on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$.

The γ-filtration

Consider the γ-filtration (II.4.7) on $K_0(A) \times K_n(A)$; If $n > 0$ then $F_\gamma^k K_n(A)$ is generated by all $\gamma^{k'}(x)$ and $a\cdot\gamma^j(x)$ with $k' \geq k$, $a \in F_\gamma^i K_0(A)$, $x \in K_n(A)$, $i > 0$ and $i + j \geq k$. (There are other possible definitions, using the ring structure on $K_*(A)$, but they coincide up to torsion [174].) For this reason, we shall ignore torsion and deal with the γ-filtration on $K_n(A) \otimes \mathbb{Q}$.

Because $x = \gamma^1(x)$ we have $K_n(A) = F_\gamma^1 K_n(A)$ for $n > 0$. The next layer F_γ^1/F_γ^2 of the filtration is also small.

IV.5.10 **Proposition 5.10.** *(Kratzer) For all commutative A, $SK_1(A) = F_\gamma^2 K_1(A)$, and $F_\gamma^1 K_1(A)/F_\gamma^2 K_1(A) = A^\times$, and for $n \geq 2$: $K_n(A) = F_\gamma^2 K_n(A)$.*

Proof. It suffices to compute in $\pi_n BSL(A)^+$, which equals $SK_1(A)$ for $n = 1$ and $K_n(A)$ for $n > 1$ (see Ex. 1.8(a)). For $G = SL_N(A)$ the identity $\det(\mathrm{id}_N) = 1$ in $R(G)$ may be written in terms of $\rho = \mathrm{id}_N - N$ as $\gamma^1(\rho) + \gamma^2(\rho) + \cdots + \gamma^N(\rho)$. Because $\gamma^i(\rho) = 0$ for $i > N$ (Exercise 5.4), this yields the identity $\sum_1^\infty \gamma^i(x) = 0$ for $x \in \pi_n BSL_N(A)^+$. Since $x = \gamma^1(x)$, this shows that $x \in F_\gamma^2 \pi_n BSL_N(A)$. ☐

IV.5.10.1 **Remark 5.10.1.** Soulé has proven [174, Thm. 1] that if A has stable range $sr(A) < \infty$ (I, Ex. 1.5) then γ^k vanishes on $K_n(A)$ for all $k \geq n + sr(A)$. This is a useful bound because $sr(R) \leq \dim(A) + 1$ for noetherian A. If F is a field, $\psi^k = k^n$ and $\gamma^n = (-1)^{n-1}(n-1)!$ on the image of $K_n^M(F) \to K_n(F)$, by 5.9.1, so the bound is best possible. The proof uses Volodin's construction of K-theory.

IV.5.11 **Theorem 5.11.** *For $n > 0$, the eigenvalues of ψ^k on $K_n(A) \otimes \mathbb{Q}$ are a subset of $\{1, k, k^2, \dots\}$, and the subspace $K_n^{(i)}(A)$ of eigenvectors for $\psi^k = k^i$ is independent of k. Finally, the ring $K_*(A) \otimes \mathbb{Q}$ is isomorphic to the bigraded ring $\oplus_{n,i} K_n^{(i)}(A)$.*

Proof. Since every element of $K_n(A)$ comes from the K-theory of a finitely generated subring, we may assume that $sr(A) < \infty$. As in the proof of II.4.10, the linear operator $\prod_1^N(\psi^k - k^i)$ is trivial on each $F_\gamma^i/F_\gamma^{i+1}$ for large N, and this implies that $K_n(A) \otimes \mathbb{Q}$ is the direct sum of the eigenspaces for $\psi^k = k^i$, $1 \leq i \leq N$. Since ψ^k and ψ^ℓ commute, it follows by downward induction on i that they have the same eigenspaces, *i.e.*, $K_n^{(i)}(A)$ is independent of k. Finally, the bigraded ring structure follows from 5.9. ☐

IV.5.11.1 **Example 5.11.1.** (Geller-Weibel) Let $A = \mathbb{C}[x_1, \dots, x_n]/(x_i x_j = 0, i \neq j)$ be the coordinate ring of the coordinate axes in $\mathbb{C}^n$. Then the Loday symbol $\langle\langle x_1, ..., x_n\rangle\rangle$ of Ex. 1.22 projects nontrivially into $K_n^{(i)}(A)$ for all i in the range $2 \leq i \leq n$. In particular, $K_n^{(i)}(A) \neq 0$ for each of these i. As $sr(A) = 2$, these are the only values of i allowed by Soulé's bound in 5.10.1.

The ring of Example 5.11.1 is not regular. In contrast, it is widely believed that the following conjecture is true for all regular rings; it may be considered to be *the* outstanding problem in algebraic K-theory. It is due to Beilinson and Soulé [174]. (See Exercise VI.4.6 for the connection to motivic cohomology.)

IV.5.12 **Vanishing Conjecture 5.12.** *(Beilinson-Soulé) If $i < n/2$ and A is regular then $K_n(A) = F_\gamma^i K_n(A)$.*

EXERCISES

EIV.5.1 **5.1.** Show that the composition of the cofibration $B\,\mathrm{Aut}(P) \to B\,\mathrm{Aut}(P \oplus Q)$ with $\lambda^k_{P\oplus Q}$ is homotopic to the map λ^k_P. By modifying $\lambda^k_{P\oplus Q}$, we can make the composition equal to λ^k_P. Using the free modules A^n and induction, conclude that we have maps $\lambda^k : BGL(A) \to BGL(A)^+$ and hence operations λ^k on $BGL(A)^+$, well defined up to homotopy.

One could use 4.1.1(c), 4.10 and 4.11.1 to consider the limit over $\mathrm{Aut}(P)$ for all projective modules P; the same construction will work except that there will be more bookkeeping.

EIV.5.2 **5.2.** Modify the construction of 5.3.2 to construct operations ψ^k and γ^k on the ring $K_0(A) \times [X, BGL(A)^+]$ for all X. (See II.4.4 and II.4.5.)

EIV.5.3 **5.3.** Show that the λ-operations are compatible with $K_1(A[t, 1/t]) \xrightarrow{\partial} K_0(A)$, the map in the Fundamental Theorem III.3.6, in the sense that for every $x \in K_0(A)$, $t \cdot x \in K_1(A[t, 1/t])$ satisfies $\partial\lambda^k(t \cdot x) = (-1)^{k-1}\psi^k(x)$.

EIV.5.4 **5.4.** (γ-dimension) Consider the γ-filtration (II.4.7) on $K_0(A) \times K_n(A)$, and show that every element of $K_n(A)$ has finite γ-dimension (II.4.5). *Hint:* Because S^n is a finite complex, each $x \in K_n(A)$ comes from some $\pi_n BGL_n(A)^+$. If $i > n$, show that γ^i kills the representation $[\mathrm{id}_n] - n$ and apply the map q.

EIV.5.5 **5.5.** For any commutative ring A, show that the ring structure on $R_A(G)$ induces a ring structure on $[X, K_0(A) \times BGL(A)^+]$.

EIV.5.6 **5.6.** Suppose that a commutative A-algebra B is finitely generated and projective as an A-module. Use 5.3 to show that the restriction of scalars map $R_B(G) \to R_A(G)$ induces a "transfer" map $BGL(B)^+ \to BGL(A)^+$. Show that it agrees on homotopy groups with the transfer maps for K_1 and K_2 in III.1.7.1 and III.5.6.3, respectively. We will encounter other constructions of the transfer in 6.3.2.

EIV.5.7 **5.7.** Use Ex. 5.3 to give an example of a regular ring A such that $K_3^{(2)}(A)$ and $K_3^{(3)}(A)$ are both nonzero.

6 Quillen's Q–construction for exact categories

The higher K-theory groups of a small exact category $\mathcal{A}$ are defined to be the homotopy groups $K_n(\mathcal{A}) = \pi_{n+1}(BQ\mathcal{A})$ of the geometric realization of a certain auxiliary category $Q\mathcal{A}$, which we now define. This category has the same objects as $\mathcal{A}$, but morphisms are harder to describe. Here is the formal definition; we refer the reader to Exercise IV.6.1 6.1 for a more intuitive interpretation of morphisms in terms of subquotients.

IV.6.1 **Definition 6.1.** Let $\mathcal{A}$ be an exact category. A morphism from A to B in $Q\mathcal{A}$ is an equivalence class of diagrams

$$A \overset{j}{\twoheadleftarrow} B_2 \overset{i}{\rightarrowtail} B, \qquad (6.1.1)$$ IV.6.1.1

where j is an admissible epimorphism and i is an admissible monomorphism in $\mathcal{A}$. Two such diagrams are equivalent if there is an isomorphism between them which is the identity on A and B. The composition of the above morphism with a morphism $B \twoheadleftarrow C_2 \rightarrowtail C$ is $A \twoheadleftarrow C_1 \rightarrowtail C$, where $C_1 = B_2 \times_B C_2$.

$$\begin{array}{ccccccc} & & C_1 & \rightarrowtail & C_2 & \rightarrowtail & C \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\ A & \twoheadleftarrow & B_2 & \rightarrowtail & B & & \end{array}$$

Two distinguished types of morphisms play a special role in $Q\mathcal{A}$: the admissible monics $A \rightarrowtail B$ (take $B_2 = A$) and the oppositely oriented admissible epis $A \twoheadleftarrow B$ (take $B_2 = B$). Both types are closed under composition, and the composition of $A \twoheadleftarrow B_2$ with $B_2 \rightarrowtail B$ is the morphism (IV.6.1.1 6.1.1). In fact, every morphism in $Q\mathcal{A}$ factors as such a composition in a way that is unique up to isomorphism.

IV.6.1.2 **Subobjects 6.1.2.** Recall from Mac [116] that (in any category) a *subobject* of an object B is an equivalence class of monics $B_2 \rightarrowtail B$, two monics being equivalent if they factor through each other. In an exact category $\mathcal{A}$, we call a subobject *admissible* if any (hence every) representative $B_2 \rightarrowtail B$ is an admissible monic.

By definition, every morphism from A to B in $Q\mathcal{A}$ determines a unique admissible subobject of B in $\mathcal{A}$. If we fix a representative $B_2 \rightarrowtail B$ for each subobject in $\mathcal{A}$, then a morphism in $Q\mathcal{A}$ from A to B is a pair consisting of an admissible subobject B_2 of B and an admissible epi $B_2 \twoheadrightarrow A$.

In particular, this shows that morphisms from 0 to B in $Q\mathcal{A}$ are in 1–1 correspondence with admissible subobjects of B.

Isomorphisms in $Q\mathcal{A}$ are in 1–1 correspondence with isomorphisms in $\mathcal{A}$. To see this, note that every isomorphism $i\colon A \cong B$ in $\mathcal{A}$ gives rise to an isomorphism in $Q\mathcal{A}$, represented either by $A \overset{i}{\rightarrowtail} B$ or by $A \overset{i^{-1}}{\twoheadleftarrow} B$. Conversely, since the subobject determined by an isomorphism in $Q\mathcal{A}$ must be the maximal subobject $B \overset{=}{\rightarrowtail} B$, every isomorphism in $Q\mathcal{A}$ arises in this way.

IV.6.1.3 **Remark 6.1.3.** Some set-theoretic restriction is necessary for $Q\mathcal{A}$ to be a category in our universe. It suffices for $\mathcal{A}$ to be *well-powered*, *i.e.,* for each object of $\mathcal{A}$ to have a set of subobjects. We shall tacitly assume this, since we will soon need the stronger assumption that $\mathcal{A}$ is a small category.

We now consider the geometric realization $BQ\mathcal{A}$ as a based topological space, the basepoint being the vertex corresponding to the object 0. In fact, $BQ\mathcal{A}$ is a connected CW complex, because the morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$ give paths in $BQ\mathcal{A}$ from the basepoint 0 to every vertex A. (See Lemma 3.3.) The morphisms $0 \twoheadleftarrow A$ also give paths from 0 to A in $Q\mathcal{A}$.

IV.6.2 **Proposition 6.2.** *The geometric realization $BQ\mathcal{A}$ is a connected CW complex with $\pi_1(BQ\mathcal{A}) \cong K_0(\mathcal{A})$. The element of $\pi_1(BQ\mathcal{A})$ corresponding to $[A] \in K_0(\mathcal{A})$ is represented by the based loop composed of the two edges from 0 to A:*

$$0 \rightarrowtail A \twoheadrightarrow 0.$$

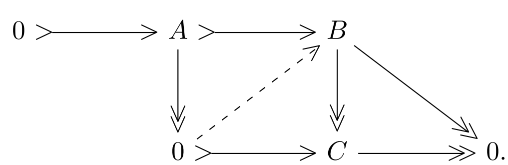
Proof. Let T denote the family of all morphisms $0 \rightarrowtail A$ in $Q\mathcal{A}$. Since each nonzero vertex occurs exactly once, T is a maximal tree. By Lemma 3.4, $\pi_1(BQ\mathcal{A})$ has the following presentation: it is generated by the morphisms in $Q\mathcal{A}$, modulo the relations that $[0 \rightarrowtail A] = 1$ and $[f]\cdot[g] = [f \circ g]$ for every pair of composable arrows in $Q\mathcal{A}$. Moreover, the element of $\pi_1(BQ\mathcal{A})$ corresponding to a morphism from A to B is the based loop following the edges $0 \rightarrowtail A \to B \leftarrowtail 0$.

Since the composition $0 \rightarrowtail B_2 \rightarrowtail B$ is in T, this shows that $[B_2 \rightarrowtail B] = 1$ in $\pi_1(BQ\mathcal{A})$. Therefore $[A \twoheadleftarrow B_2 \rightarrowtail B] = [A \twoheadleftarrow B_2]$. Similarly, the composition $0 \twoheadleftarrow A \twoheadleftarrow B$ yields the relation $[A \twoheadleftarrow B][0 \twoheadleftarrow A] = [0 \twoheadleftarrow B]$. Since every morphism (6.1.1) factors, this shows that $\pi_1(BQ\mathcal{A})$ is generated by the morphisms $[0 \twoheadleftarrow A]$.

If $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathcal{A}$, then the composition $0 \rightarrowtail C \twoheadleftarrow B$ in $Q\mathcal{A}$ is $0 \twoheadleftarrow A \rightarrowtail B$. This yields the additivity relation

$$[0 \twoheadleftarrow B] = [C \twoheadleftarrow B]\,[0 \twoheadleftarrow C] = [0 \twoheadleftarrow A]\,[0 \twoheadleftarrow C] \qquad (6.2.1)$$ IV.6.2.1

in $\pi_1(BQ\mathcal{A})$, represented by the following picture in $BQ\mathcal{A}$:



Since every relation $[f]\cdot[g] = [f \circ g]$ may be rewritten in terms of the additivity relation, $\pi_1(BQ\mathcal{A})$ is generated by the $[0 \twoheadleftarrow A]$ with (6.2.1) as the only relation. Therefore $K_0(\mathcal{A}) \cong \pi_1(BQ\mathcal{A})$. □

IV.6.2.2 **Example 6.2.2.** The presentation for $\pi_1(BQ\mathcal{A})$ in the above proof yields a function from morphisms in $Q\mathcal{A}$ to $K_0(\mathcal{A})$. It sends $[A \twoheadleftarrow B_2 \rightarrowtail B]$ to $[B_1]$, where B_1 is the kernel of $B_2 \twoheadrightarrow A$.

IV.6.3 **Definition 6.3.** Let $\mathcal{A}$ be a small exact category. Then $K\mathcal{A}$ denotes the space $\Omega BQ\mathcal{A}$, and we set

$$K_n(\mathcal{A}) = \pi_n K\mathcal{A} = \pi_{n+1}(BQ\mathcal{A}) \qquad \text{for} \quad n \geq 0.$$

Proposition 6.2 shows that this definition of $K_0(\mathcal{A})$ agrees with the one given in Chapter II. Note that any exact functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a functor $Q\mathcal{A} \to Q\mathcal{B}$, hence maps $BQ\mathcal{A} \to BQ\mathcal{B}$ and $K_n(\mathcal{A}) \to K_n(\mathcal{B})$. Thus the space $K\mathcal{A} = \Omega\, BQ\mathcal{A}$ and all the groups $K_n(\mathcal{A})$ are functors from exact categories and exact functors to spaces and abelian groups, respectively. Moreover, isomorphic functors induce the same map on K-groups, because they induce isomorphic functors $Q\mathcal{A} \to Q\mathcal{A}'$.

IV.6.3.1 **Remark 6.3.1.** If an exact category $\mathcal{A}$ is not small but has a set of isomorphism classes of objects then we define $K_n(\mathcal{A})$ to be $K_n(\mathcal{A}')$, where $\mathcal{A}'$ is a small subcategory equivalent to $\mathcal{A}$. By Ex. 6.2 this is independent of the choice of $\mathcal{A}'$. From now on, whenever we talk about the K-theory of a large exact category $\mathcal{A}$ we will use this device, assuming tacitly that we have replaced it by a small $\mathcal{A}'$. For example, this is the case in the following definitions.

IV.6.3.2 **Definition 6.3.2.** Let R be a ring with unit, and set $K(R) = K\mathbf{P}(R)$, where $\mathbf{P}(R)$ denotes the exact category of finitely generated projective R-modules. We define the K-groups of R by $K_n(R) = K_n\mathbf{P}(R)$. For $n = 0$, Lemma 6.2 shows that this agrees with the definition of $K_0(R)$ in Chapter II. For $n \geq 1$, agreement with the (nonfunctorial) +–construction definition 1.1.1 of $K(R)$ will have to wait until Section 7.

Let $f : R \to S$ be a ring homomorphism such that S is finitely generated and projective as an R-module. Then there is a forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ and hence a "transfer" functor $f_* : K_*(S) \to K_*(R)$.

IV.6.3.3 **Definition 6.3.3.** If R is noetherian, let $\mathbf{M}(R)$ denote the category of fin. gen. R-modules. Otherwise, $\mathbf{M}(R)$ is the category of pseudo-coherent modules defined in II.7.1.4. We set $G(R) = K\mathbf{M}(R)$ and define the G-groups of R by $G_n(R) = K_n\mathbf{M}(R)$. For $n = 0$, this also agrees with the definition in Chapter II.

Let $f : R \to S$ be a ring map. When S is finitely generated as an R-module (and S is in $\mathbf{M}(R)$), there is a contravariant "transfer" map $f_* : G(S) \to G(R)$, induced by the forgetful functor $f_* : \mathbf{M}(S) \to \mathbf{M}(R)$, as in II.6.2.

On the other hand, if S is flat as an R-module, the exact base change functor $\otimes_R S : \mathbf{M}(R) \to \mathbf{M}(S)$ induces a covariant map $f^* : G(R) \to G(S)$ hence maps $f^* : G_n(R) \to G_n(S)$ for all n. This generalizes the base change map $G_0(R) \to G_0(S)$ of II.6.2. We will see in V.3.5 that the base change map is also defined when S has finite flat dimension over R.

IV.6.3.4 **Definition 6.3.4.** Similarly, if X is a scheme which is quasi-projective (over a commutative ring), we define $K(X) = K\mathbf{VB}(X)$ and $K_n(X) = K_n\mathbf{VB}(X)$. If X is noetherian, we define $G(X) = K\mathbf{M}(X)$ and $G_n(X) = K_n\mathbf{M}(X)$. For $n = 0$, this agrees with the definition of $K_0(X)$ and $G_0(X)$ in Chapter II.

IV.6.3.5 **Morita Invariance 6.3.5.** Recall from II.2.7 that if two rings R and S are Morita equivalent then there are equivalences $\mathbf{P}(R) \cong \mathbf{P}(S)$ and $\mathbf{M}(R) \cong \mathbf{M}(S)$. It follows that $K_n(R) \cong K_n(S)$ and $G_n(R) \cong G_n(S)$ for all n.

IV.6.4 **Elementary properties 6.4.** Here are some elementary properties of the above definition.

If $\mathcal{A}^{op}$ denotes the opposite category of $\mathcal{A}$, then $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$ by Ex. 6.3, so we have $K_n(\mathcal{A}^{op}) = K_n(\mathcal{A})$. For example, if R is a ring then $\mathbf{P}(R^{op}) \cong \mathbf{P}(R)^{op}$ by $P \mapsto \mathrm{Hom}_R(P, R)$, so we have $K_n(R) \cong K_n(R^{op})$.

The product or direct sum $\mathcal{A} \oplus \mathcal{A}'$ of two exact categories is exact by Example II.7.1.6, and $Q(\mathcal{A} \oplus \mathcal{A}') = Q\mathcal{A} \times Q\mathcal{A}'$. Since the geometric realization preserves products by 3.1(4), we have $BQ(\mathcal{A} \oplus \mathcal{A}') = BQ\mathcal{A} \times BQ\mathcal{A}'$ and hence $K_n(\mathcal{A} \oplus \mathcal{A}') \cong K_n(\mathcal{A}) \oplus K_n(\mathcal{A}')$. For example, if R_1 and R_2 are rings then $\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \oplus \mathbf{P}(R_2)$ and we have $K_n(R_1 \times R_2) \cong K_n(R_1) \oplus K_n(R_2)$. (Cf. Ex. 1.7.) Similarly, if a quasi-projective scheme X is the disjoint union of two components X_i, then $\mathbf{VB}(X)$ is the sum of the $\mathbf{VB}(X_i)$ and we have $K_n(X) \cong K_n(X_1) \oplus K_n(X_2)$.

The direct sum $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an exact functor, and its restriction to either factor is an isomorphism. It follows that $B\oplus : BQ\mathcal{A} \times BQ\mathcal{A} \to BQ\mathcal{A}$ endows $BQ\mathcal{A}$ with the structure of a homotopy-commutative H-space. (It is actually an infinite loop space; see 6.5.1).

Finally, suppose that $i \mapsto \mathcal{A}_i$ is a functor from some small filtering category I to exact categories and exact functors. Then the filtered colimit $\mathcal{A} = \varinjlim \mathcal{A}_i$ is an exact category (Ex. II.7.9), and $Q\mathcal{A} = \varinjlim Q\mathcal{A}_i$. Since geometric realization preserves filtered colimits by 3.1(3), we have $BQ\mathcal{A} = \varinjlim BQ\mathcal{A}_i$ and hence $K_n(\mathcal{A}) = \varinjlim K_n(\mathcal{A}_i)$. The K_0 version of this result was given in chapter II, as 6.2.7 and 7.1.7.

For example, if a ring R is the filtered union of subrings R_i we have $K_n(R) \cong \varinjlim K_n(R_i)$. However, $i \mapsto \mathbf{P}(R_i)$ is not a functor. One way to fix this is to replace the category $\mathbf{P}(R_i)$ by the equivalent category $\mathbf{P}'(R_i)$ whose objects are idempotent matrices over R_i; $\mathbf{P}(R)$ is equivalent to the category $\mathbf{P}'(R) = \varinjlim \mathbf{P}_i$. Alternatively one could use the Kleisli rectification, which is described in Ex. 6.5.

IV.6.4.1 **Example 6.4.1** (Cofinality)**.** Let $\mathcal{B}$ be an exact subcategory of $\mathcal{A}$ which is closed under extensions in $\mathcal{A}$, and which is cofinal in the sense that for every A in $\mathcal{A}$ there is an A' in $\mathcal{A}$ so that $A \oplus A'$ is in $\mathcal{B}$. Then $BQ\mathcal{B}$ is homotopy equivalent to the covering space of $BQ\mathcal{A}$ corresponding to the subgroup $K_0(\mathcal{B})$ of $K_0(\mathcal{A}) = \pi_1(BQ\mathcal{A})$. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n > 0$.

A special case of this is sketched in Exercise 6.6; the general case follows from this case using the version 8.9.1 of Waldhausen Cofinality 8.9 below. Note that $K_0(\mathcal{B})$ is a subgroup of $K_0(\mathcal{A})$ by II.7.2.

Waldhausen constructed a delooping of $BQ\mathcal{A}$ in [214, p. 194], using the "QQ" construction. This in turn provides a context for products.

IV.6.5 **Definition 6.5.** When $\mathcal{A}$ is a small exact category, $QQ\mathcal{A}$ is the bicategory whose bimorphisms are equivalence classes of commutative diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccc} \bullet & \twoheadleftarrow & \bullet & \rightarrowtail & \bullet \\ \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\ \bullet & \twoheadleftarrow & \bullet & \rightarrowtail & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \twoheadleftarrow & \bullet & \rightarrowtail & \bullet \end{array}$$

in which the four little squares can be embedded in a 3×3 diagram with short exact rows and columns. Two such diagrams are equivalent if they are isomorphic by an isomorphism which restricts to the identity on each corner object.

Waldhausen proved that the loop space $\Omega QQ\mathcal{A}$ is homotopy equivalent to $BQ\mathcal{A}$ (see [214, p. 196] and Ex. 6.8). Thus we have

$$K_n(\mathcal{A}) = \pi_{n+1}BQ\mathcal{A} \cong \pi_{n+2}BQQ\mathcal{A}.$$

IV.6.5.1 **Remark 6.5.1.** There are also n-fold categories $Q^n\mathcal{A}$, defined exactly as in 6.5, with $\Omega BQ^{n+1}\mathcal{A} \simeq BQ^n\mathcal{A}$. The sequence of the $BQ^n\mathcal{A}$ (using $\Omega BQ\mathcal{A}$ if $n=0$) forms an Ω-spectrum $\mathbf{K}(\mathcal{A})$, making $K(\mathcal{A})$ into an infinite loop space.

Products

IV.6.6 **Definition 6.6.** If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are exact categories, a functor $\otimes : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ is called *biexact* if (i) each partial functor $A\otimes - : \mathcal{B}\to\mathcal{C}$ and $-\otimes B : \mathcal{A}\to\mathcal{C}$ is exact, and (ii) $A\otimes 0 = 0\otimes B = 0$ for the distinguished zero objects of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

This is the same as the definition of biexact functor in II.7.4. Note that condition (ii) can always be arranged by replacing $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ by equivalent exact categories.

Given such a biexact functor, the bicategory map $Q\mathcal{A}\otimes Q\mathcal{B}\to \mathrm{bi}(Q\mathcal{C})$ of 3.10.2 factors through the forgetful functor $QQ\mathcal{C}\to\mathrm{bi}(Q\mathcal{C})$. The functor $Q\mathcal{A}\otimes Q\mathcal{B}\to QQ\mathcal{C}$ sends a pair of morphisms $A_0 \twoheadleftarrow A_1 \rightarrowtail A_2$, $B_0 \twoheadleftarrow B_1 \rightarrowtail B_2$ to the bimorphism

$$\begin{array}{ccccc} A_0\otimes B_0 & \twoheadleftarrow & A_1\otimes B_0 & \rightarrowtail & A_2\otimes B_0 \\ \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\ A_0\otimes B_1 & \twoheadleftarrow & A_1\otimes B_1 & \rightarrowtail & A_2\otimes B_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_0\otimes B_2 & \twoheadleftarrow & A_1\otimes B_2 & \rightarrowtail & A_2\otimes B_2 \end{array} \qquad (6.6.1)$$ IV.6.6.1

Now the geometric realization IV.3.6 3.6 of the bifunctor $\otimes : Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ is a map $BQ\mathcal{A} \times BQ\mathcal{B} \to BQQ\mathcal{C}$ by IV.3.10.1 3.10.1. Since $\otimes$ sends $Q\mathcal{A} \otimes 0$ and $0 \otimes Q\mathcal{B}$ to 0, by the technical condition (ii), $B\otimes$ sends $BQ\mathcal{A} \times 0$ and $0 \times BQ\mathcal{B}$ to the basepoint, and hence factors through a map

$$BQ\mathcal{A} \wedge BQ\mathcal{B} \to BQQ\mathcal{C}, \tag{6.6.2}$$ IV.6.6.2

and in fact a pairing $\mathbf{K}(\mathcal{A}) \wedge \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C})$ of spectra; see Gillet [67, 7.12]. The reduced join operation Wh [228, p. 480] yields bilinear maps

$$K_i(\mathcal{A}) \otimes K_j(\mathcal{B}) = \pi_{i+1}(BQ\mathcal{A}) \otimes \pi_{j+1}(BQ\mathcal{B}) \to \\ \pi_{i+j+2}(BQ\mathcal{A} \wedge BQ\mathcal{B}) \to \pi_{i+j+2}(BQ\mathcal{C}) \cong K_{i+j}(\mathcal{C}). \tag{6.6.3}$$ IV.6.6.3

IV.6.6.4 **Remark 6.6.4.** We say that $\mathcal{A}$ *acts upon* $\mathcal{B}$ if there is a biexact $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$. If there is an object A_0 of $\mathcal{A}$ so that $A_0 \otimes -$ is the identity on $\mathcal{B}$, the map $S^1 = B(0 \rightrightarrows 1) \to BQ\mathcal{A}$ given by the diagram $0 \rightarrowtail A_0 \twoheadrightarrow 0$ of IV.6.2 6.2 induces a map $S^1 \wedge BQ\mathcal{A} \to B(Q\mathcal{A} \otimes Q\mathcal{B}) \to BQQ\mathcal{B}$. Its adjoint $BQ\mathcal{A} \to \Omega BQQ\mathcal{A}$ is the natural map of IV.6.5 6.5 (see Ex. EIV.6.8 6.8).

When there is an associative pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $K_*(\mathcal{A})$ becomes a graded ring; it has a unit $[A_0] \in K_0(\mathcal{A})$ if $A_0 \otimes - = - \otimes A_0 = \mathrm{id}_{\mathcal{A}}$, by the preceeding paragraph, and $\mathbf{K}(\mathcal{A})$ is a ring spectrum. When $\mathcal{A}$ acts on $\mathcal{B}$ and the two evident functors $\mathcal{A} \times \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ agree up to natural isomorphism, the pairing makes $K_*(\mathcal{B})$ into a left $K_*(\mathcal{A})$-module.

IV.6.6.5 **Example 6.6.5.** These remarks apply in particular to the category $\mathcal{A} = \mathbf{P}(R)$ over a commutative ring R, and $\mathbf{VB}(X)$ over a scheme X. Tensor product makes $K_*(R) = K_*\mathbf{P}(R)$ and $K_*(X) = K_*\mathbf{VB}(X)$ into graded-commutative rings with unit. For every R-algebra A, $K_*(A)$ and $G_*(A)$ are 2-sided graded $K_*(R)$-modules, and $G_*(X)$ is a graded $K_*(X)$-module.

If $f : A \to B$ is an R-algebra map, and B is finite over A, the finite transfer $f_* : G(B) \to G(A)$ is a $K_*(R)$-module homomorphism: $f_*(x \cdot y) = f_*(x) \cdot y$ for $x \in G_*(B)$ and $y \in K_*(R)$. This fact is sometimes referred to as the *projection formula*, and holds because $f_*(x \cdot y)$ and $x \cdot f_*(y)$ arise from the isomorphic functors $M \otimes_B (B \otimes_R P) \cong M \otimes_A (A \otimes_R P)$ of functors $\mathbf{M}(B) \times \mathbf{P}(R) \to \mathbf{M}(A)$.

The $W(R)$-module $NK_*(A)$

IV.6.7 **6.7.** Let k be a commutative ring. We saw in II.II.7.4.3 7.4.3 that the exact endomorphism category $\mathbf{End}(k)$ of pairs (P, α) has an associative, symmetric biexact pairing with itself, given by $\otimes_k$. This makes $K_*\mathbf{End}(k)$ into a graded-commutative ring. As in *loc. cit.*, the functors $\mathbf{P}(k) \to \mathbf{End}(k) \to \mathbf{P}(k)$ decompose this ring as a product of $K_*(k)$ and another graded-commutative ring which we call $\mathrm{End}_*(k)$.

If R is an k-algebra, $\mathbf{End}(k)$ acts associatively by $\otimes_k$ on the exact category $\mathbf{Nil}(R)$ of nilpotent endomorphisms (II.II.7.4.4 7.4.4), and on its subcategories $F_m\mathbf{Nil}(R)$ (Ex. II.EII.7.17 7.17). As $\mathbf{Nil}(R)$ is their union, we see that $K_*\mathbf{Nil}(R) = \operatorname{colim} K_*F_m\mathbf{Nil}(R)$ is a filtered $K_*\mathbf{End}(k)$-module.

Let $\mathrm{Nil}(R)$ denote the fiber of the forgetful functor $K\mathbf{Nil}(R) \to K(R)$; since this is split, we have $K\mathbf{Nil}(R) \simeq K(R) \times \mathrm{Nil}(R)$ and $K_*\mathbf{Nil}(R) \cong K_*(R) \times \mathrm{Nil}_*(R)$, where $\mathrm{Nil}_*(R) = \pi_* \mathrm{Nil}(R)$ is a graded $\mathrm{End}_*(k)$-module.

By Almkvist's Theorem II.7.4.3, $\mathrm{End}_0(k)$ is isomorphic to the subgroup of $W(k) = (1 + tk[[t]])^\times$ consisting of all quotients $f(t)/g(t)$ of polynomials in $1 + tR[t]$. Stienstra observed in [177] (cf. [176]) that the $\mathrm{End}_0(k)$-module structure extended to a $W(k)$-module structure by the following device. There are exact functors $F_m, V_m : \mathbf{Nil}(R) \to \mathbf{Nil}(R)$ defined by $F_m(P, \nu) = (P, \nu^m)$ and $V_m(P, \nu) = (P[t]/(t^m - \nu), t)$ (see Ex. II.7.16). Stienstra proved in [176] that $(V_m\alpha) \cdot \nu = V_m(\alpha \cdot F_m(\nu))$ for $\alpha \in \mathrm{End}_0(k)$ and $\nu \in \mathrm{Nil}_*(R)$. Since F_m is zero on $F_m\mathbf{Nil}(R)$, the elements $V_m(\alpha)$ act as zero on the image $F_m \mathrm{Nil}_*(R)$ of $K_*F_m\mathbf{Nil}(R) \to K_*\mathbf{Nil}(R) \to \mathrm{Nil}_*(R)$.

For example, the class of $\alpha = [(k, a)] - [(k, 0)])$ in $\mathrm{End}_0(k) \subseteq W(R)$ is $1 - at$, so $V_m(\alpha) = (1 - at^m)$ acts as zero. Stienstra also proves in [177] that if $g(t) = 1 + ...$ has degree $< m$ and $f(t)$ is any polynomial then the element $1 + t^m(f/g)$ of $\mathrm{End}_0(k)$ acts as zero on $F_m \mathrm{Nil}_*(R)$. Hence the ideal $\mathrm{End}_0(R) \cap (1 + t^m R[[t]])$ is zero on $F_m \mathrm{Nil}_*(R)$. Writing an element of $W(k)$ as a formal factorization $f(t) = \prod_{i=1}^\infty (1 - a_m t^m)$, the formula $f \cdot \nu = \sum (1 - a_m t^m) \cdot \nu$ makes sense as a finite sum.

IV.6.7.1 **Proposition 6.7.1.** *If $k = \mathbb{Z}/p\mathbb{Z}$, $\mathrm{Nil}_*(R)$ is a graded p-group.*
If $k = S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, $\mathrm{Nil}_*(R)$ is a graded k-module.

Proof. If $k = S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, the map $m \mapsto (1 - t)^m$ defines a ring homomorphism from k into $W(k)$, so any $W(k)$-module is a k-module. If $p = 0$ (or even $p^\nu = 0$) in k then for each n the formal factorization of $(1 - t)^{p^N}$ involves only $(1 - a_m t^m)$ for $m \geq n$. It follows that p^N annihilates the image of $K_*F_n\mathbf{Nil}(R)$ in $\mathrm{Nil}_*(R)$. Since $\mathrm{Nil}_*(R)$ is the union of these images, the result follows. □

We will see in V.8.1 that there is an isomorphism $NK_{n+1}(R) \cong \mathrm{Nil}_n(R)$, so what we have really seen is that $NK_*(R)$ is a graded $\mathrm{End}_*(k)$-module, with the properties given by 6.7.1:

IV.6.7.2 **Corollary 6.7.2.** *If $k = \mathbb{Z}/p\mathbb{Z}$, each $NK_n(R)$ is a p-group.*
If $k = S^{-1}\mathbb{Z}$, or if k is a $\mathbb{Q}$-algebra, each $NK_n(R)$ is a k-module.

IV.6.7.3 **Example 6.7.3.** If R is an algebra over the complex numbers $\mathbb{C}$, then each $NK_n(R)$ has the structure of a $\mathbb{C}$-vector space. As an abelian group, it is either zero or else uniquely divisible and uncountable.

The endofunctor $V_m(P, \alpha) = (P[t]/t^m - \alpha, t)$ of $\mathbf{End}(R)$ (Ex. II.7.16) sends $\mathbf{Nil}(R)$ to itself, and $F_m V_m(P, \nu) = \oplus_1^m (P, \nu)$. Hence V_m induces an endomorphism V_m on each $\mathrm{Nil}_n(R)$, such that $F_m V_m$ is multiplication by m.

IV.6.7.4 **Proposition 6.7.4.** *(Farrell) If any $NK_n(R)$ is nonzero, it cannot be finitely generated as an abelian group.*

Proof. Since $NK_n(R) = \operatorname{colim} K_{n-1}F_n\mathbf{Nil}(R)$, every element is killed by all sufficiently large F_m. If $NK_n(R)$ were finitely generated, there would be an integer M so that the entire group is killed by F_m for all $m > M$. Pick $\beta \neq 0$ in $NK_n(R)$ and choose $m > M$ so that $m\beta \neq 0$. But $F_m(V_m\beta) = m\beta$ is nonzero, a contradiction. □

Finite generation

The following conjecture is due to Bass.

IV.6.8 **Bass' Finiteness Conjecture 6.8.** *Let R be a commutative regular ring, finitely generated as a $\mathbb{Z}$-algebra. Then the groups $K_n(R)$ are finitely generated for all n.*

Quillen used a filtration of the Q–construction to prove in Q73 [154] that the groups $K_n(R)$ are finitely generated for any Dedekind domain R such that (1) $\mathrm{Pic}(R)$ is finite and (2) the homology groups $H_n(\mathrm{Aut}(P), st(P \otimes_R F))$ are finitely generated. He then verified (2) in Q73 [154] (number field case) and GQ82 [75] (affine curves). In other words:

IV.6.9 **Theorem 6.9.** *(Quillen) Let R be either an integrally closed subring of a number field F, finite over $\mathbb{Z}$, or else the coordinate ring of a smooth affine curve over a finite field. Then $K_n(R)$ is a finitely generated group for all n.*

EXERCISES

EIV.6.1 **6.1.** *Admissible subquotients.* Let B be an object in an exact category $\mathcal{A}$. An *admissible layer* in B is a pair of subobjects represented by a sequence $B_1 \rightarrowtail B_2 \rightarrowtail B$ of admissible monics, and we call the quotient B_2/B_1 an *admissible subquotient* of B. Show that a morphism $A \to B$ in $Q\mathcal{A}$ may be identified with an isomorphism $j\colon B_2/B_1 \cong A$ of A with an admissible subquotient of B, and that composition in $Q\mathcal{A}$ arises from the fact that a subquotient of a subquotient is a subquotient.

EIV.6.2 **6.2.** If two exact categories $\mathcal{A}$ and $\mathcal{A}'$ are equivalent (and the equivalence respects exactness), show that $Q\mathcal{A}$ and $Q\mathcal{A}'$ are equivalent. If both are small categories, conclude that $K_n(\mathcal{A}) \cong K_n(\mathcal{A}')$ for all n.

EIV.6.3 **6.3.** If $\mathcal{A}$ is an exact category, so is its opposite category $\mathcal{A}^{op}$ (see Example II.7.1.5 II.7.1.5). Show that $Q(\mathcal{A}^{op})$ is isomorphic to $Q\mathcal{A}$.

EIV.6.4 **6.4.** Let B be an object in an exact category $\mathcal{A}$. Show that the comma category $(Q\mathcal{A})/B$ is equivalent to the poset of admissible layers of B in the sense of Ex. EIV.6.1 6.1. If $\mathcal{P}$ is an exact subcategory of $\mathcal{A}$ and i denotes the inclusion $Q\mathcal{P} \subset Q\mathcal{A}$, show that i/B is equivalent to the poset of admissible layers of B with $B_2/B_1 \in \mathcal{P}$.

EIV.6.5 **6.5.** *Kleisli rectification.* Let I be a filtering category, and let $I \to CAT$ be a lax functor in the sense of Ex. 3.8. Although the family of exact categories $Q\mathcal{A}(i)$ is not filtering, the family of homotopy groups $K_n\mathcal{A}(i)$ is filtering. The following trick allows us make K-theoretic sense out of the phantom category $\mathcal{A} = \varinjlim \mathcal{A}(i)$.

Let $\mathcal{A}_i$ be the category whose objects are pairs $(A_j, j \xrightarrow{f} i)$ with A_j in $\mathcal{A}(j)$ and f a morphism in I. A morphism from $(A_j, j \xrightarrow{f} i)$ to $(A_k, k \xrightarrow{g} i)$ is a pair $(j \xrightarrow{h} k, \theta_j)$ where $f = gh$ in I and θ_j is an isomorphism $h_*(A_j) \cong A_k$ in $\mathcal{A}(k)$. Clearly $\mathcal{A}_i$ is equivalent to $\mathcal{A}(i)$, and $i \mapsto \mathcal{A}_i$ is a functor. Thus if $\mathcal{A}$ denotes $\varinjlim \mathcal{A}_i$ we have $K_n\mathcal{A} = \varinjlim K_n\mathcal{A}(i)$.

EIV.6.6 **6.6.** (Gersten) Suppose given a surjective homomorphism $\phi : K_0(\mathcal{A}) \to G$, and let $\mathcal{B}$ denote the full subcategory of all B in $\mathcal{A}$ with $\phi[B] = 0$ in G. In this exercise we show that if $\mathcal{B}$ is cofinal in $\mathcal{A}$ then $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for $n > 0$, and $K_0(\mathcal{B}) \subset K_0(\mathcal{A})$.

(a) Show that there is a functor $\psi : Q\mathcal{A} \to G$ sending the morphism (6.1.1) of $Q\mathcal{A}$ to $\phi[B_1]$, $B_1 = \ker(j)$, where G is regarded as a category with one object $*$. Using 6.2, show that the map $\pi_1(Q\mathcal{A}) \to \pi_1(G)$ is just ϕ.

(b) Show that the hypotheses of Quillen's Theorem B are satisfied by ψ, so that $B(\psi/*)$ is the homotopy fiber of $BQ\mathcal{A} \to BG$.

(c) Use Quillen's Theorem A to show that $Q\mathcal{B} \to \psi^{-1}(*)$ is a homotopy equivalence.

(d) Suppose in addition that $\mathcal{B}$ is cofinal in $\mathcal{A}$ (II.5.3), so that $K_0(\mathcal{B})$ is the subgroup $\ker(\phi)$ of $K_0(\mathcal{A})$ by II.7.2. Use Theorem A to show that $\psi^{-1}(*) \simeq \psi/*$. This proves that $BQ\mathcal{B} \to BQ\mathcal{A} \to BG$ is a homotopy fibration. Conclude that $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n \geq 1$.

EIV.6.7 **6.7.** (Waldhausen) If $\mathcal{A}$ is an exact category, let $q\mathcal{A}$ denote the bicategory (3.10) with the same objects as $\mathcal{A}$, admissible monomorphisms and epimorphisms as the horizontal and vertical morphisms, respectively; the bimorphisms in $q\mathcal{A}$ are those bicartesian squares in $\mathcal{A}$ whose horizontal edges are admissible monomorphisms, and whose vertical edges are admissible epimorphisms.

$$\begin{array}{ccc} A_{11} & \rightarrowtail & A_{10} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A_{01} & \rightarrowtail & A_{00} \end{array}$$

Show that the diagonal category (Ex. 3.14) of $q\mathcal{A}$ is the category $Q\mathcal{A}$.

EIV.6.8 **6.8.** (Waldhausen) Since the realization of the two-object category $0 \rightrightarrows 1$ is S^1, the realization of the bicategory $(0 \rightrightarrows 1) \otimes \mathcal{A}$ is $S^1 \times B\mathcal{A}$. Given a morphism

$A_0 \twoheadleftarrow A_1 \rightarrowtail A_2$ show that the pair of bimorphisms in $Q Q \mathcal{A}$

$$\begin{array}{ccccccccccc}
A_0 & \twoheadleftarrow & A_1 & \rightarrowtail & A_2 & \qquad & A_0 & \twoheadleftarrow & A_1 & \rightarrowtail & A_2 \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \twoheadleftarrow & 0 & \rightarrowtail & 0, & & 0 & \twoheadleftarrow & 0 & \rightarrowtail & 0
\end{array}$$

describe a map $S^1 \wedge BQ\mathcal{A} \to BQQ\mathcal{A}$. Waldhausen observed in [214, p. 197] that this map is adjoint to the homotopy equivalence $BQ\mathcal{A} \simeq \Omega BQQ\mathcal{A}$.

EIV.6.9 **6.9.** For every biexact $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$, show that the pairing $K_0(\mathcal{A}) \otimes K_0(\mathcal{B}) \to K_0(\mathcal{C})$ of (6.6.3) agrees with the product of II.7.4.

EIV.6.10 **6.10.** Show that the functor $Q\mathcal{A} \otimes Q\mathcal{B} \to QQ\mathcal{C}$ of 6.6 is a map of symmetric monoidal categories (the operation on $QQ\mathcal{C}$ is slotwise direct sum). Conclude that $BQ\mathcal{A} \times BQ\mathcal{B} \to BQQ\mathcal{C}$ is an H-space map. (In fact, it is an infinite loop space map.)

EIV.6.11 **6.11.** Let $\mathcal{A}$ be the direct sum $\oplus_{i\in I}\mathcal{A}_i$ of exact categories. Show that $K_n(\mathcal{A}) \cong \oplus_{i\in I} K_n(\mathcal{A}_i)$.

EIV.6.12 **6.12.** If $f : R \to S$ is such that S is in $\mathbf{P}(R)$, show that the transfer map $f_* : K_0(S) \to K_0(R)$ of 6.3.2 agrees with the transfer functor for K_0 given in II.2.8.1.

EIV.6.13 **6.13.** If $f : R \to S$ and S is in $\mathbf{P}(R)$, show that $f_* f^*$ is multiplication by $[S] \in K_0(R)$, and that $f^* f_*$ is multiplication by $[S \otimes_R S] \in K_0(S)$.

If $f : k \to \ell$ is a purely inseparable field extension, show that both $f^* f_*$ and $f_* f^*$ are multiplication by $[\ell : k] = p^r$.

If $f : R \to S$ is a Galois extension with group G, show that $f^* f_* = \sum_{g\in G} g$.

EIV.6.14 **6.14.** *Quasi-exact categories.* Let $\mathcal{C}$ be a category with a distinguished zero object '0' and a coproduct $\vee$. We say that a family $\mathcal{E}$ of sequences of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \qquad (\dagger)$$

is *admissible* if the following conditions hold (cf. Ex. II.7.8): (i) Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$; (ii) If (†) is a sequence in $\mathcal{E}$ then i is a kernel for j (resp. j is a cokernel for i) in $\mathcal{C}$; (iii) the class $\mathcal{E}$ contains all of the sequences $0 \to B \to B \vee D \to D \to 0$; (iv) the class of admissible epimorphisms is closed under composition and pullback along admissible monics; (v) the class of admissible monics is closed under composition and pullback along admissible epimorphisms.

A *quasi-exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{E}$ is admissible in the above sense. If $\mathcal{C}$ is small, show that there is a category $Q\mathcal{C}$, defined exactly as in 6.1, and that $\pi_1(BQ\mathcal{C})$ is the group $K_0(\mathcal{C})$ defined exactly as in II.7.1: the group generated by the objects of $\mathcal{C}$ subject to the relations $[C] = [B] + [D]$ arising from the admissible exact sequences. (This formulation is due to Deitmar.)

EIV.6.15 **6.15.** (Waldhausen) Show that the category $\mathbf{Sets}_{\text{fin}}$ of finite pointed sets is quasi-exact, where $\mathcal{E}$ is the collection of split sequences $0 \to B \to B \vee D \to D \to 0$, and that $K_0(\mathbf{Sets}_{\text{fin}}) = \mathbb{Z}$, exacty as in II.5.2.1. The opposite category $\mathbf{Sets}_{\text{fin}}^{op}$ is not quasi-exact, because $0 \to B \to B \wedge D \to D \to 0$ is not in $\mathcal{E}^{op}$.

EIV.6.16 **6.16.** A monoid M with identity 1 is *pointed* if it has an element 0 with $0 \cdot m = m \cdot 0 = 0$ for all $m \in M$. A pointed M-set is a pointed set X on which M acts and $0 \cdot x = *$ for all $x \in X$. Show that the category of finitely generated pointed M-sets, and its subcategory of free pointed M-sets, are quasi-exact. Here the sequence (†) in Ex. 6.14 is admissible if i is an injection and j identifies D with C/B.

7 The "$+ = Q$" Theorem

Suppose that $\mathcal{A}$ is an additive category. One way to define the K-theory of $\mathcal{A}$ is to consider the symmetric monoidal category $S = \operatorname{iso}\mathcal{A}$ (where $\Box = \oplus$) and use the $S^{-1}S$ construction: $K_n^{\oplus}\mathcal{A} = \pi_n B(S^{-1}S)$ and $K^{\oplus}\mathcal{A} = K(S) = B(S^{-1}S)$.

Another way is to suppose that $\mathcal{A}$ has the structure of an exact category and form the Q–construction on $\mathcal{A}$. Comparing the definitions of $K_0^{\oplus}\mathcal{A}$ and $K_0\mathcal{A}$ in II.5.1.2 and II.7.1, we see that the K_0 groups are not isomorphic in general, unless perhaps every exact sequence splits in $\mathcal{A}$, *i.e.,* unless $\mathcal{A}$ is a split exact category in the sense of II.7.1.2.

Here is the main theorem of this section.

IV.7.1 **Theorem 7.1.** *(Quillen) If $\mathcal{A}$ is a split exact category and $S = \operatorname{iso}\mathcal{A}$, then $\Omega BQ\mathcal{A} \simeq B(S^{-1}S)$. Hence $K_n(\mathcal{A}) \cong K_n(S)$ for all $n \geq 0$.*

In fact, $B(S^{-1}S)$ is the group completion of BS by Theorem 4.8 and Exercise 7.1. In some circumstances (see 4.9, 4.10 and 4.11.1), the $S^{-1}S$ construction is a +–construction. In these cases, Theorem 7.1 shows that the Q–construction is also a +–construction. For $\mathcal{A} = \mathbf{P}(R)$, this yields the "$+ = Q$" theorem:

IV.7.2 **Corollary 7.2** ($+ = Q$ Theorem)**.** *For every ring R,*

$$\Omega BQ\mathbf{P}(R) \simeq K_0(R) \times BGL(R)^+.$$

Hence $K_n(R) \cong K_n\mathbf{P}(R)$ for all $n \geq 0$.

IV.7.3 **Definition 7.3.** Given an exact category $\mathcal{A}$, we define the category $\mathcal{E}\mathcal{A}$ as follows. The objects of $\mathcal{E}\mathcal{A}$ are admissible exact sequences in $\mathcal{A}$. A morphism from $E' : (A' \rightarrowtail B' \twoheadrightarrow C')$ to $E : (A \rightarrowtail B \twoheadrightarrow C)$ is an equivalence class of diagrams of the following form, where the rows are exact sequences in $\mathcal{A}$:

$$\begin{array}{ccccccc}
E': & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\
\Big\downarrow & & \alpha \uparrow & & \| & & \uparrow \\
 & & A & \rightarrowtail & B' & \twoheadrightarrow & C'' \\
 & & \| & & \downarrow \beta & & \downarrow \\
E: & & A & \rightarrowtail & B & \twoheadrightarrow & C.
\end{array} \qquad (7.3.1) \quad \text{IV.7.3.1}$$

Two such diagrams are equivalent if there is an isomorphism between them which is the identity at all vertices except for the C'' vertex.

Notice that the right column in (7.3.1) is just a morphism φ in $Q\mathcal{A}$ from C' to C, so the target C is a functor $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$: $t(A \rightarrowtail B \twoheadrightarrow C) = C$. In order to improve legibility, it is useful to write $\mathcal{E}_C$ for the fiber category $t^{-1}(C)$.

IV.7.4 **Example 7.4** (Fiber categories)**.** If we fix φ as the identity map of $C = C'$, we see that the fiber category $\mathcal{E}_C = t^{-1}(C)$ of exact sequences with target C has for its morphisms all pairs (α, β) of isomorphisms fitting into a commutative diagram:

$$\begin{array}{ccccc} A' & \rightarrowtail & B' & \twoheadrightarrow & C \\ {\scriptstyle\alpha}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\beta} & & \| \\ A & \rightarrowtail & B & \twoheadrightarrow & C. \end{array}$$

In particular, every morphism in $\mathcal{E}_C$ is an isomorphism.

IV.7.4.1 **Example 7.4.1.** The fiber category $\mathcal{E}_0 = t^{-1}(0)$ is homotopy equivalent to $S = \operatorname{iso}\mathcal{A}$. To see this, consider the functor from $\operatorname{iso}\mathcal{A}$ to $\mathcal{E}_0$ sending A to the trivial sequence $A \overset{\mathrm{id}}{\rightarrowtail} A \twoheadrightarrow 0$. This functor is a full embedding. Moreover, every object of $\mathcal{E}_0$ is naturally isomorphic to such a trivial sequence, whence the claim.

IV.7.5 **Lemma 7.5.** *For any C in $\mathcal{A}$, $\mathcal{E}_C$ is a symmetric monoidal category, and there is a faithful monoidal functor $\eta_C : S \to \mathcal{E}_C$ sending A to the sequence $A \rightarrowtail A \oplus C \twoheadrightarrow C$.*

Proof. Given $E_i = (A_i \rightarrowtail B_i \twoheadrightarrow C)$ in $\mathcal{E}_C$, set $E_1 * E_2$ equal to

$$A_1 \oplus A_2 \rightarrowtail (B_1 \times_C B_2) \twoheadrightarrow C. \qquad (7.5.1)$$ IV.7.5.1

This defines a symmetric product on $\mathcal{E}_C$ with identity $e : 0 \rightarrowtail C \twoheadrightarrow C$. It is now routine to check that $S \to \mathcal{E}_C$ is a monoidal functor, and that it is faithful. □

IV.7.5.2 **Remark 7.5.2.** If $\mathcal{A}$ is split exact then every object of $\mathcal{E}_C$ is isomorphic to one coming from S. In particular, the category $\langle S, \mathcal{E}_C \rangle$ of 4.7.1 is connected. This fails if $\mathcal{A}$ has a non-split exact sequence.

IV.7.6 **Proposition 7.6.** *If $\mathcal{A}$ is split exact, each $S^{-1}S \to S^{-1}\mathcal{E}_C$ is a homotopy equivalence.*

Proof. By Ex. 4.7 and Ex. 7.1, $S^{-1}S \to S^{-1}\mathcal{E}_C \to \langle S, \mathcal{E}_C \rangle$ is a fibration, so it suffices to prove that $L = \langle S, \mathcal{E}_C \rangle$ is contractible. First, observe that the monoidal product on $\mathcal{E}_C$ induces a monoidal product on L, so BL is an H-space (as in 4.1). We remarked in 7.5.2 that L is connected. By [228, X.2.2], BL is group-like, *i.e.,* has a homotopy inverse.

For every exact sequence E, there is a natural transformation $\delta_E : E \to E{*}E$ in L, where $*$ is defined by (7.5.1), given by the diagonal.

$$\begin{array}{ccccccc}
E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \\
\downarrow & & \downarrow & & \downarrow & & \| \\
E*E: & & A\oplus A & \rightarrowtail & B\times_C B & \twoheadrightarrow & C
\end{array}$$

Now δ induces a homotopy between the identity on BL and multiplication by 2. Using the homotopy inverse to subtract the identity, this gives a homotopy between zero and the identity of BL. Hence BL is contractible. □

We also need a description of how $\mathcal{E}_C$ varies with C.

IV.7.7 **Lemma 7.7.** *For each morphism $\varphi : C' \to C$ in $Q\mathcal{A}$, there is a canonical functor $\varphi^* : \mathcal{E}_C \to \mathcal{E}_{C'}$ and a natural transformation $\eta_E : \varphi^*(E) \to E$ from φ^* to the inclusion of $\mathcal{E}_C$ in $\mathcal{E}\mathcal{A}$.*

In fact, $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is a fibered functor with base change φ^* (Ex. 7.2). It follows (from 3.7.5 that $C \mapsto \mathcal{E}_C$ is a contravariant functor from $Q\mathcal{A}$ to CAT.

Proof. Choose a representative $C' \twoheadleftarrow C'' \rightarrowtail C$ for φ and choose a pullback B' of B and C'' along C. This yields an exact sequence $A \rightarrowtail B' \twoheadrightarrow C''$ in $\mathcal{A}$. (Why?) The composite $B' \twoheadrightarrow C'' \twoheadrightarrow C'$ is admissible; if A' is its kernel then set

$$\varphi^*(A \rightarrowtail B \twoheadrightarrow C) = (A' \rightarrowtail B' \twoheadrightarrow C').$$

Since every morphism in $\mathcal{E}_C$ is an isomorphism, it is easy to see that φ^* is a functor, independent (up to isomorphism) of the choices made. Moreover, the construction yields a diagram (7.3.1), natural in E; the map β in the diagram is an admissible monic because $A \rightarrowtail B' \xrightarrow{\beta} B'$ is. Hence (7.3.1) constitutes the natural map $\eta_E : E \to \varphi^*(E)$. □

Now the direct sum of sequences defines an operation $\oplus$ on $\mathcal{E}\mathcal{A}$, and S acts on $\mathcal{E}\mathcal{A}$ via the inclusion of S in $\mathcal{E}\mathcal{A}$ given by 7.4.1. That is, $A' \square (A \rightarrowtail B \twoheadrightarrow C)$ is the sequence $A' \oplus A \rightarrowtail A' \oplus B \twoheadrightarrow C$. Since $t(A'\square E) = t(E)$ we have an induced map $T = S^{-1}t : S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$. This is also a fibered functor (Ex. 7.2).

IV.7.8 **Theorem 7.8.** *If $\mathcal{A}$ is a split exact category and $S = iso\,\mathcal{A}$, then the sequence $S^{-1}S \to S^{-1}\mathcal{E}\mathcal{A} \xrightarrow{T} Q\mathcal{A}$ is a homotopy fibration.*

Proof. We have to show that Quillen's Theorem B applies, *i.e.,* that the base changes φ^* of 7.7 are homotopy equivalences. It suffices to consider φ of the form $0 \rightarrowtail C$ and $0 \twoheadleftarrow C$. If φ is $0 \rightarrowtail C$, the composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 7.6 with φ^* is the identity by Ex. 7.5, so φ^* is a homotopy equivalence.

Now suppose that φ is $0 \twoheadleftarrow C$. The composition of the equivalence $S^{-1}S \to S^{-1}\mathcal{E}_C$ of 7.6 with φ^* sends A to $A \oplus C$ by Ex. 7.5. Since there is a natural transformation $A \to A \oplus C$ in $S^{-1}S$, this composition is a homotopy equivalence. Hence φ^* is a homotopy equivalence. □

Proof of Theorem 7.1. This will follow from Theorem 7.8, once we show that $S^{-1}\mathcal{E}\mathcal{A}$ is contractible. By Ex. 7.3, $\mathcal{E}\mathcal{A}$ is contractible. Any action of S on a contractible category must be invertible (4.7.1). By Ex. 4.6 and Ex. 7.1, $\mathcal{E}\mathcal{A} \to S^{-1}\mathcal{E}\mathcal{A}$ is a homotopy equivalence, and therefore $S^{-1}\mathcal{E}\mathcal{A}$ is contractible. $\square$

Agreement of Product Structures

Any biexact pairing $\mathcal{A}_1 \times \mathcal{A}_2 \xrightarrow{\otimes} \mathcal{A}_3$ of split exact categories (6.6) induces a pairing $S_1 \times S_2 \xrightarrow{\square} S$ of symmetric monoidal categories, where $S_i = \operatorname{iso} \mathcal{A}_i$. We now compare the resulting pairings $K(\mathcal{A}_1) \wedge K(\mathcal{A}_2) \to K(\mathcal{A}_3)$ of 6.6 and $K(S_1) \wedge K(S_2) \to K(S_3)$ of 4.6. Waldhausen's Lemma [214, 9.2.6] implies the following result; the details of the implication are given in [218, 4.3]:

IV.7.9 **Theorem 7.9.** *The homotopy equivalences $B(S_i^{-1}S_i) \to \Omega BQ\mathcal{A}_i$ of Theorem 7.1 fit into a diagram which commutes up to basepoint-preserving homotopy:*

$$\begin{array}{ccc}
B(S_1^{-1}S_1) \wedge B(S_2^{-1}S_2) & \xrightarrow{\square} & B(S_3^{-1}S_3) \\
\simeq \downarrow & & \downarrow \simeq \\
(\Omega BQ\mathcal{A}_1) \wedge (\Omega BQ\mathcal{A}_2) & \xrightarrow{\gamma} & (\Omega BQ\mathcal{A}_3) \\
\simeq \downarrow & & \downarrow \simeq \\
\Omega^2(BQ\mathcal{A}_1 \wedge BQ\mathcal{A}_2) & \xrightarrow{\Omega^2 \otimes} & \Omega^2(BQQ\mathcal{A}_3).
\end{array}$$

Hence there are commutative diagrams:

$$\begin{array}{ccc}
K_p(S_1) \otimes K_q(S_2) & \xrightarrow{\square} & K_{p+q}(S_3) \\
\cong \downarrow & & \downarrow \cong \\
K_p(\mathcal{A}_1) \otimes K_q(\mathcal{A}_2) & \xrightarrow{\otimes} & K_{p+q}(\mathcal{A}_3).
\end{array}$$

The middle map γ is induced from the H-space map $\otimes : \Omega BQ\mathcal{A}_1 \times \Omega BQ\mathcal{A}_2 \to \Omega^2 BQ\mathcal{A}_3$ of Ex. 6.10, since it sends $x \otimes 0$ and $0 \otimes y$ to 0.

EXERCISES

EIV.7.1 **7.1.** If $\mathcal{A}$ is an additive category, $S = \operatorname{iso} \mathcal{A}$ is equivalent to the disjoint union of one-object categories $\operatorname{Aut}(A)$, one for every isomorphism class in $\mathcal{A}$. Show that the translations $\operatorname{Aut}(A) \to \operatorname{Aut}(A \oplus B)$ are injections. Then conclude using Theorem 4.8 that $B(S^{-1}S)$ is the group completion of the H-space $BS = \coprod \operatorname{Aut}(A)$.

EIV.7.2 **7.2.** Show that the target functor $t : \mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is a fibered functor in the sense of Definition 3.7.3, with base change φ^* given by 7.7. Then show that the action of S on $\mathcal{E}\mathcal{A}$ is cartesian (Ex. 4.11), so that the induced functor $S^{-1}\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ is also fibered, with fiber $S^{-1}S$ over 0.

EIV.7.3 **7.3.** Let $iQ\mathcal{A}$ denote the subcategory of $Q\mathcal{A}$ whose objects are those of $\mathcal{A}$ but whose morphisms are admissible monomorphisms. Show that the category $\mathcal{E}\mathcal{A}$ of 7.3 is equivalent to the subdivision category $Sub(iQ\mathcal{A})$ of Ex. 3.9. Conclude that the category $\mathcal{E}\mathcal{A}$ is contractible.

EIV.7.4 **7.4.** Show that Quillen's Theorem B *can not* apply to $\mathcal{E}\mathcal{A} \to Q\mathcal{A}$ unless $\mathcal{A} \cong 0$. *Hint:* Compare $\pi_0 S$ to $K_0\mathcal{A}$.

EIV.7.5 **7.5.** If φ is the map $0 \rightarrowtail C$ (resp. $0 \twoheadleftarrow C$), show that $\varphi^* : \mathcal{E}_C \to \mathcal{E}_0 \cong S$ sends $A \rightarrowtail B \twoheadrightarrow C$ to A (resp. to B).

EIV.7.6 **7.6.** Describe $\mathcal{E}'\mathcal{A} = (\mathcal{E}\mathcal{A})^{op}$, which is cofibered over $(Q\mathcal{A})^{op}$ by Ex. 3.6 and 7.2. Use $\mathcal{E}'\mathcal{A}$ to prove the $+ = Q$ Theorem 7.1 and 7.2. *Hint:* There is a new action of S. Use pushout instead of pullback in (7.5.1) to prove the analogue of Proposition 7.6.

EIV.7.7 **7.7.** *Finite Sets.* Let $\mathbf{Sets}_{\text{fin}}$ denote the category of finite pointed sets, and form the category $Q\mathbf{Sets}_{\text{fin}}$ by copying the Q–construction 6.1 as in Ex. 6.14 and Ex. 6.15.
(a) Show that there is an extension category $\mathcal{E}'\mathbf{Sets}_{\text{fin}}$, defined as in Ex. 7.6, which is cofibered over $(Q\mathbf{Sets}_{\text{fin}})^{op}$ with $S = \text{iso}\,\mathbf{Sets}_{\text{fin}}$ as the fiber over the basepoint.
(b) Modify the proof of the $+ = Q$ theorem to prove that $\Omega BQ\mathbf{Sets}_{\text{fin}} \simeq S^{-1}S$.
(c) If G is a group, let $\mathcal{F}$ be the category of finitely generated free pointed G-sets, and $Q\mathcal{F}$ as in Ex. 6.16. Using 4.10.1, show that $\Omega BQ\mathcal{F} \simeq S^{-1}S \simeq \mathbb{Z} \times \Omega^\infty S^\infty(BG_+)$.

EIV.7.8 **7.8.** ($\pi_1 BQ\mathcal{A}$) Given an object A in $\mathcal{A}$, lift the morphisms $0 \rightarrowtail A \twoheadrightarrow 0$ in $Q\mathcal{A}$ to morphisms in $\mathcal{E}\mathcal{A}$, $0 \to \eta_A(0) \leftarrow \eta_0(A)$. Conclude that the isomorphism between $K_0(\mathcal{A}) = \pi_1 BQ\mathcal{A}$ and $K_0(S) = \pi_0(S^{-1}S)$ of Theorem 7.1 is the canonical isomorphism of II.7.1.2, identifying $[A]$ with $[A]$.

EIV.7.9 **7.9.** ($\pi_2 BQ\mathcal{A}$) Given an automorphism α of an object A in $\mathcal{A}$, consider the continuous map $[0,1]^2 \to BQ\mathcal{A}$ given by the commutative diagram:

$$\begin{array}{ccccc} 0 & \rightarrowtail & A & \twoheadrightarrow & 0 \\ \| & & \downarrow{\scriptstyle\alpha} & & \| \\ 0 & \rightarrowtail & A & \twoheadrightarrow & 0. \end{array}$$

Identifying the top and bottom edges to each other, the fact that the left and right edges map to the basepoint (0) means that we have a continuous function $S^2 \to BQ\mathcal{A}$, *i.e.,* an element $[\alpha]$ of $K_1(\mathcal{A}) = \pi_2(BQ\mathcal{A})$.

(a) Show that $[\alpha] + [\alpha'] = [\alpha\alpha']$ for every pair of composable automorphisms. Conclude that $\alpha \mapsto [\alpha]$ is a homomorphism $\text{Aut}(A) \to K_1(\mathcal{A})$.

(b) If $\beta \in \text{Aut}(B)$, show that the automorphism $\alpha \oplus \beta$ of $A \oplus B$ maps to $[\alpha] + [\beta]$. Using 4.8.1, this given a map from $K_1(\text{iso}\,\mathcal{A})$ to $K_1(\mathcal{A})$.

(c) Finally, lift this diagram to $\mathcal{EA}$ using Ex. 7.8, representing a map $I^2 \to B\mathcal{EA}$, and conclude that the isomorphism between $K_1(\mathcal{A}) = \pi_2 BQ\mathcal{A}$ and $K_1(S) = \pi_1(S^{-1}S)$ of Theorem 7.1 identifies $[\alpha]$ with the class of α given by III.1.6.3 and 4.8.1.

EIV.7.10 **7.10.** (Canonical involution) Let R be a commutative ring. The isomorphism $\mathbf{P}(R) \to \mathbf{P}(R)^{op}$ sending P to $\mathrm{Hom}_R(P, R)$ induces an involution on $Q\mathbf{P}(R)$ and hence $K_*(R)$ by 6.4; it is called the *canonical involution*. Show that the involution is a ring automorphism.

On the other hand, the "transpose inverse" involution of $GL(R)$ ($g \mapsto {}^t g^{-1}$) induces a homotopy involution on $BGL(R)^+$ and an involution on $K_n(R)$ for $n > 0$. Show that these two involutions agree via the '$+ = Q$' Theorem 7.2.

8 Waldhausen's $wS_\bullet$ construction

Our last construction of K-theory applies to Waldhausen categories, *i.e.,* "categories with cofibrations and weak equivalences." Unfortunately, this will occur only after a lengthy list of definitions, and we ask the reader to be forgiving.

Recall from Chapter II, Section 9 that a *category with cofibrations* is a category $\mathcal{C}$ with a distinguished zero object '0' and a subcategory co($\mathcal{C}$) of morphisms in $\mathcal{C}$ called "cofibrations" (indicated with feathered arrows $\rightarrowtail$). Every isomorphism in $\mathcal{C}$ is to be a cofibration, and so are the unique arrows $0 \rightarrowtail A$ for every object A in $\mathcal{C}$. In addition, the pushout $C \rightarrowtail B \cup_A C$ of any cofibration $A \rightarrowtail B$ is a cofibration. (See Definition II.9.1 for more precise statements.) These axioms imply that two constructions make sense: the coproduct $B \amalg C = B \cup_0 C$ of any two objects, and every cofibration $A \rightarrowtail B$ fits into a *cofibration sequence* $A \rightarrowtail B \twoheadrightarrow B/A$, where B/A is the cokernel of $A \rightarrowtail B$. The following is a restatement of Definition II.9.1.1:

IV.8.1 **Definition 8.1.** A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the "Glueing axiom" (W3) must be satisfied, which says that the pushout of weak equivalences is a weak equivalence (see II.9.1.1).

A functor $f : \mathcal{A} \to \mathcal{C}$ between two Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and the pushouts along a cofibration.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in A are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

In order to describe Waldhausen's $wS_\bullet$ construction for K-theory, we need a sequence of Waldhausen categories $S_n\mathcal{C}$. $S_0\mathcal{C}$ is the zero category, and $S_1\mathcal{C}$ is the category $\mathcal{C}$, but whose objects A are thought of as the cofibrations $0 \rightarrowtail A$. The category $S_2\mathcal{C}$ is the extension category $\mathcal{E}$ of II.9.3. For convenience, we repeat its definition here.

IV.8.2 **Extension Categories 8.2.** The objects of the extension category $S_2\mathcal{C}$ are the cofibration sequences $A_1 \rightarrowtail A_2 \twoheadrightarrow A_{12}$ in $\mathcal{C}$. A morphism $E \to E'$ in $S_2\mathcal{C}$ is a commutative diagram:

$$\begin{array}{ccccccc}
E: & & A_1 & \rightarrowtail & A_2 & \twoheadrightarrow & A_{12} \\
\downarrow & & \downarrow u_1 & & \downarrow u_2 & & \downarrow u_3 \\
E': & & A'_1 & \rightarrowtail & A'_2 & \twoheadrightarrow & A'_{12}.
\end{array}$$

We make $S_2\mathcal{C}$ into a Waldhausen category as follows. A morphism $E \to E'$ in $S_2\mathcal{C}$ is a cofibration if $A_1 \to A'_1$, $A_{12} \to A'_{12}$ and $A'_1 \cup_{A_1} A_2 \to A'_2$ are cofibrations in $\mathcal{C}$. A morphism in $S_2\mathcal{C}$ is a weak equivalence if its component maps $u_i : A_i \to A'_i$ $(i = 1, 2, 12)$ are weak equivalences in $\mathcal{C}$.

A Waldhausen category $\mathcal{C}$ is called *extensional* if it satisfies this following technically convenient axiom: weak equivalences are "closed under extensions."

IV.8.2.1 **Remark 8.2.1** (Extension axiom)**.** Suppose that $f : E \to E'$ is a map between cofibration sequences, as in IV.8.2 8.2. If the source and quotient maps of f $(A \to A'$ and $C \to C')$ are weak equivalences, so is the total map of f $(B \to B')$.

IV.8.3 **Definition 8.3.** $(S_\bullet\mathcal{C})$ If $\mathcal{C}$ is a category with cofibrations, let $S_n\mathcal{C}$ be the category whose objects $A_\bullet$ are sequences of n cofibrations in $\mathcal{C}$:

$$A_\bullet : \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$

together with a choice of every subquotient $A_{ij} = A_j/A_i$ $(0 < i < j \le n)$. These choices are to be compatible in the sense that there is a commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & A_{n-1,n} \\
 & & & & & & \twoheaduparrow \\
 & & & & & & \cdots \\
 & & & & A_{23} & \twoheadrightarrow \cdots \twoheadrightarrow & A_{2n} \\
 & & & & \twoheaduparrow & & \twoheaduparrow \\
 & & A_{12} & \twoheadrightarrow & A_{13} & \twoheadrightarrow \cdots \twoheadrightarrow & A_{1n} \\
 & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & A_3 & \rightarrowtail \cdots \rightarrowtail & A_n
\end{array} \tag{8.3.0}$$ IV.8.3.0

The conventions $A_{0j} = A_j$ and $A_{jj} = 0$ will be convenient at times. A morphism $A_\bullet \to B_\bullet$ in $S_n\mathcal{C}$ is a natural transformation of sequences.

If we forget the choices of the subquotients A_{ij} we obtain the higher extension category $\mathcal{E}_n(\mathcal{C})$ constructed in II.II.9.3.2 9.3.2. Since we can always make such choices, it follows that the categories $S_n\mathcal{C}$ and $\mathcal{E}_n(\mathcal{C})$ are equivalent. By Ex. II.EII.9.4 9.4, when $\mathcal{C}$ is a Waldhausen category, so is $\mathcal{E}_n(\mathcal{C})$ and hence $S_n\mathcal{C}$. Here are the relevant definitions for S_n, translated from the definitions II.II.9.3.2 9.3.2 for $\mathcal{E}_n$.

A weak equivalence in $S_n\mathcal{C}$ is a map $A_\bullet \to B_\bullet$ such that each $A_i \to B_i$ (hence, each $A_{ij} \to B_{ij}$) is a weak equivalence in $\mathcal{C}$. A map $A_\bullet \to B_\bullet$ is a cofibration when for every $0 \le i < j < k \le n$ the map of cofibration sequences

$$\begin{array}{ccccc}
A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \rightarrowtail & B_{ik} & \twoheadrightarrow & B_{jk}
\end{array}$$

is a cofibration in $S_2\mathcal{C}$.

The reason for including choices in the definition of the categories $S_n\mathcal{C}$ is that we can form a simplicial Waldhausen category. The maps ∂_0, ∂_1 from $\mathcal{C} = S_1\mathcal{C}$ to $0 = S_0\mathcal{C}$ are trivial; the maps $\partial_0, \partial_1, \partial_2$ from $S_2\mathcal{C}$ to $\mathcal{C}$ are q_*, t_* and s_*, respectively.

IV.8.3.1 **Definition 8.3.1.** For each $n \geq 0$, the exact functor $\partial_0 : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is defined by deletion of the bottom row of (8.3.0). That is, ∂_0 is defined by the formula

$$\partial_0(A.): \quad 0 = A_{11} \rightarrowtail A_{12} \rightarrowtail A_{13} \rightarrowtail \cdots \rightarrowtail A_{1n}$$

together with the choices $\partial_0(A.)_{ij} = A_{i+1,j+1}$. By Ex. 8.1, $\partial_0(A.)$ is in $S_{n-1}\mathcal{C}$.

For $0 < i \leq n$ we define the exact functors $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by omitting the row A_{i*} and the column containing A_i in (8.3.0), and reindexing the A_{jk} as needed. Similarly, we define the exact functors $s_i : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ by duplicating A_i, and reindexing with the normalization $A_{i,i+1} = 0$. (Exactness is checked in Ex. 8.2.)

By Ex. 8.2, the $S_n\mathcal{C}$ fit together to form a simplicial Waldhausen category $S_\bullet(\mathcal{C})$, and the subcategories $wS_n\mathcal{C}$ of weak equivalences fit together to form a simplicial category $wS_\bullet\mathcal{C}$. Hence their geometric realizations $B(wS_n\mathcal{C})$ fit together to form a simplicial topological space $BwS_\bullet\mathcal{C}$, and we write $|wS_\bullet\mathcal{C}|$ for the realization of $BwS_\bullet\mathcal{C}$. Since $S_0\mathcal{C}$ is trivial, $|wS_\bullet\mathcal{C}|$ is a connected space.

IV.8.3.2 **Remark 8.3.2.** In the realization of $BwS_\bullet\mathcal{C}$, the spaces $B(wS_n\mathcal{C}) \times \Delta^n$ are glued together along the face maps. In particular, the suspension $\Sigma B(w\mathcal{C})$ is a subspace of $|wS_\bullet\mathcal{C}|$; the adjoint map is $B(w\mathcal{C}) \to \Omega|wS_\bullet\mathcal{C}|$. In this way, each object of $\mathcal{C}$ yields an element of $\pi_1|wS_\bullet\mathcal{C}|$, and each weak equivalence $A \simeq A$ in $\mathcal{C}$ yields an element of $\pi_2|wS_\bullet\mathcal{C}|$.

Recall from chapter II, 9.1.2, that $K_0(\mathcal{C})$ is defined as the group generated by the set of weak equivalence classes [A] of objects of $\mathcal{C}$ with the relations that $[B] = [A] + [B/A]$ for every cofibration sequence

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

IV.8.4 **Proposition 8.4.** *If $\mathcal{C}$ is a Waldhausen category then $\pi_1|wS_\bullet\mathcal{C}| \cong K_0(\mathcal{C})$.*

Proof. If $X_\bullet$ is any simplicial space with X_0 a point, then $|X_\bullet|$ is connected and $\pi_1|X_\bullet|$ is the free group on $\pi_0(X_1)$ modulo the relations $\partial_1(x) = \partial_2(x)\partial_0(x)$ for every $x \in \pi_0(X_2)$. For $X_\bullet = BwS_\bullet\mathcal{C}$, $\pi_0(BwS_1\mathcal{C})$ is the set of weak equivalence classes of objects in $\mathcal{C}$, $\pi_0(BwS_2\mathcal{C})$ is the set of equivalence classes of cofibration sequences, and the maps $\partial_i : S_2\mathcal{C} \to S_1\mathcal{C}$ of 8.3.1 send $A \rightarrowtail B \twoheadrightarrow B/A$ to B/A, B and A, respectively. □

IV.8.5 **Definition 8.5.** If $\mathcal{C}$ is a small Waldhausen category, its *algebraic K-theory space* $K(\mathcal{C}) = K(\mathcal{C}, w)$ is the loop space

$$K(\mathcal{C}) = \Omega|wS_\bullet\mathcal{C}|.$$

The K-groups of $\mathcal{C}$ are defined to be its homotopy groups:

$$K_n(\mathcal{C}) = \pi_n K(\mathcal{C}) = \pi_{n+1}|wS_\bullet\mathcal{C}| \qquad \text{if } n \geq 0.$$

As we saw in Remark 8.3.2, there is a canonical map $B(w\mathcal{C}) \to K(\mathcal{C})$.

IV.8.5.1 **Remark 8.5.1.** Since the subcategory $w\mathcal{C}$ is closed under coproducts in $\mathcal{C}$ by axiom (W3), the coproduct gives an H-space structure to $|wS_\bullet\mathcal{C}|$ via the map

$$|wS_\bullet\mathcal{C}| \times |wS_\bullet\mathcal{C}| \cong |wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C}| \xrightarrow{\amalg} |wS_\bullet\mathcal{C}|.$$

IV.8.5.2 **Example 8.5.2** (Simplicial Model)**.** Suppose that $\mathcal{C}$ is a small Waldhausen category in which the isomorphisms $i\mathcal{C}$ are the weak equivalences. Let $s_n\mathcal{C}$ denote the set of objects of $S_n\mathcal{C}$; as n varies, we have a simplicial set $s_\bullet\mathcal{C}$. Waldhausen proved in [215, 1.4] that the inclusion $|s_\bullet\mathcal{C}| \to |iS_\bullet\mathcal{C}|$ is a homotopy equivalence. Therefore $\Omega|s_\bullet\mathcal{C}|$ is a simplicial model for the space $K(\mathcal{C})$.

IV.8.5.3 **Example 8.5.3** (Relative K-theory spaces)**.** If $f : \mathcal{B} \to \mathcal{C}$ is an exact functor, let $S_n f$ denote the category $S_n\mathcal{B} \times_{S_n\mathcal{C}} S_{n+1}\mathcal{C}$ whose objects are pairs

$$(B_*, C_*) = (B_1 \rightarrowtail \cdots \rightarrowtail B_n, C_0 \rightarrowtail \cdots \rightarrowtail C_n)$$

such that $f(B_*)$ is $\partial_0 C_* : C_1/C_0 \rightarrowtail \cdots \rightarrowtail C_n/C_0$. Each $S_n f$ is a Waldhausen category in a natural way, containing $\mathcal{C}$ as the (Waldhausen) subcategory of all $(0, C = \cdots = C)$, and the projection $S_n f \to S_n\mathcal{B}$ is exact. We can apply the $S_\bullet$ (and $wS_\bullet$) construction degreewise to the sequence $\mathcal{C} \to S_\bullet f \to S_\bullet\mathcal{B}$ of simplicial Waldhausen categories, obtaining a sequence of bisimplicial Waldhausen categories $S_\bullet\mathcal{C} \to S_\bullet(S_\bullet f) \to S_\bullet(S_\bullet\mathcal{B})$, and a sequence $wS_\bullet\mathcal{C} \to wS_\bullet(S_\bullet f) \to wS_\bullet(S_\bullet\mathcal{B})$ of bisimplicial categories. We will see in V.1.7 (using 8.5.4) that the realization of the bisimplicial category sequence

$$wS_\bullet\mathcal{B} \to wS_\bullet\mathcal{C} \to wS_\bullet(S_\bullet f) \to wS_\bullet(S_\bullet\mathcal{B}),$$

is a homotopy fibration sequence. Thus we may regard $K(f) = \Omega^2|wS_\bullet(S_\bullet f)|$ as a relative K-theory space; the groups $K_n(f) = \pi_n K(f)$ fit into a long exact sequence involving $f_* : K_n(\mathcal{B}) \to K_n(\mathcal{C})$, ending $K_0(\mathcal{B}) \to K_0(\mathcal{C}) \to K_{-1}(f) \to 0$ (Ex. 8.11).

IV.8.5.4 **Lemma 8.5.4.** *If $f : \mathcal{C} \to \mathcal{C}$ is the identity, $wS_\bullet f$ is contractible.*

Proof. In this case the simplicial category $S_\bullet f$ is just the simplicial path space construction of $S_\bullet\mathcal{C}$, and $wS_\bullet S_\bullet f$ is the simplicial path space construction of $wS_\bullet S_\bullet\mathcal{C}$ (see [223, 8.3.14]). These are contractible since $S_0 f = 0$ and $wS_\bullet S_0 f$ are. $\square$

IV.8.5.5 **Remark 8.5.5** (Infinite Loop Structure)**.** Lemma 8.5.4 implies that there are natural homotopy equivalences $|wS_\bullet\mathcal{C}| \simeq \Omega|wS_\bullet S_\bullet\mathcal{C}|$, and of course $K(\mathcal{C}) \simeq \Omega^2|wS_\bullet S_\bullet\mathcal{C}|$. In fact $K(\mathcal{C})$ is an infinite loop space.

To see this we just iterate the construction, forming the multisimplicial Waldhausen categories $S^n_\bullet\mathcal{C} = S_\bullet S_\bullet \cdots S_\bullet\mathcal{C}$ and the multisimplicial categories $wS^n_\bullet\mathcal{C}$ of their weak equivalences. By 8.5.4, we see that $|wS^n_\bullet\mathcal{C}|$ is the loop space of $|wS^{n+1}_\bullet\mathcal{C}|$, and that the sequence of spaces

$$\Omega|wS_\bullet\mathcal{C}|, |wS_\bullet\mathcal{C}|, |wS_\bullet S_\bullet\mathcal{C}|, \ \dots, \ \Omega|wS^n_\bullet\mathcal{C}|, \ \dots$$

forms a connective Ω-spectrum $\mathbf{K}\mathcal{C}$, called the *K-theory spectrum of $\mathcal{C}$*. Many authors think of the K-theory of $\mathcal{C}$ in terms of this spectrum. This does not affect the K-groups, because:

$$\pi_i(\mathbf{K}\mathcal{C}) = \pi_i K(\mathcal{C}) = K_i(\mathcal{C}), \qquad i \geq 0.$$

An exact functor f induces a map $f_* : K(\mathcal{B}) \to K(\mathcal{C})$ of spaces, and spectra, and of their homotopy groups $K_i(\mathcal{B}) \to K_i(\mathcal{C})$.

IV.8.6 **Exact Categories 8.6.** We saw in II.9.1.3 that any exact category $\mathcal{A}$ becomes a Waldhausen category in which the cofibration sequences are just the admissible exact sequences, and the weak equivalences are just the isomorphisms. We write $i(\mathcal{A})$ for the family of isomorphisms, so that we can form the K-theory space $K(\mathcal{A}) = \Omega|iS_\bullet\mathcal{A}|$. Waldhausen proved in [215, 1.9] that there is a homotopy equivalence between $|iS_\bullet\mathcal{A}|$ and $BQ\mathcal{A}$, so that this definition is consistent with the definition of $K(\mathcal{A})$ in Definition 6.3. His proof is given in Exercises 8.5 and 8.6 below.

Another important example of a Waldhausen category is $\mathcal{R}_f(X)$, introduced in II.9.1.4 and Ex. II.9.1. The so-called *K-theory of spaces* refers to the corresponding K-theory spaces $A(X)$, and their homotopy groups $A_n(X) = \pi_n A(X)$.

IV.8.7 **Example 8.7** ($A(*)$)**.** Recall from II.9.1.4 that the category $\mathcal{R}_f = \mathcal{R}_f(*)$ of finite based CW complexes is a Waldhausen category in which the family $h\mathcal{R}_f$ of weak equivalences is the family of weak homotopy equivalences. This category is saturated (II.9.1.1) and satisfies the extension axiom 8.2.1. Following Waldhausen [215], we write $A(*)$ for the space $K(\mathcal{R}_f) = \Omega|hS_\bullet\mathcal{R}_f|$. We have $A_0(*) = K_0\mathcal{R}_f = \mathbb{Z}$ by II.9.1.5.

IV.8.7.1 **Example 8.7.1** ($A(X)$)**.** More generally, let X be a CW complex. The category $\mathcal{R}(X)$ of CW complexes Y obtained from X by attaching cells, and having X as a retract, is a Waldhausen category in which cofibrations are cellular inclusions (fixing X) and weak equivalences are homotopy equivalences (see Ex. II.9.1). Consider the Waldhausen subcategory $\mathcal{R}_f(X)$ of those Y obtained by attaching only finitely many cells. Following Waldhausen [215], we write $A(X)$ for the space $K(\mathcal{R}_f(X)) = \Omega|hS_\bullet\mathcal{R}_f(X)|$. Thus $A_0(X) = K_0\mathcal{R}_f(X)$ is $\mathbb{Z}$ by Ex. II.9.1.

Similarly, we can form the Waldhausen subcategory $\mathcal{R}_{fd}(X)$ of those Y which are finitely dominated. We write $A^{fd}(X)$ for $K(\mathcal{R}_{fd}(X)) = \Omega|hS_\bullet\mathcal{R}_{fd}(X)|$. Note that $A_0^{fd}(X) = K_0\mathcal{R}_{fd}(X)$ is $\mathbb{Z}[\pi_1(X)]$ by Ex. II.9.1.

Cylinder Functors

When working with Waldhausen categories, it is often technically convenient to have mapping cylinders. Recall from Ex. 3.12 that the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ has the morphisms of $\mathcal{C}$ as its objects, and a map $(a,b) : f \to f'$ in $\mathcal{C}/\mathcal{C}$ is a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{8.8.0}$$ IV.8.8.0

The source $s(f) = A$ and target $t(f) = B$ of f define functors $s, t : \mathcal{C}/\mathcal{C} \to \mathcal{C}$.

IV.8.8 **Definition 8.8** (Cylinders)**.** Let $\mathcal{C}$ be a Waldhausen category. A (mapping) *cylinder functor* on $\mathcal{C}$ is a functor T from the category $\mathcal{C}/\mathcal{C}$ of arrows in $\mathcal{C}$ to the category $\mathcal{C}$, together with natural transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$ and $p : T \Rightarrow t$ so that for every $f : A \to B$ the diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & B \\ & {\scriptstyle f}\searrow & {\scriptstyle p}\downarrow & \swarrow{\scriptstyle =} & \\ & & B & & \end{array}$$

commutes in $\mathcal{C}$. The following conditions must also hold:

(i) $T(0 \rightarrowtail A) = A$, with p and j_2 the identity map, for all $A \in \mathcal{C}$.

(ii) $j_1 \amalg j_2 : A \amalg B \rightarrowtail T(f)$ is a cofibration for all $f : A \to B$.

(iii) Given a map $(a,b) : f \to f'$ in $\mathcal{C}/\mathcal{C}$, *i.e.,* a commutative square (8.8.0), if a and b are weak equivalences in $\mathcal{C}$ then so is $T(f) \to T(f')$.

(iv) Given a map $(a,b) : f \to f'$ in $\mathcal{C}/\mathcal{C}$, if a and b are cofibrations in $\mathcal{C}$, then so is $T(f) \to T(f')$, and the following map, induced by condition (ii), is also a cofibration in $\mathcal{C}$.

$$A' \amalg_A T(f) \amalg_B B' \to T(f')$$

We often impose the following extra axiom on the weak equivalences of $\mathcal{C}$.

IV.8.8.1 **Cylinder Axiom 8.8.1.** All maps $p : T(f) \to B$ are weak equivalences in $\mathcal{C}$.

Suppose $\mathcal{C}$ has a cylinder functor T. The *cone* of an object A is $\operatorname{cone}(A) = T(A \twoheadrightarrow 0)$, and the *suspension* of A is $\Sigma A = \operatorname{cone}(A)/A$. The cylinder axiom implies that $\operatorname{cone}(A) \xrightarrow{\sim} 0$ is a weak equivalence. Since $A \rightarrowtail \operatorname{cone}(A) \twoheadrightarrow \Sigma A$ is a cofibration sequence it follows from the description of $K_0(\mathcal{C})$ in II.9.1.2 that $[\Sigma A] = -[A]$ in $K_0(\mathcal{C})$. (Cf. Lemma II.9.2.1.) In fact, the Additivity Theorem (see V.1.2 below) implies that the map $\Sigma : K(\mathcal{C}) \to K(\mathcal{C})$ is a homotopy inverse with respect to the H-space structure on $K(\mathcal{C})$, because $\Sigma_* + 1 = \operatorname{cone}_* = 0$.

The name 'cylinder functor' comes from the following two paradigms.

IV.8.8.2 **Example 8.8.2.** The Waldhausen categories $\mathcal{R}_f(*)$ and $\mathcal{R}_f(X)$ of Examples 8.7 and 8.7.1 have a cylinder functor: $T(f)$ is the usual (based) mapping cylinder of f. By construction, the mapping cylinder satisfies the cylinder axiom 8.8.1. Because of this paradigm, j_1 and j_2 are sometimes called the *front* and *back* inclusions.

IV.8.8.3 **Example 8.8.3.** Let **Ch** be the Waldhausen category of chain complexes and quasi-isomorphisms constructed from an abelian (or exact) category $\mathcal{C}$; see II.9.2. The mapping cylinder of $f : A_\bullet \to B_\bullet$ is the usual mapping cylinder chain complex [223, 1.5.5], in which

$$T(f)_n = A_n \oplus A_{n-1} \oplus B_n.$$

The suspension functor $\Sigma(A_\bullet)$ is the shift operator $A_\bullet \mapsto A_\bullet[-1]$: $\Sigma(A_\bullet)_n = A_{n-1}$.

IV.8.8.4 **Example 8.8.4.** Exact categories usually do not have cylinder functors. This is reflected by the fact that for some A in $\mathcal{A}$ there may be no B such that $[A \oplus B] = 0$ in $K_0(\mathcal{A})$. However, the Waldhausen category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes does have a cylinder functor, and we used it to prove that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A})$ in II.9.2.2. In fact, $K(\mathcal{A}) \simeq K(\mathbf{Ch}^b(\mathcal{A}))$ by the Gillet-Waldhausen theorem presented in V.2.2. Thus many results requiring mapping cylinders in Waldhausen K-theory can be translated into results for Quillen K-theory.

Cofinality

A Waldhausen subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to be *cofinal* if for all C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \amalg C'$ is in $\mathcal{B}$. The K_0 version of the following theorem was proven in II.9.4. We will prove a stronger cofinality result in V.2.3 below.

IV.8.9 **Waldhausen Cofinality 8.9.** *If $\mathcal{B}$ is a cofinal Waldhausen subcategory of $\mathcal{C}$, closed under extensions, and such that $K_0(\mathcal{B}) = K_0(\mathcal{C})$. Then $wS_\bullet\mathcal{B} \to wS_\bullet\mathcal{C}$ and $K(\mathcal{B}) \to K(\mathcal{C})$ are homotopy equivalences. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{C})$ for all n.*

IV.8.9.1 **Remark 8.9.1.** By Grayson's Trick (see Ex. II.9.14), the assumption that $K_0(\mathcal{B}) = K_0(\mathcal{C})$ is equivalent to saying that $\mathcal{B}$ is *strictly cofinal* in $\mathcal{C}$, meaning that for every C in $\mathcal{C}$ there is a B in $\mathcal{B}$ so that $B \coprod C$ is in $\mathcal{B}$.

Proof. By 8.5.3, it suffices to show that the "relative" bisimplicial category $wS_\bullet(S_\bullet f)$ is contractible, where $f : \mathcal{B} \to \mathcal{C}$ is the inclusion. For this it suffices to show that each $wS_n(S_\bullet f)$ is contractible. Switching simplicial directions, we can rewrite $wS_n(S_m f)$ as $wS_m(S_n f_n)$, where $f_n : S_n\mathcal{B} \to S_n\mathcal{C}$ and $S_n f_n$ is defined in 8.5.3. Since $S_n\mathcal{B}$ is equivalent to $\mathcal{E}_n(\mathcal{C})$ (see 8.3), we see from Ex. II.9.4 that $K_0(S_n\mathcal{B}) \cong K_0(S_n\mathcal{C})$. Hence the hypothesis also applies to f_n. Replacing f by f_n, we have a second reduction: it suffices to show that the simplicial category $wS_\bullet f$ is contractible.

Let $\mathcal{B}(m,w)$ denote the category of diagrams $B_0 \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} B_m$ in $\mathcal{B}$ whose maps are weak equivalences, and $f_{(m,w)}$ the inclusion of $\mathcal{B}(m,w)$ in $\mathcal{C}(m,w)$. Then the bidegree (m,n) part $w_m S_n f$ of $wS_\bullet f$ is the set $s_n f_{(m,w)}$ of objects of $S_n f_{(m,w)}$. Working with the nerve degreewise, it suffices to show that each $w_m S_\bullet f = s_\bullet f_{(m,w)}$ is contractible. Since $\mathcal{B}$ is strictly cofinal in $\mathcal{C}$ (by Grayson's trick), this implies that $f_{(m,w)}$ is also strictly cofinal by Ex. 8.12(b). The theorem now follows from Lemma 8.9.2 below. □

IV.8.9.2 **Lemma 8.9.2.** *If $f : \mathcal{B} \to \mathcal{C}$ is strictly cofinal then $s_\bullet f$ is contractible, where the elements of $s_n f$ are the objects of $S_n f$.*

Proof. Strict cofinality implies that for each finite set X of objects (B^i_*, C^i_*) of $S_{n_i} f$, there is an object B' of $\mathcal{B}$ such that each $(B' \amalg B^i_*, B' \amalg C^i_*)$ is in $S_{n_i}\mathrm{id}_{\mathcal{B}}$, because each $B' \amalg C^i_*$ is in $\mathcal{B}$.

We saw in 8.5.4 that $s_\bullet \mathrm{id}_{\mathcal{B}}$ is the simplicial path space construction of $s_\bullet \mathcal{B}$, and is contractible because $s_0\mathcal{B}$ is a point. We will show that $s_\bullet f$ is contractible by showing that it is homotopy equivalent to $s_\bullet \mathrm{id}_{\mathcal{B}}$. For this we need to show that for any finite subcomplex L of $s_\bullet f$ there is a simplicial homotopy h (in the sense of [223, 8.3.11]) from the inclusion $L \subset s_\bullet f$ to a map $L \to s_\bullet \mathrm{id}_{\mathcal{B}} \subset s_\bullet f$, such that each component of h sends $L \cap s_\bullet \mathrm{id}_{\mathcal{B}}$ into $s_\bullet \mathrm{id}_{\mathcal{B}}$.

If X is the set of nondegenerate elements of L, we saw above that there is a B' so that $B' \amalg X$ (and hence $B' \amalg L$) is in $s_\bullet \mathrm{id}_{\mathcal{B}}$. The desired simplicial homotopy is given by the restriction of the maps $h_i : s_n f \to s_{n+1} f$, sending (B_*, C_*) to

$$(\cdots \rightarrowtail B_j \rightarrowtail B' \amalg B_j \rightarrowtail \cdots \rightarrowtail B' \amalg B_n,$$
$$\cdots \rightarrowtail C_j \rightarrowtail B' \amalg C_j \rightarrowtail \cdots \rightarrowtail B' \amalg C_n). \quad \square$$

IV.8.9.3 **Question 8.9.3.** If $\mathcal{B}$ is a cofinal Waldhausen subcategory of $\mathcal{C}$, but is not closed under extensions, is $K(\mathcal{B}) \simeq K(\mathcal{C})$? Using Ex. 8.12(a), the above proof shows that this is true if $\mathcal{B}$ is strictly cofinal in $\mathcal{C}$.

At the other extreme of cofinality, we have the following theorem of Thomason, which shows that by changing the weak equivalences in $\mathcal{A}$ we can force all the higher K-groups to vanish. Let $(\mathcal{A}, \mathrm{co})$ be any category with cofibrations; recall from II.9.1.3 that the group $K_0(\mathcal{A}) = K_0(\mathrm{iso}\,\mathcal{A})$ is defined in this context.

Suppose we are given a surjective homomorphism $\pi : K_0(\mathcal{A}) \to G$. Let $w(\mathcal{A})$ denote the family of morphisms $A \to A'$ in $\mathcal{A}$ such that $\pi[A] = \pi[A']$ in G. As observed in II.9.6.2, $(\mathcal{A}, w)$ is a Waldhausen category with $K_0(\mathcal{A}, w) = G$.

IV.8.10 **Theorem 8.10.** *There is a homotopy equivalence $wS_\bullet(\mathcal{A}, w) \to BG$. Hence $K(\mathcal{A})$ is homotopic to the discrete set G, and $K_n(\mathcal{A}, w) = 0$ for all $n \neq 0$.*

Proof. (Thomason) By construction of w, the category $w\mathcal{A}$ is the disjoint union of the full subcategories $\pi^{-1}(g)$ on the objects A with $\pi[A] = g$. For each g, fix an object A_g with $\pi[A_g] = g$. For $n > 1$, consider the function $\pi : ws_n\mathcal{A} \to G^n$ sending the object $A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$ of $wS_n\mathcal{A}$ to

$(\pi[A_1], \pi[A_{12}], \pi[A_{23}], \dots, \pi[A_{n-1,n}])$. By the construction of w, it induces a decomposition of $wS_n\mathcal{A}$ into the disjoint union (indexed by G^n) of the full subcategories $\pi^{-1}(g_1, ..., g_n)$ of objects mapping to $(g_1, ..., g_n)$. We will show that each of these components is contractible.

Given $\mathbf{g} = (g_1, ..., g_n)$, $\pi^{-1}(\mathbf{g})$ is not empty because it contains the object

$$A_{\mathbf{g}} : A_{g_1} \rightarrowtail (A_{g_1} \amalg A_{g_2}) \rightarrowtail (A_{g_1} \amalg A_{g_2} \amalg A_{g_3}) \rightarrowtail \cdots \rightarrowtail (\amalg_{i=1}^{n} A_{g_i})$$

of $wS_n\mathcal{A}$. The subcategory $\pi^{-1}(0)$ is contractible because it has initial object 0. For other $\mathbf{g}$, there is a natural transformation from the identity of $\pi^{-1}(\mathbf{g})$ to the functor $F(B) = A_{\mathbf{g}} \amalg A_{-\mathbf{g}} \amalg B$, given by the coproduct with the weak equivalence $0 \to A_{\mathbf{g}} \amalg A_{-\mathbf{g}}$. But F is null-homotopic because it factors as the composite of $F' : \pi^{-1}(\mathbf{g}) \to \pi^{-1}(0)$, $F'(B) = A_{-\mathbf{g}} \amalg B$, and $F'' : \pi^{-1}(0) \to \pi^{-1}\mathbf{g}$, $F''(C) = A_{\mathbf{g}} \amalg C$. It follows that $\pi^{-1}(\mathbf{g})$ is contractible, as desired. □

Products

IV.8.11 **8.11.** Our discussion in 6.6 about products in exact categories carries over to the Waldhausen setting. The following construction is taken from [215, just after 1.5.3]. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Waldhausen categories; recall from II.9.5.2 that a functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *biexact* if each $F(A, -)$ and $F(-, B)$ is exact, and the following condition is satisfied:

> For every pair of cofibrations ($A \rightarrowtail A'$ in $\mathcal{A}$, $B \rightarrowtail B'$ in $\mathcal{B}$) the following map is a cofibration in $\mathcal{C}$:
>
> $$F(A', B) \cup_{F(A,B)} F(A, B') \rightarrowtail F(A', B').$$

We saw in II.9.5.1 that a biexact functor induces a bilinear map $K_0(\mathcal{A}) \otimes K_0(\mathcal{B}) \to K_0(\mathcal{C})$. It also induces a morphism of bisimplicial bicategories

$$wS_\bullet\mathcal{A} \times wS_\bullet\mathcal{B} \to wwS_\bullet S_\bullet\mathcal{C}$$

which resembles (6.6.1). Upon passage to geometric realization, this factors

$$K(\mathcal{A}) \wedge K(\mathcal{B}) \to K(\mathcal{C}).$$

As observed in [200, 3.15], this pairing induces a pairing $\mathbf{K}(\mathcal{A}) \wedge \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C})$ of spectra. If $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is not only biexact but associative up to natural isomorphism, the pairing makes $\mathbf{K}(\mathcal{A})$ into a ring spectrum; it is a commutative ring spectrum if the pairing is commutative up to natural isomorphism. If in addition, $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$ is biexact and $\mathcal{A} \times \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ is associative up to natural isomorphism, then $\mathbf{K}(\mathcal{B})$ is a module spectrum over $\mathbf{K}(\mathcal{A})$.

In particular, if R is a commutative ring then $\mathbf{K}(R) = \mathbf{K}(\mathbf{P}(R))$ is a commutative ring spectrum, and $G(R)$ is a module spectrum over it. Similarly, if X is a quasi-projective scheme then $\mathbf{K}(X) = \mathbf{K}(\mathbf{VB}(X))$ is also a commutative ring spectrum, and $G(X)$ is a module spectrum over it.

EXERCISES

EIV.8.1 **8.1.** Show that for every $0 \le i < j < k \le n$ the diagram $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}$ is a cofibration sequence, and this gives an exact functor from $S_n\mathcal{C}$ to $S_2\mathcal{C}$.

EIV.8.2 **8.2.** Show that each functor $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is exact in 8.3.1. Then show that $S_\bullet\mathcal{C}$ is a simplicial category.

EIV.8.3 **8.3.** Let $f, f' : \mathcal{A} \to \mathcal{B}$ be exact functors. A natural transformation $\eta : f \to f'$ is called a *weak equivalence* if each $f(A) \xrightarrow{\sim} f'(A)$ is a weak equivalence in $\mathcal{B}$. Show that a weak equivalence induces a homotopy between the two maps $K(\mathcal{A}) \to K(\mathcal{B})$. *Hint:* Show that the maps $wS_n\mathcal{A} \to wS_n\mathcal{B}$ are homotopic in a compatible way.

EIV.8.4 **8.4.** We saw in 8.3.2 that there is a canonical map from $Bw(\mathcal{C})$ to $K(\mathcal{C})$, and hence maps $\pi_i B(w\mathcal{C}) \to K_i(\mathcal{C})$. The map $\pi_0 B(w\mathcal{C}) \to K_0(\mathcal{C})$ is described in 8.4.
(a) Every weak self-equivalence $\alpha : A \xrightarrow{\sim} A$ determines an element $[\alpha]$ of $K_1(\mathcal{C})$, by 3.4. If β is a weak self-equivalence of B, show that $[\alpha] + [\beta] = [\alpha \vee \beta]$. If $A = B$, show that $[\alpha] + [\beta] = [\beta\alpha]$
(b) If $\mathcal{A}$ is an exact category, considered as a Waldhausen category, show that the map $B(\text{iso}\,\mathcal{A}) \to K(\mathcal{A})$ induces a map from the group $K_1^{\oplus}\mathcal{A}$ of 4.8.1 to $K_1(\mathcal{A})$.
(c) In the notation of 8.2, show that a weak equivalence in $S_2\mathcal{C}$ with $A_i = A'_i$ determines a relation $[u_1] - [u_2] + [u_{12}] = 0$ in $K_1(\mathcal{C})$.
(d) Show that every pair of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ (with the same objects) determines an element of $K_1(\mathcal{C})$. (See 9.5 below.)

EIV.8.5 **8.5.** (Waldhausen) Let $\mathcal{A}$ be a small exact category. In this exercise we produce a map from $|iS_\bullet\mathcal{A}| \simeq |s_\bullet\mathcal{A}|$ to $BQ\mathcal{A}$, where $s_\bullet\mathcal{A}$ is defined in 8.5.2.
(a) Show that an object $A_\bullet$ of $iS_3\mathcal{A}$ determines a morphism $A_{12} \to A_3$ in $Q\mathcal{A}$.
(b) Show that an object $A_\bullet$ of $iS_5\mathcal{A}$ determines a sequence $A_{23} \to A_{14} \to A_5$ of row morphisms in $Q\mathcal{A}$.
(c) Recall from Ex. 3.10 that the Segal subdivision $Sub(s_\bullet\mathcal{A})$ is homotopy equivalent to $s_\bullet\mathcal{A}$. Show that (a) and (b) determine a simplicial map $Sub(s_\bullet\mathcal{A}) \to Q\mathcal{A}$. Composing with $|iS_\bullet\mathcal{A}| \simeq Sub(s_\bullet\mathcal{A})$, this yields a map $|iS_\bullet\mathcal{A}| \to BQ\mathcal{A}$.

EIV.8.6 **8.6.** We now show that the map $|iS_\bullet\mathcal{A}| \to BQ\mathcal{A}$ constructed in the previous exercise is a homotopy equivalence. Let $iQ_n\mathcal{A}$ denote the category whose objects are the degree n elements of the nerve of $Q\mathcal{A}$, *i.e.*, sequences $A_0 \to \cdots \to A_n$ in $Q\mathcal{A}$, and whose morphisms are isomorphisms.
(a) Show that $iQ_\bullet\mathcal{A}$ is a simplicial category, and that the nerve of $Q\mathcal{A}$ is the simplicial set of objects. Waldhausen proved in [215, 1.6.5] that $BQ\mathcal{A} \to |iQ_\bullet\mathcal{A}|$ is a homotopy equivalence.
(b) Show that there is an equivalence of categories $Sub(iS_n\mathcal{A}) \xrightarrow{\sim} iQ_n\mathcal{A}$ for each n, where $Sub(iS_n\mathcal{A})$ is the Segal subdivision category of Ex. 3.9. Then show that the equivalences form a map of simplicial categories $Sub(iS_\bullet\mathcal{A}) \to iQ_\bullet\mathcal{A}$. This map must be a homotopy equivalence, because it is a homotopy equivalence in each degree. The typical case $Sub(iS_3\mathcal{A}) \to iQ_3\mathcal{A}$ is illustrated in [215, 1.9].

(c) Show that the map of the previous exercise fits into a diagram

$$\begin{array}{ccccc} |s_\bullet\mathcal{A}| & \xrightarrow{\simeq} & |Sub(s_\bullet\mathcal{A})| & \longrightarrow & BQ\mathcal{A} \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ |iS_\bullet\mathcal{A}| & \xrightarrow{\simeq} & |Sub(iS_\bullet\mathcal{A})| & \xrightarrow{\simeq} & |iQ_\bullet\mathcal{A}|. \end{array}$$

Conclude that the map $|iS_\bullet\mathcal{A}| \to BQ\mathcal{A}$ of Ex. 8.5 is a homotopy equivalence.

EIV.8.7 **8.7.** Recall from 8.5.2 that $K_1(\mathcal{A}) \cong \pi_2|s_\bullet\mathcal{A}|$ for any exact category $\mathcal{A}$. Given an automorphism α of an object A in $\mathcal{A}$, show that the two 2-cells in $|s_\bullet\mathcal{A}|$ corresponding to the extensions $0 \rightarrowtail A \overset{\alpha}{\twoheadrightarrow} A$ and $A \overset{\alpha}{\rightarrowtail} A \twoheadrightarrow 0$ fit together to define an element of $\pi_2|s_\bullet\mathcal{A}|$. Then show that the map of Ex. 8.5 identifies it with the element $[\alpha]$ of $\pi_2 BQ\mathcal{A}$ described in Ex. 7.9.

EIV.8.8 **8.8.** *Finite Sets.* Show that the category $\mathbf{Sets}_{\text{fin}}$ of finite pointed sets is a Waldhausen category, where the cofibrations are the injections and the weak equivalences are the isomorphisms. Then mimick Exercises 8.5 and 8.6 to show that the space $BQ\mathbf{Sets}_{\text{fin}}$ of Ex. 6.15 is homotopy equivalent to the Waldhausen space $iS_\bullet\mathbf{Sets}_{\text{fin}}$. Using Theorem 4.9.3 and Ex. 7.7, conclude that the Waldhausen K-theory space $K(\mathbf{Sets}_{\text{fin}})$ is $\mathbb{Z}\times(B\Sigma_\infty)^+ \simeq \Omega^\infty S^\infty$. Thus $K_n(\mathbf{Sets}_{\text{fin}}) \cong \pi_n^s$ for all n.

EIV.8.9 **8.9.** *G-Sets.* If G is a group, show that the category $G{-}\mathbf{Sets}_+$ of finitely generated pointed G-sets, and its subcategory $\mathcal{F}$ of free pointed G-sets, are Waldhausen categories. Then mimick Exercises 8.5 and 8.6 to show that the spaces $BQ(G{-}\mathbf{Sets}_+)$ and $BQ\mathcal{F}$ of Ex. 6.16 are homotopy equivalent to the Waldhausen spaces $iS_\bullet(G{-}\mathbf{Sets}_+)$ and $iS_\bullet\mathcal{F}$. Using Ex. 7.7, conclude that the Waldhausen K-theory space $K(\mathcal{F})$ is homotopy equivalent to $\Omega^\infty S^\infty(BG_+)$.

EIV.8.10 **8.10.** Given a category with cofibrations $\mathcal{C}$, let $\mathcal{E} = \mathcal{E}(\mathcal{C})$ denote the category of extensions in $\mathcal{C}$ (see II.9.3), and $s_\bullet\mathcal{C}$ the simplicial set of 8.5.2. In this exercise we show that the source and quotient functors $s, q : \mathcal{E} \to \mathcal{C}$ induce $s_\bullet\mathcal{E} \simeq s_\bullet\mathcal{C} \times s_\bullet\mathcal{C}$.
(a) Recall from Ex. 3.11 that for A in $s_n\mathcal{C}$ the simplicial set $s/(n,A)$ is the pullback of $s_\bullet\mathcal{E}$ and Δ^n along s and $A : \Delta^n \to s_\bullet\mathcal{C}$. Show that $s/(0,0)$ is equivalent to $s_\bullet\mathcal{C}$.
(b) For every vertex α of Δ^n and every A in $s_n\mathcal{C}$ that the map $s/(0,0) \to s/(n,A)$ of Ex. 3.11 is a homotopy equivalence.
(c) Use (b) to show that $s : s_\bullet\mathcal{E} \to s_\bullet\mathcal{C}$ satisfies the hypothesis of Ex. 3.11(b).
(d) Use Ex. 3.11(b) to show that there is a homotopy fibration $s_\bullet\mathcal{C} \to s_\bullet\mathcal{E} \to s_\bullet\mathcal{C}$
Conclude that $s \times q : s_\bullet\mathcal{E} \to s_\bullet\mathcal{C} \times s_\bullet\mathcal{C}$ is a homotopy equivalence.

EIV.8.11 **8.11.** Given an exact functor $f : \mathcal{B} \to \mathcal{C}$, mimick the proof of 8.4 to show that the group $K_{-1}(f) = \pi_1(wS_\bullet S_\bullet f)$ of 8.5.3 is the cokernel of $K_0(\mathcal{B}) \to K_0(\mathcal{C})$.

EIV.8.12 **8.12.** Suppose that $\mathcal{B}$ is a strictly cofinal Waldhausen subcategory of $\mathcal{C}$.
(a) Show that $S_n\mathcal{B}$ is strictly cofinal in $S_n(\mathcal{C})$.
(b) Show that, in the proof of 8.9, $\mathcal{B}(w,m)$ is strictly cofinal in $\mathcal{C}(w,m)$.

EIV.8.13 **8.13.** Any exact category $\mathcal{A}$ is cofinal in its idempotent completion $\hat{\mathcal{A}}$, by definition (see II.7.3). Let $\mathcal{A}'$ be the subcategory of $\hat{\mathcal{A}}$ consisting of all B in $\hat{\mathcal{A}}$ such that $[B]$ lies in the subgroup $K_0(\mathcal{A})$ of $K_0(\hat{\mathcal{A}})$. Show that $\mathcal{A}'$ is an exact category, closed under admissible epimorphisms in $\hat{\mathcal{A}}$, and that $\mathcal{A}$ is strictly cofinal in $\mathcal{A}'$. Hence $K(\mathcal{A}) \simeq K(\mathcal{A}')$.

EIV.8.14 **8.14.** Let $\mathbf{Ch}(\mathcal{C})$ be the Waldhausen category of chain complexes in an exact category $\mathcal{C}$, as in 8.8.3. Show that $\mathbf{Ch}(\mathcal{C})$ and $\mathbf{Ch}^b(\mathcal{C})$ are saturated and satisfy the Extension axiom 8.2.1, and the Cylinder Axiom 8.8.1.

EIV.8.15 **8.15.** If $(\mathcal{C}, \mathrm{co}, w)$ is a saturated Waldhausen category with a cylinder functor, satisfying the cylinder axiom, show that the category $\mathrm{co}\, w\mathcal{C}$ of "trivial cofibrations" (cofibrations which are weak equivalences) is homotopy equivalent to $w\mathcal{C}$. *Hint:* Use the cylinder to show that each i/C is contractible, and apply Theorem A.

9 The Gillet-Grayson construction

Let $\mathcal{A}$ be an exact category. Following Grayson and Gillet [68], we define a simplicial set $G_\bullet = G_\bullet\mathcal{A}$ as follows.

IV.9.1 **Definition 9.1.** If $\mathcal{A}$ is a small exact category, $G_\bullet$ is the simplicial set defined as follows. The set G_0 of vertices consists of all pairs of objects (A, B) in $\mathcal{A}$. The set G_1 of edges consists of all pairs of short exact sequences with the same cokernel:

$$A_0 \rightarrowtail A_1 \twoheadrightarrow A_{01}, \quad B_0 \rightarrowtail B_1 \twoheadrightarrow A_{01}. \tag{9.1.0}$$ IV.9.1.0

The degeneracy maps $G_1 \to G_0$ send (9.1.0) to (A_1, B_1) and (A_0, B_0), respectively.

The set G_n consists of all pairs of triangular commutative diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccccccccccc}
 & & & & A_{n-1,n} & \qquad & & & & & A_{n-1,n} \\
 & & & & \twoheaduparrow & & & & & & \twoheaduparrow \\
 & & & & \cdots & & & & & & \cdots \\
 & & A_{12} & \rightarrowtail \cdots \rightarrowtail & A_{1n} & & & & A_{12} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\
 & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow & & \twoheaduparrow \\
 A_{01} & \twoheadrightarrow & A_{02} & \rightarrowtail \cdots \rightarrowtail & A_{0n} & & A_{01} & \twoheadrightarrow & A_{02} & \rightarrowtail \cdots \rightarrowtail & A_{0n} \\
 \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\
 A_0 \rightarrowtail A_1 & \twoheadrightarrow & A_2 & \rightarrowtail \cdots \rightarrowtail & A_n & & B_0 \rightarrowtail B_1 & \twoheadrightarrow & B_2 & \rightarrowtail \cdots \rightarrowtail & B_n
\end{array} \tag{9.1.1}$$ IV.9.1.1

so that each sequence $A_i \rightarrowtail A_j \twoheadrightarrow A_{ij}$ and $B_i \rightarrowtail B_j \twoheadrightarrow A_{ij}$ is exact. As in the definition of $S_\bullet\mathcal{A}$ (8.3.1), the face maps $\partial_i : G_n \to G_{n-1}$ are obtained by deleting the row $A_{i\,\bullet}$ and the columns containing A_i and B_i, while the degeneracy maps $\sigma_i : G_n \to G_{n+1}$ are obtained by duplicating A_i and B_i, and reindexing.

Suppressing the choices A_{ij} for the cokernels, we can abbreviate (9.1.1) as:

$$\frac{A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n}{B_0 \rightarrowtail B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_n}\,. \tag{9.1.2}$$ IV.9.1.2

IV.9.1.3 **Remark 9.1.3.** $|G_\bullet|$ is a homotopy commutative and associative H-space. Its product $|G_\bullet| \times |G_\bullet| \to |G_\bullet|$ arises from the simplicial map $G_\bullet \times G_\bullet \to G_\bullet$ whose components $G_n \times G_n \to G_n$ are termwise $\oplus$.

Note that for each isomorphism $A \cong A'$ in $\mathcal{A}$ there is an edge in G_1 from $(0, 0)$ to (A, A'), represented by $(0 \rightarrowtail A \twoheadrightarrow A, 0 \rightarrowtail A' \twoheadrightarrow A)$. Hence (A, A') represents zero in the group $\pi_0|G_\bullet|$.

IV.9.2 **Lemma 9.2.** *There is a group isomorphism $\pi_0|G_\bullet| \cong K_0(\mathcal{A})$.*

Proof. As in 3.3, $\pi_0|G_\bullet|$ is presented as the set of elements (A, B) of G_0, modulo the equivalence relation that for each edge (9.1.0) we have

$$(A_1, B_1) = (A_0, B_0).$$

It is an abelian group by 9.1.3, with operation $(A, B) \oplus (A', B') = (A \oplus A', B \oplus B')$. Since $(A \oplus B, B \oplus A)$ represents zero in $\pi_0|G_\bullet|$, it follows that (B, A) is the inverse of (A, B). From this presentation, we see that there is a map $K_0(\mathcal{A}) \to \pi_0|G_\bullet|$ sending $[A]$ to $(A, 0)$, and a map $\pi_0|G_\bullet| \to K_0(\mathcal{A})$, sending (A, B) to $[A] - [B]$. These maps are inverses to each other. □

IV.9.3 **9.3.** We now compare $G_\bullet$ with the loop space of the simplicial set $s_\bullet\mathcal{A}$ of 8.5.2. If we forget the bottom row of either of the two triangular diagrams in (9.1.1), we get a triangular commutative diagram of the form (8.3.0), *i.e.,* an element of $s_n\mathcal{A}$. The resulting set maps $G_n \to s_n\mathcal{A}$ fit together to form a simplicial map $\partial_0 : G_\bullet \to s_\bullet\mathcal{A}$.

IV.9.3.1 **Path Spaces 9.3.1.** Recall from [223, 8.3.14] that the *path space* $PX_\bullet$ of a simplicial set $X_\bullet$ has $PX_n = X_{n+1}$, its ith face operator is the ∂_{i+1} of $X_\bullet$, and its ith degeneracy operator is the σ_{i+1} of $X_\bullet$. The forgotten face maps $\partial_0 : X_{n+1} \to X_n$ form a simplicial map $PX_\bullet \to X_\bullet$, and $\pi_0(PX_\bullet) \cong X_0$. In fact, σ_0 induces a canonical simplicial homotopy equivalence from $PX_\bullet$ to the constant simplicial set X_0; see [223, Ex. 9.3.7]. Thus $PX_\bullet$ is contractible exactly when X_0 is a point.

Now there are two maps $G_n \to s_{n+1}\mathcal{A}$, obtained by forgetting one of the two triangular diagrams (9.1.1) giving an element of G_n. The face and degeneracy maps of $G_\bullet$ are defined so that these yield two simplicial maps from $G_\bullet$ to the path space $P_\bullet = P(s_\bullet\mathcal{A})$. Clearly, either composition with the canonical map $P_\bullet \to s_\bullet\mathcal{A}$ yields the map $\partial_0 : G_\bullet \to s_\bullet\mathcal{A}$. Thus we have a commutative diagram

$$\begin{array}{ccc} G_\bullet & \longrightarrow & P_\bullet \\ \downarrow & & \downarrow \\ P_\bullet & \longrightarrow & s_\bullet\mathcal{A}. \end{array} \tag{9.3.2}$$ IV.9.3.2

Since $s_0\mathcal{A}$ is a point, the path space $|P_\bullet|$ is canonically contractible. Therefore this diagram yields a canonical map $|G_\bullet| \to \Omega|s_\bullet\mathcal{A}|$. On the other hand, we saw in 8.5.2 and 8.6 that $|s_\bullet\mathcal{A}| \simeq BQ\mathcal{A}$, so $\Omega|s_\bullet\mathcal{A}| \simeq \Omega BQ\mathcal{A} = K(\mathcal{A})$.

We cite the following result from [68, 3.1]. Its proof uses simplicial analogues of Quillen's Theorems A and B.

IV.9.4 **Theorem 9.4.** *(Gillet-Grayson) Let $\mathcal{A}$ be a small exact category. Then the map of (9.3) is a homotopy equivalence:*

$$|G_\bullet| \simeq \Omega|s_\bullet\mathcal{A}| \simeq K(\mathcal{A}).$$

Hence $\pi_i|G_\bullet| = K_i(\mathcal{A})$ for all $i \geq 0$.

IV.9.5 **Example 9.5.** A *double s.e.s.* in $\mathcal{A}$ is a pair ℓ of short exact sequences in $\mathcal{A}$ on the same objects:

$$\ell : \qquad A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C, \quad A \overset{f'}{\rightarrowtail} B \overset{g'}{\twoheadrightarrow} C.$$

Thus ℓ is an edge (in G_1) from (A, A) to (B, B). To ℓ we attach the element $[\ell]$ of $K_1(\mathcal{A}) = \pi_1|G_\bullet|$ given by the following 3-edged loop.

$$\begin{array}{ccc} (A,A) & \xrightarrow{\ \ell\ } & (B,B) \\ {\scriptstyle e_A}\nwarrow & & \nearrow{\scriptstyle e_B} \\ & (0,0) & \end{array}$$

where e_A denotes the canonical double s.e.s. $(0 \rightarrowtail A \twoheadrightarrow A,\ 0 \rightarrowtail A \twoheadrightarrow A)$.

The following theorem was proven by A. Nenashev in [143].

IV.9.6 **Nenashev's Theorem 9.6.** *$K_1(\mathcal{A})$ may be described as follows.*

(a) Every element of $K_1(\mathcal{A})$ is represented by the loop $[\ell]$ of a double s.e.s.;

(b) $K_1(\mathcal{A})$ is presented as the abelian group with generators the double s.e.s. in $\mathcal{A}$, subject to two relations:

(i) If E is a short exact sequence, the loop of the double s.e.s. (E, E) is zero;

(ii) for any diagram of six double s.e.s. (9.6.1) such that the "first" diagram commutes, and the "second" diagram commutes, then

$$[r_0] - [r_1] + [r_2] = [c_0] - [c_1] + [c_2],$$

where r_i is the i^{th} row and c_i is the i^{th} column of (9.6.1).

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ C' & \rightrightarrows & C & \rightrightarrows & C''. \end{array} \qquad (9.6.1)$$ IV.9.6.1

IV.9.6.2 **Example 9.6.2.** If α is an automorphism of A, the class $[\alpha] \in K_1(\mathcal{A})$ is the class of the double s.e.s. $(0 \rightarrowtail A \xrightarrow{\alpha} A, 0 \rightarrowtail A \xrightarrow{=} A)$. If β is another automorphism of A, the relation $[\alpha\beta] = [\alpha] + [\beta]$ comes from relation (ii) for

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ 0 & \rightrightarrows & A & \overset{\alpha}{\underset{1}{\rightrightarrows}} & A \\ \downarrow\downarrow & & {\scriptstyle 1}\downarrow\downarrow{\scriptstyle 1} & & {\scriptstyle \beta}\downarrow\downarrow{\scriptstyle 1} \\ 0 & \rightrightarrows & A & \overset{\alpha\beta}{\underset{1}{\rightrightarrows}} & A. \end{array}$$

EXERCISES

EIV.9.1 **9.1.** Verify that condition 9.6(i) holds in $\pi_1|G_\bullet|$.

EIV.9.2 **9.2.** Show that omitting the choice of quotients A_{ij} from the definition of $G_\bullet\mathcal{A}$ yields a homotopy equivalent simplicial set $G'_\bullet\mathcal{A}$. An element of $G'_n\mathcal{A}$ is a diagram (9.1.2) together with a compatible family of isomorphisms $A_j/A_i \cong B_j/B_i$.

EIV.9.3 **9.3.** Consider the involution on $G_\bullet$ which interchanges the two diagrams in (9.1.1). We saw in 9.2 that it induces multiplication by -1 on $K_0(\mathcal{A})$. Show that this involution is an additive inverse map for the H-space structure 9.1.3 on $|G_\bullet|$.

EIV.9.4 **9.4.** If $\alpha : A \cong A$ is an isomorphism, use relation (ii) in Nenashev's presentation 9.6 to show that $[\alpha^{-1}] \in K_1(\mathcal{A})$ is represented by the loop of the double s.e.s.:

$$\begin{array}{c} A \xrightarrow{\alpha} A \longrightarrow 0 \\ \hline\hline A \xrightarrow[=]{} A \longrightarrow 0 \end{array}$$

EIV.9.5 **9.5.** If $\mathcal{A}$ is a split exact category, use Nenashev's presentation 9.6 to show that $K_1(\mathcal{A})$ is generated by automorphisms (9.6.2).

10 Non-connective spectra in K-theory

In §III.4 we introduced the negative K-groups of a ring using Bass' Fundamental Theorem III.3.7 for $K_0(R[t,t^{-1}])$. For many applications, it is useful to have a spectrum-level version of this construction, viz., a non-connective "Bass K-theory spectrum" $\mathbf{K}^B(R)$ with $\pi_n\mathbf{K}^B(R) = K_n(R)$ for all $n < 0$. In this section we constuct such a non-connective spectrum starting from any one of the functorial models of a connective K-theory spectrum $\mathbf{K}(R)$. (See 1.9(iii), 4.5.2 and 8.5.5.)

Let $\mathbf{E}$ be a functor from rings to spectra. Since the inclusions of $\mathbf{E}(R)$ in $\mathbf{E}(R[x])$ and $\mathbf{E}(R[x^{-1}])$ split, the homotopy pushout $\mathbf{E}(R[x]) \vee_{\mathbf{E}(R)} \mathbf{E}(R[x^{-1}])$ is the wedge of $\mathbf{E}(R)$ and these two complementary factors.

IV.10.1 **Definition 10.1.** Write $L\mathbf{E}(R)$ for the spectrum homotopy cofiber of the map f_0 from this homotopy pushout to $\mathbf{E}(R[x,x^{-1}])$, and $\Lambda\mathbf{E}(R)$ for the desuspension $\Omega L\mathbf{E}(R)$.

Since the mapping cone is natural, $L\mathbf{E}$ and $\Lambda\mathbf{E}$ are functors and there is a cofibration sequence, natural in $\mathbf{E}$ and R:

$$\Lambda\mathbf{E}(R) \to \mathbf{E}(R[x]) \vee_{\mathbf{E}(R)} \mathbf{E}(R[x^{-1}]) \xrightarrow{f_0} \mathbf{E}(R[x,x^{-1}]) \to L\mathbf{E}(R).$$

The algebraic version of the Fundamental Theorem of higher K-theory, established in V.6.2 and V.8.2, states that there is a split exact sequence

$$0 \to K_n(R) \to K_n(R[x]) \oplus K_n(R[x^{-1}]) \to K_n(R[x,x^{-1}]) \overset{\leftarrow}{\longrightarrow} K_{n-1}(R) \to 0,$$

in which the splitting is multiplication by $x \in K_1(\mathbb{Z}[x,x^{-1}])$. Applying π_n to the case $\mathbf{E} = \mathbf{K}$ of Definition 10.1 shows that $\pi_n L\mathbf{K}(R) \cong K_{n-1}(R)$ for all $n > 0$. The Fundamental Theorems for K_1 and K_0 (III, 3.6 and 3.7) imply that $\pi_0\Lambda\mathbf{K}(R) = K_0(R)$, $\pi_{-1}\Lambda\mathbf{K}(R) = K_{-1}(R)$ and that $\pi_n\Lambda\mathbf{K}(R) = 0$ for $n < -1$.

We will need the following topological version of the Fundamental Theorem, also established in the next chapter (in V.8.4). Fix a map $S^1 \to \mathbf{K}(\mathbb{Z}[x,x^{-1}])$, represented by the element $x \in K_1(\mathbb{Z}[x,x^{-1}])$. Recall from 1.10.2 and Ex. 4.14 that this map induces a product map $\mathbf{K}(R) \xrightarrow{\cup x} \Omega\mathbf{K}(R[x,x^{-1}])$, natural in the ring R. Composing with $\Omega\mathbf{K}(R[x,x^{-1}]) \to \Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda\mathbf{K}(R)$ yields a map of spectra $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$.

IV.10.2 **Fundamental Theorem 10.2.** *For any ring R, the map $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ induces a homotopy equivalence between $\mathbf{K}(R)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(R)$. In particular, $K_n(R) \cong \pi_n\Lambda\mathbf{K}(R)$ for all $n \geq 0$.*

By induction on k, we have natural maps

$$\Lambda^{k-1}\mathbf{K}(R) \xrightarrow{\cup x} \Lambda^{k-1}\Omega\mathbf{K}(R[x,x^{-1}]) \to \Lambda^k\Omega L\mathbf{K}(R) \xleftarrow{\simeq} \Lambda^k\mathbf{K}(R).$$

IV.10.3 **Corollary 10.3.** *For $k>0$ the map $\Lambda^{k-1}\mathbf{K}(R) \to \Lambda^k\mathbf{K}(R)$ induces a homotopy equivalence between $\Lambda^{k-1}\mathbf{K}(R)$ and the $(-k)$-connective cover of $\Lambda^k\mathbf{K}(R)$, with $K_n(R) \cong \pi_n\Lambda^{k-1}\mathbf{K}(R) \cong \pi_n\Lambda^k\mathbf{K}(R)$ for $n > -k$, and $K_{-k}(R) \cong \pi_{-k}\Lambda^k\mathbf{K}(R)$.*

Proof. We proceed by induction on k, the case $k=1$ being Theorem IV.10.2 10.2. Set $\mathbf{E}=\Lambda^{k-1}\mathbf{K}$; we have a natural isomorphism $K_n(R)\cong\pi_n\mathbf{E}(R)$ for $n>-k$, such that $\cup x: K_n(R)\to K_{n+1}(R[x,x^{-1}])$ agrees with π_n of $\mathbf{E}(R)\to\Omega\mathbf{E}(R[x,x^{-1}])$ up to isomorphism. The map $\pi_n\mathbf{E}(R[x])\vee_{\mathbf{E}(R)}\mathbf{E}(R[x^{-1}])\to\pi_n\mathbf{E}(R[x,x^{-1}])$ in IV.10.1 10.1 is an injection for all n, being either the injection from $K_n(R[x])\oplus K_n(R[x^{-1}])/K_n(R)$ to $K_n(R[x,x^{-1}])$ of III.III.4.1.2 4.1.2 and IV.10.2 10.2 (for $n>-k$) or $0\to 0$ (for $n\le -k$). It follows from III.III.4.1.2 4.1.2 that for $n>-k$ the maps $K_n(R)\cong\pi_n\Lambda^{k-1}\mathbf{K}(R)\to\pi_n\Lambda^k\mathbf{K}(R)$ are isomorphisms, and that the composite

$$K_{-k}(R)\xrightarrow{\cup x}K_{1-k}(R[x,x^{-1}])\cong\pi_{1-k}\Lambda^{k-1}\mathbf{K}(R[x,x^{-1}])\to\pi_{-k}\Lambda^k\mathbf{K}(R)$$

is an isomorphism. Since $\Lambda^{k-1}\mathbf{K}(R)$ is $(-k)$-connected, it is the $(-k)$-connected cover of $\Lambda^k\mathbf{K}(R)$. It is also clear from IV.10.1 10.1 that $\pi_n\Lambda^k\mathbf{K}(R)=0$ for $n<-k$. □

IV.10.4 **Definition 10.4.** We define $\mathbf{K}^B(R)$ to be the homotopy colimit of the diagram

$$\mathbf{K}(R)\to\Omega L\mathbf{K}(R)\xleftarrow{\simeq}\Lambda\mathbf{K}(R)\to\cdots\Lambda^{k-1}\mathbf{K}(R)\to\Lambda^k\Omega L\mathbf{K}(R)\xleftarrow{\simeq}\Lambda^k\mathbf{K}(R)\to\cdots$$

(The homotopy colimit may be obtained by inductively replacing each portion $\bullet\xleftarrow{\simeq}\bullet\to\bullet$ by a pushout and then taking the direct limit of the resulting sequence of spectra.)

By IV.10.3 10.3, the canonical map $\mathbf{K}(R)\to\mathbf{K}^B(R)$ induces isomorphisms $K_n(R)\cong\pi_n\mathbf{K}^B(R)$ for $n\ge 0$, and $K_n(R)\cong\pi_n\mathbf{K}^B(R)$ for all $n\le 0$ as well.

IV.10.4.1 **Variant 10.4.1.** The "suspension ring" $S(R)$ of R provides an alternative way of constructing a non-connective spectrum for K-theory. Recall from III, Ex. EIII.1.15 1.15, that $S(R)$ is defined to be $C(R)/M(R)$. In III, Ex. EIII.4.10 4.10, we saw that there are isomorphisms $K_n(R)\cong K_0S^{|n|}(R)$ for $n\le 0$. In fact, Gersten and Wagoner proved that $K_0(R)\times BGL(R)^+\simeq\Omega BGL(S(R))^+$ so that $K_n(R)\cong K_{n+1}S(R)$ for all $n\ge 0$. It follows that the sequence of spaces $\mathbf{K}^{GW}(R)_i=K_0(S^i(R))\times BGL(S^i(R))^+$ form a nonconnective spectrum with $\pi_n\mathbf{K}^{GW}(R)\cong K_n(R)$ for all n. We leave it as an exercise to show that a homotopy equivalence between the 0th space of $\mathbf{K}(R)$ and $K_0(R)\times BGL(R)^+$ induces a homotopy equivalence of spectra $\mathbf{K}^B(R)\simeq\mathbf{K}^{GW}(R)$.

We now introduce a delooping of Quillen's space $K(\mathcal{A})=\Omega BQ\mathcal{A}$ (or spectrum) associated to an exact category $\mathcal{A}$, as $K(S\mathcal{A})$ for a different exact category $S\mathcal{A}$. Iterating this yields a non-connective spectrum with connective cover $K(\mathcal{A})$, which agrees with the construction of Definition IV.10.1 10.1 when $\mathcal{A}=\mathbf{P}(R)$.

IV.10.5 **Big vector bundles 10.5.** Many constructions require that $K(X)$ be strictly functorial in X. For this we introduce the notion of big vector bundles, which I learned from Thomason; see Ex. IV.10.3 10.3. Let $\mathcal{V}$ be a small category of schemes over a fixed scheme X. By a *big vector bundle* over X we will mean the choice of a vector bundle $\mathcal{E}_Y$ on Y for each morphism $Y\to X$ in $\mathcal{V}$, equipped with an isomorphism $f^*\mathcal{E}_Y\to\mathcal{E}_Z$ for every $f:Z\to Y$ over X such that: (i) to the identity on Y we associate the identity on $\mathcal{E}_Y$, and (ii) for each composition $W\xrightarrow{g}Z\xrightarrow{f}Y$, the map $(fg)^*$ is the composition $g^*f^*\mathcal{E}_Y\to g^*\mathcal{E}_Z\to\mathcal{E}_W$.

Let $\mathbf{VB}_{\mathcal{V}}(X)$ denote the category of big vector bundles over X. The obvious forgetful functor $\mathbf{VB}_{\mathcal{V}}(X) \to \mathbf{VB}(X)$ is an equivalence of categories, and $X \mapsto \mathbf{VB}_{\mathcal{V}}(X)$ is clearly a contravariant functor from $\mathcal{V}$ to exact categories. Since K-theory is a functor on exact categories, $X \mapsto K\mathbf{VB}_{\mathcal{V}}(X)$ is a presheaf of spectra on $\mathcal{V}$.

If $\mathcal{V}$ is a small category of noetherian schemes and flat maps, a big coherent module over X for $\mathcal{V}$ is the choice of a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ on Y for each $Y \to X$, equipped with a natural isomorphism $f^*\mathcal{F}_Y \to \mathcal{F}_Z$ for every (flat) $f : Z \to Y$ over X, subject to the usual conditions on identity maps and compositions. Let $\mathbf{M}_{\mathcal{V}}(X)$ denote the category of big coherent modules over X. The obvious forgetful functor $\mathbf{M}_{\mathcal{V}}(X) \to \mathbf{M}(X)$ is an equivalence of categories, and $X \mapsto \mathbf{M}_{\mathcal{V}}(X)$ is clearly a contravariant functor from $\mathcal{V}$ to exact categories. Since K-theory is a functor on exact categories, $X \mapsto K\mathbf{M}_{\mathcal{V}}(X)$ is a presheaf of spectra on $\mathcal{V}$.

IV.10.6 **Non-connective K-theory of schemes 10.6.** Let $\mathcal{V}$ be a small category of quasi-projective schemes such that, whenever X is in $\mathcal{V}$, then so are $X \times \mathbb{A}^1$ and $X \times \operatorname{Spec}(\mathbb{Z}[x, x^{-1}])$. Using big vector bundles on $\mathcal{V}$, we may arrange that $X \mapsto \mathbf{K}(X)$ is a functor from $\mathcal{V}$ to spectra. In this way, Construction 10.1 may be made functorial in X.

There is also a Fundamental Theorem like 10.2 for the algebraic K-theory of a quasi-projective scheme X (and even for quasi-compact, quasi-separated schemes), due to Thomason and Trobaugh [200, 6.1]. Using this and functoriality of $\Lambda^k\mathbf{K}(X)$, the proof of 10.3 goes through, and we define $\mathbf{K}^B(X)$ to be the homotopy colimit of the $\Lambda^k\mathbf{K}(X)$. If $X = \operatorname{Spec}(R)$ then $\mathbf{K}(X)$ is homotopy equivalent to $\mathbf{K}(R)$ and hence $\mathbf{K}^B(X)$ is homotopy equivalent to $\mathbf{K}^B(R)$. As for rings, the canonical map $\mathbf{K}(X) \to \mathbf{K}^B(X)$ induces isomorphisms $K_n(X) \cong \pi_n\mathbf{K}^B(X)$ for $n \geq 0$, and $K_n(X) \cong \pi_n\mathbf{K}^B(X)$ for all $n \leq 0$ as well.

EXERCISES

EIV.10.1 **10.1.** Let I be an ideal in a ring R, and write $\mathbf{K}^B(R, I)$ for the homotopy fiber of $\mathbf{K}^B(R) \to \mathbf{K}^B(R/I)$. Let $\mathbf{K}^{\leq 0}(R, I)$ denote the homotopy cofiber of the 0-connected cover $\mathbf{K}^B(R, I)\langle 0\rangle \to \mathbf{K}^B(R, I)$, as in 4.11.2. Thus $\pi_n\mathbf{K}^{\leq 0}(R, I) = 0$ for $n > 0$, and $\pi_0\mathbf{K}^{\leq 0}(R, I) \cong K_0(I)$ by Ex. 1.15. Use III.2.3 to show that $\pi_n\mathbf{K}^{\leq 0}(R, I) \cong K_n(I)$ for all $n < 0$.

EIV.10.2 **10.2.** Let $\mathcal{A}$ be the category $\mathbf{VB}(X)$. Use the method of 10.4.1 to produce a non-connective spectrum with connective cover $\mathbf{K}(X)$.

EIV.10.3 **10.3.** Let $\mathcal{V}$ be a small category of schemes, so that $X \mapsto \mathbf{VB}(X)$ is a contravariant lax functor on $\mathcal{V}$. Recall the Kleisli rectification of $\mathbf{VB}_X$ in Exercise 6.5, whose objects are pairs $(Y \to X, \mathcal{E}_Y)$, and whose morphisms are pairs $(Z \xrightarrow{h} Y, h^*(\mathcal{E}_Y) \cong \mathcal{E}_Z)$. Given a morphism $f : T \to X$ in $\mathcal{V}$, use the natural isomorphism $h^*f^* \cong f^*h^*$ to construct an exact functor $f : \mathbf{VB}_X \to \mathbf{VB}_T$. Compare this with the construction of big vector bundles in 10.5.

11 Karoubi-Villamayor K-theory

Following Gersten, we say that a functor F from rings (or rings without unit) to sets is *homotopy invariant* if $F(R) \cong F(R[t])$ for every R. Similarly, a functor F from rings to CW complexes (spaces) is called *homotopy invariant* if for every ring R the natural map $R \to R[t]$ induces a homotopy equivalence $F(R) \simeq F(R[t])$. Note that each homotopy group $\pi_n F(R)$ also forms a homotopy invariant functor.

Of course, this notion may be restricted to functors defined on any subcategory of rings which is closed under polynomial extensions and contains the evaluations as well as the inclusion $R \subset R[t]$. For example, we saw in II, 6.5 and 7.9.3 that $G_0(R)$ is a homotopy invariant functor defined on noetherian rings (and schemes) and maps of finite flat dimension.

Conversely, recall from III.3.4 that R is called *F-regular* if $F(R) \cong F(R[t_1, ..., t_n])$ for all n. Clearly, any functor F from rings to sets becomes homotopy invariant when restricted to the subcategory of F-regular rings. For example, we see from II.7.8 that K_0 becomes homotopy invariant when restricted to regular rings. The Fundamental Theorem in Chapter V, 6.3 implies that the functors K_n are also homotopy invariant when restricted to regular rings.

There is a canonical way to make F into a homotopy invariant functor.

IV.11.1 **Definition 11.1** (Strict homotopization)**.** Let F be a functor from rings to sets. Its *strict homotopization* $[F]$ is defined as the coequalizer of the evaluations at $t = 0, 1$: $F(R[t]) \rightrightarrows F(R)$. In fact, $[F]$ is a homotopy invariant functor and there is a universal transformation $F(R) \to [F](R)$; see Ex. 11.1. Moreover, if F takes values in groups then so does $[F]$; see Ex. 11.3.

IV.11.1.1 **Example 11.1.1.** Recall that a matrix is called *unipotent* if it has the form $1+\nu$ for some nilpotent matrix ν. Let $Unip(R)$ denote the subgroup of $GL(R)$ generated by the unipotent matrices. This is a normal subgroup of $GL(R)$, because the unipotent matrices are closed under conjugation. Since every elementary matrix $e_{ij}(r)$ is unipotent, this contains the commutator subgroup $E(R)$ of $GL(R)$.

We claim that $[E]R = [Unip]R = 1$ for every R. Indeed, if $1+\nu$ is unipotent, $(1+t\nu)$ is a matrix in $Unip(R[t])$ with $\partial_0(1+t\nu) = 1$ and $\partial_1(1+t\nu) = (1+\nu)$. Since $Unip(R)$ is generated by these elements, $[Unip]R$ must be trivial. The same argument applies to the elementary group $E(R)$.

We now consider $GL(R)$ and its quotient $K_1(R)$. A priori, $[GL]R \to [K_1]R$ is a surjection. In fact, it is an isomorphism.

IV.11.2 **Lemma 11.2.** *Both* $[GL]R$ *and* $[K_1]R$ *are isomorphic to* $GL(R)/Unip(R)$.

IV.11.2.1 **Definition 11.2.1.** For each ring R, we define $KV_1(R)$ to be $GL(R)/Unip(R)$. Thus $KV_1(R)$ is the strict homotopization of $K_1(R) = GL(R)/E(R)$.

Proof. The composite $Unip(R) \to GL(R) \to [GL]R$ is trivial, because it factors through $[Unip]R = 1$. Hence $[GL]R$ (and $[K_1]R$) are quotients of

$GL(R)/Unip(R)$. By Higman's trick III.3.5.1, if $g \in GL(R[t])$ is in the kernel of ∂_0 then $g \in Unip(R[t])$ and hence $\partial_1(g) \in Unip(R)$. Hence $\partial_1(NGL(R)) = Unip(R)$. Hence $GL(R)/Unip(R)$ is a strictly homotopy invariant functor; universality implies that the induced maps $[GL]R \to [K_1]R \to GL(R)/Unip(R)$ must be isomorphisms. □

To define the higher Karoubi-Villamayor groups, we introduce the simplicial ring $R[\Delta^\bullet]$, and use it to define the notion of homotopization. The simplicial ring $R[\Delta^\bullet]$ also plays a critical role in the construction of higher Chow groups and motivic cohomology, which is used in Chapter VI.

IV.11.3 **Definition 11.3.** For each ring R the coordinate rings of the standard simplices form a simplicial ring $R[\Delta^\bullet]$. It may described by the diagram

$$R \leftleftarrows R[t_1] \Lleftarrow R[t_1,t_2] \;\overset{\longleftarrow}{\underset{\longleftarrow}{\leftleftarrows}}\; \cdots R[t_1,\dots,t_n]\cdots$$

with $R[\Delta^n] = R[t_0,t_1,\cdots,t_n]/(\sum t_i = 1) \cong R[t_1,\cdots,t_n]$. The face maps ∂_i are given by: $\partial_i(t_i)=0$; $\partial_i(t_j)$ is t_j for $j<i$ and t_{j-1} for $j>i$. Degeneracies σ_i are given by: $\sigma_i(t_i)=t_i+t_{i+1}$; $\sigma_i(t_j)$ is t_j for $j<i$ and t_{j+1} for $j>i$.

IV.11.4 **Definition 11.4.** Applying the functor GL to $R[\Delta^\bullet]$ gives us a simplicial group $GL_\bullet = GL(R[\Delta^\bullet])$. For $n \ge 1$, we define the Karoubi-Villamayor groups to be $KV_n(R) = \pi_{n-1}(GL_\bullet) = \pi_n(BGL_\bullet)$.

Since $\pi_0(GL_\bullet)$ is the coequalizer of $GL(R[t]) \rightrightarrows GL(R)$, we see from Lemma 11.2 that Definitions 11.2.1 and 11.4 of $KV_1(R)$ agree: $KV_1(R) = GL(R)/Unip(R) \cong \pi_0(GL_\bullet)$.

The proof in Ex. 1.11 that $BGL(R)^+$ is an H-space also applies to $BGL(R[\Delta^\bullet])$ (Exercise 11.9). From the universal property in Theorem 1.8 we deduce the following elementary result.

IV.11.4.1 **Lemma 11.4.1.** *The map $BGL(R) \to BGL(R[\Delta^\bullet])$ factors through an H-map $BGL(R)^+ \to BGL(R[\Delta^\bullet])$. Thus there are canonical maps $K_n(R) \to KV_n(R)$, $n \ge 1$.*

IV.11.4.2 **Remark 11.4.2.** In fact, $BGL(R[\Delta^\bullet])^+$ is an infinite loop space; it is the 0^{th} space of the geometric realization $\mathbf{KV}(R)$ of the simplicial spectrum $\mathbf{K}(R[\Delta^\bullet])\langle 0\rangle$ of 4.11.2. (For any (-1)-connected simplicial spectrum $\mathbf{E}^\cdot$, the 0^{th} space of $|\mathbf{E}^\cdot|$ is the realization of the 0^{th} simplicial space.) Since $R[\Delta^0]=R$, there is a canonical morphism of spectra $\mathbf{K}(R) \to \mathbf{KV}(R)$. This shows that the map $BGL(R)^+ \to BGL(R[\Delta^\bullet])$ of 11.4.1 is in fact an infinite loop space map.

It is useful to put the definition of KV_* into a more general context:

IV.11.5 **Definition 11.5** (Homotopization)**.** Let F be a functor from rings to CW complexes. Its *homotopization* $F^h(R)$ is the geometric realization of the simplicial space $F(R[\Delta^\bullet])$. Thus F^h is also a functor from rings to CW complexes, and there is a canonical map $F(R) \to F^h(R)$.

IV.11.5.1 **Lemma 11.5.1.** *Let F be a functor from rings to CW complexes. Then:*

(1) F^h is a homotopy invariant functor;

(2) $\pi_0(F^h)$ is the strict homotopization $[F_0]$ of the functor $F_0(R) = \pi_0 F(R)$;

(3) If F is homotopy invariant then $F(R) \simeq F^h(R)$ for all R.

IV.11.5.2 **Corollary 11.5.2.** *The abelian groups $KV_n(R)$ are homotopy invariant:*

$$KV_n(R) \cong KV_n(R[x]) \quad \textit{for every } n \geq 1.$$

Proof of 11.5.1. We claim that the inclusion $R[\Delta^\bullet] \subset R[x][\Delta^\bullet]$ is a simplicial homotopy equivalence, split by evaluation at $x = 0$. For this, we define ring maps $h_i : R[x][\Delta^n] \to R[x][\Delta^{n+1}]$ by: $h_i(f) = \sigma_i(f)$ if $f \in R[\Delta^n]$ and $h_i(x) = x(t_{i+1} + \cdots + t_{n+1})$. These maps define a simplicial homotopy (see [223]) between the identity map of $R[x][\Delta^\bullet]$ and the composite

$$R[x][\Delta^\bullet] \xrightarrow{x=0} R[\Delta^\bullet] \subset R[x][\Delta^\bullet].$$

Applying F gives a simplicial homotopy equivalence between $F^h(R[\Delta^\bullet])$ and $F^h(R[x][\Delta^\bullet])$. Geometric realization converts this into a topological homotopy equivalence between $F^h(R)$ and $F^h(R[x])$.

Part (2) follows from the fact that, for any simplicial space $X_\bullet$, the group $\pi_0(|X_\bullet|)$ is the coequalizer of $\partial_0, \partial_1 : \pi_0(X_1) \rightrightarrows \pi_0(X_0)$. In this case $\pi_0(X_0) = \pi_0 F(R)$ and $\pi_0(X_1) = \pi_0 F(R[t])$.

Finally, if F is homotopy invariant then the map from the constant simplicial space $F(R)$ to $F(R[\Delta^\bullet])$ is a homotopy equivalence in each degree. It follows (see [214]) that their realizations $F(R)$ and $F^h(R)$ are homotopy equivalent. □

It is easy to see that $F \to F^h$ is universal (up to homotopy equivalence) for natural transformations from F to homotopy invariant functors. A proof of this fact is left to Ex. 11.2.

IV.11.6 **Example 11.6.** Suppose that $G(R)$ is a group-valued functor. Then $G^h(R)$ is the realization of the simplicial group $G(R[\Delta^\bullet])$. This shows that G^h may have higher homotopy groups even if G does not.

In fact, the groups $\pi_n(G_\bullet)$ of any simplicial group $G_\bullet$ may be calculated using the formula $\pi_p(G_\bullet) = H_n(N^*G_\bullet)$, where $N^*G_\bullet$ is the *Moore complex*; see [223, 11.3.6] [118, 17.3]. By definition, the Moore complex of a simplicial group $G_\bullet$ is the chain complex of groups with $N^0G_\bullet = G_0$, $N^1G_\bullet = \ker(\partial_0 : G_1 \to G_0)$ and $N^nG_\bullet = \cap_{i=0}^{n-1} \ker(\partial_i)$ for $n \geq 1$, with differential $(-1)^n\partial_n$. See Ex. 11.4.

In the case that $G_\bullet(R) = G(R[\Delta^\bullet])$, $N^1G_\bullet(R)$ is the group $NG(R)$ of III.3.3, and $N^nG_\bullet(R) \subset G(R[t_1, ..., t_n])$ is the n^{th} iterate of this functor.

A related situation arises when $F(R) = BG(R)$. Then $|G(R[\Delta^\bullet])|$ is the loop space of $F^h(R)$, which is a connected space with $\pi_{n+1}F^h(R) = H_n(N^*G(R[\Delta^\bullet]))$.

IV.11.6.1 **Example 11.6.1.** Suppose that $F(R) = |G_\bullet(R)|$ for some functor $G_\bullet$ from rings to simplicial groups. Then $F^h(R)$ is the geometric realization of a bisimplicial group $G_{pq} = G_q(R[\Delta^p])$. We can calculate the homotopy groups of any bisimplicial space $G_{\bullet\bullet}$ using the standard spectral sequence [151]

$$E^1_{pq} = \pi_q(G_{p\bullet}) \Rightarrow \pi_{p+q}|G_{\bullet\bullet}|.$$

As a special case, if $F(R) \simeq F(R[t_1, ..., t_n])$ for all n then $G_{p\bullet} \simeq G_\bullet(R)$ for all p, so the spectral sequence degenerates to yield $F(R) \simeq F^h(R)$.

IV.11.7 **Theorem 11.7.** *If $F(R)$ is any functorial model of $BGL(R)^+$ then we also have $KV_n(R) = \pi_n F^h(R)$ for all $n \geq 1$. Moreover, there is a first quadrant spectral sequence (for $p \geq 0, q \geq 1$):*

$$E^1_{pq} = K_q(R[\Delta^p]) \Rightarrow KV_{p+q}(R). \qquad (11.7.1)$$ IV.11.7.1

Proof. (Anderson) We may assume (by Ex. 11.2) that $F(R) = |G_\bullet(R)|$ for a functor $G_\bullet$ from rings to simplicial groups which is equipped with a natural transformation $BGL \to G_\bullet$ such that $BGL(R) \to |G_\bullet(R)|$ identifies $|G_\bullet(R)|$ with $BGL(R)^+$. Such functors exist; see 1.9. The spectral sequence of 11.6.1 becomes (11.7.1) once we show that $KV_n(R) = \pi_n G_\bullet(R[\Delta^\bullet])$. Thus it suffices to prove that $BGL^h(R) \simeq |G_\bullet(R)|^h$. Since $BGL^h(R)$ is an H-space (Ex. 11.9), the proof is standard, and relegated to Exercise 11.10. □

IV.11.8 **Theorem 11.8.** *If R is regular, then $K_p(R) \cong KV_p(R)$ for all $p \geq 1$.*

Proof. If R is regular, then each simplicial group $K_p(R[\Delta^\bullet])$ is constant (by the Fundamental Theorem in Chapter V, 6.3). Thus the spectral sequence (11.7.1) degenerates at E^2 to yield the result. □

We now quickly develop the key points in KV-theory.

IV.11.9 **Definition 11.9.** We say that a ring map $f : R \to S$ is a *GL-fibration* if

$$GL(R[t_1, ..., t_n]) \times GL(S) \to GL(S[t_1, ..., t_n])$$

is onto for every n. Note that we do not require R and S to have a unit.

IV.11.9.1 **Example 11.9.1.** If I is a nilpotent ideal in R, then $R \to R/I$ is a GL-fibration. This follows from Ex. I.1.12(iv), because each $I[t_1, ..., t_n]$ is also nilpotent.

IV.11.9.2 **Remark 11.9.2.** Any GL-fibration must be onto. That is, $S \cong R/I$ for some ideal I of R. To see this, consider the $(1, 2)$ entry α_{12} of a preimage of the elementary matrix $e_{12}(st)$. Since $f(\alpha_{12}) = st$, evaluation at $t = 1$ gives an element of R mapping to $s \in S$. However, not every surjection is a GL-fibration; see Ex. 11.6(d).

IV.11.9.3 **Example 11.9.3.** Any ring map $R \to S$ is homotopic to a GL-fibration. Indeed, the inclusion of R into the graded ring $R' = R \oplus xS[x] = R \times_S S[x]$ induces a homotopy equivalence $GL(R[\Delta^\bullet]) \simeq GL(R'[\Delta^\bullet])$ by Ex. 11.5, so that $KV_*(R) \cong KV_*(R')$. Moreover, the map $R' \to S$ sending x to 1 is a GL-fibration by Ex. 11.6(a,c).

The definition of $KV_n(I)$ makes sense if I is a ring without unit using the group $GL(I)$ of §III.2: $KV_n(I) = \pi_n BGL(I[\Delta^\bullet])$. Since $GL(R \oplus I)$ is the semidirect product of $GL(R)$ and $GL(I)$, we clearly have $KV_n(R \oplus I) \cong KV_n(R) \oplus KV_n(I)$. This generalizes as follows.

IV.11.10 **Theorem 11.10** (Excision)**.** *If $R \to R/I$ is a GL-fibration, there is a long exact sequence*

$$\begin{aligned} KV_{n+1}(R/I) \to KV_n(I) \to KV_n(R) \to KV_n(R/I) \to \cdots \\ \to KV_1(I) \to KV_1(R) \to KV_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I). \end{aligned}$$

Theorem 11.10 is called an "Excision Theorem" because it says that (whenever $R \to R/I$ is a GL-fibration) $KV_n(R, I) \cong KV_n(I)$ for all $n \geq 1$.

Proof. Let $G_n \subset GL(R/I[\Delta^n])$ denote the image of $GL(R[\Delta^n])$. Then there is an exact sequence of simplicial groups

$$1 \to GL(I[\Delta^\bullet]) \to GL(R[\Delta^\bullet]) \to G_\bullet \to 1. \qquad (11.10.1)$$ IV.11.10.1

Now any short exact sequence of simplicial groups is a fibration sequence, meaning there is a long exact sequence of homotopy groups. Moreover, the quotient $GL(R/I[\Delta^\bullet])/G_\bullet$ is a constant simplicial group, by Ex. 11.7. It is now a simple matter to splice the long exact sequences together to get the result. The splicing details are left to Ex. 11.7. □

IV.11.10.2 **Corollary 11.10.2.** *For any ring map $\phi : R \to S$, set $I' = R \times_S xS[x] = \{(r, x\,f(x)) \in R \times xS[x] : \phi(r) = f(1)\}$. Then there is a long exact sequence*

$$\cdots \to KV_{n+1}(S) \to KV_n(I') \to KV_n(R) \to KV_n(S) \to \cdots$$

ending in $KV_1(I') \to KV_1(R) \to KV_1(S) \to K_0(I')$.

Proof. Set $R' = R \oplus xS[x]$ and note that $R' \to S$ is a GL-fibration with kernel I' by 11.9.3. Since R' is graded, $KV_n(R) \cong KV_n(R')$ for all $n \geq 1$. The desired long exact sequence comes from Theorem 11.10. □

IV.11.10.3 **Remark 11.10.3.** When $R \to R/I$ is a GL-fibration, then $KV_*(I) \cong KV_*(I')$, and the long exact sequences of 11.10 and 11.10.2 coincide (with $S = R/I$). This follows from the 5-lemma, since the map ϕ factors through $R' \to R/I$ yielding a morphism of long exact sequences.

Theorem 11.10 fails if $R \to R/I$ is not a GL-fibration. Not only does the extension of Theorem 11.10 to K_0 fail (as the examples $\mathbb{Z} \to \mathbb{Z}/8$ and $\mathbb{Z} \to \mathbb{Z}/25$ show), but we need not even have $KV_*(I) \cong KV_*(I')$, as Exercise 11.14 shows.

IV.11.11 **Corollary 11.11.** *If I is a nilpotent ideal in a ring R, then $KV_n(I) = 0$ and $KV_n(R) \cong KV_n(R/I)$ for all $n \geq 1$.*

Proof. By 11.9.1, 11.10 and Lemma II.2.2, it suffices to show that $KV_n(I) = \pi_n GL(I[\Delta^\bullet]) = 0$. (A stronger result, that $GL(I[\Delta^\bullet])$ is simplicially contractible, is relegated to Ex. 11.11.) By Exercise I.1.12(iii), $GL_m(I[\Delta^n])$ consists of the matrices $1+x$ in $M_m(I[\Delta^n])$, so if $T = (t_0 t_1 \cdots t_{n-1})$ then the degree $n+1$ part of the Moore complex (11.6) consists of matrices $1+xT$, and $\partial_n(1+xT) = 1$ exactly when $x = t_n y$ for some matrix y. Regarding y as a matrix over $I[t_0, ..., t_{n-1}]$, the element $1 + yTt_n$ in $GL(I[\Delta^{n+1}])$ belongs to the Moore complex and ∂_{n+1} maps $1 + yTt_n$ to $1 + xT$. □

IV.11.12 **Theorem 11.12.** *[Mayer-Vietoris] Let $\varphi : R \to S$ be a map of rings, sending an ideal I of R isomorphically onto an ideal of S. If $S \to S/I$ is a GL-fibration, then $R \to R/I$ is also a GL-fibration, and there is a long exact Mayer-Vietoris sequence*

$$\cdots \to KV_{n+1}(S/I) \to KV_n(R) \to KV_n(R/I) \oplus KV_n(S) \to KV_n(S/I) \to \cdots$$
$$\to KV_1(R/I) \oplus KV_1(S) \to KV_1(S/I) \to K_0(R) \to K_0(R/I) \oplus K_0(S).$$

It is compatible with the Mayer-Vietoris sequence for K_1 and K_0 in III.2.6.

Proof. To see that $R \to R/I$ is a GL-fibration, fix $\bar{g} \in GL(R/I[t_1, ..., t_n])$ with $\bar{g}(0) = I$. Since $S \to S/I$ is a GL-fibration, there is a $g' \in GL(S[t_1, ..., t_n])$ which is $\varphi(\bar{g})$ modulo I. Since R is the pullback of S and R/I, there is a g in $GL(R[t_1, ..., t_n])$ mapping to g' and $\bar{g}$. Hence $R \to R/I$ is a GL-fibration.

As in the proof of Theorem III.5.8, there is a morphism from the (exact) chain complex of 11.10 for (R, I) to the corresponding chain complex for (S, I). Since every third term of this morphism is an isomorphism, the required Mayer-Vietoris sequence follows from a diagram chase. □

Here is an application of this result. Since $R[x] \to R$ has a section, it is a GL-fibration. By homotopy invariance, it follows that $KV_n(xR[x]) = 0$ for all $n \geq 1$. (Another proof is given in Ex. 11.5.)

IV.11.13 **Definition 11.13.** For any ring R (with or without unit), define ΩR to be the ideal $(x^2 - x)R[x]$ of $R[x]$. Iterating yields $\Omega^2 R = (x_1^2 - x_1)(x_2^2 - x_2)R[x_1, x_2]$, etc.

The following corollary of 11.10 shows that, for $n \geq 2$, we can also define $KV_n(R)$ as $KV_1(\Omega^{n-1}R)$, and hence in terms of K_0 of the rings $\Omega^n R$ and $\Omega^n R[x]$.

IV.11.13.1 **Corollary 11.13.1.** *For all R, $KV_1(R)$ is isomorphic to the kernel of the map $K_0(\Omega R) \to K_0(xR[x])$, and $KV_n(R) \cong KV_{n-1}(\Omega R)$ for all $n \geq 2$.*

Proof. The map $xR[x] \xrightarrow{x=1} R$ is a GL-fibration by Ex. 11.6(c), with kernel ΩR. The result now follows from Theorem 11.10. □

We conclude with an axiomatic treatment, due to Karoubi and Villamayor.

IV.11.14 **Definition 11.14.** A *positive homotopy K-theory* (for rings) consists of a sequence of functors K_n^h, $n \geq 1$, on the category of rings without unit, together with natural connecting maps $\delta_n : K_{n+1}^h(R/I) \to K_n^h(I)$ and $\delta_0 : K_1^h(R/I) \to K_0(I)$, defined for every GL-fibration $R \to R/I$, satisfying the following axioms:

(1) The functors K_n^h are homotopy invariant;

(2) For every GL-fibration $R \to R/I$ the resulting sequence is exact:

$$\to K_{n+1}^h(R/I) \xrightarrow{\delta} K_n^h(I) \to K_n^h(R) \to K_n^h(R/I) \xrightarrow{\delta} K_{n-1}^h(I) \to \cdots$$
$$\to K_1^h(R) \to K_1^h(R/I) \xrightarrow{\delta} K_0(I) \to K_0(R) \to K_0(R/I).$$

IV.11.14.1 **Theorem 11.14.1.** *Up to isomorphism, there is a unique positive homotopy K-theory, namely $K_n^h = KV_n$.*

Proof. The fact that KV_n form a positive homotopy K-theory is given by 11.4, 11.5.2 and 11.10. The axioms imply that any other positive homotopy K-theory must satisfy the conclusion of 11.13.1, and so must be isomorphic to KV-theory. □

EXERCISES

EIV.11.1 **11.1.** Let F be a functor from rings to sets. Show that $[F]$ is a homotopy invariant functor, and that every natural transformation $F(R) \to H(R)$ to a homotopy invariant functor H factors uniquely through $F(R) \to [F](R)$.

EIV.11.2 **11.2.** Let F and H be functors from rings to CW complexes, and assume that H is homotopy invariant. Show that any natural transformation $F(R) \to H(R)$ factors through maps $F^h(R) \to H(R)$ such that for each ring map $R \to S$ the map $F^h(R) \to F^h(S) \to H(S)$ is homotopy equivalent to $F^h(R) \to H(R) \to H(S)$.

EIV.11.3 **11.3.** If G is a functor from rings to groups, let $NG(R)$ denote the kernel of the map $t = 0 : G(R[t]) \to G(R)$. Show that the image $G_0(R)$ of the induced map $t = 1 : NG(R) \to G(R)$ is a normal subgroup of $G(R)$, and that $[G]R = G(R)/G_0(R)$. Thus $[G]R$ is a group.

EIV.11.4 **11.4.** (Moore) If $G_\bullet$ is a simplicial group and $N^nG_\bullet = \cap_{i=0}^{n-1} \ker(\partial_i)$ as in 11.6, show that $\partial_{n+1}(N^{n+1}G_\bullet)$ is a normal subgroup of G_n. Conclude that $\pi_n(G_\bullet)$ is also a group. *Hint:* conjugate elements of $N^{n+1}G$ by elements of s_nG_n.

EIV.11.5 **11.5.** Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Show that for every homotopy invariant functor F on rings we have $F(R_0) \simeq F(R)$. In particular, if F is defined on rings without unit then $F(xR[x]) \simeq F(0)$ for every R. *Hint:* Copy the proof of III.3.4.1.

EIV.11.6 **11.6.** *GL-fibrations.* Let $f : R \to S$ be a GL-fibration with kernel I. Show that:

(a) If f factors as $R \to R' \to S$, then $R' \to S$ is a GL-fibration.

(b) Both $xR[x] \to xS[x]$ and $\Omega R \to \Omega S$ are GL-fibrations.

(c) The map $xR[x] \to R$, $f(x) \mapsto f(1)$, is a GL-fibration with kernel ΩR.

(d) $\mathbb{Z} \to \mathbb{Z}/4$ is not a GL-fibration, but $GL(\mathbb{Z}) \to GL(\mathbb{Z}/4)$ is onto.

(e) If S is a regular ring (with unit), then every surjection $R \to S$ is a GL-fibration. *Hint:* $K_1(S) \cong K_1(S[x])$ by III.3.8.

EIV.11.7 **11.7.** Let $f : R \to S$ be a GL-fibration with kernel I, and define $G_\bullet$ as in the proof of Theorem 11.10. Show that $GL(S[\Delta^\bullet])/G_\bullet$ is a constant simplicial group. Use this to show that $\pi_i(G_\bullet) = KV_{i+1}(S)$ for all $i > 0$, but that the cokernel of $\pi_0(G_\bullet) \to \pi_0 GL(S[\Delta^\bullet])$ is the image of $K_1(S)$ in $K_0(I)$ under the map of III.2.3. Combining this with the long exact sequence of homotopy groups for (11.10.1), finish the proof of 11.10.

EIV.11.8 **11.8.** Consider the unit functor U on rings. The identity $U(M_m(R)) = GL_m(R)$ implies that $U^h(M(R)) = KV(R)$. If R is commutative, use I.3.12 to show that $U^h(R) \simeq U(R_{\text{red}})$.

EIV.11.9 **11.9.** Show that $BGL(R[\Delta^\bullet])$ is an H-space. This fact is used to prove Lemma 11.4.1. *Hint:* See Ex. 1.11; the permutation matrices lie in $E(R)$, and $E(R[\Delta^\bullet])$ is path connected.

EIV.11.10 **11.10.** (Anderson) Use Exercise IV.11.9 to complete the proof of Theorem 11.7.

EIV.11.11 **11.11.** If I is nilpotent, show that the simplicial sets $GL_m(I[\Delta^\bullet])$ and $GL(I[\Delta^\bullet])$ have a simplicial contraction [223, 8.4.6]. *Hint:* multiply by t_n.

EIV.11.12 **11.12.** If R is K_i-regular for all $i \le n$, show that $K_i(R) \cong KV_i(R)$ for all $i \le n$, and that $KV_{n+1}(R) \cong [K_n]R$.

EIV.11.13 **11.13.** (Strooker) Consider the ring $R = \mathbb{Z}[x]/(x^2-4)$. In this exercise, we use two different methods to show that the map $K_2(R) \to KV_2(R)$ is not onto, and that its cokernel is $\mathbb{Z}/2$. Note that $KV_1(R) = K_1(R) = R^\times = \{\pm 1\}$ by Ex. III.5.13.

(a) Compare the Mayer-Vietoris sequences III.5.8 and 11.12 to show that the natural map $K_2(R) \to K_2(\mathbb{Z})^2$ has cokernel $\mathbb{Z}/2$, yet $KV_2(R) \cong K_2(\mathbb{Z})^2 = (\mathbb{Z}/2)^2$.

(b) Use III.5.8 and Ex. III.5.15 to compute $K_1(R[t])$ and $K_1(R[t_1,t_2])$. Then show that the sequence $N^2K_1(R) \to NK_1(R) \to K_1(R)$ is not exact. Use the spectral sequence (11.7.1) to conclude that the map $K_2(R) \to KV_2(R)$ is not onto, and that its cokernel is $\mathbb{Z}/2$.

EIV.11.14 **11.14.** Let k be a field of characteristic 0, and set $S = k[x,(x+1)^{-1}]$

(a) Show that $K_1(\mathbb{Z}\oplus I, I) \cong K_1(S,I)$, where $I = x^2S$. *Hint:* Use the obstruction described in III, Ex. 2.6, showing that $3\psi(a\,dx\otimes x^2) = 0$.

(b) Use III, Ex. 5.14(c) to show that $K_1(S,I)$ is the cokernel of $k^\times \xrightarrow{\text{dlog}} \Omega_k$, $\text{dlog}(a) = da/a$.

(c) Show that $KV_1(I) = 0$, and conclude that the sequence $KV_1(I) \to KV_1(S) \to KV_1(S/I)$ is not exact.

12 Homotopy K-theory

In order to define a truly homotopy invariant version of algebraic K-theory, we need to include K_0 and even the negative K-groups. This is most elegantly done at the level of spectra, and that approach begins by constructing the non-connective "Bass K-theory spectrum" $\mathbf{K}^B(R)$ out of any one of the functorial models of a connective K-theory spectrum $\mathbf{K}(R)$. (See 1.9(iii), 4.5.2, 6.5.1 and 8.5.5.)

Let R be an associative ring with unit. By 11.3 there is a simplicial ring $R[\Delta^\bullet]$ and hence a simplicial spectrum $\mathbf{K}^B(R[\Delta^\bullet])$.

IV.12.1 **Definition 12.1.** Let $KH(R)$ denote the (fibrant) geometric realization of the simplicial spectrum $\mathbf{K}^B(R[\Delta^\bullet])$. For $n\in\mathbb{Z}$, we write $KH_n(R)$ for $\pi_n KH(R)$.

It is clear from the definition that $KH(R)$ commutes with filtered colimits of rings, and that there are natural transformations $K_n(R)\to KH_n(R)$ which factor through $KV_n(R)$ when $n\ge 1$. Indeed, the spectrum map $\mathbf{K}(R)\langle 0\rangle \to \mathbf{K}^B(R) \to KH(R)$ factors through the spectrum $\mathbf{KV}(R) = \mathbf{K}(R[\Delta^\bullet])\langle 0\rangle$ of 11.4.2.

IV.12.2 **Theorem 12.2.** *Let R be a ring. Then:*

(1) $KH(R)\simeq KH(R[x])$, i.e., $KH_n(R)\cong KH_n(R[x])$ for all n.

(2) $KH(R[x,x^{-1}])\simeq KH(R)\times\Omega^{-1}KH(R)$, i.e.,

$$KH_n(R[x,x^{-1}])\cong KH_n(R)\oplus KH_{n-1}(R)\quad\textit{for all } n.$$

(3) If $R = R_0\oplus R_1\oplus\cdots$ is a graded ring then $KH(R)\simeq KH(R_0)$.

Proof. Part (1) is a special case of 11.5.1. Part (2) follows from the Fundamental Theorem 10.2 and (1). Part (3) follows from (1) and Ex. 11.5. □

The homotopy groups of a simplicial spectrum are often calculated by means of a standard right half-plane spectral sequence. In the case at hand, *i.e.,* for $KH(R)$, the edge maps are the canonical maps $K_q(R)\to KH_q(R)$, induced by $\mathbf{K}^B(R)\to KH(R)$, and the spectral sequence specializes to yield:

IV.12.3 **Theorem 12.3.** *For each ring R there is an exhaustive convergent right half-plane spectral sequence:*

$$E^1_{p,q} = N^pK_q(R) \Rightarrow KH_{p+q}(R).$$

The edge map from $E^1_{0,q} = K_q(R)$ to $KH_q(R)$ identifies $E^2_{p,0}$ with the strict homotopization $[K_p](R)$ of $K_p(R)$, defined in 11.1.

The phrase "exhaustive convergent" in 12.3 means that for each n there is a filtration $0 \subseteq F_0KH_n(R) \subseteq \cdots \subseteq F_{p-1}KH_n(R) \subseteq F_pKH_n(R) \subseteq \cdots$ with union $KH_n(R)$, zero for $p < 0$, and isomorphisms $E^\infty_{p,q} \cong F_pKH_n(R)/F_{p-1}KH_n(R)$ for $q = n - p$. (A discussion of convergence may be found in [223, 5.2.11].)

As pointed out in 11.8, we will see in chapter V, 6.3 that regular rings are K_q-regular for all q, *i.e.,* that $N^pK_q(R) = 0$ for every q and every $p > 0$. For such rings, the spectral sequence 12.3 degenerates at E^1, showing that the edge maps are isomorphisms. We record this as follows:

IV.12.3.1 **Corollary 12.3.1.** *If R is regular noetherian, then $\mathbf{K}(R) \simeq KH(R)$. In particular, $K_n(R) \simeq KH_n(R)$ for all n.*

For the next application, we use the fact that if R is K_i-regular for some i, then it is K_q-regular for all $q \le i$. If $i \le 0$, this was proven in III, 4.2.3. For $i = 1$ it was shown in III, Ex. 3.9. In the remaining case $i > 1$, the result will be proven in Chapter V, Theorem 8.6.

IV.12.3.2 **Corollary 12.3.2.** *Suppose that the ring R is K_i-regular for some fixed i. Then $KH_n(R) \cong K_n(R)$ for all $n \le i$, and $KH_{i+1}(R) = [K_{i+1}]R$.*

If R is K_0-regular then $KV_n(R) \cong KH_n(R)$ for all $n \ge 1$, and $KH_n(R) \cong K_n(R)$ for all $n \le 0$. In this case the spectral sequences of (11.7.1) and 12.3 coincide.

Proof. In this case, the spectral sequence degenerates below the line $q = i$, yielding the first assertion. If R is K_0-regular, the morphism $\mathbf{KV}(R) \to KH(R)$ induces a morphism of spectral sequences, from (11.7.1) to 2.3, which is an isomorphism on $E^1_{p,q}$ (except when $p = 0$ and $q \le 0$). The comparison theorem yields the desired isomorphism $\mathbf{KV}(R) \to KH(R)\langle 0\rangle$. □

IV.12.3.3 **Theorem 12.3.3.** *If $1/\ell \in R$ then $KH_n(R;\mathbb{Z}/\ell) \cong K_n(R;\mathbb{Z}/\ell)$ for all n.*

Proof. The proof of 12.3 goes through with finite coefficients to yield a spectral sequence with $E^1_{p,q} = N^pK_q(R;\mathbb{Z}/\ell) \Rightarrow KH_{p+q}(R;\mathbb{Z}/\ell)$. When $1/\ell \in R$ and $p > 0$, the groups $N^pK_q(R)$ are $\mathbb{Z}[1/\ell]$-modules (uniquely ℓ-divisible groups) by 6.7.2. By the Universal Coefficient Theorem (2.5) we have $N^pK_q(R;\mathbb{Z}/\ell) = 0$, so the spectral sequence degenerates to yield the result. □

If I is a non-unital ring, we define $KH(I)$ to be $KH(\mathbb{Z}\oplus I)/KH(\mathbb{Z})$ and set $KH_n(I) = \pi_nKH(I)$. If I is an ideal in a ring R, recall (from 1.11 or Ex. 10.1) that $\mathbf{K}^B(R, I)$ denotes the homotopy fiber of $\mathbf{K}^B(R) \to \mathbf{K}^B(R/I)$; it depends upon R. The following result, which shows that the KH-analogue does not depend upon R, is one of the most important properties of KH-theory.

IV.12.4 **Theorem 12.4** (Excision)**.** *Let I be an ideal in a ring R. Then the sequence $KH(I) \to KH(R) \to KH(R/I)$ is a homotopy fibration. Thus there is a long exact sequence*

$$\cdots \to KH_{n+1}(R/I) \to KH_n(I) \to KH_n(R) \to KH_n(R/I) \to \cdots$$

Proof. Let $KH(R,I)$ denote the homotopy fiber of $KH(R) \to KH(R/I)$. By standard simplicial homotopy theory, $KH(R,I)$ is homotopy equivalent to $|\mathbf{K}^B(R[\Delta^\bullet], I[\Delta^\bullet])|$. It suffices to prove that $KH(I) \to KH(R,I)$ is a homotopy equivalence.

We first claim that $KH_n(I) \to KH_n(R,I)$ is an isomorphism for $n \leq 0$. For each $p \geq 0$, let $\mathbf{K}^B(R[\Delta^p], I[\Delta^p])\langle 0\rangle$ be the 0-connected cover of $\mathbf{K}^B(R[\Delta^p], I[\Delta^p])$, and define $\mathbf{K}^{\leq 0}(R[\Delta^p], I[\Delta^p])$ by the termwise "Postnikov" homotopy fibration:

$$\mathbf{K}^B(R[\Delta^p], I[\Delta^p])\langle 0\rangle \to \mathbf{K}^B(R[\Delta^p], I[\Delta^p]) \to \mathbf{K}^{\leq 0}(R[\Delta^p], I[\Delta^p]).$$

Let C_R denote the geometric realization of $\mathbf{K}^{\leq 0}(R[\Delta^\bullet], I[\Delta^\bullet])$. Comparing the standard spectral sequence for C_R and the spectral sequence of Theorem 12.3, we see that $KH_n(R,I) \cong \pi_n(C_R)$ for all $n \leq 0$. By Exercise 10.1, the ring map $A = \mathbb{Z} \oplus I \to R$ induces $\pi_n\mathbf{K}^{\leq 0}(A[\Delta^p], I[\Delta^p]) \cong \pi_n\mathbf{K}^{\leq 0}(R[\Delta^p], I[\Delta^p])$ for all n and p. Hence we have homotopy equivalences for each p and hence a homotopy equivalence on realizations, $C_A \simeq C_R$. The claim follows.

For $n > 0$, we consider the homotopy fiber sequence

$$\mathbf{KV}(R,I) \to KH(R,I) \to C_R,$$

where $\mathbf{KV}(R,I)$ is the geometric realization of $\mathbf{K}^B(R[\Delta^\bullet], I[\Delta^\bullet])\langle 0\rangle$. Comparing with the spectrum $\mathbf{KV}(R)$ defined in 11.4.2, we see that $\mathbf{KV}(R,I)$ is the 0-connected cover of the homotopy fiber of $\mathbf{KV}(R) \to \mathbf{KV}(R/I)$.

The theorem now follows when $R \to R/I$ is a GL-fibration, since in this case $KV_n(I) \cong KV_n(R,I)$ for all $n \geq 1$ by 11.10. Combining this with the above paragraph, the 5-lemma shows that in this case $KH(I) \simeq KH(R,I)$, as required.

An important GL-fibration is given by the non-unital map $xR[x] \to R$ (or the unital $\mathbb{Z} \oplus xR[x] \to \mathbb{Z} \oplus R$) with kernel ΩR; see Ex. 11.6(c). In the following diagram, the bottom two rows are homotopy fibration sequences by the previous paragraph, and the terms in the top row are defined so that the columns are homotopy fibrations:

$$\begin{array}{ccccc}
KH(\mathbb{Z} \oplus \Omega R, \Omega I) & \to & KH(\mathbb{Z} \oplus xR[x], xI[x]) & \to & KH(R,I) \\
\downarrow & & \downarrow & & \downarrow \\
KH(\Omega R) & \longrightarrow & KH(xR[x]) & \longrightarrow & KH(R) \\
\downarrow & & \downarrow & & \downarrow \\
KH(\Omega R/I) & \longrightarrow & KH(xR/I[x]) & \longrightarrow & KH(R/I).
\end{array}$$

Since $KH(xR[x])$ is contractible (by 12.2), the top middle term is contractible, and we have a natural homotopy equivalence $\Omega KH(R) \simeq KH(\Omega R)$. Since the top row must also be a homotopy fibration, we also obtain a natural homotopy equivalence $\Omega KH(R,I) \to KH(\mathbb{Z} \oplus \Omega R, \Omega I)$. Applying π_n yields isomorphisms $KH_{n+1}(R,I) \cong KH_n(\mathbb{Z} \oplus \Omega R, \Omega I)$ for all n.

Now suppose by induction on $n \geq 0$ that, for all rings R' and ideals I', the canonical map $I' \to R'$ induces an isomorphism $KH_n(I') \cong KH_n(R', I')$. In particular, $\Omega I \to \Omega R$ induces $KH_n(\Omega I) \cong KH_n(\mathbb{Z} \oplus \Omega R, \Omega I)$. It follows that the map from $A = \mathbb{Z} \oplus I$ to R induces a commutative diagram of isomorphisms:

$$\begin{array}{ccccc} KH_{n+1}(I) & = & KH_{n+1}(A,I) & \xrightarrow{\simeq} & KH_n(\mathbb{Z} \oplus \Omega A, \Omega I) \\ & & \downarrow & & \downarrow \simeq \\ & & KH_{n+1}(R,I) & \xrightarrow{\simeq} & KH_n(\mathbb{Z} \oplus \Omega R, \Omega I). \end{array}$$

This establishes the inductive step. We have proven that for all R and I, $KH_n(I) \cong KH_n(R,I)$ for all n, and hence $KH(I) \simeq KH(R,I)$, as required. □

IV.12.5 **Corollary 12.5.** *If I is a nilpotent ideal in a ring R, then the spectrum $KH(I)$ is contractible and $KH(R) \simeq KH(R/I)$. In particular, $KH_n(I) = 0$ and $KH_n(R) \cong KH_n(R/I)$ for all integers n.*

Proof. By Ex. II.2.5, I is K_0-regular, and $K_n(I) = 0$ for $n \leq 0$ (see III, Ex. 4.3). By 12.3.2 and 11.11, we have $KH_n(I) \cong KV_n(I) = 0$ for $n > 0$, and $KH_n(I) = 0$ for $n \leq 0$. Since $KH_n(I) = 0$ for all n, $KH(I)$ is contractible. The remaining assertions now follow from Excision 12.4. □

IV.12.5.1 **Example 12.5.1.** Let R be a commutative Artinian ring, with associated reduced ring $R_{\text{red}} = R/\text{nilradical}(R)$. As R_{red} is regular, we see from 12.5 and 12.3.1 that $KH_n(R) \cong K_n(R_{\text{red}})$ for all n. In particular, $KH_0(R) = K_0(R) = H^0(R)$ and $KH_n(R) = 0$ for $n < 0$.

IV.12.5.2 **Example 12.5.2.** Let R be a 1-dimensional commutative noetherian ring. Then $KH_0(R) \cong H^0(R) \oplus [\text{Pic}]R$, $KH_{-1}(R)$ is torsionfree, and $KH_n(R) = 0$ for all $n \leq -2$. This follows from 12.3.2 and Exercise III.4.4, which states that R is K_{-1}-regular and computes $K_n(R)$ for $n \leq 0$. An example in which $KV_1(R) \to KH_1(R)$ is not onto is given in Ex. 12.2.

If in addition R is seminormal, then R is Pic-regular by Traverso's Theorem I.3.12. In this case we also have $KH_0(R) = K_0(R) = H^0(R) \oplus \text{Pic}(R)$ and $KV_1(R) \cong KH_1(R)$.

IV.12.6 **Corollary 12.6** (Closed Mayer-Vietoris)**.** *Let $R \to S$ be a map of commutative rings, sending an ideal I of R isomorphically onto an ideal of S. Then there is a long exact Mayer-Vietoris sequence (for all integers n):*

$$\cdots \to KH_{n+1}(S/I) \to KH_n(R) \to KH_n(R/I) \oplus KH_n(S) \to KH_n(S/I) \to \cdots$$

Recall (10.4) that $\mathbf{K}^B(R)$ is the homotopy colimit of a diagram of spectra $L^q\mathbf{K}(R)$. Since geometric realization commutes with homotopy colimits, at least up to weak equivalence, we have $KH(R) = \operatorname{colim}_q |L^q\mathbf{K}(R[\Delta^\bullet])|$.

IV.12.7 **Definition 12.7.** Let X be a scheme. Using the functorial nonconnective spectrum $\mathbf{K}^B$ of 10.6, let $KH(X)$ denote the (fibrant) geometric realization of the simplicial spectrum $\mathbf{K}^B(X \times \Delta^\bullet)$, where $\Delta^\bullet = \operatorname{Spec}(R[\Delta^\bullet])$ as in 11.3. For $n \in \mathbb{Z}$, we write $KH_n(X)$ for $\pi_n KH(X)$.

IV.12.8 **Lemma 12.8.** *For any quasi-projective scheme X we have:*

(1) $KH(X) \simeq KH(X \times \mathbb{A}^1)$.

(2) $KH(X \times \operatorname{Spec}(\mathbb{Z}[x, x^{-1}]) \simeq KH(X) \times \Omega^{-1}KH(R)$*, i.e.,*

$$KH_n(X[x, x^{-1}]) \cong KH_n(X) \oplus KH_{n-1}(X) \quad \textit{for all } n.$$

(3) If X is regular noetherian, then $\mathbf{K}(X) \simeq KH(X)$. In particular, $K_n(X) \simeq KH_n(X)$ for all n.

Proof. The proof of 11.5.1 goes through to show (1). From the Fundamental Theorem (see 10.6), we get (2). We will see in V.6.13.2 that if X is a regular noetherian scheme then $K(X) \simeq K(X \times \mathbb{A}^1)$ and hence $\mathbf{K}^B(X) \simeq \mathbf{K}^B(X \times \mathbb{A}^1)$. It follows that $KH(X) = \mathbf{K}^B(X \times \Delta^\bullet)$ is homotopy equivalent to the constant simplicial spectrum $\mathbf{K}^B(X)$. □

EXERCISES

EIV.12.1 **12.1.** *Dimension shifting.* Fix a ring R, and let $\Delta^d(R)$ denote the coordinate ring $R[t_0, \dots, t_d]/(f)$, $f = t_0 \cdots t_d(1 - \sum t_i)$ of the d-dimensional tetrahehron over R. Show that for all n, $KH_n(\Delta^d(R)) \cong KH_n(R) \oplus KH_{d+n}(R)$. If R is regular, conclude that $KH_n(\Delta^d(R)) \cong K_n(R) \oplus K_{d+n}(R)$, and that $K_0(\Delta^d(R)) \cong K_0(R) \oplus K_d(R)$. *Hint:* Use the Mayer-Vietoris squares of III.4.3.1, where we saw that $K_j(\Delta^n(R)) \cong K_{j+1}(\Delta^{n-1}(R))$ for $j < 0$. In III, Ex. 4.8 we saw that each $\Delta^n(R)$ is K_0-regular if R is.

EIV.12.2 **12.2.** (KV_1 need not map onto KH_1.) Let k be a field of characteristic 0, I the ideal of $S = k[x, (x+1)^{-1}]$ generated by x^2, and $R = k \oplus I$. Show that $KH_n(R) \cong KH_n(S)$ for all n, but that there is an exact sequence $0 \to KV_1(R) \to KH_1(R) \to \mathbb{Z} \to 0$. *Hint:* Use the Mayer-Vietoris sequence for $R \to S$ and apply I.3.12 to show that $K_0(R) = \mathbb{Z} \oplus k/\mathbb{Z}$, $NK_0(R) \cong tk[t]$ and $N^2K_0(R) \cong t_1t_2k[t_1, t_2]$. Alternatively, note that $KV_1(I) = 0$ by Ex. 11.14.

EIV.12.3 **12.3.** The seminormalization R^+ of a reduced commutative ring R was defined in I, Ex. 3.15. Show that $KH_n(R) \cong KH_n(R^+)$ for all n. *Hint:* show that KH is invariant under subintegral extensions.

Chapter V

The Fundamental Theorems of higher K-theory

We now restrict our attention to exact categories and Waldhausen categories, where the extra structure enables us to use the following types of comparison theorems: Additivity (1.2), Cofinality (2.3), Approximation (2.4), Resolution (3.1), Devissage (4.1), and Localization (2.1, 2.5, 5.1 and 7.1). These are the extensions to higher K-theory of the corresponding theorems of chapter II. The highlight of this chapter is the so-called "Fundamental Theorem" of K-theory (6.3 and 8.2), comparing $K(R)$ to $K(R[t])$ and $K(R[t,t^{-1}])$, and its analogue (6.13.2 and 8.3) for schemes.

1 The Additivity theorem

If $F' \to F \to F''$ is a sequence of exact functors $F', F, F'' : \mathcal{B} \to \mathcal{C}$ between two exact categories (or Waldhausen categories), the Additivity Theorem tells us when the induced maps $K(\mathcal{B}) \to K(\mathcal{C})$ satisfy $F_* = F'_* + F''_*$. To state it, we need to introduce the notion of a short exact sequence of functors, which was mentioned briefly in II(9.1.8).

V.1.1 **Definition 1.1.** (a) If $\mathcal{B}$ and $\mathcal{C}$ are exact categories, we say that a sequence $F' \to F \to F''$ of exact functors and natural transformations from $\mathcal{B}$ to $\mathcal{C}$ is a *short exact sequence* of exact functors, and write $F' \rightarrowtail F \twoheadrightarrow F''$, if

$$0 \to F'(B) \to F(B) \to F''(B) \to 0$$

is an exact sequence in $\mathcal{C}$ for every $B \in \mathcal{B}$.
(b) If $\mathcal{B}$ and $\mathcal{C}$ are Waldhausen categories, we say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence*, or a *cofibration sequence* of exact functors if each $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofibration sequence and if for every cofibration $A \rightarrowtail B$ in $\mathcal{B}$, the evident map $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathcal{C}$.

When exact categories are regarded as Waldhausen categories, these two notions of "short exact sequence" of exact functors between exact categories are easily seen to be the same.

V.1.1.1 **Universal Example 1.1.1.** Recall from chapter II, 9.3, that the extension category $\mathcal{E} = \mathcal{E}(\mathcal{C})$ is the category of all exact sequences $E : A \rightarrowtail B \twoheadrightarrow B/A$ in $\mathcal{C}$. If $\mathcal{C}$ is an exact category, or a Waldhausen category, so is $\mathcal{E}$. The source $s(E) = A$, target $t(E) = B$, and quotient $q(E) = C$ of such a sequence are exact functors from $\mathcal{E}$ to $\mathcal{C}$, and $s \rightarrowtail t \twoheadrightarrow q$ is a short exact sequence of functors. This example is universal in the sense that giving an exact sequence of exact functors from $\mathcal{B}$ to $\mathcal{C}$ is the same thing as giving an exact functor $\mathcal{B} \to \mathcal{E}(\mathcal{C})$.

V.1.2 **Additivity Theorem 1.2.** *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors from $\mathcal{B}$ to $\mathcal{C}$, either between exact categories or between Waldhausen categories. Then $F_* \simeq F'_* + F''_*$ as H-space maps $K(\mathcal{B}) \to K(\mathcal{C})$. Therefore on the homotopy groups we have $F_* = F'_* + F''_* : K_i(\mathcal{B}) \to K_i(\mathcal{C})$.*

Proof. By universality of $\mathcal{E}$, we may assume that $\mathcal{B}$ is $\mathcal{E}$ and prove that $t_* = s_* + q_*$. The map $s_* + q_*$ is induced by the exact functor $s \coprod q : \mathcal{E} \to \mathcal{C}$ which sends $A \rightarrowtail B \twoheadrightarrow C$ to $A \amalg C$, because the H-space structure on $K(\mathcal{C})$ is induced from $\amalg$ (see IV, 6.4 and 8.5.1). The compositions of t and $s \coprod q$ with the coproduct functor $\amalg : \mathcal{C} \times \mathcal{C} \to \mathcal{E}$ agree:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\amalg} \mathcal{E} \underset{s \vee q}{\overset{t}{\rightrightarrows}} \mathcal{C}$$

and hence give the same map on K-theory. The Extension Theorem 1.3 below proves that $K(\amalg)$ is a homotopy equivalence, from which we conclude that $t \simeq s \coprod q$, as desired. □

Given an exact category (or a Waldhausen category) $\mathcal{A}$, we say that a sequence $0 \to A_n \to \cdots \to A_0 \to 0$ is *admissibly exact* if each map decomposes as $A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$, and each $B_i \rightarrowtail A_i \twoheadrightarrow B_{i-1}$ is an exact sequence.

V.1.2.1 **Corollary 1.2.1.** *(Additivity for characteristic exact sequences). If*

$$0 \to F^0 \to F^1 \to \cdots \to F^n \to 0$$

is an admissibly exact sequence of exact functors $\mathcal{B} \to \mathcal{C}$, then $\sum(-1)^p F^p_* = 0$ as maps from $K_i(\mathcal{B})$ to $K_i(\mathcal{C})$.

Proof. This follows from the Additivity Theorem 1.2 by induction on n. □

V.1.2.2 **Remark 1.2.2.** Suppose that F' and F are the same functor in the Additivity Theorem 1.2. Using the H-space structure we have $F''_* \simeq F_* - F_* \simeq 0$. It follows that the homotopy fiber of $F'' : K(\mathcal{B}) \longrightarrow K(\mathcal{C})$ is homotopy equivalent to $K(\mathcal{B}) \vee \Omega K(\mathcal{C})$.

V.1.2.3 **Example 1.2.3.** Let $\mathcal{C}$ be a Waldhausen category with a cylinder functor (IV.8.8). Then the definition of cone and suspension imply that $1 \rightarrowtail \text{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors: each $A \rightarrowtail \text{cone}(A) \twoheadrightarrow \Sigma A$ is exact. If $\mathcal{C}$ satisfies the Cylinder Axiom IV.8.8.1, the cone is null-homotopic. The Additivity Theorem implies that $\Sigma_* + 1 = \text{cone}_* = 0$. It follows that $\Sigma : K(\mathcal{C}) \to K(\mathcal{C})$ is a homotopy inverse with respect to the H-space structure on $K(\mathcal{C})$.

The following calculation of $K(\mathcal{E})$, used in the proof of the Additivity Theorem 1.2, is due to Quillen [153] for exact categories and to Waldhausen [215] for Waldhausen categories. (The K_0 version of this Theorem was presented in II.9.3.1.)

V.1.3 **Extension Theorem 1.3.** *The exact functor* $(s,q) : \mathcal{E} = \mathcal{E}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ *induces homotopy equivalences* $wS.\mathcal{E} \simeq (wS.\mathcal{C})^2$ *and* $K(\mathcal{E}) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathcal{C})$. *The coproduct functor* $\amalg$*, sending* (A,B) *to the sequence* $A \rightarrowtail A \amalg B \twoheadrightarrow B$*, is a homotopy inverse.*

Proof for Waldhausen categories. Let $\mathcal{C}^w_m$ denote the category of sequences $A \xrightarrow{\simeq} B \xrightarrow{\simeq} \cdots$ of m weak equivalences; this is a category with cofibrations (defined termwise). The set $s_n\mathcal{C}^w_m$ of sequences of n cofibrations in $\mathcal{C}^w_m$ (IV.8.5.2)

$$\begin{array}{ccccccc}
A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_n \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_n \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
\cdots & & \cdots & & & & \cdots
\end{array}$$

is naturally isomorphic to the m-simplices in the nerve of the category $wS_n\mathcal{C}$. That is, the bisimplicial sets $wS.\mathcal{C}$ and $s.\mathcal{C}^w_.$ are isomorphic. By Ex. IV.8.10 applied to $\mathcal{C}^w_m$, each of the maps $s.\mathcal{E}(\mathcal{C}^w_m) \to s.\mathcal{C}^w_m \times s.\mathcal{C}^w_m$ is a homotopy equivalence. As m varies, we get a bisimplicial map $s.\mathcal{E}(\mathcal{C}^w_.) \to s.\mathcal{C}^w_. \times s.\mathcal{C}^w_.$, which must then be a homotopy equivalence. But we have just seen that this is isomorphic to the bisimplicial map $wS.\mathcal{E}(\mathcal{C}) \to wS.\mathcal{C} \times wS.\mathcal{C}$ of the Extension Theorem. $\square$

We include a proof of the Extension Theorem 1.3 for exact categories, because it is short and uses a different technique.

Proof of 1.3 for Exact Categories. By Quillen's Theorem A (IV.3.7), it suffices to show that, for every pair (A,C) of objects in $\mathcal{C}$, the comma category $\mathcal{T} = (s,q)/(A,C)$ is contractible. A typical object in this category is a triple $T = (u,E,v)$, where E is an extension $A_0 \rightarrowtail B_0 \twoheadrightarrow C_0$ and both $u : A_0 \to A$ and $v : C_0 \to C$ are morphisms in $Q\mathcal{C}$. We will compare $\mathcal{T}$ to its subcategories $\mathcal{T}_A$ and $\mathcal{T}_C$, consisting of those triples T such that: u is an admissible epi, respectively, v is an admissible monic. The contraction of $B\mathcal{T}$ is illustrated in the following diagram.

$$
\begin{array}{lccccccccc}
T: & A & \xleftarrow{u} & A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & C_0 & \xrightarrow{v} & C \\
\downarrow{\scriptstyle \eta_T} & \| & & \downarrow{\scriptstyle i} & & \downarrow & & \| & & \| \\
p(T): & A & \overset{j}{\twoheadrightarrow} & A_1 & \rightarrowtail & B & \twoheadrightarrow & C_0 & \xrightarrow{v} & C \\
\uparrow{\scriptstyle \pi_{p(T)}} & \| & & \| & & \uparrow & & \uparrow & & \| \\
qp(T): & A & \twoheadrightarrow & A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & C_1 & \rightarrowtail & C \\
\uparrow & \| & & \downarrow & & \uparrow & & \uparrow & & \| \\
0 & A & \twoheadrightarrow & 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 & \rightarrowtail & C
\end{array}
$$

Given a triple T, choose a factorization of u as $A_0 \overset{i}{\rightarrowtail} A_1 \overset{j}{\twoheadleftarrow} A$, and let B be the pushout of A_1 and B_0 along A_0; B is in $\mathcal{C}$ and $p(E): A_1 \rightarrowtail B \twoheadrightarrow C_0$ is an exact sequence by II, Ex. 7.8(2). Thus $p(T) = (j, p(E), v)$ is in the subcategory $\mathcal{T}_A$. The construction shows that p is a functor from $\mathcal{T}$ to $\mathcal{T}_A$, and that there is a natural transformation $\eta_T : T \to p(T)$. This provides a homotopy $B\eta$ between the identity of $B\mathcal{T}$ and the map $Bp : B\mathcal{T} \to B\mathcal{T}_A \subset B\mathcal{T}$.

By duality, if we choose a factorization of v as $C_0 \twoheadleftarrow C_1 \rightarrowtail C$ and let B_1 be the pullback of B_0 and C_1 along C_0, then we obtain a functor $q : \mathcal{T} \to \mathcal{T}_C$, and a natural transformation $\pi_T : q(T) \to T$. This provides a homotopy $B\pi$ between $Bq : B\mathcal{T} \to B\mathcal{T}_C \subset B\mathcal{T}$ and the identity of $B\mathcal{T}$.

The composition of $B\eta$ and the inverse of $B\pi$ is a homotopy $B\mathcal{T} \times I \to B\mathcal{T}$ between the identity and $Bqp : B\mathcal{T} \to B(\mathcal{T}_A \cap \mathcal{T}_C) \subset B\mathcal{T}$. Finally, the category $(\mathcal{T}_A \cap \mathcal{T}_C)$ is contractible because it has an initial object: $(A \twoheadrightarrow 0, 0, 0 \rightarrowtail C)$. Thus Bqp (and hence the identity of $B\mathcal{T}$) is a contractible map, providing the contraction $B\mathcal{T} \simeq 0$ of the comma category. □

It is useful to have a variant of the Extension Theorem involving two Waldhausen subcategories $\mathcal{A}, \mathcal{C}$ of a Waldhausen category $\mathcal{B}$. Recall from II.9.3 that the extension category $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\mathcal{C}$ by $\mathcal{A}$ is the Waldhausen subcategory of $\mathcal{E}(\mathcal{B})$ consisting of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ with A in $\mathcal{A}$ and C in $\mathcal{C}$.

V.1.3.1 **Corollary 1.3.1.** *Let $\mathcal{A}$ and $\mathcal{C}$ be Waldhausen subcategories of a Waldhausen category $\mathcal{B}$, and $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ the extension category. Then $(s, q) : \mathcal{E} \to \mathcal{A} \times \mathcal{C}$ induces a homotopy equivalence $K(\mathcal{E}) \to K(\mathcal{A}) \times K(\mathcal{C})$.*

Proof. Since (s, q) is a left inverse to $\amalg : \mathcal{A} \times \mathcal{C} \to \mathcal{E}$, it suffices to show that the identity of $K(\mathcal{E})$ is homotopic to $\amalg(s, q)_* = \amalg(s, 0)_* + \amalg(0, t)_*$. This follows from Additivity applied to the short exact sequence $\amalg(s, 0) \rightarrowtail \mathrm{id}_{\mathcal{E}} \twoheadrightarrow \amalg(0, t)$ of functors displayed in the proof of the corresponding Extension Theorem (Proposition II.9.3.1) for K_0. □

The rest of this section is devoted to applications of the Additivity Theorem.

V.1.4 **Exact Sequences 1.4.** Let $\mathcal{A}^{[0,n]}_{\text{exact}}$ denote the category of admissibly exact sequences of length n. If equivalent to the category $\mathcal{A}$; for $n = 2$ it is the category $\mathcal{E}$ of cofibration sequences of 1.1.1. In fact, $\mathcal{A}^{[0,n]}_{\text{exact}}$ is a Waldhausen category in a way which extends the structure in 1.1.1: $A_* \to A'_*$ is a weak equivalence if each $A_i \to A'_i$ is, and is a cofibration if the $B_i \rightarrowtail B'_i$ and $B'_i \cup_{B_i} A_i \rightarrowtail A^i_1$ are all cofibrations. Since $\mathcal{A}^{[0,n]}_{\text{exact}}$ is the extension category $\mathcal{E}$ of $\mathcal{A} \cong \mathcal{A}^{[0,0]}_{\text{exact}}$ by $\mathcal{A}^{[1,n]}_{\text{exact}}$ (1.3.1), the Extension Theorem 1.3 implies that the functors $A_* \mapsto B_i$ $(i = 0, ..., n-1)$ from $\mathcal{A}^{[0,n]}_{\text{exact}}$ to $\mathcal{A}$ induce a homotopy equivalence $K\mathcal{A}^{[0,n]}_{\text{exact}} \cong \prod_{i=1}^{n} K(\mathcal{A})$.

Projective bundles

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-projective scheme X, and consider the projective space bundle $\mathbb{P} = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$. We saw in II.8.5 that $K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis $\{[\mathcal{O}(-i)] : i = 0 \dots, r\}$. The summands of this decomposition arise from the exact functors

$$u_i : \mathbf{VB}(X) \to \mathbf{VB}(\mathbb{P}), \quad u_i(\mathcal{N}) = \pi^*(\mathcal{N})(-i).$$

V.1.5 **Projective Bundle Theorem 1.5.** *Let $\mathbb{P}(\mathcal{E})$ be a projective bundle over a quasi-projective scheme X. Then the u_i induce an equivalence $K(X)^{r+1} \simeq K(\mathbb{P}(\mathcal{E}))$. Thus $K_*(X) \otimes_{K_0(X)} K_0(\mathbb{P}(\mathcal{E})) \longrightarrow K_*(\mathbb{P}(\mathcal{E}))$ is a ring isomorphism.*

When $\mathcal{E}$ is a trivial bundle, so $\mathbb{P}(\mathcal{E}) = \mathbb{P}^n_X$, we have the following special case.

V.1.5.1 **Corollary 1.5.1.** *As a ring, $K_*(\mathbb{P}^n_X) \cong K_*(X) \otimes_{\mathbb{Z}} K_0(\mathbb{P}^n_{\mathbb{Z}}) \cong K_*(X)[z]/(z^{r+1})$.*

To prove Theorem 1.5, recall from II.8.7.1 that we call a vector bundle $\mathcal{F}$ *Mumford-regular* if $R^q\pi_*\mathcal{F}(-q) = 0$ for all $q > 0$. We write $\mathbf{MR}$ for the exact category of all Mumford-regular vector bundles. The direct image $\pi_* : \mathbf{MR} \to \mathbf{VB}(X)$ is an exact functor, so the following lemma allows us to define the transfer map $\pi_* : K(\mathbb{P}) \to K(X)$.

V.1.5.2 **Lemma 1.5.2.** $\mathbf{MR} \subset \mathbf{VB}(\mathbb{P})$ *induces an equivalence* $K\mathbf{MR} \simeq K(\mathbb{P})$.

Proof. Write $\mathbf{MR}(n)$ for the category of all $\mathcal{F}$ for which $\mathcal{F}(-n)$ is Mumford-regular. Then $\mathbf{MR} = \mathbf{MR}(0) \subseteq \mathbf{MR}(-1) \subseteq \cdots$, and $\mathbf{VB}(\mathbb{P}(\mathcal{E}))$ is the union of the $\mathbf{MR}(n)$ as $n \to -\infty$. Thus $K(\mathbb{P}) = \varinjlim K\mathbf{MR}(n)$. Hence it suffices to show that each inclusion $\iota_n : \mathbf{MR}(n) \subset \mathbf{MR}(n-1)$ induces a homotopy equivalence on K-theory. We saw in the proof of II.8.7.10 that the exact functors $\lambda_i : \mathbf{MR}(n-1) \to \mathbf{MR}(n)$, $\lambda_i(\mathcal{F}) = \mathcal{F}(i) \otimes \pi^*(\wedge^i\mathcal{E})$, fit into a Koszul resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{F}(1) \otimes \pi^*\mathcal{E} \to \cdots \to \mathcal{F}(r+1) \otimes \pi^* \wedge^{r+1} \mathcal{E} \to 0.$$

By Additivity (1.2.1), ι_n has $\sum_{i=1}^{r}(-1)^i\lambda_i$ as a homotopy inverse. $\square$

Proof of Theorem 1.5. By II.8.7.9 we have exact functors $T_i : \mathbf{MR} \to \mathbf{VB}(X)$, with $T_0 = \pi_*$, which assemble to form a map $t : \mathbf{MR} \to \coprod \mathbf{VB}(X)$. By Quillen's canonical resolution of a Mumford-regular bundle (II.8.7.8) is an exact sequence of exact functors:

$$0 \to \pi^*(T_r)(-r) \xrightarrow{\varepsilon(-r)} \cdots \to \pi^*(T_i)(-i) \xrightarrow{\varepsilon(-i)} \cdots \xrightarrow{\varepsilon(-1)} \pi^*(T_0) \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Again by Additivity (1.2.1), $\sum(-1)^i u_i T_i$ is homotopic to the identity on $K\mathbf{MR}$, so u_* is split up to homotopy.

Define $v_i : \mathbf{MR} \to \mathbf{VB}(X)$ by $v_i(\mathcal{F}) = \pi_*(\mathcal{F}(i))$. Then by II.8.7.2:

$$v_i u_j(\mathcal{N}) = \begin{cases} 0, & i < j; \\ \mathcal{N}, & i = j; \\ Sym_{i-j}\mathcal{E} \otimes \pi^*(\mathcal{N}), & i > j. \end{cases}$$

It follows that $v_* \circ u_* : K(X)^{r+1} \to K(X)^{r+1}$ is given by a triangular matrix whose diagonal entries are homotopic to the identity. Thus $v_* \circ u_*$ is a homotopy equivalence, as desired. □

V.1.5.3 **Variant 1.5.3.** Theorem 1.5 remains valid if X is noetherian instead of quasiprojective, but the proof is more intricate because in this case $K(X)$ is defined to be $K\mathbf{Ch}_{\mathrm{perf}}(X)$ (see 2.7.3). This generalization was proven by Thomason in [200, 4.1] by (a) replacing $\mathbf{VB}(X)$ by the category of perfect complexes of coherent sheaves; (b) replacing $\mathbf{MR}$ by the category of all Mumford-regular coherent sheaves (II.8.7.1) and passing to the Waldhausen category of bounded perfect complexes of Mumford-regular sheaves (Lemma 1.5.2 remains valid); and (c) observing that Quillen's canonical resolution (II.8.7.8) makes sense for Mumford-regular sheaves. With these modifications, the proof we have given for Theorem 1.5 goes through. By Ex. 1.10, Theorem 1.5 even remains valid for all quasi-compact and quasi-separated schemes X.

The projective line over a ring

Let R be any associative ring. We define $\mathbf{mod}\text{-}\mathbb{P}^1_R$ to be the abelian category of triples $\mathcal{F} = (M_+, M_-, \alpha)$, where $M_\pm$ is in $\mathbf{mod}\text{-}R[t^{\pm 1}]$ and α is is an isomorphism $M_+ \otimes_{R[t]} R[t, 1/t] \xrightarrow{\simeq} M_- \otimes_{R[1/t]} R[t, 1/t]$. It has a full (exact) subcategory $\mathbf{VB}(\mathbb{P}^1_R)$ consisting of triples where $M_\pm$ are finitely generated projective modules, and we write $K(\mathbb{P}^1_R)$ for $K\mathbf{VB}(\mathbb{P}^1_R)$.

If R is commutative, it is well known that $\mathbf{mod}\text{-}\mathbb{P}^1_R$ is equivalent to the category of quasi-coherent sheaves on $\mathbb{P}^1_R$, and $\mathbf{VB}(\mathbb{P}^1_R)$ is equivalent to the usual category of vector bundles on the line $\mathbb{P}^1_R$; thus $K(\mathbb{P}^1_R)$ agrees with the definition in IV.6.3.4, and both π_* and $R^1\pi_*$ have their usual meanings.

We define the functors $\pi_*, R^1\pi_* : \mathbf{mod}\text{-}\mathbb{P}^1_R \to \mathbf{mod}\text{-}R$ via the exact sequence

$$0 \to \pi_*(\mathcal{F}) \to M_+ \times M_- \xrightarrow{d} M_- \otimes_{R[1/t]} R[t, 1/t] \to R^1\pi_*(\mathcal{F}) \to 0.$$

where $d(x, y) = \alpha(x) - y$. If R is commutative, these are the usual functors π_* and $R^1\pi_*$.

There are exact functors $u_i : \mathbf{P}(R) \to \mathbf{VB}(\mathbb{P}^1_R)$, sending P to the triple $(P[t], P[1/t], t^i)$; for commutative R the u_i are the functors $u_i(P) = \pi^*(P) \otimes \mathcal{O}(-i)$ of Theorem 1.5.

V.1.5.4 **Theorem 1.5.4.** *The functors u_0, u_1 induce an equivalence $K(R) \oplus K(R) \simeq K(\mathbb{P}^1_R)$. In addition, $(u_{i+1})_* + (u_{i+1})_* \simeq (u_i)_* + (u_{i+2})_*$ for all i.*

Proof. If $\mathcal{F} = (M_+, M_-, \alpha)$, we define $\mathcal{F}(n)$ to be $(M_+, M_-, t^{-n}\alpha)$, and let $X_0, X_1 : \mathcal{F}(n-1) \to \mathcal{F}(n)$ be the maps $(1, 1/t)$ and $(t, 1)$, respectively. Then we have an exact sequence (the Koszul resolution of $\mathcal{F}$).

$$0 \to \mathcal{F}(-2) \xrightarrow{(X_1, -X_0)} \mathcal{F}(-1)^2 \xrightarrow{(X_0, X_1)} \mathcal{F} \to 0.$$

Applying this to $u_i(P)$ and using $u_i(P)(n) = u_{i-n}(P)$ yields the exact sequence $u_{i+2} \rightarrowtail u_{i+1}^2 \twoheadrightarrow u_i$ of functors, and the relations follow from Additivity 1.2. The proof of Theorem 1.5 now goes through to prove Theorem 1.5.4 (see Ex. 1.3). $\square$

Severi-Brauer schemes

V.1.6 **1.6.** Let A be a central simple algebra over a field k, and ℓ a maximal subfield of A. Then $A \otimes_k \ell \cong M_r(\ell)$ for some r, and the set of minimal left ideals of $M_r(\ell)$ correspond to the ℓ-points of the projective space $\mathbb{P}^{r-1}_\ell$; if I is a minimal left ideal corresponding to a line L of ℓ^r then the rows of matrices in I all lie on L. The Galois group $Gal(\ell/k)$ acts on this set, and it is well known that there is a variety X, defined over k, such that $X_\ell = X \times_k \ell$ is $\mathbb{P}^{r-1}_\ell$ with this Galois action. The variety X is called the *Severi-Brauer* variety of A. For example, the Severi-Brauer variety associated to $A = M_r(k)$ is just $\mathbb{P}^{r-1}_k$.

Historically, these varieties arose in the 1890's (over $\mathbb{R}$) as forms of a complex variety, together with a real structure given by an involution with no fixed points.

V.1.6.1 **Example 1.6.1.** Let X be a non-singular projective curve over k defined by an irreducible quadratic $aX^2 + bY^2 = Z^2$ $(a, b \in k)$. Then $X \cong \mathbb{P}^1_k$ if and only if X has a k-point, which holds if and only if the quadratic form $q(x, y) = ax^2 + by^2$ has a solution to $q(x, y) = 1$ in k. The associated algebra is the quaternion algebra $A(a, b)$ of III.6.9. For example, if X is the plane curve $X^2 + Y^2 + Z^2 = 0$ over $\mathbb{R}$ then A is the usual quaternions $\mathbb{H}$.

Here are some standard facts about Severi-Brauer varieties. By faithfully flat descent, the vector bundle $\mathcal{O}^r(-1)$ on $\mathbb{P}^{r-1}_\ell$ descends to a vector bundle J on X of rank r, and $A \cong H^0(X, \mathrm{End}_X(J))$ because $\mathrm{End}_{\mathbb{P}^{r-1}_\ell}(\mathcal{O}^r(-1))$ is a sheaf of matrix algebras having $M_r(\ell)$ as its global sections. Moreover, if $\pi : X \to S = \mathrm{Spec}(k)$ is the structure map then $\pi^*(A) \cong \mathrm{End}_X(J)$, as can be checked by pulling back to ℓ.

There is a canonical surjection $\mathcal{O}^r_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}$ (Ex. II.6.14); by descent it defines a surjection $J \twoheadrightarrow \mathcal{O}_X$. Hence there is a Koszul resolution:

$$0 \to \wedge^r J \to \cdots \to \wedge^2 J \to J \to \mathcal{O}_X \to 0.$$

The n-fold tensor product $A^{\otimes n}$ of A over k is also a central simple algebra, isomorphic to $\mathrm{End}_X(J^{\otimes n})$. Moreover, since $J^{\otimes n}$ is a right module over $A^{\otimes n} = \mathrm{End}_X(J^{\otimes n})$ there is an exact functor $J^{\otimes n}\otimes : \mathbf{P}(A^{\otimes n}) \to \mathbf{VB}(X)$ sending P to $J^{\otimes n} \otimes_{A^{\otimes n}} P$.

V.1.6.2 **Theorem 1.6.2.** *(Quillen) If X is the Severi-Brauer variety of A, the functors $J^{\otimes n}\otimes$ define an equivalence $\prod_{n=0}^{r-1} K(A^{\otimes n}) \xrightarrow{\sim} K(X)$, and an isomorphism*

$$\bigoplus_{n=0}^{r-1} K_*(A^{\otimes n}) \xrightarrow{\simeq} K_*(X).$$

V.1.6.3 **Example 1.6.3.** If X is the nonsingular curve $aX^2 + bY^2 + Z^2$ associated to the quaternion algebra $A = A(a, b)$, then $K_*(X) \cong K_*(k) \oplus K_*(A)$.

The proof of Theorem 1.6.2 is a simple modification of the proof of Theorem 1.5. First, we define a vector bundle $\mathcal{F}$ to be *regular* if $\mathcal{F} \otimes_k \ell$ is Mumford-regular on $X_\ell = \mathbb{P}_\ell^{r-1}$. The regular bundles form an exact subcategory of $\mathcal{O}_X$-**mod**, and $\mathcal{F} \mapsto \pi_*\mathcal{F} = H^0(X, \mathcal{F})$ is an exact functor from regular bundles to k-modules, as one checks by passing to ℓ and applying II.8.7.4. To get the analogue of the Quillen Resolution Theorem II.8.7.8, we modify Definition II.8.7.6 using J.

V.1.6.4 **Definition 1.6.4** (T_n)**.** Given a regular $\mathcal{O}_X$-module $\mathcal{F}$, we define a natural sequence of k-modules $T_n = T_n(\mathcal{F})$ and $\mathcal{O}_X$-modules $Z_n = Z_n(\mathcal{F})$, starting with $T_0(\mathcal{F}) = \pi_*\mathcal{F}$ and $Z_{-1} = \mathcal{F}$. Let Z_0 be the kernel of the natural map $J \otimes_A \pi_*\mathcal{F} \to \mathcal{F}$. Inductively, we define $T_n(\mathcal{F}) = \pi_* \mathrm{Hom}_X(J^{\otimes n}, Z_{n-1})$ and define Z_n to be the kernel of $J^{\otimes n} \otimes_{A^{\otimes n}} T_n(\mathcal{F}) \to Z_{n-1}(\mathcal{F})$.

These fit together to give a natural sequence of $\mathcal{O}_X$-modules

$$0 \to J^{\otimes r-1} \otimes_{A^{\otimes r-1}} T_{r-1}(\mathcal{F}) \to \cdots \to \mathcal{O}_X \otimes_k T_0(\mathcal{F}) \to \mathcal{F} \to 0. \qquad (1.6.5)$$ V.1.6.5

When lifted to X_ℓ, it is easy to see that these are exactly the functors T_n and Z_n of II.8.7.6 for $\mathcal{F} \otimes_k \ell$. By faithfully flat descent, (1.6.5) is exact, *i.e.*, a resolution of the regular bundle $\mathcal{F}$.

Thus all the tools used in the Projective Bundle Theorem 1.5 are available for Severi-Brauer varieties. The rest of the proof is routine (Exercise 1.4).

Theorem 1.6.2 may be generalized over any base scheme S. Here are the key points. An *Azumaya algebra* over S is a sheaf of rings $\mathcal{A}$ which is locally isomorphic to $M_r(\mathcal{O}_S)$ for the étale topology. That is, there is a faithfully flat map $T \to S$ so that $\mathcal{A} \otimes_S \mathcal{O}_T$ is the sheaf of rings $M_r(\mathcal{O}_T)$. Let $\mathbf{VB}(\mathcal{A})$ denote the exact category of vector bundles on S which are left modules for $\mathcal{A}$, and define

$$K(\mathcal{A}) = K\mathbf{VB}(\mathcal{A}); \quad K_n(\mathcal{A}) = K_n\mathbf{VB}(\mathcal{A}).$$

By a *Severi-Brauer* scheme over S we mean a scheme X which is locally isomorphic to projective space for the étale topology, *i.e.*, such that $X \times_S T \cong \mathbb{P}_T^{r-1}$ for some faithful étale map $T \to S$. In this situation, we may define a vector bundle J on X by faithfully flat descent so that $J \otimes_S T = \mathcal{O}_T^r(-1)$, as above, and then $\mathcal{A} = \pi_* \mathrm{End}_X(J)$ will be an Azumaya algebra over S. Again, each $J^{\otimes n}$ is a right module over $\mathcal{A}^{\otimes n}$ and we have exact functors $J^{\otimes n}\otimes_{\mathcal{A}^{\otimes n}}$: $\mathbf{VB}(\mathcal{A}^{\otimes n}) \to \mathbf{VB}(X)$. Replacing $H^0(X,-)$ with π_*, Definition 1.6.4 makes sense and (1.6.5) is admissibly exact. Therefore the proof still works in this generality, and we have:

V.1.6.6 **Theorem 1.6.6.** *(Quillen) If X is a Severi-Brauer variety over S, with associated Azumaya algebra A, the functors $J^{\otimes n}\otimes_{\mathcal{A}^{\otimes n}}$ define an isomorphism*

$$\coprod_{n=0}^{r-1} K_*(\mathcal{A}^{\otimes n}) \xrightarrow{\simeq} K_*(X).$$

Our next application of Additivity was used in IV, 8.5.3–8.5.5 to show that $K(\mathcal{C})$ is an infinite loop space. To do this, we defined the relative K-theory space to be $K(f) = \Omega^2|wS.(S.f)|$, and invoked the following result.

V.1.7 **Proposition 1.7.** *If $f : \mathcal{B} \to \mathcal{C}$ is an exact functor, the following sequence is a homotopy fibration:*

$$\Omega|wS.(S.\mathcal{B})| \to |wS.\mathcal{C}| \to |wS.(S.f)| \to |wS.(S.\mathcal{B})|.$$

Proof. Each category S_nf of IV.8.5.3 is equivalent to the extension category $\mathcal{E}(\mathcal{B}, S_nf, S_n\mathcal{C})$ of $\mathcal{B}$ by $S_n\mathcal{C}$ (see II.9.3). By 1.3.1, the map

$$(s,q) : wS.(S_nf.) \xrightarrow{\simeq} wS.\mathcal{B} \times wS.(S_n\mathcal{C})$$

is a homotopy equivalence. That is, $|wS.\mathcal{B}| \to |wS.(S_nf.)| \to |wS.(S_n\mathcal{C})|$ is a (split) fibration of connected spaces for each n. But if $X. \to Y. \to Z.$ is any sequence of simplicial spaces, and each $X_n \to Y_n \to Z_n$ is a fibration with Z_n connected, then $\Omega|Z.| \to |X.| \to |Y.| \to |Z.|$ is a homotopy fibration sequence; see [214, 5.2]. This applies to our situation by realizing in the $wS.$ direction first (so that $X_n = |wS.\mathcal{B}|$ for all n), and the result follows. □

V.1.7.1 **Remark 1.7.1.** As observed in IV.8.5.4, $wS.S.f$ is contractible when f is the identity of $\mathcal{B}$. It follows that $|wS.\mathcal{B}| \simeq \Omega|wS.(S.\mathcal{B})|$, yielding the formulation of 1.7 given in IV.8.5.3.

Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories. An *admissible filtration* of F is a sequence $0 \rightarrowtail F_1 \rightarrowtail F_2 \rightarrowtail \cdots \rightarrowtail F_n = F$ of functors and admissible monomorphisms, sending an object A in $\mathcal{A}$ to the sequence

$$0 = F_0(A) \rightarrowtail F_1(A) \rightarrowtail \cdots \rightarrowtail F_n(A) = F(A).$$

It follows that the quotient functors F_p/F_{p-1} exist, but they may not be exact.

V.1.8 **Proposition 1.8.** *(Admissible Filtrations) If $0 \rightarrowtail F_1 \rightarrowtail F_2 \rightarrowtail \cdots \rightarrowtail F_n = F$ is an admissible filtration of F, and the quotient functors F_i/F_{i-1} are exact, then $F : K\mathcal{A} \to K\mathcal{B}$ is homotopic to $\sum F_i/F_{i-1}$. In particular,*

$$F_* = \sum (F_i/F_{i-1})_* : K_*(\mathcal{A}) \to K_*(\mathcal{B}).$$

Proof. Apply the Additivity Theorem 1.2 to $F_{i-1} \rightarrowtail F_i \twoheadrightarrow F_i/F_{i-1}$, and use induction on n. □

Here is a simple application of Proposition 1.8. Let $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ be a graded ring, and consider the category $\mathbf{P}_{gr}(S)$ of finitely generated graded projective S-modules. Its K-groups are naturally modules over $\mathbb{Z}[\sigma, \sigma^{-1}]$, where σ acts by the shift automorphism $\sigma(P) = P[-1]$ of graded modules. If $S = R$, $K_*(R)[\sigma, \sigma^{-1}] \cong K_*(\mathbf{P}_{gr}(R))$ by Ex. IV.6.11. Thus the base change map $\otimes_R S : \mathbf{P}_{gr}(R) \to \mathbf{P}_{gr}(S)$ induces a morphism $K_*(R)[\sigma, \sigma^{-1}] \to K_*(\mathbf{P}_{gr}(S))$.

V.1.8.1 **Corollary 1.8.1.** *If $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is graded then the base change map induces an isomorphism $K_*(R)[\sigma, \sigma^{-1}] \cong K_*(\mathbf{P}_{gr}(S))$.*

Proof. For each $a \le b$, let $\mathbf{P}_{[a,b]}(S)$ denote the (exact) subcategory of $\mathbf{P}_{gr}(S)$ consisting of graded modules P generated by the P_i with $i \le b$, and with $P_i = 0$ for $i < a$. By Ex. 1.9, the identity functor on this category has an admissible filtration: $0 = F_a \rightarrowtail F_{a+1} \rightarrowtail \cdots \rightarrowtail F_b = \mathrm{id}$, where F_nP denotes the submodule of P generated by the P_i with $i \le n$. Moreover, there is a natural isomorphism between F_n/F_{n-1} and the degree n part of the exact functor $\otimes_S R : \mathbf{P}_{gr}(S) \to \mathbf{P}_{gr}(R)$. By Proposition 1.8, the homomorphism $\oplus_{n=a}^{b} K_*(R) \otimes \sigma^n \cong K_*\mathbf{P}_{[a,b]}(R) \to K_*\mathbf{P}_{[a,b]}(S)$ is an isomorphism with inverse $\otimes_S R$. Since $\mathbf{P}_{gr}(S)$ is the filtered colimit of the $\mathbf{P}_{[a,b]}(S)$, the result follows from IV.6.4. □

V.1.9 **Flasque Categories 1.9.** Call an exact (or Waldhausen) category $\mathcal{A}$ *flasque* if there is an exact functor $\infty : \mathcal{A} \to \mathcal{A}$ and a natural isomorphism $\infty(A) \cong A \amalg \infty(A)$, *i.e.,* $\infty \cong 1 \amalg \infty$, 1 being the identity functor. By additivity, $\infty_* = 1_* + \infty_*$, and hence the identity map $1_* : K(\mathcal{A}) \to K(\mathcal{A})$ is null-homotopic. Therefore $K(\mathcal{A})$ is contractible, and $K_i(\mathcal{A}) = 0$ for all i.

V.1.9.1 **The Eilenberg Swindle 1.9.1.** For example, the category of *countably* generated R-modules is flasque, so its K-theory is trivial. To see this, let $\infty(M) = M^\infty$ be the direct sum $M \oplus M \oplus \cdots$ of infinitely many copies of M. The isomorphism $M^\infty \cong M \oplus M^\infty$ is the shift

$$(M \oplus M \oplus \cdots) \cong M \oplus (M \oplus M \oplus \cdots).$$

This infinite shifting trick is often called the "Eilenberg swindle" (see I.2.8, II.6.1.4 and II.9.1.4); it is why we restrict to finitely generated modules in defining $K(R)$.

V.1.9.2 **Flasque rings 1.9.2.** Here is another example of a flasque category, due to Karoubi. Recall from II.2.1.3 that a ring R is called *flasque* if there is an R-bimodule M_∞, finitely generated projective as a right module, and a bimodule isomorphism $\theta : R \oplus M_\infty \cong M_\infty$. If R is flasque, then $\mathbf{P}(R)$ and $\mathbf{M}(R)$ are flasque categories in the sense of 1.9, with $\infty(M) = M \otimes_R M_\infty$. The contractibility of $K(R) = K_0(R) \times BGL(R)^+$ for flasque rings, established in Ex. IV.1.17, may be viewed as an alternative proof that $K\mathbf{P}(R) = \Omega BQ\mathbf{P}(R)$ is contractible, via the '$+ = Q$' Theorem IV.7.2.

As mentioned in IV.10.4.1, this contractibility was used by Karoubi, Gersten and Wagoner to define deloopings of $K(R)$ in terms of the suspension ring $S(R)$ of IV.1.11.2, forming a non-connective spectrum $\mathbf{K}^{GW}(R)$ homotopy equivalent to the spectrum $\mathbf{K}^B(R)$ of IV.10.4.

EXERCISES

EV.1.1 **1.1.** In the proof of the Extension Theorem 1.3, show that the functors p and q are left and right adjoint, respectively, to the inclusions of $\mathcal{T}_A$ and $\mathcal{T}_C$ in $\mathcal{T}$. This proves that Bp and Bq are homotopy equivalences.

EV.1.2 **1.2.** Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of vector bundles on a scheme (or a ringed space). Show that the map $(- \otimes \mathcal{E})_* : K(X) \to K(X)$ given by the exact functor $\mathcal{F} \to \mathcal{F} \otimes \mathcal{E}$ satisfies $(-\otimes\mathcal{E})_* = (-\otimes\mathcal{E}')_* + (-\otimes\mathcal{E}'')_*$.

EV.1.3 **1.3.** Complete the proof of Theorem 1.5.4, modifying the proof of 1.5 for $\mathbb{P}^1_R$. (See Quillen [153, 8.4.1].)

EV.1.4 **1.4.** Complete the proof of Theorems 1.6.2 and 1.6.6, by modifying the proof of 1.5. (See [153, 8.4.1].) For extra credit, describe the ring structure on $K_*(X)$ using the pairings $\textbf{mod-}A^{\otimes i} \times \textbf{mod-}A^{\otimes j} \to \textbf{mod-}A^{\otimes i+j}$ (tensor product over k) and the Morita equivalence of $A^{\otimes i}$ and $A^{\otimes i+n}$.

EV.1.5 **1.5.** Given an exact category $\mathcal{A}$ and integers $a \le b$, let $\operatorname{iso}\mathbf{Ch}^{[a,b]}(\mathcal{A})$ denote the category of chain complexes $C_b \to \cdots \to C_a$ in $\mathcal{A}$. We may consider it as a Waldhausen category whose cofibrations are degreewise admissible monics, with isomorphisms as the weak equivalences (II.9.1.3). Use the Additivity Theorem 1.2 to show that the "forget differentials" functor $\operatorname{iso}\mathbf{Ch}^{[a,b]}(\mathcal{A}) \to \prod_{i=a}^{b} \mathcal{A}$ induces a homotopy equivalence on K-theory.

EV.1.6 **1.6.** If $\mathcal{A}$ is an exact category, the category $\mathcal{A}^{[0,n]}_{\text{exact}}$ of admissibly exact sequences (Example 1.4) may be viewed as a subcategory of the category $\operatorname{iso}\mathbf{Ch}^{[0,n]}(\mathcal{A})$ of the previous exercise. Use the Additivity Theorem to show that the "forget differentials" functor $\mathcal{A}^{[0,n]}_{\text{exact}} \to \prod_{i=0}^{n} \mathcal{A}$ and the functor $A_* \mapsto (B_1, B_1 \oplus B_2, ..., B_{n-1} \oplus B_n, B_n)$ induce homotopy equivalent maps on K-theory.

EV.1.7 **1.7.** If $f : \mathcal{B} \to \mathcal{C}$ is exact, show that the composite $wS.\mathcal{B} \to \Omega|wS.S.\mathcal{B}| \to |wS.\mathcal{C}|$ in Proposition 1.7 is the map induced by f. *Hint:* Use $wS.S.\mathrm{id}_{\mathcal{B}} \to wS.S.f$.

EV.1.8 **1.8.** Recall from IV.8.7 that $A(*)$ is the K-theory of the category $\mathcal{R}_f(*)$ of finite based CW complexes. Let $\mathcal{R}^{(2)}_f(*)$ be the subcategory of simply connected complexes. Show that $K\mathcal{R}^{(2)}_f(*) \xrightarrow{\sim} K\mathcal{R}_f(*) = A(*)$ is a homotopy equivalence, with $Y \mapsto \Sigma^2 Y$ as inverse. Then formulate a version for $A(X)$.

EV.1.9 **1.9.** (Swan) If $S = R \oplus S_1 \oplus \cdots$ is a graded ring and P is a graded projective S-module, show that the map $(P \otimes_S R) \otimes_R S \to P$ is an isomorphism. If F_nP is the submodule of P generated by the P_i with $i \le n$, show that F_nP and P/F_nP are graded projective modules, and that F_n and $P \mapsto P/F_nP$ are exact functors from $\mathbf{P}_{gr}(S)$ to itself. Conclude that $\cdots \rightarrowtail F_nP \rightarrowtail F_{n+1}P \rightarrowtail \cdots$ is an admissible filtration of P. Is there a natural isomorphism $P \cong F_nP \oplus P/F_nP$?

EV.1.10 **1.10.** Let X be a quasi-compact, quasi-separated scheme. Show that the variant 1.5.3 of the Projective Bundle Theorem 1.5 holds for X. *Hint:* X is the inverse limit of an inverse system of noetherian schemes X_α with affine bonding maps by [200, C.9]. Show that any vector bundle $\mathcal{E}$ on X is the pullback of a vector bundle $\mathcal{E}_\alpha$ over some X_α.

2 Waldhausen Localization and Approximation

Here are two fundamental results about Waldhausen K-theory that, although technical in nature, have played a major role in the development of K-theory.

Waldhausen Localization

The first fundamental result involves a change in the category of weak equivalences, with the same underlying category of cofibrations. The K_0 version of this result, which needed fewer hypotheses, was presented in II.9.6.

V.2.1 **Waldhausen Localization Theorem 2.1.** *Let $\mathcal{A}$ be a category with cofibrations, equipped with two categories of weak equivalences, $v(\mathcal{A}) \subset w(\mathcal{A})$, such that $(\mathcal{A}, v)$ and $(\mathcal{A}, w)$ are both Waldhausen categories. In addition, we suppose that $(\mathcal{A}, w)$ has a cylinder functor satisfying the Cylinder Axiom (IV.8.8.1), and that $w(\mathcal{A})$ satisfies the Saturation and Extension Axioms (II.9.1.1 and IV.8.2.1). Then*

$$K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{A}, w)$$

is a homotopy fibration, where $\mathcal{A}^w$ denotes the Waldhausen subcategory of $(\mathcal{A}, v)$ consisting of all A in $\mathcal{A}$ for which $0 \to A$ is in $w(\mathcal{A})$. In particular, there is a long exact sequence:

$$\cdots \to K_{i+1}(\mathcal{A}, w) \to K_i(\mathcal{A}^w) \to K_i(\mathcal{A}, v) \to K_i(\mathcal{A}, w) \to \cdots,$$

ending in the exact sequence $K_0(\mathcal{A}^w) \to K_0(\mathcal{A}, v) \to K_0(\mathcal{A}, w) \to 0$ of II.9.6.

Proof. Consider the bicategory $v.w.\mathcal{C}$ (IV.3.10) whose bimorphisms are commutative squares in $\mathcal{C}$

$$\begin{array}{ccc} \cdot & \xrightarrow{w'} & \cdot \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ \cdot & \xrightarrow{w} & \cdot \end{array}$$

in which the vertical maps are in $v\mathcal{C}$ and the horizontal maps are in $w\mathcal{C}$. Considering $w\mathcal{C}$ as a bicategory which is vertically constant, we saw in IV, 3.10.2 and Ex. 3.13 that $w\mathcal{C} \to v.w.\mathcal{C}$ is a homotopy equivalence. Applying this construction to $S_n\mathcal{C}$, we get equivalences $wS_n\mathcal{C} \simeq v.wS_n\mathcal{C}$ and hence $wS.\mathcal{C} \simeq v.wS.\mathcal{C}$.

Let $v.\operatorname{co} w.\mathcal{C}$ denote the sub-bicategory of those squares in $v.w.\mathcal{C}$ whose horizontal maps are also cofibrations. We claim that the inclusions $v.\operatorname{co} w.\mathcal{C} \subset v.w.\mathcal{C}$ are homotopy equivalences. To see this, we fix m and consider the column category $v_m w.\mathcal{C}$ to be the category of weak equivalences in the category $\mathcal{C}(m, v)$ of diagrams $C_0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C_m$ in $\mathcal{C}$ whose maps are in $v\mathcal{C}$. Because $(\mathcal{C}(m,v),w)$ inherits the saturation and cylinder axioms from $(\mathcal{C},w)$, it follows from IV, Ex. 8.15 that the inclusion $v_m \operatorname{co} w.\mathcal{C} \subset v_m w.\mathcal{C}$ is a homotopy equivalence. Since this is true for each m, the claim follows.

Now each $S_n\mathcal{C}$ inherits a cylinder functor from $\mathcal{C}$. Replacing $\mathcal{C}$ by the $S_n\mathcal{C}$ shows that the simplicial bicategory $v.w.S.\mathcal{C}$ contains a simplicial bicategory $v.\operatorname{co} w.S.\mathcal{C}$, and that the inclusion $v.\operatorname{co} w.S.\mathcal{C} \subset v.w.S.\mathcal{C}$ is a homotopy equivalence. This means that the right vertical map is a homotopy equivalence in the following diagram; the bottom horizontal map is a homotopy equivalence by the first paragraph of this proof.

$$\begin{array}{ccccc} vS.\mathcal{C}^w & \longrightarrow & vS.\mathcal{C} & \longrightarrow & v.\operatorname{co} w.S.\mathcal{C} \\ & & \downarrow & & \downarrow \simeq \\ & & wS.\mathcal{C} & \xrightarrow{\simeq} & v.w.S.\mathcal{C} \end{array}$$

Thus it suffices to show that the top row is a homotopy fibration. We will identify it with the homotopy fibration $vS.\mathcal{C}^w \to vS.\mathcal{C} \to vS.(S.f)$ arising from the relative K-theory space construction (IV.8.5.3), applied to the inclusion $f : (\mathcal{C}^w, v) \to (\mathcal{C}, v)$.

By the extension axiom, a trivial cofibration in $(\mathcal{C},w)$ is just a cofibration whose quotient lies in $\mathcal{C}^w$. In particular, there is an equivalence $S_1 f \to \operatorname{co} w\mathcal{C}$. Forgetting the choices of the C_i/C_j yields an equivalence $S_n f \to \operatorname{co} w_n\mathcal{C}$, where $\operatorname{co} w_n\mathcal{C}$ is the category of all trivial cofibration sequences $C_0 \overset{\sim}{\rightarrowtail} C_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} C_n$, and an equivalence between $vS_n f$ and the vertical category $v.\operatorname{co} w_n\mathcal{C}$ of the bicategory $v.\operatorname{co} w.\mathcal{C}$. Similarly, forgetting choices yields an equivalence between the categories $vS_m(S_n f)$ and $v.\operatorname{co} w_n(S_m\mathcal{C})$, and thus a homotopy equivalence $vS.S.f \to v.\operatorname{co} w.S.\mathcal{C}$.

Now $vS_m\mathcal{C} \to v.\operatorname{co} w_n(S_m\mathcal{C})$ factors through $vS_m(S_n f)$, so $vS.\mathcal{C} \to v.\operatorname{co} w.S.\mathcal{C}$ factors through the homotopy equivalence $vS.S.f \to v.\operatorname{co} w.S.\mathcal{C}$, as required. □

For us, the most important application of Waldhausen Localization is the following theorem, which allows us to replace the K-theory of any exact category $\mathcal{A}$ by the K-theory of the category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes, which is a Waldhausen category with a cylinder functor. This result was first worked out by Waldhausen in special cases, and generalized by Gillet. Our presentation is taken from [200, 1.11.7].

Let $\mathcal{A}$ be an exact category, and consider the category $\mathbf{Ch}^b(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$. We saw in II.9.2 that $\mathbf{Ch}^b(\mathcal{A})$ is a Waldhausen category; the cofibrations are degreewise admissible monomorphisms, and the weak equivalences are quasi-isomorphisms (as computed in a specified ambient abelian category). The isomorphism $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A})$ of Theorem II.9.2.2 generalizes as follows.

V.2.2 **Theorem 2.2.** *(Gillet-Waldhausen) Let $\mathcal{A}$ be an exact category, closed under kernels of surjections in an abelian category (in the sense of II.7.0.1)*

Then the exact inclusion $\mathcal{A} \subset \mathbf{Ch}^b(\mathcal{A})$ *induces a homotopy equivalence* $K(\mathcal{A}) \xrightarrow{\sim} K\mathbf{Ch}^b(\mathcal{A})$*.*

In particular, $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^b(\mathcal{A})$ *for all* n*.*

Proof. We will apply Waldhausen's Localization Theorem 2.1 to the following situation. For $a \le b$, let $\mathbf{Ch}^{[a,b]}$ denote the full subcategory of all complexes A_* in $\mathbf{Ch}(\mathcal{A})$ for which the A_i are zero unless $a \le i \le b$. This is a Waldhausen subcategory of $\mathbf{Ch}^b(\mathcal{A})$ with w the quasi-isomorphisms. We write $\operatorname{iso}\mathbf{Ch}^{[a,b]}$ for the Waldhausen category with the same underlying category with cofibrations, but with isomorphisms as weak equivalences. Because $\mathcal{A}$ is closed under kernels of surjections, the subcategory of quasi-isomorphisms in $\operatorname{iso}\mathbf{Ch}^{[a,b]}$ is just the Waldhausen category $\mathcal{A}^{[a,b]}_{\text{exact}}$ of Example 1.4 (see Ex. 2.4). We claim that there is a homotopy fibration

$$K\mathcal{A}^{[a,b]}_{\text{exact}} \to K\operatorname{iso}\mathbf{Ch}^{[a,b]} \xrightarrow{\chi} K(\mathcal{A}).$$

By Example 1.4 and Ex. 1.5, the first two spaces are products of $n = b - a$ and $n+1$ copies of $K(\mathcal{A})$, respectively. By Ex. 1.6, the induced map $\prod_{a+1}^{b} K(\mathcal{A}) \to \prod_{a}^{b} K(\mathcal{A})$ is equivalent to that induced by the exact functor

$$(B_{a+1}, ..., B_b) \mapsto (B_{a+1}, B_{a+1} \oplus B_{a+2}, ..., B_{b-1} \oplus B_b, B_b).$$

The homotopy cofiber of this map is $K(\mathcal{A})$, with the map $\prod_a^b K(\mathcal{A}) \to K(\mathcal{A})$ being the alternating sum of the factors, *i.e.,* the Euler characteristic χ. This shows that $K\mathbf{Ch}^{[a,b]} \simeq K(\mathcal{A})$ for each a and b.

Taking the direct limit as $a \to -\infty$ and $b \to +\infty$ yields a homotopy fibration

$$K\mathcal{A}^{[-\infty,\infty]}_{\text{exact}} \to K\operatorname{iso}\mathbf{Ch}^{[-\infty,\infty]} \xrightarrow{\chi} K(\mathcal{A}),$$

where χ is the Euler characteristic. But by Waldhausen Localization 2.1, the cofiber is $K\mathbf{Ch}^b(\mathcal{A})$. □

V.2.2.1 **Remark 2.2.1.** When $\mathcal{A}$ is not closed under kernels in its ambient abelian category, $K_0(\mathcal{A})$ may not equal $K_0\mathbf{Ch}^b(\mathcal{A})$; see Ex. II.9.11. However, the following trick shows that the extra assumption is harmless in Theorem 2.2, provided that we allow ourselves to change the ambient notion of quasi-isomorphism slightly in $\mathbf{Ch}^b(\mathcal{A})$. Consider the Yoneda embedding of $\mathcal{A}$ in the abelian category $\mathcal{L}$ of contravariant left exact functors (Ex. II.7.8). As pointed out in *loc. cit.*, the idempotent completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (II.7.3) is closed under surjections in $\mathcal{L}$.

Let $\mathcal{A}'$ be the full subcategory of $\widehat{\mathcal{A}}$ consisting of all B with $[B]$ in the subgroup $K_0(\mathcal{A})$ of $K_0(\widehat{\mathcal{A}})$. We saw in Ex. IV.8.13 that $\mathcal{A}'$ is exact and closed under admissible epis in $\hat{\mathcal{A}}$ (and hence in $\mathcal{L}$), so that Theorem 2.2 applies to $\mathcal{A}'$. By K_0-cofinality (II, 7.2 and 9.4), $K_0(\mathcal{A}) = K_0(\mathcal{A}') = K_0\mathbf{Ch}^b(\mathcal{A}) = K_0\mathbf{Ch}^b(\mathcal{A}')$. By Waldhausen Cofinality (IV.8.9), $K(\mathcal{A}) \simeq K(\mathcal{A}')$ and $K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}')$. Hence $K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A})$.

V.2.3 **Cofinality Theorem 2.3.** *Let $(\mathcal{A}, v)$ be a Waldhausen category with a cylinder functor satisfying the cylinder axiom (IV.8.8.1). Suppose that we are given a surjective homomorphism $\pi : K_0(\mathcal{A}) \to G$, and let $\mathcal{B}$ denote the full Waldhausen subcategory of all B in $\mathcal{A}$ with $\pi[B] = 0$ in G.*

Then $vs.\mathcal{B} \to vs.\mathcal{A} \to BG$ and its delooping $K(\mathcal{B}) \to K(\mathcal{A}) \to G$ are homotopy fibrations. In particular, $K_n(\mathcal{B}) \cong K_n(\mathcal{A})$ for all $n > 0$ and (as in II.9.6.2) there is a short exact sequence:

$$0 \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \xrightarrow{\pi} G \to 0.$$

Proof. (Thomason) As in II.9.6.2, we can form the Waldhausen category $(\mathcal{A}, w)$, where $w(\mathcal{A})$ is the set of maps $A \to A'$ in $\mathcal{A}$ with $\pi[A] = \pi[A']$. It is easy to check that $w(\mathcal{A})$ is saturated (II.9.1.1), $\mathcal{B} = \mathcal{A}^w$, and that $(\mathcal{A}, w)$ satisfies the Extension Axiom IV.8.2.1. By the Waldhausen Localization Theorem 2.1, there is a homotopy fibration

$$K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}, w).$$

By IV.8.10, $ws.(\mathcal{A}, w) \simeq BG$ and hence $K(\mathcal{A}, w) \simeq \Omega(BG) = G$, as required. □

Combining this with the Waldhausen Cofinality Theorem IV.8.9.1, we obtain the following variation. Recall from Theorem II.9.4 that a Waldhausen subcategory $\mathcal{B}$ is said to be *cofinal* in $\mathcal{A}$ if for each A in $\mathcal{A}$ there is an A' so that $A \amalg A'$ is in $\mathcal{B}$, and that this implies that $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ is an injection.

V.2.3.1 **Corollary 2.3.1.** *Let $\mathcal{B}$ be a cofinal Waldhausen subcategory of $\mathcal{A}$ closed under extensions. Suppose that $\mathcal{A}$ has a cylinder functor satisfying the cylinder axiom (IV.8.8.1), and restricting to a cylinder functor on $\mathcal{B}$.*

Then for $G = K_0(\mathcal{A})/K_0(\mathcal{B})$ there is a homotopy fibration sequence

$$K(\mathcal{B}) \to K(\mathcal{A}) \to G.$$

Proof. Clearly $\mathcal{B}$ is contained in the Waldhausen subcategory $\mathcal{A}^w$ associated to $K_0(\mathcal{A}) \to G$. By the Waldhausen Cofinality Theorem IV.8.9.1, $K(\mathcal{B}) \simeq K(\mathcal{A}^w)$. The result now follows from Theorem 2.3. □

Waldhausen Approximation

The second fundamental result is the Approximation Theorem, whose K_0 version was presented in II.9.7. Consider the following "approximate lifting property," which is to be satisfied by an exact functor $F : \mathcal{A} \to \mathcal{B}$:

(App) Given any map $b : F(A) \to B$ in $\mathcal{B}$, there is a map $a : A \to A'$ in $\mathcal{A}$ and a weak equivalence $b' : F(A') \simeq B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.

Roughly speaking, this axiom says that every object and map in $\mathcal{B}$ lifts up to weak equivalence to $\mathcal{A}$. Note that if we replace A' by the mapping cylinder $T(a)$ of IV.8.8, a by $A \rightarrowtail T(a)$ and b' by $F(T(a)) \simeq F(A') \simeq B$, then we may assume that a is a cofibration. The following result is taken from [215, 1.6.7].

V.2.4 **Theorem 2.4** (Waldhausen Approximation Theorem)**.** *Suppose that $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between saturated Waldhausen categories, satisfying the conditions:*

(a) A morphism f in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a weak equivalence in $\mathcal{B}$.

(b) $\mathcal{A}$ has a cylinder functor satisfying the cylinder axiom.

(c) The approximate lifting property (App) is satisfied.

Then $wS.\mathcal{A} \xrightarrow{\sim} wS.\mathcal{B}$ and $K(\mathcal{A}) \xrightarrow{\sim} K(\mathcal{B})$ are homotopy equivalences. In particular, the groups $K_*\mathcal{A}$ and $K_*\mathcal{B}$ are isomorphic.

Proof. (Waldhausen) Each of the exact functors $S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfies (App); see Ex. 2.1. Applying Proposition 2.4.1 below to these functors, we see that each $wS_n\mathcal{A} \to wS_n\mathcal{B}$ is also a homotopy equivalence. It follows that the bisimplicial map $wS_.\mathcal{A} \to wS_.\mathcal{B}$ is also a homotopy equivalence, as required. □

V.2.4.1 **Proposition 2.4.1.** *(Waldhausen) Suppose that $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between Waldhausen categories, satisfying the three hypotheses of Theorem 2.4. Then $wF : w\mathcal{A} \xrightarrow{\sim} w\mathcal{B}$ is a homotopy equivalence.*

Proof. By Quillen's Theorem A (IV.3.7), it suffices to show that the comma categories wF/B are contractible. The condition (App) states that, given any object (A, b) of $F/\mathcal{B}$ there is a map a in $F/\mathcal{B}$ to an object (A', b') of $wF/\mathcal{B}$. Applying it to $(0, 0)$ shows that $wF/\mathcal{B}$ is nonempty. For any finite set of objects (A_i, b_i) in $wF/\mathcal{B}$, the maps $A_i \to \oplus A_i$ yield maps in $F/\mathcal{B}$ to $(\oplus A_i, b)$, where $b : F(\oplus A_i) \cong \oplus F(A_i) \to B$, and hence maps a_i from each (A_i, b_i) to an object (A', b') in $wF/\mathcal{B}$; the $F(a_i)$ are in $w\mathcal{B}$ by saturation, so the a_i are in $w\mathcal{A}$ and represent maps in $wF/\mathcal{B}$. The same argument shows that if we are given any finite diagram D on these objects in $wF/\mathcal{B}$, an object (A, b) in $F/\mathcal{B}$ and maps $(A_i, b_i) \to (A, b)$ forming a (larger) commutative diagram D_+ in $F/\mathcal{B}$, then by composing with $(A, b) \to (A', b')$, we embed D into a diagram D'_+ in $wF/\mathcal{B}$ with a terminal object. This implies that $|D|$ is contractible in $|wF/\mathcal{B}|$.

The rest of the proof consists of finding such a diagram D_+ for every "non-singular" finite subcomplex of $|wF/\mathcal{B}|$, using simplicial methods. We omit this part of the proof, which is lengthy (5 pages), and does not seem relevant to this book, and refer the reader to [215, 1.6.7]. □

V.2.4.2 **Remark 2.4.2.** The Approximation Theorem can fail in the absence of a cylinder functor. For example, if $\mathcal{A}$ is an exact category then $\mathcal{A}^{\oplus} \subset \mathcal{A}$ satisfies (App), yet $K_0(\mathcal{A}^{\oplus})$ and $K_0(\mathcal{A})$ are often different; see II.7.1.

Combining Waldhausen Localization 2.1 and Approximation 2.4 yields the following useful result, applicable to exact functors $F : \mathcal{A} \to \mathcal{B}$ which are onto up to weak equivalence. Let $\mathcal{A}^w$ denote the Waldhausen subcategory of $\mathcal{A}$ consisting of all A such that $F(A)$ is weak equivalent to 0 in $\mathcal{B}$.

V.2.5 **Theorem 2.5.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between two saturated Waldhausen categories having cylinder functors, with $\mathcal{B}$ extensional (IV.8.2.1). If every object B and every map $F(A) \to B$ in $\mathcal{B}$ lifts to $\mathcal{A}$ up to weak equivalence, then $K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{B})$ is a homotopy fibration sequence, and there is a long exact sequence:*

$$\cdots \xrightarrow{F} K_{n+1}(\mathcal{B}) \to K_n(\mathcal{A}^w) \to K_n(\mathcal{A}) \xrightarrow{F} K_n(\mathcal{B}) \to \cdots,$$

ending in the exact sequence $K_0(\mathcal{A}^w) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$.

Proof. (Thomason) Let $v(\mathcal{A})$ and $w(\mathcal{B})$ denote the respective categories of weak equivalences in $\mathcal{A}$ and $\mathcal{B}$, and set $w(\mathcal{A}) = F^{-1}(w(\mathcal{B}))$. Replacing $v(\mathcal{A})$ with $w(\mathcal{A})$ yields a new Waldhausen category, which we write as $(\mathcal{A}, w)$ for clarity. The Approximation Theorem 2.4 states that $K(\mathcal{A}, w) \simeq K(\mathcal{B})$. Since $(\mathcal{A}, w)$ inherits the extension axiom from $\mathcal{B}$, Waldhausen Localization 2.1 applies to give the fibration $K(\mathcal{A}^w) \to K(\mathcal{A}, v) \to K(\mathcal{B})$ and hence the displayed long exact sequence. □

V.2.5.1 **Changing cofibrations 2.5.1.** (Hinich-Shektman). Let $\mathcal{A} = (\mathcal{A},\ co\mathcal{A},\ w\mathcal{A})$ be a saturated Waldhausen category with a cylinder functor, satisfying the cylinder axiom. Suppose that $co\mathcal{A} \subset co_1\mathcal{A} \subset \mathcal{A}$ is such that $\mathcal{A}_1 = (\mathcal{A}, co_1\mathcal{A}, w\mathcal{A})$ is also a Waldhausen category. Then $K(\mathcal{A}) \simeq K(\mathcal{A}_1)$, by Waldhausen Approximation.

V.2.6 **2.6.** Combining Theorem 2.5 with the Gillet-Waldhausen Theorem 2.2 yields several useful localization sequences.

V.2.6.1 ***G*-theory Localization for rings 2.6.1.** Localization at a central multiplicatively closed set S in a ring R induces an exact functor $\mathbf{M}(R) \to \mathbf{M}(S^{-1}R)$ satisfying (App). Passing to $\mathbf{Ch}^b\mathbf{M}(R) \to \mathbf{Ch}^b\mathbf{M}(S^{-1}R)$ does not change the K-theory (by 2.2) but does add a cylinder functor, so (App) still holds (see II, Ex. 9.2). Hence Theorem 2.5 applies, with $\mathcal{A}^w$ being the category $\mathbf{Ch}^b_S\mathbf{M}(R)$ of complexes E such that $S^{-1}E$ is exact.

We define $G(R \text{ on } S)$ to be $K\mathbf{Ch}^b_S\mathbf{M}(R)$, and $G_n(R \text{ on } S) = K_n\mathbf{Ch}^b_S\mathbf{M}(R)$, so that we get a homotopy fibration $G(R \text{ on } S) \to G(R) \to G(S^{-1}R)$, and a long exact sequence

$$\cdots \to G_{n+1}(S^{-1}R) \to G_n(R \text{ on } S) \to G_n(R) \to G_n(S^{-1}R) \to \cdots$$

ending in the surjection $G_0(R) \to G_0(S^{-1}R)$ of II.6.4.1. When R is noetherian, we will identify $G(R \text{ on } S) = K\mathbf{Ch}^b_S\mathbf{M}(R)$ with $K\mathbf{M}_S(R)$ in 6.1 below.

V.2.6.2 ***G*-theory Localization for schemes 2.6.2.** If Z is a closed subscheme of a noetherian scheme X, we define $G(X \text{ on } Z)$ to be $K\mathbf{Ch}^b_Z\mathbf{M}(X)$, where $\mathbf{Ch}^b_Z\mathbf{M}(X)$ is the (Waldhausen) category of bounded complexes which are acyclic on $X - Z$.

Now $G(X) = K\mathbf{M}(X)$ by IV.6.3.4, and the localization $\mathbf{M}(X) \to \mathbf{M}(X - Z)$ satisfies (App). Since $\mathcal{A}^w = \mathbf{Ch}^b_Z\mathbf{M}(X)$, Theorems 2.5 and 2.2 yield a homotopy fibration $G(X \text{ on } Z) \to G(X) \to G(X - Z)$ and a long exact sequence

$$\cdots \to G_{n+1}(X - Z) \to G_n(X \text{ on } Z) \to G_n(X) \to G_n(X - Z) \to \cdots$$

ending in the surjection $G_0(X) \to G_0(X-Z)$ of II.6.4.2. Later on (in 3.10.2, 6.11 and Ex. 4.3), we will identify $G(X \text{ on } Z)$ with $K\mathbf{M}_Z(X)$ and $G(Z)$.

Let S be a central multiplicatively closed set of central elements in a noetherian ring R. If S consists of nonzerodivisors, we will see in Theorem 7.1 that the analogue of $K\mathbf{M}_S(R)$ for projective modules is the K-theory of the category $\mathbf{H}_S(R)$ of S-torsion perfect modules (generalizing II.7.7.4). Otherwise, this is not correct; see Exercises 2.9 and 7.3 below. Instead, as in II.9.8, we define $K(R \text{ on } S)$ to be the K-theory of $\mathbf{Ch}^b_S\mathbf{P}(R)$, the Waldhausen category of bounded complexes P of finitely generated projective modules such that $S^{-1}P$ is exact.

V.2.6.3 **Theorem 2.6.3.** *If S is a central multiplicatively closed set in a ring R, there is a homotopy fibration $K(R \text{ on } S) \to K(R) \to K(S^{-1}R)$, and hence a long exact sequence*

$$\cdots K_{n+1}(S^{-1}R) \to K_n(R \text{ on } S) \to K_n(R) \to K_n(S^{-1}R)\cdots$$

ending in the exact sequence $K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$ of II.9.8.

Proof. As in the proof of II.9.8, we consider the category $\mathcal{P}$ of $S^{-1}R$-modules of the form $S^{-1}P$ for P in $\mathbf{P}(R)$. We saw in II.9.8.1 that (by clearing denominators in the maps), the localization from $\mathcal{A} = \mathbf{Ch}^b(\mathbf{P}(R))$ to $\mathcal{B} = \mathbf{Ch}^b(\mathcal{P})$ satisfies (App), so Theorem 2.5 applies with $\mathcal{A}^w = \mathbf{Ch}^b_S\mathbf{P}(R)$. Thus we have a homotopy fibration $K\mathbf{Ch}^b_S\mathbf{P}(R) \to K(R) \to K(\mathcal{P})$. By Cofinality (IV.6.4.1), $K(\mathcal{P}) \to K(S^{-1}R) \to G$ is a homotopy fibration, and the result follows. □

We conclude with a few useful models for K-theory, arising from the Waldhausen Approximation Theorem 2.4.

V.2.7.1 **Homologically bounded complexes 2.7.1.** If $\mathcal{A}$ is an abelian category, let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen category of homologically bounded chain complexes of objects in $\mathcal{A}$, and $\mathbf{Ch}^{hb}_{\pm}(\mathcal{A})$ the subcategory of bounded below (resp., bounded above) complexes. We saw in II.9.7.4 that $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^{hb}_-(\mathcal{A})$ and $\mathbf{Ch}^{hb}_+(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ satisfy (App), by good truncation; dually, $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^{hb}_+(\mathcal{A})$ and $\mathbf{Ch}^{hb}_+(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ satisfy the dual of (App). By Waldhausen Approximation (and 2.2), this yields

$$K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}_-(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}_+(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}(\mathcal{A}).$$

We gave the K_0 version of the resulting isomorphism $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^b(\mathcal{A}) \cong K_n\mathbf{Ch}^{hb}(\mathcal{A})$ in II.9.7.4. We will see another argument for this in 3.8.1 below.

V.2.7.2 **Perfect complexes 2.7.2.** A *perfect complex* of R-modules is a complex M which is quasi-isomorphic to a bounded complex of finitely generated projective R-modules, *i.e.,* to a complex in $\mathbf{Ch}^b(\mathbf{P}(R))$. We saw in II.9.7.5 that the perfect complexes of R-modules form a Waldhausen subcategory $\mathbf{Ch}_{\text{perf}}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$, and that (App) holds for the inclusions $\mathbf{Ch}^b(\mathbf{P}(R)) \subset \mathbf{Ch}^-_{\text{perf}}(R) \subset \mathbf{Ch}_{\text{perf}}(R)$. Thus (invoking Theorems 2.2 and 2.4) we have that

$$K(R) \simeq K\mathbf{Ch}^b(\mathbf{P}(R)) \simeq K\mathbf{Ch}^-_{\text{perf}}(R) \simeq K\mathbf{Ch}_{\text{perf}}(R).$$

If S is a central multiplicatively closed set in R, then $K(R \text{ on } S) = K\mathbf{Ch}^b_S\mathbf{P}(R)$ is also the K-theory of the category $\mathbf{Ch}_{\text{perf},S}(R)$ of perfect complexes P with $S^{-1}P$ exact. This follows from Waldhausen Approximation; the Approximation Property for the inclusion $\mathbf{Ch}^b_S\mathbf{P}(R) \subset \mathbf{Ch}_{\text{perf},S}(R)$ was established in II, Ex. 9.2.

V.2.7.3 **K-theory of schemes 2.7.3.** If X is any scheme, we define $K(X)$ to be $K\mathbf{Ch}_{\text{perf}}(X)$. Thus our $K_0(X)$ is the group $K_0^{der}(X)$ of II, Ex. 9.10.

When X is a quasi-projective scheme over a commutative ring, we defined $K(X) = K\mathbf{VB}(X)$ in IV.6.3.4. These definitions agree; in fact they also agree when X is a separated regular noetherian scheme (II.8.2), or more generally a (quasi-compact, quasi-separated) scheme such that every coherent sheaf is a quotient of a vector bundle. Indeed, $K\mathbf{VB}(X) \simeq K\mathbf{Ch}_{\text{perf}}(X)$, by Waldhausen Approximation applied to $\mathbf{Ch}^b(\mathbf{VB}(X)) \subset \mathbf{Ch}_{\text{perf}}(X)$. The condition (App) is given in [SGA6, II] or [200, 2.3.1]: given a map $P \to C$ from a bounded complex of vector bundles to a perfect complex, there is a factorization $P \to Q \xrightarrow{\sim} C$ in these settings.

V.2.7.4 **$G(R)$ and Pseudo-coherent complexes 2.7.4.** If R is a noetherian ring, the discussion of 2.7.1 applies to the abelian category $\mathbf{M}(R)$ of finitely generated R-modules. Thus we have:

$$G(R) = K\mathbf{M}(R) \simeq K\mathbf{Ch}^b\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}_+\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}\mathbf{M}(R).$$

Instead of $\mathbf{Ch}^{hb}_+\mathbf{M}(R)$, we could consider the (Waldhausen) category $\mathbf{Ch}^{hb}_+\mathbf{P}(R)$ of bounded below, homologically bounded chain complexes of finitely generated projective modules, or even the category $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$ of homologically bounded *pseudo-coherent* complexes (R-module complexes which are quasi-isomorphic to a bounded complex of finitely generated modules; see II.9.7.6). By II, Ex. 9.7, Waldhausen Approximation applies to the inclusions of $\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_+\mathbf{P}(R)$ in $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$. Hence we also have $G(R) \simeq K\mathbf{Ch}^{hb}_+\mathbf{P}(R) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$.

If R is not noetherian, we can consider the exact category $\mathbf{M}(R)$ of pseudo-coherent modules (II.7.1.4), which we saw is closed under kernels of surjections, and the Waldhausen category $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$ of homologically bounded pseudo-coherent complexes (II.9.7.6). Since (App) holds by Ex. II.9.7, the same proof gives:

$$G(R) = K\mathbf{M}(R) \simeq K\mathbf{Ch}^b\mathbf{M}(R) \simeq K\mathbf{Ch}^{hb}_+\mathbf{P}(R) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(R).$$

Now suppose that S is a multiplicatively closed set of central elements in R. Anticipating Theorem 5.1 below, we consider the category $\mathbf{M}_S(R)$ of S-torsion modules in $\mathbf{M}(R)$. If R is noetherian, this is an abelian category by II.6.2.8; if not, it is the exact category of pseudo-coherent S-torsion modules (II.7.1.4). By Theorem 2.2, $K\mathbf{M}_S(R) \simeq K\mathbf{Ch}^b\mathbf{M}_S(R)$.

EXERCISES

EV.2.1 **2.1.** If an exact functor $F : \mathcal{A} \to \mathcal{B}$ satisfies the approximate lifting property (App), show (by induction on n) that each $S_nF : S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfies (App).

EV.2.2 **2.2.** If $\mathcal{A}$ is a strictly cofinal exact subcategory of $\mathcal{A}'$, show that $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A}')$ satisfies (App), and that $K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}')$.

EV.2.3 **2.3.** Let split$\mathbf{Ch}^b(\mathcal{A})$ denote the category $\mathbf{Ch}^b(\mathcal{A})$, made into a Waldhausen category by restricting the cofibrations to be the degreewise split monomorphisms whose quotients lie in $\mathcal{A}$ (a priori they lie in $\hat{\mathcal{A}}$; see II.7.3). Generalize II.9.2.4 by showing that split$\mathbf{Ch}^b(\mathcal{A}) \to \mathbf{Ch}^b(\mathcal{A})$ induces a homotopy equivalence on K-theory, so that $K_n(\text{split}\mathbf{Ch}^b(\mathcal{A})) \cong K_n(\mathcal{A})$ for all n.

EV.2.4 **2.4.** If $\mathcal{A}$ is an exact subcategory of an abelian category $\mathcal{M}$, the Waldhausen category $\mathcal{A}^{[0,n]}_{\text{exact}}$ of admissibly exact complexes of length n (Example 1.4) is contained in the category $\mathbf{Ch}^{[0,n]}(\mathcal{A})^{qiso}$ of complexes in $\mathcal{A}$ which are acyclic as complexes in $\mathcal{M}$. If $\mathcal{A}$ is closed under kernels of surjections in $\mathcal{M}$, show that these categories are the same.

EV.2.5 **2.5.** Consider the exact category $\mathbb{F}(R)$ of finite free R-modules (II.5.4.1). Analyze Remark 2.2.1 to show that $K\mathbf{F}(R) \simeq K\mathbf{Ch}^b(\mathbf{F}(R))$. If S is a central multiplicative set in R, compare $K\mathbf{Ch}^b_S(\mathbf{F}(R))$ to $K(R \text{ on } S)$. Is $\mathbf{Ch}^b_S(\mathbf{F}(R))$ cofinal in $\mathbf{Ch}^b_S(\mathbf{P}(R))$?

EV.2.6 **2.6.** Let S be a central multiplicative set in a ring R. Mimick the proof of 2.7.4 to show that $K\mathbf{Ch}^b_S\mathbf{M}(R)$ is equivalent to $K\mathbf{Ch}^+_{\text{pcoh},S}(R)$ and $K\mathbf{Ch}^{hb}_{\text{pcoh},S}(R)$. A fancier proof of this equivalence will be given in 3.10.1 below.

EV.2.7 **2.7.** Let X be a noetherian scheme. Show that $G(X) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(X)$, generalizing II, Ex. 9.8. *Hint:* Mimick the proof of II, Ex. 9.7. A fancier proof of this equivalence will be given in 3.10.2 below.

EV.2.8 **2.8.** Let X be a noetherian scheme, and $\mathcal{F}$ the Waldhausen category of bounded above perfect cochain complexes of flat $\mathcal{O}_X$-modules. Show that $\mathbf{VB}(X) \subset \mathcal{F}$ induces an equivalence $K(X) \simeq K(\mathcal{F})$.

EV.2.9 **2.9.** Let $R = k[s,t]/(st)$ and $S = \{s^n\}$, where k is a field, so that $S^{-1}R = k[s, 1/s]$. Show that every R-module M with $s^nM = 0$ for some n has infinite projective dimension, so that the category $\mathbf{H}_S(R)$ consists only of 0. Then use the Mayer-Vietoris sequence III.2.6 to show that $K_0(R \text{ on } S) = \mathbb{Z}$. Conclude that $K\mathbf{H}_S(R)$ is not the homotopy fiber of $K(R) \to K(S^{-1}R)$.

3 The Resolution Theorems and transfer maps

In this section we establish the Resolution Theorems for exact categories (3.1) and Waldhausen categories of chain complexes (3.9). We first give the version for exact categories, and some of its important applications. The second Resolution Theorem 3.9 requires the properties of derived categories which are listed in 3.8.

The Fundamental Theorem 3.3–3.4 that $K_* \cong G_*$ for regular rings and schemes (proven for K_0 in II, 7.8 and 8.2), and the existence of transfer maps f_* (3.3.2, 3.5 and 3.7), are immediate consequences of the first Resolution Theorem, as applied to $\mathbf{P}(R) \subseteq \mathbf{H}(R)$ and $\mathbf{VB}(X) \subseteq \mathbf{H}(X)$.

Recall from II.7.0.1 that $\mathcal{P}$ is said to be *closed under kernels of admissible surjections* in an exact category $\mathcal{H}$ if whenever $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{H}$ is an exact sequence with B, C in $\mathcal{P}$ then A is also in $\mathcal{P}$. (A prototype is $\mathcal{P} = \mathbf{H}_n(R)$, $\mathcal{H} \subseteq \mathbf{mod}\text{-}R$.)

V.3.1 **Resolution Theorem 3.1.** *Let $\mathcal{P}$ be a full exact subcategory of an exact category $\mathcal{H}$, such that $\mathcal{P}$ is closed under extensions and under kernels of admissible surjections in $\mathcal{H}$. Suppose in addition that every object M of $\mathcal{H}$ has a finite $\mathcal{P}$-resolution:*

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Then $K(\mathcal{P}) \simeq K(\mathcal{H})$, and thus $K_i(\mathcal{P}) \cong K_i(\mathcal{H})$ for all i.

The proof will reduce the theorem to the special case in which objects of $\mathcal{H}$ have a $\mathcal{P}$-resolution of length one, which will be handled in Proposition 3.1.1.

Proof. The category $\mathcal{H}$ is the union of the subcategories $\mathcal{H}_n$ of objects with resolutions of length at most n, and $\mathcal{H}_0 = \mathcal{P}$. Since the kernel of any admissible $P \twoheadrightarrow P'$ is also in $\mathcal{P}$, $\mathcal{H}_{n-1} \subseteq \mathcal{H}_n$ is closed under admissible subobjects and extensions (see Ex. 3.1). Applying 3.1.1, we see that each $K(\mathcal{P}) \to K(\mathcal{H}_{n-1}) \to K(\mathcal{H}_n)$ is a homotopy equivalence. Taking the colimit over n yields the result. □

V.3.1.1 **Proposition 3.1.1.** *Let $\mathcal{P} \subset \mathcal{H}$ be as in Theorem 3.1, and suppose that every M in $\mathcal{H}$ fits into an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ with the P_i in $\mathcal{P}$. Then $K(\mathcal{P}) \to K(\mathcal{H})$ is a homotopy equivalence, and $K_*(\mathcal{P}) \cong K_*(\mathcal{H})$.*

Proof. (Quillen) The inclusion $Q\mathcal{P} \subset Q\mathcal{H}$ is not full, so we consider the full subcategory $\mathcal{Q}$ on the objects of $Q\mathcal{P}$, and write i for the inclusion $Q\mathcal{P} \subset \mathcal{Q}$. For each P in $Q\mathcal{P}$, the objects of the comma category i/P are pairs (P_2, u) with u of the form $P_2 \twoheadleftarrow P_1 \rightarrowtail P$ and P/P_1 in $\mathcal{H}$. Set $z(P_2, u) = (P_1, P_1 \rightarrowtail P)$ and note that $P_2 \twoheadleftarrow P_1$ and $0 \rightarrowtail P_1$ are morphisms of $Q\mathcal{P}$. They define natural transformations $(P_2, u) \to z(P_2, u) \leftarrow (0, 0 \rightarrowtail P)$ in i/P. This shows that i/P is contractible, and hence by Theorem A (IV.3.7) that $i : Q\mathcal{P} \to \mathcal{Q}$ is a homotopy equivalence.

It now suffices to show that the inclusion $j : \mathcal{Q} \to Q\mathcal{H}$ is a homotopy equivalence. We shall resort to the dual of Theorem A, so we need to show that for each M in $\mathcal{H}$, the comma category $M\backslash j$ is contractible. The objects of $M\backslash j$ are pairs $(P, u : M \twoheadleftarrow P_1 \rightarrowtail P)$ with P in $\mathcal{P}$; it is nonempty by the assumption that some $P_0 \twoheadrightarrow M$ exists. Let $\mathcal{C}$ denote the full subcategory on the pairs $(P, M \twoheadleftarrow P)$. The inclusion $\mathcal{C} \subset (M\backslash j)$ is a homotopy equivalence because it has a right adjoint, namely $r(P, u) = (P_1, M \twoheadleftarrow P_1)$. And the category $\mathcal{C}$ is contractible because if we fix any $(P_0, u_0 : M \twoheadleftarrow P_0)$, then $p(P, u) = (P \times_M P_0, M \twoheadleftarrow P \times_M P_0)$ is in $\mathcal{C}$ (because $\mathcal{P}$ is subobject-closed) and there are natural transformations $(P, u) \leftarrow p(P, u) \rightarrow (P_0, u_0)$. □

V.3.1.2 **Remark 3.1.2.** It is not known how to generalize the Resolution Theorem to Waldhausen categories. Other proofs of the Resolution Theorem for exact categories, using Waldhausen K-theory, have been given in Gra87 [76] and Staf [175].

Here is the main application of the Resolution Theorem. It is just the special case in which $\mathcal{P} = \mathbf{P}(R)$ and $\mathcal{H} = \mathbf{H}(R)$.

V.3.2 **Theorem 3.2.** *For every ring R, the inclusion of $\mathbf{P}(R)$ in $\mathbf{H}(R)$ induces an equivalence $K(R) = K\mathbf{P}(R) \simeq K\mathbf{H}(R)$, so $K_*(R) = K_*\mathbf{P}(R) \cong K_*\mathbf{H}(R)$.*

If S is a multiplicatively closed set of central nonzero-divisors of R, we introduced the categories $\mathbf{H}_S(R)$ and $\mathbf{H}_{1,S}(R)$ in II.II.7.7.3 7.7.3. The proof there using Resolution applies verbatim to yield:

V.3.2.1 **Corollary 3.2.1.** *$K\mathbf{H}_{1,S}(R) \simeq K\mathbf{H}_S(R)$, and $K_*\mathbf{H}_{1,S}(R) \cong K_*\mathbf{H}_S(R)$.*

By definition, a ring R is *regular* if every R-module has a finite projective resolution, *i.e.,* finite projective dimension (see I.I.3.7.1 3.7.1.). We say R is *coherent* if the category $\mathbf{M}(R)$ of pseudo-coherent R-modules (II.II.7.1.4 7.1.4) is abelian

V.3.3 **Theorem 3.3.** *[Fundamental Theorem] If R is a noetherian (or coherent) regular ring, $K(R) \simeq G(R)$. Thus for every n we have $K_n(R) \cong G_n(R)$.*

Proof. In either case, $\mathbf{H}(R)$ is the category $\mathbf{M}(R)$. The Resolution Theorem V.3.2 3.2 gives the identification. □

V.3.3.1 **Corollary 3.3.1.** *If $f : R \to S$ is a homomorphism, R is regular noetherian and S is finite as an R-module, then there is a transfer map $f_* : K_*(S) \to K_*(R)$, defined by the G-theory transfer map (IV.IV.6.3.3 6.3.3):*

$$K(S) \to G(S) = K\mathbf{M}(S) \xrightarrow{f_*} K\mathbf{M}(R) = G(R) \simeq K(R).$$

V.3.3.2 **Transfer Maps for $K_*(R)$ 3.3.2.** Let $f \colon R \to S$ be a ring homomorphism such that S has a finite R-module resolution by finitely generated projective R-modules. Then the restriction of scalars defines a functor $\mathbf{P}(S) \to \mathbf{H}(R)$. By V.3.2 3.2, we obtain a transfer map $f_* \colon K(S) \to K\mathbf{H}(R) \simeq K(R)$, and hence maps $f_* : K_n(S) \to K_n(R)$. If S is projective as an R-module, f_* is the transfer map of IV.IV.6.3.2 6.3.2.

The *projection formula* states that f_* is a $K_*(R)$-module homomorphism when R is commutative. That is, if $x \in K_*(S)$ and $y \in K_*(R)$ then $f_*(x \cdot$

$f^*(y)) = f_*(x) \cdot y$ in $K_*(R)$. To see this, we note that the biexact functor $\mathbf{H}(S) \times \mathbf{P}(R) \to \mathbf{H}(R)$, $(M, P) \mapsto M \otimes_R P$, produces a pairing $K\mathbf{H}(S) \wedge K(R) \to K\mathbf{H}(R)$ representing the right side via Theorem 3.2. Since $M \otimes_R P \cong M \otimes_S (S \otimes_R P)$, it is naturally homotopic to the pairing representing the left side.

We have already seen special cases of the transfer map f_*. It was defined for K_0 in II.7.9, and for K_1 in a special case in III.1.7 and III, Ex. 1.11. If S is projective as an R-module then f_* was also constructed for K_2 in III.5.6.3, and for all K_n in IV.1.1.3.

Recall from II.8.2 that a separated noetherian scheme X is *regular* if every coherent $\mathcal{O}_X$-module $\mathcal{F}$ has a finite resolution by vector bundles; see [SGA6, II, 2.2.3 and 2.2.7.1] or II.8.2–8.3. The Resolution Theorem applies to $\mathbf{VB}(X) \subset \mathbf{M}(X)$, and we have:

V.3.4 **Theorem 3.4.** *If X is a separated regular noetherian scheme, then $K(X) = K\mathbf{VB}(X)$ satisfies:*

$$K(X) \simeq G(X) \quad \textit{and} \quad K_*(X) \cong G_*(X).$$

V.3.4.1 **Variant 3.4.1.** If X is quasi-projective (over a commutative ring), we defined $K(X)$ to be $K\mathbf{VB}(X)$ in IV.6.3.4. We saw in II.8.3.1 that the Resolution Theorem applies to $\mathbf{VB}(X) \subset \mathbf{H}(X)$ so we have $K(X) \simeq K\mathbf{H}(X)$.

V.3.4.2 **Remark 3.4.2.** Theorem 3.4 does not hold for non-separated regular noetherian schemes. This is illustrated when X is the affine line with a double origin over a field, since (as we saw in II, 8.2.4 and Ex. 9.10) $G_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ but $K_0\mathbf{VB}(X) = \mathbb{Z}$. The analogue of Theorem 3.4 for quasi-compact regular schemes is given in Exercise 3.9.

V.3.5 **Base change maps for $G_*(R)$ 3.5.** Let $f : R \to S$ be a homomorphism of noetherian rings such that S has finite flat dimension $\mathrm{fd}_R\, S$ as a right R-module. Let $\mathcal{F} \subset \mathbf{M}(R)$ be the full subcategory of all R-modules M which are Tor-independent of S in the sense that

$$\mathrm{Tor}_n^R(S, M) = 0 \qquad \text{for } n \neq 0.$$

As observed in II.7.9, the usual properties of Tor show that every finitely generated R-module M has a finite resolution by objects of $\mathcal{F}$, that $\mathcal{F}$ is an exact subcategory closed under kernels, and that $M \mapsto M \otimes_R S$ is an exact functor from $\mathcal{F}$ to $\mathbf{M}(S)$. By the Resolution Theorem 3.1, there is a natural map

$$G(R) \xleftarrow{\simeq} K(\mathcal{F}) \to G(S),$$

giving maps $f^* : G_*(R) \to G_*(S)$. If $g : S \to T$ is another map, and T has finite flat dimension over S, then the natural isomorphism $(M \otimes_R S) \otimes_S T \cong M \otimes_R T$ shows that $g^* f^* \simeq (gf)^*$.

Note that if the ring S is finite over R then the forgetful functor $\mathbf{M}(S) \to \mathbf{M}(R)$ is exact and induces a contravariant "finite transfer" map $f_* : G(S) \to G(R)$ (see II.6.2 and IV.6.3.3). The seemingly strange notation (f^* and f_*) is chosen with an eye towards schemes: if $Y = \operatorname{Spec}(R)$ and $X = \operatorname{Spec}(S)$ then f maps X to Y, so f^* is contravariant and f_* is covariant as functors on schemes.

V.3.5.1 **Example 3.5.1.** Let $i : R \to R[s]$ be the inclusion and $f : R[s] \to R$ the map $f(s) = 0$. Since i is flat, we have the flat base change $i^* : G(R) \to G(R[s])$. Now f has finite flat dimension, since $\operatorname{Tor}_n^{R[s]}(R, -) = 0$ for $n \geq 2$, so we also have a base change map $f^* : G(R[s]) \to G(R)$ by 3.5. Since fi is the identity on R, the composite $f^*i^* : G(R) \to G(R[s]) \to G(R)$ is homotopic to the identity map. The Fundamental Theorem for $G(R)$ (6.2 below) will show that these are inverse homotopy equivalences.

In contrast, the transfer maps $K(R) \xrightarrow{f_*} K(R[s])$ and $G(R) \xrightarrow{f_*} G(R[s])$ are zero. This follows from the Additivity Theorem applied to the sequence of functors $i^* \rightarrowtail i^* \twoheadrightarrow f_*$ sending an R-module M to

$$0 \to M[s] \xrightarrow{s} M[s] \to M \to 0.$$

V.3.5.2 **Example 3.5.2.** Suppose that $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is a graded noetherian ring, and let $\mathbf{M}_{gr}(S)$ be the abelian category of finitely generated graded S-modules. Its K-groups are naturally modules over $\mathbb{Z}[\sigma, \sigma^{-1}]$, where σ acts by the shift automorphism $\sigma(M) = M(-1)$ of graded modules. (See Exercises II.6.12 and II.7.14.)

Now assume that S is flat over R, so that tensoring with S gives a functor from $\mathbf{M}(R)$ to $\mathbf{M}_{gr}(S)$, and hence a $\mathbb{Z}[\sigma, \sigma^{-1}]$-module map

$$\beta : G_i(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] \to K_i\mathbf{M}_{gr}(S).$$

In the special case $S = R$, $\mathbf{M}_{gr}(R)$ is just a coproduct of copies of $\mathbf{M}(R)$, and the map β is an isomorphism: $G_*(R)[\sigma, \sigma^{-1}] \cong K_*\mathbf{M}_{gr}(R)$. If R has finite flat dimension over S (via $S \to R$) then the Resolution Theorem 3.1, applied to the category $\mathcal{P}_{gr}$ of graded S-modules Tor-independent of R, induces a map

$$K_i\mathbf{M}_{gr}(S) \to K_i\mathbf{M}_{gr}(R) \cong G_i(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}]$$

which is a left inverse to β, because $\otimes_R$ sends $\mathbf{M}_{gr}(R)$ to $\mathcal{P}_{gr}$ and there is a natural isomorphism $(M \otimes_R S) \otimes_S R \cong M$. In fact, β is an isomorphism (see Ex. 3.3).

Similarly, if $\mathbf{M}_{gr,\geq 0}(S)$ is the subcategory of positively graded S-modules, there is a natural map $\beta : G_i(R) \otimes \mathbb{Z}[\sigma] \to K_i\mathbf{M}_{gr,\geq 0}(S)$. If R has finite flat dimension over S, it is an isomorphism (see Ex. 3.3).

V.3.5.3 **Example 3.5.3** (Projection Formula)**.** Let $f : R \to S$ be a homomorphism of commutative noetherian rings such that S is a finitely generated right R-module of finite projective dimension. Then $G_*(S)$ and $G_*(R)$ are $K_*(R)$-modules by IV.6.6.5. The *projection formula* states that $f_*(x \cdot f^*y) = f_*(x) \cdot y$ in $G_*(R)$, provided that either (i) $x \in G_*(S)$ and $y \in K_*(R)$ or (ii) $x \in K_*(S)$ and $y \in G_*(R)$.

For (i), observe that the functor $\otimes_R : \mathbf{M}(S) \times \mathbf{P}(R) \to \mathbf{M}(S) \to \mathbf{M}(R)$ is biexact, so it induces a pairing $G(S) \wedge K(R) \to G(S) \to G(R)$ representing the right side $f_*(x) \cdot y$. Since $M \otimes_R P \cong M \otimes_S (S \otimes_R P)$, it also represents the left side.

For (ii), let $\mathcal{F} \subset \mathbf{M}(R)$ be as in 3.5 and observe that the functor $\mathbf{P}(S) \times \mathcal{F} \to \mathbf{M}(S) \to \mathbf{M}(R)$, $(P, M) \mapsto P \otimes_R M$ is biexact. Hence it produces a pairing $K(S) \wedge K(\mathcal{F}) \to G(S) \to G(R)$, representing the left side $f_*(x \cdot f^* y)$ of the projection formula. But this pairing also factors through $\mathbf{P}(S) \to \mathbf{H}(R)$ followed by the tensor product pairing $\mathbf{H}(R) \times \mathcal{F} \to \mathbf{M}(R)$ representing the right side.

V.3.6 **Base change maps for $G_*(X)$ 3.6.** If $f : X \to Y$ is a morphism of noetherian schemes such that $\mathcal{O}_X$ has finite flat dimension over $f^{-1}\mathcal{O}_Y$, there is also a contravariant map f^* from $G(Y)$ to $G(X)$. This is because every coherent $\mathcal{O}_Y$-module has a finite resolution by coherent modules which are Tor-independent of $f_*\mathcal{O}_X$, locally on X, and f^* is an exact functor on the category $\mathbf{L}(f)$ of these modules. If $g : W \to X$ is another map of finite flat dimension, then $g^* f^* \simeq (fg)^*$ by the natural isomorphism $g^*(f^*\mathcal{F}) \cong (fg)^*\mathcal{F}$.

V.3.6.1 **Example 3.6.1.** If X is a noetherian scheme we can consider the flat structure map $p : X[s] \to X$ and the zero-section $f : X \to X[s]$, where $X[s] = X \times \operatorname{Spec}(\mathbb{Z}[s])$ as in II.6.5.1. As in 3.5.1, f has finite flat dimension and pf is the identity on X, so $f^* : G(X[s]) \to G(X)$ is defined and the composition $f^* p^*$ is homotopic to the identity on $G(X)$. The Fundamental Theorem 6.13 will show that $p^* : G(X) \simeq G(X[s])$ is a homotopy equivalence.

In contrast the finite transfer map $f_* : G(X) \to G(X[s])$ is zero. This follows from the Additivity Theorem applied to the sequence of functors $p^* \rightarrowtail p^* \twoheadrightarrow f_*$ from $\mathbf{M}(X)$ to $\mathbf{M}(X[s])$, analogous to the one in 3.5.1.

V.3.7 **Proposition 3.7.** *If $f : X \to Y$ is a proper morphism of noetherian schemes, there is a "proper transfer" map $f_* : G(X) \to G(Y)$. This induces homomorphisms $f_* : G_n(X) \to G_n(Y)$ for each n. The transfer map makes $G_n(X)$ functorial for proper maps.*

The G_0 version of Proposition 3.7, $f_*([\mathcal{F}]) = \sum(-1)^i [R^i f_*(\mathcal{F})]$, is given in II.6.2.6.

Proof. By Serre's "Theorem B" (see II.6.2.6), the higher direct images $R^i f_*(\mathcal{F})$ of a coherent module are coherent, and vanish for i large. They are obtained by replacing $\mathcal{F}$ by a flasque resolution, applying f_* and taking cohomology. The map $f_* : K\mathbf{M}(X) \to K\mathbf{M}(Y)$ exists by Ex. 3.2. □

V.3.7.1 **Proposition 3.7.1.** *If $f : X \to Y$ is a proper morphism of finite flat dimension. Then there is a "transfer" map $f_* : K(X) \to K(Y)$. This induces homomorphisms $f_* : K_n(X) \to K_n(Y)$ for each n. The transfer map makes $K_n(X)$ functorial for projective maps.*

Proof. As in II.8.4, let $\mathbf{P}(f)$ be the category of vector bundles E on X such that $R^i f_*(E) = 0$ for $i > 0$. We saw in *loc. cit.* that $f_* : \mathbf{P}(f) \to \mathbf{H}(Y)$ is an exact functor, *i.e.,* that the $\mathcal{O}_Y$-module $f_*(E)$ is perfect (II, Ex. 9.10). By Ex. 3.6(c) and left exactness of f_*, the hypotheses of the Resolution Theorem 3.1 are satisfied, so we have $K(X) \simeq K\mathbf{P}(f)$. Thus we can define the transfer map to be the composite

$$K(X) \simeq K\mathbf{P}(f) \xrightarrow{f_*} K\mathbf{H}(Y) \simeq K\mathbf{VB}(Y) = K(Y).$$

Given a second map $g : Y \to Z$ of finite flat dimension, we can replace $\mathbf{P}(f)$ by $\mathcal{P} = \mathbf{P}(f \times gf)$, so that E in $\mathcal{P}$ satisfy $R^i g_*(f_*(E)) = R^i(gf)_*(E) = 0$ for $i > 0$. Thus $f_*(\mathcal{P})$ lies in the subcategory $\mathbf{H}(g)$ of perfect g_*-acyclic modules, on which g_* is exact, and $g_* f_* \simeq (gf)_*$ because of the natural isomorphism $(gf)_*(E) \cong g_* f_*(E)$ for E in $\mathcal{P}$. Functoriality is now straightforward (Ex. 3.5). □

V.3.7.2 **Base change Theorem 3.7.2.** *Let $f : X \to Y$ be a proper morphism of quasi-projective schemes and $g : Y' \to Y$ a morphism of finite flat dimension,* Tor-*independent of X, and set $X' = X \times_Y Y'$ so there is a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then $g^* f_* \simeq f'_* g'^*$ as maps $G(X) \to G(Y')$.

If in addition f has finite flat dimension, so that $f_* : K(X) \to K(Y)$ is defined then $g^* f_* \simeq f'_* g'^*$ as maps $K(X) \to K(Y')$.

The idea of the proof is to use the following base change formula of [SGA6, IV.3.1.0]: if E is homologically bounded with quasi-coherent cohomology, then

$$Lg^*(Rf_* E) \xrightarrow{\sim} Rf'_* L(g')^* E.$$

Proof. Let $\mathcal{A}$ be the category of $\mathcal{O}_X$-modules which are f_*-acyclic and Tor-independent of $\mathcal{O}_{X'}$. For E in $\mathcal{A}$, the base change formula implies that for all $i \in \mathbb{Z}$:

$$\mathrm{Tor}_i(f_* E, \mathcal{O}_{Y'}) = L_i g^*(f_* E) = R^{-i} f'_*(g'^* E).$$

These groups must vanish unless $i = 0$, because Tor_i and $R^i f'_*$ vanish for $i < 0$. That is, $f_*(E)$ is Tor-independent of $\mathcal{O}_{Y'}$, and $(g')^* E$ is f'_*-acyclic, and we have $g^* f_*(E) \cong f'_*(g')^* E$. Therefore $g^* f_* = f'_*(g')^*$ as exact functors on $\mathcal{A}$. It remains to apply the Resolution Theorem twice to show that $\mathcal{A} \subset \mathbf{L}(f) \subset \mathbf{M}(Y)$ induce equivalences on K-theory. The second was observed in 3.6, and the first follows from Ex. 3.6(e).

The proof is easier for $K(X) \to K(Y')$ when f has finite flat dimension, using the category $\mathbf{P}(f)$; $f_* : \mathbf{P}(f) \to \mathbf{H}(Y)$ and $(g')^* : \mathbf{P}(f) \to \mathbf{VB}(X')$ are exact, and we saw in the proof of 3.7.1 that $K\mathbf{P}(f) \simeq K(X)$. □

V.3.7.3 **Corollary 3.7.3.** *(Projection formula) If $f : X \to Y$ is a projective map of finite flat dimension, then for $x \in K_0(X)$ and $y \in G_n(Y)$ we have*

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \textit{in } G_n(Y).$$

The G_0 version of this projection formula was given in Ex. II.8.3(b). (Cf. Ex. 3.10.) We will generalize the projection formula to higher K-theory in 3.12 below.

Proof. As in the proof of 3.7.1, let $\mathbf{P}(f)$ be the category of f_*-acyclic vector bundles E on X. By Ex. 3.6(c) and left exactness of f_*, the hypotheses of the Resolution Theorem 3.1 are satisfied and we have $K_*(X) \cong K_*\mathbf{P}(f)$. Thus it suffices to show that the projection formula holds when $x = [E]$ for E in $\mathbf{P}(f)$. Let $\mathbf{L}_E$ denote the full subcategory of $\mathbf{M}(Y)$ consisting of modules which are Tor-independent of f_*E and $\mathcal{O}_X$. By the Resolution Theorem, $K(\mathbf{L}_E) \simeq G(Y)$. The functor $\mathbf{L}_E \to \mathbf{M}(Y)$ given by $F \mapsto f_*E \otimes F$ is exact, and induces $y \mapsto f_*(x) \cdot y$.

Similarly, the exact functors $\mathbf{L}_E \to \mathbf{M}(X)$, sending F to $f^*(F)$ and $E \otimes f^*(F)$, induce $y \mapsto f^*(y)$ and $y \mapsto x \cdot f^*(y)$, respectively. The projection formula of [SGA6, III.3.7] shows that $R^i f_*(E \otimes f^*F) = 0$ for $i > 0$ and that $f_*(E \otimes f^*F) \cong f_*(E) \otimes F$. Hence $F \mapsto f_*(E \otimes f^*F)$ is an exact functor $\mathbf{L}_E \to \mathbf{M}(Y)$, and the projection formula follows. □

Derived Approximation

The third fundamental result for Waldhausen categories is an Approximation Theorem for the K-theory of categories based upon chain complexes, and is proven using Waldhausen Approximation 2.4. Roughly speaking, it says that the K-theory of $\mathcal{C}$ only depends on the derived category of $\mathcal{C}$, defined as localization $w^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set w of quasi-isomorphisms in $\mathcal{C}$.

In order for the statement of this result to make more sense, $\mathcal{C}$ will be a subcategory of $\mathbf{Ch}(\mathcal{M})$ for some abelian category $\mathcal{M}$; recall that the derived category $\mathbf{D}(\mathcal{M})$ is the localization of the category $\mathbf{Ch}(\mathcal{M})$ at the family of quasi-isomorphisms. For basic facts about derived categories and triangulated categories, we refer the reader to the Appendix to Chapter II, and to the standard references [208], [86] and [223, 10].

V.3.8 **Triangulated and Localizing Categories 3.8.** Let $\mathcal{C}$ be a full additive subcategory of $\mathbf{Ch}(\mathcal{M})$ which is closed under all the shift operators $C \mapsto C[n]$ and mapping cones. The localization $w^{-1}\mathcal{C}$ of $\mathcal{C}$ is the category obtained from $\mathcal{C}$ by formally inverting the multiplicatively closed set $w = w(\mathcal{C})$ of all quasi-isomorphisms in $\mathcal{C}$. (The usual construction, detailed in (II.A.5), uses a calculus of fractions to compose maps.) It is a triangulated category by [223, 10.2.5].

Here are two key observations that make it possible for us to better understand these triangulated categories, and to even see that they exist. One is that chain homotopic maps in $\mathcal{C}$ are identified in $w^{-1}\mathcal{C}$; see II, Ex. A.5 or [223, 10.1.2]. Another is that if w is saturated then C is isomorphic to zero in $w^{-1}\mathcal{C}$ if and only if $0 \to C$ is in w; see II.A.3.2 or [223, 10.3.10].

We say that $\mathcal{C}$ is a *localizing subcategory* of $\mathbf{Ch}(\mathcal{M})$ if the natural map $w^{-1}\mathcal{C} \to w^{-1}\mathbf{Ch}(\mathcal{M}) = \mathbf{D}(\mathcal{M})$ is an embedding. This will be the case whenever the following condition holds: given any quasi-isomorphism $C \to B$ with C in $\mathcal{C}$, there is a quasi-isomorphism $B \to C'$ with C' in $\mathcal{C}$. (See II.A.3 or [223, 10.3.13].)

For example, if $\mathcal{B}$ is a Serre subcategory of $\mathcal{M}$, the category $\mathbf{Ch}_{\mathcal{B}}(\mathcal{M})$ of complexes in $\mathbf{Ch}(\mathcal{M})$ with homology in $\mathcal{B}$ is localizing ([223, 10.4.3]), so $\mathbf{D}_{\mathcal{B}}(\mathcal{M}) = w^{-1}\mathbf{Ch}_{\mathcal{B}}(\mathcal{M})$ is a subcategory of $\mathbf{D}(\mathcal{M})$. The functor $\mathbf{D}(\mathcal{B}) \to D_{\mathcal{B}}(\mathcal{A})$ need not be an equivalence; see [223, Ex. 10.4.3].

V.3.8.1 **Example 3.8.1.** If $\mathcal{B} \subset \mathcal{C} \subseteq \mathbf{Ch}(\mathcal{M})$, the derived categories $w^{-1}\mathcal{B}$ and $w^{-1}\mathcal{C}$ are equivalent if for every complex C in $\mathcal{C}$ there is a quasi-isomorphism $B \xrightarrow{\sim} C$ with B in $\mathcal{B}$. (See [200, 1.9.7].)

For example, if $\mathcal{A}$ is an abelian category, the inclusion $\mathbf{Ch}^b(\mathcal{A}) \to \mathbf{Ch}^{hb}(\mathcal{A})$ induces an equivalence of derived categories, where $\mathbf{Ch}^{hb}(\mathcal{A})$ is the category of homologically bounded complexes (II.9.7.4), because (as we saw in 2.7.1) every homologically bounded complex is quasi-isomorphic to a bounded complex.

If R is any ring, $\mathbf{Ch}^b\mathbf{M}(R)$ has the same derived category as $\mathbf{Ch}^{hb}_{\text{pcoh}}(R)$ the homologically bounded pseudo-coherent complexes; see [223, Ex. 10.4.6]. Similarly, the inclusions $\mathbf{Ch}^b_{\text{perf}}(R) \subset \mathbf{Ch}^+_{\text{perf}}(R) \subset \mathbf{Ch}_{\text{perf}}(R)$ induce equivalence on derived categories.

V.3.8.2 **Homotopy Commutative Diagrams 3.8.2.** As pointed out in II.A.4, the chain homotopy category of $\mathcal{C}$ satisfies a calculus of fractions, allowing us to perform constructions in $w^{-1}\mathcal{C}$. Here is a typical example: any homotopy commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} A' & \longrightarrow & B_1 & \longleftarrow & B' \\ & \searrow & \downarrow f & \swarrow & \\ & & B & & \end{array}$$

can be made into a commutative diagram by replacing B_1 by the homotopy pullback of $B_1 \xrightarrow{f} B \xleftarrow{=} B$, which is defined as the shifted mapping cone of $B \to \text{cone}(f)$, and is quasi-isomorphic to B_1.

We are now ready for the Waldhausen version of the Resolution Theorem, which is due to Thomason and Trobaugh [200]. Fix an ambient abelian category $\mathcal{M}$ and consider the Waldhausen category $\mathbf{Ch}(\mathcal{M})$ of all chain complexes over $\mathcal{M}$.

V.3.9 **Thomason-Trobaugh Resolution Theorem 3.9.** *Let $\mathcal{A} \subset \mathcal{B}$ be saturated Waldhausen subcategories of* $\mathbf{Ch}(\mathcal{M})$*, closed under mapping cones and all shift maps $A \mapsto A[n]$. If $w^{-1}\mathcal{A} \xrightarrow{\simeq} w^{-1}\mathcal{B}$ (i.e., the derived categories of $\mathcal{A}$ and $\mathcal{B}$ are equivalent), then $K\mathcal{A} \xrightarrow{\simeq} K\mathcal{B}$ is a homotopy equivalence.*

V.3.9.1 **Remark 3.9.1.** A map in $\mathcal{A}$ is a weak equivalence in $\mathcal{A}$ if and only if it is a weak equivalence in $\mathcal{B}$. This is because, by saturation, both conditions are equivalent to the mapping cone being zero in the (common) derived category.

Proof. (Thomason-Trobaugh) Let $\mathcal{A}^+$ be the comma category whose objects are weak equivalences $w : A \xrightarrow{\sim} B$ with A in $\mathcal{A}$ and B in $\mathcal{B}$; morphisms in $\mathcal{A}^+$ are commutative diagrams in $\mathcal{B}$. It is a Waldhausen category in a way that makes $\mathcal{A} \to \mathcal{A}^+ \to \mathcal{B}$ into exact functors; a morphism $(A \xrightarrow{\sim} B) \to (A' \xrightarrow{\sim} B')$ is a cofibration (resp., weak equivalence) if both its component maps $A \to A'$ and $B \to B'$ are. The forgetful functor $\mathcal{A}^+ \to \mathcal{A}$ is right adjoint to the inclusion $\mathcal{A} \to \mathcal{A}^+$, and exact, so $K(\mathcal{A}) \simeq K(\mathcal{A}^+)$.

We will show that the Approximation Theorem 2.4 (or, rather, its dual) applies to the exact functor $\mathcal{A}^+ \to \mathcal{B}$ sending $A \xrightarrow{\sim} B$ to B. This will imply that $K(\mathcal{A}^+) \simeq K(\mathcal{B})$, proving the theorem.

Condition 2.4(a) is satisfied, because given a map $(A \xrightarrow{\sim} B) \to (A' \xrightarrow{\sim} B')$ in $\mathcal{A}^+$ and a weak equivalence $B \xrightarrow{\sim} B'$, the map $A \to A'$ is a weak equivalence in $\mathcal{B}$ by the saturation axiom, and hence is in $w\mathcal{A}$ by 3.9.1. Condition 2.4(b) holds, because the cylinder functor on $\mathcal{B}$ induces one on $\mathcal{A}^+$. Thus it suffices to check that the dual $(\mathrm{App})^{op}$ of the approximation property holds for $\mathcal{A}^+ \to \mathcal{B}$; this will be a consequence of the hypothesis that $\mathcal{A}$ and $\mathcal{B}$ have the same derived category.

Using the Gabriel-Zisman Theorem (II.A.3), the approximation property $(\mathrm{App})^{op}$ states that given $B' \xrightarrow{b} B \xleftarrow{\sim} A$ with A in $\mathcal{A}$, there is a commutative diagram

$$\begin{array}{ccccc} & & A'' & \longrightarrow & A \\ & & \sim\downarrow & & \sim\downarrow \\ B' & \longrightarrow & B'' & \longrightarrow & B \end{array} \qquad (3.9.2)$$ V.3.9.2

with A'' in $\mathcal{A}$, such that the bottom composite is b.

By assumption, B' is quasi-isomorphic to an object A_1 of $\mathcal{A}$; by calculus of fractions this is represented by a chain $A_1 \xrightarrow{\sim} B_1 \xleftarrow{\sim} B'$. Composing with $B' \to B \xleftarrow{\sim} A$ yields a map from A_1 to A in the derived category, which must be represented by a chain $A_1 \to A_2 \xleftarrow{\sim} A$ with A_2 in $\mathcal{A}$. Composing $B_1 \xleftarrow{\sim} A_1 \to A \to B$ (and using 3.8.2) yields a commutative diagram

$$\begin{array}{ccccccc} & & A_1 & \longrightarrow & A_2 & \xleftarrow{\sim} & A \\ & & \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ B' & \longrightarrow & B_1 & \longrightarrow & B_2 & \xleftarrow{\sim} & B \end{array} \qquad (3.9.3)$$ V.3.9.3

whose bottom composite is chain homotopic to b. Let A'' and B'' denote the shifted mapping cylinders of $A \oplus A_1 \to A_2$ and $B \oplus B_1 \to B_2$, respectively; by the 5-lemma (the extension axiom), the induced map $A'' \to B''$ is a quasi-isomorphism. By the universal property of mapping cylinders [223, 1.5.3], the chain homotopic maps $B' \xrightarrow{b} B \to B_2$ and $B \to B_1 \to B_2$ lift to a map $B \to B''$. We have now constructed a diagram like (3.9.2), which commutes up to chain homotopy. By 3.8.2, this suffices to construct a commutative diagram of this type. $\square$

Applications

The Thomason-Trobaugh Resolution Theorem 3.9 provides a more convenient criterion than the Waldhausen Approximation Theorem 2.4 in many cases, because of the simplicity of the criterion 3.8.1: every complex in the larger category must be quasi-isomorphic to a complex in the smaller category.

V.3.10.1 **Homologically bounded complexes 3.10.1.** If $\mathcal{A}$ is abelian, we saw in 3.8.1 that $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ have the same derived categories, so Theorem 3.9 applies applies, and (using Theorem 2.2) we recover the computation of 2.7.1: $K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}(\mathcal{A})$, and $K_n(\mathcal{A}) \cong K_n\mathbf{Ch}^{hb}(\mathcal{A})$ for all n.

If R is a ring, we saw in 3.8.1 that $\mathbf{Ch}^b\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ have the same derived categories. Again by Theorem 3.9, we recover the computation of 2.7.4: $G(R) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$.

If S is a central multiplicative set in R, it is easy to see by truncating that $\mathbf{Ch}^b_S\mathbf{M}(R) \to \mathbf{Ch}^{hb}_{\mathrm{pcoh},S}(R)$ and $\mathbf{Ch}^b_S\mathbf{P}(R) \to \mathbf{Ch}_{\mathrm{perf},S}(R)$ induce equivalences on derived categories. By Theorem 3.9, we obtain the calculation of 2.7.4 that they induce homotopy equivalences on K-theory.

V.3.10.2 **Example 3.10.2.** If X is a noetherian scheme, the discussion of 3.10.1 applies to the abelian category $\mathbf{M}(X)$ of coherent $\mathcal{O}_X$-modules. We saw in Ex. II.9.8 that if a complex E has only finitely many nonzero cohomology sheaves, and these are coherent, then E is pseudo-coherent (*i.e.,* it is quasi-isomorphic to a bounded above complex of vector bundles); by truncating below, it is quasi-isomorphic to a bounded complex of coherent modules. By 3.8.1, this proves that $\mathbf{Ch}^b\mathbf{M}(X)$ and $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ have the same derived categories, and hence $G(X) \simeq K\mathbf{Ch}^b\mathbf{M}(X) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$.

Let Z be a closed subscheme of X. We saw in 2.6.2 that the relative K-theory of $G(X) \to G(X-Z)$ is the K-theory of the category $\mathbf{Ch}^b_Z\mathbf{M}(X)$ of complexes of coherent modules which are acyclic on $X-Z$. It is contained in the category $\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$ of homologically bounded pseudo-coherent complexes acyclic on $X-Z$. The truncation argument in the previous paragraph shows that these two categories have the same derived categories, and hence the same K-theory: $K\mathbf{Ch}^b_Z\mathbf{M}(X) \simeq K\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$.

This argument works if X is quasi-compact but not noetherian, provided that we understand $\mathbf{M}(X)$ to be pseudo-coherent modules (see [200, 3.11]); this is the case when X is quasi-projective over a commutative ring). However, it does not work for general X; see [SGA6, I]. The following definition generalizes the definition of $G_0(X)$ given in II, Ex. 9.8 and [SGA6, IV(2.2)].

V.3.10.3 **Definition 3.10.3.** If a scheme X is not noetherian, then we define $G(X)$ to be $K\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$. If Z is closed in X, we define $G(X \text{ on } Z)$ to be $K\mathbf{Ch}^{hb}_{\mathrm{pcoh},Z}(X)$.

By Theorem 2.5, $G(X \text{ on } Z) \to G(X) \to G(X-Z)$ is a homotopy fibration, and we get a long exact sequence on homotopy groups, exactly as in 2.6.2.

V.3.10.4 **Example 3.10.4.** (Thomason) If X is any quasi-compact scheme, the inclusion $\mathbf{Ch}^b_{\mathrm{perf}}(X) \subset \mathbf{Ch}_{\mathrm{perf}}(X)$ induces an equivalence on derived categories by [200, 3.5]. Comparing with Definition 2.7.3, we see that $K(X) \simeq K\mathbf{Ch}^b_{\mathrm{perf}}(X)$.

If X also has an ample family of line bundles (for example if X is quasi-projective), the inclusion of $\mathbf{Ch}^b\mathbf{VB}(X)$ in $\mathbf{Ch}_{\text{perf}}(X)$ induces an equivalence on derived categories, by [200, 3.6 and 3.8]. In this case, we get a fancy proof that $K(X) \simeq K\mathbf{VB}(X)$, which was already observed in 2.7.3.

Proper transfer f_* and the Projection Formula

V.3.11 **Proper Transfer 3.11.** Here is a homological construction of the transfer f_* of 3.7, associated to a proper map $f : X \to Y$ of noetherian schemes. It is based upon the fact (see [SGA4, V.4.9]) that the direct image f_* sends flasque sheaves to flasque sheaves. Let $\mathcal{F}_X$ denote the Waldhausen category of homologically bounded complexes of flasque $\mathcal{O}_X$-modules whose stalks have cardinality at most κ for a suitably large κ; f_* is an exact functor from flasque modules to flasque modules, and from $\mathcal{F}_X$ to $\mathcal{F}_Y$. Moreover, $K(\mathcal{F}_X)$ is independent of κ by Approximation 2.4, and $\mathcal{F}_X \subset \mathbf{Ch}^{hb}_{\text{pcoh}}(X)$ induces a homotopy equivalence on K-theory by Resolution 3.9 (see Ex. 3.8). Hence we obtain the transfer map f_* as the composite

$$G(X) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(X) \simeq K(\mathcal{F}_X) \xrightarrow{f_*} K(\mathcal{F}_Y) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(Y) \simeq G(Y).$$

Given another proper map $g : Y \to Z$ (with Z noetherian), the composition $g_*f_* : \mathcal{F}_X \to \mathcal{F}_Z$ equals $(gf)_*$. Thus $X \mapsto K(\mathcal{F}_X)$ and hence $X \mapsto G(X)$ is a functor on the category of noetherian schemes and proper maps.

Suppose in addition that X has finite flat dimension over Y, so that f_* sends perfect complexes to perfect complexes. (We say that f is a *perfect map.*) Then we have an exact functor $f_* : \mathbf{Ch}_{\text{perf}}(X) \to \mathbf{Ch}_{\text{perf}}(Y)$ and hence a proper transfer $K(X) \simeq K\mathbf{Ch}_{\text{perf}}(X) \xrightarrow{f_*} K\mathbf{Ch}_{\text{perf}}(Y) \simeq K(Y)$. The same argument shows that $X \mapsto K(X)$ is a functor on the category of noetherian schemes and perfect proper maps.

V.3.11.1 **Variant 3.11.1.** We can alter f_* using the "Godement resolution" functor T, from complexes of $\mathcal{O}_X$-modules to complexes of flasque sheaves; one takes the direct sum total complex of the Godement resolutions of the individual sheaves, as in [SGA4, XVII.4.2]. Since the direct image f_* is exact on flasque sheaves, the functor $Rf_* = f_* \circ T$ is exact on all complexes of $\mathcal{O}_X$-modules. The resulting map $Rf_* : K\mathbf{Ch}^{hb}_{\text{pcoh}}(X) \to K(\mathcal{F}_Y) \simeq K\mathbf{Ch}^{hb}_{\text{pcoh}}(Y)$ is homotopic to the functorial construction of $G(X) \to G(Y)$ in 3.11, so $f_*(x) = Rf_*(x)$ for all $x \in G_m(X)$.

V.3.12 **Projection Formula 3.12.** *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Then $f_* : G_*(X) \to G_*(Y)$ is a graded $K_*(Y)$-module homomorphism: for all $x \in G_m(X)$ and $y \in K_n(Y)$:*

$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

Suppose in addition that f has finite flat dimension. Then $f_* : K_*(X) \to K_*(Y)$ exists and is a graded $K_*(Y)$-module homomorphism: the same formula holds in $K_{m+n}(Y)$ for all $x \in K_m(X)$, where $f_*(x) \in K_m(Y)$.

Proof. We will express each side as the pairing on K-theory arising from the construction of IV.8.11 applied to a biexact pairing of Waldhausen categories $\mathbf{M}(X) \times \mathbf{VB}(Y) \to \mathcal{A}_X \times \mathcal{F} \to \mathcal{A}_Y$ (II.9.5.2). Let $\mathcal{A}_X$ denote the category $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ of homologically bounded pseudo-coherent complexes of $\mathcal{O}_X$-modules, and let $\mathcal{F}$ denote the category of bounded above perfect complexes of flat $\mathcal{O}_Y$-modules. Note that $\mathbf{M}(X) \subset \mathcal{A}_X$ induces $G(X) \simeq K(\mathcal{A}_X)$ by 2.7.4 and $\mathbf{VB}(X) \subset \mathcal{F}$ induces $K(Y) \simeq K(\mathcal{F})$ by Ex. 2.8. Then the functors $(E, F) \mapsto (Rf_*E) \otimes_Y F$ and $(E, F) \mapsto Rf_*(E \otimes_X f^*F)$ are biexact, where Rf_* is defined in 3.11.1. By IV.8.11, they induce maps $K(\mathcal{A}_X) \wedge K(\mathcal{F}) \to K(\mathcal{A}_Y)$ which on homotopy groups are the pairings sending (x, y) to $f_*(x \cdot f^*(y))$ and $f_*(x) \cdot y$, respectively. The canonical map

$$(Rf_*E) \otimes_Y F \to f_*(TE \otimes_Y f^*F) \to Rf_*(E \otimes_X f^*F)$$

is a natural quasi-isomorphism of pseudo-coherent complexes; see [SGA6, III.3.7]. Hence it induces a homotopy between the two maps, as desired.

If in addition X has finite flat dimension, we merely replace $\mathcal{A}_X$ (resp., $\mathcal{A}_Y$) by the category of perfect complexes on X (resp., on Y), and the same proof works. □

EXERCISES

EV.3.1 **3.1.** Under the hypothesis of the Resolution Theorem 3.1, show that $\mathcal{H}_{n-1}$ is closed under admissible subobjects and extensions in $\mathcal{H}_n$. (It suffices to consider $n = 1$.)

EV.3.2 **3.2.** Let $f : \mathcal{B} \to \mathcal{C}$ be a left exact functor between two abelian categories such that the right derived functors R^if exist, and suppose that every object of $\mathcal{B}$ embeds in an f-acyclic object (an object for which R^if vanishes when $i > 0$). Let $\mathcal{A}$ be the full subcategory of f-acyclic objects, and $\mathcal{B}_f$ the full subcategory of objects B such that only finitely many $R^if(B)$ are nonzero.

(a) Show that $\mathcal{A}$ and $\mathcal{B}_f$ are exact subcategories of $\mathcal{B}$, closed under cokernels of admissible monomorphisms in $\mathcal{B}$, and that f is an exact functor on $\mathcal{A}$.

(b) Show that $K(\mathcal{A}) \simeq K(\mathcal{B}_f)$. In this way we can define $f_* : K_n(\mathcal{B}_f) \to K_n(\mathcal{C})$ by $K(\mathcal{B}_f) \simeq K(\mathcal{A}) \to K(\mathcal{C})$.

EV.3.3 **3.3.** Show that β is an isomorphism in Example 3.5.2. To do this, let $F_m(M)$ be the submodule of M generated by $M_{-m} \oplus \cdots \oplus M_m$, and consider the subcategory $\mathbf{M}_m(S)$ of graded B-modules M with $M = F_m(M)$. Show that $K_i\mathbf{M}_m(R)$ is a sum of copies of $G_i(R)$ and use the admissible filtration

$$0 \subset F_0(M) \subset \cdots \subset F_m(M) = M$$

of modules in $\mathbf{M}_m(S)$ (see 1.8) to show that $K_i\mathbf{M}_m(S) \cong K_i\mathbf{M}_m(R)$ for all m.

EV.3.4 **3.4.** If $S = R \oplus S_1 \oplus \cdots$ is a graded noetherian ring, and both S/R and R/S have finite flat dimension, modify the previous exercise to show that Example 3.5.2 still holds. *Hint:* Consider S-modules which are acyclic for both $\otimes_S R$ and $\otimes_S S$.

EV.3.5 **3.5.** Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms of finite flat dimension. Show that $f^*g^* = (gf)^*$ as maps $G_*(Z) \to G_*(X)$.

EV.3.6 **3.6.** (Quillen) In this exercise, we give another construction of the transfer map $f_* : G(X) \to G(Y)$ associated to a projective morphism $f : X \to Y$ of noetherian schemes. Let $\mathcal{A} \subset \mathbf{M}(X)$ denote the (exact) subcategory consisting of all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $Rf^i(\mathcal{F}) = 0$ for $i > 0$. Because X is projective, for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer n_0 such that, for all $n \geq n_0$, the modules $\mathcal{F}(n)$ are in $\mathcal{A}$, and are generated by global sections; see [85, III.8.8].

(a) Show that any coherent $\mathcal{F}$ embeds in $\mathcal{F}(n)^r$ for large n and r. *Hint:* To prove it for $\mathcal{F} = \mathcal{O}_X$, apply $\mathrm{Hom}(\mathcal{O}_X(n), -)$ to a surjection $\mathcal{O}_X^r \to \mathcal{O}_X(n)$; now apply $\otimes\mathcal{F}$.

(b) Show that every coherent module $\mathcal{F}$ has a finite $\mathcal{A}$-resolution, starting with (a).

(c) Show that every vector bundle has a finite resolution by vector bundles in $\mathcal{A}$.

(d) Use Ex. 3.2 to define f_* to be the composition $f_* : G(X) \simeq K(\mathcal{A}) \to G(Y)$. Then show that this definition of f_* is homotopy equivalent to the map defined in 3.7, so that the maps $f_* : G_n(X) \to G_n(Y)$ agree.

(e) Given $g : Z \to X$, let $\mathcal{B} \subset \mathbf{M}(X)$ be the subcategory of modules Tor-independent of $\mathcal{O}_Z$. Show that every $\mathcal{F}$ in $\mathcal{B}$ has a finite $\mathcal{A}\cap\mathcal{B}$-resolution. *Hint:* If $\mathcal{F}$ is in $\mathcal{B}$, so is $\mathcal{F}(n)$.

EV.3.7 **3.7.** Fix a noetherian ring R, and let $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(R)$ be the category of all bounded below chain complexes of R-modules which are quasi-isomorphic to a (bounded) complex in $\mathbf{Ch}^b(\mathbf{M}(R))$. For example, injective resolutions of finitely generated R-modules belong to $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(R)$. Let $\mathcal{I} \subset \mathbf{Ch}^{hb,+}_{\mathbf{M}}(R)$ denote the subcategory of complexes of injective R-modules. Show that the derived categories of $\mathcal{I}$, $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(R)$ and $\mathbf{Ch}^b(\mathbf{M}(R))$ are isomorphic. Conclude that

$$K(\mathcal{I}) \simeq K\mathbf{Ch}^{hb,+}_{\mathbf{M}}(R) \simeq K\mathbf{Ch}^b(\mathbf{M}(R) \simeq G(R).$$

EV.3.8 **3.8.** Fix a noetherian scheme X, and let $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(X)$ be the category of all bounded below chain complexes of $\mathcal{O}_X$-modules which are quasi-isomorphic to a (bounded) complex in $\mathbf{Ch}^b(\mathbf{M}(X))$.

(a) Show that the derived categories of $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(X)$ and $\mathbf{Ch}^b(\mathbf{M}(X))$ are isomorphic, and conclude that $\mathbf{Ch}^{hb,+}_{\mathbf{M}}(X) \simeq K\mathbf{Ch}^b(\mathbf{M}(X) \simeq G(X)$.

(b) Let $\mathcal{I} \subset \mathcal{F} \subset \mathbf{Ch}^{hb,+}_{\mathbf{M}}(X)$ denote the subcategories of complexes of injective and flasque $\mathcal{O}_X$-modules, respectively. Show that $K(\mathcal{I}) \simeq K(\mathcal{F}) \simeq G(X)$ as well.

EV.3.9 **3.9.** If X is a quasi-compact regular scheme, show that every pseudo-coherent module is perfect (II, Ex. 9.10). Then show that $\mathbf{Ch}_{\mathrm{perf}}(X)$ is the same as $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}$. Conclude that $K(X) \simeq G(X)$ by definition (2.7.3 and 3.10.3).

EV.3.10 **3.10.** *(Projection Formula)* Suppose that $f\colon X \to Y$ is a proper map between quasi-projective schemes, both of which have finite flat dimension. Use a modification of the proof of 3.7.3 to show that the proper transfer map $f_* : G_m(X) \to G_m(Y)$ is a $K_0(Y)$-module map, *i.e.,* that $f_*(x \cdot f^*y) = f_*(x) \cdot y$ for $y \in K_0(X)$ and $x \in G_m(X)$. (The conclusion is a special case of 3.12; see [SGA6, IV.2.11.1].)

EV.3.11 **3.11.** (Thomason) Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $g : Y' \to Y$ be a map Tor-independent of f. Show that the Base Change Theorem 3.7.2 remains valid in this context: if g has finite flat dimension then $g^*f_* \simeq f'_*g'^*$ as maps $G(X) \to G(Y')$, while if f has finite flat dimension then $g^*f_* \simeq f'_*g'^*$ as maps $K(X) \to K(Y')$

(a) First consider the category $\mathcal{C}$ of bounded above pseudo-coherent complexes of flat modules on X. Then g'^* is exact on $\mathcal{C}$ and takes values in $\mathcal{C}'$. Show that the Godement resolution T of 3.11.1 sends $\mathcal{C}$ to itself, and that $E \to TE$ is a quasi-isomorphism. Then show that the inclusion of $\mathcal{C}$ in $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ is an equivalence of derived categories, so $G(X) \simeq K(\mathcal{C})$.

(b) Consider the category $\mathcal{A}$ whose objects consist of an E in $\mathcal{C}$, a bounded above complex F of flat modules on Y with a quasi-isomorphism $F \to f_*E$, a bounded below complex G of flasque modules on X' with a quasi-isomorphism $(g')^*E \to G$. Show that $\mathcal{A}$ has the same derived category as $\mathcal{C}$, and conclude that $G(X) \simeq K(\mathcal{A})$.

(c) Show that g^*f_* is represented by the exact functor on $\mathcal{A}$ sending (E, F, G) to g^*F, and that $f'_*(g')^*$ is represented by the exact functor sending (E, F, G) to f'_*G.

(d) The canonical base change of [SGA4, XVII.4.2.12] is the natural isomorphism

$$g^*F \to g^*f_*(E) \to g^*f_*\, g'_*\, g'^*E = g^*g_*\, f'_*\, g'^*E \to f'_*\, g'^*E \to f'_*G.$$

Show that this base change induces the desired homotopy $g^*f_* \simeq f'_*\, g'^*$.

EV.3.12 **3.12.** Suppose that $F : \mathcal{M}^1 \to \mathcal{M}^2$ is an additive functor between abelian categories, and that $\mathcal{A}^i \subset \mathbf{Ch}(\mathcal{M}^i)$ $(i = 1, 2)$ are saturated Waldhausen subcategories, closed under mapping cones and shifts. If F sends $\mathcal{A}^1$ to $\mathcal{A}^2$ and induces an equivalence of derived categories, modify the proof of Theorem 3.9 to show that $K(\mathcal{A}^1) \simeq K(\mathcal{A}^2)$.

EV.3.13 **3.13.** Let $\mathbb{P}^1_R$ denote the projective line over an associative ring R, as in 1.5.4, and let $\mathbf{H}_n$ denote the subcategory of $\mathbf{mod}\text{-}\,\mathbb{P}^1_R$ of modules having a resolution of length n by vector bundles. Show that $K(\mathbb{P}^1_R) \simeq K\mathbf{H}_n$ for all n.

EV.3.14 **3.14.** Let S be a set of central nonzerodivisors in a ring R.

(a) Let $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ denote the category of perfect chain complexes of S-torsion R-modules. Show that the inclusion of $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ into $\mathbf{Ch}_{\text{perf},S}(R)$ induces an equivalence $K\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S) \simeq K(R \text{ on } S)$. (See 2.7.2.) *Hint:* Consider the functor $\varinjlim \operatorname{Hom}(R/sR, -)$ from $\mathbf{Ch}^b_S\mathbf{P}(R)$ to $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$.

(b) Let $\mathcal{H}$ be the additive subcategory of $\mathbf{H}_S(R)$ generated by the projective R/sR-modules. Show that $K\mathbf{Ch}^b(\mathcal{H}) \simeq K\mathbf{H}_S(R)$.

(c) Show that the inclusion of $\mathbf{Ch}^b(\mathcal{H})$ in $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_S)$ satisfies property (App), and conclude that $K(R \text{ on } S) \simeq K\mathbf{H}_S(R)$. By Theorem 2.6.3, this yields a long exact sequence

$$\cdots K_{n+1}(S^{-1}R) \xrightarrow{\partial} K_n\mathbf{H}_S(R) \to K_n(R) \to K_n(S^{-1}R) \xrightarrow{\partial} \cdots.$$

EV.3.15 **3.15.** Let s be a nonzerodivisor in a commutative noetherian ring R, and assume that R is flat over a subring R_0, with R_0 isomorphic to R/sR. Use the projection formula 3.5.3 to show that the transfer map $G_*(R/sR) \to G_*(R)$ is zero.

EV.3.16 **3.16.** (Thomason [200, 5.7]) Let X be a quasi-projective scheme, and let Z be a subscheme defined by an invertible ideal I of $\mathcal{O}_X$. Let $\mathbf{Ch}_{\text{perf},Z}(X)$ denote the category of perfect complexes on X which are acyclic on $X - Z$. For reasons that will become clear in §7, we write $K(X \text{ on } Z)$ for $K\mathbf{Ch}_{\text{perf},Z}(X)$.

(a) Let $\mathbf{M}_Z(X)$ denote the category of $\mathcal{O}_X$-modules supported on Z, and let $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_Z)$ denote the category of perfect complexes of modules in $\mathbf{M}_Z(X)$. Show that the inclusion into $\mathbf{Ch}_{\text{perf},Z}(X)$ induces an equivalence $K\mathbf{Ch}_{\text{perf}}(\mathbf{M}_Z) \simeq K(X \text{ on } Z)$.

(b) Let $\mathbf{H}_Z(X)$ denote the subcategory of modules in $\mathbf{H}(X)$ supported on Z. If $\mathcal{H}$ is the additive subcategory of $\mathbf{H}_Z(X)$ generated by the $\mathcal{O}_X/I^n\mathcal{O}_X(n)$, show that $K(\mathcal{H}) \cong K\mathbf{H}_Z(X)$.

(c) Show that the inclusion of $\mathbf{Ch}^b(\mathcal{H})$ in $\mathbf{Ch}_{\text{perf}}(\mathbf{M}_Z)$ satisfies (App), so that Waldhausen Approximation 2.4 (and 2.2) imply that $K(X \text{ on } Z) \simeq K\mathbf{H}_Z(X)$.

4 Devissage

This is a result that allows us to perform calculations like $G_*(\mathbb{Z}/p^r) \cong K_*(\mathbb{F}_p)$, which arise from the inclusion of the abelian category $\mathbf{M}(\mathbb{F}_p)$ of all finite elementary abelian p-groups into the abelian category $\mathbf{M}(\mathbb{Z}/p^r)$ of all finite abelian p-groups of exponent p^r. It is due to Quillen and taken from [153].

V.4.1 **Devissage Theorem 4.1.** *Let $i : \mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories such that $\mathcal{A}$ is an exact abelian subcategory of $\mathcal{B}$ (II.6.1.5) and $\mathcal{A}$ is closed in $\mathcal{B}$ under subobjects and quotients. Suppose that every object B of $\mathcal{B}$ has a finite filtration*

$$0 = B_r \subset \cdots \subset B_1 \subset B_0 = B$$

by objects in $\mathcal{B}$ such that every subquotient B_i/B_{i-1} lies in $\mathcal{A}$. Then

$$K(\mathcal{A}) \simeq K(\mathcal{B}) \text{ and } K_*(\mathcal{A}) \cong K_*(\mathcal{B}).$$

Proof. By Quillen's Theorem A (IV.3.7), it suffices to show that the comma categories Qi/B are contractible for every B in $\mathcal{B}$. If B is in $\mathcal{A}$, then $Qi/B \simeq *$ because B is a terminal object. Since B has a finite filtration, it suffices to show that the inclusion $i/B' \to i/B$ is a homotopy equivalence for each $B' \rightarrowtail B$ in $\mathcal{B}$ with B/B' in $\mathcal{A}$.

By Ex. IV.6.1, we may identify the objects of Qi/B, which are pairs $(A, A \to B)$, with admissible layers $u : B_1 \rightarrowtail B_2 \rightarrowtail B$ such that B_2/B_1 is in $\mathcal{A}$. Let J denote the subcategory of Qi/B consisting of all admissible layers of B with $B_1 \subseteq B'$. We have functors $s : Qi/B \to J$, $s(u) = (B_1 \cap B' \rightarrowtail B_2 \rightarrowtail B)$ and $r : J \to Qi/B'$, $r(u) = (B_1 \rightarrowtail B_2 \cap B' \rightarrowtail B)$, because $\mathcal{A}$ is closed under subobjects. The natural transformations $u \to s(u) \leftarrow rs(u)$ defined by

$$(B_1 \rightarrowtail B_2 \rightarrowtail B) \to (B_1 \cap B' \rightarrowtail B_2 \rightarrowtail B) \leftarrow (B_1 \cap B' \rightarrowtail B_2 \cap B' \rightarrowtail B)$$

show that s is left adjoint to the inclusion $J \subset Qi/B$ and that r is right adjoint to the inclusion $Qi/B' \subset J$. It follows (IV.3.2) that $Qi/B' \subset J \subset Qi/B$ are homotopy equivalences, as desired. □

V.4.1.1 **Open Problem 4.1.1.** Generalize the Devissage Theorem 4.1 to Waldhausen categories. Such a result should yield the above Devissage Theorem when applied to $\mathbf{Ch}^b(\mathcal{A})$.

V.4.2 **Corollary 4.2.** *If I is a nilpotent ideal in a noetherian ring R, then $G(R/I) \simeq G(R)$ and hence $G_*(R/I) \cong G_*(R)$.*

Proof. As in II.6.3.1, devissage applies to $\mathbf{M}(R/I) \subset \mathbf{M}(R)$, because every finitely generated R-module M has a finite filtration by submodules MI^n. □

V.4.2.1 **Example 4.2.1.** Let R be an artinian local ring with maximal ideal $\mathfrak{m}$ ($\mathfrak{m}^r = 0$) and quotient field $k = R/\mathfrak{m}$. (E.g., $R = \mathbb{Z}/p^r$ and $k = \mathbb{F}_p$). Then we have $G_*(R) \cong K_*(k)$.

It is instructive to deconstruct the argument slightly. In this case, $\mathbf{M}(k)$ is an exact subcategory of $\mathbf{M}(R)$ and every R-module M has the natural filtration

$$0 = \mathfrak{m}^r M \subset \mathfrak{m}^{r-1} M \subset \cdots \subset \mathfrak{m}M \subset M.$$

Note that the Admissible Filtrations Proposition 1.8 does not apply because for example $F(M) = M/\mathfrak{m}M$ is not an exact functor.

4.2.2 **Open Problem 4.2.2.** Compute the K-groups $K_*(R)$ of an artinian local ring R, assuming $K_*(k)$ is known. If $\mathrm{char}(k) = 0$, this can be done using cyclic homology (Goodwillie's Theorem): the relative groups $K_n(R, \mathfrak{m})$ and $HC_{n-1}(R, \mathfrak{m})$ are isomorphic. If $\mathrm{char}(k) \neq 0$, this can be done using topological cyclic homology (McCarthy's Theorem): $K_n(R, \mathfrak{m})$ and $TC_n(R, \mathfrak{m})$ are isomorphic. In terms of generators and relations, though, we only know K_0, K_1, K_2 and sometimes K_3 at present. (See II.2.2, III.2.4 and III.5.11.1.)

V.4.3 **Application 4.3.** (Quillen) Let $\mathcal{A}$ be an abelian category such that every object has finite length. By devissage, $K(\mathcal{A})$ is equivalent to $K(\mathcal{A}_{ss})$, where $\mathcal{A}_{ss}$ is the subcategory of semisimple objects. By Schur's Lemma, $\mathcal{A}_{ss} \cong \oplus \mathbf{M}(D_i)$, where the D_i are division rings. (The D_i are the endomorphism rings of non-isomorphic simple objects A_i.) It follows from Ex. IV.6.11 that

$$K_*(\mathcal{A}) \simeq \oplus_i K_*(D_i).$$

This applies to finitely generated torsion modules over Dedekind domains and curves, and more generally to finitely generated modules of finite support over any commutative ring or scheme.

V.4.4 **Application 4.4** (R-modules with support)**.** Given a central element s in a noetherian ring R, let $\mathbf{M}_s(R)$ denote the abelian subcategory of $\mathbf{M}(R)$ consisting of all M such that $Ms^n = 0$ for some n. We saw in II.6.3.3 that these modules have finite filtrations with subquotients in $\mathbf{M}(R/sR)$. By devissage, $K\mathbf{M}_s(R) \simeq K\mathbf{M}(R/sR)$, so we have $K_*\mathbf{M}_s(R) \cong G_*(R/sR)$. More generally, given any ideal I we can form the exact category $\mathbf{M}_I(R)$ of all M such that $MI^n = 0$ for some n. Again by devissage, $K\mathbf{M}_I(R) \simeq K\mathbf{M}(R/I)$ and we have $K_*\mathbf{M}_I(R) \cong G_*(R/I)$. The case $I = \mathfrak{p}$ ($K_*\mathbf{M}_\mathfrak{p}(R) \cong G_*(R/\mathfrak{p})$) will be useful in section 6 below.

If S is a central multiplicatively closed set in R, the exact category $\mathbf{M}_S(R)$ is the filtered colimit over $s \in S$ of the $\mathbf{M}_s(R)$. By IV.6.4, $K\mathbf{M}_S(R) = \Omega BQ\mathbf{M}_S(R)$ is $\varinjlim K\mathbf{M}_s(R)$ and hence

$$K_n\mathbf{M}_S(R) = \varinjlim K_n\mathbf{M}_s(R) = \varinjlim G_n(R/sR).$$

EXERCISES

EV.4.1 **4.1.** (Jordan-Hölder) Given a ring R, describe the K-theory of the category of R-modules of finite length. (K_0 is given by Ex. II.6.1.)

EV.4.2 **4.2.** If X is a noetherian scheme, show that $G(X_{\rm red}) \simeq G(X)$ and hence that $G_*(X_{\rm red}) \cong G_*(X)$. This generalizes the result II.6.3.2 for G_0.

EV.4.3 **4.3.** Let Z be a closed subscheme of a noetherian scheme X, and let $\mathbf{M}_Z(X)$ be the exact category of all coherent X-modules supported on Z. Generalize 4.4 and II.6.3.4 by showing that

$$G(Z) = K\mathbf{M}(Z) \simeq K\mathbf{M}_Z(X).$$

EV.4.4 **4.4.** Let $S = R \oplus S_1 \oplus \cdots$ be a graded noetherian ring, and let $\mathbf{M}^b_{gr}(S)$ denote the category of graded modules M with $M_n = 0$ for all but finitely many n. (Ex. II.6.14) Via $S \to R$, any R-module in $\mathbf{M}^b_{gr}(R) = \mathbf{M}_{gr}(R)$ is a graded S-module in $\mathbf{M}^b_{gr}(S)$.

(a) Use devissage to show that $\mathbf{M}_{gr}(R) \subset \mathbf{M}^b_{gr}(S)$ induces an equivalence in K-theory, so that (in the notation of Example 3.5.2) $K_*\mathbf{M}^b_{gr}(S) \simeq G_*(R)[\sigma, \sigma^{-1}]$.

(a) If $t \in S_1$, write $\mathbf{M}_{gr,t}(S)$ for the category of t-torsion modules in $\mathbf{M}_{gr}(S)$. Show that $\mathbf{M}_{gr}(S/tS) \subset \mathbf{M}_{gr,t}(S)$ induces $K\mathbf{M}_{gr}(S/tS) \simeq K\mathbf{M}_{gr,t}(S)$.

5 The Localization Theorem for abelian categories

The K_0 Localization Theorems for abelian categories (II.6.4) and certain exact categories (II.7.7.4) generalize to higher K-theory, in a way we shall now describe. It is also due to Quillen.

Recall from the discussion before II.6.4 that a *Serre subcategory* of an abelian category $\mathcal{A}$ is an abelian subcategory $\mathcal{B}$ which is closed under subobjects, quotients and extensions. The quotient abelian category $\mathcal{A}/\mathcal{B}$ exists; Gabriel's construction of $\mathcal{A}/\mathcal{B}$ using the Calculus of Fractions is also described just before II.6.4.

V.5.1 **Abelian Localization Theorem 5.1.** *Let $\mathcal{B}$ be a Serre subcategory of a (small) abelian category $\mathcal{A}$. Then*

$$K(\mathcal{B}) \to K(\mathcal{A}) \xrightarrow{loc} K(\mathcal{A}/\mathcal{B})$$

is a homotopy fibration sequence. Thus there is a long exact sequence of homotopy groups

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n(\mathcal{B}) \to K_n(\mathcal{A}) \xrightarrow{loc} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{n-1}(\mathcal{B}) \to \cdots \qquad (5.1.1)$$ V.5.1.1

ending in $K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$, the exact sequence of II.6.4.

Proof. (Quillen) For any A in $\mathcal{A}$, let us write $\bar{A}$ for the object $\mathrm{loc}(A)$ of $\mathcal{A}/\mathcal{B}$. Recall from IV.3.2.3 that for any object L in $\mathcal{A}/\mathcal{B}$ the comma category $L\backslash Q\mathrm{loc}$ consists of pairs (A, u) with A in $\mathcal{A}$ and $u : L \to \bar{A}$ a morphism in $Q\mathcal{A}/\mathcal{B}$. We will deduce the result from Theorem B (IV.3.7), so we need to show that $0\backslash Q\mathrm{loc}$ is homotopy equivalent to $Q\mathcal{B}$, and that for every $L \to L'$ the map $L'\backslash Q\mathrm{loc} \to L\backslash Q\mathrm{loc}$ is a homotopy equivalence.

To do this, we introduce the full subcategory F_L of $L\backslash Q\mathrm{loc}$ consisting of pairs (A, u) in which u is an isomorphism in $\mathcal{A}/\mathcal{B}$. (By IV.6.1.2, isomorphisms in $Q\mathcal{A}/\mathcal{B}$ are in 1–1 correspondence with isomorphisms in $\mathcal{A}/\mathcal{B}$, and are described in II.A.1.2.) In particular, the category F_0 ($L = 0$) is equivalent to the subcategory $Q\mathcal{B}$ because, by construction (II.6.4), the objects of $\mathcal{B}$ are exactly the objects of $\mathcal{A}$ isomorphic to 0 in $\mathcal{A}/\mathcal{B}$. Thus the functor $Q\mathcal{B} \to Q\mathcal{A}$ factors as $Q\mathcal{B} \cong F_0 \hookrightarrow 0\backslash Q\mathrm{loc} \to Q\mathcal{A}$.

V.5.1.2 **Claim 5.1.2.** The inclusion $i : F_0 \to 0\backslash Q\mathrm{loc}$ is a homotopy equivalence.

This will follow from Quillen's Theorem A (IV.3.7) once we observe that for every A in $\mathcal{A}$ the comma category $i/(A, u : 0 \to \bar{A})$ is contractible. By IV.6.1.2, the morphism u in $Q\mathcal{A}/\mathcal{B}$ is the same as a subobject L of $\bar{A}$ in $\mathcal{A}/\mathcal{B}$, and this is represented by a subobject of A. By Ex. IV.6.1, we can represent each object of $i/(A, u)$ by an admissible layer $A_1 \rightarrowtail A_2 \rightarrowtail A$ in $\mathcal{A}$ such that $\bar{A}_1 = \bar{A}_2 = L$, and it is easy to see that $i/(A, u)$ is equivalent to the poset of such layers. But this poset is directed by the very construction of $\mathcal{A}/\mathcal{B}$: $A_1 \rightarrowtail A_2 \rightarrowtail A$ and $A_1' \rightarrowtail A_2' \rightarrowtail A$ both map to $(A_1 \cap A_1') \rightarrowtail (A_2 + A_2') \rightarrowtail A$. This verifies our claim, showing that $Q\mathcal{B} \to 0\backslash Q\mathrm{loc}$ is a homotopy equivalence.

V.5.1.3 **Claim 5.1.3.** The inclusion $F_L \xhookrightarrow{i} L\backslash Q\mathrm{loc}$ is a homotopy equivalence for every L.

This is proven with the same argument used to prove Claim 5.1.2. The only difference is that now the category $i/(A, u)$ is equivalent to the poset of admissible layers $A_1 \rightarrowtail A_2 \rightarrowtail \bar{A}$ with $\overline{A_2/A_1} \cong L$; the construction works in this context to shows that it is directed.

We now introduce several auxiliary categories. Fix N in $\mathcal{A}$ and let $\mathcal{E}_N$ be the category of pairs $(A, h : A \to N)$ for which $\bar{h}$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. By definition, a morphism from (A, h) to (A', h') is a morphism $A \twoheadleftarrow A'' \rightarrowtail A'$ in $Q\mathcal{A}$ such that the two composites $A'' \to N$ agree. It is easily checked that there is a well defined functor $k : \mathcal{E}_N \to Q\mathcal{B}$ sending (A, h) to $\ker(h)$.

Let $\mathcal{E}_N'$ denote the full subcategory of $\mathcal{E}_N$ on the (A, h) with h onto. We will show that $\mathcal{E}_N' \to \mathcal{E}_N$ and $\mathcal{E}_N' \to Q\mathcal{B}$ are homotopy equivalences.

V.5.1.4 **Claim 5.1.4.** For each N in $\mathcal{A}$, $k' : \mathcal{E}_N' \to Q\mathcal{B}$ is a homotopy equivalence.

This will follow from Theorem A once we show that k'/T is contractible for each T in $Q\mathcal{B}$. An object of this comma category is a datum (A, h, u), where (A, h) is in $\mathcal{E}_N$, and $u : \ker(h) \to T$ is in $Q\mathcal{B}$. The subcategory $\mathcal{C}$ of all (A, h, u) with u surjective is contractible because it has $(N, 1_N, 0 \twoheadleftarrow T)$ as initial object.

And the inclusion of $\mathcal{C}$ in k'/T is a homotopy equivalence because it has a left adjoint (IV.3.2), sending $(A, h, \ker(h) \rightarrowtail T_0 \twoheadleftarrow T)$ to $(A_0, h_0, T_0 \twoheadleftarrow T)$, where A_0 is the pushout of $\ker(h) \to A$ along $\ker(h) \rightarrowtail T_0$ and h_0 is the induced map. The claim follows.

V.5.1.5 **Claim 5.1.5.** For each N, $\mathcal{E}'_N \hookrightarrow \mathcal{E}_N$ is a homotopy equivalence.

Let I_N denote the partially ordered set of objects N_i of N such that N/N_i is in $\mathcal{B}$, and consider the functor $\mathcal{E}_N \to I_N$ sending (A, h) to the image $h(A)$. This is a fibered functor (IV.3.7.3); the fiber over i is $\mathcal{E}'_{N_i}$, and the base change for $N_j \subset N_i$ sends (A, h) to $(h^{-1}(N_j), h)$. Since $k_N(A, h) = k_{N_i}(A, h)$, it follows from Claim 5.1.4 that the base change maps $\mathcal{E}'_{N_i} \to \mathcal{E}'_{N_j}$ are homotopy equivalences. By Theorem B (IV.3.7), $\mathcal{E}'_N \to \mathcal{E}_N \to I_N$ is a homotopy fibration. Since I_N has a final object (N), it is contractible and we get $\mathcal{E}'_N \simeq \mathcal{E}_N$ as claimed.

V.5.1.6 **Claim 5.1.6.** If $g : N \to N'$ is a map in $\mathcal{A}$ which is an isomorphism in $\mathcal{A}/\mathcal{B}$, then $g_* : \mathcal{E}_N \to \mathcal{E}_{N'}$ is a homotopy equivalence.

Indeed, if (A, h) is in $\mathcal{E}_N$ then $\ker(h) \subseteq \ker(gh)$ defines a natural transformation from $k : \mathcal{E}_N \to Q\mathcal{B}$ to $k\, g_* : \mathcal{E}_N \to \mathcal{E}_{N'} \to Q\mathcal{B}$. As k is a homotopy equivalence (by Claims 5.1.4 and 5.1.5), so is g_*.

Now fix L in $\mathcal{A}/\mathcal{B}$, and let I_L be the category of pairs $(N, \bar{N} \xrightarrow{\sim} L)$; morphisms are maps $g : N \to N'$ in $\mathcal{A}$ such that $\bar{N} \cong \bar{N}'$. This category is filtering by II.A.1.2, and there is a functor from I_L to categories sending $(N, \bar{N} \xrightarrow{\sim} L)$ to $\mathcal{E}_N$ and g to g_*.

V.5.1.7 **Claim 5.1.7.** F_L is isomorphic to the colimit over I_L of the categories $\mathcal{E}_N$.

For each $n = (N, \bar{N} \xrightarrow{\sim} L)$ we have a functor $p_n : \mathcal{E}_N \to F_L$ sending (A, h) to $L \xrightarrow{\sim} \bar{N} \xrightarrow{\sim} \bar{A}$. Since $p_n = p_{n'} g_*$ for each morphism g, there is a functor $p : \operatorname{colim} \mathcal{E}_N \to F_L$.

Consider the composite functor $\mathcal{E}_N \xrightarrow{k} Q\mathcal{B} \xrightarrow{\sim} F_0 \xrightarrow{\sim} 0\backslash Q\text{loc}$, sending (A, h) to $(\ker(h), 0)$; it is a homotopy equivalence by Claims 5.1.2–5.1.5. Using the map $i : 0 \rightarrowtail L$ in $Q(\mathcal{A}/\mathcal{B})$, we have a second functor

$$\mathcal{E}_N \xrightarrow{p} F_L \hookrightarrow L\backslash Q\text{loc} \xrightarrow{i^*} 0\backslash Q\text{loc},$$

sending (A, h) to $(A, 0 \rightarrowtail \bar{A})$. There is a natural transformation between them, given by the inclusion of $\ker(h)$ in A. Hence we have a homotopy commutative diagram:

$$\begin{array}{ccccc} \mathcal{E}_N & \xrightarrow{p_n} & F_L & \hookrightarrow & L\backslash Q\text{loc} \\ {\scriptstyle k}\downarrow & & & & \downarrow{\scriptstyle i^*} \\ Q\mathcal{B} & \xrightarrow{\sim} & F_0 & \hookrightarrow & 0\backslash Q\text{loc}. \end{array}$$

From Claims 5.1.6 and 5.1.7, it follows that each $p_n : \mathcal{E}_N \to F_L$ is a homotopy equivalence. From Claim 5.1.3, $F_L \hookrightarrow L\backslash Q\mathrm{loc}$ is a homotopy equivalence. It follows that $i^* : L\backslash Q\mathrm{loc} \to 0\backslash Q\mathrm{loc}$ is also a homotopy equivalence. This finishes the proof of the Localization Theorem 5.1. □

Taking homotopy groups with coefficients mod ℓ also converts homotopy fibration sequences into long exact sequences (IV.2.1.1), so we immediately obtain a finite coefficient analogue of (5.1.1).

V.5.2 **Corollary 5.2.** *For each ℓ there is also a long exact sequence*

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) \xrightarrow{\partial} K_n(\mathcal{B};\mathbb{Z}/\ell) \to K_n(\mathcal{A};\mathbb{Z}/\ell) \to K_n(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) \xrightarrow{\partial} \cdots$$

V.5.3 **Open Problem 5.3.** Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$ (II.6.4), and let $\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})$ denote the category of all bounded complexes in $\mathcal{A}$ whose cohomology lies in $\mathcal{B}$ (II, Ex. 9.5). Is $K(\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})) \simeq K(\mathcal{B})$? Such a result would make the Localization Theorem 5.1 for abelian categories an immediate consequence of Theorems 2.5 and 2.2.

EXERCISES

EV.5.1 **5.1.** Suppose that $\alpha : A \to A$ is a morphism in $\mathcal{A}$ which is an isomorphism in $\mathcal{A}/\mathcal{B}$, and so determines an element $[\alpha]$ of $K_1(\mathcal{A}/\mathcal{B})$. Show that $\partial : K_1(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{B})$ sends $[\alpha]$ to $[\mathrm{coker}(\alpha)] - [\ker(\alpha)]$. *Hint:* Use the representative of $[\alpha]$ in $\pi_2 BQ(\mathcal{A}/\mathcal{B})$ given in IV, Ex. 7.9, and IV.6.2.

EV.5.2 **5.2.** Show that the map $S^n \to P^{n+1}(\mathbb{Z}/\ell)$ of IV.2.1.1 applied to the homotopy fibration of Theorem 5.1 yields a commutative diagram comparing the localization sequences of (5.1) and (5.2):

$$\begin{array}{ccccccc}
K_{n+1}(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) & \xrightarrow{\partial} & K_n(\mathcal{B};\mathbb{Z}/\ell) & \to & K_n(\mathcal{A};\mathbb{Z}/\ell) & \to & K_n(\mathcal{A}/\mathcal{B};\mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_n(\mathcal{A}/\mathcal{B}) & \xrightarrow{\partial} & K_{n-1}(\mathcal{B}) & \longrightarrow & K_{n-1}(\mathcal{A}) & \longrightarrow & K_{n-1}(\mathcal{A}/\mathcal{B}).
\end{array}$$

EV.5.3 **5.3.** Suppose that there is a biexact functor $\mathcal{A} \times \mathcal{C} \to \mathcal{A}'$ which induces biexact functors $\mathcal{B} \times \mathcal{C} \to \mathcal{B}'$ and $\mathcal{A}/\mathcal{B} \times \mathcal{C} \to \mathcal{A}'/\mathcal{B}'$. Use Ex. IV.1.23 to show that for $x \in K_j(\mathcal{A}/\mathcal{B})$ and $y \in K_n(\mathcal{C})$ the element $\{x, y\} \in K_{n+i}(\mathcal{A}'/\mathcal{B}')$ satisfies $\partial(\{x, y\}) = \{\partial(x), y\}$ in $K_{n+i}(\mathcal{B}')$.

6 Applications of the Localization Theorem

In this section, we give two families of applications of the Localization Theorem 5.1, to $G(R)$ and to $G(X)$. All rings and schemes will be noetherian in this section, so that $\mathbf{M}(R)$ and $\mathbf{M}(X)$ are abelian categories.

V.6.1 **Application 6.1.** Let S be a central multiplicatively closed set in a noetherian ring R. We saw in II.6.4.1 that the category $\mathbf{M}_S(R)$ of finitely generated S-torsion modules is a Serre subcategory of $\mathbf{M}(R)$ with quotient category $\mathbf{M}(S^{-1}R)$. By the Localization Theorem 5.1, there is a homotopy fibration

$$K\mathbf{M}_S(R) \to G(R) \to G(S^{-1}R).$$

We observed in 4.4 that $\mathbf{M}_S(R)$ is the colimit over all $s \in S$ of the $\mathbf{M}(R/sR)$, and that $K_*\mathbf{M}_S(R) \cong \varinjlim G_*(R/sR)$. Comparing this to the Waldhausen localization sequence (2.6.1), we see that $\mathbf{M}_S(R) \to \mathbf{Ch}^b_S\mathbf{M}(R)$ induces a homotopy equivalence $K\mathbf{M}_S(R) \xrightarrow{\sim} K\mathbf{Ch}^b_S\mathbf{M}(R) = G(R \text{ on } S)$.

The prototype is the case when $S = \{s^n\}$. Here the maps $G(R/sR) \to G(R/s^nR)$ are homotopy equivalences by devissage, so $G(R/sR) \simeq K\mathbf{M}_S(R)$; see 4.4. By inspection, the map $G(R/sR) \to G(R)$ identifying it with the homotopy fiber of $G(R) \to G(R[1/s])$ is the transfer i_* (IV.6.3.3) associated to $i : R \to R/sR$. Thus the long exact Localization sequence (5.1.1) becomes:

$$\cdots \to G_{n+1}(R[s^{-1}]) \xrightarrow{\partial} G_n(R/sR) \xrightarrow{i_*} G_n(R) \to G_n(R[s^{-1}]) \xrightarrow{\partial} \cdots . \qquad (6.1.1)$$ V.6.1.1

This is a sequence of $K_*(R)$-modules, because $\mathbf{P}(R)$ acts on the sequence of abelian categories $\mathbf{M}_S(R) \to \mathbf{M}(R) \to \mathbf{M}(S^{-1}R)$.

V.6.1.2 **Example 6.1.2.** It is useful to observe that any $s \in S$ determines an element $[s]$ of $K_1(R[1/s])$ and hence $G_1(R[1/s])$, and that $\partial(s) \in G_0(R/sR)$ is $[R/sR] - [I]$, where $I = \{r \in R : sr = 0\}$. This formula is immediate from Ex. 5.1. In particular, when R is a domain we have $\partial(s) = [R/sR]$.

V.6.1.3 **Example 6.1.3.** If $R = \mathbb{Z}[s, 1/f(s)]$ with $f(0) = 1$, the maps i_* are zero in (6.1.1) and hence the maps $G_n(R) \to G_n(R[1/s])$ are injections. Indeed, the vanishing of i_* follows from the projection formula (3.5.3): $i_*(i^*y) = i_*(1 \cdot i^*y) = i_*(1) \cdot y$ together with the observation in 6.1.2 that $i_*([R/sR]) = i_*\partial(s) = 0$. (Cf. Ex. 3.15.)

For the following result, we adopt the notation that f denotes the inclusion of R into $R[s]$, j is $R[s] \hookrightarrow R[s, s^{-1}]$, and $(s = 1)$ denotes the map from either $R[s]$ or $R[s, s^{-1}]$ to R, obtained by sending s to 1. Because R has finite flat dimension over $R[s]$ and $R[s, s^{-1}]$, there are base change maps $(s = 1)^* : G(R[s]) \to G(R)$ and $(s = 1)^* : G(R[s, s^{-1}]) \to G(R)$.

V.6.2 **Fundamental Theorem for $G(R)$ 6.2.** *Let R be a noetherian ring. Then:*

(i) The flat base change $f^*: G(R) \to G(R[s])$ is a homotopy equivalence, split by $(s=1)^*$. Hence

$$f^*: G_n(R) \cong G_n(R[s]) \quad \textit{for all } n.$$

(ii) The flat base change $j^*: G(R[s]) \to G(R[s,s^{-1}])$ induces isomorphisms

$$G_n(R[s,s^{-1}]) \cong G_n(R) \oplus G_{n-1}(R).$$

Proof. We first observe that (i) implies (ii). Indeed, because $(s=0)_*: G(R) \to G(R[s])$ is zero by 3.5.1, the localization sequence (6.1.1) for $j: R[s] \to R[s,s^{-1}]$ splits into short exact sequences

$$0 \to G_n(R[s]) \xrightarrow{j^*} G_n(R[s,s^{-1}]) \xrightarrow{\partial} G_{n-1}(R) \to 0.$$

By (i) we may identify $(jf)^*: G_n(R) \to G_n(R[s,s^{-1}])$ with j^*. Because $(s=1)\circ jf$ is the identity, $(jf)^*(s=1)^*$ is homotopic to the identity of $G(R)$. Assertion (ii) follows.

To prove (i), we introduce the graded subring $S = R[st,t]$ of $R[s,t]$ where $\deg(s)=0$, $\deg(t)=1$. Let $\mathbf{M}^b_{gr}(S)$ denote the Serre subcategory of all graded modules M in $\mathbf{M}_{gr}(S)$ with only finitely many nonzero M_n, *i.e.,* graded t-torsion modules. By devissage (Ex. 4.4), $\mathbf{M}^b_{gr}(S)$ has the same K-theory as its subcategory $\mathbf{M}_{gr}(S/tS)$. There is also an equivalence of quotient categories

$$\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(R[s])$$

induced by the exact functor $M \mapsto M/(t-1)M$ from $\mathbf{M}_{gr}(S)$ to $\mathbf{M}(R[s])$. Note that both S and $S/tS = R[st]$ are flat over R and that R has finite flat dimension over both, so the K-theory of both $\mathbf{M}_{gr}(S)$ and $\mathbf{M}^b_{gr}(S)$ are isomorphic to $G(R)[\sigma,\sigma^{-1}]$ by Example 3.5.2. Hence the localization sequence (5.1.1) gives us the following diagram with exact rows:

$$\begin{array}{ccccc}
\cdots \to K_n\mathbf{M}_{gr}(S/tS) & \longrightarrow & K_n\mathbf{M}_{gr}(S) & \longrightarrow & G_n(R[s]) \to \cdots \\
\uparrow \scriptstyle{3.5.2} & & \uparrow \scriptstyle{3.5.2} & & \uparrow \\
0 \to G_n(R)\otimes\mathbb{Z}[\sigma,\sigma^{-1}] & \xrightarrow{h} & G_n(R)\otimes\mathbb{Z}[\sigma,\sigma^{-1}] & \longrightarrow & G_n(R) \to 0.
\end{array}$$

To describe the map h, recall from 3.5.2 that it is induced by the functor from $\mathbf{M}(R)$ to $K\mathbf{M}_{gr}(S)$ sending M to $M \otimes S/tS$. We have an exact sequence of functors from $\mathbf{M}(R)$ to $\mathbf{M}_{gr}(S)$:

$$0 \to M \otimes S(-1) \xrightarrow{t} M \otimes S \to M \otimes S/tS \to 0.$$

As pointed out in Example 3.5.2, the first two maps induce the maps σ and 1, respectively, from $G_n(R)$ to $G(R)[\sigma,\sigma^{-1}]$. Hence $h = 1-\sigma$, proving (i). ☐

Because $R[s]$ and $R[s, s^{-1}]$ are regular whenever R is, combining Theorems 6.2 and 3.3 yields the following important consequence.

V.6.3 **Theorem 6.3.** *[Fundamental Theorem] If R is a regular noetherian ring, then the base change $K(R) \to K(R[s])$ is a homotopy equivalence, so $K_n(R) \cong K_n(R[s])$ for all n. In addition,*

$$K_n(R[s, s^{-1}]) \cong K_n(R) \oplus K_{n-1}(R) \quad \textit{for all } n.$$

In particular, regular rings are K_n-regular for all n (in the sense of III.3.4).

V.6.3.1 **Corollary 6.3.1.** *For any regular ring R, there is a split exact sequence*

$$0 \to K_n(R) \to K_n(R[s]) \oplus K_n(R[s^{-1}]) \to K_n(R[s, s^{-1}]) \overset{\partial}{\underset{\leftarrow}{\to}} K_{n-1}(R) \to 0,$$

in which the splitting is multiplication by $s \in K_1(\mathbb{Z}[s, s^{-1}])$.

Proof. This is obtained from the direct sum of the split exact sequences (6.1.1) for $R[s] \to R[s, s^{-1}]$ and $R[s^{-1}] \to R[s, s^{-1}]$, and the isomorphism $K_n(R) \cong K_n(R[s])$. By 6.1, the map ∂ in the localization sequence (6.1.1) is $K_*(R)$-linear. Thus for $x \in K_{n-1}(R)$ and $s \in K_1(R[s, s^{-1}])$ we have $\partial\{s, x\} = \{\partial s, x\} = x$. □

The technique used in 6.2 to prove that $G(R) \simeq G(R[s])$ applies more generally to filtered rings. By a *filtered ring* we mean a ring R equipped with an increasing filtration $\{F_i R\}$ such that $1 \in F_0 R$, $F_i R \cdot F_j R \subseteq F_{i+j} R$ and $R = \cup F_i R$. The associated graded ring is $\mathrm{gr}(R) = \oplus F_i R / F_{i-1} R$. It is easy to see that if $\mathrm{gr}(R)$ is noetherian then so is R.

For example, any positively graded ring such as $R[s]$ is filtered with $F_i(R) = R_0 \oplus \cdots \oplus R_i$; in this case $gr(R) = R$.

V.6.4 **Theorem 6.4.** *Let R be a filtered ring such that $gr(R)$ is noetherian and of finite flat dimension d over $k = F_0 R$, and such that k has finite flat dimension over $gr(R)$. Then $k \subset R$ induces $G(k) \simeq G(R)$.*

Proof. The hypotheses imply that R has flat dimension $\mathrm{fd}_k A \le d$. Indeed, since each of the $F_i R / F_{i-1} R$ has flat dimension at most d it follows by induction that $\mathrm{fd}_k F_i R \le d$ and hence that $\mathrm{fd}_k R \le d$. Thus the map $G(k) \to G(R)$ is defined. Let S denote the graded subring $\oplus (F_i R) t^i$ of $R[t]$. Then $S/tS \cong \mathrm{gr}(R)$, S is noetherian, $\mathrm{fd}_k S \le d$ and k has finite flat dimension over S. Finally, the category $\mathbf{M}_{gr}(S)$ is abelian (as S is noetherian) and we have $\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(R)$.

It follows that $K\mathbf{M}_{gr}(S)$ and $K\mathbf{M}_{gr}(S/tS) \cong K\mathbf{M}^b_{gr}(S)$ are both isomorphic to $G(k)[\sigma, \sigma^{-1}]$ by Example 3.5.2 (and Ex. 3.4). The rest of the proof is the same as the proof of the Fundamental Theorem 6.2. (See Ex. 6.12). □

V.6.4.1 **Remark 6.4.1.** If in addition k is regular, then so is R. (This follows from the fact that for $M \in \mathbf{M}(R)$, $\mathrm{pd}_R(M) = \mathrm{fd}_R(M)$; see [153, p. 112].) It follows from Theorem 3.3 that $K(k) \simeq K(R)$.

V.6.4.2 **Example 6.4.2.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. Then the universal enveloping algebra $R = U(\mathfrak{g})$ is filtered, with $F_1R = k \oplus \mathfrak{g}$, and its associated graded algebra is a polynomial ring (the symmetric algebra of the vector space $\mathfrak{g}$). Thus Theorem 6.4 implies that $K(k) \simeq K(U(\mathfrak{g}))$.

V.6.5 **Theorem 6.5.** *(Gersten) Let $A = k\{X\}$ be a free k-algebra on a set X, where k is noetherian regular. Then $K_*(k\{X\}) \cong G_*(k\{X\}) \cong K_*(k)$.*

Proof. It is known (see [41]) that A is a coherent regular ring, *i.e.,* that the category $\mathbf{M}(A)$ of pseudo-coherent modules is abelian. Hence $K_*(A) \cong G_*(A)$ by Theorem 3.3. Replacing k by $k[t]$, we see that the same is true of $A[t]$.

Now A has the set of all words as a k-basis; we grade A by word length and consider the category of graded pseudo-coherent A-modules. The proof of 6.4 goes through (Exercise 6.13) to prove that $G_*(k) \cong G_*(A)$, as desired. □

As remarked in IV.1.9(iv), the calculation $K_*(\mathbb{Z}\{X\}) = K_*(\mathbb{Z})$ was used by Anderson to show that Swan's definition $K^{Sw}(R)$ of the higher K-theory of R agrees with Quillen's definitions $K_0(R) \times BGL(R)^+$ and $\Omega BQ\mathbf{P}(R)$ (which we saw were equivalent in IV.7.2).

Dedekind Domains

Suppose that R is a Dedekind domain with fraction field F. Then R and F and regular, as are the residue fields $R/\mathfrak{p}$, so $K_*(R) \cong G_*(R)$, etc. Hence the localization sequence of 6.1 with $S = R - \{0\}$ becomes the long exact sequence:

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_n(R/\mathfrak{p}) \xrightarrow{\oplus (i_{\mathfrak{p}})_*} K_n(R) \longrightarrow K_n(F) \xrightarrow{\partial} \cdots . \quad (6.6)$$ V.6.6

Here $\mathfrak{p}$ runs over the nonzero prime ideals of R, and the maps $(i_{\mathfrak{p}})_* : K_n(R/\mathfrak{p}) \to K_n(R)$ are the transfer maps of 3.3.2.

Writing $K_1(R) = R^\times \oplus SK_1(R)$ (see III.1.1.1), the formula 6.1.2 allows us to identify the ending with the sequence $1 \to R^\times \to F^\times \xrightarrow{\text{div}} D(R) \to K_0(R) \to \mathbb{Z} \to 0$ of I.3.6. Therefore we may extract the following exact sequence, which we have already studied in III.6.5:

$$\oplus_{\mathfrak{p}} K_2(R/\mathfrak{p}) \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \xrightarrow{\oplus (i_{\mathfrak{p}})_*} SK_1(R) \to 1. \quad (6.6.1)$$ V.6.6.1

We claim that ∂ is the tame symbol of III.6.3 and that the above continues the sequence of III.6.5. Since the $\mathfrak{p}$-component of ∂ factors through the localization $K_2(R) \to K_2(R_{\mathfrak{p}})$ and the localization sequence for $R_{\mathfrak{p}}$, we may suppose that R is a DVR with parameter π. In this case, we know that ∂ in $K_*(R)$-linear, so if $u \in R^\times$ has image $\bar{u} \in R/\mathfrak{p}$ then $\partial\{\pi, u\} = [\bar{u}]$ in $R/\mathfrak{p}^\times$. Similarly, ∂ sends $\{\pi, \pi\} = \{\pi, -1\}$ to $\{\partial\pi, -1\} = [R/\pi] \cdot [-1]$, which is the class of the unit -1. Since every element of $K_2(F)$ is a product of such terms modulo $K_2(R)$, it follows that ∂ is indeed the tame symbol, as claimed.

Here are two special cases of (6.6) which arose in chapter III. First, if all of the residue fields $R/\mathfrak{p}$ are finite, then $K_2(R/\mathfrak{p}) = 0$ and we obtain the exact sequence:

$$0 \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \xrightarrow{\oplus (i_{\mathfrak{p}})_*} SK_1(R) \to 1.$$

V.6.6.2 **Corollary 6.6.2.** *If R is a semilocal Dedekind domain, $K_3(F) \xrightarrow{\partial} \oplus K_2(R/\mathfrak{p})$ is onto, and we obtain the exact sequence*

$$0 \to K_2(R) \to K_2(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \to 1.$$

Proof. It suffices to lift a symbol $\{\bar{a}, \bar{b}\} \in K_2(R/\mathfrak{p})$. We can lift $\bar{a}, \bar{b}$ to units a, b of R as R is semilocal. Choose $s \in R$ so that $R/sR = R/\mathfrak{p}$. Since ∂ is a $K_*(R)$-module homomorphism, we have:

$$\partial(\{a, b, s\}) = \{a, b\}\partial(s) = \{a, b\} \cdot [R/sR] = (i_{\mathfrak{p}})_*\{a, b\} = \{\bar{a}, \bar{b}\}. \qquad \square$$

Suppose that $R \subset R'$ is an inclusion of Dedekind domains, with R' finitely generated as an R-module. Then the fraction field F' of R' is finite over F, so the exact functors $\mathbf{M}(R') \to \mathbf{M}(R)$ and $\mathbf{M}(F') \to \mathbf{M}(F)$ inducing the transfer maps (IV.6.3.3) are compatible. Thus we have a homotopy commutative diagram

$$\begin{array}{ccccc} K(\mathbf{M}_{\text{tors}}R') & \longrightarrow & G(R') & \longrightarrow & G(F') \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbf{M}_{\text{tors}}R) & \longrightarrow & G(R) & \longrightarrow & G(F). \end{array} \tag{6.6.3}$$ V.6.6.3

Taking homotopy groups yields the morphism of localization sequences (6.6):

$$\begin{array}{ccccccc} K_n(R') & \longrightarrow & K_n(F') & \xrightarrow{\oplus\partial_{\mathfrak{p}'}} & \oplus_{\mathfrak{p}'} K_{n-1}(R'/\mathfrak{p}') & \longrightarrow & K_{n-1}(R') \\ \downarrow{\scriptstyle N_{R'/R}} & & \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \bar{N} = \oplus N_{\mathfrak{p}'/\mathfrak{p}}} & & \downarrow \\ K_n(R) & \longrightarrow & K_n(F) & \xrightarrow{\partial} & \oplus_{\mathfrak{p}} K_{n-1}(R/\mathfrak{p}) & \longrightarrow & K_{n-1}(R). \end{array} \tag{6.6.4}$$ V.6.6.4

If R is a discrete valuation ring with fraction field F, parameter s and residue field $R/sR = k$, we define the *specialization map* $\lambda_s : K_n(F) \to K_n(k)$ by $\lambda_s(a) = \partial(\{s, a\})$. Because ∂ is $K_*(R)$-linear (6.1.1), a different choice of parameter will yield a different specialization map: if $u \in R^\times$ is such that $s' = us$ then $\lambda_{s'}(a) = \lambda_s(a) + (-1)^n\{u, \partial a\}$.

V.6.7 **Theorem 6.7.** *If a discrete valuation ring R contains a field k_0, and $[k : k_0]$ is finite, then the localization sequence (6.6) breaks up into split exact sequences:*

$$0 \longrightarrow K_n(R) \longrightarrow K_n(F) \underset{\leftarrow}{\xrightarrow{\partial}} K_{n-1}(k) \longrightarrow 0.$$

Moreover, the canonical map $K_n(R) \to K_n(k)$ factors through the specialization map $K_n(F) \to K_n(k)$. A similar assertion holds for K-theory with coefficients.

Proof. Suppose first that $k \subset R$, so that (6.6) is a sequence of $K_*(k)$-modules. Consider the map $K_n(k) \to K_{n+1}(F)$ sending a to $\{s, a\}$; we have $\partial(\{s, a\}) = \{\partial(s), a\} = [k] \cdot a = a$. Hence ∂ is a split surjection, and the result follows.

In general, there is a finite field extension F' of F so that the integral closure R' of R contains k, and $R \to k$ extends to a map $R' \to k$ with kernel $\mathfrak{p} = tR'$. Then the map $\partial' : K_{n+1}(R'[1/t]) \to K_n(k)$ is a surjection, split by $a \mapsto \{t, a\}$, by the same argument. Now the composition $\mathbf{M}(R'/tR') \to \mathbf{M}(R') \to \mathbf{M}(R)$ is just $\mathbf{M}(k) \to \mathbf{M}(R)$ so, as in (6.6.4), we have a morphism of localization sequences

$$\begin{array}{ccccc} K_n(R') & \longrightarrow & K_n(R'[1/t]) & \overset{\partial'}{\rightleftarrows} & K_{n-1}(k) \\ \downarrow & & \downarrow & & \| \\ K_n(R) & \longrightarrow & K_n(F) & \xrightarrow{\partial} & K_{n-1}(k). \end{array}$$

It follows that ∂ is also a split surjection. □

V.6.7.1 **Corollary 6.7.1.** *If k is a field, the localization sequence (6.6) for $k[t] \subset k(t)$ breaks up into split short exact sequences:*

$$0 \longrightarrow K_n(k) \underset{\leftarrow}{\longrightarrow} K_n(k(t)) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_{n-1}(k[t]/\mathfrak{p}) \longrightarrow 0.$$

Proof. When $R = k[t]_{(t)}$, 6.7 says that the localization sequence for $R \subset k(t)$ breaks up into split exact sequences $0 \to K_n(R) \to K_n(k(t)) \to K_{n-1}(k) \to 0$. In addition, the map $K_n(k) \cong K_n(k[t]) \to K_n(R)$ is split by $(t = 0)^*$, so the localization sequence for $k[t] \to R$ also breaks up into the split exact sequences $0 \to K_n(k) \to K_n(R) \xrightarrow{\partial} \oplus_{\mathfrak{p} \neq 0} K_{n-1}(k[t]/\mathfrak{p}) \to 0$. Combining these split exact sequences yields the result. □

We remark that if R contains any field then it always contains a field k_0 so that k is algebraic over k_0. Thus the argument used to prove 6.7 actually proves more.

V.6.7.2 **Corollary 6.7.2.** *If a discrete valuation ring R contains a field k_0, then each $K_{n+1}(F) \xrightarrow{\partial} K_n(k)$ is onto, and the localization sequence (6.6) breaks up into short exact sequences. (The sequences may not split.)*

Proof. We may assume that k is algebraic over k_0, so that every element $a \in K_n(k)$ comes from $a \in K_n(k')$ for some finite field extension k' of k_0. Passing to a finite field extension F' of F containing k', such that $R \to k$ extends to R' as above, consider the composite $\gamma : K_n(k') \to K_{n+1}(F') \to K_{n+1}(F)$ sending a' to the transfer of $\{t, a'\}$. As in the proof of Theorem 6.7, $\partial\gamma : K_n(k') \to K_n(k)$ is the canonical map, so $a = \partial\gamma(a')$. □

Write i and π for the maps $R \to F$ and $R \to k$, respectively.

V.6.7.3 **Lemma 6.7.3.** *The composition of $i^* : K_n(R) \to K_n(F)$ and λ_s is the natural map $\pi^* : K_n(R) \to K_n(k)$. A similar assertion holds for K-theory with coefficients.*

Proof. Because the localization sequence (6.6) is a sequence of $K_*(R)$-modules, for every $a \in K_n(R)$ we have

$$\lambda_s(i^*a) = \partial(\{s, a\}) = \{\partial(s), a\} = [k] \cdot a = \pi^*(a). \qquad \square$$

V.6.7.4 **Corollary 6.7.4.** *If k is an algebraically closed field, and A is any commutative k-algebra, the maps $K_n(k) \to K_n(A)$ and $K_n(k; \mathbb{Z}/m) \to K_n(A; \mathbb{Z}/m)$ are injections.*

Proof. Choosing a map $A \to F$ to a field, we may assume that $A = F$. Since F is the union of its finitely generated subfields F_α and $K_*(F) = \varinjlim K_*(F_\alpha)$, we may assume that F is finitely generated over k. We may now proceed by induction on the transcendence degree of F over k. It is a standard fact that F is the fraction field of a discrete valuation ring R, with residue field $E = R/sR$. By Lemma 6.7.3, the composition of $K_*(k) \to K_*(F)$ with the specialization map λ_s is the natural map $K_n(k) \to K_n(E)$, which is an injection by the inductive hypothesis. It follows that $K_*(k) \to K_*(F)$ is an injection. $\square$

To prove that $K_*(\mathbb{Z})$ injects into $K_*(\mathbb{Q})$ it is useful to generalize to integers in arbitrary global fields. Recall that a *global field* is either a number field or the function field of a curve over a finite field.

V.6.8 **Theorem 6.8.** *(Soulé [171]) Let R be a Dedekind domain whose field of fractions F is a global field. Then $K_n(R) \cong K_n(F)$ for all odd $n \geq 3$; for even $n \geq 2$ the localization sequence breaks up into exact sequences:*

$$0 \to K_n(R) \to K_n(F) \to \bigoplus_{\mathfrak{p}} K_{n-1}(R/\mathfrak{p}) \to 0.$$

Proof. Let $SK_n(R)$ denote the kernel of $K_n(R) \to K_n(F)$; from (6.6), it suffices to prove that $SK_n(R) = 0$ for $n \geq 1$. For $n = 1$ this is the Bass-Milnor-Serre Theorem III.2.5 (and III.2.5.1). From the computation of $K_n(\mathbb{F}_q)$ in IV.1.13 and the fact that $K_n(R)$ is finitely generated (IV.6.9), we see that $SK_n(R)$ is 0 for $n > 0$ even, and is finite for n odd. Fixing $n = 2i - 1 \geq 1$, we may choose a positive integer ℓ annihilating the (finite) torsion subgroup of $K_n(R)$. Thus $SK_n(R)$ injects into the subgroup $K_n(R)/\ell$ of $K_n(R; \mathbb{Z}/\ell)$, which in turn injects into $K_n(F; \mathbb{Z}/\ell)$ by Proposition 6.8.1 below. Since $SK_n(R)$ vanishes in $K_n(F)$ and hence in $K_n(F; \mathbb{Z}/\ell)$, this forces $SK_n(R)$ to be zero. $\square$

V.6.8.1 **Proposition 6.8.1.** *(Soulé) Let R be a Dedekind domain whose field of fractions F is a global field. Then for each ℓ and each even $n \geq 2$, the boundary map $\partial : K_n(F; \mathbb{Z}/\ell) \to \oplus K_{n-1}(R/\mathfrak{p}; \mathbb{Z}/\ell)$ is onto in the localization sequence with coefficients $\mathbb{Z}/\ell$.*

The conclusion of 6.8.1 is false for $n = 1$. Indeed, the kernel of $K_0(R) \to K_0(F)$ is the finite group $\text{Pic}(R)$, so $K_1(F; \mathbb{Z}/\ell) \to \oplus K_0(R/\mathfrak{p}; \mathbb{Z}/\ell)$ is not onto.

For $n = 2$, it is easy to see that the map $\partial : K_2(F; \mathbb{Z}/\ell) \to \oplus K_1(R/\mathfrak{p}; \mathbb{Z}/\ell)$ is onto, because $K_1(R; \mathbb{Z}/\ell)$ injects into $K_1(F; \mathbb{Z}/\ell)$ by Ex. IV.2.3.

Proof. Since $K_{n-1}(R/\mathfrak{p})/2\ell = K_{n-1}(R/\mathfrak{p};\mathbb{Z}/2\ell)$ surjects onto $K_{n-1}(R/\mathfrak{p})/\ell = K_{n-1}(R/\mathfrak{p};\mathbb{Z}/\ell)$ for each $\mathfrak{p}$, we may increase ℓ to assume that $\ell \not\equiv 2 \pmod 4$, so that the product (IV.2.8) is defined on $K_*(R;\mathbb{Z}/\ell)$.

Suppose first that R contains a primitive ℓ^{th} root of unity, ζ_ℓ. If $\beta \in K_2(R;\mathbb{Z}/\ell)$ is the Bott element (IV.2.5.2), then multiplication by β^{i-1} induces an isomorphism $\oplus\mathbb{Z}/\ell \cong \oplus K_1(R/\mathfrak{p})/\ell \to \oplus K_{2i-1}(R/\mathfrak{p})$ by IV.1.13. That is, every element of $\oplus K_{2i-1}(R/\mathfrak{p})$ has the form $\beta^{i-1}a$ for a in $\oplus K_1(R/\mathfrak{p})$. Lifting a to $s \in K_2(F)$, the element $x = \beta^{i-1}s$ of $K_{2i}(F;\mathbb{Z}/\ell)$ satisfies $\partial(x) = \beta^{i-1}\partial(x) = \beta^{i-1}a$, as desired.

In the general case, we pass to the integral closure R' of R in the field $F' = F(\zeta_\ell)$. Every prime ideal $\mathfrak{p}$ of R has a prime ideal $\mathfrak{p}'$ of R' lying over it, and the transfer maps $K_{2i-1}(R'/\mathfrak{p}') \to K_{2i-1}(R/\mathfrak{p})$ are all onto by IV.1.13. The surjectivity of ∂ follows from the commutative diagram

$$\begin{array}{ccc} K_{2i}(F';\mathbb{Z}/\ell) & \xrightarrow[\text{onto}]{\partial'} & \oplus_{\mathfrak{p}'} K_{2i-1}(R'/\mathfrak{p}';\mathbb{Z}/\ell) \\ \downarrow & & \downarrow \text{onto} \\ K_{2i}(F;\mathbb{Z}/\ell) & \xrightarrow{\partial} & \oplus_{\mathfrak{p}} K_{2i-1}(R/\mathfrak{p};\mathbb{Z}/\ell). \end{array} \quad \square$$

V.6.8.2 **Wild Kernels 6.8.2.** For even n, the map $K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell)$ need not be an injection. In fact the subgroup $\operatorname{div}K_n(F)$ of elements of $K_n(F)$ which map to zero in *each* quotient $K_n(F)/\ell \cdot K_n(F)$ lies in the torsion subgroup T of $K_n(R)$ (by 6.8.1) and if $\ell \cdot T = 0$ then $\operatorname{div}K_n(F)$ is the kernel of $K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell)$. Tate observed that $\operatorname{div}K_n(F)$ can be nonzero even for K_2. In fact, $\operatorname{div}K_{2i}(F)$ is isomorphic to the *wild kernel*, defined as the intersection (over all valuations v on F) of the kernels of all maps $K_{2i}(F) \to K_{2i}(F_v)$. This is proven in [225].

V.6.9 **Gersten's DVR Conjecture 6.9.** *Suppose that R is a discrete valuation domain with maximal ideal $\mathfrak{m} = sR$, residue field $k = R/\mathfrak{m}$ and field of fractions $F = R[s^{-1}]$. The localization sequence (6.6) becomes*

$$\cdots \xrightarrow{\partial} K_n(k) \xrightarrow{i_*} K_n(R) \to K_n(F) \xrightarrow{\partial} K_{n-1}(k) \xrightarrow{i_*} K_{n-1}(R) \to \cdots.$$

Gersten conjectured that this sequence splits up (for every discrete valuation ring) into short exact sequences

$$0 \to K_n(R) \to K_n(F) \to K_{n-1}(k) \to 0.$$

This conjecture is known for $n = 0, 1, 2$ (see 6.6.2), when $\operatorname{char}(F) = \operatorname{char}(k)$ (6.7 ff) or when k is algebraic over $\mathbb{F}_p$ (6.9.2 and Ex. 6.11). It is not known in the general mixed characteristic case, *i.e.,* when $\operatorname{char}(F) = 0$ and $\operatorname{char}(k) = p$. However, it does hold for K-theory with coefficients $\mathbb{Z}/\ell$, as we now show.

V.6.9.1 **Theorem 6.9.1.** *If R is a discrete valuation ring with residue field k, then the localization sequence with coefficients breaks up:*

$$0 \to K_n(R;\mathbb{Z}/\ell) \to K_n(F;\mathbb{Z}/\ell) \xrightarrow{\partial} K_{n-1}(k;\mathbb{Z}/\ell) \to 0.$$

Proof. We begin with the case when $1/\ell \in k$, which is due to Gillet. The *henselization* R^h of R is a union of localizations $R'_{\mathfrak{m}'}$ over semilocal Dedekind domains R' which are étale over R such that $R'/\mathfrak{m}' \cong k$; see [127, I.4.8]. By Gabber Rigidity IV.2.10, $K_*(k;\mathbb{Z}/\ell) \cong K_*(R^h;\mathbb{Z}/\ell) \cong \varinjlim K_*(R'_{\mathfrak{m}'};\mathbb{Z}/\ell)$. Hence for each $a \in K_{n-1}(k;\mathbb{Z}/\ell)$ there is an $(R',\mathfrak{m}')$ with $R'/\mathfrak{m}' \cong k$ and an $a' \in K_{n-1}(R'_{\mathfrak{m}'};\mathbb{Z}/\ell)$ mapping to a. If $s' \in R'$ generates $\mathfrak{m}'$ then $F' = R'[1/s']$ is the field of fractions of R', and the product $b = \{a', s'\}$ is an element of $K_n(F';\mathbb{Z}/\ell)$. By (6.6.3), the functor $\mathbf{M}(R') \to \mathbf{M}(R)$ induces a map of localization sequences

$$\begin{array}{ccccc}
K_n(R';\mathbb{Z}/\ell) & \longrightarrow & K_n(F';\mathbb{Z}/\ell) & \xrightarrow{\oplus\partial_{\mathfrak{p}}} & \oplus_{\mathfrak{p}} K_{n-1}(R'/\mathfrak{p};\mathbb{Z}/\ell) \\
\downarrow N_{R'/R} & & \downarrow N_{F'/F} & & \downarrow \bar{N} = \oplus N_{(R'/\mathfrak{p})/k} \\
K_n(R;\mathbb{Z}/\ell) & \longrightarrow & K_n(F;\mathbb{Z}/\ell) & \xrightarrow{\partial} & K_{n-1}(k;\mathbb{Z}/\ell).
\end{array}$$

Because the top row is a sequence of $K_*(R;\mathbb{Z}/\ell)$-modules (6.1), we have $\partial_{\mathfrak{p}}\{a', s'\} = \{a', \partial_{\mathfrak{p}} s'\}$ for all $\mathfrak{p}$. Since $\partial(s') = [R'/s'R'] = [k]$ by III.3.1.1, we see that $\partial_{\mathfrak{m}'}\{a', s'\} = \{a', [k]\} = a' \cdot 1 = a$, while if $\mathfrak{p} \neq \mathfrak{m}'$ we have $\partial_{\mathfrak{p}}\{a', s'\} = 0$. Hence $\bar{N} \oplus \partial_{\mathfrak{p}}$ sends $\{a', s'\}$ to a. By commutativity of the right square in the above diagram, we see that $b = N_{F'/F}(\{a', s'\})$ is an element of $K_n(F;\mathbb{Z}/\ell)$ with $\partial(b) = a$. This shows that ∂ is onto, and hence that the sequence breaks up as stated.

Next we consider the case that $\ell = p^\nu$ and k has characteristic p, which is due to Geisser and Levine [62, 8.2]. We will need the following facts from VI.4.7 below, which were also proven in *op. cit.*: the group $K_{n-1}(k)$ has no ℓ-torsion, and $K_n^M(k)/\ell^\nu \to K_n(k;\mathbb{Z}/\ell^\nu)$ is an isomorphism for all ν. By the Universal Coefficient Theorem IV.2.5, this implies that every element of $K_n(k;\mathbb{Z}/\ell)$ is the image of a symbol $\{a_1, \dots, a_n\}$ with a_i in $k^\times$. If $a'_i \in R^\times$ is a lift of a_i and $s \in R$ is a parameter then $\partial : K_1(F) \to K_0(k)$ sends s to $[k]$ by III.1.1. It follows that $\partial : K_{n+1}(F;\mathbb{Z}/\ell) \to K_n(k;\mathbb{Z}/\ell)$ is onto, because it sends $\{s, a'_1, \dots, a'_n\}$ to $\{a_1, \dots, a_n\}$. □

We can now prove Gersten's DVR Conjecture 6.9 when k is finite; the same proof works if k is algebraic over a finite field (Ex. 6.11).

V.6.9.2 **Corollary 6.9.2.** *If R is a discrete valuation domain whose residue field k is finite, then for all $i > 0$: $K_{2i-1}(R) \cong K_{2i-1}(F)$ and there is a split exact sequence*

$$0 \to K_{2i}(R) \to K_{2i}(F) \underset{\leftarrow}{\longrightarrow} K_{2i-1}(k) \to 0.$$

Proof. It suffices to show that the map $K_{2i}(F) \xrightarrow{\partial} K_{2i-1}(k)$ is a split surjection in the localization sequence (6.6), because $K_{2i}(k) = 0$ by IV.1.13. Set $\ell = |k|^i - 1$ and recall from IV.1.13 that $K_{2i-1}(k) \cong \mathbb{Z}/\ell$, and hence $K_{2i}(k;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$. Pick a generator b of this group; using the surjection $K_{2i+1}(F;\mathbb{Z}/\ell) \to K_{2i}(k;\mathbb{Z}/\ell)$ of 6.9.1, lift b to an element a of $K_{2i+1}(F;\mathbb{Z}/\ell)$. By compatibility of localization sequences (Ex. 5.2), the image of a under $K_{2i+1}(F;\mathbb{Z}/\ell) \to K_{2i}(F)$ is sent by ∂ to b. $\square$

We can now determine the K-theory with coefficients of local fields.

V.6.10 **Proposition 6.10.** *Let E be a local field, finite over $\mathbb{Q}_p$, and let π be a parameter for the ring of integers in E. If the residue field is $\mathbb{F}_q$ and $q \equiv 1 \pmod{m}$, then $K_*(E;\mathbb{Z}/m)$ is a free $\mathbb{Z}/m[\beta]$-module on generators 1 and z, where β is the Bott element and z is the class of π in $K_1(E;\mathbb{Z}/m) = E^\times/E^{\times m}$.*

Since $\{\pi,\pi\} = \{-1,\pi\}$ in $K_2(E)$ (III.5.10.2), the ring structure is given by $z^2 = [-1]\cdot z$. If $q = 2^\nu$ or if $q \equiv 1 \pmod{2m}$ then $[-1] = 0$ in $K_1(E)/m$ so $z^2 = 0$.

Proof. Since the ring R of integers in E is a complete DVR and $R/\pi R \cong \mathbb{F}_q$, Gabber rigidity (IV.2.10) implies that $K_*(R;\mathbb{Z}/m) \cong K_*(\mathbb{F}_q;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$. By 6.9.2, $K_*(E;\mathbb{Z}/m)$ is a free $\mathbb{Z}/m[\beta]$-module on generators 1 and z. $\square$

V.6.10.1 **Remark 6.10.1.** We will see in VI.7.4.1 that when $n = p^\nu$ we have

$$K_{2i-1}(E;\mathbb{Z}/n) \cong (\mathbb{Z}/n)^d \oplus \mathbb{Z}/m_i \oplus \mathbb{Z}/m_{i-1},$$

where $d = [E:\mathbb{Q}_p]$, $m_i = \min\{n, w_i^{(p)}\}$ and the numbers $w_i^{(p)}$ are defined in VI, 2.2 and 2.3. By Moore's Theorem III.6.2.4, the torsion subgroup of $K_2(E)$ is the group of roots of unity in E, cyclic of order w_1, so $K_3(E)_{(p)}$ must be the direct sum of $\mathbb{Z}/w_2^{(p)}$ and an extension of a p-divisible group by $\mathbb{Z}^d_{(p)}$.

I do not know how to reconstruct the other groups $K_*(E)$ from the above information; there might be a $\mathbb{Z}^d_{(p)}$ in all odd degrees, or there might be divisible p-torsion in even degrees.

V.6.10.2 **Example 6.10.2.** Consider the union E_q of the local fields E over $\mathbb{Q}_p$ whose ring of integers R has residue field $\mathbb{F}_q$. For each such E, $K_*(E;\mathbb{Z}/m)$ is described by Proposition 6.10. If $E \subset E'$ is a finite field extension, with ramification index e, the map $K_*(E;\mathbb{Z}/m) \to K_*(E';\mathbb{Z}/m)$ sends z_E to $e\,z_{E'}$. If m divides e, the map sends z_E to 0. Taking the direct limit over all E, we see that $K_*(E_q;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$.

Localization for Schemes

V.6.11 **Example 6.11.** Let X be a noetherian scheme, $j : U \subset X$ an open subscheme, and $Z = X - U$ the closed complement. In this case we take $\mathcal{A} = \mathbf{M}(X)$ and $\mathcal{B} = \mathbf{M}_Z(X)$ the category of coherent X-modules supported on Z, *i.e.,* modules whose restriction to U is zero. Gabriel has shown that $\mathbf{M}(X)/\mathbf{M}_Z(X)$ is the category $\mathbf{M}(U)$ of coherent U-modules; see II.6.4.2. By devissage (Ex. 4.3), $K\mathbf{M}_Z(X) \simeq G(Z)$. Hence we have a homotopy fibration $G(Z) \to G(X) \to G(U)$, and the localization sequence becomes:

$$\cdots \xrightarrow{\partial} G_n(Z) \to G_n(X) \xrightarrow{j^*} G_n(U) \xrightarrow{\partial} G_{n-1}(Z) \to \cdots$$

ending in the exact sequence $G_0(Z) \to G_0(X) \to G_0(U) \to 0$ of II.6.4.2. This is a sequence of $K_*(X)$-modules, because $\otimes$ is a biexact pairing of $\mathbf{VB}(X)$ with the sequence $\mathbf{M}(Z) \to \mathbf{M}(X) \to \mathbf{M}(U)$.

V.6.11.1 **Example 6.11.1.** If R is commutative noetherian, $U = \mathrm{Spec}(R[s])$ is an open subset of the line $\mathbb{P}^1_R$ with complement $\infty : \mathrm{Spec}(R) \hookrightarrow \mathbb{P}^1_R$. Since $\mathbb{P}^1_R \xrightarrow{\pi} \mathrm{Spec}(R)$ is flat, the map $\pi^* : G(R) \to G(\mathbb{P}^1_R)$ exists and $j^*\pi^* \simeq p^*$ is the homotopy equivalence of Theorem 6.2. It follows that π^* splits the localization sequence $G(R) \xrightarrow{\infty_*} G(\mathbb{P}^1_R) \xrightarrow{j^*} G(R)$ and hence we have $G_n(\mathbb{P}^1_R) \cong G_n(R) \oplus G_n(R)$.

V.6.11.2 **Mayer-Vietoris Sequences 6.11.2.** If $X = U \cup V$ then $Z = X - U$ is contained in V and $V - Z = U \cap V$. Comparing the two localization sequences, we see that the square

$$\begin{array}{ccc} G(X) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ G(V) & \longrightarrow & G(U \cap V). \end{array}$$

is homotopy cartesian, *i.e.,* $G(X) \xrightarrow{\Delta} G(U) \times G(V) \xrightarrow{\pm} G(U \cap V)$ is a homotopy fibration sequence; on homotopy groups it yields the long exact "Mayer-Vietoris" sequence

$$\cdots \to G_{n+1}(U \cap V) \xrightarrow{\partial} G_n(X) \xrightarrow{\Delta} G_n(U) \times G_n(V) \xrightarrow{\pm} G_n(U \cap V) \xrightarrow{\partial} \cdots.$$

V.6.12 **Smooth Curves 6.12.** Suppose that X is an irreducible curve over a field k, with function field F. For each closed point $x \in X$, the field $k(x)$ is a finite field extension of k. The category $\mathbf{M}_0(X)$ of coherent torsion modules (modules of finite length) is a Serre subcategory of $\mathbf{M}(X)$, and $\mathbf{M}(F) \cong \mathbf{M}(X)/\mathbf{M}_0(X)$ (II.6.4.2). By devissage (4.3), $K_*\mathbf{M}_0(X) \cong \oplus_x K_*(k(x)) = \oplus K_*(x)$. In this case, the Localization sequence (5.1.1) becomes:

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_x K_n(k(x)) \xrightarrow{\oplus(i_x)_*} G_n(X) \to K_n(F) \xrightarrow{\partial} \cdots.$$

Here the maps $(i_x)_* : K_n(x) \to G_n(X)$ are the finite transfer maps of 3.6 and 3.6 associated to the inclusion of $x = \mathrm{Spec}(k(x))$ into X. If X is regular, then $K_n(X) \cong G_n(X)$ and this sequence tells us about $K_n(X)$.

The residue fields $k(x)$ of X are finite field extensions of k. As such, we have transfer maps $N_{k(x)/k} : K_n(k(x)) \to K_n(k)$. The following result, due to Gillet, generalizes the Weil Reciprocity of III.6.5.3 for symbols $\{f, g\} \in K_2(F)$. We write ∂_x for the component $K_{n+1}(F) \to K_n(x)$ of the map ∂ in 6.12.

V.6.12.1 **Weil Reciprocity Formula 6.12.1.** Let X be a projective curve over a field k, with function field F. For every $a \in K_{n+1}(F)$ we have the following formula in $K_n(k)$:

$$\sum\nolimits_{x \in X} N_{k(x)/k}\partial_x(a) = 0.$$

Proof. Consider the proper transfer $\pi_* : G_n(X) \to G_n(k)$ of 3.7 associated to the structure map $\pi : X \to \mathrm{Spec}(k)$. By functoriality, $\pi_*(i_x)_* = N_{k(x)/k}$ for each closed point $x \in X$. Because $\oplus_x (i_x)_* \partial_x = (\oplus (i_x)_*)\partial = 0$, we have:

$$\sum N_{k(x)/k}\partial_x(a) = \sum \pi_*(i_x)_*\partial_x(a) = \pi_* \sum (i_x)_*\partial_x(a) = 0. \qquad \square$$

V.6.13 **Fundamental Theorem for $G(X)$ 6.13.** *If X is a noetherian scheme, the flat maps $X[s,s^{-1}] \xhookrightarrow{j} X[s] \xrightarrow{p} X$ induce a homotopy equivalence $p^* : G(X) \simeq G(X[s])$ and isomorphisms*

$$G_n(X[s,s^{-1}]) \cong G_n(X) \oplus G_{n-1}(X).$$

Proof. If $z : X \to X[s]$ is the zero-section, we saw in 3.6.1 that $z_* = 0$. Hence the Localization Sequence (6.11) splits into short exact sequences

$$0 \xrightarrow{z_*} G_n(X[s]) \xrightarrow{j^*} G_n(X[s,s^{-1}]) \xrightarrow{\partial} G_{n-1}(X) \xrightarrow{z_*} 0.$$

If $f : X \to X[s,s^{-1}]$ is the section $s = 1$, we saw in 3.6.1 that f defines a map f^* and that $f^*j^*p^*$ is homotopic to the identity of $G(X)$. Hence it suffices to prove that $p^* : G(X) \simeq G(X[s])$.

We first suppose that X is separated, so that the intersection of affine opens is affine, and proceed by induction on the number of affine opens. If X is affine, this is Theorem 6.2, so suppose that $X = U_1 \cup U_2$ with U_2 affine. The inductive hypothesis applies to U_1 and $U_{12} = U_1 \cap U_2$, and we have a map of Mayer-Vietoris sequences (6.11.2):

$$\begin{array}{ccccccccc}
* & \longrightarrow & G_{n+1}(U_{12}) & \xrightarrow{\partial} & G_n(X) & \longrightarrow & G_n(U_1) \times G_n(U_2) & \longrightarrow & G_n(U_{12}) \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow & & \simeq\downarrow & & \simeq\downarrow \\
* & \longrightarrow & G_{n+1}(U_{12}[s]) & \xrightarrow{\partial} & G_n(X[s]) & \to & G_n(U_1[s]) \times G_n(U_2[s]) & \to & G_n(U_{12}[s]).
\end{array}$$

The 5-lemma implies that $G(X) \simeq G(X[s])$ for X separated. Another induction establishes the result for non-separated X, since any open subscheme of a separated scheme is separated. $\square$

V.6.13.1 **Corollary 6.13.1.** *If X is noetherian, then $X \times \mathbb{P}^1 \xrightarrow{\pi} X$ and $X \xrightarrow{\infty} X \times \mathbb{P}^1$ induce homotopy equivalences $(\pi^*, \infty_*) : G(X) \oplus G(X) \xrightarrow{\simeq} G(X \times \mathbb{P}^1)$.*

Proof. As in 6.11.1, the complement of $\infty(X)$ is $X \times \mathbb{A}^1$ and the map π^* is split by $j^* : G(X \times \mathbb{P}^1) \to G(X \times \mathbb{A}^1) \simeq G(X)$. Hence π^* splits the localization sequence $G(X) \xrightarrow{\infty_*} G(X \times \mathbb{P}^1) \xrightarrow{j^*} G(X)$. $\square$

V.6.13.2 **Corollary 6.13.2.** *If X is regular noetherian then for all n:*

$$K_n(X) \cong K_n(X[s]), \quad K_n(X[s,s^{-1}]) \cong K_n(X) \oplus K_{n-1}(X)$$

and $K_n(X \times \mathbb{P}^1) \cong K_n(X) \oplus K_n(X)$.

We now generalize 6.11.1 and 6.13.1 from $\mathbb{P}^1$ to $\mathbb{P}^r$; the G_0 version of the following result was given in II, Ex. 6.14. If X is regular, Theorem 6.14 and Ex. 6.7 provide another proof of the Projective Bundle Theorem 1.5 and 1.5.1.

V.6.14 **Theorem 6.14.** *If X is a noetherian scheme, the functors $\mathbf{M}(X) \xrightarrow{\mathcal{O}(i)} \mathbf{M}(\mathbb{P}^r_X)$ sending M to $M \otimes \mathcal{O}(i)$ induce an isomorphism $\sum \mathcal{O}(i) : \coprod_{i=0}^r G(X) \xrightarrow{\simeq} G(\mathbb{P}^r_X)$. That is, $G_n(\mathbb{P}^r_X) \cong G_n(X) \otimes K_0(\mathbb{P}^r_{\mathbb{Z}})$.*

Proof. By localization, we may assume $X = \mathrm{Spec}(R)$; set $S = R[x_0, \cdots, x_r]$. The standard description of coherent sheaves on $\mathbb{P}^r_R$ ([85, Ex. II.5.9]) amounts to the equivalence of quotient categories

$$\mathbf{M}(\mathbb{P}^r_R) \cong \mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S),$$

where $\mathbf{M}^b_{gr}(S)$ is the Serre subcategory of graded modules M with $M_m = 0$ for all but finitely many m. By devissage (Ex. 4.4), $\mathbf{M}^b_{gr}(S)$ has the same K-theory as its subcategory $\mathbf{M}_{gr}(R)$, *i.e.,* $G_*(R)[\sigma, \sigma^{-1}]$. Thus the localization sequence (5.1.1) and Example 3.5.2 give us the following diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_n(\mathbf{M}_{gr}(R)) & \longrightarrow & K_n(\mathbf{M}_{gr}(R[x_0, \cdots, x_r])) & \longrightarrow & G_n(\mathbb{P}^r_R) & \longrightarrow \cdots \\
 & \simeq \uparrow & & \simeq \uparrow & & \uparrow & \\
0 \longrightarrow & G_n(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] & \xrightarrow{h} & G_n(R) \otimes \mathbb{Z}[\sigma, \sigma^{-1}] & \longrightarrow & G_n(R) & \longrightarrow 0.
\end{array}$$

Now σ is induced by $\otimes_S S(-1)$. Setting $E = S^{r+1}$, consider the Koszul resolution for R (I.5.4):

$$0 \to \wedge^{r+1} E(-r-1) \to \cdots \to \wedge^2 E(-2) \to E(-1) \to S \to R \to 0.$$

The Additivity Theorem for $\otimes_S S(i) : \mathbf{M}_{gr}(S) \to \mathbf{M}_{gr}(S)$ shows that h is multiplication by the map $\sum \binom{r+1}{i} (-\sigma)^i = (1-\sigma)^{r+1}$. Since this is an injection with the prescribed cokernel, we have proven the first assertion. The second is immediate from II.8.6. $\square$

EXERCISES

EV.6.1 **6.1.** Suppose that R is a 1-dimensional commutative noetherian domain with fraction field F. Show that there is a long exact sequence

$$\cdots \to K_{n+1}(F) \xrightarrow{\partial} \oplus_{\mathfrak{p}} K_n(R/\mathfrak{p}) \to G_n(R) \to K_n(F) \xrightarrow{\partial} \cdots.$$

ending in the Heller-Reiner sequence of Ex. II.6.8. If R is regular, i.e., a Dedekind domain, this is the sequence (6.6).

Now let R be the local ring of $\mathbb{R} + x\mathbb{C}[x]$ at the maximal ideal (x, ix). Show that $G_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and conclude that the localization sequence (6.1.1) does not split up into short exact sequences.

EV.6.2 **6.2.** If S is the seminormalization of R (I, Ex. 3.15–3.16), and S is finite over R, show that the transfer $G(S) \to G(R)$ is an equivalence.

EV.6.3 **6.3.** If R is noetherian, show that the splitting in Theorem 6.2 is given by the map $G_n(R) \to G_{n+1}(R[s, s^{-1}])$ which is multiplication by $s \in K_1(\mathbb{Z}[s, s^{-1}])$. *Hint:* Use Ex. 5.3 with $\mathcal{C} = \mathbf{M}(R)$.

EV.6.4 **6.4.** Let φ be an automorphism of a commutative noetherian ring R, and let $R_\varphi[t]$ be the twisted polynomial ring of all polynomials $\sum r_i t^i$ with multiplication determined by $tr = \varphi(r)t$. It is well known that $R_\varphi[t]$ is a regular ring when R is. Show that the hypotheses of Theorem 6.4 are satisfied, so that $G(R) \simeq G(R_\varphi[t])$. Then show that $\{t^n\}$ is a multiplicative system (II.A.1.1), so that the localization $R_\varphi[t, 1/t]$ is a well defined ring. Then show that there is an exact sequence:

$$\cdots \to G_n(R) \xrightarrow{\varphi_*} G_n(R) \to G_n(R_\varphi[t, 1/t]) \to \cdots .$$

EV.6.5 **6.5.** Let R be commutative noetherian, and $S = \mathrm{Sym}(P)$ the symmetric algebra (I.5.8) of a finitely generated projective R-module P. Show that $R \subset S$ induces an equivalence $G(R) \simeq G(S)$. *Hint:* Locally, $S \cong R[s_1, ..., s_n]$.

EV.6.6 **6.6.** *Jouanolou's Trick.* Let X be a noetherian scheme. If $p : E \to X$ is a vector bundle, show that $p^* : G(X) \to G(E)$ is an equivalence. More generally, show that $p^* : G(X) \to G(E)$ is an equivalence for any flat map $p : E \to X$ whose fibers are affine spaces (such as a torsor under a vector bundle E).

Jouanolou proved that if X is quasi-projective over a field then there is an affine scheme $E = \mathrm{Spec}(R)$ and a torsor $E \to X$ under a vector bundle. It follows that $G(X) \simeq G(R)$. This trick reduces the study of $G(X)$ for quasi-projective X to the study of $G(R)$.

EV.6.7 **6.7.** (Quillen) If E is any vector bundle over X of rank $r+1$, we can form the projective bundle $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}(E))$. Modify the proof of Theorem 6.14 to show that once again the maps $M \mapsto M \otimes \mathcal{O}(i)$ induce an isomorphism

$$G_n(\mathbb{P}(E)) \cong K_0(\mathbb{P}(E)) \otimes G_n(X) \cong \coprod_{i=0}^{r} G_n(X).$$

EV.6.8 **6.8.** Let X be the affine plane with a double origin over a field k, the standard example of a (quasi-compact) scheme which is not separated. Generalize II.8.2.4 by showing that $G(X) \simeq G(k) \times G(k)$. Then show that every vector bundle is trivial and $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^2)$, so $K\mathbf{VB}(X) \simeq K(k)$. Since $G(X) \simeq K(X)$ (by Ex. 3.9), this shows that $K\mathbf{VB}(X) \neq K(X)$. Note that $G(X) = K\mathbf{VB}(X)$ for separated regular schemes by Resolution (see 3.4.2).

EV.6.9 **6.9.** (Roberts) If $\pi : X' \to X$ is a finite birational map, there is a closed $Z \subset X$ (nowhere dense) such that $\pi : X' - Z' \cong X - Z$, where $Z' = Z \times_X X'$.
(a) Show that there is a fibration $G(Z') \to G(Z) \times G(X') \to G(X)$. This yields a Mayer-Vietoris sequence

$$G_{n+1}(X) \xrightarrow{\partial} G_n(Z') \xrightarrow{\Delta} G_n(Z) \times G_n(X') \xrightarrow{\pm} G_n(X) \xrightarrow{\partial} G_{n-1}(Z').$$

(b) If X is the node $\mathrm{Spec}(k[x,y]/(y^2-x^3-x))$, show that $G_n(X)\cong G_n(k)\oplus G_{n-1}(k)$ for all n. This contrasts with $K_0(X)\cong K_0(k)\oplus K_1(k)$; see II.2.9.1.

EV.6.10 **6.10.** Suppose that a curve X is a union of n affine lines, meeting in a set Z of m rational points (isomorphic to $\mathrm{Spec}(k)$), and that no three lines meet in a point. Show that $G_n(X)\cong G_n(k)^n\oplus G_{n-1}(k)^m$.

EV.6.11 **6.11.** Let R be a DVR whose residue field k is infinite, but algebraic over $\mathbb{F}_p$. Modify the proof of 6.9.2 to show that Gersten's DVR conjecture 6.9 holds for R.

EV.6.12 **6.12.** Complete the proof of Theorem 6.4 for filtered rings, by mimicking the proof of Theorem 6.2.

EV.6.13 **6.13.** Let $R=k\oplus R_1\oplus\cdots$ be a graded ring such that R and $R[t]$ are coherent (3.3) and of finite flat dimension over k, and k has finite flat dimension over R. Modify the proof of Theorem 6.4 to show that $k\subset R$ induces $G(k)\simeq G(R)$. *Hint:* $\mathbf{M}_{gr}(S)$ is still an abelian category.

EV.6.14 **6.14.** The formula $\lambda_s(a)=\partial(\{s,a\})$ defines a specialization map for K-theory with coefficients $\mathbb{Z}/m$. Now suppose that k is algebraically closed, and show that: (i) the specialization $\lambda_s: K_n(F;\mathbb{Z}/m)\to K_n(k;\mathbb{Z}/m)$ is independent of the choice of $s\in R$; (ii) for $F=k(t)$, the specialization $\lambda_s: K_n(F;\mathbb{Z}/m)\to K_n(k;\mathbb{Z}/m)$ is independent of R and s. *Hint:* If $f\in F$ then $\lambda(f)=0$ as $K_1(k;\mathbb{Z}/m)=0$.

7 Localization for $K_*(R)$ and $K_*(X)$.

When R is a ring and $s\in R$ is a central element, we would like to say something about the localization map $K(R)\to K(R[\frac{1}{s}])$. If R is regular, we know the third term in the long exact sequence from the localization theorem for G_*:

$$\cdots\xrightarrow{\partial}G_n(R/sR)\xrightarrow{i_*}K_n(R)\to K_n(R[1/s])\xrightarrow{\partial}\cdots.$$

Because R is regular, every R/sR-module has finite projective dimension over R, and i_* is induced from the inclusion $\mathbf{M}(R/sR)\subset\mathbf{M}(R)=\mathbf{H}(R)$. More generally, suppose that S is a central multiplicatively closed set of nonzerodivisors in R, and consider the category $\mathbf{H}_S(R)$ of all S-torsion R-modules M in $\mathbf{H}(R)$.

We saw in II.7.7.4 that the sequence $K_0\mathbf{H}_S(R)\to K_0(R)\to K_0(S^{-1}R)$ is exact. This K_0 exact sequence has the following extension to higher K-theory.

V.7.1 **Theorem 7.1** (Localization for Nonzerodivisors)**.** *Let S be a central multiplicatively closed subset of R consisting of nonzerodivisors. Then*

$$K\mathbf{H}_S(R)\to K(R)\to K(S^{-1}R)$$

is a homotopy fibration. Thus there is a long exact sequence

$$\cdots \to K_{n+1}(S^{-1}R) \xrightarrow{\partial} K_n\mathbf{H}_S(R) \to K_n(R) \to K_n(S^{-1}R) \xrightarrow{\partial} \cdots$$

ending in $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$*, the sequence of II.7.7.4.*

In this section, we will give a direct proof of Theorem 7.1, following Quillen. We note that an indirect proof is given in Exercise 3.14 above, using Theorem 2.6.3. In addition, we saw in Ex. 2.9 that the nonzerodivisor hypothesis is necessary. In the general case, the third term $K(R \text{ on } S)$ is more complicated; see Theorem 2.6.3.

V.7.1.1 **Caveat 7.1.1.** The map $K_0(R) \to K_0(S^{-1}R)$ is not onto. Instead, the sequence continues with $K_{-1}\mathbf{H}_S(R)$ etc., using negative K-groups. In order to get a spectrum-level fibration, therefore, one needs to use the non-connective spectra $\mathbf{K}^B(R)$ of IV.10.4 to get nontrivial negative homotopy groups, or else replace $K(S^{-1}R)$ with $K(\mathcal{P})$, as we shall now do (and did in the proofs of II.9.8 and 2.6.3).

V.7.2 **Definition 7.2.** Let $\mathcal{P}$ denote the exact subcategory of $\mathbf{P}(S^{-1}R)$ consisting of projective modules which are localizations of projective R-modules.

By Cofinality (IV.6.4.1) and Ex. IV.6.6, $K_n\mathcal{P} \cong K_n(S^{-1}R)$ for $n > 0$, and $K_0(R) \to K_0(S^{-1}R)$ factors as a surjection $K_0(R) \to K_0\mathcal{P}$ followed by an inclusion $K_0\mathcal{P} \to K_0(S^{-1}R)$. Hence we may replace $K(S^{-1}R)$ by $K\mathcal{P}$ in Theorem 7.1.

Let $\mathcal{E}$ denote the category of admissible exact sequences in $\mathcal{P}$, as defined in IV.7.3; morphisms are described in (IV.7.3.1), and the target of $A \rightarrowtail B \twoheadrightarrow C$ yields a functor $t : \mathcal{E} \to Q\mathcal{P}$. We write $\mathcal{F}$ for the pullback category $Q\mathbf{P}(R) \times_{Q\mathcal{P}} \mathcal{E}$ whose objects are pairs $(P, A \rightarrowtail B \twoheadrightarrow S^{-1}P)$ and whose morphisms are pairs of compatible morphisms. Since t is a fibered functor (by Ex. IV.7.2), so is $\mathcal{F} \to Q\mathbf{P}(R)$.

The monoidal category $T = \operatorname{iso}\mathbf{P}(R)$ acts fiberwise on $\mathcal{E}$ via the inclusion of T in $\mathcal{E}$ given by IV.7.4.1. Hence T also acts on $\mathcal{F}$, and we may localize at T (in the sense of IV.4.7.1).

V.7.2.1 **Lemma 7.2.1.** *The sequence* $T^{-1}\mathcal{F} \to Q\mathbf{P}(R) \to Q\mathcal{P}$ *is a homotopy fibration.*

Proof. By IV, Ex. 4.11, the fibers of $T^{-1}\mathcal{F} \to Q\mathbf{P}(R)$ and $T^{-1}\mathcal{E} \to Q\mathcal{P}$ are equivalent. By Quillen's Theorem B (IV.3.8.1), we have a homotopy cartesian square:

$$\begin{array}{ccc} T^{-1}\mathcal{F} & \longrightarrow & T^{-1}\mathcal{E} \text{ (contractible)} \\ \big\downarrow & & \big\downarrow t \\ Q\mathbf{P}(R) & \longrightarrow & Q\mathcal{P}. \end{array}$$

We saw in the proof of Theorem IV.7.1 that $T^{-1}\mathcal{E}$ is contractible (because $\mathcal{E}$ is, by Ex. IV.7.3), whence the result. □

To prove Theorem 7.1, we need to identify $T^{-1}\mathcal{F}$ and $K\mathbf{H}_S(R)$. We first observe that we may work with the exact subcategory $\mathbf{H}_{1,S}$ of $\mathbf{H}_S(R)$ consisting of S-torsion modules of projective dimension ≤ 1. Indeed, the Resolution Theorem (see 3.2.1) implies that $K_*\mathbf{H}_S(R) \cong K_*\mathbf{H}_{1,S}$.

Next, we construct a diagram of categories on which T acts, of the form

$$Q\mathbf{H}_{1,S} \xleftarrow{h} \mathcal{G} \xrightarrow{f} \mathcal{F}$$

and whose localization is $Q\mathbf{H}_{1,S} \xleftarrow{T^{-1}h} T^{-1}\mathcal{G} \xrightarrow{T^{-1}f} T^{-1}\mathcal{F}$.

V.7.3 **Definition 7.3.** Let $\mathcal{G}$ denote the category whose objects are exact sequences $0 \to K \rightarrowtail P \twoheadrightarrow M \oplus Q$ with P, Q in $\mathbf{P}(R)$ and M in $\mathbf{H}_{1,S}$. The morphisms in $\mathcal{G}$ are isomorphism classes of diagrams (in which the maps in the bottom row are direct sums of maps):

$$\begin{array}{ccccc}
K' & \leftarrowtail & K & = & K \\
\downarrow & & \downarrow & & \downarrow \\
P' & = & P' & \rightarrowtail & P \\
\downarrow & & \downarrow & & \downarrow \\
M' \oplus Q' & \twoheadleftarrow & M_1 \oplus Q_1 & \rightarrowtail & M \oplus Q.
\end{array}$$

We let $T = \mathrm{iso}\,\mathbf{P}(R)$ act on $\mathcal{G}$ by $T\Box(P \twoheadrightarrow M \oplus Q) = T \oplus P \twoheadrightarrow P \twoheadrightarrow M \oplus Q$.

The functor $f : \mathcal{G} \to \mathcal{F}$ sends $K \rightarrowtail P \twoheadrightarrow M \oplus Q$ to $(Q, T{-}1P \twoheadrightarrow T^{-1}M)$. By inspection, f is compatible with the action of T, so $T^{-1}f$ is defined.

The functor $h : \mathcal{G} \to Q\mathbf{H}_{1,S}$ sends $K \rightarrowtail P \twoheadrightarrow M \oplus Q$ to M. The action of T is fiberwise for h (IV, Ex. 4.11) so h induces a functor $T^{-1}h : T^{-1}\mathcal{G} \to Q\mathcal{P}$.

Proof of Theorem 7.1. The maps $Q\mathbf{H}_{1,S} \xrightarrow{\simeq} T^{-1}\mathcal{G} \xrightarrow{\simeq} T^{-1}\mathcal{F}$ are homotopy equivalences by Lemmas 7.3.1–7.3.2. Let g be the composite of these equivalences with $T^{-1}\mathcal{F} \to Q\mathbf{P}(R)$, so that $Q\mathbf{H}_{1,S} \xrightarrow{g} Q\mathbf{P}(R) \to Q\mathcal{P}$ is a homotopy fibration by 7.2.1. In Lemma 7.4, we identify $-g$ with the canonical map, proving the theorem. $\Box$

V.7.3.1 **Lemma 7.3.1.** *The functor $h : \mathcal{G} \to Q\mathbf{H}_{1,S}$ is a homotopy equivalence. It follows that $\mathcal{G} \to T^{-1}\mathcal{G}$ and $T^{-1}\mathcal{G} \to Q\mathbf{H}_{1,S}$ are also homotopy equivalences.*

Proof. For each M, let $\mathcal{G}_M$ denote the category whose objects are surjections $P \twoheadrightarrow M$, and whose morphisms are admissible monics $P' \rightarrowtail P$ in $\mathbf{P}(R)$ compatible with the maps to M. Choosing a basepoint $P_0 \twoheadrightarrow M$, $P \times_M P'$ is projective and there are natural transformations $(P \twoheadrightarrow M) \leftarrow (P \times_M P' \twoheadrightarrow M) \to (P_0 \twoheadrightarrow M)$ giving a contracting homotopy for each $\mathcal{G}_M$.

The Segal subdivision $Sub(\mathcal{G}_M)$ of $\mathcal{G}_M$ (IV, Ex. 3.9) is equivalent to the fiber $h^{-1}(M)$ (by Ex. 7.2), so the fibers of h are contractible. Since h is fibered (by Ex. 7.1), Quillen's Theorem A (variation IV.3.7.4) applies to show that h is a homotopy equivalence. In particular, since T acts trivially on $Q\mathbf{H}_{1,S}$ it acts invertibly on $\mathcal{G}$ in the sense of IV.4.7. It follows from Ex. IV.4.6 that $\mathcal{G} \to T^{-1}\mathcal{G}$ is also a homotopy equivalence. $\square$

For the following lemma, we need the following easily checked fact: if P is a projective R-module, then P is a submodule of $S^{-1}P$, because S consists of central nonzerodivisors. In addition, $S^{-1}P$ is the union of the submodules $\frac{1}{s}P$, $s \in S$.

V.7.3.2 **Lemma 7.3.2.** *The functor $f : \mathcal{G} \to \mathcal{F}$ is a homotopy equivalence, and hence $T^{-1}f : T^{-1}\mathcal{G} \to T^{-1}\mathcal{F}$ is too.*

Proof. We will show that for each Q in $\mathbf{P}(R)$ the map $(pf)^{-1}(Q) \to p^{-1}(Q)$ is a homotopy equivalence of fibers. Since $p : \mathcal{F} \to Q\mathbf{P}(R)$ and $pf : \mathcal{G} \to Q\mathbf{P}(R)$ are fibered (Ex. 7.1), the result will follow from Quillen's Theorem B (IV.3.8.1).

Fix Q and let T_Q be the category whose objects are maps $P \twoheadrightarrow Q$ with P in $\mathbf{P}(R)$, and whose morphisms are module injections $P' \rightarrowtail P$ over Q whose cokernel is S-torsion. Then the functor $Sub(T_Q) \to (pf)^{-1}(Q)$ which sends $P' \rightarrowtail P$ over Q to $P \twoheadrightarrow Q \oplus (P/P')$ is an equivalence of categories (by Ex. 7.3).

Set $W = S^{-1}Q$, so that $p^{-1}(Q) = \mathcal{E}_W$. Then $Sub(T_Q) \xrightarrow{\sim} (pf)^{-1}(Q) \to \mathcal{E}_W$ sends $P' \rightarrowtail P$ over Q to $S^{-1}P \to W$; this factors through the target functor $Sub(T_Q) \to T_Q$ (which is a homotopy equivalence by Ex. IV.3.9), and the functor $w : T_Q \to \mathcal{E}_W$ sending $P \twoheadrightarrow Q$ to $S^{-1}P \twoheadrightarrow W$. Thus we are reduced to showing that w is an equivalence. By Quillen's Theorem A (IV.3.7), it suffices to fix V in $\mathcal{P}$ and $E : V \twoheadrightarrow W$ in $\mathcal{E}_W$ and show that w/E is contractible.

Consider the poset Λ of projective R-submodules P of V such that $S^{-1}P \cong V$, and whose image under $V \to W$ is Q. The evident map from Λ to w/E sending P to $(P, S^{-1}P \twoheadrightarrow W)$ is an equivalence, so we are reduced to showing that Λ is contractible. Fix P and P' in Λ and let K, K' denote the kernels of $P \twoheadrightarrow Q$ and $P' \twoheadrightarrow Q$. Then $S^{-1}K = S^{-1}K'$ and $S^{-1}K + P = S^{-1}K' + P'$ as submodules of V. Thus for some $s \in S$, both P and P' are contained in the submodule $P'' = P + \frac{1}{s}K$ of V. But P'' is easily seen to be in Λ, proving that Λ is a filtering poset. Thus Λ and hence w/E are contractible. $\square$

V.7.4 **Lemma 7.4.** *The canonical map $Q\mathbf{H}_{1,S} \to Q\mathbf{P}(R)$ is homotopic to the additive inverse of $g : Q\mathbf{H}_{1,S} \xleftarrow{T^{-1}h} T^{-1}\mathcal{G} \xrightarrow{T^{-1}f} T^{-1}\mathcal{F} \xrightarrow{p} Q\mathbf{P}(R)$.*

Proof. Recall from II.7.7 that $\mathbf{H}_1 = \mathbf{H}_1(R)$ denotes the category of R-modules M having a resolution $0 \to P_1 \to P_0 \to M \to 0$ with the P_i in $\mathbf{P}(R)$. By the Resolution Theorem 3.1.1, $K(R) \simeq K\mathbf{H}_1$ and $Q\mathbf{P}(R) \simeq Q\mathbf{H}_1$. We claim that the following diagram commutes up to sign.

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{p \circ f} & Q\mathbf{P}(R) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \simeq} \\
Q\mathbf{H}_{1,S}(R) & \longrightarrow & Q\mathbf{H}_1.
\end{array}$$

The two functors from $\mathcal{G}$ to $Q\mathbf{H}_1$ in this diagram send $\pi = (K \rightarrowtail P \twoheadrightarrow M \oplus Q)$ to M and Q, respectively. By Additivity 1.2, their sum is homotopic to the target functor t. It suffices to show that t is null homotopic.

Let s be the functor $\mathcal{G} \to Q\mathbf{P}(R)$ sending π to P; since s maps all morphisms to injections, $0 \rightarrowtail P$ defines a natural transformation $0 \Rightarrow s$. Since $P \twoheadrightarrow M \oplus Q$ is a natural transformation $t \Rightarrow s$, we have a homotopy $t \sim s \sim 0$, as desired. □

The following result generalizes Exercises III.3.10 and III.4.11.

V.7.5 **Proposition 7.5.** *(Karoubi) Let $f : A \to B$ be a ring homomorphism and S a central multiplicatively closed set of nonzerodivisors in A such that $f(S)$ is a central set of nonzerodivisors in B. Assume that $f : A/sA \cong B/sB$ for all $s \in S$. Then: (1) the functor $\mathbf{H}_S(A) \to \mathbf{H}_S(B)$ is an equivalence;*

(2) the square

$$\begin{array}{ccc} \mathbf{K}^B(A) & \longrightarrow & \mathbf{K}^B(B) \\ \downarrow & & \downarrow \\ \mathbf{K}^B(S^{-1}A) & \longrightarrow & \mathbf{K}^B(S^{-1}B) \end{array}$$

is homotopy cartesian, and (3) there is a Mayer-Vietoris sequence

$$\cdots K_{n+1}(S^{-1}B) \xrightarrow{\partial} K_n(A) \xrightarrow{\Delta} K_n(B) \times K_n(S^{-1}A) \xrightarrow{\pm} K_n(S^{-1}B) \xrightarrow{\partial} \cdots.$$

Proof. The assumption implies that $\hat{A} = \varprojlim A/sA$ and $\hat{B} = \varprojlim B/sB$ are isomorphic. Therefore, in order to prove part (1) we may assume that $B = \hat{A}$. Then B is faithfully flat over A, and $\otimes_A B$ is a faithful exact functor from $\mathbf{H}_S(A)$ to $\mathbf{H}_S(B)$. To show that $\mathbf{H}_S(A) \to \mathbf{H}_S(B)$ is an equivalence, it suffices to show that every M in $\mathbf{H}_S(B)$ is isomorphic to a module coming from $\mathbf{H}_S(A)$.

Choose a B-module resolution $0 \to Q_1 \xrightarrow{f} Q_0 \to M \to 0$, and $s \in S$ so that $sM = 0$. Then the inclusion $sQ_0 \subset Q_0$ factors through a map $f' : sQ_0 \to Q_1$. The cokernel $M' = Q_1/sQ_0$ is also in $\mathbf{H}_S(B)$, and if we choose Q' so that $Q_0 \oplus Q_1 \oplus Q' \cong B^n$ then we have a resolution

$$0 \to B^n \xrightarrow{\gamma=(f,f',1)} B^n \to M \oplus M' \to 0.$$

Replacing M by $M \oplus M'$, we may assume that $Q_1 = Q_0 = B^n$. Then there is a matrix γ' over B such that $\gamma\gamma' = sI$. Now the assumption that $A/sA \cong B/sB$ implies that there is a matrix $\alpha : A^n \to A^n$ of the form $\alpha = \gamma + s^2\beta$. Since $\alpha = \gamma(I + s\gamma'\beta)$ and $I + s\gamma'\beta$ is invertible over $B = \hat{A}$, we see that the A-module $\operatorname{coker}(\alpha)$ is in $\mathbf{H}_S(A)$ and that $\operatorname{coker}(\alpha) \otimes_A B \cong \operatorname{coker}(\gamma) = M$.

Part (2) follows from Part(1) and Theorem 7.1; the Mayer-Vietoris sequence follows formally from the square (as in 6.11.2). □

V.7.5.1 **Example 7.5.1.** The proposition applies to $S = \{p^n\}$ and the rings $\mathbb{Z} \to \mathbb{Z}_{(p)} \to \hat{\mathbb{Z}}_p$ (p-adics). If G is a group, it also applies to $\mathbb{Z}[G] \to \mathbb{Z}_{(p)}[G] \to \hat{\mathbb{Z}}_p[G]$.

That is, $K_*(\mathbb{Z}[G])$ fits into a Mayer-Vietoris sequence involving $K_*(\mathbb{Z}_{(p)}[G])$, $K_*(\mathbb{Z}[\frac{1}{p}][G])$ and $K_*(\mathbb{Q}[G])$, as well as a Mayer-Vietoris sequence of the form:

$$K_{n+1}(\hat{\mathbb{Q}}_p[G]) \xrightarrow{\partial} K_n(\mathbb{Z}[G]) \to K_n(\hat{\mathbb{Z}}_p[G]) \oplus K_n(\mathbb{Z}[\frac{1}{p}][G]) \to K_n(\hat{\mathbb{Q}}_p[G]) \xrightarrow{\partial} \cdots.$$

More generally, let $S \subset \mathbb{Z}$ be generated by a finite set P of primes; Proposition 7.5 applies to $\mathbb{Z}[G] \to \prod_{p\in P} \hat{\mathbb{Z}}_p[G]$, and there is a similar Mayer-Vietoris sequence relating $K_*(\mathbb{Z}[G])$ to the product of the $K_*(\hat{\mathbb{Z}}_p[G])$. This sequence is particularly useful when G is finite and P is the set of primes dividing $|G|$, because the rings $S^{-1}\mathbb{Z}[G]$ and $\hat{\mathbb{Q}}_p[G]$ are semisimple.

Localization for vector bundles

Here is the analogue of Theorem 7.1 for vector bundles on a scheme X, which generalizes the exact sequence in Ex. II.8.1. As discussed in II.8.3, we assume that X is quasi-compact in order that every module in $\mathbf{H}(X)$ has a finite resolution by vector bundles. We define $K(X \text{ on } Z)$ to be the K-theory space of the Waldhausen category $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ of perfect complexes on X which are exact on U.

V.7.6 **Theorem 7.6.** *(Thomason-Trobaugh [200, 5.1]) Let X be a quasi-compact, quasi-separated scheme, and U a quasi-compact open in X with complement Z. Then $K(X \text{ on } Z) \to K(X) \xrightarrow{j^*} K(U)$ is a homotopy fibration, and there is a long exact sequence*

$$\cdots \to K_{n+1}(U) \xrightarrow{\partial} K_n(X \text{ on } Z) \to K_n(X) \to K_n(U) \xrightarrow{\partial} \cdots$$

ending in $K_0(X \text{ on } Z) \to K_0(X) \to K_0(U)$, the sequence of Ex. II.8.1.

Recall that $\mathbf{H}_Z(X)$ denotes the category of modules in $\mathbf{H}(X)$ supported on Z.

V.7.6.1 **Corollary 7.6.1.** *If X is quasiprojective and Z is defined by an invertible ideal, then $\mathbf{H}_Z(X) \subset \mathbf{Ch}_{\mathrm{perf},Z}(X)$ induces an equivalence on K-theory. Thus there is a long exact sequence*

$$\cdots \to K_{n+1}(U) \xrightarrow{\partial} K_n\mathbf{H}_Z(X) \to K_n(X) \to K_n(U) \xrightarrow{\partial} \cdots.$$

Proof. (Thomason) The Approximation Theorem implies that $K\mathbf{H}_Z(X) \simeq K(X \text{ on } Z)$, as we saw in Ex. 3.16 □

V.7.6.2 **Remark 7.6.2.** Since the model $K\mathbf{Ch}_{\mathrm{perf},Z}(X)$ for the fiber $K(X \text{ on } Z)$ of $K(X) \to K(U)$ is complicated, it would be nice to have a simpler model. A naïve guess for such a model would be the K-theory of the category $\mathbf{H}_Z(X)$. This is correct if X is regular by the localization sequence (6.11) for G-theory, if Z is a divisor (7.6.1), and even if Z is locally a complete intersection in X [200, 5.7]. Exercise 7.4 shows that this cannot be right in general, even if $X = \mathrm{Spec}(A)$.

V.7.6.3 **Definition 7.6.3.** Let us define the (non-connective) spectrum $\mathbf{K}^B(X \text{ on } Z)$ to be the homotopy fiber of the morphism $\mathbf{K}^B(X) \to \mathbf{K}^B(X - Z)$. Then Theorem 7.6 states that the K-theory spectrum $\mathbf{K}(X \text{ on } Z)$ of $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ is the (-1)-connected cover of $\mathbf{K}^B(X \text{ on } Z)$. In particular, $K_n(X \text{ on } Z)$ is $\pi_n\mathbf{K}^B(X \text{ on } Z)$ for all $n \geq 0$. For $n < 0$ we define $K_n(X \text{ on } Z) = \pi_n\mathbf{K}^B(X \text{ on } Z)$; since $K_n(X) = \pi_n\mathbf{K}^B(X)$ in this range (by IV.10.6), the sequence of Theorem 7.6 may be continued:

$$K_0(X) \to K_0(U) \to K_{-1}(X \text{ on } Z) \to K_{-1}(X) \to K_{-1}(U) \to \cdots .$$

Proof of 7.6. (Thomason) For simplicity, we write $\mathcal{A}$ for the Waldhausen category $\mathbf{Ch}_{\mathrm{perf}}(X)$ of II.9.7.5, in which weak equivalences are quasi-isomorphisms. Let w be the class of weak equivalences such that $\mathcal{F} \xrightarrow{w} \mathcal{G}$ if and only if $\mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U$. By the Waldhausen Localization Theorem 2.1, we have a fibration $K(X \text{ on } Z) \to K(X) \to K(w\mathcal{A})$. Let G be the cokernel of $K_0(X) \to K_0(U)$ and let $\mathcal{B}$ denote the full Waldhausen category of all perfect complexes $\mathcal{F}$ on U with $[\mathcal{F}] = 0$ in G. By the Cofinality Theorem 2.3, $K_n(\mathcal{B}) \simeq K_n(U)$ for all $n > 0$ and $K_0(\mathcal{B})$ is the image of $K_0(X) \to K_0(U)$. Thus the proof reduces to showing that $K(w\mathcal{A}) \to K(\mathcal{B})$ is a homotopy equivalence. By the Approximation Theorem 2.4, this reduces to showing that $\mathcal{A} \to \mathcal{B}$ induces an equivalence of derived categories. This is the conclusion of the following theorem of Thomason and Trobaugh. □

V.7.7 **Theorem 7.7.** *$w^{-1}\mathbf{Ch}_{\mathrm{perf}}(X) \to w^{-1}\mathcal{B}$ is an equivalence of triangulated categories. In more detail:*

(a) For every perfect complex $\mathcal{F}$ on U with $[\mathcal{F}]$ in the image of $K_0(X) \to K_0(U)$, there is a perfect complex $\mathcal{E}$ on X such that $\mathcal{E}|_U \simeq \mathcal{F}$ in $\mathbf{D}(U)$.

(b) Given perfect complexes $\mathcal{E}$, $\mathcal{E}'$ on X and a map $b : \mathcal{E}|_U \to \mathcal{E}'|_U$ in $\mathbf{D}(U)$, there is a diagram $\mathcal{E} \xleftarrow{a} \mathcal{E}'' \xrightarrow{a'} \mathcal{E}'$ of perfect complexes on X so that $a|_U$ is an isomorphism and $b = a'|_U(a|_U)^{-1}$ in $\mathbf{D}(U)$.

(c) If $a : \mathcal{E} \to \mathcal{E}'$ is a map of perfect complexes on X which is 0 in $\mathbf{D}(U)$, there is a perfect complex $\mathcal{E}''$ and a map $s : \mathcal{E}'' \to \mathcal{E}$ which is a quasi-isomorphism on U such that $a\,s = 0$ in $\mathbf{D}(X)$.

Proof. The proof of this theorem is quite deep, and beyond the level of this book; we quote [200, 5.2.2–4] for the proof. As a side-note, it is part (a) that was suggested to Thomason by the Trobaugh simulacrum. □

V.7.8 **7.8.** We shall now give an elementary proof (due to Quillen) of Corollary 7.6.1 in the easier case when Z is a divisor and U is affine. We need assume that X is quasi-projective (over a commutative ring R), in order to use the definition $K(X) = K\mathbf{VB}(X)$ in IV.6.3.4. We write $\mathbf{H}_{1,Z}(X)$ for $\mathbf{H}_Z(X) \cap \mathbf{H}_1(X)$, the (exact) subcategory of modules in $\mathbf{H}(X)$ with a length one resolution by vector bundles, which are supported on Z.

For our proof of 7.6.1, let $\mathcal{P}$ denote the exact category of vector bundles on U of the form $j^*(P)$, where P is a vector bundle on X. The assumption that U is affine guarantees $\mathcal{P}$ is cofinal in $\mathbf{VB}(U)$. Therefore, as with the $\mathcal{P}$ of 7.2, we may replace $K(U)$ by $K\mathcal{P}$. We also need the following analogue of the easily checked fact about the relation between P and $S^{-1}P$. If P is a vector bundle on X then $j^*(P)$ is a vector bundle on U, and its direct image $j_*j^*(P)$ is a quasicoherent module on X. If $X = \mathrm{Spec}(R)$ and $I = sR$ then $j_*j^*(P)$ is the R-module $P[1/s]$.

V.7.8.1 **Lemma 7.8.1.** *If P is a vector bundle on X, then P is a submodule of $j_*j^*(P)$, and $j_*j^*(P)$ is the union of its submodules $I^{-n}P$.*

Proof of Corollary 7.6.1. The proof of Theorem 7.1 goes through formally, replacing $\mathbf{P}(R)$ by $\mathbf{VB}(X)$, $S^{-1}R$ by U and $\mathbf{H}_{1,S}$ by $\mathbf{H}_{1,Z}(X)$, the (exact) subcategory of modules in $\mathbf{H}(X)$ with a length one resolution by vector bundles, which are supported on Z. The definitions of $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{G}$ make sense in this context, and every exact sequence in $\mathcal{P}$ splits because U is affine. Now everything goes through immediately except for the proof that Λ is a filtering poset. That proof used the structure of $S^{-1}P$; the argument still goes through formally if we use the corresponding structure of $j_*j^*(P)$ which is given by Lemma 7.8.1. □

V.7.9 **Proposition 7.9.** *(Excision) Let X and Z be as in Theorem 7.6, and let $i : V \subset X$ be the inclusion of an open subscheme containing Z. Then the restriction $i^* : \mathbf{Ch}_{\mathrm{perf},Z}(X) \to \mathbf{Ch}_{\mathrm{perf},Z}(V)$ is exact and induces a homotopy equivalence*

$$\mathbf{K}^B(X \text{ on } Z) \xrightarrow{\simeq} \mathbf{K}^B(V \text{ on } Z).$$

Proof. ([200, 3.19]) Let $T(\mathcal{F})$ denote the functorial Godement resolution of $\mathcal{F}$, and represent the functor Ri_* as $i_* \circ T : \mathbf{Ch}_{\mathrm{perf},Z}(V) \to \mathbf{Ch}_{\mathrm{perf},Z}(X)$. This is an exact functor, taking values in the Waldhausen subcategory $\mathcal{A}$ of perfect complexes which are strictly zero on $U = X - Z$. The left exact functor Γ_Z (subsheaf with supports in Z) induces an exact functor $\Gamma_Z \circ T$ from $\mathbf{Ch}_{\mathrm{perf},Z}(X)$ to $\mathcal{A}$ which is an inverse of the inclusion up to natural quasi-isomorphism. Thus we have a homotopy equivalence $K\mathcal{A} \simeq K\mathbf{Ch}_{\mathrm{perf},Z}(X)$.

If $\mathcal{F}$ is a perfect complex on V, the natural map $i^*Ri_*\mathcal{F} \to \mathcal{F}$ is a quasi-isomorphism since $\mathcal{F} \to T(\mathcal{F})$ is. If $\mathcal{G}$ is a perfect complex on X, strictly zero on U, then $\mathcal{G} \to Ri_*(i^*\mathcal{G})$ is a quasi-isomorphism because it is an isomorphism at points of Z and a quasi-isomorphism at points of U. Thus $Ri_* : K\mathbf{Ch}_{\mathrm{perf},Z}(V) \to K(\mathcal{A})$ is a homotopy inverse of i^*. This proves that $K(X \text{ on } Z) \xrightarrow{\simeq} K(V \text{ on } Z)$. Given the Fundamental Theorem for schemes (8.3 below), it is a routine matter of bookkeeping to deduce the result for the non-connective Bass spectra $\mathbf{K}^B$. □

V.7.10 **Corollary 7.10.** *(Mayer-Vietoris) Let X be a quasi-compact, quasi-separated scheme, and U, V quasi-compact open subschemes with $X = U \cup V$. Then the* *square*

$$\begin{array}{ccc} \mathbf{K}^B(X) & \longrightarrow & \mathbf{K}^B(U) \\ \downarrow & & \downarrow \\ \mathbf{K}^B(V) & \longrightarrow & \mathbf{K}^B(U \cap V) \end{array}$$

is homotopy cartesian, i.e., there is a homotopy fibration sequence

$$\mathbf{K}^B(X) \to \mathbf{K}^B(U) \times \mathbf{K}^B(V) \xrightarrow{\pm} \mathbf{K}^B(U \cap V).$$

On homotopy groups, this yields the long exact "Mayer-Vietoris" sequence:

$$\cdots \to K_{n+1}(U \cap V) \xrightarrow{\partial} K_n(X) \xrightarrow{\Delta} K_n(U) \times K_n(V) \xrightarrow{\pm} K_n(U \cap V) \xrightarrow{\partial} \cdots .$$

Proof. Take $Z = X - U$ in Theorem 7.6, and apply Proposition 7.9. □

V.7.11 **Corollary 7.11.** *Let X be a quasi-compact, quasi-separated scheme, and U, V quasi-compact open subschemes with $X = U \cup V$. Then the square*

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(U) \\ \downarrow & & \downarrow \\ KH(V) & \longrightarrow & KH(U \cap V) \end{array}$$

is homotopy cartesian. On homotopy groups, this yields the long exact "Mayer-Vietoris" sequence:

$$\cdots KH_{n+1}(U \cap V) \xrightarrow{\partial} KH_n(X) \xrightarrow{\Delta} KH_n(U) \times KH_n(V) \xrightarrow{\pm} KH_n(U \cap V) \xrightarrow{\partial} \cdots.$$

Proof. Recall from IV.12.7 that $KH(X)$ is defined to be the realization of $\mathbf{K}^B(X \times \Delta^\bullet)$. The square of simplicial spectra is degreewise homotopy cartesian by 7.10, and hence homotopy cartesian. □

EXERCISES

EV.7.1 **7.1.** Show that the functors $h : \mathcal{G} \to Q\mathcal{P}$ and $pf : \mathcal{G} \to Q\mathbf{P}(R)$ of Definition 7.3 are fibered, with base change ϕ^* constructed as in Lemma IV.7.7. (The proofs for h and pf are the same.)

EV.7.2 **7.2.** Show that there is a functor from the Segal subdivision $Sub(\mathcal{G}_M)$ of $\mathcal{G}_M$ (IV, Ex. 3.9) to the fiber $h^{-1}(M)$, sending $P' \rightarrowtail P$ to $P \to (P/P') \oplus M$. Then show that it is an equivalence of categories.

EV.7.3 **7.3.** Show that the functor $Sub(T) \to (pf)^{-1}(Q)$ which sends $P' \rightarrowtail P \twoheadrightarrow Q$ to $P \twoheadrightarrow Q \oplus (P/P')$ is an equivalence of categories. (See IV, Ex. 7.3.)

EV.7.4 **7.4.** (Gersten) Let A be the homogeneous coordinate ring of a smooth projective curve X over an algebraically closed field. Then A is a 2-dimensional graded domain such that the "punctured spectrum" $U = \mathrm{Spec}(A) - \{\mathfrak{m}\}$ is regular, where $\mathfrak{m}$ is the maximal ideal at the origin. The blowup of $\mathrm{Spec}(A)$ at the origin is a line bundle over X, and U is isomorphic to the complement of the zero-section of this bundle.

(a) Show that $K_n(U) \cong K_n(X) \oplus K_{n-1}(X)$.

(b) Suppose now that A is not a normal domain. We saw in II, Ex. 8.1 that $\mathbf{H}_{\mathfrak{m}}(A) \cong 0$. Show that the image of $K_1(A) \to K_1(U)$ is $k^\times$, and conclude that $K_*\mathbf{H}_{\mathfrak{m}}(A)$ cannot be the third term $K_*(\mathrm{Spec}(A) \text{ on } U)$ in the localization sequence for $K_*(A) \to K_*(U)$.

EV.7.5 **7.5.** Let $\mathbb{P}^1_R$ denote the projective line over an associative ring R, as in 1.5.4, let $\mathbf{H}_1$ denote the subcategory of modules which have a resolution of length 1 by vector bundles, as in Ex. 3.13, and let $\mathbf{H}_{1,t}$ denote the subcategory $\mathbf{H}_1$ consisting of modules $\mathcal{F} = (M, 0, 0)$.

(a) Show that $\mathbf{H}_{1,T}(R[t]) \to \mathbf{H}_{1,t}$, $M \mapsto (M, 0, 0)$, is an equivalence of categories.

(b) Show that there is an exact functor $\mathbf{VB}(\mathbb{P}^1_R) \xrightarrow{j^*} \mathbf{P}(R[1/t])$, $j^*(\mathcal{F}) = M_-$.

(c) Using Ex. 3.13, show that the inclusion $\mathbf{H}_{1,t} \subset \mathbf{H}_1$ induces a homotopy fibration sequence $K\mathbf{H}_T(R[t]) \to K(\mathbb{P}^1_R) \xrightarrow{i^*} K(R[s])$.

(d) If R is commutative, show that the fibration sequence in (c) is the same as the sequence in 7.6.1 with Z the origin of $X = \mathbb{P}^1_R$.

For the next few exercises, we fix an automorphism φ of a ring R, and form the twisted polynomial ring $R_\varphi[t]$ and its localization $R_\varphi[t, 1/t]$, as in Ex. 6.4. Similarly, there is a twisted polynomial ring $R_{\varphi^{-1}}[s]$ and an isomorphism $R_\varphi[t, 1/t] \cong R_{\varphi^{-1}}[s, 1/s]$ obtained by identifying $s = 1/t$. Let $\mathbf{H}_{1,t}(R_\varphi[t])$ denote the exact category of t-torsion modules in $\mathbf{H}_1(R[t])$.

EV.7.6 **7.6.** Show that $K\mathbf{H}_{1,t}(R_\varphi[t]) \to K(R_\varphi[t]) \to K(R_\varphi[t, 1/t])$ is a homotopy fibration. This generalizes Theorem 7.1 and parallels Ex. 6.4. (This result is due to Grayson; the induced K_1–K_0 sequence was discovered by Farrell and Hsiang.)

EV.7.7 **7.7.** (Twisted projective line $\mathbb{P}^1_{R,\varphi}$) We define **mod**-$\mathbb{P}^1_{R,\varphi}$ to be the abelian category of triples $\mathcal{F} = (M_t, M_s, \alpha)$, where M_t (resp., M_s) is a module over $R_\varphi[t]$ (resp. $R_{\varphi^{-1}}[s]$) and α is is an isomorphism $M_t[1/t] \xrightarrow{\cong} M_s[1/s]$. It has a full (exact) subcategory $\mathbf{VB}(\mathbb{P}^1_{R,\varphi})$ consisting of triples where M_t and M_s are finitely generated projective modules, and we write $K(\mathbb{P}^1_{R,\varphi})$ for $K\mathbf{VB}(\mathbb{P}^1_{R,\varphi})$.

(a) Show that $\mathcal{F}(-1) = (M_t, \varphi^* M_s, \alpha t)$ is a vector bundle whenever $\mathcal{F}$ is.

(b) Generalize Theorem 1.5.4 to show that $K(R) \times K(R) \simeq K(\mathbb{P}^1_{R,\varphi})$.

(c) Show that there is an exact functor $\mathbf{VB}(\mathbb{P}^1_{R,\varphi}) \xrightarrow{j^*} \mathbf{P}(R_{\varphi^{-1}}[s])$, $j^*(\mathcal{F}) = M_s$.

(d) Show that $\mathbf{H}_{1,t}(R_\varphi[t])$ is equivalent to the category of modules $\mathcal{F}$ having a resolution of length 1 by vector bundles, and with $j^*(\mathcal{F}) = 0$. (Cf. Ex. 7.5(a).)

(e) Generalize Ex. 7.5(c) to show that there is a homotopy fibration

$$K\mathbf{H}_T(R[t]) \to K(\mathbb{P}^1_{R,\varphi}) \xrightarrow{j^*} K(R[s]).$$

8 The Fundamental Theorem for $K_*(R)$ and $K_*(X)$

The main goal of this section is to prove the Fundamental Theorem, which gives a decomposition of $K_*(R[t,1/t])$. For regular rings, the decomposition simplifies to the formulas $NK_n(R) = 0$ and $K_n(R[t,1/t]) \cong K_n(R) \oplus K_{n-1}(R)$ of Theorem 6.3.

Let R be a ring, set $T = \{t^n\} \subset R[t]$, and consider the category $\mathbf{H}_{1,T}$ of t-torsion $R[t]$-modules M in $\mathbf{H}_1(R[t])$. On the one hand, we know from II.7.8.2 that $\mathbf{H}_{1,T}$ is equivalent to the category $\mathbf{Nil}(R)$ of nilpotent endomorphisms of projective R-modules (II.7.4.4). We also saw in IV.6.7 that the forgetful functor induces a decomposition $K\mathbf{Nil}(R) \simeq K(R) \times \mathrm{Nil}(R)$.

V.8.1 **Theorem 8.1.** *For every R and every n,* $\mathrm{Nil}_n(R) \cong NK_{n+1}(R)$

Theorem 8.1 was used in IV.6.7.2 to derive several properties of the group $NK_*(R)$, including: if R is a $\mathbb{Z}/p\mathbb{Z}$-algebra then each $NK_n(R)$ is a p-group, and if $\mathbb{Q} \subset R$ then $NK_n(R)$ is a uniquely divisible abelian group.

Proof. We know from Ex. 7.5 that $\mathbf{Nil}(R)$ is equivalent to the category $\mathbf{H}_{1,t}$ of modules $\mathcal{F}$ on the projective line $\mathbb{P}^1_R$ with $j^*\mathcal{F} = 0$, and which have a length 1 resolution by vector bundles. Substituting this into Corollary 7.6.1 (or Ex. 7.5 if R is not commutative) yields the exact sequence

$$\cdots K_{n+1}(R[1/t]) \to K_n(R) \oplus \mathrm{Nil}_n(R) \to K_n(\mathbb{P}^1_R) \xrightarrow{j^*} K_n(R[1/t]) \cdots . \quad (8.1.1)$$ V.8.1.1

Now the composition $\mathbf{P}(R) \subset \mathbf{Nil}(R) \to \mathbf{H}(\mathbb{P}^1_R)$ sends P to $(P,0,0)$, and there is an exact sequence $u_1(P) \rightarrowtail u_0(P) \twoheadrightarrow (P,0,0)$ obtained by tensoring P with the standard resolution $0 \to \mathcal{O}(-1) \to \mathcal{O} \to (R,0,0) \to 0$. By Additivity 1.2, the corresponding map $K(R) \to K(\mathbb{P}^1_R)$ in (8.1.1) is $u_0 - u_1$. By the Projective Bundle Theorem 1.5 (or 1.5.4 if R is not commutative), the map $(u_0, u_0 - u_1) : K(R) \times K(R) \to K(\mathbb{P}^1_R)$ is an equivalence. Since $j^*u_0(P) \cong P \otimes_R R[1/t]$, the composition $j^* \circ u_0 : K(R) \to K(R[1/t])$ is the standard base change map inducing the decomposition $K_n(R[1/t]) \cong K_n(R) \oplus NK_n(R)$. Thus (8.1.1) splits into the split extension $0 \to K_n(R) \xrightarrow{u_0 - u_1} K_n(\mathbb{P}^1_R) \to K_n(R) \to 0$ and the desired isomorphism $NK_{n+1}(R) \cong K_{n+1}(R[t])/K_{n+1}(R) \xrightarrow{\cong} \mathrm{Nil}_n(R)$. □

V.8.2 **Theorem 8.2.** *[Fundamental Theorem] There is a canonically split exact sequence*

$$0 \to K_n(R) \xrightarrow{\Delta} K_n(R[t]) \oplus K_n(R[1/t]) \xrightarrow{\pm} K_n(R[t,1/t]) \overset{\partial}{\rightleftarrows} K_{n-1}(R) \to 0.$$

in which the splitting of ∂ is given by multiplication by $t \in K_1(\mathbb{Z}[t,t^{-1}])$.

Proof. Because the base change $\mathbf{mod}\text{-}\mathbb{P}^1_R \to \mathbf{mod}\text{-}R[t]$ is exact, there is a map between the localization sequences for T in Theorem 7.1 and (8.1.1), yielding the commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \to & K_n\mathbf{H}_{1,T} & \longrightarrow & K_n(\mathbb{P}^1_R) & \xrightarrow{j^*} & K_n(R[1/t]) & \longrightarrow & K_{n-1}\mathbf{H}_{1,T} & \to \cdots \\
& \| & & \downarrow & & \downarrow & & \| & \\
\cdots \to & K_n\mathbf{H}_{1,T} & \xrightarrow{f_*\oplus 0} & K_n(R[t]) & \xrightarrow{j^*} & K_n(R[t,1/t]) & \longrightarrow & K_{n-1}\mathbf{H}_{1,T} & \to \cdots .
\end{array}$$

Now $K_n(R) \to K_n\mathbf{Nil}(R) \to K_n(R[t])$ is the transfer map f_* of (3.3.2), induced from the ring map $f : R[t] \to R$, and $\mathrm{Nil}_n(R) \to K_n(\mathbb{P}^1_R)$ is zero by the proof of Theorem 8.1, Since f_* is zero by 3.5.1, the diagram yields the exact sequence $Seq(K_n, R)$ displayed in the Theorem, for $n \geq 1$. The exact sequence $Seq(K_n, R)$ for $n \leq 0$ was constructed in III.4.1.2.

To see that $Seq(K_n, R)$ is split exact, we only need to show that ∂ is split by the cup product with $[t] \in K_1(R[t, 1/t])$. But the maps in the localization sequence commute with multiplication by $K_*(R)$, by Exercise 8.1 (or Ex. IV.1.23). Hence we have the formula: $\partial(\{t, x\}) = \partial(t) \cdot x = [R[t]/tR[t]] \cdot x$. This shows that $x \mapsto \{t, x\}$ is a right inverse to ∂; the maps $t \mapsto 1$ from $K_n(R[t])$ and $K_n(R[1/t])$ to $K_n(R)$ yield the rest of the splitting. □

There is of course a variant of the Fundamental Theorem 8.2 for schemes. For every scheme X, let $X[t]$ and $X[t, t^{-1}]$ denote the schemes $X \times \mathrm{Spec}(\mathbb{Z}[t])$ and $X \times \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ respectively.

V.8.3 **Theorem 8.3.** *For every quasi-projective scheme X we have canonically split exact sequences for all n, where the splitting of ∂ is by multiplication by t.*

$$0 \to K_n(X) \xrightarrow{\Delta} K_n(X[t]) \oplus K_n(X[1/t]) \xrightarrow{\pm} K_n(X[t, 1/t]) \underset{\longleftarrow}{\xrightarrow{\partial}} K_{n-1}(X) \to 0.$$

in which the splitting of ∂ is given by multiplication by $t \in K_1(\mathbb{Z}[t, t^{-1}])$.

Proof. Consider the closed subscheme $X_0 = X \times 0$ of $\mathbb{P}^1_X$. The open inclusion $X[t] \hookrightarrow \mathbb{P}^1_X$ is a flat map, and induces a morphism of homotopy fibration sequences for $n \geq 1$ from Corollary 7.6.1:

$$\begin{array}{ccccccc}
K_n\mathbf{H}_{X_0}(\mathbb{P}^1_X) & \longrightarrow & K_n(\mathbb{P}^1_X) & \xrightarrow{j^*} & K_n(X[1/t]) & \xrightarrow{\partial} & K_{n-1}\mathbf{H}_{X_0}(\mathbb{P}^1_X) \\
\| & & \downarrow & & \downarrow & & \| \\
K_n\mathbf{H}_{X_0}(X[t]) & \longrightarrow & K_n(X[t]) & \xrightarrow{j^*} & K_n(X[t, 1/t]) & \xrightarrow{\partial} & K_{n-1}\mathbf{H}_{X_0}(X[t]).
\end{array} \qquad (8.3.1)$$ V.8.3.1

As in the proof of Theorem 8.1, there is an exact sequence $u_1 \rightarrowtail u_0 \twoheadrightarrow i_*$ of functors from $\mathbf{VB}(X)$ to $\mathbf{H}(\mathbb{P}^1_X)$, where $u_i(E) = E \otimes \mathcal{O}(i)$ and i_* is the restriction of scalars associated to $i : X_0 \hookrightarrow \mathbb{P}^1_X$. By Additivity 1.2, $i_* = u_0 - u_1$. Since $j^* u_0$ is the base change, we see that the top row splits in the same way that (8.1.1) does. Since the map $K_n(X) \to K_n\mathbf{H}_{X_0}(X[t]) \to K_n(X[t])$ is zero by 3.6.1, the bottom left map is zero on homotopy groups. Via a diagram chase on (8.3.1) using the Projective Bundle Theorem 1.5, the exact sequence of Theorem 8.3 follows formally for $n \geq 1$, and for $n = 0$ provided we define $K_{-1}(X)$ to be the cokernel of the displayed map '$\pm$'.

Since the maps in (8.3.1) are $K_*(X)$-module maps (by Ex. 8.2), we have $\partial(\{t, x\}) = \partial(t) \cdot x$. But the base change $X[t] \xrightarrow{\pi} \mathrm{Spec}(\mathbb{Z}[t])$ induces a morphism of localization sequences, and t lifts to $K_1(\mathbb{Z}[t, 1/t])$, so by naturality we have $\partial_X(t) = \pi^* \partial_{\mathbb{Z}}(t) = \pi^*([\mathbb{Z}[t]/t]) = 1$ in the subgroup $K_0(X)$ of $K_0\mathbf{H}_{X_0}(X[t])$. Hence $\partial(\{t, x\}) = x$, regarded as an element of the subgroup $K_*(X)$ of $K_*\mathbf{H}_{X_0}(X[t])$.

Finally, given the result for $n = 1$, the result follows formally for $n \leq 0$, where $K_{n-1}(X)$ is given by Definition IV.10.6; see Ex. 8.3. □

V.8.3.2 **Remark 8.3.2.** $K_n(X)$ was defined for $n \leq -1$ to be $\pi_n \mathbf{K}^B(X)$ in Definition IV.10.6, where $\mathbf{K}^B(X)$ was defined using Theorem 8.4 below. Unravelling that definition, we see that the groups $K_n(X)$ may also be inductively defined to be the cokernel $LK_{n+1}(X)$ of $K_{n+1}(X[t]) \oplus K_{n+1}(X[1/t]) \to K_{n+1}(R[t, 1/t])$. As with Definition III.4.1.1, this definition is concocted so that Theorem 8.3 remains true for $n = 0$ and also for all negative n; see Ex. 8.3.

Theorems 8.2 and 8.3 have versions which involve the (connective) spectra $\mathbf{K}(R)$ and $\mathbf{K}(X)$ associated to $K(R)$ and $K(X)$. Recall from IV.10.1 that the spectrum $\Lambda\mathbf{K}(R)$ is defined so that we have a cofibration sequence

$$\Lambda\mathbf{K}(R) \to \mathbf{K}(R[t]) \vee_{\mathbf{K}(R)} \mathbf{K}(R[1/t]) \xrightarrow{f_0} \mathbf{K}(R[t, 1/t]) \xrightarrow{\partial} \Omega^{-1}\Lambda\mathbf{K}(R).$$

Replacing R by X in Definition IV.10.1 yields a spectrum $\Lambda\mathbf{K}(X)$ fitting into the cofibration sequence obtained from this one by replacing R with X throughout. Fixing a spectrum map $S^1 \to \mathbf{K}(\mathbb{Z}[t, 1/t])$ representing $[x] \in [S^1, \mathbf{K}(\mathbb{Z}[t, 1/t])] = K_1(\mathbb{Z}[t, 1/t])$, the product yields a map $\mathbf{K}(R)$ to $\mathbf{K}(R[t, 1/t])$; composed with $\Omega\partial$, it yields morphisms of spectra $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$, and $\mathbf{K}(X) \to \Lambda\mathbf{K}(X)$. The following result was used in Section IV.10 to define the non-commutative "Bass K-theory spectra" $\mathbf{K}^B(R)$ and $\mathbf{K}^B(X)$ as the homotopy colimit of the iterates $\Lambda^k\mathbf{K}(R)$ and $\Lambda^k\mathbf{K}(X)$.

V.8.4 **Theorem 8.4.** *For any ring R, the map $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ induces a homotopy equivalence between $\mathbf{K}(R)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(R)$. In particular, $K_n(R) \cong \pi_n\Lambda\mathbf{K}(R)$ for all $n \geq 0$.*

Similarly, for any quasi-projective scheme X, the map $\mathbf{K}(X) \to \Lambda\mathbf{K}(X)$ induces a homotopy equivalence between $\mathbf{K}(X)$ and the (-1)-connective cover of the spectrum $\Lambda\mathbf{K}(X)$. In particular, $K_n(X) \cong \pi_n\Lambda\mathbf{K}(X)$ for all $n \geq 0$.

Proof. We give the proof for R; the proof for X is the same. By Theorem 8.3, there is a morphism $\mathbf{K}(R) \to \Lambda\mathbf{K}(R)$ which is an isomorphism on π_n for all $n \geq 0$, and the only other nonzero homotopy group is $\pi_{-1}\Lambda\mathbf{K}(R) = K_{-1}(R)$. The theorem is immediate. □

V.8.4.1 **Corollary 8.4.1.** *For every ring R, the spectrum $\mathbf{K}^B(R[t])$ decomposes as $\mathbf{K}^B(R) \vee N\mathbf{K}^B(R)$ and the spectrum $\mathbf{K}^B(R[t,1/t])$ splits as*

$$\mathbf{K}^B(R[t,1/t]) \simeq \mathbf{K}^B(R) \vee N\mathbf{K}^B(R) \vee N\mathbf{K}^B(R) \vee \Omega^{-1}\mathbf{K}^B(R).$$

K_n-regularity

The following material is due to Vorst [212] and van der Kallen. Let s be a central nonzerodivisor in R, and write $[s] : R[x] \to R[x]$ for the substitution $f(x) \mapsto f(sx)$. Let $NK_*(R)_{[s]}$ denote the colimit of the directed system

$$NK_*(R) \xrightarrow{[s]} NK_*(R) \xrightarrow{[s]} NK_*(R) \xrightarrow{[s]} \cdots.$$

V.8.5 **Lemma 8.5.** *For any central nonzerodivisor s, $NK_*(R)_{[s]} \xrightarrow{\cong} NK_*(R[1/s])$. In particular, if $NK_n(R) = 0$ then $NK_n(R[1/s]) = 0$.*

Proof. Let C be the colimit of the directed system $R[x] \xrightarrow{[s]} R[x] \xrightarrow{[s]} R[x] \cdots$ of ring homomorphisms. There is an evident map $C \to R$ splitting the inclusion, and the kernel is the ideal $I = xR[1/s][x] \subset R[1/s][x]$. By IV.6.4, $K_*(C,I)$ is the filtered colimit of the groups $NK_n(R) = K_n(R[x],x)$ along the maps $[s]$, *i.e.*, $NK_*(R)_{[s]}$. Moreover, $C[1/s] \cong R[1/s][x]$.

We can apply Proposition 7.5 to $R \to C$, since $I/sI = 0$ implies $R/s^iR \cong C/s^iC$, to obtain the Mayer-Vietoris sequence

$$\cdots \to K_n(R) \xrightarrow{\Delta} K_n(C) \times K_n(R[1/s]) \xrightarrow{\pm} K_n(R[1/s][x]) \xrightarrow{\partial} \cdots.$$

Splitting off $K_n(R)$ from $K_n(C)$ and $K_n(R[1/s])$ from $K_n(R[1/s][x])$ yields the desired isomorphism of $NK_n(R)_{[s]} \cong K_n(C,I)$ with $NK_n(R[1/s])$. □

V.8.5.1 **Example 8.5.1.** If R is commutative and reduced, then $NK_n(R) = 0$ implies that $NK_n(R_P) = 0$ for every prime ideal P of R. Vorst [212] also proved the converse: if $NK_n(R_\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ then $NK_n(R) = 0$. Van der Kallen proved the stronger result that NK_n is a Zariski sheaf on Spec(R).

Recall from III.3.4 that R is called *K_n-regular* if $K_n(R) \cong K_n(R[t_1,\dots,t_m])$ for all m. By Theorem 6.3, every noetherian regular ring is K_n-regular for all n.

V.8.6 **Theorem 8.6.** *If R is K_n-regular, then it is K_{n-1}-regular. More generally, if $K_n(R) \cong K_n(R[s,t])$ then $K_{n-1}(R) \cong K_{n-1}(R[s])$.*

Proof. It suffices to suppose that $K_n(R) \cong K_n(R[s,t])$, so that $NK_n(R[s]) = 0$, and prove that $NK_{n-1}(R) = 0$. By Lemma 8.5, $NK_n(R[s,1/s]) = 0$. But $NK_{n-1}(R)$ is a summand of $NK_n(R[s,1/s])$ by the Fundamental Theorem 8.2. □

The following partial converse was proven in [47].

V.8.7 **Theorem 8.7.** *Let R be a commutative ring containing $\mathbb{Q}$. If R is K_{n-1}-regular and $NK_n(R) = 0$, then R is K_n-regular.*

V.8.7.1 **Remark 8.7.1.** There are rings R for which $NK_n(R) = 0$ but R is not K_n-regular. For example, the ring $R = \mathbb{Q}[x,y,z]/(z^2+y^3+x^{10}+x^7y)$ has $K_0(R) \cong K_0(R[t])$ but $K_0(R) \ncong K_0(R[s,t])$, and $K_{-1}(R) \ncong K_{-1}(R[t])$; see [47].

EXERCISES

EV.8.1 **8.1.** For any central nonzerodivisor $s \in R$, multiplication by $[s] \in K_1(R[1/s])$ yields a map $K_n(R) \to K_{n+1}(R[1/s])$. Show that the boundary map $\partial : K_{n+1}(R[1/s]) \to K_n\mathbf{H}_s(R)$ satisfies $\partial(\{s,x\}) = \bar{x}$ for every $x \in K_n(R)$, where $\bar{x}$ is the image of x under the natural map $K_n(R) \to K_n(R/sR) \to K_n\mathbf{H}_s(R)$. *Hint:* Use Exercises 5.3 and IV.1.23; the ring map $\mathbb{Z}[t] \to R$, $t \mapsto s$, induces compatible pairings $\mathbf{P}(\mathbb{Z}[t]) \times \mathbf{P}(R) \to \mathbf{P}(R)$ and $\mathbf{P}(\mathbb{Z}[t,1/t]) \times \mathbf{P}(R) \to \mathbf{P}(R[1/s])$.

EV.8.2 **8.2.** Show that $\mathbf{VB}(X)$ acts on the terms in the localization sequence of Corollary 7.6.1 in the sense of IV.6.6.4. Deduce that the map $\partial : K_n(U) \to K_{n-1}\mathbf{H}_Z(X)$ satisfies $\partial(\{u,x\}) = \partial(u) \cdot x$ for $u \in K_*(U)$ and $x \in K_*(X)$. *Hint:* Mimick Ex. 5.3, using Ex. IV.1.23.

EV.8.3 **8.3.** Given a scheme X, show that $F_n(R) = K_n(X \times \operatorname{Spec} R)$ is a contracted functor in the sense of III.4.1.1. Then show that the functor $K_{-n}(X)$ of 8.3.2 is the contracted functor $L^nF_0(\mathbb{Z})$.

EV.8.4 **8.4.** *Twisted* Nil *groups*, Let φ be an automorphism of a ring R, and consider the category $\mathbf{Nil}(\varphi)$ whose objects are pairs (P,ν), where P is a finitely generated projective R-module and ν is a nilpotent endomorphism of P which is semilinear in the sense that $\nu(xr) = \nu(x)\varphi(r)$. If $\varphi = \mathrm{id}_R$, this is the category $\mathbf{Nil}(R)$ of II.7.4.4. As in *loc. cit.*, we define $\mathrm{Nil}_n(\varphi)$ to be the kernel of the map $K_n\mathbf{Nil}(\varphi) \to K_n(R)$ induced by $(P,\nu) \mapsto P$. Similarly, we define $NK_n(\varphi)$ to be the cokernel of the natural map $K_n(R) \to K_n(R_\varphi[t])$.

(a) Show that $K_n\mathbf{Nil}_\varphi(R) \cong K_n(R) \oplus \mathrm{Nil}_n(\varphi)$, and $K_n(R_\varphi[t]) \cong K_n(R) \oplus NK_n(\varphi)$.

(b) Show that $\mathbf{Nil}_\varphi(R)$ is equivalent to the category $K\mathbf{H}_{1,t}(R_\varphi[t])$ of Ex. 7.6.

(c) Show that $NK_n(\varphi) \cong NK_n(\varphi^{-1})$. *Hint:* $\mathbf{P}(R^{op}) \cong \mathbf{P}(R)^{op}$ by IV.6.4.

(d) Prove the twisted analogue of Theorem 8.1: $\mathrm{Nil}_n(\varphi) \cong NK_{n+1}(\varphi)$ for all n.

EV.8.5 **8.5.** (Grayson) Let $K^\varphi(R)$ be the homotopy fiber of $K(R) \xrightarrow{1-\varphi^*} K(R)$, and set $K^\varphi_n(R) = \pi_n K^\varphi(R)$. If R is regular, use Ex. 6.4 to show that $K_*(R_\varphi[t,1/t]) \cong K^\varphi_*(R)$. Then show that there is a canonical isomorphism for any R:

$$K_n(R_\varphi[t,1/t]) \cong K^\varphi_n(R) \oplus \mathrm{Nil}_{n-1}(\varphi) \oplus \mathrm{Nil}_{n-1}(\varphi^{-1}).$$

9 The coniveau spectral sequence of Gersten and Quillen

In this section we give another application of the Localization Theorem 5.1, which reduces the calculation of $G_n(X)$ to a knowledge of the K-theory of fields, up to extensions. The prototype of the extension problem is illustrated by the exact sequences in 6.6 and 6.12, for Dedekind domains and smooth curves.

Suppose first that R is a finite-dimensional commutative noetherian ring. We let $\mathbf{M}^i(R)$ denote the subcategory of $\mathbf{M}(R)$ consisting of those R-modules M whose associated prime ideals all have height $\geq i$. We saw in II.6.4.3 and Ex. II.6.9 that each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$ and that $\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \bigoplus_{ht(\mathfrak{p})=i} \mathbf{M}_{\mathfrak{p}}(R_{\mathfrak{p}})$, where $\mathbf{M}_{\mathfrak{p}}(R_{\mathfrak{p}})$ is the category of $R_{\mathfrak{p}}$-modules of finite length.

By devissage (Application 4.4) we have $K\mathbf{M}_{\mathfrak{p}}(R_{\mathfrak{p}}) \simeq G(k(\mathfrak{p})) \simeq K(k(\mathfrak{p}))$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, so $K_*\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \oplus_{ht(\mathfrak{p})=i} G_*(k(\mathfrak{p}))$. The Localization Theorem yields long exact sequences

$$\xrightarrow{\partial} K_n\mathbf{M}^{i+1}(R) \to K_n\mathbf{M}^i(R) \to \bigoplus_{ht(\mathfrak{p})=i} K_n(k(\mathfrak{p})) \xrightarrow{\partial} K_{n-1}\mathbf{M}^{i+1}(R) \to \cdots$$

and, writing $\mathbf{M}^{i-1}/\mathbf{M}^{i+1}$ for $\mathbf{M}^{i-1}(R)/\mathbf{M}^{i+1}(R)$,

$$\bigoplus_{ht(\mathfrak{p})=i-1} K_{n+1}(k(\mathfrak{p})) \xrightarrow{\partial} \bigoplus_{ht(\mathfrak{p})=i} K_n(k(\mathfrak{p})) \to K_n(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to \bigoplus_{ht(\mathfrak{p})=i-1} K_n(k(\mathfrak{p})) \xrightarrow{\partial} \tag{9.1}$$

V.9.1

ending in an extension to K_1 of the K_0 sequence of II.6.4.3:

$$\bigoplus_{ht(\mathfrak{p})=i-1} k(\mathfrak{p})^\times \xrightarrow{\Delta} D^i(R) \to K_0(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to D^{i-1}(R) \to 0.$$

Here $D^i(R)$ is the free abelian group on the height i primes, and Δ sends $r/s \in k(\mathfrak{p})^\times$ to $[R/(r,\mathfrak{p})] - [R/(s,\mathfrak{p})]$ by Example 6.1.2. (This is the formula of II, Ex. 6.8.)

Recall from II.6.4.3 that the *generalized Weil divisor class group* $CH^i(R)$ is defined to be the image of $D^i(R) \to K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R)$. As the kernel of this map is the image of Δ, this immediately gives the interpretation, promised in II.6.4.3:

V.9.1.1 **Lemma 9.1.1.** *$CH^i(R)$ is the quotient of $D^i(R)$ by the relations that $\Delta(r/s) = 0$ for each $r/s \in k(\mathfrak{p})^\times$ and each prime ideal $\mathfrak{p}$ of height $i-1$.*

This equivalence relation, that the length of $R_{\mathfrak{p}}/(r,\mathfrak{p})$ is zero in $D^i(R)$ for each $r \in R$ and each prime $\mathfrak{p}$ of height $i-1$, is called *rational equivalence*; see Proposition 9.4.1.

For general R, the localization sequences (9.1) cannot break up. Indeed, we saw in I.3.6 and II.6.4.3 that even the map $\oplus k(y)^\times \xrightarrow{\Delta} D^i(X)$ can be nonzero. Instead, the sequences assemble to form a spectral sequence converging to $G_*(R)$.

V.9.2 **Proposition 9.2.** *If R is noetherian and $\dim(R) < \infty$, there is a convergent 4^{th} quadrant cohomological spectral sequence*

$$E_1^{p,q} = \bigoplus\nolimits_{ht(\mathfrak{p})=p} K_{-p-q}(k(\mathfrak{p})) \Rightarrow G_{-p-q}(R).$$

The edge maps $G_n(R) \to E_1^{0,-n} = \oplus G_n(k(\mathfrak{p}))$ associated to the minimal primes $\mathfrak{p}$ of R are induced by the localizations $R \to R_{\mathfrak{p}}$ followed by the isomorphism $G(k(\mathfrak{p})) \simeq G(R_{\mathfrak{p}})$ of 4.2.1.

Along the line $p+q=0$, we have $E_1^{p,-p} \cong D^p(R)$ and $E_2^{p,-p} \cong CH^p(R)$.

$\mathbb{Z}$	0	0	0
$F^\times$	$\to D^1(R)$	0	0
$K_2(F)$	$\to \oplus k(x_1)^\times$	$\to D^2(R)$	0
$K_3(F)$	$\to \oplus K_2(x_1)$	$\to \oplus k(x_2)^\times$	$\to D^3(R)$

The E_1 page of the spectral sequence

Proof. Setting $D_1^{p,q} = \oplus_i K_{-p-q}\mathbf{M}^i(R)$, the localization sequences (9.1) yield an exact couple (D_1, E_1). Because $\mathbf{M}(R) = \mathbf{M}^0(R)$ and $\mathbf{M}^p(R) = \emptyset$ for $p > \dim(R)$, the resulting spectral sequence is bounded and converges to $K_*(R)$ (see [223, 5.9.7]). Since $E_1^{p,-p}$ is the divisor group $D^p(R)$, and $d_1^{p-1,p}$ is the map $\oplus k(x)^\times \xrightarrow{\Delta} D^p(R)$ of (9.1), the group $E_2^{p,-p}$ is isomorphic to $CH^p(R)$ by 9.1.1. Finally, the edge maps are given by $G_n(R) = K_n\mathbf{M}(R) \to K_n\mathbf{M}(R)/\mathbf{M}^1(R) \cong \oplus K_n(k(\eta))$; the component maps are induced by $\mathbf{M}(R) \to \mathbf{M}(k(\eta))$. □

V.9.2.1 **Coniveau Filtration 9.2.1.** The p^{th} term in the filtration on the abutment $G_n(R)$ is defined to be the image of $K_n\mathbf{M}^p(R) \to K_n\mathbf{M}(R)$. In particular, the filtration on $G_0(R)$ is the coniveau filtration of II.6.4.3.

V.9.2.2 **Remark 9.2.2.** If $f : R \to S$ is flat then $\otimes_R S$ sends $\mathbf{M}^i(R)$ to $\mathbf{M}^i(S)$. It follows that the spectral sequence (9.2) is covariant for flat maps.

V.9.2.3 **Remark 9.2.3.** If $\dim(R) = \infty$, it follows from [223, Ex. 5.9.2] that the spectral sequence (9.2) converges to $\varprojlim K_*\mathbf{M}(R)/\mathbf{M}^i(R)$.

Motivated by his work with Brown, Gersten made the following conjecture for regular local rings. (If $\dim(R) = 1$, this is Gersten's conjecture 6.9.) His conjecture was extended to semilocal rings by Quillen in [153], who then established the important special case when R is essentially of finite type over a field (9.6 below).

V.9.3 **Gersten-Quillen Conjecture 9.3.** *If R is a semilocal regular ring, the maps $K_n\mathbf{M}^{i+1}(R) \to K_n\mathbf{M}^i(R)$ are zero for every n and i.*

This conjecture implies that, for R a regular semilocal domain, (9.1) breaks into short exact sequences

$$0 \to K_n\mathbf{M}^i(R) \to \oplus_{ht=i} K_n(k(\mathfrak{p})) \to K_{n-1}\mathbf{M}^{i+1}(R) \to 0,$$

which splice to yield exact sequences (with F the field of fractions of R):

$$0 \to K_n(R) \to K_n(F) \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p})) \to \cdots \to \bigoplus_{ht(\mathfrak{p})=i} K_{n-i}(k(\mathfrak{p})) \to \cdots \tag{9.3.1}$$ V.9.3.1

It follows of course that the spectral sequence (9.2) collapses at E_2, with $E_2^{p,q} = 0$ for $p > 0$. Hence the Gersten-Quillen conjecture 9.3 implies that $K_n(R)$ is the kernel of $K_n(F) \to \oplus K_{n-1}(k(\mathfrak{p}))$ for all n.

The coniveau filtration for schemes

Of course, the above discussion extends to modules over a noetherian scheme X. Here we let $\mathbf{M}^i(X)$ denote the category of coherent $\mathcal{O}_X$-modules whose support has codimension $\geq i$. If $i < j$ we write $\mathbf{M}^i/\mathbf{M}^j$ for the quotient abelian category $\mathbf{M}^i(X)/\mathbf{M}^j(X)$. Then $\mathbf{M}^i/\mathbf{M}^{i+1}$ is equivalent to the direct sum, over all points x of codimension i in X, of the $\mathbf{M}_x(\mathcal{O}_{X,x})$; by devissage this category has the same K-theory as its subcategory $\mathbf{M}(k(x))$. Thus the Localization Theorem yields a long exact sequence

$$\bigoplus_y K_{n+1}(k(y)) \xrightarrow{\partial} \bigoplus_x K_n(k(x)) \to K(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to \bigoplus_y K_n(k(y)), \tag{9.4}$$ V.9.4

where y runs over all points of codimension $i-1$ and x runs over all points of codimension i. This ends in the K_1–K_0 sequence of II.6.4.3:

$$\bigoplus_{\mathrm{codim}(y)=i-1} k(y)^\times \xrightarrow{\Delta} D^i(X) \to K_0(\mathbf{M}^{i-1}/\mathbf{M}^{i+1}) \to D^{i-1}(X) \to 0,$$

where $D^i(X)$ denotes the free abelian group on the set of points of X having codimension i (see II.6.4.3). If $r/s \in k(\mathfrak{p})^\times$, the formula for Δ on $k(y)^\times$ is determined by the formula in (9.1): choose an affine open $\mathrm{Spec}(R) \subset X$ containing y; then $\Delta(r/s)$ is $[R/(r,\mathfrak{p})] - [R/(s,\mathfrak{p})]$ in $D^i(R) \subset D^i(X)$. This gives a presentation for the Weil divisor class group $CH^i(X)$ of II.6.4.3, defined as the image of $D^i(X)$ in $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}$: it is the cokernel of Δ.

Now the usual Chow group $A^i(X)$ of codimension i cycles on X modulo rational equivalence, as defined in [58], is the quotient of $D^i(X)$ by the following relation: for every irreducible subvariety W of $X \times \mathbb{P}^1$ having codimension i, meeting $X \times \{0,\infty\}$ properly, the cycle $[W \cap X \times 0]$ is equivalent to $[W \cap X \times \infty]$. We have the following identification, which was promised in II.6.4.3.

V.9.4.1 **Lemma 9.4.1.** *$CH^i(X)$ is the usual Chow group $A^i(X)$.*

Proof. The projection $W \to \mathbb{P}^1$ defines a rational function, *i.e.,* an element $t \in k(W)$. Let Y denote the image of the projection $W \to X$, and y its generic point; the proper intersection condition implies that Y has codimension $i+1$ and W is finite over Y. Hence the norm $f \in k(y)$ of t exists. Let $x \in X$ be a point of codimension i; it defines a discrete valuation ν on $k(y)$, and it is well known [58] that the multiplicity of x in the cycle $[W \cap X \times 0] - [W \cap X \times \infty]$ is $\nu(f)$. □

As observed in 9.1 (and II.6.4.3), the sequences in (9.4) do not break up for general X. The proof of Proposition 9.2 generalizes to this context to prove the following:

V.9.5 **Proposition 9.5.** *(Gersten) If X is noetherian and $\dim(X) < \infty$, there is a convergent 4^{th} quadrant cohomological spectral sequence (zero unless $p+q \le 0$):*

$$E_1^{p,q} = \bigoplus\nolimits_{codim(x)=p} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X).$$

If X is reduced, the components $G_n(X) \to G_n(k(\eta))$ of the edge maps $G_n(X) \to E_1^{0,-n}$ associated to the generic points η of X are induced by the flat maps $\mathrm{Spec}(k(\eta)) \to X$.

Along the line $p+q=0$, we have $E_2^{p,-p} \cong CH^p(X)$.

V.9.5.1 **Coniveau Filtration 9.5.1.** The p^{th} term in the filtration on the abutment $G_n(X)$ is defined to be the image of $K_n\mathbf{M}^p(X) \to K_n\mathbf{M}(X)$. In particular, the filtration on $G_0(X)$ is the coniveau filtration of II.6.4.3.

V.9.5.2 **Remark 9.5.2.** The spectral sequence in 9.5 is covariant for flat morphisms, for the reasons given in Remark 9.2.2. If X is noetherian but not finite-dimensional, it converges to $\varprojlim K_*\mathbf{M}(X)/\mathbf{M}^i(X)$.

We shall now prove the Gersten-Quillen Conjecture 9.3 for algebras over a field.

V.9.6 **Theorem 9.6.** *(Quillen) Let R be an algebra of finite type over a field, and let $A = S^{-1}R$ be the semilocal ring of R at a finite set of prime ideals. Then Conjecture 9.3 holds for A: for each i the map $K\mathbf{M}^{i+1}(A) \to K\mathbf{M}^i(A)$ is zero.*

Proof. We may replace R by $R[1/f]$, $f \in S$, to assume that R is smooth. Because $\mathbf{M}^{i+1}(S^{-1}R)$ is the direct limit (over $s \in S$) of the $\mathbf{M}^{i+1}(R[1/s])$, which in turn is the direct limit (over nonzerodivisors t) of the $\mathbf{M}^i(R[1/s]/tR[1/s])$, it suffices to show that for every nonzerodivisor t of R that there is an $s \in S$ so that the functor $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ is null homotopic on K-theory spaces. This is the conclusion of Proposition 9.6.1 below. □

V.9.6.1 **Proposition 9.6.1.** *(Quillen) Let R be a smooth domain over a field and $S \subset R$ a multiplicative set so that $S^{-1}R$ is semilocal. Then for each $t \neq 0$ in R with $t \notin S$ there is an $s \in S$ so that each base change $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ induces a null-homotopic map on K-spaces.*

Proof. Suppose first that R contains a subring B mapping isomorphically onto R/tR, and that R is smooth over B. We claim that the kernel I of $R \to R/tR$ is locally principal. To see this, we may assume that B is local, and even (by Nakayama's Lemma) that B is a field. But then R is a Dedekind domain and every ideal is locally principal, whence the claim. We choose s so that $I[1/s] \cong R[1/s]$.

Now for any B-module M we have the characteristic split exact sequence of $R[1/s]$-modules:

$$0 \to I[1/s] \otimes_B M \to R[1/s] \otimes_B M \to M[1/s] \to 0.$$

Since R is flat, if M is in $\mathbf{M}^i(B)$ then this is an exact sequence in $\mathbf{M}^i(R[1/s])$. That is, we have a short exact sequence of exact functors $\mathbf{M}^i(B) \to \mathbf{M}^i(R[1/s])$. By the Additivity Theorem V.1.2 1.2, we see that $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R[1/s])$ is null homotopic.

For the general case of V.9.6.1 9.6.1, we need the following algebraic lemma, due to Quillen, and we refer the reader to Q341 [153, 5.12] for the proof.

V.9.6.2 **Lemma 9.6.2.** *Suppose that $X = \mathrm{Spec}(R)$, for a finitely generated ring R over an infinite field k. If $Z \subset X$ is closed of dimension r and $T \subset X$ is a finite set of closed points, then there is a projection $X \to \mathrm{Spec}(k[t_1, ..., t_r])$ which is finite on Z and smooth at each point of T.*

Resuming the proof of V.9.6 9.6, we next assume that k is an infinite field. Let $A = k[t_1, ..., t_r]$ be the polynomial subalgebra of R given by Lemma V.9.6.1 9.6.1 so that $B = R/tR$ is finite over A and R is smooth over A at the primes of R not meeting S. Set $R' = R \otimes_A B$; then R'/R is finite and hence $S^{-1}R'$ is also semilocal, and R' is smooth over B at the primes of R' not meeting S. In particular, there is an $s \in S$ so that $R'[1/s]$ is smooth (hence flat) over B. For such s, each $\mathbf{M}^i(B) \to \mathbf{M}^i(R[1/s])$ factors through exact functors $\mathbf{M}^i(B) \to \mathbf{M}^i(R'[1/s]) \to \mathbf{M}^i(R[1/s])$. But $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R'[1/s])$ is zero by the first part of the proof. It follows that $K\mathbf{M}^i(B) \to K\mathbf{M}^i(R[1/s])$ is zero as well.

Finally, if k is a finite field we invoke the following standard transfer argument. Given $x \in K_n\mathbf{M}^i(R/tR)$ and a prime p, let k'' denote the infinite p-primary algebraic extension of k; as the result is true for $R \otimes_k k''$ then there is a finite subextension k' with $[k' : k] = p^r$ so that $p^r x$ maps to zero over $R \otimes_k k'$. Applying the transfer from $K_n\mathbf{M}^i(R \otimes k')$ to $K_n\mathbf{M}^i(R)$, it follows that $F^i_*(x)$ is killed by a power of p. Since this is true for all p, we must have $F^i_*(x) = 0$. □

V.9.6.3 **Definition 9.6.3** (Pure exactness)**.** A subgroup B of an abelian group A is said to be *pure* if $B/nB \to A/nA$ is an injection for every n. More generally, an exact sequence A_* of abelian groups is *pure exact* if the image of each A_{n+1} is a pure subgroup of A_n; it follows that each sequence A_*/nA_* is exact. If A^i_* is a filtered family of pure exact sequences, it is easy to see that colim A^i_* is pure exact.

V.9.6.4 **Corollary 9.6.4.** *If R is a semilocal ring, essentially of finite type over a field, then the sequence (9.3.1) is also pure exact.*

In particular, we have exact sequences (see Ex. 9.4):

$$0 \to K_n(R)/\ell \to K_n(F)/\ell \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p}))/\ell \to \cdots ,$$

$$0 \to {}_\ell K_n(R) \to {}_\ell K_n(F) \to \bigoplus_{ht(\mathfrak{p})=1} {}_\ell K_{n-1}(k(\mathfrak{p})) \to \cdots .$$

Proof. (Grayson) Because the maps in 9.6.1 are null homotopic, we see from 1.2.2 that if $F^i_{s,t}$ is the homotopy fiber of $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ then the exact sequences $0 \to K_{n+1}\mathbf{M}^i(R[1/s]) \to \pi_n F^i_{s,t} \to K_n\mathbf{M}^i(R/tR) \to 0$ splits. Taking the direct limit, we see that the sequences $0 \to K_{n+1}\mathbf{M}^i(S^{-1}R) \to \pi_n F^i \to K_n\mathbf{M}^{i+1}(R) \to 0$ are pure exact. Splicing these sequences together yields the assertion. □

In mixed characteristic, Gillet and Levine proved the following result; we refer the reader to [69] for the proof. Let Λ be a discrete valuation domain with parameter π, residue field $k = \Lambda/\pi$ of characteristic $p > 0$ and field of fractions of characteristic 0.

V.9.7 **Theorem 9.7.** *Let A be a smooth algebra of finite type over Λ and $S \subset A$ a multiplicative set so that $R = S^{-1}A$ is semilocal. If $t \in A$ and A/tA is flat over Λ then every base change $K\mathbf{M}^i(A/tA) \to K\mathbf{M}^i(R)$ is null-homotopic.*

Now the set $T = \{t \in A : R/tR \text{ is flat over } \Lambda\}$ is multiplicatively closed, generated by the height 1 primes other than πR. Hence the localization $D = T^{-1}R$ is the discrete valuation ring $D = R_{(\pi R)}$ whose residue field is the quotient field of $R/\pi R$. By Theorem 9.7 with $i = 0$, $K_n(R) \to K_n(D)$ is an injection.

V.9.7.1 **Corollary 9.7.1.** *Let R be a regular semilocal Λ-algebra, as in 9.7. Then:*

(a) For $i \geq 1$, the transfer map $K\mathbf{M}^{i+1}(R) \to K\mathbf{M}^i(R)$ is null homotopic.

(b) Each $K_0\mathbf{M}^i(R)$ is generated by the classes $[R/\mathbf{x}R]$, where each $\mathbf{x} = (x_1, \dots, x_i)$ is a regular sequence in R of length i.

(c) Sequence (9.3.1) is exact except possibly at $K_n(R)$ and $\oplus_{ht=1} K_{n-1}(k(\mathfrak{p}))$.

(d) If Gersten's DVR conjecture 6.9 is true for D, then the Gersten-Quillen conjecture 9.3 is true for R.

Proof. (a) As $i + 1 > 1 = \dim(D)$, each module in $K\mathbf{M}^{i+1}(R)$ vanishes over D, so it is killed by a t for which R/tR is flat. Thus $K\mathbf{M}^{i+1}(R)$ is the direct limit over $t \in T$ and $s \in S$ of the $K\mathbf{M}^i(A[1/s]/t)$; since the maps $K\mathbf{M}^i(A[1/s]/t) \to K\mathbf{M}^i(R)$ are zero by Theorem 9.7, (a) follows. We prove (b) by induction on i, the case $i = 1$ being immediate from the localization sequence

$$R^\times \to \text{frac}(R)^\times \xrightarrow{\Delta} K_0\mathbf{M}^1(R) \to 0,$$

together with the formula $\Delta(r/s) = [R/rR] - [R/sR]$ of 6.1.2. For $i > 1$ the localization sequence (9.1), together with (a), yield an exact sequence

$$\oplus_{ht(\mathfrak{p})=i-1} k(\mathfrak{p})^\times \xrightarrow{\partial} K_0\mathbf{M}^i(R) \xrightarrow{0} K_0(R) \xrightarrow{\cong} K_0(\text{frac}(R)) \to 0.$$

If $0 \neq r, s \in R/\mathfrak{p}$ then the formula (Ex. 5.1) for $\partial_\mathfrak{p} : k(\mathfrak{p})^\times \to K_0\mathbf{M}^i(R)$ yields $\partial_\mathfrak{p}(r/s) = [R/\mathfrak{p}+rR]-[R/\mathfrak{p}+sR]$. Thus it suffices to show that each $[R/\mathfrak{p}+rR]$ is a sum of terms $R/\mathbf{x}R$. Since the height 1 primes of R are principal we have $\mathfrak{p} = xR$ and $R/\mathfrak{p} + aR = R/(x,r)R$ when $i = 2$. By induction there are regular sequences $\mathbf{x}$ of length $i-1$ such that $[R/\mathfrak{p}] = \sum[R/\mathbf{x}R]$ in $K_0\mathbf{M}^{i-1}(R)$. Given $r \in R - \mathfrak{p}$ we can choose (by prime avoidance) an r' so that $\mathfrak{p} + rR = \mathfrak{p} + r'R$ and $(\mathbf{x}, r')$ is a regular sequence in R. Because $R/\mathbf{x}$ is Cohen-Macaulay, all its associated primes have height $i-1$. Hence we have the desired result (by Ex. II.6.16): $[R/\mathfrak{p} + rR] = \sum[R/(\mathbf{x}, r')R]$ in $K_0\mathbf{M}^i(R)$.

Part (c) is immediate from exactness of $K_n(R) \to K_n(E) \to K_{n-1}\mathbf{M}^1(R)$, where F is the field of fractions of R, and part (a), which yields the exact sequence:

$$0 \to K_n\mathbf{M}^1(R) \to \bigoplus_{ht(\mathfrak{p})=1} K_{n-1}(k(\mathfrak{p})) \to \cdots \to \bigoplus_{ht(\mathfrak{p})=i} K_{n-i}(k(\mathfrak{p})) \to \cdots$$

It follows that (9.3.1) is exact for R (the Gersten-Quillen conjecture 9.3 holds for R) if and only if each $K_n(R) \to K_n(F)$ is an injection. Suppose now that $K_*(D) \to K_*(F)$ is an injection (conjecture 6.9 holds for D). Then the composition $G_n(R) \hookrightarrow G_n(D) \hookrightarrow G_n(F)$ is an injection, proving (d). $\square$

K-cohomology

Fix a noetherian scheme X. For each point x, let us write i_x for the inclusion of x in X. Then we obtain a skyscraper sheaf $(i_x)_*A$ on X (for the Zariski topology) for each abelian group A. If we view the coniveau spectral sequence (9.5) as a presheaf on X and sheafify, the rows of the E_1 page assemble to form the following chain complexes of sheaves on X:

$$0 \to \mathcal{K}_n \to \oplus_{\mathrm{cd}=1}(i_x)_*K_n(x) \to \oplus_{\mathrm{cd}=2}(i_y)_*K_n(y) \to \cdots \to 0. \qquad (9.8)$$ V.9.8

Here $\mathcal{K}_n$ is the sheaf associated to the presheaf $U \mapsto K_n(U)$; its stalk at $x \in X$ is $K_n(\mathcal{O}_{X,x})$. Since the stalk sequence of (9.8) at $x \in X$ is the row $q = -n$ of the coniveau spectral sequence for $R = \mathcal{O}_{X,x}$, we see that if (9.3.1) is exact for every local ring of X then (9.8) is an exact sequence of sheaves. Since each $(i_y)_*K_n(y)$ is a flasque sheaf, (9.8) is a flasque resolution of the sheaf $\mathcal{K}_n$. In summary, we have proven:

V.9.8.1 **Proposition 9.8.1.** *Assume that X is a regular quasi-projective scheme, or more generally that the Gersten-Quillen conjecture 9.3 holds for the local rings of X. Then (9.8) is a flasque resolution of $\mathcal{K}_n$, and the E_2 page of the coniveau spectral sequence (9.5) is*

$$E_2^{p,q} \cong H^p(X, \mathcal{K}_{-q}).$$

In addition, we have $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ for all $p > 0$.

The isomorphism $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ is often referred to as *Bloch's formula*, since it was first discovered for $p = 2$ by Spencer Bloch in [24].

V.9.8.2 **Example 9.8.2.** When X is the projective line $\mathbb{P}^1_k$ over a field k, the direct image $\pi_*: K_n(\mathbb{P}^1_k) \to K_n(k)$ is compatible with a morphism of spectral sequences (see Ex. 9.3); the only nontrivial maps are from $E_1^{1,-n-1}(\mathbb{P}^1_k) = \oplus K_n(k(x))$ to $E_1^{0,-n}(k) = K_n(k)$. As these are the transfer maps for the field extensions $k(x)/k$, this is a split surjection. Comparing with Corollary 1.5.1, we see that

$$H^0(\mathbb{P}^1_k, \mathcal{K}_n) \cong K_n(k), \quad H^1(\mathbb{P}^1_k, \mathcal{K}_n) \cong K_{n-1}(k).$$

EXERCISES

EV.9.1 **9.1.** Let k be a field and S a multiplicative set in a domain $R = k[x_1, \dots, x_m]/J$ so that $S^{-1}R$ is regular. Modify the proof of Proposition 9.6 to show that for each $t \neq 0$ in R there is an $s \in S$ so that each $\mathbf{M}^i(R/tR) \to \mathbf{M}^i(R[1/s])$ induces the zero map on K-groups.

EV.9.2 **9.2.** (Quillen) Let $R = k[[x_1, ..., x_n]]$ be a power series ring over a field. Modify the proof of Theorem 9.6 to show that the maps $K\mathbf{M}^{i+1}(R) \to K\mathbf{M}^i(R)$ are zero, so that (9.3.1) is exact.

If k is complete with respect to a nontrivial valuation, show that the above holds for the subring A of convergent power series in R.

EV.9.3 **9.3.** (Gillet) Let $f: X \to Y$ be a proper map of relative dimension d. Show that $f_*: \mathbf{M}^i(X) \to \mathbf{M}^{i-d}(Y)$ for all i, and deduce that there is a homomorphism of spectral sequences $f_*: E_r^{p,q}(X) \to E_r^{p-d,q+d}(Y)$, compatible with the proper transfer map $f_*: G_*(X) \to G_*(Y)$ of 3.7, and the pushforward maps $CH^i(X) \to CH^{i-d}(Y)$.

EV.9.4 **9.4.** Let $F: \mathbf{Ab} \to \mathbf{Ab}$ be any additive functor which commutes with filtered direct limits. Show that F sends pure exact sequences to pure exact sequences. In particular, this applies to $F(A) = {}_\ell A = \{a \in A : \ell a = 0\}$ and $F(A) = A \otimes B$.

10 Descent and Mayer-Vietoris properties

In recent decades, the fact that K-theory has the Mayer-Vietoris property with respect to special cartesian squares of schemes has played an important role in understanding its structure. For cartesian squares describing open covers (see 10.1 below), it is equivalent to the assertion that K-theory satisfies "Zariski descent," and this in turn is related to features such as the coniveau spectral sequence of Propositions 9.2 and 9.5. These notions generalize from K-theory to other presheaves, either of simplicial sets or of spectra, on a finite-dimensional noetherian scheme X.

As in 10.6 below, a presheaf F is said to satisfy descent if the fibrant replacement $F(U) \to \mathbb{H}_{\mathrm{zar}}(U, F)$ is a weak equivalence for all U, where "fibrant replacement" is with respect to the local injective model structure (defined in 10.5 below). From a practical viewpoint, the recognition criterion for descent is the Mayer-Vietoris property.

V.10.1 **Definition 10.1.** Let F be a presheaf of simplicial sets (or spectra) on a scheme X. We say that F has the *Mayer-Vietoris property* (for the Zariski topology on X) if for every pair of open subschemes U and V the following square is homotopy cartesian.

$$\begin{array}{ccc} F(U\cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U\cap V). \end{array}$$

Before defining Zariski descent, we motivate it by stating the following theorem, due to Brown and Gersten. We postpone the proof of this theorem until later in this section.

V.10.2 **Theorem 10.2.** *Let F be a presheaf of simplicial sets (or spectra) on X. Then F satisfies Zariski descent if and only if it has the Mayer-Vietoris property.*

V.10.3 **Example 10.3.** Recall that $G(X)$ denotes the K-theory space for the category of coherent sheaves on X. Since restriction of sheaves is functorial, $U \mapsto G(U)$ is a presheaf on X for the Zariski topology. In 6.11.2 we saw that G has the Mayer-Vietoris property on any noetherian scheme X. This was the original example of a presheaf satisfying Zariski descent, discovered in 1972 by Brown and Gersten in [36].

On the other hand, let $\mathbf{K}^B$ denote the non-connective K-theory spectrum of vector bundles (IV.10.6). As above, $U \mapsto \mathbf{K}^B(U)$ is a presheaf on X. Using big vector bundles IV.10.5, we may even arrange that it is a presheaf on schemes of finite type over X. Corollary 7.10 states that $\mathbf{K}^B$ has the Mayer-Vietoris property on any noetherian scheme X. (This fails for the connective spectrum $\mathbf{K}$ because the map $K_0(U) \oplus K_0(V) \to K_0(U \cap V)$ may not be onto. Corollary 7.11 states that KH has the Mayer-Vietoris property on any noetherian scheme X.

Model categories

In order to define Zariski descent in the category of presheaves on X, we need to consider two model structures on this category. Recall that a map $i : A \to B$ is said to have the *left lifting property* relative to a class $\mathcal{F}$ of morphisms if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{\;b\;} & Y \end{array}$$

with p in $\mathcal{F}$ there is a lift $h : B \to X$ such that $hi = a$ and $ph = b$.

V.10.4 **Definition 10.4.** Let $\mathcal{C}$ be a complete and cocomplete category. A *model structure* on $\mathcal{C}$ consists of three subcategories whose morphisms are called *weak equivalences*, *fibrations* and *cofibrations*, together with two functorial factorizations of each morphism, satisfying the following properties.

(1) (2-out-of-3) If f and g are composable morphisms in $\mathcal{C}$ and two of f, g, fg are weak equivalences, so is the third.

(2) (Retracts) If f is a retract of g, and g is a weak equivalence (resp., a fibration, resp., a cofibration) then so is f.

A map which is both a cofibration and a weak equivalence is called a *trivial cofibration*, and a map which is both a fibration and a weak equivalence is called a *trivial fibration*. These notions are used in the next two axioms.

(3) (Lifting) A trivial cofibration has the left lifting property with respect to fibrations. Similarly, cofibrations have the left lifting property with respect to trivial fibrations.

(4) (Factorization) One of the two functorial factorizations of a morphism is as a trivial cofibration followed by a fibration; the other is as a cofibration followed by a trivial fibration.

A *model category* is a category $\mathcal{C}$ with a model structure. The *homotopy category* of $\mathcal{C}$, $\operatorname{Ho}\mathcal{C}$, is defined to be the localization of $\mathcal{C}$ with respect to the class of weak equivalences. (This kind of localization is described in the Appendix to Chapter II.)

We refer the reader to [90] for more information about model categories. The main point is that a model structure provides us with a calculus of fractions for the homotopy category of $\mathcal{C}$.

V.10.4.1 **Example 10.4.1.** The standard example is the model structure on the category of simplicial sets, in which cofibrations are injections, fibrations are Kan fibrations; a map $X \to Y$ is a weak equivalence if its geometric realization $|X| \to |Y|$ is a homotopy equivalence. ($|X|$ is defined in IV.3.1.4.)

There is also a model structure on the category of spectra. Since the details depend on the exact definition of spectrum used, we will not dwell on this point. The model category of symmetric spectra is introduced and studied in [91].

V.10.5 **Definition 10.5.** A morphism $A \to B$ of presheaves (of either simplicial sets or spectra) is called a *global weak equivalence* if each $A(U) \to B(U)$ is a weak equivalence; it is called a (Zariski) *local weak equivalence* if it induces an isomorphism on (Zariski) sheaves of homotopy groups (resp., stable homotopy groups).

We say that $A(U) \to B(U)$ is a *cofibration* if each $A(U) \to B(U)$ is a cofibration; a *local injective fibration* is a map which has the right lifting property with respect to cofibrations which are local weak equivalences. Jardine showed that the Zariski-local weak equivalences, cofibrations and local injective fibrations determine model structures on the categories of presheaves of simplicial sets, and of spectra (see [97, 2.3]). We shall call these the *local injective* model structures, to distinguish them from other model structures in the literature.

In any model structure, an object B is called *fibrant* if the terminal map $B \to *$ is a fibration. A *fibrant replacement* of an object A is a trivial cofibration $A \to B$ with B fibrant. By the Factorization axiom 10.4(4), there is a functorial fibrant replacement. For the local injective model structure on presheaves, we write the fibrant replacement as $A \to \mathbb{H}_{\mathrm{zar}}(-, A)$, and write $\mathbb{H}^n_{\mathrm{zar}}(X, A)$ for $\pi_{-n}\mathbb{H}_{\mathrm{zar}}(X, A)$.

V.10.6 **Definition 10.6.** Let A be a presheaf of either simplicial presheaves or spectra. We say that A satisfies *Zariski descent* on a scheme X if the fibrant replacement $A \to \mathbb{H}_{\mathrm{zar}}(-, A)$ is a global weak equivalence. That is, if $A(U) \xrightarrow{\simeq} \mathbb{H}_{\mathrm{zar}}(U, A)$ is a weak equivalence for every open U in X.

Note that although inverse homotopy equivalences $\mathbb{H}_{\mathrm{zar}}(U, A) \to A(U)$ exist, they are only natural in U up to homotopy (unless A is fibrant).

Given any abelian group A and a positive integer n, the *Eilenberg-Mac Lane spectrum* $K(A, n)$ is a functorial spectrum satisfying $\pi_n K(A, n) = A$, and $\pi_i K(A, n) = 0$ for $i \neq n$. More generally, given a chain complex A_* of abelian groups, the *Eilenberg-Mac Lane spectrum* $K(A_*, n)$ is a functorial spectrum satisfying $\pi_k K(A_*, n) = H_{k-n}(A_*)$. See [223, 8.4.1, 10.9.19 and Ex. 8.4.4] for one construction.

V.10.6.1 **Example 10.6.1.** Let $\mathcal{A}$ be a Zariski sheaf on X. The *Eilenberg-Mac Lane spectrum* $K(\mathcal{A}, n)$ is the sheaf $U \mapsto K(\mathcal{A}(U), n)$. If $\mathcal{A} \to \mathcal{I}$ is an injective resolution, then $\mathbb{H}(-, \mathcal{A})$ is weak equivalent to the sheaf of Eilenberg-Mac Lane spectra $K(\mathcal{I}, 0)$, so

$$\mathbb{H}^n(X, \mathcal{A}) = \pi_{-n}\mathbb{H}(X, \mathcal{A}) \cong \pi_{-n}K(\mathcal{I}, 0)(X) = H^n\mathcal{I}(X)$$

is the usual sheaf cohomology group $H^n(X, \mathcal{A})$. This observation, that the fibrant replacement is the analogue of an injective resolution, is due to Brown-Gersten [36] and dubbed the "Great Enlightenment" by Thomason.

V.10.6.2 **Remark 10.6.2.** Given any Grothendieck topology t, we obtain the notion of *t-local weak equivalence* of presheaves by replacing Zariski sheaves by t-sheaves in the above definition. Jardine also showed that the t-local weak equivalences and cofibrations determine a model structure on the category of presheaves, where the t-local injective fibrations are defined by the right lifting property. Thus we have other fibrant replacements $A \to \mathbb{H}_t(-, A)$.

We say that A *satisfies t-descent* (on the appropriate site) if the fibrant replacement $A \to \mathbb{H}_t(-, A)$ is a global weak equivalence. The most commonly used versions are: Zariski descent, étale descent, Nisnevich descent and *cdh* descent.

If A is a presheaf of simplicial sets, we can sheafify it to form a Zariski sheaf, $a_{\text{zar}}A$. Since $A \to a_{\text{zar}}A$ is a weak equivalence, their fibrant replacements are weak equivalent. This is the idea behind our next lemma.

V.10.7 **Lemma 10.7.** *Let A, B be presheaves of simplicial sets on schemes. If B satisfies Zariski descent, then any natural transformation $\eta_R : A(\text{Spec } R) \to B(\text{Spec } R)$ (from commutative rings to simplicial sets) extends to a natural transformation $\eta_U : A(U) \to \mathbb{H}(U, B)$.*

The composite $A(U) \to B(U)$ with a homotopy equivalence $\mathbb{H}(U, B) \overset{\simeq}{\leftrightarrows} B(U)$ exists, but (unless B is fibrant) is only well defined up to weak equivalence.

Proof. Define the presheaf A_{aff} by $A_{\text{aff}}(U) = A(\text{Spec } \mathcal{O}(U))$, and define B_{aff} similarly; the natural map $A_{\text{aff}} \to A$ is a weak equivalence because it is so locally. Clearly η extends to $\eta_{\text{aff}} : A_{\text{aff}} \to B_{\text{aff}}$. The composite map in the diagram

$$\begin{array}{ccccc} A(U) & \longrightarrow & \mathbb{H}_{\text{zar}}(U, A) & \underset{\to}{\overset{\simeq}{\longleftarrow}} & \mathbb{H}_{\text{zar}}(U, A_{\text{aff}}) \\ & & & & \downarrow{\scriptstyle \eta_{\text{aff}}} \\ B(U) & \overset{\simeq}{\longrightarrow} & \mathbb{H}_{\text{zar}}(U, B) & \overset{\simeq}{\longleftarrow} & \mathbb{H}_{\text{zar}}(U, B_{\text{aff}}) \end{array}$$

is the natural transformation we desire. Note that the top right horizontal map is a trivial fibration, evaluated at U; the existence of a natural inverse, *i.e.,* a presheaf map $\mathbb{H}_{\text{zar}}(-, A) \to \mathbb{H}_{\text{zar}}(-, A_{\text{aff}})$ splitting it, follows from the lifting axiom V.10.4 10.4(3). Thus the map $A(U) \to \mathbb{H}_{\text{zar}}(U, B)$ is natural in U. □

V.10.7.1 **Remark 10.7.1.** In Lemma V.10.7 10.7, η only needs to be defined on a category of commutative rings containing the rings $\mathcal{O}(U)$ and maps $\mathcal{O}(U) \to \mathcal{O}(V)$.

Zariski descent and the Mayer-Vietoris property

If F is a presheaf of spectra which satisfies Zariski descent, then F has the Mayer-Vietoris property, by Exercise EV.10.5 10.5. In order to show that the converse holds, we need the following result, originally proven for simplicial presheaves in BG [36, Thm. 1′].

V.10.8 **Proposition 10.8.** *Let X be a finite-dimensional noetherian space, and F a presheaf of spectra on X which has the Mayer-Vietoris property (for the Zariski topology). If all the presheaves $\pi_q F$ have zero as associated sheaves, then $\pi_q F(X) = 0$ for all q.*

Proof. It suffices to prove the following assertion for all $d \geq 0$: for all open $X' \subseteq X$, all q and all $a \in \pi_q F(X')$, there exists an open $U \subseteq X'$ with $a|_U = 0$ and $\mathrm{codim}_X(X - U) \geq d$. Indeed, when $d > \dim(X)$ we must have $U = X'$ and hence $a|_{X'} = 0$ as required.

The assertion is clear for $d = 0$ (take $U = \emptyset$), so suppose that it holds for d and consider $a \in \pi_q F(X')$ as in the assertion. By induction, $a|_U = 0$ for an open U whose complement Z has codimension $\geq d$. Let $x_1, \dots, x_n$ be the generic points of Z of codimension d. By assumption, there is a neighborhood V of these points such that $a|_V = 0$, and $X - V$ has codimension at least d. The Mayer-Vietoris property gives an exact sequence

$$\pi_{q+1}F(U \cap V) \xrightarrow{\partial} \pi_q F(U \cup V) \to \pi_q F(U) \oplus \pi_q F(V).$$

If $a|_{U \cup V}$ vanishes we are done. If not, there is a $z \in \pi_{q+1}F(U \cap V)$ with $a = \partial(z)$. By induction, with $X'' = U \cap V$, there is an open W in X'' whose complement has codimension $\geq d$ and such that $z|_W = 0$. Let $y_1, \dots$ be the generic points of $X'' - W$, Y the closure of these points in V and set $V' = V - Y$. Looking at codimensions, we see that V' is also a neighborhood of the x_i and that $U \cap V' = W$. Mapping the above sequence to the exact sequence

$$\pi_{q+1}F(W) \xrightarrow{\partial} \pi_q F(U \cup V') \to \pi_q F(U) \oplus \pi_q F(V'),$$

we see that $a|_{U \cup V'} = \partial(z|_W) = 0$. As $U \cup V'$ contains all the x_i, its complement has codimension $> d$. This completes the inductive step, proving the desired result. □

Proof of Theorem V.10.2 *10.2.* We have already noted that the 'only if' direction holds by Ex. EV.10.5 10.5. In particular, since $\mathbb{H}(-, E)$ is fibrant, it always has the Mayer-Vietoris property. Now assume that E has the Mayer-Vietoris property, and let $F(U)$ denote the homotopy fiber of $E(U) \to \mathbb{H}(U, E)$. Then F is a presheaf, and F has the Mayer-Vietoris property by Exercise EV.10.1 10.1. Since $E \to \mathbb{H}(-, E)$ is a local weak equivalence, the stalks of the presheaves $\pi_q F$ are zero. By Proposition V.10.8 10.8, $\pi_q F(U) = 0$ for all U. From the long exact homotopy sequence of a fibration, this implies that $\pi_* E(U) \cong \pi_* \mathbb{H}(U, E)$ for all U, *i.e.,* $E \to \mathbb{H}(-, E)$ is a global weak equivalence. □

Nisnevich descent

To discuss descent for the Nisnevich topology, we need to introduce some terminology. A commutative square of schemes of the form

$$\begin{array}{ccc} U_Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{\ i\ } & X \end{array}$$

is called *upper distinguished* if $U_Y = U \times_X Y$, U is open in X, f is étale and $(Y - U_Y) \to (X - U)$ is an isomorphism of the underlying reduced closed subschemes. The *Nisnevich topology* on the category of schemes of finite type over X is the Grothendieck topology generated by coverings $\{U \to X, Y \to X\}$ for the upper distinguished squares. In fact, a presheaf F is a sheaf if and only if F takes upper distinguished squares to cartesian (*i.e.,* pullback) squares; see [122, 12.7].

For each point $x \in X$, there is a canonical map from the hensel local scheme $\operatorname{Spec} \mathcal{O}^h_{X,x}$ to X; these form a conservative family of points for the Nisnevich topology. Thus a sheaf $\mathcal{F}$ is zero if and only if its stalks are zero at these points, and if $a \in \mathcal{F}(X)$ is zero then for every point x there is a point y in an étale $Y \to X$ such that $k(x) \cong k(y)$ and $a|_Y = 0$.

V.10.9 **Definition 10.9.** A presheaf of simplicial sets on X is said to have the Mayer-Vietoris property for the Nisnevich topology if it sends upper distinguished squares to homotopy cartesian squares. (Taking $Y = V$ and $X = U \cup V$, this implies that it also has the Mayer-Vietoris property for the Zariski topology.)

V.10.10 **Theorem 10.10.** *A presheaf has the Mayer-Vietoris property for the the Nisnevich topology if and only if it satisfies Nisnevich descent.*

Proof. (Thomason) The proof of Theorem 10.2 goes through, replacing Ex. 10.5 by Ex. 10.6 and Proposition 10.8 by Ex. 10.7. □

V.10.10.1 **Example 10.10.1.** As in 10.3, the localization sequence 6.11 for $G(X) \to G(U)$ and devissage (Ex. 4.2) show that G satisfies Nisnevich descent.

The functor $\mathbf{K}^B$ satisfies Nisnevich descent, by [200, 10.8]. It follows formally from this, and the definition of KH (IV.12.1) that homotopy K-theory KH also satisfies Nisnevich descent.

The descent spectral sequence

The descent spectral sequence, due to Brown and Gersten [36], is a method for computing the homotopy groups of a presheaf F of spectra on X. It is convenient to write F_n for the presheaf of abelian groups $U \mapsto \pi_n F(U)$, and let $\mathcal{F}_n$ denote the Zariski sheaf associated to the presheaf F_n.

V.10.11 **Theorem 10.11.** *Let X be a noetherian scheme with* $\dim(X) < \infty$*. Then for every presheaf of spectra F having the Mayer-Vietoris property (for the Zariski topology on X) there is a spectral sequence:*

$$E_2^{pq} = H^p_{zar}(X, \mathcal{F}_{-q}) \Rightarrow F_{-p-q}(X).$$

Proof. For each U, we have a Postnikov tower $\cdots P_nF(U) \xrightarrow{p_n} P_{n-1}F(U) \cdots$ for $F(U)$; each p_n is a fibration, $\pi_q P_n F(U) = \pi_q F(U)$ for $n \le q$ and $\pi_q P_n F(U) = 0$ for all $n > q$. It follows that the fiber of p_n is an Eilenberg-Mac Lane space $K(\pi_n F(U), n)$. Since the Postnikov tower is functorial, we get a tower of presheaves of spectra. Since the fibrant replacement is also functorial, we have a tower of spectra $\cdots \mathbb{H}(-, P_nF) \xrightarrow{p'_n} \mathbb{H}(-, P_{n-1}F) \cdots$. By Ex. 10.2, the homotopy fiber of p'_n is $K(\pi_n \mathcal{F}, n)$. Thus we have an exact couple with

$$D^1_{pq} = \oplus \pi_{p+q} \mathbb{H}(X, P_pF) \quad \text{and} \quad E^1_{pq} = \oplus \pi_{p+q} K(\pi_p \mathcal{F}, p)(X) = \oplus H^{-q}(X, \mathcal{F}_p).$$

Since the spectral sequence is bounded, it converges to $\varinjlim D^1_* = \pi_* F(X)$; see [223, 5.9.7]. Re-indexing as in [223, 5.4.3], and rewriting E^2 as E_2 yields the desired cohomological spectral sequence. □

V.10.11.1 **Remark 10.11.1.** The proof of Theorem 10.11 goes through for the Nisnevich topology. That is, if F has the Mayer-Vietoris property for the Nisnevich topology, there is a spectral sequence:

$$E_2^{pq} = H^p_{\text{nis}}(X, \mathcal{F}_{-q}) \Rightarrow F_{-p-q}(X).$$

V.10.12 **Example 10.12.** As in Example 10.2, the G-theory presheaf $\mathbf{G}$ satisfies the Mayer-Vietoris property, so there is a fourth quadrant spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{G}_{-q})$ converging to $G_*(X)$. Here $\mathcal{G}_n$ is the sheaf associated to the presheaf G_n. This is the original Brown-Gersten spectral sequence of [36].

Similarly, the presheaf $U \mapsto \mathbf{K}^B$ satisfies the Mayer-Vietoris property, so there is a spectral sequence with $E_2^{p,q} = H^p(X, \mathcal{K}_{-q})$ converging to $K_*(X)$. This spectral sequence lives mostly in the fourth quadrant. For example for $F = \mathbf{K}$ we have $\mathcal{K}_0 = \mathbb{Z}$ and $\mathcal{K}_1 = \mathcal{O}_X^\times$ (see Section II.2 and III.1.4), so the rows $q = 0$ and $q = -1$ are the cohomology groups $H^p(X, \mathbb{Z})$ and $H^p(X, \mathcal{O}_X^\times)$. This spectral sequence first appeared in [200].

If X is regular, of finite type over a field, then Gillet and Soulé proved in [70, 2.2.4] that the descent spectral sequence (10.11) is isomorphic to the spectral sequence of Proposition 9.8.1, which arises from the coniveau spectral sequence of 9.5. This assertion holds more generally if X is regular and the Gersten-Quillen Conjecture 9.3 holds for the local rings of X.

EXERCISES

EV.10.1 **10.1.** Let $F \to E \to B$ be a sequence of presheaves which is a homotopy fibration sequence when evaluated at any U. If E and B have the Mayer-Vietoris property, for either the Zariski or Nisnevich topology, show that F does too.

EV.10.2 **10.2.** Suppose that $F \to E \to B$ is a sequence of presheaves such that each $F(U) \to E(U) \to B(U)$ is a fibration sequence. Show that each $\mathbb{H}(U, F) \to \mathbb{H}(U, E) \to \mathbb{H}(U, B)$ is a homotopy fibration sequence.

EV.10.3 **10.3.** Let F be a simplicial presheaf which is fibrant for the local injective model structure 10.5. Show that $F(V) \to F(U)$ is a Kan fibration for every $U \subset V$.

EV.10.4 **10.4.** Let F be a simplicial presheaf which is fibrant for the local injective model structure 10.5. Show that F has the Mayer-Vietoris property. *Hint:* ([138, 3.3.1]) If F is fibrant, $\hom(-, F)$ takes homotopy cocartesian squares to homotopy cartesian squares of simplicial sets. Apply this to the square involving U, V and $U \cap V$.

EV.10.5 **10.5.** (Jardine) Let F be a presheaf of spectra which satisfies Zariski descent on X. Show that F has the Mayer-Vietoris property. *Hint:* We may assume that F is fibrant for the local injective model structure of 10.5. In that case, F is a sequence of simplicial presheaves F^n, fibrant for the local injective model structure on simplicial presheaves, such that all the bonding maps $F^n \to \Omega F^{n+1}$ are global weak equivalences; see [97].

EV.10.6 **10.6.** Let F be a presheaf of spectra satisfying Nisnevich descent. Show that F has the Mayer-Vietoris property for the Nisnevich topology.

EV.10.7 **10.7.** Show that the Nisnevich analogue of Proposition 10.8 holds: if F has the Mayer-Vietoris property for the Nisnevich topology, and the Nisnevich sheaves associated to the $\pi_q F$ are zero, then $\pi_q F(X) = 0$ for all q. *Hint:* Consider the $f^{-1}(x_i) \subset Y$.

EV.10.8 **10.8.** (Gillet) Let G be a sheaf of groups on a scheme X, and $\rho : G \to \mathrm{Aut}(\mathcal{F})$ a representation of G on a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank n. Show that ρ determines a homotopy class of maps from $B_\bullet G$ to $B_\bullet GL_n(\mathcal{O}_X)$. *Hint:* Use the Mayer-Vietoris property for $\mathbb{H}(-, B_\bullet GL_n(\mathcal{O}_X))$ and induction on the number of open subschemes in a cover trivializing $\mathcal{F}$.

EV.10.9 **10.9.** Suppose that X is a 1-dimensional noetherian scheme, with singular points $x_1, \dots, x_s$. If A_i is the local ring of X at x_i, show that $K_{-1}(X) \cong \oplus K_{-1}(A_i)$. (Note that $K_n(X) = K_n(A_i) = 0$ for all $n \le -2$, by Ex. III.4.4.)

Using the fact (III.4.4.3) that K_{-1} vanishes on hensel local rings, show that $K_{-1}(X) \cong H^1_{\mathrm{nis}}(X, \mathbb{Z})$. This group is isomorphic to $H^1_{\mathrm{et}}(X, \mathbb{Z})$; see III.4.1.4.

EV.10.10 **10.10.** Suppose that X is an irreducible 2-dimensional noetherian scheme, with isolated singular points $x_1, \dots, x_s$. If A_i is the local ring of X at x_i, show that there is an exact sequence

$$0 \to H^2_{\mathrm{zar}}(X, \mathcal{O}_X^\times) \to K_{-1}(X) \to \oplus K_{-1}(A_i) \to 0.$$

If X is normal, it is well known that $H^1_{\mathrm{nis}}(X, \mathbb{Z}) = 0$. In this case, use III.4.4.3 to show that $K_{-1}(X) \cong H^2_{\mathrm{nis}}(X, \mathcal{O}_X^\times)$. This can be nonzero, as Ex. III.4.13 shows.

11 Chern classes

The machinery involved in defining Chern classes on higher K-theory can be overwhelming, so we begin with two simple constructions over rings.

V.11.1 **Dennis Trace map 11.1.** Let R be a ring. It is a classical fact that $H_*(G;R)$ is a direct summand of $HH_*(R[G])$ for any group G; see [223, 9.1.2]. Using the ring maps $R[GL_m(R)] \to M_m(R)$ and Morita invariance of Hochschild homology [223, 9.5], we have natural maps

$$HH_*(R[GL_m(R)]) \to HH_*(M_m(R)) \xrightarrow{\cong} HH_*(R).$$

Via the Hurewicz map $\pi_n BGL(R)^+ \xrightarrow{h} H_n(BGL(R)^+, \mathbb{Z})$, this yields homomorphisms:

$$K_n(R) \xrightarrow{h} H_n(GL(R), \mathbb{Z}) = \varinjlim_m H_n(GL_m(R), \mathbb{Z}) \to HH_n(R).$$

They are called the *Dennis trace maps*, having been discovered by R. K. Dennis around 1975. In fact, they are ring homomorphisms; see [94, p. 133].

Given any representation $\rho : G \to GL_m(R)$, there is a natural map

$$H_*(G,\mathbb{Z}) \to HH_*(R[G]) \xrightarrow{\rho} HH_*(R[GL_m(R)]) \to HH_*(M_m(R)) \xrightarrow{\cong} HH_*(R).$$

This map, regarded as an element of $\mathrm{Hom}(H_*(G), HH_*(R))$, naturally lifts to an element $c_1(\rho)$ of $\mathbb{H}^0(G, HH)$. Set $c_i(\rho) = 0$ for $i \geq 2$; the verification that the $c_i(\rho)$ form Chern classes (in the sense of 11.2 below) is left to Exercise 11.4.

Let Ω^*_R denote the exterior algebra of Kähler differentials of R over $\mathbb{Z}$ [223, 9.4.2]. If $\mathbb{Q} \subset R$, there is a projection $HH_n(R) \to \Omega^n_R$ sending $r_0 \otimes \cdots \otimes r_n$ to $(r_0/n!)\, dr_1 \wedge \cdots \wedge dr_n$. Gersten showed that, up to the factor $(-1)^{n-1}n$, this yields Chern classes $c_n : K_n(R) \to \Omega^n_R$. We encountered them briefly in Chapter III (Ex. 6.10 and 7.7): if x, y are units of R then $c_2(\{x,y\}) = -\frac{dx}{x} \wedge \frac{dy}{y}$.

Chern classes for rings

Suppose that $A = A(0) \oplus A(1) \oplus \cdots$ is a graded-commutative ring. Then for any group G, the group cohomology $\oplus H^i(G, A(i))$ is a graded-commutative ring (see [223, 6.7.11]). More generally we may suppose that each $A(i)$ is a chain complex, so that $A = A(0) \oplus A(1) \oplus \cdots$ is a graded dg ring; the group hypercohomology $\oplus H^i(G, A(i))$ is still a graded-commutative ring

V.11.2 **Definition 11.2.** A *theory of Chern classes* for a ring R with coefficients A is a rule assigning to every group G, and every representation $G \xrightarrow{\rho} \mathrm{Aut}_R(P)$ with P in $\mathbf{P}(R)$, elements $c_i(\rho) \in H^i(G, A(i))$ with $c_0(\rho) = 1$ and satisfying the following axioms. The formal power series $c_t(\rho) = \sum c_i(\rho)t^i$ in $1 + \prod H^i(G, A(i))$ is called the *total Chern class* of ρ.

(1) *Functoriality.* For each homomorphism $\phi : H \to G$, $c_i(\rho \circ \phi) = \phi^* c_i(\rho)$.

(2) *Triviality.* The trivial representation ε of G on R has $c_i(\varepsilon) = 0$ for $i > 0$. If $\rho : G \to \mathrm{Aut}(P)$ then $c_i(\rho) = 0$ for $i > \mathrm{rank}(P)$.

(3) *Sum Formula.* For all ρ_1 and ρ_2, $c_n(\rho_1 \oplus \rho_2) = \sum_{i+j=n} c_i(\rho_1)c_j(\rho_2)$. Alternatively, $c_t(\rho_1 \oplus \rho_2) = c_t(\rho_1)c_t(\rho_2)$.

(4) *Multiplicativity.* For all ρ_1 and ρ_2, $c_t(\rho_1 \otimes \rho_2) = c_t(\rho) * c_t(\rho_2)$. Here $*$ denotes the product in the (non-unital) λ-subring $1 + \prod_{i>0} H^i(G, A(i))$ of $W(H^*(G, A))$; see II.4.3. (We ignore this axiom if R is non-commutative.)

V.11.2.1 **Example 11.2.1.** Taking $G = 1$, the Chern classes $c_i(\rho)$ belong to the cohomology $H^iA(i)$ of the complex $A(i)$. Since $\rho : 1 \to \mathrm{Aut}(P)$ depends only on P, we write $c_i(P)$ for $c_i(\rho)$. The Sum Formula for $c_t(P \oplus Q)$ shows that the total Chern class c_t factors through $K_0(R)$, with $c_t([P]) = c_t(P)$. Comparing the above axioms to the axioms given in II.4.11, we see that a theory of Chern classes for R induces Chern classes $c_i : K_0(R) \to H^iA(i)$ in the sense of II.4.11.

V.11.2.2 **Variant 11.2.2.** A common trick is to replace $A(i)$ by its cohomological shift $B(i) = A(i)[i]$. Since $H^nB(i) = H^n\mathcal{A}(i)[i] = H^{i+n}A(i)$, a theory of Chern classes for B is a rule assigning elements $c_i(\rho) \in H^{2i}(G, A(i))$.

V.11.2.3 **Universal Elements 11.2.3.** In any such theory, the tautological representations $\mathrm{id}_n : GL_n(R) \to \mathrm{Aut}(R^n)$ play a distinguished role. They are interconnected by the natural inclusions $\rho_{mn} : GL_n(R) \hookrightarrow GL_m(R)$ for $m \geq n$, and if $m \geq n$ then $c_n(\mathrm{id}_n) = \rho^*_{mn}c_n(\mathrm{id}_m)$. These elements form an inverse system, and we set

$$c_n(\mathrm{id}) = \varprojlim_m \; c_n(\mathrm{id}_m) \in \varprojlim_m \; H^n(GL_m(R), A(n)).$$

The elements $c_n(\mathrm{id})$ are universal in the sense that, if $P \oplus Q \cong R^m$ and we extend $G \xrightarrow{\rho} \mathrm{Aut}(P)$ to $\rho \oplus 1_Q : G \to GL_m(R)$, then we have $c_n(\rho) = (\rho \oplus 1_Q)^*c_n(\mathrm{id}_m)$.

To get Chern classes, we need to recall how the cohomology $H^*(G, A)$ is computed. Let B_* denote the bar resolution of G (it is a projective $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$), and set $C = B_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, so that $H_*(G, \mathbb{Z}) = H_*C$. Since G acts trivially on A, $H^*(G, A)$ is the cohomology of $\mathrm{Hom}_{\mathbb{Z}[G]}(B_*, A) = \mathrm{Hom}_{\mathbb{Z}}(C, A)$. The i^{th} cohomology of this Hom complex is the group of chain homotopy equivalence classes of maps $C \to A[i]$, and each such class determines a map from $H_n(G, \mathbb{Z}) = H_n(C)$ to $H_{n-i}A = H^{i-n}A$; see [223, 2.7.5].

When A is a complex of $\mathbb{Z}/m$-modules, we could replace the bar construction B_* with B_*/mB_* in the above: $H^*(G, A)$ is the cohomology of $\mathrm{Hom}(C/mC, A)$. Since $H_*(G, \mathbb{Z}/m) = H_*(C/mC)$, we get maps $H_*(G, \mathbb{Z}/m) \to H^{i-n}A$.

V.11.3 **Lemma 11.3.** *A theory of Chern classes for a ring R with coefficients A yields homomorphisms for all $n, i > 0$, called Chern classes:*

$$c_{i,n} : K_n(R) \to H_{n-i}A(i) = H^{i-n}A(i).$$

If A is a graded ring, the maps are zero for $i \neq n$ and we simply have classes

$$c_i : K_i(R) \to A(i).$$

When R is commutative, we have the product rule: if $a_m \in K_m(R)$ and $a_n \in K_n(R)$ then

$$c_{m+n}(a_m a_n) = \frac{-(m+n-1)!}{(m-1)!(n-1)!} c_m(a_m) c_n(a_n).$$

In particular, $c_n(\{x_1, \dots, x_n\}) = (-1)^{n-1}(n-1)! c_1(x_1) \cdots c_1(x_n)$ for all $x_1, \dots, x_n$ in $K_1(R)$.

Proof. By the above remarks, each element $c_i(\mathrm{id}_m)$ determines a homomorphism $H_*(GL_m(R), \mathbb{Z}) \to H_*A(i)$. By functoriality, these maps are compatible as m varies and yield a homomorphism c_i from $H_*(GL(R), \mathbb{Z}) = \varinjlim H_*(GL_m(R), \mathbb{Z})$ to $H_*A(i)$. Via the Hurewicz map h, this yields homomorphisms for $n > 0$:

$$c_{i,n} : K_n(R) \xrightarrow{h} H_n(GL(R), \mathbb{Z}) \xrightarrow{c_i} H^{i-n}A(i).$$

In the special case that A is a ring, the target vanishes for $i \neq n$. The product rule follows from Exercise 11.3, which states that the right side is $c_m(a_m) * c_n(a_n)$. ☐

V.11.3.1 **Construction 11.3.1.** Here is a homotopy-theoretic construction for the maps $c_{i,n}$ of Lemma 11.3, due to Quillen. Recall that the Dold-Kan correspondence applied to the chain complex $A(i)[i]$ produces an H-space $K(A(i), i)$, called a *generalized Eilenberg-Mac Lane space*, with $\pi_n K(A(i), i) = H_{n-i}A(i)$. It classifies cohomology in the sense that for any topological space X, elements of $H^n(X, A(i))$ are in 1–1 correspondence with homotopy classes of maps $X \to K(A(i), n)$; for $X = BG$, elements of $H^n(G, A(i)) = H^n(BG, A(i))$ correspond to maps $BG \to K(A(i), n)$. See [223, 6.10.5 and 8.6.4].

In this way, each $c_i(\mathrm{id})$ determines a system of maps $BGL_m(R) \to K(A(i), i)$, compatible up to homotopy. As $K(A(i), i)$ is an H-space, these maps factor through the spaces $BGL_m(R)^+$; via the telescope construction, they determine a map $BGL(R)^+ \to K(A(i), i)$. The Chern classes of 11.3 may also be defined as:

$$c_{i,n} : K_n(R) = \pi_n BGL(R)^+ \to \pi_n K(A(i), i) = H^{i-n}A(i).$$

The map $BGL(R)^+ \to K(A(i), i)$ can be made functorial in R by the following trick. Apply the integral completion functor $\mathbb{Z}_\infty$ (IV.1.9.ii) to get a functorial map $\mathbb{Z}_\infty BGL(R) \to \mathbb{Z}_\infty K(A(i), i)$, and choose a homotopy inverse for the homotopy equivalence $K(A(i), i) \to \mathbb{Z}_\infty K(A(i), i)$.

V.11.3.2 **Finite coefficients 11.3.2.** If A is a complex of $\mathbb{Z}/m$-modules, then the $c_i(\mathrm{id}_m)$ determine a homomorphism from $H_*(GL(R), \mathbb{Z}/m)$ to $H_*A(i)$, by the remarks before 11.3. Via the Hurewicz map h (Ex. IV.2.4), this yields homomorphisms for $n > 0$:

$$c_{i,n} : K_n(R;\mathbb{Z}/m) \xrightarrow{h} H_n(GL(R),\mathbb{Z}/m) \xrightarrow{c_i} H^{i-n}A(i).$$

The product rule holds in this case if m is odd or if $8|m$; see IV.2.8.

V.11.4 **Example 11.4.** Suppose that $\mathcal{A}$ is a complex of Zariski sheaves on $X = \mathrm{Spec}(R)$. If $\mathcal{A} \to \mathcal{I}$ is an injective resolution (or the total complex of a Cartan-Eilenberg resolution if $\mathcal{A}$ is a bounded below complex), then $H^*\mathcal{I}(X) = H^*(X,\mathcal{A})$. If $\mathcal{A} = \oplus\mathcal{A}(i)$, a theory of Chern classes for R with coefficients $A = \oplus\mathcal{I}(i)(X)$ yields Chern classes $c_{i,n} : K_n(R) \to H^{i-n}(X,\mathcal{A}(i))$. We will see many examples of this construction below.

Chern classes for schemes

One way to get a theory of Chern classes for a scheme X is to consider equivariant cohomology groups $H^*(X,G,\mathcal{A})$. The functors $\mathcal{F} \mapsto H^n(X,G,\mathcal{F})$ are defined to be the right derived functors of the functor $H^0(X,G,\mathcal{F}) = \mathrm{Hom}_{X,G}(\mathbb{Z}_X,\mathcal{F}) = H^0(X,\mathcal{F}^G)$, where $\mathcal{F}$ belongs to the abelian category of sheaves of G-modules on X. Since it is useful to include open subschemes of X, we want to extend the definition in a natural way over a category of schemes including X.

V.11.5 **Definition 11.5.** Let $\mathcal{V}$ be a category of schemes, and $\mathcal{A} = \oplus\mathcal{A}(i)$ a graded complex of Zariski sheaves of abelian groups on $\mathcal{V}$, forming a graded sheaf of dg rings. A *theory of Chern classes* on $\mathcal{V}$ is a rule associating to every X in $\mathcal{V}$, every sheaf G of groups on X and every representation ρ of G in a locally free $\mathcal{O}_X$-module $\mathcal{F}$, elements $c_i(\rho) \in H^i(X,G,\mathcal{A}(i))$ for $i \geq 0$, with $c_0(\rho) = 1$, satisfying the following axioms (resembling those of 11.2). Here $c_t(\rho)$ denotes the total Chern class $\sum c_i(\rho)t^i$ in $1 + \prod H^i(X,G,\mathcal{A}(i))$.

(1) *Functoriality.* For each compatible system ϕ of morphisms $X \xrightarrow{f} X'$, $G \to f^*G'$ and $\mathcal{F} \to f^*\mathcal{F}'$ we have $c_i(\phi^*\rho) = \phi^*c_i(\rho)$.

(2) *Triviality.* The trivial representation ε of G on $\mathcal{O}_X$ has $c_i(\varepsilon) = 0$ for $i > 0$. If $\rho : G \to \mathrm{Aut}(\mathcal{F})$ then $c_i(\rho) = 0$ for $i > \mathrm{rank}(\mathcal{F})$.

(3) *Whitney sum formula.* If $0 \to \rho' \to \rho \to \rho'' \to 0$ is a short exact sequence of representations of G in locally free $\mathcal{O}_X$-modules, then $c_t(\rho) = c_t(\rho')c_t(\rho'')$.

(4) *Multiplicativity.* Given representations ρ_1 and ρ_2 of G on locally free $\mathcal{O}_X$-modules, $c_t(\rho_1 \otimes \rho_2) = c_t(\rho) * c_t(\rho_2)$. Here $*$ denotes the product in the non-unital λ-subring $\prod_{i>0} H^i(X,G,\mathcal{A}(i))$ of $W(H^*(X,G,\mathcal{A}))$; see II.4.3.

V.11.5.1 **Example 11.5.1.** Taking $G = 1$, the Chern classes $c_i(\rho)$ belong to the cohomology $H^i(X,\mathcal{A}(i))$ of the complex $\mathcal{A}(i)$. By the Whitney sum formula, the total Chern class c_t factors through $K_0(X)$; cf. V.11.2.1 11.2.1. Comparing the above axioms to the axioms given in II.4.11, we see that a theory of Chern classes for X induces Chern classes $c_i : K_0(X) \to H^i(X, \mathcal{A}(i))$ in the sense of II.4.11.

V.11.6 **Lemma 11.6.** *Let G be a sheaf of groups and $\mathcal{A}$ a complex of Zariski sheaves on a scheme X, with injective resolution $\mathcal{A} \to \mathcal{I}$. Then there are natural maps*

$$H^n(X, G, \mathcal{A}) \to H^n(G, \mathcal{I}(X)) \to \bigoplus_{i+j=n} \operatorname{Hom}\left(H_i(G, \mathbb{Z}), H^j(X, \mathcal{A})\right).$$

Proof. Since G acts trivially on $\mathcal{A}$, we can compute the equivariant cohomology of $\mathcal{A}$ as hyperExt groups. Take an injective sheaf resolution $\mathcal{A} \to \mathcal{I}$ and let B_* denote the bar resolution of G, with $C = B_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ as before. Then

$$H^n(X, G, A) = H^n \operatorname{Hom}_{\mathbb{Z}[G]}(B_*, \mathbb{Z}) \otimes \mathcal{I}(X) = H^n \operatorname{Hom}_{\mathbb{Z}}(C, \mathbb{Z}) \otimes \mathcal{I}(X).$$

The cohomology of the natural map $\operatorname{Hom}(C, \mathbb{Z}) \otimes \mathcal{I}(X) \to \operatorname{Hom}(C, \mathcal{I}(X))$ is the desired map $H^n(X, G, A) \to H^n(G, \mathcal{I}(X))$. The second map in the Lemma comes from the discussion before Lemma V.11.3 11.3, since $H^*\mathcal{I}(X) = H^*(X, \mathcal{A})$. □

V.11.7 **Corollary 11.7.** *A theory of Chern classes for* $\operatorname{Spec}(R)$ *with coefficients $\mathcal{A}$ (in the sense of Definition V.11.5 11.5) determines a theory of Chern classes for the ring R with coefficients $\mathcal{I}(X)$ (in the sense of Definition V.11.2 11.2), and hence (by V.11.3 11.3) Chern class homomorphisms*

$$c_{i,n} : K_n(R) \to H^{i-n}(\operatorname{Spec}(R), \mathcal{A}(i)).$$

V.11.8 **Proposition 11.8.** *A theory of Chern classes for X with coefficients $\mathcal{A}$ determines Chern class homomorphisms*

$$c_{i,n} : K_n(X) \to H^{i-n}(X, \mathcal{A}(i)).$$

Proof. By V.11.7 11.7, there is a theory of Chern classes for rings, and by Construction V.11.3.1 11.3.1 there are natural transformations $c_i : K_0(R) \times BGL(R)^+ \to K(\mathcal{A}(i)(R), i))$. Let $K(\mathcal{O}_X)$ denote the presheaf sending X to $K_0(R) \times BGL(R)^+$, where $R = \mathcal{O}(X)$; the map $K \to K(\mathcal{O}_X)$ is a local weak equivalence (V.10.5 10.5). By construction, the presheaf $\mathbb{H}(-, K(\mathcal{A}(i), i))$ satisfies Zariski descent (V.10.6 10.6). By Lemma V.10.7 10.7, each of the c_i extends to a morphism of presheaves $c_i : \mathbb{H}_{\text{zar}}(-, K(\mathcal{O}_X)) \to \mathbb{H}(-, K(\mathcal{A}(i), i))$. Composing with $K(X) \to \mathbb{H}_{\text{zar}}(X, K) \to \mathbb{H}_{\text{zar}}(X, K(\mathcal{O}_X))$ and taking homotopy groups yields the desired Chern classes on X. □

V.11.9 **Example 11.9.** Let $\mathcal{K}_i$ be the Zariski sheaf on a scheme X associated to the presheaf $U \mapsto K_i(U)$. (This sheaf was discussed in V.9.8 9.8 and V.10.12 10.12.) Gillet showed in his 1978 thesis (see Gillet [67, §8]) that there is a theory of Chern classes with coefficients $\oplus\mathcal{K}_i$, defined on the category of (smooth) varieties of finite type over a field. By V.11.7 11.7 and V.11.8 11.8, this yields Chern classes for each algebra, $c_{i,n} : K_n(R) \to H^{i-n}(\operatorname{Spec}(R), \mathcal{K}_i)$, and more generally Chern classes $K_n(X) \to$

$H^{i-n}(X,\mathcal{K}_i)$. Via the isomorphism $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}^\times)$ of I.5.10.1, the Chern class $c_{1,0} : K_0(X) \to \mathrm{Pic}(X)$ is the determinant II.2.6. The Splitting Principle II.8.8.1 implies that the Chern classes $c_{i,0} : K_0(X) \to H^i(X,\mathcal{K}_i) \cong CH^i(X)$ are the same as the Chern classes of II.8.9 discovered by Grothendieck (where the last isomorphism is Bloch's formula 9.8.1).

If x is a unit of R, the map $c_{1,0} : R^\times \to H^0(R,\mathcal{K}_1) = R^\times$ is the natural identification, so by the product rule the map $c_{n,n} : K_n(R) \to H^0(R,\mathcal{K}_n)$ satisfies

$$c_{n,n}(\{x_1,\dots,x_n\}) = (-1)^{n-1}(n-1)!\{x_1,\dots,x_n\}.$$

Étale Chern classes

V.11.10 **Example 11.10.** In [81, p. 245], Grothendieck constructed a theory of étale Chern classes $c_i(\rho) \in H^{2i}_{et}(X,G,\mu_m^{\otimes i})$ for every X over $\mathrm{Spec}(\mathbb{Z}[1/m])$. This is the case of Definition 11.5 in which $\mathcal{A}(i) = R\pi_*\mu_m^{\otimes i}[i]$ (using the shift trick of 11.2.2), where $R\pi_*$ is the direct image functor from étale sheaves to Zariski sheaves: $H^{i-n}(X,\mathcal{A}(i)) = H^{2i-n}_{\mathrm{et}}(X,\mu_m^{\otimes i})$. This yields a theory of étale Chern classes and hence (by 11.8) Chern class maps

$$c_{i,n} : K_n(X;\mathbb{Z}/m) \to H^{2i-n}(X,R\pi_*\mu_m^{\otimes i}) = H^{2i-n}_{\mathrm{et}}(X,\mu_m^{\otimes i}).$$

Soulé introduced the étale Chern classes $K_n(R;\mathbb{Z}/m) \xrightarrow{c_{i,n}} H^{2i-n}_{\mathrm{et}}(\mathrm{Spec}(R),\mu_m^{\otimes i})$ in [171], using them to detect many of the elements of $K_n(\mathbb{Z})$ described in Chapter VI. Suslin used the étale class $c_{2,2}$ to describe $K_3(F)$ in [187]; see VI.5.19 below.

By construction, $c_1(\rho)$ is the boundary of the element in $H^1_{\mathrm{et}}(X,G,\mathbb{G}_m)$ classifying the G-line bundle of $\det(\rho)$. Letting G be $R^\times$ and $\rho : G \to \mathrm{Aut}(R)$ the canonical isomorphism, we see that $c_{1,1} : R^\times \to H^1_{\mathrm{et}}(\mathrm{Spec}(R),\mu_m)$ is the Kummer map $R^\times/R^{\times n} \subset H^1_{\mathrm{et}}(\mathrm{Spec}(R),\mu_m)$. Therefore the Chern class $c_{1,1} : K_1(R) \to H^1_{\mathrm{et}}(\mathrm{Spec}(R),\mu_m)$ is the determinant $K_1(R) \to R^\times$ followed by the Kummer map.

V.11.10.1 **Lemma 11.10.1.** *If R contains a primitive m^{th} root of unity ζ, with corresponding Bott element $\beta \in K_2(R;\mathbb{Z}/m)$, then $c_{1,2} : K_2(R;\mathbb{Z}/m) \to H^0_{et}(R,\mu_m)$ sends $K_2(R)/m$ to 0 and satisfies $c_{1,2}(\beta) = \zeta$.*

Proof. Recall that $H^1_{\mathrm{et}}(\mathrm{Spec}(R),\mathbb{G}_m) \cong R^\times$. We claim that the left square in the following diagram is commutative. (The right square clearly commutes.)

$$\begin{array}{ccccccc}
K_2(R) \longrightarrow & K_2(R;\mathbb{Z}/m) & \xrightarrow{\text{Bockstein}} & K_1(R) & \xrightarrow{m} & K_1(R) \\
 & \downarrow{\scriptstyle c_{1,2}} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\
0 \quad \to & H^0_{\mathrm{et}}(R,\mu_m) & \longrightarrow & H^0_{\mathrm{et}}(R,\mathbb{G}_m) & \xrightarrow{m} & H^0_{\mathrm{et}}(R,\mathbb{G}_m).
\end{array}$$

Since the Bockstein sends β to ζ, and $K_2(R)$ to 0, the lemma will follow.

To see the claim, we fix injective resolutions $\mathbb{G}_m \to \mathcal{J}$ and $\mu_m \to \mathcal{I}$; we have a distinguished triangle

$$\mathcal{I} \longrightarrow \mathcal{J} \xrightarrow{m} \mathcal{J} \xrightarrow{\delta} \mathcal{I}[1].$$

If C is the standard chain complex for G introduced before Lemma 11.3, then δ induces a map $\mathrm{Hom}(C,\mathbb{Z}) \otimes \mathcal{J}(X) \to \mathrm{Hom}(C,\mathbb{Z}) \otimes \mathcal{I}(X)[1]$. By construction, the boundary $H^1_{\mathrm{et}}(X,G,\mathbb{G}_m) \to H^2_{\mathrm{et}}(X,G,\mu_m)$ is the cohomology of this map. By Lemma 11.6, this induces a map

$$\mathrm{Hom}(H_1(G,\mathbb{Z}), H^0_{\mathrm{et}}(X,\mathbb{G}_m)) \to \mathrm{Hom}(H_2(G,\mathbb{Z}/m), H^0_{\mathrm{et}}(X,\mu_m)).$$

For $G = GL_n(R)$, $c_{1,2}$ is the image of det, and the result follows. □

Motivic Chern classes

Let $\mathbb{Z}(i)$, $i \geq 0$ denote the motivic complexes (of Zariski sheaves), defined on the category $\mathcal{V}$ of smooth quasi-projective varieties over a fixed field k, as in [122]. The *motivic cohomology* of X, $H^{n,i}(X,\mathbb{Z})$, is defined to be $H^n(X,\mathbb{Z}(i))$. Here is our main result, due to S. Bloch [25]:

V.11.11 **Theorem 11.11.** *There is a theory of Chern classes with coefficients in motivic cohomology. The associated Chern classes are* $c_{i,n} : K_n(X) \to H^{2i-n,i}(X)$.

V.11.11.1 **Remark 11.11.1.** The Chern classes $c_{i,0} : K_0(X) \to H^{2i,i}(X) \cong CH^i(X)$ are the same as the Chern classes of II.8.9, discovered by Grothendieck in 1957. The class $c_{1,1} : K_1(R) \to H^{1,1}(\mathrm{Spec}(R)) \cong R^\times$ is the same as the determinant map of III.1.1.1.

Before proving Theorem 11.11, we use it to generate many other applications.

V.11.12 **Examples 11.12.** A. Huber has constructed natural multiplicative transformations from motivic cohomology to other cohomology theories; see [92]. Applying them to the motivic classes $c_i(\rho)$ yields a theory of Chern classes, and hence Chern classes, for these other cohomology theories. Here is an enumeration.

(1) *de Rham.* If k is a field of characteristic 0, and X is smooth, the de Rham cohomology groups $H^*_{dR}(X)$ are defined using the de Rham complex Ω^*_X. Thus we have Chern classes $c_i : K_n(X) \to H^{2i-n}_{dR}(X)$.

(2) *Betti.* If X is a smooth variety over $\mathbb{C}$, there is a topological space $X(\mathbb{C})$ and its Betti (=topological) cohomology is $H^*_{\mathrm{top}}(X(\mathbb{C}),\mathbb{Z})$. The Betti Chern classes are $c_i : K_n(X) \to H^{2i-n}_{\mathrm{top}}(X(\mathbb{C}),\mathbb{Z})$, arising from the realization $H^{n,i}(X) \to H^n_{\mathrm{top}}(X(\mathbb{C}),\mathbb{Z})$. Beilinson has pointed out that the image of the Betti Chern classes in $H^*_{\mathrm{top}}(X(\mathbb{C}),\mathbb{C})$ are compatible with the de Rham Chern classes, via the Hodge structure on $H^*_{\mathrm{top}}(X(\mathbb{C}),\mathbb{C})$.

Of course, if R is a $\mathbb{C}$-algebra the natural maps $GL(R) \to GL(R)^{\rm top}$ (described in IV.3.9.2) induce the transformations $H_*(GL(R)) \longrightarrow H_*^{\rm top}(GL(R)^{\rm top})$, as well as $H^{n,i}(X, GL_n(R)) \to H^n_{\rm top}(X(\mathbb{C}), GL_n(R)^{\rm top})$. It should come as no surprise that these Chern classes are nothing more than the maps $K_n(X) \to KU^{-n}(X(\mathbb{C}))$ followed by the topological Chern classes $c_i : KU^{-n}(X(\mathbb{C})) \to H^{2i-n}_{\rm top}(X(\mathbb{C}))$ of II.3.7.

(3) *Deligne-Beilinson.* If X is a smooth variety defined over $\mathbb{C}$, the Deligne-Beilinson cohomology groups $H^*_{\mathcal{D}}(X, \mathbb{Z}(*))$ are defined using truncations of the augmented de Rham complex $\mathbb{Z} \to \Omega^*_X$. The map $H^{n,i}(X) \to H^n_{\mathcal{D}}(X, \mathbb{Z}(i))$ induces Chern classes $K_n(X) \to H^{2i-n}_{\mathcal{D}}(X, \mathbb{Z}(i))$. These were first described by Beilinson.

(4) *ℓ-adic.* If k is a field of characteristic $\neq \ell$, the ℓ-adic cohomology of a variety makes sense and there are maps $H^{n,i}(X) \to H^n_\ell(X, \mathbb{Z}_\ell(i))$. This gives ℓ-adic Chern classes

$$c_i : K_n(X) \to H^{2i-n}_{\rm et}(X, \mathbb{Z}_\ell(i)).$$

The projection to $H^{2i-n}_{\rm et}(X, \mu_{\ell^\nu}^{\otimes i})$ recovers the étale Chern classes of 11.10.

We shall now use the motivic Chern classes to prove that $K^M_*(F) \to K_*(F)$ is an injection modulo torsion for any field F. For this we use the ring map $K^M_*(F) \xrightarrow{\cong} \oplus_i H^{i,i}(\mathrm{Spec}(F))$, induced by the isomorphism $K^M_1(F) = F^\times \cong H^{1,1}(F)$ and the presentation of $K^M_*(F)$. It is an isomorphism, by a theorem of Totaro and Nesterenko-Suslin [144].

V.11.13 **Lemma 11.13.** *Let F be a field. Then the composition of $K^M_i(F) \to K_i(F)$ and $c_{i,i} : K_i(F) \to H^{i,i}(\mathrm{Spec}(F)) \cong K^M_i(F)$ is multiplication by $(-1)^{i-1}(i-1)!$.*

In particular, $K^M_*(F) \to K_*(F)$ is an injection modulo torsion.

Proof. Since $c_{1,1}(a) = a$, the product rule 11.3 implies that $c_{i,i} : K_i(F) \to K^M_i(F)$ takes $\{a_1, \dots, a_i\}$ to $(-1)^{i-1}(i-1)!\{a_1, \dots, a_i\}$. □

The outlines of the following proof are due to Grothendieck, and use only formal properties of motivic cohomology; these properties will be axiomatized in 11.14.

Proof of Theorem 11.11. Let $B_\bullet G$ denote the simplicial group scheme

$$\mathrm{Spec}(k) \leftleftarrows G \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, G \times G \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots,$$

and similarly for $E_\bullet G$. Given a representation ρ of G in a locally free $\mathcal{O}_X$-module $\mathcal{F}$, G acts on the geometric vector bundle $\mathbb{A}$ of rank n associated to $\mathcal{F}$, and we can construct the projective bundle $\mathbb{P} = \mathbb{P}(\mathbb{A}) \times_G E_\bullet G$ over $B_\bullet G$. There is a canonical line bundle $\mathcal{L}$ on $\mathbb{P}$, and an element $\xi \in H^{2,1}(\mathbb{P}) \cong H^1(\mathbb{P}, \mathcal{O}^\times) = \mathrm{Pic}(\mathbb{P})$ associated to $\mathcal{L}$. By the Projective Bundle Theorem [122, 15.12], $H^{*,*}(\mathbb{P})$ is a free $H^{*,*}(B_\bullet G)$-module with basis the elements $\xi^i \in H^{2i,i}(\mathbb{P})$ $(0 \le i < n)$. That is, we have an isomorphism

$$\oplus_{i=0}^{n-1} H^{p-2i,q-i}(B_\bullet G) \xrightarrow{(1,\xi,\dots,\xi^{n-1})} H^{p,q}(\mathbb{P}).$$

The Chern classes $c_i = c_i^G(\rho) \in H^{2i,i}(B_\bullet G)$ are defined by the coefficients of $x^n \in H^{2n,n}(\mathbb{P})$ relative to this basis, by the following equation in $H^{2n,n}(\mathbb{P})$:

$$\xi^n - c_1\xi^{n-1} + \cdots + (-1)^i c_i \xi^{n-i} + \cdots + (-1)^n c_n = 0.$$

By definition, we have $c_i^G(\rho) = 0$ for $i > n$. For each n, this construction is natural in G and ρ. Moreover, if $\epsilon : G \to GL_n$ is the trivial representation, then $\mathbb{P}$ is $\mathbb{P}^{n-1} \times B_\bullet G$; since $\xi^n = 0$ in $H^{2n,n}(\mathbb{P}^{n-1})$, we have $c_i^G(\epsilon) = 0$ for all $i > 0$.

We now fix n and X. For every sheaf $\mathcal{A}$ and sheaf G of groups on X, there is a canonical map $H^*(B_\bullet G, \mathcal{A}) \to H^*(X, G, \mathcal{A})$; setting $\mathcal{A} = \mathbb{Z}(i)$, we define the classes $c_i(\rho) \in H^{2i}(X, G, \mathbb{Z}(i))$ to be the images of the $c_i \in H^{2i,i}(B_\bullet G)$ under this map.

It remains to verify the axioms in Definition 11.5. The Functoriality and Triviality axioms have been already checked. For the Whitney sum formula, suppose that we are given representations ρ' and ρ'' in locally free sheaves $\mathcal{F}'$, $\mathcal{F}''$ of rank n' and n'', respectively, and a representation ρ on an extension $\mathcal{F}$. The projective space bundle $\mathbb{P}(\mathcal{F})$ contains $\mathbb{P}(\mathcal{F}')$, and $\mathbb{P}(\mathcal{F}) - \mathbb{P}(\mathcal{F}')$ is a vector bundle over $\mathbb{P}(\mathcal{F}'')$. The localization exact sequence of [122, (14.5.5)] becomes:

$$H^{2n-2n'',n'}(\mathbb{P}(\mathcal{F}'')) \xrightarrow{i_*} H^{2n,n}(\mathbb{P}(\mathcal{F})) \xrightarrow{j^*} H^{2n,n}(\mathbb{P}(\mathcal{F}'')).$$

Now both $i_*(1)$ and $f'' = \xi^{n''} - c_1(\rho'') + \cdots \pm c_{n''}$ generate the kernel of j^*. By the projection formula, $f' = \xi^{n'} - c_1(\rho') + \cdots \pm c_{n'}$ satisfies $0 = i_*(i^* f') = i_*(1) f'$. Hence $0 = f'' f'$ in $H^{2n,n}(\mathbb{P}(\mathcal{F}))$. Since this is a monic polynomial in ξ, we must have

$$\xi^n - c_1(\rho) + \cdots + \pm c_n = (\xi^{n''} - c_1(\rho'') + \cdots \pm c_{n''})(\xi^{n'} - c_1(\rho') + \cdots \pm c_{n'}).$$

Equating the coefficients give the Whitney sum formula.

For the Multiplicativity axiom, we may use the splitting principle to assume both sheaves have filtrations with line bundles as quotients. By the Whitney sum formula, we may further reduce to the case in which $\mathcal{F}_1$ and $\mathcal{F}_2$ are lines bundles. In the case the formula is just the isomorphism $\mathrm{Pic}(X) \cong H^{2,1}(X)$, $\mathcal{L} \mapsto c_1(\mathcal{L})$; see [122, 4.2]. □

Twisted duality theory

Although many cohomology theories (such as those listed in 11.12) come equipped with a natural map from motivic cohomology, and thus have induced Chern classes, it is possible to axiomatize the proof of Theorem 11.11 to avoid the need from such a natural map. This was done by Gillet in [67], using the notion of a twisted duality theory. Fix a category $\mathcal{V}$ of schemes over a base S, and a graded sheaf $\oplus\mathcal{A}(i)$ of dg rings, as in 11.5. We assume that $\mathcal{A}$ satisfies the homotopy invariance property: $H^*(X, \mathcal{A}(*)) \cong H^*(X \times \mathbb{A}^1, \mathcal{A}(*))$.

V.11.14 **Definition 11.14.** A *twisted duality theory* on $\mathcal{V}$ with coefficients $\mathcal{A}$ consists of a covariant bigraded "homology" functor $H_n(-,\mathcal{A}(i))$ on the subcategory of proper maps in $\mathcal{V}$, equipped with contravariant maps j^* for open immersions and (for X flat over S of relative dimension d) a distinguished element η_X of $H_{2d}(X,\mathcal{A}(d))$ called the *fundamental classes* of X, subject to the following axioms.

(i) For every open immersion $j : U \hookrightarrow X$ and every proper map $p : Y \to X$, the following square commutes:

$$\begin{array}{ccc} H_n(Y,\mathcal{A}(i)) & \xrightarrow{j_Y^*} & H_n(U\times_X Y,\mathcal{A}(i)) \\ {\scriptstyle p_!}\downarrow & & \downarrow{\scriptstyle p_{U\,!}} \\ H_n(X,\mathcal{A}(i)) & \xrightarrow{j^*} & H_n(U,\mathcal{A}(i)). \end{array}$$

(ii) If $j : U \to X$ is an open immersion with closed complement $\iota : Z \to X$, there is a long exact sequence (natural for proper maps)

$$\cdots H_n(Z,\mathcal{A}(i)) \xrightarrow{\iota_!} H_n(X,\mathcal{A}(i)) \xrightarrow{j^*} H_n(U,\mathcal{A}(i)) \xrightarrow{\partial} H_{n-1}(Z,\mathcal{A}(i))\cdots$$

(iii) For each (X,U,Z) as in (ii), there is a *cap product*

$$H_p(X,\mathcal{A}(r))\otimes H_Z^q(X,\mathcal{A}(s)) \xrightarrow{\cap} H_{p-q}(Z,\mathcal{A}(r-s)),$$

which is a pairing of presheaves on each X, and such that for each proper map $p : Y \to X$ the *projection formula* holds:

$$p_!(y)\cap z = p_{Z\,!}(y\cap p^*(z)), \qquad y\in H_p(Y,\mathcal{A}(r)), z\in H_Z^q(X,\mathcal{A}(s)).$$

(iv) If X is smooth over S of relative dimension d, and $Z \hookrightarrow X$ is closed, the cap product with η_X is an isomorphism: $H_Z^{2d-n}(X,\mathcal{A}(d-s)) \xrightarrow{\cong} H_n(Z,\mathcal{A}(s))$. (This axiom determines $H_*(Z,\mathcal{A}(*))$.) In addition:
- the isomorphism $H^0(X,\mathcal{A}(0)) \xrightarrow{\eta_X\cap} H_{2d}(X,\mathcal{A}(d))$ sends 1 to η_X;
- If Z has codimension 1, the fundamental class η_Z corresponds to an element $[Z]$ of $H_Z^2(X,\mathcal{A}(1))$, *i.e.*, $\eta_X\cap[Z]=\eta_Z$. Writing cycle(Z) for the image of $[Z]$ in $H^2(X,\mathcal{A}(1))$, we require that these cycle classes extend to a natural transformation $\mathrm{Pic}(X)\to H^2(X,\mathcal{A}(1))$.

(v) If $Z \xrightarrow{\iota} X$ is a closed immersion of smooth schemes (of codimension c) then the isomorphism $H^n(Z,\mathcal{A}(r)) \cong H_Z^{n+2c}(X,\mathcal{A}(r+c))$ in (iv) is induced by a map $\iota_! : \mathcal{A}(r) \xrightarrow{\cong} R\iota^!\mathcal{A}(r+c)[2c]$ in the derived category of Z, where $\iota^!$ is the "sections with support" functor. The projection formula of (iii) for $Z\to X$ is represented in the derived category of X by the commutative diagram

$$\begin{array}{ccccc} R\iota_!\mathcal{A}(r)\otimes^{\mathbb{L}}\mathcal{A}(s) & \xrightarrow{R\iota_!(1\otimes\iota^!)} & R\iota_!\big(\mathcal{A}(r)\otimes^{\mathbb{L}}\mathcal{A}(s)\big) & \longrightarrow & R\iota_!\mathcal{A}(r+s) \\ \simeq\downarrow \iota_!\otimes 1 & & & & \simeq\downarrow \iota_! \\ R\iota^!\mathcal{A}(r+c)[2c]\otimes^{\mathbb{L}}\mathcal{A}(s) & & \longrightarrow & & R\iota^!\mathcal{A}(r+s+c)[2c]. \end{array}$$

(vi) For all $n \geq 1$ and X in $\mathcal{V}$, let $\xi \in H^2(\mathbb{P}^n_X, \mathcal{A}(1))$ be the cycle class of a hyperplane. Then the map $\pi^* : H_*(X, \mathcal{A}(*)) \to H^*(\mathbb{P}^n_X, \mathcal{A}(*))$, composed with the cap product with powers of ξ, is onto, and the cap product with powers of ξ induces an isomorphism:

$$\bigoplus_{i=0}^{n} H_{n+2i}(X, \mathcal{A}(q+i)) \xrightarrow{(1,\xi,\dots,\xi^n)\cap\pi^*} H_n(\mathbb{P}^n_X, \mathcal{A}(q)).$$

Gillet [67, 2.2] *If there is a twisted duality theory on $\mathcal{V}$ with coefficients $\mathcal{A}$, then there is a theory of Chern classes on $\mathcal{V}$ with coefficients $\mathcal{A}$.*

EXERCISES

EV.11.1 **11.1.** Suppose given a theory of Chern classes for a ring A with coefficients in a cochain complex C, as in V.11.2 11.2, and let $R_A(G)$ denote the representation ring of G over A (see Ex. II.EII.4.2 4.2). Show that there are well defined functions $c_n : R_A(G) \to H^i(G, C(i))$ forming Chern classes on the λ-ring $R_A(G)$ in the sense of II.II.4.11 4.11. If K is the space $K(C) = \prod K(C(n), n)$, show that they determine a natural transformation $R_A(G) \to [BG, K(C)]$. By the universal property of the $+$-construction (IV.IV.5.7 5.7), this yields a unique element of $[BGL(A)^+, K(C)]$. Compare this with the construction in V.11.3 11.3.

EV.11.2 **11.2.** Consider the unital λ-ring $\mathbb{Z} \times (1 + \prod H^n(G, A(n)))$ with multiplication $(a, f) * (b, g) = (ab, f^b g^a (f * g))$. Show that $(\text{rank}, c_t) : K_*(X) \to \mathbb{Z} \times (1 + \prod H^n(G, A(n)))$ is a homomorphism of unital λ-rings.

EV.11.3 **11.3.** (Grothendieck) Let $H = \oplus H^n$ be a graded ring and write W for the non-unital subalgebra $1 + \prod H^n$ of $W(H)$. Show that for $x \in H^m$, $y \in H^n$ we have

$$(1-x) * (1-y) = 1 - \frac{-(m+n-1)!}{(m-1)!(n-1)!} xy + \cdots .$$

Hint: In the universal case $H = \mathbb{Z}[x, y]$, $W(H)$ embeds in $W(H \otimes \mathbb{Q})$. Now use the isomorphism $W(H \otimes \mathbb{Q}) \cong \prod H \otimes \mathbb{Q}$ and compute in the mn coordinate.

EV.11.4 **11.4.** *Dennis trace as a Chern class.* Let $C_*(R)$ denote the standard Hochschild chain complex with $H_n(C_*(R)) = HH_n(R)$; see WHomo [223, 9.1]. In the notation of V.11.2.3 11.2.3, there is a natural chain map $B_* \otimes \mathbb{Z} \to C_*(\mathbb{Z}[G])$, representing an element of $H^0 \operatorname{Hom}(B_* \otimes \mathbb{Z}, C_*(\mathbb{Z}[G]))$, which on homology is the $H_*(G, \mathbb{Z}) \to HH_*(\mathbb{Z}[G])$

of 11.1; see [223, 9.7.5]. Since the $C_*(M_m(R)) \to C_*(R)$ are chain maps [223, 9.5.7], any representation ρ determines a chain map and an associated element $c_1(\rho)$ of $H^0(G, C_*(R)) = H^0 \operatorname{Hom}(B_* \otimes \mathbb{Z}, C_*(R))$.

Setting $A(0) = \mathbb{Z}$, $A(1) = C_*[-1]$ and $A(n) = 0$ for $n > 1$, we may regard $c_1(\rho)$ as an element of $H^1(G, A(1))$. Show that $c_1(\rho)$ determines a theory of Chern classes with $c_i(\rho) = 0$ for $i \geq 2$. *Hint:* the product $c_1(\rho_1)c_1(\rho_2)$ is in $H^2(G, A(2)) = 0$, and the method of 11.2.3 applies.

EV.11.5 **11.5.** Suppose that R contains both $1/m$ and the group μ_m of primitive m^{th} roots of unity. Let $\rho : \mu_m \to GL_2(R)$ be the representation $\rho(\zeta)(x, y) = (\zeta x, \zeta^{-1} y)$. Show that the étale Chern class $c_{2,2}(\rho) : H_4(\mu_m, \mathbb{Z}/m) \to H^0_{\text{et}}(R, \mu_m^{\otimes 2}) \cong \mathbb{Z}/m$ is an isomorphism. *Hint:* if $\lambda : \mu_m \to R^\times$ is the tautological representation, show that $c_{2,2} = c_{1,1}(\lambda)^2$.

EV.11.6 **11.6.** Show that a theory of Chern classes with coefficients $\mathcal{A}(i)$ yields Chern classes with coefficients $\mathcal{B}(i) = \mathcal{A}(i) \otimes^{\mathbb{L}} \mathbb{Z}/m$, and that there is a commutative diagram

$$\begin{array}{ccccc}
K_n(X) & \longrightarrow & K_n(X; \mathbb{Z}/m) & \longrightarrow & K_{n-1}(R) \\
\downarrow c_i & & \downarrow c_i & & \downarrow c_i \\
H^{i-n}(X, \mathcal{A}(i)) & \to & H^{i-n}(X, \mathcal{B}(i)) & \to & H^{i+1-n}(X, \mathcal{A}(i)).
\end{array}$$

Hint: $\operatorname{Hom}(C_*, \mathcal{A})[1] \to \operatorname{Hom}(C_*, \mathcal{A} \otimes^{\mathbb{L}} \mathbb{Z}/m) \to \operatorname{Hom}(C_*, \mathcal{A}) \to$ is a triangle.

As an application of this, let $\mathcal{A}(i)$ be the complex for Deligne-Beilinson cohomology of a complex variety; $\mathcal{B}(i)$ computes étale cohomology, and we have a commutative diagram, showing that the étale and Deligne-Beilinson Chern classes are compatible.

$$\begin{array}{ccccc}
K_n(X) & \longrightarrow & K_n(X; \mathbb{Z}/m) & \longrightarrow & K_{n-1}(R) \\
\downarrow c_i & & \downarrow c_i & & \downarrow c_i \\
H^{2i-n}_{\mathcal{D}}(X, \mathbb{Z}(i)) & \to & H^{2i-n}_{\text{et}}(X, \mu_m^{\otimes i}) & \to & H^{2i+1-n}_{\mathcal{D}}(X, \mathbb{Z}(i)).
\end{array}$$

EV1.1.7 **11.7.** By Kummer theory, $H^1_{\text{et}}(X, \mu_m) \cong {}_m\operatorname{Pic}(X) \oplus U(X)/mU(X)$, where $U(X) = \mathcal{O}(X)^\times$. Modifying Example 11.10, show that the Chern class $c_{1,1} : K_1(X; \mathbb{Z}/m) \to H^1_{\text{et}}(X, \mu_m)$ sends $K_1(X)/m$ to $U(X)/mU(X)$ by the determinant, and that the induced quotient map $c_{1,1} : {}_mK_0(X) \to {}_m\operatorname{Pic}(X)$ may be identified with the canonical map of II.8.1.

EV.11.8 **11.8.** (Suslin) Suppose given a theory of Chern classes for a field F with coefficients A. Show that the Chern class $c_n : K_n(F) \to H^0 A(n)$ factors through Suslin's map $K_n(F) \to K_n^M(F)$ described in IV.1.15. In particular, the K-cohomology Chern class $c_n : K_n(F) \to H^0(\operatorname{Spec}(F), \mathcal{K}_n) \cong K_n(F)$ factors through $K_n(F) \to K_n^M(F)$. *Hint:* Use $K_n^M(F) \cong H_n(GL_n(F))/H_n(GL_{n-1}(F))$ to reduce to the Triviality axiom 11.2(2) that $c_n(\mathrm{id}_{n-1}) = 0$.

Chapter VI

The higher K-theory of Fields

The problem of computing the higher K-groups of fields has a rich history, beginning with Quillen's calculation for finite fields (IV.1.13), and Borel's calculation of $K_*(F)\otimes\mathbb{Q}$ for number fields (IV.1.18), Tate's calculations of the Milnor K-groups of number fields (III.7.2(a)) and Quillen's observation that the image of the stable homotopy groups π_*^s in $K_*(\mathbb{Z})$ contained the image of the J-homomorphism, whose orders are described by Bernoulli numbers. In the early 1970's, a series of conjectures were made concerning the K-theory of number fields, and the structure of Milnor K-theory. After decades of partial calculations and further conjectures, the broad picture is now in place. The goal of this chapter is to explain what we now know about the K-theory of fields, and especially number fields.

1 K-theory of algebraically closed fields

We begin by calculating the K-theory of algebraically closed fields. The results in this section are due to Suslin [182, 184].

Let C be a smooth curve over an algebraically closed field k, with function field F. The local ring of C at any closed point $c\in C$ is a discrete valuation ring, and we have a specialization map $\lambda_c : K_*(F,\mathbb{Z}/m)\to K_*(k,\mathbb{Z}/m)$ (see V.6.7 and Ex. V.6.14). If $C=\mathbb{P}^1$, we saw in V, Ex. 6.14 that all of the specialization maps λ_c agree. The following result, due to Suslin [182], shows that this holds more generally.

VI.1.1 **Theorem 1.1.** *(Rigidity) Let C be a smooth curve over an algebraically closed field k, with function field $F=k(C)$. If c_0,c_1 are two closed points of C, the specializations $K_*(F,\mathbb{Z}/m)\to K_*(k,\mathbb{Z}/m)$ coincide.*

Proof. There is no loss of generality in assuming that C is a projective curve. Suppose that $f : C \to \mathbb{P}^1$ is a finite map. Let R_0 and R' be the local ring in $k(t)$ at $t = 0$ and its integral closure in $F = k(C)$, respectively. If s_c is a parameter at c and e_c is the ramification index at c, so that $t = u \prod s_c^{e_c}$ in R', then for $a \in K_*(F, \mathbb{Z}/m)$ we have $\partial_c(\{t, a\}) = e_c \partial_c\{s_c, a\} = e_c \lambda_c(a)$. Since $k(c) = k$ for all closed $c \in C$, we see from Chapter V, (6.6.4) that

$$\lambda_0(N_{F/k(t)}a) = \partial_0 N_{F/k(t)}(\{t, a\}) = \sum_{f(c)=0} N_c \partial_c(\{t, a\}) = \sum_{f(c)=0} e_c \lambda_c(a).$$

A similar formula holds for any other point of $\mathbb{P}^1$. In particular, since $\lambda_0(Na) = \lambda_\infty(Na)$ we have the formula

$$\sum\nolimits_{f(c)=0} e_c \lambda_c(a) = \sum\nolimits_{f(c)=\infty} e_c \lambda_c(a).$$

We can assemble this information as follows. Let A denote the abelian group $\mathrm{Hom}(K_*(F; \mathbb{Z}/m), K_*(k, \mathbb{Z}/m))$, and recall that $\mathrm{Cart}(C)$ denotes the free abelian group on the closed points of C. There is a homomorphism $\lambda : \mathrm{Cart}(C) \to A$ sending $[c]$ to the specialization map λ_c of V.6.7. If we regard f as an element of $F^\times$, its divisor is $\sum_{f(c)=0} e_c[c] - \sum_{f(c)=\infty} e_c[c]$. The displayed equation amounts to the formula $\lambda \circ \mathrm{div} = 0$. Now the Picard group $\mathrm{Pic}(C)$ is the cokernel of the divisor map $F^\times \to \mathrm{Cart}(C)$ (see I.5.12), so λ factors through $\mathrm{Pic}(C)$. Since A is a group of exponent m, λ factors through $\mathrm{Pic}(C) \otimes \mathbb{Z}/m$. However, the kernel $J(C)$ of the degree map $\mathrm{Pic}(C) \to \mathbb{Z}$ is a divisible group (I.5.16, so λ is zero on $J(C)$. Since $[c_0] - [c_1] \in J(C)$, this implies that $\lambda_{c_0} = \lambda_{c_1}$. □

VI.1.2 **Corollary 1.2.** *If A is any finitely generated smooth integral k-algebra, and $h_0, h_1 : A \to k$ are any k-algebra homomorphisms, then the induced maps $h_i^* : K_*(A; \mathbb{Z}/m) \to K_*(k; \mathbb{Z}/m)$ coincide.*

Proof. The kernels of the h_i are maximal ideals, and it is known that there is a prime ideal $\mathfrak{p}$ of A contained in their intersection such that $A/\mathfrak{p}$ is 1-dimensional. If R is the normalization of $A/\mathfrak{p}$ then the h_i factor through $A \to R$. Therefore we can replace A by R. Since $C = \mathrm{Spec}(R)$ is a smooth curve, the specializations λ_i on $F = k(C)$ agree by Theorem 1.1. The result follows, since by Theorem V.6.7, the induced maps h_i^* factor as $K_*(R; \mathbb{Z}/m) \to K_*(F; \mathbb{Z}/m) \xrightarrow{\lambda_i} K_*(k; \mathbb{Z}/m)$. □

VI.1.3 **Theorem 1.3.** *If $k \subset F$ is an inclusion of algebraically closed fields, the maps $K_n(k; \mathbb{Z}/m) \to K_n(F; \mathbb{Z}/m)$ are isomorphisms for all m.*

Proof. We saw in V.6.7.4 that both $K_n(k) \to K_n(F)$ and $K_n(k; \mathbb{Z}/m) \to K_n(F; \mathbb{Z}/m)$ are injections. To see surjectivity, we write F as the union of its finitely generated subalgebras A. Therefore every element of $K_n(F; \mathbb{Z}/m)$ is the image of some element of $K_n(A; \mathbb{Z}/m)$ under the map induced from the inclusion $h_0 : A \hookrightarrow F$. Since the singular locus of A is closed, some localization $A[1/s]$ is smooth, so we may assume that A is smooth.

But for any maximal ideal $\mathfrak{m}$ of A we have a second map $h_1 : A \to A/\mathfrak{m} = k \hookrightarrow F$. Both h_0 and h_1 factor through the basechange $A \to A \otimes_k F$ and the induced maps $h_i : A \otimes_k F \to F$. Since $A \otimes_k F$ is smooth over F, the map $h_0^* : K_n(A;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ coincides with $h_1^* : K_n(A;\mathbb{Z}/m) \to K_n(k;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ by Corollary 1.2. This finishes the proof. □

Finite characteristic. For algebraically closed fields of characteristic $p > 0$, we may take $k = \bar{\mathbb{F}}_p$ to determine $K_*(F)$. Recall from IV.1.13 that $K_n(\bar{\mathbb{F}}_p) = 0$ for even $n > 0$, and that $K_{2i-1}(\bar{\mathbb{F}}_p) = \cup K_{2i-1}(\mathbb{F}_{p^\nu})$ is isomorphic as an abelian group to $\bar{\mathbb{F}}_p^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$. In particular, $K_n(F;\mathbb{Z}/p) = K_n(\bar{\mathbb{F}}_p;\mathbb{Z}/p) = 0$; this implies that $K_n(F)$ is uniquely p-divisible. (We saw in IV.5.6 that this is true more generally for perfect fields of characteristic p.)

We also saw in IV.1.13.2 that if $p \nmid m$ and $\beta \in K_2(\bar{\mathbb{F}}_p;\mathbb{Z}/m) \cong \mu_m(\bar{\mathbb{F}}_p)$ is the Bott element (whose Bockstein is a primitive m^{th} root of unity), then $K_*(\bar{\mathbb{F}}_p;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$ as a graded ring. The action of the Frobenius automorphism $\phi(x) = x^p$ on F induces multiplication by p^i on $K_{2i-1}(\bar{\mathbb{F}}_p)$; we say that the action is *twisted* i times. The following corollary to Theorem 1.3 is immediate from these remarks.

VI.1.3.1 **Corollary 1.3.1.** *Let F be an algebraically closed field of characteristic $p > 0$.*
(i) If n is even and $n > 0$, $K_n(F)$ is uniquely divisible.
(ii) If $n = 2i-1$ is odd, $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and the torsion group $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$. In particular, it is divisible with no p-torsion, and the Frobenius automorphism acts on the torsion subgroup as multiplication by p^i.
(iii) When $p \nmid m$, the choice of a Bott element $\beta \in K_2(F;\mathbb{Z}/m)$ determines a graded ring isomorphism $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$.

Recall that any divisible abelian group is the direct sum of a uniquely divisible group and a divisible torsion group, a divisible torsion group is the sum of its Sylow subgroups, and an ℓ-primary divisible group is a direct sum of copies of $\mathbb{Z}/\ell^\infty$. Therefore $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and $\oplus_{\ell \neq p} \mathbb{Z}/\ell^\infty \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$.

VI.1.3.2 **Remark 1.3.2.** If F is any separably closed field of characteristic p, then $K_n(F)$ is non-canonically a summand of $K_n(\bar{F})$ by a transfer argument (as it is uniquely p-divisible for all $n > 0$, by IV.5.6). Therefore $K_n(F)$ also has the structure described in Corollary 1.3.1; see Exercise 1.1.

We now turn to the structure of $K_*(F)$ when F has characteristic zero.

VI.1.4 **Proposition 1.4.** *If F is an algebraically closed field of characteristic 0 then for every $m > 0$ the choice of a Bott element $\beta \in K_2(F;\mathbb{Z}/m)$ determines a graded ring isomorphism $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$.*

Proof. Pick a primitive m^{th} root of unity ζ in $\bar{\mathbb{Q}}$, and let β be the corresponding Bott element in $K_2(\mathbb{Q}(\zeta);\mathbb{Z}/m)$. We use this choice to define a Bott element $\beta \in K_2(E;\mathbb{Z}/m)$ for all fields containing $\mathbb{Q}(\zeta)$, natural in E. By Theorem 1.3, it suffices to show that the induced ring map $\mathbb{Z}/m[\beta] \to K_*(F;\mathbb{Z}/m)$ is an isomorphism for some algebraically closed field F containing $\bar{\mathbb{Q}}$.

Fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, where $p \nmid m$. For each $q = p^\nu$, let E_q denote the maximal algebraic extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$ with residue field $\mathbb{F}_q$; $\bar{\mathbb{Q}}_p$ is the union of the E_q. For each $q \equiv 1 \pmod{m}$, we saw in Example V.6.10.2 (which uses Gabber rigidity) that $K_*(E_q;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$. If $q|q'$, the map $K_*(E_q;\mathbb{Z}/m) \to K_*(E_{q'};\mathbb{Z}/m)$ is an isomorphism, by naturality of β. Taking the direct limit over q, we have $K_*(\bar{\mathbb{Q}}_p;\mathbb{Z}/m) = \mathbb{Z}/m[\beta]$, as desired. □

VI.1.4.1 **Remark 1.4.1.** There is a map $K_*(\mathbb{C};\mathbb{Z}/m) \to \pi_*(BU;\mathbb{Z}/m)$ arising from the change of topology; see IV.4.12.3. Suslin proved in [184] that this map is an isomorphism. We can formally recover this result from Proposition 1.4, since both rings are polynomial rings in one variable, and the generator $\beta \in K_2(\mathbb{C};\mathbb{Z}/m)$ maps to a generator of $\pi_2(BU;\mathbb{Z}/m)$ by IV.1.13.2.

To determine the structure of $K_*(F)$ when F has characteristic 0, we need a result of Harris and Segal [84, 3.1]. Let $m = \ell^\nu$ be a prime power and R a ring containing the group μ_m of m^{th} roots of unity. The group $\mu_m \wr \Sigma_n = (\mu_m)^n \rtimes \Sigma_n$ embeds into $GL_n(R)$ as the group of invertible $n \times n$ matrices with only one nonzero entry in each row and column, every nonzero entry being in μ_m. Taking the union over n, the group $G = \mu_m \wr \Sigma_\infty$ embeds into $GL(R)$.

There is an induced map $\pi_*(BG^+) \to K_*(R)$ when $\mu_m \subset R$. Given a finite field $\mathbb{F}_q$ with $\ell \nmid q$, transfer maps for $R = \mathbb{F}_q(\zeta_{2\ell})$ give maps $\pi_*(BG^+) \to K_*(R) \to K_*(\mathbb{F}_q)$. Let $\mu_{(\ell)}(F)$ denote the ℓ-primary subgroup of $\mu(F)$.

VI.1.5 **Theorem 1.5.** *(Harris-Segal) If $\ell \nmid q$ and $|\mu_{(\ell)}(\mathbb{F}_q(\zeta_\ell))| = m$, then each group $\pi_{2i-1}B(\mu_m \wr \Sigma_\infty)^+$ contains a cyclic summand which maps isomorphically to the ℓ-Sylow subgroup of $\pi_{2i-1}BGL(\mathbb{F}_q)^+ = K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i-1)$.*

Proof. Fix i and let $N = \ell^\nu$ be the largest power of ℓ dividing q^i-1. Theorem 3.1 of [84] states that both $\pi_{2i-1}(BG^+)_{(\ell)} \to K_{2i-1}(\mathbb{F}_q)_{(\ell)}$ and $\pi_{2i}(BG^+;\mathbb{Z}/N) \to K_{2i}(\mathbb{F}_q;\mathbb{Z}/N) \cong \mathbb{Z}/N$ are onto. Since $\pi_{2i}(BG^+;\mathbb{Z}/N)$ is a $\mathbb{Z}/N$-module (IV.2.2), the latter map splits, *i.e.,* there is a $\mathbb{Z}/N$ summand in $\pi_{2i}(BG^+;\mathbb{Z}/N)$ and hence in $\pi_{2i-1}(BG^+)$ mapping isomorphically onto $K_{2i-1}(\mathbb{F}_q;\mathbb{Z}/N) \cong \mathbb{Z}/N$. □

VI.1.5.1 **Corollary 1.5.1.** *For each prime $\ell \neq p$ and each $i > 0$, $K_{2i-1}(\bar{\mathbb{Q}}_p)$ contains a nonzero torsion ℓ-group.*

Proof. Fix a $q = p^\nu$ such that $q \equiv 1 \pmod{\ell}$, and $q \equiv 1 \pmod 4$ if $\ell = 2$, and let m be the largest power of ℓ dividing $q-1$. We consider the set of all local fields E over $\mathbb{Q}_p$ (contained in a fixed common $\bar{\mathbb{Q}}_p$) whose ring of integers R has residue field $\mathbb{F}_q$. For each such E, we have $K_{2i-1}(R) \cong K_{2i-1}(E)$ by V.6.9.2. If R_q denotes the union of these R, and E_q is the the union E, this implies that $K_{2i-1}(R_q) \cong K_{2i-1}(E_q)$.

Because G is a torsion group, the homology groups $H_*(G;\mathbb{Q})$ vanish for $* > 0$, so the groups $\pi_*(BG^+)$ are torsion groups by the Hurewicz theorem. Since $G \to GL(\mathbb{F}_q)$ factors through $GL(R_q)$, the surjection $\pi_*(BG^+) \to$

$\pi_* BGL(\mathbb{F}_q)^+ = K_*(\mathbb{F}_q)$ factors through $\pi_* BGL(R_q)^+ = K_*(R_q)$ and hence through $K_{2i-1}(E_q)$. It follows that $K_{2i-1}(E_q)$ contains a torsion group mapping onto the ℓ-torsion subgroup of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q-1)$. Taking the direct limit over q, it follows that the ℓ-torsion subgroup of $K_{2i-1}(\bar{\mathbb{Q}}_p;\mathbb{Z}/m)$ maps onto the ℓ-torsion subgroup of $K_{2i-1}(\bar{\mathbb{F}}_p)$. □

VI.1.5.2 **Corollary 1.5.2.** *If $q \equiv 1 \pmod{\ell}$ and m is the order of $\mu_{(\ell)}(\mathbb{F}_q)$ then each group $K_{2i-1}(\mathbb{Z}[\zeta_m]) \cong K_{2i-1}(\mathbb{Q}(\zeta_m))$ contains a cyclic summand mapping isomorphically onto the ℓ-primary component of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i-1)$.*

In fact, the summand is the torsion subgroup of $K_{2i-1}(\mathbb{Q}(\zeta_m))$, as we shall see in Theorem VI.8.2 8.2.

Proof. We have $K_{2i-1}(\mathbb{Z}[\zeta_m]) \cong K_{2i-1}(\mathbb{Q}(\zeta_m))$ by V.V.6.8 6.8, and there is a canonical ring map $\mathbb{Z}[\zeta_m] \to \mathbb{F}_q$. Since $\mu_m \subset \mathbb{Z}[\zeta_m]$, the split surjection of Theorem VI.1.5 1.5 factors through $\pi_{2i-1}(BG^+) \to K_{2i-1}(\mathbb{Z}[\zeta_m]) \to K_{2i-1}(\mathbb{F}_q)$. □

VI.1.5.3 **Remark 1.5.3.** Let S denote the symmetric monoidal category of finite free μ_m-sets (IV.IV.4.1.1 4.1.1). The space $K(S)$ is $\mathbb{Z} \times BG^+$, and the Barratt-Priddy Theorem identifies it with the zeroth space of the spectrum $\Sigma^\infty(BG_+)$. As pointed out in IV.IV.4.10.1 4.10.1, the map $BG^+ \to GL(R)^+$ arises from the free R-module functor $S \to \mathbf{P}(R)$, and therefore $K(S) \to K(R)$ extends to a map of spectra $\mathbf{K}(S) \to \mathbf{K}(R)$.

If $k \subset F$ is an inclusion of algebraically closed fields, $K_*(k) \to K_*(F)$ is an injection by V.V.6.7.4 6.7.4. The following result implies that it is an isomorphism on torsion subgroups, and that $K_n(F)$ is divisible for $n \neq 0$.

VI.1.6 **Theorem 1.6.** *Let F be an algebraically closed field of characteristic 0. Then*
(i) If n is even and $n > 0$, $K_n(F)$ is uniquely divisible.
(ii) If $n = 2i-1$ is odd, $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and a torsion group isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Proof. Fix $i > 0$. For each prime ℓ, the group $K_{2i-1}(F;\mathbb{Z}/\ell)$ is zero by Proposition VI.1.4 1.4. By the universal coefficient sequence IV.IV.2.5 2.5, $K_{2i-2}(F)$ has no ℓ-torsion and $K_{2i-1}(F)/\ell = 0$. That is, $K_{2i-1}(F)$ is ℓ-divisible for all ℓ and hence divisible, while $K_{2i-2}(F)$ is torsionfree. We now consider the universal coefficient sequence

$$0 \to K_{2i}(F)/\ell \to K_{2i}(F;\mathbb{Z}/\ell) \to {}_\ell K_{2i-1}(F) \to 0.$$

The middle group is $\mathbb{Z}/\ell$ by Proposition VI.1.4 1.4. By Corollary VI.1.5.1 1.5.1, the exponent ℓ subgroup of $K_{2i-1}(F)$ is nonzero, and hence cyclic of order ℓ. This implies that $K_{2i}(F)/\ell = 0$, *i.e.,* the torsionfree group $K_{2i}(F)$ is divisible.

Since a divisible abelian group is the direct sum of a uniquely divisible group and a divisible torsion group, a divisible torsion group is the sum of its Sylow subgroups, and an ℓ-primary divisible group is a direct sum of copies of $\mathbb{Z}/\ell^\infty$, it follows that $K_{2i-1}(F)$ is the direct sum of a uniquely divisible group and $\oplus_\ell \mathbb{Z}/\ell^\infty \cong \mathbb{Q}/\mathbb{Z}$. □

We conclude this section with a description of the torsion module of $K_{2i-1}(F)$ as a representation of the group $\mathrm{Aut}(F)$ of field automorphisms of F. For this we need some elementary remarks. If F is algebraically closed, there is a tautological action of $\mathrm{Aut}(F)$ on the group $\mu = \mu(F)$ of roots of unity in F: $g \in \mathrm{Aut}(F)$ sends ζ to $g(\zeta)$. This action gives a surjective homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(\mu)$, called the *cyclotomic* representation. To describe $\mathrm{Aut}(\mu)$, recall that the group $\mu(F)$ is isomorphic to either $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$, according to the characteristic of F.

Since any endomorphism of $\mathbb{Q}/\mathbb{Z}$ induces an endomorphism of its exponent m subgroup $\mathbb{Z}/m$, and is equivalent to a compatible family of such, $\mathrm{End}(\mathbb{Q}/\mathbb{Z})$ is isomorphic to $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m$. It is easy to see that $\hat{\mathbb{Z}}$ is the product over all primes ℓ of the ℓ-adic integers $\hat{\mathbb{Z}}_\ell$, so $\mathrm{Aut}(\mu) \cong \prod \hat{\mathbb{Z}}_\ell^\times$. A similar argument, with $p \nmid m$, shows that $\mathrm{End}(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$ is isomorphic to $\prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell$, and $\mathrm{Aut}(\mu) \cong \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell^\times$.

If $\mathrm{char}(F) = 0$, the subfield of F fixed by the kernel of $\mathrm{Aut}(F) \to \mathrm{Aut}(\mu)$ is the infinite cyclotomic extension $\mathbb{Q}(\mu) = \cup_m \mathbb{Q}(\zeta_m)$, by elementary Galois theory, and $\mathrm{Aut}(F)$ surjects onto $\mathrm{Aut}(\mathbb{Q}(\mu)) = \mathrm{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \cong \mathrm{Aut}(\mu) \cong \hat{\mathbb{Z}}^\times$. If F is algebraically closed of characteristic $p > 0$, the situation is similar: $\mathrm{Aut}(F)$ surjects onto $\mathrm{Aut}(\bar{\mathbb{F}}_p) = \mathrm{Gal}(\bar{\mathbb{F}}_p/F_p) \cong \mathrm{Aut}(\mu)$; the Frobenius is topologically dense in this group.

VI.1.7 **Definition 1.7.** For all $i \in \mathbb{Z}$, we shall write $\mu(i)$ for the abelian group μ, made into a $\mathrm{Aut}(F)$-module by letting $g \in \mathrm{Aut}(F)$ act as $\zeta \mapsto g^i(\zeta)$. (This modified module structure is called the i^{th} *Tate twist* of the cyclotomic module μ.) If M is any $\mathrm{Aut}(F)$-submodule of μ, we write $M(i)$ for the abelian group M, considered as a submodule of $\mu(i)$. In particular, its Sylow decomposition is $\mu(i) = \oplus \mathbb{Z}/\ell^\infty(i)$.

VI.1.7.1 **Proposition 1.7.1.** *If F is algebraically closed and $i > 0$, the torsion submodule of $K_{2i-1}(F)$ is isomorphic to $\mu(i)$ as an* $\mathrm{Aut}(F)$*-module.*

Proof. It suffices to show that the submodule ${}_mK_{2i-1}(F)$ is isomorphic to $\mu_m(i)$ for all $m > 0$ prime to the characteristic. Fix a primitive m^{th} root of unity ζ in F, and let β be the corresponding Bott element. Then $K_*(F;\mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$, by either 1.3.1 or 1.6; Since ${}_mK_{2i-1}(F) \cong K_{2i}(F;\mathbb{Z}/m)$ as $\mathrm{Aut}(F)$-modules, and the abelian group $K_{2i}(F;\mathbb{Z}/m)$ is isomorphic to $\mathbb{Z}/m$ on generator β^i.

By naturality of the product (IV, 1.10 and 2.8), the group $\mathrm{Aut}(F)$ acts on $K_*(F;\mathbb{Z}/m)$ by ring automorphisms. For each $g \in \mathrm{Aut}(F)$ there is an $a \in \mathbb{Z}/m^\times$ such that $g(\zeta) = \zeta^a$. Thus g sends β to $a\beta$, and g sends β^i to $(a\beta)^i = a^i\beta^i$. Since $\mu_m(i)$ is isomorphic to the abelian group $\mathbb{Z}/m$ with g acting as multiplication by a^i, we have $\mu_m(i) \cong K_{2i}(F;\mathbb{Z}/m)$. □

EXERCISES

EVI.1.1 **1.1.** Show that the conclusion of 1.3.1 holds for any separably closed field of characteristic p: if $n > 0$ is even then $K_n(F)$ is uniquely divisible, while if n is odd then $K_n(F)$ is the sum of a uniquely divisible group and $(\mathbb{Q}/\mathbb{Z})_{(p)}$. *Hint:* By IV.5.6, $K_n(F)$ is uniquely p-divisible for all $n > 0$.

EVI.1.2 **1.2.** (Suslin) Let $k \subset F$ be an extension of algebraically closed fields, and let X be an algebraic variety over k. Write X_F for the corresponding variety $X \otimes_k F$ over F. In this exercise we show that the groups $G_*(X;\mathbb{Z}/m)$ (defined in IV.6.3.4) are independent of k.

(i) If R is the local ring of a smooth curve C at a point c, show that there is a specialization map $\lambda_c : G_*(X_{k(C)};\mathbb{Z}/m) \to G_*(X;\mathbb{Z}/m)$.

(ii) *Rigidity.* Show that the specialization λ_c is independent of the choice of c.

(iii) If $h_1, h_2 : A \to k$ are as in 1.2, show that the maps $h_i^* : G_*(X_A;\mathbb{Z}/m) \to G_*(X;\mathbb{Z}/m)$ exist and coincide.

(iv) Show that the base-change $G_*(X;\mathbb{Z}/m) \to G_*(X_F;\mathbb{Z}/m)$ is an isomorphism.

EVI.1.3 **1.3.** Let E be a local field, finite over $\mathbb{Q}_p$ and with residue field $\mathbb{F}_q$. Use Theorem 1.6 and the proof of 1.5.1 to show that $K_{2i-1}(E)_{\text{tors}}$ is the direct sum of $\mathbb{Z}/(q^i-1)$ and a p-group, and that $K_{2i-1}(E)_{\text{tors}} \to K_{2i-1}(\bar{\mathbb{Q}}_p)$ is an injection modulo p-torsion.

EVI.1.4 **1.4.** The Galois group $\Gamma = \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ acts on μ_m and hence on the group G of Theorem 1.5. Show that the induced action of Γ on $\pi_*(BG^+)$ is trivial, and conclude that the summand of $K_{2i-1}(\mathbb{Q}(\zeta))$ in 1.5.2 is invariant under Γ.

2 The e-invariant of a field

The odd-indexed K-groups of any field F have a canonical torsion summand, discovered by Harris and Segal in [84]. It is detected by a map called the e-invariant, which we now define.

Let $\bar{F}$ be a separably closed field, and $\mu = \mu(\bar{F})$ the group of its roots of unity. We saw in Proposition 1.7.1 (and Ex. 1.1) that $K_{2i-1}(\bar{F})_{\text{tors}}$ is isomorphic to the Tate twist $\mu(i)$ of μ as an $\text{Aut}(\bar{F})$-module (see Definition 1.7). The target group $\mu(i)^G$ is always the direct sum of its ℓ-primary Sylow subgroups $\mu_{(\ell)}(i)^G \cong \mathbb{Z}/\ell^\infty(i)^G$.

VI.2.1 **Definition 2.1.** Let F be a field, with separable closure $\bar{F}$ and Galois group $G = \text{Gal}(\bar{F}/F)$. Since $K_*(F) \to K_*(\bar{F})$ is a homomorphism of G-modules, with G acting trivially on $K_n(F)$, it follows that there is a natural map

$$e : K_{2i-1}(F)_{\text{tors}} \to K_{2i-1}(\bar{F})^G_{\text{tors}} \cong \mu(i)^G.$$

We shall call e the *e-invariant.*

If $\mu(i)^G$ is a finite group it is cyclic, and we write $w_i(F)$ for its order, so that $\mu(i)^G \cong \mathbb{Z}/w_i(F)$. If ℓ is a prime, we write $w_i^{(\ell)}(F)$ for the order of $\mu_{(\ell)}(i)^G$. Thus the target of the e-invariant is $\bigoplus_\ell \mathbb{Z}/w_i^{(\ell)}(F)$, and $w_i(F) = \prod w_i^{(\ell)}(F)$.

VI.2.1.1 **Example 2.1.1** (finite fields)**.** It is a pleasant exercise to show that $w_i(\mathbb{F}_q) = q^i - 1$ for all i. Since this is the order of $K_{2i-1}(\mathbb{F}_q)$ by IV.1.13, we see that in this case, the e-invariant is an isomorphism. (See Exercise IV.1.26.)

VI.2.1.2 **Example 2.1.2.** If i is odd, $w_i(\mathbb{Q}) = 2$ and $w_i(\mathbb{Q}(\sqrt{-1})) = 4$. If i is even then $w_i(\mathbb{Q}) = w_i(\mathbb{Q}(\sqrt{-1}))$, and $\ell | w_i(\mathbb{Q})$ exactly when $(\ell - 1)$ divides i. We have: $w_2 = 24$, $w_4 = 240$, $w_6 = 504 = 2^3 \cdot 3^2 \cdot 7$, $w_8 = 480 = 2^5 \cdot 3 \cdot 5$, $w_{10} = 1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$, and $w_{12} = 65,520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$. These formulas may be derived from Propositions 2.2 and 2.3 below.

In [108], Lee and Szczarba used a variant of the formula $K_3(R) = H_3(St(R); \mathbb{Z})$ (Ex. IV.1.9) to show that $K_3(\mathbb{Z}) \cong K_3(\mathbb{Q}) \cong \mathbb{Z}/48$. It follows that the e-invariant $K_3(\mathbb{Q}) \to \mathbb{Z}/24$ cannot be an injection. (We will see in Remark 2.1.3 that it vanishes on the nonzero symbol $\{-1,-1,-1\}$.)

VI.2.1.3 **Remark 2.1.3.** The complex Adams e-invariant for stable homotopy is a map from π^s_{2i-1} to $\mathbb{Z}/w_i(\mathbb{Q})$, whence our terminology. Quillen observed in [155] that the Adams e-invariant is the composition $\pi^s_{2i-1} \to K_{2i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_i(\mathbb{Q})$. (Adams defined his e-invariant using $\pi_{2i}(BU)/w_i(\mathbb{Q})$; Quillen's assertions have been translated using Remark 1.4.1.)

If i is positive and divisible by 4, the real (Adams) e-invariant coincides with the complex e-invariant. If $i \equiv 2 \pmod 4$, the real e-invariant is a map

$$\pi^s_{2i-1} \to \pi_{2i}(BO)/2w_i(\mathbb{Q}) = \mathbb{Z}/2w_i(\mathbb{Q})$$

For all even $i > 0$, Adams proved in 1966 that the real e-invariant restricts to an injection on the image of $J : \pi_{2i-1}O \to \pi^s_{2i-1}$ and induces an isomorphism $(\mathrm{im}J)_{2i-1} \cong \mathbb{Z}/w_i(\mathbb{Q})$. His proof used the "Adams Conjecture," which was later verified by Quillen. Quillen showed in [155] that the real e-invariant factors through $K_{2i-1}(\mathbb{Z}) = K_{2i-1}(\mathbb{Q})$, so $(\mathrm{im}J)_{2i-1}$ injects into $K_{2i-1}(\mathbb{Z})$. In particular, the image $\{-1,-1,-1\}$ of $\eta^3 \in \pi^s_3$ is nonzero in $K_3(\mathbb{Z})$ (see Ex. IV.1.12). Since the map from $\pi_{8k+3}(BO) = \mathbb{Z}$ to $\pi_{8k+3}(BU) = \mathbb{Z}$ has image $2\mathbb{Z}$, it follows that the e-invariant $K_{8k+3}(\mathbb{Q}) \to \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of Definition 2.1 is not an injection on $(\mathrm{im}J)_{8k+3}$.

Not all of the image of J injects into $K_*(\mathbb{Z})$. If $n \equiv 0, 1 \pmod 8$ then $J(\pi_n O) \cong \mathbb{Z}/2$, but Waldhausen showed (in 1982) that these elements map to zero in $K_n(\mathbb{Z})$.

Formulas for $w_i(F)$

We now turn to formulas for the numbers $w_i^{(\ell)}(F)$. Let ζ_m denote a primitive m^{th} root of unity. For odd ℓ, we have the following simple formula.

VI.2.2 **Proposition 2.2.** *Fix a prime $\ell \neq 2$, and let F be a field of characteristic $\neq \ell$. Let $a \leq \infty$ be maximal such that $F(\zeta_\ell)$ contains a primitive ℓ^ath root of unity and set $r = [F(\zeta_\ell) : F]$. If $i = c\ell^b$, where $\ell \nmid c$, then the numbers $w_i^{(\ell)} = w_i^{(\ell)}(F)$ are ℓ^{a+b} if $r \mid i$, and 1 otherwise. That is:*
(a) If $\zeta_\ell \in F$ then $w_i^{(\ell)} = \ell^{a+b}$;

(b) If $\zeta_\ell \notin F$ and $i \equiv 0 \pmod{r}$ then $w_i^{(\ell)} = \ell^{a+b}$;
(c) If $\zeta_\ell \notin F$ and $i \not\equiv 0 \pmod{r}$ then $w_i^{(\ell)} = 1$.

Proof. Since ℓ is odd, $G = \mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ is a cyclic group of order $r\ell^{\nu-a}$ for all $\nu \geq a$. If a generator of G acts on μ_{ℓ^ν} by $\zeta \mapsto \zeta^g$ for some $g \in (\mathbb{Z}/\ell^\nu)^\times$ then it acts on $\mu^{\otimes i}$ by $\zeta \mapsto \zeta^{g^i}$. Now use the criterion of Lemma 2.2.1; if $r \mid i$ then $\mathrm{Gal}(F(\zeta_{\ell^{a+b}})/F)$ is cyclic of order $r\ell^b$, while if $r \nmid i$ the exponent r of $\mathrm{Gal}(F(\zeta_\ell)/F)$ does not divide i. □

VI.2.2.1 **Lemma 2.2.1.** $w_i^{(\ell)}(F) = \max\{\ell^\nu \mid \mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ *has exponent dividing* $i\}$

Proof. Set $\zeta = \zeta_{\ell^\nu}$. Then $\zeta^{\otimes i}$ is invariant under $g \in \mathrm{Gal}(\bar{F}/F)$ precisely when $g^i(\zeta) = \zeta$, and $\zeta^{\otimes i}$ is invariant under all of G precisely when the group $\mathrm{Gal}(F(\zeta_{\ell^\nu})/F)$ has exponent i. □

VI.2.2.2 **Example 2.2.2.** Consider $F = \mathbb{Q}(\zeta_{p^a})$. If $i = cp^b$ then $w_i^{(p)}(F) = p^{a+b}$ $(p \neq 2)$. If $\ell \neq 2, p$ then $w_i^{(\ell)}(F) = w_i^{(\ell)}(\mathbb{Q})$ for all i. This number is 1 unless $(\ell-1) \mid i$; if $(\ell-1)|i$ but $\ell \nmid i$ then $w_i^{(\ell)}(F) = \ell$. In particular, if $\ell = 3$ and $p \neq 3$ then $w_i^{(3)}(F) = 1$ for odd i, and $w_i^{(3)}(F) = 3$ exactly when $i \equiv 2, 4 \pmod{6}$.

The situation is more complicated when $\ell = 2$, because $\mathrm{Aut}(\mu_{2^\nu}) = (\mathbb{Z}/2^\nu)^\times$ contains two involutions if $\nu \geq 3$. We say that a field F is *exceptional* if $\mathrm{char}(F) = 0$ and the Galois groups $\mathrm{Gal}(F(\zeta_{2^\nu})/F)$ are not cyclic for large ν. If F is not exceptional, we say that it is *non-exceptional.*

VI.2.3 **Proposition 2.3.** *($\ell = 2$) Let F be a field of characteristic $\neq 2$. Let a be maximal such that $F(\sqrt{-1})$ contains a primitive 2^ath root of unity. If $i = c2^b$, where $2 \nmid c$, then the 2-primary numbers $w_i^{(2)} = w_i^{(2)}(F)$ are:*
(a) If $\sqrt{-1} \in F$ then $w_i^{(2)} = 2^{a+b}$ for all i.
(b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)} = 2$.
(c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)} = 2^{a+b}$.
(d) If $\sqrt{-1} \notin F$, F is non–exceptional and i is even then $w_i^{(2)} = 2^{a+b-1}$.

The proof of Proposition 2.3(a,b,d) is almost identical to that of 2.2 with $r = 1$. The proof in the exceptional case (c) is relegated to Exercise 2.2.

Both $\mathbb{R}$ and $\mathbb{Q}_2$ are exceptional, and so are each of their subfields. In particular, real number fields (like $\mathbb{Q}$) are exceptional, and so are some totally imaginary number fields, like $\mathbb{Q}(\sqrt{-7})$.

VI.2.3.1 **Example 2.3.1.** (local fields) Let E be a local field, finite over $\mathbb{Q}_p$ and with residue field $\mathbb{F}_q$. Then $w_i(E)$ is $w_i(\mathbb{F}_q) = q^i - 1$ times a power of p. (The precise power of p is given in Exercise 2.3 when $E = \mathbb{Q}_p$.) This follows from Propositions 2.2 and 2.3, using the observation that (for $\ell \neq p$) the number of ℓ-primary roots of unity in $E(\zeta_\ell)$ is the same as in $\mathbb{F}_q(\zeta_\ell)$.

By Exercise 1.3, the map $K_{2i-1}(E)_{\mathrm{tors}} \xrightarrow{e} \mathbb{Z}/w_i(E)$ is a surjection up to p-torsion, and induces an isomorphism on ℓ-primary torsion subgroups

$K_{2i-1}(E)\{\ell\} \cong \mathbb{Z}/w_i^{(\ell)}$ for $\ell \neq p$. We will see in Proposition 7.3 that the torsion subgroup of $K_{2i-1}(E)$ is exactly $\mathbb{Z}/w_i(E)$.

Bernoulli numbers

The numbers $w_i(\mathbb{Q})$ are related to the *Bernoulli numbers* B_k. These were defined by Jacob Bernoulli in 1713 as coefficients in the power series

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty}(-1)^{k+1} B_k \frac{t^{2k}}{(2k)!}.$$

(We use the topologists' B_k from [135], all of which are positive. Number theorists would write it as $(-1)^{k+1}B_{2k}$.) The first few Bernoulli numbers are:

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, B_6 = \frac{691}{2730}, B_7 = \frac{7}{6}, B_8 = \frac{3617}{510}.$$

The denominator of B_k is always squarefree, divisible by 6, and equal to the product of all primes with $(p-1)|2k$. Moreover, if $(p-1) \nmid 2k$ then p is not in the denominator of B_k/k even if $p|k$; see [135]. From this information, it is easy to verify the following identity. Recall from 2.1.2 that $w_i(\mathbb{Q}) = 2$ when i is odd.

VI.2.4 **Lemma 2.4.** *If $i = 2k$ is even then $w_i(\mathbb{Q})$ is the denominator of $B_k/4k$. The prime ℓ divides $w_i(\mathbb{Q})$ exactly when $(\ell - 1)$ divides i.*

Although the numerator of B_k is difficult to describe, it is related to the notion of irregular primes, which we now define.

VI.2.4.1 **Example 2.4.1** (Irregular Primes)**.** A prime p is called *irregular* if p divides the order h_p of $\mathrm{Pic}(\mathbb{Z}[\mu_p])$; if p is not irregular it is called *regular.* Iwasawa proved that a prime p is regular if and only if $\mathrm{Pic}(\mathbb{Z}[\mu_{p^\nu}])$ has no p-torsion for all ν. The smallest irregular primes are $37, 59, 67, 101, 103, 131$ and 149. Siegel conjectured that asymptotically about 39% of all primes are irregular; about 39% of the primes less than 4 million are irregular.

Kummer proved that p is irregular if and only if p divides the numerator of one of the Bernoulli numbers B_k, $k \leq (p-3)/2$ (see Washington [216, 5.34]). By Kummer's congruences ([216, 5.14]), a regular prime p does not divide the numerator of *any* B_k/k (but $5|B_5$). Thus only irregular primes can divide the numerator of B_k/k.

The historical interest in regular primes is Kummer's 1847 proof of Fermat's Last Theorem (case I) for regular primes: $x^p + y^p = z^p$ has no solution in which $p \nmid xyz$. For us, certain calculations of K-groups become easier at regular primes. (See Example 8.3.2 and Proposition 10.5.)

VI.2.4.2 **Remark 2.4.2.** Bernoulli numbers also arise as values of the Riemann zeta function. Euler proved (in 1735) that $\zeta_{\mathbb{Q}}(2k) = B_k(2\pi)^{2k}/2(2k)!$. By the functional equation, we have $\zeta_{\mathbb{Q}}(1-2k) = (-1)^k B_k/2k$. By Lemma 2.4, the denominator of $\zeta_{\mathbb{Q}}(1-2k)$ is $\frac{1}{2}w_{2k}(\mathbb{Q})$.

VI.2.4.3 **Remark 2.4.3.** The Bernoulli numbers are of interest to topologists because if $n = 4k-1$ the image of $J : \pi_n SO \to \pi_n^s$ is cyclic of order equal to the denominator of $B_k/4k$, and the numerator determines the number of exotic $(4k-1)$-spheres which bound parallelizable manifolds; see [135, App. B].

VI.2.5 **Harris-Segal Theorem 2.5.** *Let F be a field with $1/\ell \in F$; if $\ell = 2$, we also suppose that F is non-exceptional. Set $w_i = w_i^{(\ell)}(F)$. Then each $K_{2i-1}(F)$ has a direct summand isomorphic to $\mathbb{Z}/w_i$, detected by the e-invariant.*

If F is the field of fractions of an integrally closed domain R then $K_{2i-1}(R)$ also has a direct summand isomorphic to $\mathbb{Z}/w_i(F)$, detected by the e-invariant.

The splitting $\mathbb{Z}/w_i \to K_{2i-1}(R)$ is called the Harris-Segal map, and its image is called the (ℓ-primary) Harris-Segal summand of $K_{2i-1}(R)$.

We will see in Theorem 8.2 below that $\mathbb{Z}/w_i$ is the torsion subgroup of $K_{2i-1}(\mathbb{Z}[\zeta_{\ell^a}])$. It follows that the Harris-Segal map is unique, and hence so is the Harris-Segal summand of $K_{2i-1}(R)$. This uniqueness was originally established by Kahn and others.

Proof. Suppose first that either $\ell \neq 2$ and $\zeta_\ell \in R$, or that $\ell = 2$ and $\zeta_4 \in R$. If R has $m = \ell^a$ ℓ-primary roots of unity, then $w_i^{(\ell)}(\mathbb{Q}(\zeta_m))$ equals $w_i = w_i^{(\ell)}(F)$ by 2.2 and 2.3. Thus there is no loss in generality in assuming that $R = \mathbb{Z}[\zeta_m]$. Pick a prime p with $p \not\equiv 1 \pmod{\ell^{a+1}}$. Then $\zeta_{\ell^{a+1}} \notin \mathbb{F}_p$, and if $\mathfrak{p}$ is any prime ideal of $R = \mathbb{Z}[\zeta_m]$ lying over p then the residue field $R/\mathfrak{p}$ is $\mathbb{F}_q = \mathbb{F}_p(\zeta_m)$. We have $w_i = w_i^{(\ell)}(\mathbb{F}_q)$ by Example 2.3.1.

The quotient map $R \to R/\mathfrak{p}$ factors through the $\mathfrak{p}$-adic completion $\hat{R}_\mathfrak{p}$, whose field of fractions is the local field $E = \mathbb{Q}_p(\zeta_m)$. By Example 2.3.1, $w_i^{(\ell)}(E) = w_i$ and $K_{2i-1}(E)\{\ell\} \xrightarrow{e} \mathbb{Z}/w_i$ is an isomorphism. Now the e-invariant for the finite group $K_{2i-1}(R)$ is the composite

$$K_{2i-1}(R)_{(\ell)} \cong K_{2i-1}(\mathbb{Q}(\zeta_m))_{(\ell)} \to K_{2i-1}(E)\{\ell\} \xrightarrow{e} \mathbb{Z}/w_i.$$

By Corollary 1.5.2, $K_{2i-1}(R)$ contains a cyclic summand A of order w_i, mapping to the summand $\mathbb{Z}/w_i$ of $K_{2i-1}(\mathbb{F}_q)$ under $K_{2i-1}(R) \to K_{2i-1}(\hat{R}_\mathfrak{p}) \to K_{2i-1}(\mathbb{F}_q)$. Therefore A injects into (and is isomorphic to) $K_{2i-1}(\hat{R}_\mathfrak{p})\{\ell\} \cong K_{2i-1}(E)\{\ell\} \cong \mathbb{Z}/w_i$. The theorem now follows in this case.

Suppose now that $\zeta_\ell \notin R$. By Exercise 2.5, we may assume that F is a subfield of $\mathbb{Q}(\zeta_m)$, that $\mathbb{Q}(\zeta_m) = F(\zeta_\ell)$, and that R is the integral closure of $\mathbb{Z}$ in F. We may suppose that $r = [\mathbb{Q}(\zeta_m) : F]$ divides i since otherwise $w_i = 1$, and set $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/F)$. By Proposition 2.2, $w_i = w_i^{(\ell)}(\mathbb{Q}[\zeta_m])$. We have just seen that there is a summand A of $K_{2i-1}(\mathbb{Q}[\zeta_m])$ mapping isomorphically to $\mathbb{Z}/w_i$ by the e-invariant. By Ex. 1.4, Γ acts trivially on A.

Since $f : R \to \mathbb{Z}[\zeta_m]$ is Galois, the map f^*f_* is multiplication by $\sum_{g\in G} g$ on $K_{2i-1}(\mathbb{Z}[\zeta_m])$, and hence multiplication by r on A (see Ex. IV.6.13). Since f_*f^* is multiplication by r on $f_*(A)$, we see that $f^* : f_*(A) \to A$ is an isomorphism with inverse f_*/r. Hence $f_*(A)$ is a summand of $K_{2i-1}(R)$, and the e-invariant $K_{2i-1}(R) \xrightarrow{f^*} K_{2i-1}(\mathbb{Z}[\zeta_m]) \xrightarrow{e} \mathbb{Z}/w_i$ maps $f_*(A)$ isomorphically to $\mathbb{Z}/w_i$.

When $\ell = 2$ and F is non-exceptional but $\sqrt{-1} \notin F$, we may again assume by Ex. 2.5 that F is a subfield of index 2 in $\mathbb{Q}(\zeta_m) = F(\sqrt{-1})$. By Proposition 2.3, $w_i^{(2)}(\mathbb{Q}(\zeta_m)) = 2w_i$ and there is a summand A of $K_{2i-1}(\mathbb{Q}(\zeta_m))$ mapping isomorphically to $\mathbb{Z}/2w_i$ by the e-invariant; by Ex. 1.4, Γ acts trivially on A and we set $\bar{A} = f_*(A)$. Since $f^* f_*$ is multiplication by 2 on A, the image of $\bar{A}$ is $2A$. From the diagram

$$\begin{array}{ccccc} \bar{A} & \rightarrowtail & K_{2i-1}(F) & \xrightarrow{e} & \mathbb{Z}/w_i \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \text{include}} & & \\ A & \rightarrowtail & K_{2i-1}(\mathbb{Q}(\zeta_m)) & \xrightarrow{e} & \mathbb{Z}/2w_i \end{array}$$

we see that $\bar{A} \cong 2A \cong \mathbb{Z}/w_i$, as desired. □

VI.2.5.1 **Remark 2.5.1.** If F is an exceptional field, a transfer argument using $F(\sqrt{-1})$ shows that there is a cyclic summand in $K_{2i-1}(F)$ whose order is either $w_i(F)$, $2w_i(F)$ or $w_i(F)/2$. (Exercise 2.4); we will also call these *Harris-Segal summands.*

When F is a totally imaginary number field, we will see in Theorem 8.4 below that the Harris-Segal summand always has order $w_i(F)$. The following theorem, extracted from Theorem 9.5 below, shows that all possibilities occur for real number fields, *i.e.,* number fields embeddable in $\mathbb{R}$.

VI.2.6 **Theorem 2.6.** *Let F be a real number field. Then the Harris-Segal summands in $K_{2i-1}(F)$ and $K_{2i-1}(\mathcal{O}_F)$ are isomorphic to:*

(1) $\mathbb{Z}/w_i(F)$, *if* $i \equiv 0 \pmod 4$ *or* $i \equiv 1 \pmod 4$, *i.e.,* $2i-1 \equiv \pm 1 \pmod 8$*;*

(2) $\mathbb{Z}/2w_i(F)$, *if* $i \equiv 2 \pmod 4$, *i.e.,* $2i-1 \equiv 3 \pmod 8$*;*

(3) $\mathbb{Z}/\frac{1}{2}w_i(F)$, *if* $i \equiv 3 \pmod 4$, *i.e.,* $2i-1 \equiv 5 \pmod 8$.

VI.2.7 **Example 2.7.** Let $F = \mathbb{Q}(\zeta + \zeta^{-1})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta)$, $\zeta^p = 1$ with p odd. Then $w_i(F) = 2$ for odd i, and $w_i(F) = w_i(\mathbb{Q}(\zeta))$ for even $i > 0$ by 2.2 and 2.3 (see Ex. 2.6). Note that $p \,|\, w_i(F(\zeta))$ for all i, $p \,|\, w_i(F)$ if and only if i is even, and $p \,|\, w_i(\mathbb{Q})$ only when $(p-1)|i$; see 2.2.2.

If $n \equiv 3 \pmod 4$, the groups $K_n(\mathbb{Z}[\zeta + \zeta^{-1}]) = K_n(F)$ are classically finite (see IV.6.9 or 8.1); the order of their Harris-Segal summands are given by Theorem 2.6. When $n \not\equiv -1 \pmod{2p-2}$, the group $K_n(F)$ has an extra p-primary factor not coming from the image of J (see 2.1.3).

EXERCISES

EVI.2.1 **2.1.** For every prime ℓ with $1/\ell \in F$, show that the following are equivalent:
(i) $F(\zeta_\ell)$ has only finitely many ℓ-primary roots of 1;
(ii) $w_i^{(\ell)}(F)$ is finite for some $i \equiv 0 \pmod{2(\ell-1)}$;
(iii) $w_i^{(\ell)}(F)$ is finite for all $i > 0$.

EVI.2.2 **2.2.** Prove Proposition 2.3(c), giving the formula $w_i^{(2)}(F) = 2^{a+b}$ when i is even and F is exceptional. *Hint:* Consider $\mu(i)^H$, $H = \mathrm{Gal}(\bar{F}/F(\sqrt{-1}))$.

EVI.2.3 **2.3.** If p is odd, show that $w_i(\mathbb{Q}_p) = p^i - 1$ unless $(p-1)|i$, and if $i = m(p-1)p^b$ $(p \nmid m)$ then $w_i(\mathbb{Q}_p) = (p^i - 1)p^{1+b}$.

For $p = 2$, show that $w_i(\mathbb{Q}_2) = 2(2^i - 1)$ for i odd; if i is even, say $i = 2^b m$ with m odd, show that $w_i(\mathbb{Q}_2) = (2^i - 1)2^{2+b}$.

EVI.2.4 **2.4.** Let $f : F \to E$ be a field extension of degree 2, and suppose $x \in K_*(E)$ is fixed by $\mathrm{Gal}(E/F)$. If x generates a direct summand of order $2m$, show that $f_*(x)$ is contained in a cyclic summand of $K_*(F)$ of order either m, $2m$ or $4m$.

EVI.2.5 **2.5.** Let F be a field of characteristic 0. If $\ell \neq 2$ and $a < \infty$ is as in 2.2, show that there is a subfield F_0 of $\mathbb{Q}(\zeta_{\ell^a})$ such that $w_i^{(\ell)}(F_0) = w_i^{(\ell)}(F)$. If $\ell = 2$ and $a < \infty$ is as in 2.3, show that there is a subfield F_0 of $\mathbb{Q}(\zeta_{2^a})$ such that $w_i^{(2)}(F_0) = w_i^{(2)}(F)$, and that F_0 is exceptional (resp., non-exceptional) if F is.

EVI.2.6 **2.6.** Let ℓ be an odd prime, and $F = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ the maximal real subfield of $\mathbb{Q}(\zeta_\ell)$. Show that $w_i(F) = 2$ for odd i, and that $w_i(F) = w_i(\mathbb{Q}(\zeta_\ell))$ for even $i > 0$. In particular, $\ell | w_i(\mathbb{Q}(\zeta_\ell))$ for all i, but $\ell | w_i(F)$ if and only if i is even.

3 The K-theory of $\mathbb{R}$

In this section, we describe the algebraic K-theory of the real numbers $\mathbb{R}$, or rather the torsion subgroup $K_n(\mathbb{R})_{\mathrm{tors}}$ of $K_n(\mathbb{R})$. Here is the punchline:

VI.3.1 **Theorem 3.1.** *(Suslin) For all $n \geq 1$,*
(a) $K_n(\mathbb{R})$ is the direct sum of a uniquely divisible group and $K_n(\mathbb{R})_{tors}$.
(b) The torsion groups and $K_n(\mathbb{R})_{tors} \to K_n(\mathbb{C})_{tors}$ are given by Table 3.1.1.
(c) The map $K_n(\mathbb{R}; \mathbb{Z}/m) \to \widetilde{KO}(S^n; \mathbb{Z}/m) = \pi_n(BO; \mathbb{Z}/m)$ is an isomorphism for all integers m.

i (mod 8)	1	2	3	4	5	6	7	8
$K_i(\mathbb{R})$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Q}/\mathbb{Z}$	0	0	0	$\mathbb{Q}/\mathbb{Z}$	0
$\downarrow$	injects	0	$a \mapsto 2a$	0	0	0	$\cong$	0
$K_i(\mathbb{C})$	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$	0
$\downarrow$	0	0	$\cong$	0	0	0	$a \mapsto 2a$	0
$K_i(\mathbb{H})$	0	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Q}/\mathbb{Z}$	0

Table 3.1.1: The torsion subgroups of $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$, $n > 0$. VI.3.1.1

VI.3.1.2 **Remark 3.1.2.** Table 3.1.1 shows that $K_n(\mathbb{R})_{\text{tors}} \cong \pi_{n+1}(BO;\mathbb{Q}/\mathbb{Z})$ is the Harris-Segal summand of $K_n(\mathbb{R})$ (in the sense of 2.5.1) for all n odd; the e-invariant (Definition 2.1) is the map from $K_n(\mathbb{R})_{\text{tors}}$ to $K_n(\mathbb{C})_{\text{tors}}$. When n is $8k+1$, the $\mathbb{Z}/2$-summand in $K_n(\mathbb{R})$ is generated by the image of Adams' element $\mu_n \in \pi_n^s$; Adams showed that μ_n is detected by the complex Adams e-invariant. Adams also showed that the elements $\mu_{8k+2} = \eta \cdot \mu_{8k+1}$ and $\mu_{8k+3} = \eta^2 \cdot \mu_{8k+1}$ are nonzero and detected by the real Adams e-invariant which, as Quillen showed in [155], maps π_n^s to $K_n(\mathbb{R})_{\text{tors}} \cong \pi_{n+1}(BO;\mathbb{Q}/\mathbb{Z})$. (See Remark 2.1.3.) It follows that when $n = 8k+3$, the kernel of the e-invariant $K_n(\mathbb{R})_{\text{tors}} \to \mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2$, and is generated by the nonzero element $\{-1,-1,\mu_{8k+1}\}$. When $n = 8k+5$, the Harris-Segal summand is zero even though the target of the e-invariant is $\mathbb{Z}/2$.

A similar calculation for the quaternions $\mathbb{H}$ is also due to Suslin [185, 3.5]. The proof uses the algebraic group $SL_n(\mathbb{H})$ in place of $SL_n(\mathbb{R})$.

VI.3.2 **Theorem 3.2.** *(Suslin) For all $n \geq 1$,*
(a) $K_n(\mathbb{H})$ is the direct sum of a uniquely divisible group and $K_n(\mathbb{H})_{tors}$.
(b) The torsion groups and $K_n(\mathbb{C})_{tors} \to K_n(\mathbb{H})_{tors}$ are given by Table 3.1.1.
(c) The map $K_n(\mathbb{H};\mathbb{Z}/m) \to \widetilde{KSp}(S^n;\mathbb{Z}/m) = \pi_n(BSp;\mathbb{Z}/m)$ is an isomorphism for all integers m.

The method of proof uses a universal homotopy construction which is of independent interest, and also gives an alternative calculation of $K_*(\mathbb{C})$ to the one we gave in Proposition 1.4. As observed in 1.4.1, the punchline of that calculation is that, for all $n \geq 1$, $K_n(\mathbb{C};\mathbb{Z}/m) \to \widetilde{KU}(S^n;\mathbb{Z}/m) = \pi_n(BU;\mathbb{Z}/m)$ is an isomorphism.

We begin with some general remarks. If G is a topological group, then it is important to distinguish between the classifying space BG^δ of the discrete group G, which is the simplicial set of IV.3.4.1, and the classifying space BG^{top} of the topological group G^{top}, which is discussed in IV.3.9. For example, the homotopy groups of $BGL(\mathbb{C})^\delta$ are zero, except for the fundamental group, while the homotopy groups of $BGL(\mathbb{C})^{\text{top}} \simeq BU$ are given by Bott periodicity II.3.1.1.

Next, suppose that G is a Lie group having finitely many components, equipped with a left invariant Riemannian metric. Given $\varepsilon > 0$, let G_ε denote the ε–ball about 1. If ε is small, then G_ε is geodesically convex; the geodesic between any two points lies in G_ε.

VI.3.3 **Definition 3.3.** Let BG_ε denote the simplicial subset of BG^δ whose p-simplices are the p-tuples $[g_1,\dots,g_p]$ such that there is a point in the intersection of G_ε with all the translates $g_1\cdots g_i G_\varepsilon$, $i \leq p$. (This condition is preserved by the face and degeneracy maps of BG^δ.)

Suslin proved the following result in [184, 4.1].

VI.3.4 **Theorem 3.4.** *(Suslin) Let G be a Lie group. If ε is small enough so that G_ε is geodesically convex, then $BG_\varepsilon \to BG^\delta \to BG^{top}$ is a homotopy fibration.*

The next step is the construction of a universal chain homotopy. Given a commutative ring R, the algebraic group GL_n is Spec(H), where H is the Hopf algebra $R[\{x_{ij}\}_{i,j=1}^n][\det(X)^{-1}]$, where $\det(X)$ is the determinant of the universal matrix $X = (x_{ij})$ in $GL_n(H)$. For every commutative R-algebra B there is a bijection $\mathrm{Hom}_R(H, B) \to GL_n(B)$ sending f to the matrix $f(X)$; the counit structure map $H \to R$ corresponds to the identity matrix of $GL_n(R)$.

For each positive integer p, let A^p denote the henselization of the p-fold tensor product $H^{\otimes p}$ along the kernel of the evident structure map $H^{\otimes p} \to R$, so that (A^p, I^p) is a hensel pair, where I^p is the kernel of $A^p \to R$. For $i = 1, \dots, p$, the coordinates $pr_i : H \to H^{\otimes p} \to A^p$ determine matrices $X^i = pr_i(X)$ in $GL_n(A^p)$, and since X^i is congruent to the identity modulo I^p we even have $X^i \in GL_n(I^p)$.

Recall that for any discrete group G and integer m, the homology groups $H_*(G, \mathbb{Z}/m)$ of G are the homology of a standard chain complex, which we will write as $C_*(G)$, whose degree p piece is $\mathbb{Z}/m[G^p]$; see [WHomo 223, 6.5]. We write u^p for the p-chain $[X^1, ..., X^p]$ in $C_p(GL_n(I^p))$. The differential d sends u^p to

$$[X^2, ..., X^p] + \sum_{i=1}^{p-1}(-1)^i[\dots, X^iX^{i+1}, \dots] + (-1)^p[X^1, ..., X^{p-1}].$$

Now the A^p fit together to form a cosimplicial R-algebra $A^\bullet$, whose cofaces $\partial^j : A^p \to A^{p+1}$ are induced by the comultiplication $\Delta : H \to H \otimes H$. Applying GL_n yields cosimplicial groups $GL_n(A^\bullet)$ and $GL_n(I^\bullet)$. We are interested in the cosimplicial chain complex $C_*(GL_n(I^\bullet))$, which we may regard as a third quadrant double chain complex, with $C_p(GL_n(I^{-q}))$ in the (p, q) spot. Thus $(u^p) = (0, u^1, u^2, \dots)$ is an element of total degree 0 in the associated product total complex, i.e., in $\prod_{p=0}^\infty C_p(GL_n(I^p))$ (see [WHomo 223, 1.2.6]). By construction, $d(u^p) = \sum(-1)^j\partial^j(u^{p-1})$, so u^p is a cycle in this total complex.

VI.3.5 **Proposition 3.5.** *For each n, the image of the cycle (u^p) in $\prod_{p=0}^\infty C_p(GL(I^p))$ is a boundary in the product total complex of $C_*(GL(I^\bullet))$. That is, there are chains $c^p \in C_{p+1}(GL(I^p))$ so that $d(c^p) + \sum(-1)^j\partial^j(c^{p-1}) = u^p$ for all $p \geq 1$.*

Proof. Since the reduced complex $\widetilde{C}_*(G)$ is the subcomplex of $C_*(G)$ obtained by setting $C_0 = 0$, and $u^p \in \widetilde{C}_p$, it suffices to show that (u^p) is a boundary in the total complex T_* of $\widetilde{C}_*(GL(I^\bullet))$. By Gabber Rigidity IV.2.10 [IV.2.10], the reduced homology $\widetilde{H}_*(GL(I^t), \mathbb{Z}/m)$ is zero for each t. Thus the rows $\widetilde{C}_*(GL(I^{-q}))$ of the double complex are exact. By the Acyclic Assembly Lemma [WHomo 223, 2.7.3], the product total complex T_* is exact, so every cycle is a boundary. □

VI.3.6 **Lemma 3.6.** *For $G = GL_n(\mathbb{R})$, if ε is small enough then the embedding*

$$BG_\varepsilon \to BG^\delta \to BGL(\mathbb{R})$$

induces the zero map on $\widetilde{H}_*(-, \mathbb{Z}/m)$.

Proof. Let B^p denote the ring of germs of continuous $\mathbb{R}$-valued functions on the topological space $G^p = G \times \cdots \times G$ defined in some neighborhood of $(1, \dots, 1)$; B^p is a hensel ring. The coordinate functions give a canonical map $H^{\otimes p} \to B^p$, whose henselization is a map $A^p \to B^p$, and we write c^p_{ctn} for the image of c^p in $C_{p+1}(GL(B^p))$. Since $GL(B^p)$ is the group of germs of continuous $GL(\mathbb{R})$-valued functions on G^p, u^p is the germ of the function $G^p \to GL(\mathbb{R})^p$ sending $(g_1, \dots, g_p)$ to $(g_1, \dots, g_p)$, and c^p_{ctn} is a $\mathbb{Z}/m$-linear combination of germs of continuous functions $\gamma : G^p \to GL(\mathbb{R})^{p+1}$. That is, we may regard each c^p_{ctn} as a continuous map of some neighborhood of $(1, \dots, 1)$ to $C_{p+1}(GL(\mathbb{R})^{\text{top}})$.

If N is a fixed integer, there is an $\varepsilon > 0$ so that the c^p_{ctn} are defined on $(G_\varepsilon)^p$ for all $p \le N$. Extending c^p_{ctn} by linearity, we get homomorphisms $s^p : C_p(BG_\varepsilon) \to C_{p+1}(BSL(\mathbb{R})^{\text{top}})$. It is clear from Proposition VI.3.5 3.5 that s is a chain contraction for the canonical embedding $\widetilde{C}_*(BG_\varepsilon) \to \widetilde{C}_*(BGL(\mathbb{R})^{\text{top}})$, defined in degrees at most N. This proves the Lemma. □

VI.3.7 **Proposition 3.7.** *For $G = SL_n(\mathbb{R})$, if ε is small enough then*
(a) the embedding $BG_\varepsilon \to BG^\delta \to BSL(\mathbb{R})$ induces the zero map on $\widetilde{H}_*(-, \mathbb{Z}/m)$.
(b) $\widetilde{H}_i(BG_\varepsilon, \mathbb{Z}/m) = 0$ for all $i \le (n-1)/2$.
(c) $H_i(BG^\delta, \mathbb{Z}/m) \to H_i(BG^{top}, \mathbb{Z}/m)$ is an isomorphism for all $i \le (n-1)/2$.

Proof. Since $H_*(BSL(\mathbb{R})) \to H_*(BGL(\mathbb{R}))$ is a split injection (see Ex. EVI.3.1 3.1), part (a) follows from Lemma VI.3.6 3.6. For (b) and (c), let q be the smallest integer such that $\widetilde{H}_q(BG_\varepsilon, \mathbb{Z}/m)$ is nonzero; we will show that $q > (n-1)/2$. Since BG_ε has only one 0-simplex, $q \ge 1$. Consider the Serre spectral sequence associated to VI.3.4 3.4:

$$E^2_{p,q} = H_p(BG^{\text{top}}, H_q(BG_\varepsilon, \mathbb{Z}/m)) \Rightarrow H_*(BG^\delta, \mathbb{Z}/m).$$

Then $H_i(BG^\delta, \mathbb{Z}/m) \to H_i(BG^{\text{top}}, \mathbb{Z}/m)$ is an isomorphism for $i < q$ and the exact sequence of low degree terms for the spectral sequence is

$$H_{q+1}(BG^\delta, \mathbb{Z}/m) \xrightarrow{\text{onto}} H_{q+1}(BG^{\text{top}}, \mathbb{Z}/m) \xrightarrow{d^q} H_q(BG_\varepsilon, \mathbb{Z}/m) \to H_q(BG^\delta, \mathbb{Z}/m).$$

Milnor proved in M-Lie [132, Thm. 1] that the left-hand map is a split surjection; it follows that the right-hand map is an injection. By Homological Stability IV.IV.1.14 1.14, $H_i(BSL_n(\mathbb{R})^\delta) \to H_i(BSL(\mathbb{R})^\delta)$ is an isomorphism for $i \le (n-1)/2$; part (a) implies that $q > (n-1)/2$. This proves parts (b) and (c). □

VI.3.8 **Corollary 3.8.** *$BSL(\mathbb{R})^\delta \to BSL(\mathbb{R})^{top}$ and $BGL(\mathbb{R})^\delta \to BGL(\mathbb{R})^{top}$ induce isomorphisms on $\widetilde{H}_*(-, \mathbb{Z}/m)$.*

Proof. Set $G = SL_n(\mathbb{R})$. Passing to the limit as $n \to \infty$ in VI.3.7 3.7 proves the assertion for SL. □

Proof of Theorem VI.3.1 *3.1.* Since $BSL(\mathbb{R})^+$ and $BSL(\mathbb{R})^{\text{top}} \simeq BSO$ are simply connected, Corollary VI.3.8 3.8 implies that $\pi_n(BSL(\mathbb{R})^+; \mathbb{Z}/m) \to \pi_n(BSO; \mathbb{Z}/m)$ is an isomorphism for all n. We saw in Chapter IV, Ex. EIV.1.8 1.8 that the map

$\pi_n(BSL(\mathbb{R})^+;\mathbb{Z}/m) \to K_n(\mathbb{R};\mathbb{Z}/m)$ is an isomorphism for $n \geq 3$, and an injection for $n = 2$ with cokernel $\mathbb{Z}/2$; the same is true for $\pi_n(BSO;\mathbb{Z}/m) \to \pi_n(BO;\mathbb{Z}/m)$. This proves part (c) for $n \geq 3$; the result for $n = 1, 2$ is classical (see III.1.5.4 and IV.2.5.1, or [131, p. 61]).

Now consider the action of complex conjugation c on $K_*(\mathbb{C})$. The image of $i^* : K_n(\mathbb{R}) \to K_n(\mathbb{C})$ lands in the invariant subgroup $K_n(\mathbb{C})^c$, which by Theorem 1.6 is the direct sum of a uniquely divisible group and a torsion group (which is either 0, $\mathbb{Z}/2$, 0 or $\mathbb{Q}/\mathbb{Z}$ depending on n modulo 4). The transfer i_* satisfies $i_* i^* = 2$ and $i^* i_* = 1 + c$; see Ex. IV.6.13. Hence (for $n \geq 1$) $K_n(\mathbb{R})$ is the direct sum of its torsion submodule $K_n(\mathbb{R})_{\text{tors}}$ and the uniquely divisible abelian group $K_n(\mathbb{C})^c \otimes \mathbb{Q}$, and the kernel and cokernel of $K_n(\mathbb{R})_{\text{tors}} \to K_n(\mathbb{C})^c_{\text{tors}}$ are elementary abelian 2-groups. By Ex. IV.2.6, we have $K_n(\mathbb{R})_{\text{tors}} \cong K_{n+1}(\mathbb{R};\mathbb{Q}/\mathbb{Z}) \cong \pi_{n+1}(BO, \mathbb{Q}/\mathbb{Z})$. These torsion groups may be read off from Bott periodicity II.3.1.1. □

EXERCISES

EVI.3.1 **3.1.** For any commutative ring R and ideal I, show that $R^\times$ acts trivially on the homology of $SL(R)$ and $GL(R)$, while $(1+I)^\times$ acts trivially on the homology of $SL(I)$ and $GL(I)$. Conclude that $SL(R) \to GL(R)$ and $SL(I) \to GL(I)$ are split injections on homology.

EVI.3.2 **3.2.** Using Exercise 3.1, show that there is a universal homotopy construction for SL_n parallel to the one in Proposition 3.5 for GL_n. Use this to prove Proposition 3.7 directly, modifying the proof of Lemma 3.6.

EVI.3.3 **3.3.** Check that 3.6, 3.7 and 3.8 go through with $\mathbb{R}$ replaced by $\mathbb{C}$. Using these, prove the analogue of Theorem 3.1 for $\mathbb{C}$ and compare it to Theorem 1.6.

EVI.3.4 **3.4.** If F is any formally real field, such as $\mathbb{R} \cap \bar{\mathbb{Q}}$, show that $K_*(F;\mathbb{Z}/m) \cong K_*(\mathbb{R};\mathbb{Z}/m)$ for all m.

4 Relation to motivic cohomology

Motivic cohomology theory, developed by Voevodsky, is intimately related to algebraic K-theory. For every abelian group A and every (n, i), the motivic cohomology of a smooth scheme X over a field consists of groups $H^n(X, A(i))$, defined as the hypercohomology of a certain chain complex $A(i)$ of Nisnevich sheaves. An introduction to motivic cohomology is beyond the scope of this book, and we refer the reader to [122] for the definitions and properties of motivic cohomology.

When $A = \mathbb{Q}$, we have isomorphisms $H^n(X, \mathbb{Q}(i)) \cong K^{(i)}_{2i-n}(X)$, where the right side refers to the eigenspace of $K_{2i-n}(X) \otimes \mathbb{Q}$ on which the Adams operations ψ^k act as multiplication by k^i, described in IV, Theorem 5.11. This fact is due to Bloch [25], and follows from 4.2 and 4.9 below; see Ex. 4.5.

Here is the fundamental structure theorem for motivic cohomology with finite coefficients, due to Rost and Voevodsky. Since the proof is scattered over 15-20 research papers, we refer the reader to the book [82] for the proof.

For any smooth X, there is a natural map $H^n(X, \mathbb{Z}/m(i)) \to H^n_{\text{et}}(X, \mu_m^{\otimes i})$ from motivic to étale cohomology. It arises from the forgetful functor a_* from étale sheaves to Nisnevich sheaves on X, via the isomorphism $H^n_{\text{et}}(X, \mu_m^{\otimes i}) \cong H^n_{nis}(X, Ra_*\mu_m^{\otimes i})$ and a natural map $\mathbb{Z}/m(i) \to Ra_*\mu_m^{\otimes i}$.

VI.4.1 **Norm Residue Theorem 4.1.** *(Rost-Voevodsky) If k is a field containing $1/m$, the natural map induces isomorphisms*

$$H^n(k, \mathbb{Z}/m(i)) \cong \begin{cases} H^n_{et}(k, \mu_m^{\otimes i}) & n \le i \\ 0 & n > i. \end{cases}$$

If X is a smooth scheme over k, the natural map $H^n(X, \mathbb{Z}/m(i)) \to H^n_{et}(X, \mu_m^{\otimes i})$ is an isomorphism for $n \le i$. For $n > i$, the map identifies $H^n(X, \mathbb{Z}/m(i))$ with the Zariski hypercohomology on X of the truncated direct image complex $\tau^{\le i} Ra_*(\mu_m^{\otimes i})$.

A result of Totaro, and Nesterenko-Suslin [144, 201] states that the i^{th} norm residue symbol (III.7.11) factors through an isomorphism $K_i^M(k) \xrightarrow{\cong} H^i(k, \mathbb{Z}(i))$, compatibly with multiplication. This means that the Milnor K-theory ring $K_*^M(k)$ (III.7.1) is isomorphic to the ring $\oplus H^i(k, \mathbb{Z}(i))$. Since $H^i(k, \mathbb{Z}(i))/m \cong H^i(k, \mathbb{Z}/m(i))$, we deduce the following special case of Theorem 4.1. This special case was once called the Bloch-Kato conjecture, and is in fact equivalent to Theorem 4.1; see [190], [63] or [82] for a proof.

VI.4.1.1 **Corollary 4.1.1.** *If k is a field containing $1/m$, the norm residue symbols are isomorphisms for all i: $K_i^M(k)/m \xrightarrow{\cong} H^i_{et}(k, \mu_m^{\otimes i})$. They form a ring isomorphism:*

$$\oplus K_i^M(k)/m \xrightarrow{\cong} \oplus H^i(k, \mathbb{Z}(i))/m \cong \oplus H^i(k, \mathbb{Z}/m(i)) \cong \oplus H^i_{et}(k, \mu_m^{\otimes i}).$$

The Merkurjev-Suslin isomorphism $K_2(k)/m \cong H^2_{\text{et}}(k, \mu_m^{\otimes 2})$ of [125], mentioned in III.6.10.4, is the case $i = 2$ of 4.1.1. The isomorphism $K_3^M(k)/m \cong H^3_{\text{et}}(k, \mu_m^{\otimes 3})$ for $m = 2^\nu$ was established by Rost and Merkurjev-Suslin; see [126].

The key technical tool which allows us to use Theorem 4.1 in order to make calculations is the *motivic-to-K-theory spectral sequence*, so-named because it goes from motivic cohomology to algebraic K-theory. The construction of this spectral sequence is given in the references cited in the Historical Remark 4.4; the final assertion in 4.2 is immediate from the first assertion and Theorem 4.1.

VI.4.2 **Theorem 4.2.** *For any coefficient group A, and any smooth scheme X over a field k, there is a spectral sequence, natural in X and A:*

$$E_2^{p,q} = H^{p-q}(X, A(-q)) \Rightarrow K_{-p-q}(X; A).$$

If $X = \operatorname{Spec}(k)$ and $A = \mathbb{Z}/m$, where $1/m \in k$, then the E_2 terms are just the étale cohomology groups of k, truncated to lie in the octant $q \le p \le 0$.

VI.4.2.1 **Addendum 4.2.1.** If $A = \mathbb{Z}/m$ and $m \not\equiv 2 \pmod 4$, the spectral sequence has a multiplicative structure which is the product in motivic cohomology on the E_2 page and the K-theory product (IV.2.8) on the abutment.

For any coefficients $\mathbb{Z}/m$, there is a pairing between the spectral sequence of (4.2) for $\mathbb{Z}$ coefficients and the spectral sequence of (4.2) with coefficients $\mathbb{Z}/m$. On the E_2 page, it is the product in motivic cohomology and on the abutment it is the pairing $K_*(X) \otimes K_*(X;\mathbb{Z}/m) \to K_*(X;\mathbb{Z}/m)$ of IV, Ex. 2.5.

VI.4.2.2 **Remark 4.2.2.** The spectral sequence of (4.2) has an analogue for non-smooth schemes over k, in which the motivic cohomology groups are replaced by higher Chow groups $CH^i(X,n)$. It is established in [56, 13.12] and [109, 18.9]. For any equidimensional quasi-projective scheme X, there is a convergent spectral sequence

$$E_2^{p,q} = CH^{-q}(X, -p-q) \Rightarrow G_{-p-q}(X).$$

If X is smooth and k is perfect, then $H^n(X,\mathbb{Z}(i)) \cong CH^i(X, 2i-n)$; see [122, 19.1]. This identifies the present spectral sequence with (4.2). Since $CH^i(X,n)$ is the same as the Borel-Moore homology group $H^{BM}_{2i+n}(X,\mathbb{Z}(i))$, this spectral sequence is sometimes cited as a homology spectral sequence with $E^2_{p,q} = H^{BM}_{p-q}(X,\mathbb{Z}(-q))$.

VI.4.3 **Edge Map 4.3.** Let k be a field. The edge map $K_{2i}(k;\mathbb{Z}/m) \to H^0_{\text{et}}(k,\mu_m^{\otimes i})$ in (4.2) is the e-invariant of 2.1, and is an isomorphism for the algebraic closure $\bar{k}$ of k; the details are given in Example 4.5(ii) below.

We now consider the other edge map, from $E_2^{0,-n} = H^n(k,\mathbb{Z}(n)) \cong K_n^M(k)$ to $K_n(k)$. Since the ring $K_*^M(k)$ is generated by its degree 1 terms, and the low degree terms of (4.2) yield isomorphisms $H^1(k,\mathbb{Z}(1)) \cong K_1(k)$, and $H^1(k,\mathbb{Z}/m(1)) \cong K_1(k)/m$, the multiplicative structure described in 4.2.1 implies that the edge maps in the spectral sequence are canonically identified with the maps $K_*^M(k) \to K_*(k)$ and $K_*^M(k)/m \to K_*(k;\mathbb{Z}/m)$ described in IV.1.10.1. (This was first observed in [62, 3.3] and later in [56, 15.5].)

By V.11.13, the kernel of the edge map $K_n^M(k) \to K_n(k)$ is a torsion group of exponent $(n-1)!$. This is not best possible; we will see in 4.3.2 that the edge map $K_3^M(k) \to K_3(k)$ is an injection.

Since $\{-1,-1,-1,-1\}$ is nonzero in $K_4^M(\mathbb{Q})$ and $K_4^M(\mathbb{R})$ but zero in $K_4(\mathbb{Q})$ (by Ex. IV.1.12), the edge map $K_4^M(k) \to K_4(k)$ is not an injection for subfields of $\mathbb{R}$. This means that the differential $d_2 : H^1(\mathbb{Q},\mathbb{Z}(3)) \to K_4^M(\mathbb{Q})$ is nonzero.

Similarly, using the étale Chern class $c_{3,3} : K_3(k,\mathbb{Z}/m) \to H^n_{\text{et}}(k,\mu_m^{\otimes n})$ of V.11.10, we see that the kernel of the edge map $K_n^M(k)/m \to K_n(k;\mathbb{Z}/m)$ has exponent $(n-1)!$. (The composition of $c_{3,3}$ with the isomorphism $H^n_{\text{et}}(k,\mu_m^{\otimes n}) \cong K_n^M(k)/m$ of Corollary 4.1.1 satisfies $c_{3,3}(x) = -2x$ for all $x \in K_3^M(k)/m$.) Since $K_3^M(\mathbb{Q}) \cong \mathbb{Z}/2$ on $\{-1,-1,-1\}$ (Remark 2.1.3), and this element dies in $K_3(\mathbb{Q})/8 \cong \mathbb{Z}/8$ and hence $K_3(\mathbb{Q};\mathbb{Z}/8)$, the edge map $K_3^M(\mathbb{Q})/8 \to K_3(\mathbb{Q};\mathbb{Z}/8)$ is not an injection.

VI.4.3.1 **Low degree terms 4.3.1.** When k is a field, the edge map $K_2^M(k) \to K_2(k)$ is an isomorphism by Matsumoto's Theorem III.6.1, so the low degree sequence $0 \to K_2^M(k)/m \to K_2(k;\mathbb{Z}/m) \to \mu_m(k) \to 0$ of (4.2) may be identified with the Universal Coefficient sequence IV.2.5. This yields the Merkurjev-Suslin formula $K_2(k)/m \cong H^2_{\text{et}}(k, \mu_m^{\otimes 2})$ of III(6.10.4). Since $H^n(X,\mathbb{Z}(0)) = 0$ for $n < 0$ and $H^n(X,\mathbb{Z}(1)) = 0$ for $n \le 0$, by [122, 4.2], we also obtain the exact sequences

$$K_4(k) \to H^0(k,\mathbb{Z}(2)) \xrightarrow{d_2} K_3^M(k) \to K_3(k) \to H^1(k,\mathbb{Z}(2)) \to 0,$$

$$K_4(k;\mathbb{Z}/m) \to H^0_{\text{et}}(k,\mu_m^{\otimes 2}) \xrightarrow{d_2} K_3^M(k)/m \to K_3(k;\mathbb{Z}/m) \to H^1_{\text{et}}(k,\mu_m^{\otimes 2}) \to 0.$$

Since the kernel of $K_3^M(k)/m \to K_3(k;\mathbb{Z}/m)$ is nonzero for $k=\mathbb{Q}$, the differential d_2 can be nontrivial with finite coefficients. The integral d_2 is always zero:

VI.4.3.2 **Proposition 4.3.2.** *The map $K_3^M(k) \to K_3(k)$ is an injection for every field k.*

Proof. We have seen that the kernel of $K_3^M(k) \to K_3(k)$ has exponent 2. If $\text{char}(k) = 2$, then $K_3^M(k)$ has no 2-torsion (Izhboldin's Theorem III.7.8) and the result holds, so we may suppose that $\text{char}(k) \ne 2$. Consider the motivic group $H(k) = H^0(k,\mathbb{Z}(2))/2$. Since the differential $d_2 : H^0(k,\mathbb{Z}(2)) \to K_3^M(k)$ in 4.3.1 factors through $H(k)$, it suffices to show that $H(k) = 0$. By universal coefficients, $H(k)$ is a subgroup of $H^0(k,\mathbb{Z}/2(2)) \cong H^0_{\text{et}}(k,\mathbb{Z}/2) = \mathbb{Z}/2$; by naturality this implies that $H(k) \subseteq H(k') \subset \mathbb{Z}/2$ for any field extension k' of k. Thus we may suppose that k is algebraically closed. In this case, $K_4(k)$ is divisible (by 1.6) and $K_3^M(k)$ is uniquely divisible (by III.7.2), so it follows from 4.3.1 that $H^0(k,\mathbb{Z}(2))$ is divisible and hence $H(k) = 0$. □

VI.4.4 **Historical Remark 4.4.** This spectral sequence (4.2) has an awkward history. In 1972, Lichtenbaum [112] made several conjectures relating the K-theory of integers in number fields to étale cohomology and (via this) to values of Zeta functions at negative integers (see 8.8 below). Expanding on these conjectures, Quillen speculated that there should be a spectral sequence like (4.2) (with finite coefficients) at the 1974 Vancouver ICM, and Beilinson suggested in 1982 that one might exist with coefficients $\mathbb{Z}$.

The existence of such a spectral sequence was claimed by Bloch and Lichtenbaum in their 1994 preprint [27], which was heavily cited for a decade, but there is a gap in their proof. Friedlander and Suslin showed in [56] that one could start with the construction of [27] to get a spectral sequence for all smooth schemes, together with the multiplicative structure of 4.2.1. The spectral sequence in [27] was also used to construct the Borel-Moore spectral sequence in 4.2.2 for quasi-projective X in [56, 13.12] and [109, 8.9]

Also in the early 1990s, Grayson constructed a spectral sequence in [78], following suggestions of Goodwillie and Lichtenbaum. Although it converged to the K-theory of regular rings, it was not clear what the E_2 terms were until 2001, when Suslin showed (in [189]) that the E_2 terms in Grayson's spectral sequence agreed with motivic cohomology for fields. Using the machinery of [56], Suslin then constructed the spectral sequence of Theorem 4.2 for all smooth varieties over a field, and also established the multiplicative structure of 4.2.1.

In 2000–1, Voevodsky observed (in [209, 210]) that the slice filtration for the motivic spectrum representing K-theory (of smooth varieties) gave rise to a spectral sequence, and showed that it had the form given in Theorem 4.2 modulo two conjectures about motivic homotopy theory (since verified). Yet a third construction was given by Levine in [109] [110]; a proof that these three spectral sequences agree is also given in [110].

VI.4.4.1 **Remark 4.4.1.** A similar motivic spectral sequence was established by Levine in [109, (8.8)] over a Dedekind domain, in which the group $H^n_M(X, A(i))$ is defined to be the $(2i-n)$th hypercohomology on X of the complex of higher Chow group sheaves $z^i \otimes A$.

We now give several examples in which the motivic-to-K-theory spectral sequence degenerates at the E_2 page, quickly yielding the K-groups.

VI.4.5 **Examples 4.5.** (i) When k is a separably closed field, $H^n_{\text{et}}(k, -) = 0$ for $n > 0$ and the spectral sequence degenerates along the line $p = q$ to yield $K_{2i}(k;\mathbb{Z}/m) \cong \mathbb{Z}/m$, $K_{2i-1}(k;\mathbb{Z}/m) = 0$. This recovers the calculations of 1.3.1 and 1.4 above. In particular, the Bott element $\beta \in K_2(k, \mathbb{Z}/m)$ (for a fixed choice of ζ) corresponds to the canonical element ζ in $E_2^{-1,-1} = H^0_{\text{et}}(k, \mu_m)$.

(ii) If k is any field containing $1/m$, and $G = \text{Gal}(\bar{k}/k)$, then $H^0_{\text{et}}(k, \mu_m^{\otimes i})$ is the subgroup of $\mu_m^{\otimes i}$ invariant under G; by Definition 2.1 it is isomorphic to $\mathbb{Z}/(m, w_i(k))$. By naturality in k and (i), the edge map of (4.2) (followed by the inclusion) is the composition $K_{2i}(k;\mathbb{Z}/m) \to K_{2i}(\bar{k};\mathbb{Z}/m) \to \mu_m^{\otimes i}$. Therefore the edge map vanishes on $K_{2i}(k)/m$ and (by the Universal Coefficient Sequence of IV.2.5) induces the e-invariant ${}_mK_{2i-1}(k) \to \text{Hom}(\mathbb{Z}/m, \mathbb{Z}/w_i(k)) = \mathbb{Z}/(m, w_i(k))$ of 2.1.

(iii) For a finite field $\mathbb{F}_q$ with m prime to q, we have $H^n_{\text{et}}(\mathbb{F}_q, -) = 0$ for $n > 1$ [168, p. 69]. There is also a duality isomorphism $H^1_{\text{et}}(\mathbb{F}_q, \mu_m^{\otimes i}) \cong H^0_{\text{et}}(\mathbb{F}_q, \mu_m^{\otimes i})$. Thus each diagonal $p + q = -n$ in the spectral sequence (4.2) has only one nonzero entry, so $K_{2i}(\mathbb{F}_q;\mathbb{Z}/m)$ and $K_{2i-1}(\mathbb{F}_q;\mathbb{Z}/m)$ are both isomorphic to $\mathbb{Z}/(m, w_i(k))$. This recovers the computation for finite fields given in 2.1.1 and IV.1.13.1.

(iv) Let F be the function field of a curve over a separably closed field containing $1/m$. Then $H^n_{\text{et}}(F, -) = 0$ for $n > 1$ (see [168, p. 119]) and $H^0_{\text{et}}(F, \mu_m^{\otimes i}) \cong \mathbb{Z}/m$ as in (i). By Kummer theory,

$$H^1_{\text{et}}(F, \mu_m^{\otimes i}) \cong H^1_{\text{et}}(F, \mu_m) \otimes \mu_m^{\otimes i-1} \cong F^\times/F^{\times m} \otimes \mu_m^{\otimes i-1}.$$

(The twist by $\mu_m^{\otimes i-1}$ is to keep track of the action of the Galois group.) As in (iii), the spectral sequence degenerates to yield $K_{2i}(F;\mathbb{Z}/m) \cong \mathbb{Z}/m$, $K_{2i-1}(F;\mathbb{Z}/m) \cong F^\times/F^{\times m}$. Since the spectral sequence is multiplicative, it follows that the map $F^\times/F^{\times m} \to K_{2i+1}(F;\mathbb{Z}/m)$ sending u to $\{\beta^i, u\}$ is an isomorphism because it corresponds to the isomorphism $E_2^{0,-1} \to E_2^{-i,-i-1}$ obtained by multiplication by the element $\zeta^{\otimes i}$ of $E_2^{-i,-i}$. Thus

$$K_n(F;\mathbb{Z}/m) \cong \begin{cases} \mathbb{Z}/m \text{ on } \beta^i, & n = 2i, \\ F^\times/F^{\times m} \text{ on } \{\beta^i, u\} & n = 2i+1. \end{cases}$$

When X has dimension $d > 0$, the spectral sequence (4.2) extends to the fourth quadrant, with terms only in columns $\le d$. This is because $H^n(X, A(i)) = 0$ for $n > i + d$; see [122, 3.6]. To illustrate this, we consider the case $d = 1$, *i.e.,* when X is a curve.

VI.4.6 **Example 4.6.** Let X be a smooth projective curve over a field k containing $1/m$, with function field F. By Theorem 4.1, $E_2^{p,q} = H^{p-q}(X, \mathbb{Z}/m(-q))$ is $H^{p-q}_{\text{et}}(X, \mu_m^{\otimes q})$ for $p \le 0$, and $E_2^{p,q} = 0$ for $p \ge 2$ by the above remarks. That is, the E_2-terms in the third quadrant of (4.2) are étale cohomology groups, but there are also modified terms in the column $p = +1$. To determine these, we note that a comparison of the localization sequences for $\text{Spec}(F) \to X$ in motivic cohomology [122, 14.5] and étale cohomology yields an exact sequence

$$0 \to H^{i+1}(X, \mathbb{Z}/m(i)) \to H^{i+1}_{\text{et}}(X, \mu_m^{\otimes i}) \to H^{i+1}_{\text{et}}(F, \mu_m^{\otimes i}).$$

In particular, $E_2^{1,0} = 0$ and $E_2^{1,-1} = E_\infty^{1,-1} = \text{Pic}(X)/m$. In this case, we can identify the group $K_0(X;\mathbb{Z}/m) = \mathbb{Z}/m \oplus \text{Pic}(X)/m$ (see II.8.2.1) with the abutment of (4.2) in total degree 0.

Now suppose that k is separably closed and $m = \ell^\nu$. Then X has (ℓ-primary) étale cohomological dimension 2, and it is well known that $H^1_{\text{et}}(X, \mu_m) \cong {}_m\text{Pic}(X)$ and $H^2_{\text{et}}(X, \mu_m) \cong \mathbb{Z}/m$; see [127, pp. 126, 175]. Thus the spectral sequence has only three diagonals ($p - q = 0, 1, 2$) with terms $\mathbb{Z}/m$, ${}_m\text{Pic}(X) \cong (\mathbb{Z}/m)^{2g}$ and $\text{Pic}(X)/m \cong \mathbb{Z}/m$ (see I.5.16); the only nonzero term in the column $p = +1$ is $\text{Pic}(X)/m \cong \mathbb{Z}/m$. By 4.5(iv), there is simply no room for any differentials, so the spectral sequence degenerates at E_2. Since the e-invariant maps $K_{2i}(k;\mathbb{Z}/m)$ isomorphically onto $\mu_m^{\otimes i} \cong E_\infty^{-i,-i}$, the extensions split and we obtain

VI.4.6.1 **Proposition 4.6.1.** *Let X be a smooth projective curve over a separably closed field containing $1/m$. Then*

$$K_n(X;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m \oplus \mathbb{Z}/m, & n = 2i, \quad n \ge 0, \\ {}_m\text{Pic}(X) \cong (\mathbb{Z}/m)^{2g}, & n = 2i-1, n > 0. \end{cases}$$

The multiplicative structure of $K_*(X;\mathbb{Z}/m)$ is given in Exercise 4.3. When $k = \bar{\mathbb{F}}_p$, the structure of $K_*(X)$ is given in Theorem 6.4 below.

Geisser and Levine proved in [62] that if k is a field of characteristic $p > 0$ then the motivic cohomology groups $H^{n,i}(X, \mathbb{Z}/p^\nu)$ vanish for all $i \ne n$. This allows us to clarify the relationship between $K_*^M(k)$ and $K_*(k)$ at the prime p. Part (b) should be compared with Izhboldin's Theorem III.7.8 that $K_n^M(k)$ has no p-torsion.

VI.4.7 **Theorem 4.7.** *Let k be a field of characteristic p. Then for all $n \geq 0$,*
(a) for all $\nu > 0$, the map $K_n^M(k)/p^\nu \to K_n(k;\mathbb{Z}/p^\nu)$ is an isomorphism;
(b) $K_n(k)$ has no p-torsion;
(c) the kernel and cokernel of $K_n^M(k) \to K_n(k)$ are uniquely p-divisible.

Proof. The Geisser-Levine result implies that the spectral sequence (4.2) with coefficients $\mathbb{Z}/p^\nu$ collapses at E_2, with all terms zero except for $E_2^{0,q} = K_{-q}^M(k)/p^\nu$. Hence the edge maps of 4.3 are isomorphisms. This yields (a). Since the surjection $K_n^M(k) \to K_n^M(k)/p \cong K_n(k;\mathbb{Z}/p)$ factors through $K_n(k)/p$, the Universal Coefficient sequence of IV.2.5 implies (b), that $K_{n-1}(k)$ has no p-torsion. Finally (c) follows from the 5-lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(k) & \xrightarrow{p} & K_n^M(k) & \longrightarrow & K_n^M(k)/p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & K_n(k) & \xrightarrow{p} & K_n(k) & \longrightarrow & K_n(k;\mathbb{Z}/p) & \longrightarrow & 0. \end{array}$$ □

VI.4.8 **Example 4.8** (Periodicity for $\ell > 2$)**.** Let β denote the Bott element in $K_2(\mathbb{Z}[\zeta_\ell];\mathbb{Z}/\ell)$ corresponding to the primitive ℓ^{th} root of unity ζ_ℓ, as in IV.2.5.2, and let $b \in K_{2(\ell-1)}(\mathbb{Z};\mathbb{Z}/\ell)$ denote the image of $-\beta^{\ell-1}$ under the transfer map i_*. Since $i^*(b) = -(\ell-1)\beta^{\ell-1} = \beta^{\ell-1}$, the e-invariant of b is the canonical generator $\zeta^{\otimes \ell-1}$ of $H^0(\mathbb{Z}[1/\ell], \mu_\ell^{\ell-1})$ by naturality. If X is any smooth variety over a field containing $1/\ell$, multiplication by b gives a map $K_n(X;\mathbb{Z}/\ell) \to K_{n+2(\ell-1)}(X;\mathbb{Z}/\ell)$; we refer to this as a *periodicity map*. Indeed, the multiplicative pairing in Addendum 4.2.1, of b with the spectral sequence converging to $K_*(X;\mathbb{Z}/\ell)$, gives a morphism of spectral sequences $E_r^{p,q} \to E_r^{p+1-\ell,q+1-\ell}$ from (4.2) to a shift of itself. On the E_2 page, these maps are isomorphisms for $p \leq 0$, induced by $\mu_\ell^{\otimes i} \cong \mu_\ell^{\otimes i+\ell-1}$.

The term 'periodicity map' comes from the fact that the periodicity map is an isomorphism $K_n(X;\mathbb{Z}/\ell) \xrightarrow{\simeq} K_{n+2(\ell-1)}(X;\mathbb{Z}/\ell)$ for all $n > \dim(X) + \mathrm{cd}_\ell(X)$, $\mathrm{cd}_\ell(X)$ being the étale cohomological dimension of X for ℓ-primary sheaves. This follows from the comparison theorem for the morphism $\cup b$ of the spectral sequence (4.2) to itself.

VI.4.8.1 **Example 4.8.1** (Periodicity for $\ell = 2$)**.** Pick a generator v_1^4 of the group $\pi_8^s(S^0;\mathbb{Z}/16) \cong \mathbb{Z}/16$; it defines a generator of $K_8(\mathbb{Z}[1/2];\mathbb{Z}/16)$ and, by the edge map in (4.2), a canonical element of $H^0_{\mathrm{et}}(\mathbb{Z}[1/2];\mu_{16}^{\otimes 4})$ which we shall also call v_1^4. If X is any scheme, smooth over $\mathbb{Z}[1/2]$, the multiplicative pairing of v_1^4 (see Addendum 4.2.1) with the spectral sequence converging to $K_*(X;\mathbb{Z}/2)$ gives a morphism of spectral sequences $E_r^{p,q} \to E_r^{p-4,q-4}$ from (4.2) to itself. For $p \leq 0$ these maps are isomorphisms, induced by $E_2^{p,q} \cong H^{p-q}_{\mathrm{et}}(X,\mathbb{Z}/2)$; we shall refer to these isomorphisms as *periodicity isomorphisms*.

VI.4.9 **Example 4.9** (Adams Operations)**.** The Adams operations ψ^k act on the spectral sequence (4.2), commuting with the differentials and converging to the action of ψ^k on $K_*(k)$ and $K_*(k;\mathbb{Z}/m)$ (§IV.5), with $\psi^k = k^i$ on the row $q=-i$. This was proven by Soulé; see [70, 7.1]. Since $(k^i - k^{i+r-1})\,d_r(x) = d_r(\psi^k x) - \psi^k(d_r x) = 0$ for all x in row $-i$, we see that the image of the differentials d_r are groups of bounded exponent. That is, the spectral sequence (4.2) degenerates modulo bounded torsion.

EXERCISES

EVI.4.1 **4.1.** If $cd_\ell(k) = d$ and $\mu_{\ell^\nu} \subset k$, show that $\cup\beta : K_n(k;\mathbb{Z}/\ell^\nu) \to K_{n+2}(k;\mathbb{Z}/\ell^\nu)$ is an isomorphism for all $n \geq d$. This is a strong form of periodicity.

EVI.4.2 **4.2.** (Browder) Let $\mathbb{F}_q$ be a finite field with $\ell \nmid q$. Show that the periodicity maps $K_n(\mathbb{F}_q;\mathbb{Z}/\ell) \to K_{n+2(\ell-1)}(\mathbb{F}_q;\mathbb{Z}/\ell)$ of 4.8 are isomorphisms for all $n \geq 0$.

EVI.4.3 **4.3.** Let X be a smooth projective curve over an algebraically closed field k, and $[x] \in \mathrm{Pic}(X)$ the class of a closed point x. By I.5.16, $\mathrm{Pic}(X)/m \cong \mathbb{Z}/m$ on $[x]$, and by II.8.2.1 we have $K_0(X)/m \cong \mathbb{Z}/m \oplus \mathbb{Z}/m$ with basis $\{1,[x]\}$. In this exercise, we clarify (4.2.1), assuming $1/m \in k$.

(i) Show that multiplication by the Bott element β^i induces an isomorphism $K_0(X)/m \xrightarrow{\cong} K_{2i}(X;\mathbb{Z}/m)$.

(ii) Show that $K_1(X)$ is divisible, so that the map $K_1(X;\mathbb{Z}/m) \to {}_m\mathrm{Pic}(X)$ in the Universal Coefficient sequence is an isomorphism.

(iii) Show that multiplication by the Bott element β^i induces an isomorphism $K_1(X;\mathbb{Z}/m) \xrightarrow{\cong} K_{2i+1}(X;\mathbb{Z}/m)$.

(iv) Conclude that the ring $K_*(X;\mathbb{Z}/m)$ is $\mathbb{Z}/m[\beta] \otimes \mathbb{Z}/m[M]$, where $M = \mathrm{Pic}(X)/m \oplus {}_m\mathrm{Pic}(X)$ is a graded ideal of square zero.

EVI.4.4 **4.4.** Use the formula $H^n(\mathbb{P}^1_k, A(i)) \cong H^n(k, A(i)) \oplus H^{n-2}(k, A(i-1))$ (see [122, 15.12]) to show that the spectral sequence (4.2) for $\mathbb{P}^1_k$ is the direct sum of two copies of the spectral sequence for k, on generators $1 \in E_2^{0,0}$ and $[L] \in E_2^{1,-1}$. Using this, re-derive the calculation of V.6.14 that $K_n(\mathbb{P}^1_k) \cong K_n(k) \otimes K_0(\mathbb{P}^1)$.

EVI.4.5 **4.5.** Use (4.2) and 4.9 to recover the isomorphism $H^n(X,\mathbb{Q}(i)) \cong K^{(i)}_{2i-n}(X)$, due to Bloch [25].

EVI.4.6 **4.6.** The *Vanishing Conjecture* in K-theory states that $K_n^{(i)}(X)$ vanishes whenever $i \leq n/2$, $n > 0$. (See [174, p. 501].) Using the Universal Coefficient sequence

$$0 \to H^j(X,\mathbb{Z}(i))/m \to H^j(X,\mathbb{Z}/m(i)) \to {}_mH^{j+1}(X,\mathbb{Z}(i)) \to 0,$$

(a) show that $H^j(X,\mathbb{Z}(i))$ is uniquely divisible for $j \leq 0$, and (b) conclude that the Vanishing Conjecture is equivalent to the assertion that $H^j(X,\mathbb{Z}(i))$ vanishes for all $j \leq 0$ ($i \neq 0$). This and Exercise 4.5 show that the Vanishing Conjecture holds for any field k whose groups $K_n(k)$ are finitely generated, such as number fields.

5 K_3 of a field

In this section, we study the group $K_3(F)$ of a field F. By Proposition 4.3.2, $K_3^M(F)$ injects into $K_3(F)$. By IV.1.20, the map $K_3(F) \to H_3(SL(F))$ is onto, and its kernel is the subgroup of $K_3^M(F)$ generated by the symbols $\{-1, a, b\}$. Assuming that $K_3^M(F)$ is known, we may use homological techniques. The focus of this section will be to relate the group $K_3^{\text{ind}}(F) := K_3(F)/K_3^M(F)$ to Bloch's group $B(F)$ of a field F, which we now define.

For any abelian group A, let $\tilde{\wedge}^2 A$ denote the quotient of the group $A \otimes A$ by the subgroup generated by all $a \otimes b + b \otimes a$. The exterior power $\wedge^2 A$ is the quotient of $\tilde{\wedge}^2 A$ by the subgroup (isomorphic to $A/2A$) of all symbols $x \wedge x$.

VI.5.1 **Definition 5.1.** For any field F, let $\mathcal{P}(F)$ denote the abelian group presented with generators symbols $[x]$ for $x \in F - \{0\}$, with relations $[1] = 0$ and

$$[x] - [y] + [y/x] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - x}{1 - y}\right] = 0, \quad x \neq y \text{ in } F - \{0, 1\}.$$

There is a canonical map $\mathcal{P}(F) \to \tilde{\wedge}^2 F^\times$ sending $[1]$ to 0 and $[x]$ to $x \wedge (1 - x)$ for $x \neq 1$, and *Bloch's group* $B(F)$ is defined to be its kernel. Thus we have an exact sequence

$$0 \to B(F) \to \mathcal{P}(F) \to \tilde{\wedge}^2 F^\times \to K_2(F) \to 0.$$

VI.5.1.1 **Remark 5.1.1.** Since the cases $B(\mathbb{F}_2) = 0$ and $B(\mathbb{F}_3) = \mathbb{Z}$ are pathological, we will tacitly assume that $|F| \geq 4$ in this section. Theorem 5.2 below implies that if $q > 3$ is odd then $B(\mathbb{F}_q)$ is cyclic of order $(q+1)/2$, while if $q > 3$ is even then $B(\mathbb{F}_q)$ is cyclic of order $q + 1$. This is easy to check for small values of q; see Ex. 5.3.

VI.5.1.2 **Remark 5.1.2.** The group $\mathcal{P}(F)$ is closely related to the *scissors congruence* group for polyhedra in hyperbolic 3-space $\mathcal{H}^3$ with vertices in $\mathcal{H}^3$ or $\partial\mathcal{H}^3$, and has its origins in Hilbert's Third Problem. It was first studied for $\mathbb{C}$ by Wigner, Bloch and Thurston and later by Dupont and Sah; see [51, 4.10].

For any finite cyclic abelian group A of even order m, there is a unique nontrivial extension $\tilde{A}$ of A by $\mathbb{Z}/2$. If A is cyclic of odd order, we set $\tilde{A} = A$. Since the group $\mu(F)$ of roots of unity is a union of finite cyclic groups, we may define $\tilde{\mu}(F)$ as the union of the $\tilde{\mu}_n(F)$. Here is the main result of this section.

VI.5.2 **Theorem 5.2.** *(Suslin) For any field F with $|F| \geq 4$, there is an exact sequence*

$$0 \to \tilde{\mu}(F) \to K_3^{ind}(F) \to B(F) \to 0.$$

The proof is taken from [187, 5.2], and will be given at the end of this section. To prepare for the proof, we introduce the element c in Lemma 5.4 and construct a map $\psi : H_3(GL(F), \mathbb{Z}) \to B(F)$ in Theorem 5.7. In Theorem 5.16 we connect ψ to the group M of monomials matrices, and the group $\tilde{\mu}(F)$ appears in 5.20 as part of the calculation of $\pi_3(BM^+)$. The proof of Theorem 5.2 is obtained by collating all this information.

VI.5.2.1 **Remark 5.2.1.** In fact, $B(\mathbb{Q}) \cong \mathbb{Z}/6$. This follows from Theorem 5.2 and the calculations that $K_3(\mathbb{Q}) \cong \mathbb{Z}/48$ (2.1.2), $K_3^M(\mathbb{Q}) \cong \mathbb{Z}/2$ (III.7.2.d) and $\tilde{\mu}(\mathbb{Q}) \cong \mathbb{Z}/4$. In fact, the element $c = [2] + [-1]$ has order exactly 6 in both $B(\mathbb{Q})$ and $B(\mathbb{R})$. This may be proven using the Rogers L-function, which is built from the dilogarithm function. See [187, pp. 219–220].

As an application, we compute K_3 of a number field F. Let r_1 and r_2 denote the number of real and complex embeddings, *i.e.,* the number of factors of $\mathbb{R}$ and $\mathbb{C}$ in the $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R}$. Then $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$ by III.7.2(d), and $K_3(F)$ is finitely generated by IV.6.9 and V.6.8. By Borel's Theorem IV.1.18, $K_3(F)$ is the sum of $\mathbb{Z}^{r_2}$ and a finite group. We can make this precise.

VI.5.3 **Corollary 5.3.** *Let F be a number field, with r_1 real embeddings and r_2 complex embeddings, and set $w = w_2(F)$. Then $K_3^{ind}(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$, and:*

(a) If F is totally imaginary then $K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w$;

(b) If F has $r_1 > 0$ embeddings into $\mathbb{R}$ then

$$K_3(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/(2w) \oplus (\mathbb{Z}/2)^{r_1-1}.$$

Proof. By 4.3.1 and Proposition 4.3.2, there is an exact sequence

$$0 \to K_3^M(F) \to K_3(F) \to H^1(F, \mathbb{Z}(2)) \to 0.$$

Therefore $H^1(F, \mathbb{Z}(2)) \cong K_3^{\text{ind}}(F)$ is the direct sum of $\mathbb{Z}^{r_2}$ and a finite group, say of order m'. Choose m divisible by m' and w. Because $H^0(F, \mathbb{Z}(2))$ is divisible by Ex. 4.6(a), the map $H^0(F, \mu_m^{\otimes 2}) \to H^1(F, \mathbb{Z}(2))_{\text{tors}}$ is an isomorphism. But $H^0(F, \mu_m^{\otimes 2}) \cong \mathbb{Z}/w$, so $K_3^{\text{ind}}(F) \cong \mathbb{Z}/w$. This establishes the result when F is totally imaginary, since in that case $K_3^M(F) = 0$.

When $F = \mathbb{Q}$ then $w = 24$ and $K_3(\mathbb{Q}) \cong \mathbb{Z}/48$ is a nontrivial extension of $\mathbb{Z}/w$ by $\mathbb{Z}/2$; $K_3(\mathbb{Q})$ embeds in $K_3(\mathbb{R})$ by Theorem 3.1 and 3.1.2. When F has a real embedding, it follows that $K_3(\mathbb{Q}) \subseteq K_3(F)$ so $\{-1,-1,-1\}$ is a nonzero element of $2K_3(F)$. Hence the extension is nontrivial, as claimed. □

VI.5.3.1 **Rigidity Conjecture 5.3.1.** *(Suslin [185, 5.4]) Let F_0 denote the algebraic closure of the prime field in F. The Rigidity Conjecture states that $K_3^{ind}(F_0) \to K_3^{ind}(F)$ is an isomorphism. If $\text{char}(F) > 0$ then $K_3^{ind}(F_0)$ is $\mathbb{Z}/w_2(F)$; if $\text{char}(F) = 0$, $K_3^{ind}(F_0)$ is given by Corollary 5.3.*

The element c of $B(F)$

The elements $c = [x] + [1-x]$ and $\langle x \rangle = [x] + [x^{-1}]$ of $B(F)$ play an important role, as illustrated by the following calculations.

VI.5.4 **Lemma 5.4.** *Assuming that $|F| \geq 4$,*

(a) $c = [x] + [1-x]$ is independent of the choice of $x \in F - \{0,1\}$.

(b) For each x in $F - \{0,1\}$, $2\langle x \rangle = 0$.

(c) There is a homomorphism $F^\times \to B(F)$ sending x to $\langle x \rangle$.

(d) $3c = \langle -1 \rangle$ and hence $6c = 0$ in $B(F)$.

Proof. Given $x \neq y$ in $F - \{0,1\}$, we have the relations in $\mathcal{P}(F)$:

$$[1-y] - [1-x] + \left[\frac{1-x}{1-y}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + [y/x] = 0;$$
$$[x^{-1}] - [y^{-1}] + [x/y] - \left[\frac{1-x}{1-y}\right] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] = 0.$$

Subtracting the first from the relation in 5.1 yields $[x]+[1-x]-[y]-[1-y]=0$, whence (a) holds. Adding the second to the relation in 5.1 yields $\langle y\rangle - \langle x\rangle = \langle y/x\rangle$. Interchanging x and y, and using $\langle y/x\rangle = \langle x/y\rangle$, we obtain $2\langle y/x\rangle = 0$. Because $|F| \geq 4$, any $z \in F - \{0,1\}$ has the form $z = zx/x$ for $x \neq 1$ and hence $2\langle z\rangle = 0$. For (d), we compute using (b) and (c):

$$\begin{aligned} 3c =& [x] + [1-x] + [x^{-1}] + [1-x^{-1}] + [(1-x)^{-1}] + [1-(1-x)^{-1}] \\ =& \langle x\rangle + \langle 1-x\rangle + \langle 1-x^{-1}\rangle = \langle -(1-x)^2\rangle = \langle -1\rangle. \end{aligned}$$ □

VI.5.4.1 **Corollary 5.4.1.** *If* $\mathrm{char}(F) = 2$ *or* $\sqrt{-1} \in F$ *then* $3c = 0$ *in* $B(F)$*; if* $\mathrm{char}(F) = 3$ *or* $\sqrt[3]{-1} \in F$ *then* $2c = 0$ *in* $B(F)$.

The map $\psi : H_3(GL_2) \to B(F)$

We will now construct a canonical map $H_3(GL_2(F), \mathbb{Z}) \to B(F)$; see Theorem 5.7. To do this, we use the group hyperhomology of $GL_2(F)$ with coefficients in the chain complex arising from the following construction (for a suitable X).

VI.5.5 **Definition 5.5.** If X is any set, let $C_*(X)$ denote the "configuration" chain complex in which C_n is the free abelian group on the set of $(n+1)$-tuples $(x_0, \dots, x_n)$ of distinct points in X, with differential

$$d(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n).$$

There is a natural augmentation $C_0(X) \to \mathbb{Z}$ sending each (x) to 1.

If a group G acts on X, then $C_*(X)$ is a complex of G-modules, and we may form its hyperhomology $\mathbb{H}_n(G, C_*(X))$; see [223, 6.1.15]. There is a canonical map $\mathbb{H}_n(G, C_*(X)) \to H_n(C_G)$, where C_G denotes $C_*(X) \otimes_G \mathbb{Z}$.

VI.5.5.1 **Lemma 5.5.1.** *If* X *is infinite then* $C_*(X) \to \mathbb{Z}$ *is a quasi-isomorphism.*

Proof. If X_0 is a proper subset of X and $z \in X - X_0$ then $s_n(x_0, \dots) = (z, x_0, \dots)$ defines a chain homotopy $s_n : C_n(X_0) \to C_{n+1}(X)$ from the inclusion $C_*(X_0) \to C_*(X)$ to the projection $C_*(X_0) \to \mathbb{Z} \to C_*(X)$, where the last map sends 1 to (z). □

If X is finite and $|X| > n+1$, we still have $H_n C_*(X) = 0$, by Exercise 5.1.

VI.5.5.2 **Corollary 5.5.2.** *If a group* G *acts on* X*, and* X *is infinite (or* $|X| > n+1$*), then* $\mathbb{H}_n(G, C_*(X)) \cong H_n(G, \mathbb{Z})$.

The case of most interest to us is the action of the group $G = GL_2(F)$ on $X = \mathbb{P}^1(F)$. If $n \leq 2$ then G acts transitively on the basis of $C_n(X)$, and $C_n(X) \otimes_G \mathbb{Z}$ is an induced module from the stabilizer subgroup G_x of an element x. By Shapiro's Lemma [223, 6.3.2] we have $H_q(C_n(X) \otimes_G \mathbb{Z}) = H_q(G_x, \mathbb{Z})$.

VI.5.6 **Lemma 5.6.** *$H_0(C_G) = \mathbb{Z}$, $H_n(C_G) = 0$ for $n = 1, 2$ and $H_3(C_G) \cong \mathcal{P}(F)$.*

Proof. By right exactness of $\otimes_G$, we have $H_0(C_*(X) \otimes_G \mathbb{Z}) = \mathbb{Z}$. The differential from $C_2 \otimes_G \mathbb{Z} \cong \mathbb{Z}$ to $C_1 \otimes_G \mathbb{Z} \cong \mathbb{Z}$ is an isomorphism, since $d(0, 1, \infty) \equiv (0, 1)$. For $n = 3$ we write $[x]$ for $(0, \infty, 1, x)$; C_3 is a free $\mathbb{Z}[G]$-module on the set $\{[x] : x \in F - \{0, 1\}\}$. Similarly, C_4 is a free $\mathbb{Z}[G]$-module of the set of all 5-tuples $(0, \infty, 1, x, y)$ and we have

$$\begin{aligned} d(0, \infty, 1, x, y) &= (\infty, 1, x, y) - (0, 1, x, y) + (0, \infty, x, y) - (0, \infty, 1, y) + (0, \infty, 1, x) \\ &= \left[\frac{1-x}{1-y}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + [y/x] - [y] + [x]. \end{aligned}$$

Thus the cokernel $H_3(C_G)$ of $d : C_4 \otimes_G \mathbb{Z} \to C_3 \otimes_G \mathbb{Z}$ is $\mathcal{P}(F)$. □

VI.5.6.1 **Remark 5.6.1.** The proof of Lemma 5.6 goes through for all finite fields, since $|\mathbb{P}^1(F)| \geq |\mathbb{P}^1(\mathbb{F}_2)| = 7$. Hence $H_n(C_*(X)) = 0$ for $n \leq 3$ by Exercise 5.1.

Let T_2 denote the diagonal subgroup (isomorphic to $F^\times \times F^\times$) of $GL_2(F)$; the semidirect product $T_2 \rtimes \Sigma_2$ is the subgroup of M_2 of monomial matrices in $GL_2(F)$ (matrices with only one nonzero term in every row and column).

VI.5.7 **Theorem 5.7.** *For all F, $H_2(GL_2(F), \mathbb{Z}) = F^\times$, and there is a map ψ and an exact sequence*

$$H_3(M_2, \mathbb{Z}) \to H_3(GL_2(F), \mathbb{Z}) \xrightarrow{\psi} B(F) \to 0.$$

To prove Theorem 5.7, we consider the hyperhomology spectral sequence

$$E^1_{p,q} = H_q(G, C_p(X)) \Rightarrow \mathbb{H}_{p+q}(G, C_*(X)) \cong H_{p+q}(G, \mathbb{Z}) \qquad (5.8)$$ VI.5.8

with $G = GL_2(F)$; see [223, 6.1.15]. By Lemma 5.6, the edge map $H_3(G, \mathbb{Z}) \to E^\infty_{3,0}$ lands in a subset of $E^1_{3,0} = H_3(C_G) \cong \mathcal{P}(F)$, which we must show is $B(F)$.

It is not hard to determine all 10 nonzero terms of total degree at most 4 in (5.8). Indeed, the stabilizer of $0 \in X$ is the group B of upper triangular matrices, so $E^1_{0,q} = H_q(G, C_0) = H_q(B, \mathbb{Z})$; the stabilizer of $(0, \infty) \in X^2$ is the diagonal subgroup $T_2 = F^\times \times F^\times$, so $E^1_{1,q} = H_q(G, C_1) = H_q(T_2, \mathbb{Z})$, and the stabilizer of $(0, \infty, 1) \in X^3$ is the subgroup $\Delta = \{(a, a^{-1})\}$ of T_2, isomorphic to $F^\times$, so $E^1_{2,q} = H_q(\Delta, \mathbb{Z})$. By [184, §3], the inclusion $T_2 \subset B$ induces an isomorphism on homology.

It is not hard to see that the differential $d^1 : H_q(\Delta) \to H_q(T_2)$ is induced by the inclusion $\Delta \subset T_2$. Since the inclusion is split (by projection onto the first component of T_2), the map $d_1 : H_q(\Delta) \to H_q(T_2)$ is a split injection, and hence $E^2_{2,q} = 0$ for all q. The following lemma is proven in Exercise 5.2.

VI.5.8.1 **Lemma 5.8.1.** *Let $\sigma : T_2 \to T_2$ be the involution $\sigma(a,b) = (b,a)$. Then the differential $d^1 : H_q(T_2) \to H_q(B) \cong H_q(T_2)$ is induced by $1-\sigma$.*

Thus $E^2_{0,q} = H_q(T_2)_\sigma$ and $E^2_{1,q} = H_q(T_2)^\sigma/H_q(F^\times)$.

Proof of Theorem 5.7. ([187, Thm. 2.1]) By Lemma 5.8.1, the row $q = 1$ in (5.8) has $E^2_{0,1} = (T_2)_\sigma = F^\times$ and $E^2_{1,1} = 0$ (because $T_2^\sigma = F^\times$). Writing $(H_n)_\sigma$ for $H_n(T_2,\mathbb{Z})_\sigma$, the low degree terms of the E^2 page are depicted in Figure 5.8.2.

$(H_3)_\sigma$			
$(H_2)_\sigma$	$(H_2)^\sigma$	0	
$F^\times$	0	0	$E^2_{3,1}$
$\mathbb{Z}$	0	0	$\mathcal{P}(F)$

Figure 5.8.2: The E^2 page of (5.8). VI.5.8.2

By the Künneth formula [223, 6.1.13], $H_n(T_2) \cong \oplus_{i+j=n} H_i(F^\times) \otimes H_j(F^\times)$, with σ interchanging the factors; if $x, y \in H_i(F^\times)$ then $\sigma(x \otimes y) = y \otimes x = (-1)^i x \otimes y$. Since $H_2(F^\times) = \wedge^2 F^\times$, the group $H_2(T_2)_\sigma$ is the direct sum of $\wedge^2 F^\times$ and $\tilde{\wedge}^2(F^\times)$. A routine but tedious calculation shows that the differential $d^3 : \mathcal{P}(F) \to H_2(T_2)_\sigma$ is the canonical map $\mathcal{P}(F) \to \tilde{\wedge}^2(F^\times)$ of 5.1 followed by the split inclusion of $\tilde{\wedge}^2(F^\times)$ into $H_2(T_2)_\sigma$; see [187, 2.4]. Thus the cokernel of d^3 is $E^3_{0,2} \cong \wedge^2 F^\times \oplus K_2(F)$. In particular, we have $E^3_{3,0} = B(F)$ and $H_2(GL_2(F),\mathbb{Z}) \cong E^3_{0,2} \cong \wedge^2 F^\times \oplus K_2(F)$.

Let K denote the kernel of the edge map $H_3(GL_2(F),\mathbb{Z}) \to B(F)$. From (5.8.2) we see that K is an extension of a quotient Q_2 of $H_2(T_2)^\sigma$ by a quotient Q_3 of $H_3(T_2)_\sigma$. Moreover, $H_3(GL_2(F),\mathbb{Z})$ is an extension of $B(F)$ by K.

Recall that $M_2 \cong T_2 \rtimes \Sigma_2$. Since $H_p(\Sigma_2, T_2) = 0$ for $p \neq 0$, the Hochschild-Serre spectral sequence $H_p(\Sigma_2, H_q T_2) \Rightarrow H_{p+q}(M_2)$ degenerates enough to show that the cokernel of $H_3(T_2) \oplus H_3(\Sigma_2) \to H_3(M_2)$ is a quotient of $H_2(T_2)_\sigma$. Analyzing the subquotient Q_2 in (5.8.2), Suslin showed in [187, p. 223] that K is the image of $H_3(M_2,\mathbb{Z}) \to H_3(GL_2(F),\mathbb{Z})$. The result follows. □

In order to extend the map $H_3(GL_2,\mathbb{Z}) \xrightarrow{\psi} B(F)$ of Theorem 5.7 to a map $\psi : H_3(GL_3,\mathbb{Z}) \to \mathcal{P}(F)$, we need a small digression.

A cyclic homology construction

Recall that under the Dold-Kan correspondence [223, 8.4], a nonnegative chain complex C_* (*i.e.*, one with $C_n = 0$ if $n < 0$) corresponds to to a simplicial abelian group $\{\widetilde{C}_n\}$. Conversely, given a simplicial abelian group $\{\widetilde{C}_n\}$, C_* is the associated reduced chain complex.

For example, the chain complex $C_*(X)$ of Definition 5.5 corresponds to a simplicial abelian group; $\widetilde{C}_n(X)$ is the free abelian group on the set X^{n+1} of all $(n+1)$-tuples $(x_0,\dots,x_n)$ in X, including duplication. In fact, $\widetilde{C}_n(X)$ is a cyclic abelian group in the following sense. (These assertions are relegated to Ex. 5.6.)

VI.5.9 **Definition 5.9.** ([223, 9.6]) A *cyclic abelian group* is a simplicial abelian group $\{\widetilde{C}_n\}$ together with an automorphism t_n of each $\widetilde{C}_n$ satisfying: $t_n^{n+1} = 1$;

$$\partial_i t_n = t_n \partial_{i-1} \text{ and } \sigma_i t_n = t_n \sigma_i \text{ for } i \neq 0; \quad \partial_0 t_n = \partial_n \text{ and } \sigma_0 t_n = t_{n+1}^2 \sigma_n.$$

The associated *acyclic complex* (C_*^a, d^a) is the complex obtained from the reduced complex C_* by omitting the last face operator; C_*^a is acyclic, and there is a chain map $N : C_* \to C_*^a$ defined by $N = \sum_{i=0}^n (-1)^i t_n^i$ on C_n. The mapping cone of N has $C_{n-1} \oplus C_n^a$ in degree n, and $(b, c) \mapsto b$ defines a natural quasi-isomorphism $\text{cone}(N)[1] \xrightarrow{\simeq} C_*$ (see [223, 1.5]). In fact, cone$(N)[1]$ is a reduced form of two columns of Tsygan's double complex [223, 9.6.6].

Since $N_0 : C_0 \to C_0^a$ is the natural identification isomorphism, we may truncate the zero terms to get a morphism of chain complexes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C_1 & \xleftarrow{d} & C_2 & \xleftarrow{d} & \cdots & \xleftarrow{d} & C_n & \xleftarrow{d} \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \downarrow N & \\ 0 & \longleftarrow & C_1^a & \xleftarrow{d^a} & C_2^a & \xleftarrow{d^a} & \cdots & \xleftarrow{d^a} & C_n^a & \xleftarrow{d^a} \end{array} \quad . \tag{5.9.1}$$ VI.5.9.1

We write D_* for the the associated mapping cone of this morphism. Thus, $D_0 = C_1^a$, and $D_n = C_{n+1}^a \oplus C_n$ for $n > 0$ with differential $(x, y) \mapsto (d^a x - Ny, -dy)$. Then $\text{cone}(N)[1] \to D_*$ is a quasi-isomorphism.

VI.5.9.2 **Example 5.9.1.** When $X = \mathbb{P}^2(F)$, let C_n denote the subgroup of $C_n(X)$ generated by the $(n+1)$-tuples of points $(x_0, \dots, x_n)$ for which no three x_i are collinear. Since C_n is closed under the operator t_n, the associated simplicial abelian subgroup $\{\widetilde{C}_n\}$ of $\{\widetilde{C}_n(X)\}$ has the structure of a cyclic abelian subgroup. The proof of Lemma 5.5.1 goes through to show that if X is infinite then $C_* \to \mathbb{Z}$ and hence $C_* \to C_*(X)$ are quasi-isomorphisms. It follows that the map $\varepsilon : D_0 \to \mathbb{Z}$ sending (x, y) to 1 induces a quasi-isomorphism $D_* \to \mathbb{Z}$.

Under the canonical action of the group $GL_3 = GL_3(F)$ on $X = \mathbb{P}^2(F)$, GL_3 sends the subcomplex C_* of $C_*(X)$ to itself, so GL_3 acts on C_* and D_*.

The map $\psi : H_3(GL_3) \to B(F)$

We shall now construct a map $H_3(GL_3, \mathbb{Z}) \xrightarrow{\psi} \mathcal{P}(F)$ whose image is $B(F)$. We will relate it to the map of Theorem 5.7 in Lemma 5.10.

Let C_* be the subcomplex of Example 5.9.1 for $X = \mathbb{P}^2(F)$. By 5.5 and Lemma 5.5.1, the hyperhomology $\mathbb{H}_n(GL_3, C_*)$ is just $H_n(GL_3, \mathbb{Z})$ when F is infinite (or $|F| > n+1$), and there is a canonical map from $H_n(GL_3, \mathbb{Z}) = \mathbb{H}_n(GL_3, D_*)$ to $H_n(D_G)$, where D_G denotes $D_* \otimes_{GL_3} \mathbb{Z}$.

The four points $p_1 = (1:0:0)$, $p_2 = (0:1:0)$, $p_3 = (0:0:1)$ and $q = (1:1:1)$ play a useful role in any analysis of the action of GL_3 on $\mathbb{P}^2(F)$.

For example, the $\mathbb{Z}[GL_3]$-module C_3 is generated by $P = (p_1, p_2, q, p_3)$. Just as in Lemma VI.5.6 5.6, $C_n \otimes_G \mathbb{Z} = \mathbb{Z}$ for $n \leq 3$, while C_4 and C_5 are free $\mathbb{Z}[G]$-modules on the set of all 5-tuples and 6-tuples

$$\begin{bmatrix} a \\ x \end{bmatrix} := \quad (p_1, p_2, q, (1 : a : x), p_3)$$

$$\begin{bmatrix} a\ b \\ x\ y \end{bmatrix} := \quad (p_1, p_2, q, (1 : a : x), (1 : b, y), p_3)\,,$$

where $a \neq x$, $b \neq y$, $a \neq b$, $x \neq y$ and $ay \neq bx$. By inspection of D_G, $d(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = P$, $d^a(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = N(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = 0$, and $d : (D_G)_3 \to (D_G)_2$ is zero. Thus the terms $n \leq 4$ of the complex D_G have the form:

$$\oplus \mathbb{Z} \begin{bmatrix} a \\ x \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} a\ b \\ x\ y \end{bmatrix} \xrightarrow{\begin{pmatrix} -1 & 0 \\ -N & d^a \end{pmatrix}} \mathbb{Z}\, P \oplus \bigoplus \mathbb{Z} \begin{bmatrix} a \\ x \end{bmatrix} \xrightarrow{0} \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \to 0.$$

The map $\psi' : (D_G)_3 \to \mathcal{P}(F)$ defined by $\psi'(P) = 2c$, $\psi'(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = a$ vanishes on the image of $(D_G)_4$, by a straightforward calculation done in Su91 [187, 3.3], so it induces a homomorphism $\psi : H_3(GL_3, \mathbb{Z}) \to H_3(D_G) \to \mathcal{P}(F)$.

VI.5.10 **Lemma 5.10.** *The composition $H_3(GL_2, \mathbb{Z}) \to H_3(GL_3, \mathbb{Z}) \xrightarrow{\psi} \mathcal{P}(F)$ is the homomorphism $H_3(GL_2, \mathbb{Z}) \to B(F) \subset \mathcal{P}(F)$ of Theorem VI.5.7 5.7.*

Proof. (Su91 [187, 3.4]) The subgroup $GL_2 = GL_2(F)$ also acts on $X = \mathbb{P}^2(F)$ and fixes the origin $p_3 = (0 : 0 : 1)$, so it acts on $X_0 = X - \{p_3\}$. The maps $f_n : C_n(X_0) \to C_{n+1}(X) = C^a_{n+1}(X) \subset D_n$ sending $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, p_3)$ satisfy $fd = d^a f$, so they form a GL_2-equivariant chain map $f : C_*(X_0) \to C^a_*(X)/C_0(X)[1]$.

If C'_n denotes the subgroup $f_n^{-1}(C_{n+1})$ of $C_n(X_0)$, then the restriction of f defines a GL_2-equivariant chain map $C'_* \to C^a_*/C^a_0[1] \subset D_*$. Therefore the composition of Lemma VI.5.10 5.10 factors as

$$H_3(GL_2, \mathbb{Z}) \to H_3(C'_{*GL_2}) \xrightarrow{f} H_4(C^a_G) \to H_3(D_G) \xrightarrow{\psi'} \mathcal{P}(F).$$

The projection from $\mathbb{P}^2(F) - \{p_3\}$ to $\mathbb{P}^1(F)$ with center p_3 is GL_2-equivariant and determines a homomorphism from C'_* to $C^1_* = C_*(\mathbb{P}^1(F))$ over $\mathbb{Z}$. By inspection, the composition $H_3(GL_2) \to H_3(C'_{*GL_2}) \to H_3(C^1_{*GL_2}) \cong \mathcal{P}(F)$ is the inclusion of Theorem VI.5.7 5.7. $\square$

VI.5.11 **Lemma 5.11.** *Let T_3 denote the diagonal subgroup $F^\times \times F^\times \times F^\times$ of $GL_3(F)$. If F is infinite, $H_n(T_3, \mathbb{Z}) \to H_n(D_{*T_3})$ is zero for $n > 0$.*

Proof. Since $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 0 : 1)$ are fixed by T_3, the augmentation $D_0 \to \mathbb{Z}$ has a T_3-equivariant section sending 1 to (p_1, p_2). Therefore $D_* \cong \mathbb{Z} \oplus D'_*$ as T_3-modules, and D'_* is acyclic. Therefore if $n > 0$ we have $\mathbb{H}_n(T_3, D'_*) = 0$. Since $H_n(D_{*T_3}) = H_n(D'_{*T_3})$ for $n > 0$, the map $H_n(T_3, \mathbb{Z}) \to H_n(D_{*T_3})$ factors through $\mathbb{H}_n(T_3, D'_*) = 0$. $\square$

VI.5.12 **Proposition 5.12.** *The image of ψ is $B(F)$, and there is an exact sequence*

$$H_3(M_2,\mathbb{Z}) \oplus H_3(T_3,\mathbb{Z}) \to H_3(GL_3(F),\mathbb{Z}) \xrightarrow{\psi} B(F) \to 0.$$

Proof. By [183, 3.4], $H_3(T_3,\mathbb{Z})$ and $H_3(GL_2,\mathbb{Z})$ generate $H_3(GL_3,\mathbb{Z})$. The restriction of ψ to $H_3(T_3,\mathbb{Z})$ is zero by Lemma 5.11, since it factors as

$$H_3(T_3,\mathbb{Z}) \to H_3(D_{T_3}) \to H_3(D_G) \xrightarrow{\psi'} \mathcal{P}(F).$$

The proposition now follows from Theorem 5.7 and Lemma 5.10. □

VI.5.12.1 **Remark 5.12.1.** The map ψ extends to a map defined on $H_3(GL(F),\mathbb{Z})$, because of the stability result $H_3(GL_3(F),\mathbb{Z}) \cong H_3(GL(F),\mathbb{Z})$; see IV.1.15.

We now consider the image of $H_3(\Sigma_\infty,\mathbb{Z})$ in $B(F)$, where we regard Σ_∞ as the subgroup of permutation matrices in $GL(F)$, i.e., as the direct limit of the permutation embeddings $\iota_n : \Sigma_n \subset GL_n(F)$. We will use the following trick.

Let S be the p-Sylow subgroup of a finite group G. Then the transfer-corestriction composition $H_n(G) \longrightarrow H_n(S) \longrightarrow H_n(G)$ is multiplication by $[G:S]$, which is prime to p. Therefore the p-primary torsion in $H_n(G,\mathbb{Z})$ is the image of $H_n(S,\mathbb{Z})$ when $n>0$.

VI.5.13 **Proposition 5.13.** *The image of $H_3(\Sigma_\infty,\mathbb{Z}) \xrightarrow{\iota_*} H_3(GL(F),\mathbb{Z}) \xrightarrow{\psi} B(F)$ is the subgroup generated by $2c$, which is trivial or cyclic of order 3.*

Proof. We saw in Ex. IV.1.13 that $H_3(\Sigma_\infty,\mathbb{Z}) \cong \mathbb{Z}/12 \oplus (\mathbb{Z}/2)^2$, so the image of $H_3(\Sigma_\infty,\mathbb{Z}) \to H_3(GL(F),\mathbb{Z})$ has at most 2- and 3-primary torsion. Nakaoka proved in [141] that $H_3(\Sigma_6,\mathbb{Z}) \cong H_3(\Sigma_\infty,\mathbb{Z})$.

Using the Sylow 2-subgroup of Σ_6, Suslin proves [183, 4.4.1] that the 2-primary subgroup of $H_3(\Sigma_6,\mathbb{Z})$ maps to zero in $B(F)$. We omit the details.

Now the 3-primary component of $H_3(\Sigma_6,\mathbb{Z})$ is the image of $H_3(S)$, where S is a Sylow 3-subgroup of Σ_6. We may take $S = A_3 \times \sigma A_3 \sigma^{-1}$, generated by the 3-cycles (123) and (456), where $\sigma = (14)(25)(36)$. Since $H_2(A_3,\mathbb{Z}) = 0$, the Künneth formula yields $H_3(S,\mathbb{Z}) = H_3(A_3,\mathbb{Z}) \oplus H_3(\sigma A_3\sigma^{-1},\mathbb{Z})$. By [223, 6.7.8], these two summands have the same image in $H_3(\Sigma_6,\mathbb{Z})$. By Lemma 5.14 below, the 3-primary component of the image of $H_3(S,\mathbb{Z})$ in $B(F)$ is generated by $2c$, as desired. □

We are reduced to the alternating group A_3, which is cyclic of order 3, embedded in $GL_3(F)$ as the subgroup of even permutation matrices. We need to analyze the homomorphism $H_3(A_3,\mathbb{Z}) \to H_3(GL_3(F),\mathbb{Z}) \xrightarrow{\psi} B(F)$.

VI.5.14 **Lemma 5.14.** *If $|F| \geq 4$, the image of $\mathbb{Z}/3 \cong H_3(A_3)$ in $B(F)$ is generated by $2c$.*

Proof. If $\mathrm{char}(F) = 3$ then the permutation representation of A_3 is conjugate to an upper triangular representation, so $H_*(A_3,\mathbb{Z}) \to H_*(GL_3(F),\mathbb{Z})$ is trivial (see [181]). Since $2c = 0$ by Corollary 5.4.1, the result follows. Thus we may assume that $\mathrm{char}(F) \neq 3$.

When $\mathrm{char}(F) \neq 3$, the permutation representation is conjugate to the representation $A_3 \to GL_2(F)$ with generator $\lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. For example, if $\mathrm{char}(F) = 2$, this representation identifies A_3 with a subgroup of $\Sigma_3 \cong GL_2(\mathbb{F}_2)$. By the Comparison Theorem [223, 2.2.6], there is a morphism between $\mathbb{Z}[A_3]$-module resolutions of $\mathbb{Z}$, from the standard periodic free resolution (see [223, 6.2.1]) to C_*:

$$\begin{array}{ccccccccc}
\mathbb{Z}[A_3] & \xleftarrow{1-\lambda} & \mathbb{Z}[A_3] & \xleftarrow{1+\lambda+\lambda^2} & \mathbb{Z}[A_3] & \xleftarrow{1-\lambda} & \mathbb{Z}[A_3] & \xleftarrow{1+\lambda+\lambda^2} & \cdots \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
C_0 & \xleftarrow{d} & C_1 & \xleftarrow{d} & C_2 & \xleftarrow{d} & C_3 & \xleftarrow{d} & \cdots .
\end{array} \tag{5.14.1}$$ VI.5.14.1

We can build the morphisms f_n by induction on n, starting with $f_0(1) = (0)$, $f_1(1) = (\infty, 0)$ and $f_2(1) = (\infty, 0, x) + (1, \infty, x) = (0, 1, x)$ for any $x \in F$ $(x \neq 0, 1)$. If we set $w = 1 - 1/x$ and choose $y \neq \infty, 0, 1, x, w$ (which is possible when $|F| \geq 4$) then we may also take

$$\begin{aligned} f_3(1) = &-(\infty, 0, x, y) - (1, \infty, x, y) - (0, 1, x, y) \\ &+ (1, \infty, w, y) + (0, 1, w, y) + (\infty, 0, w, y). \end{aligned}$$

Here we have regarded F as embedded in $\mathbb{P}^1(F)$ via $x \mapsto \left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right)$. Taking coinvariants in (5.14.1), the generator 1 of $H_3(A_3)$ in the periodic complex maps to $f_3(1)$, representing an element of $H_3(C_{*G})$. Applying ψ sends this element to

$$-\left[\frac{x}{y}\right] - \left[\frac{y-1}{x-1}\right] - \left[\frac{y(x-1)}{x(y-1)}\right] + [x(1-y)] + \left[\frac{1-x}{xy}\right] + \left[\frac{y}{(1-x)(1-y)}\right]$$

in $B(F)$. As $|F| \geq 4$, we can take $x \neq -1$ and $y = x^{-1}$, so this expression becomes $-[x^2] - 2[-x^2] + 2[x-1] + [1/x^2]$. This equals $-2c$, by Exercise 5.5. $\square$

Monomial matrices

By definition, a *monomial matrix* in $GL_n(F)$ is one which has only one nonzero entry in each row and column. We write M_n for the group of all monomial matrices in $GL_n(F)$ and M for the union of the M_n in $GL(F)$; M_n is isomorphic to the wreath product $F^\times \wr \Sigma_n = (F^\times)^n \rtimes \Sigma_n$ and $M \cong F^\times \wr \Sigma_\infty$. We encountered M_2 in the proof of Theorem 5.7 and the subgroup $\mu(F) \wr \Sigma_n$ of M_n in Theorem 1.5.

VI.5.15 **Proposition 5.15.** *Let $\iota : \Sigma_\infty \to GL(F)$ be the inclusion. Then there is an exact sequence:*

$$H_3(M, \mathbb{Z}) \longrightarrow H_3(GL(F), \mathbb{Z}) \oplus H_3(\Sigma_\infty, \mathbb{Z}) \xrightarrow{(\psi, -\psi\iota_*)} B(F) \to 0.$$

Proof. (Cf. [187, 4.3]) Let $H_3^0 M_n$ denote the kernel of $H_3(M_n, \mathbb{Z}) \to H_3(\Sigma_n, \mathbb{Z})$; it suffices to show that the image of $H_3^0 M_n \to H_3(GL(F), \mathbb{Z})$ is the kernel of ψ for large n. The image contains the kernel of ψ because, by Proposition 5.12, the kernel of ψ comes from the image of $H_3(T_3, \mathbb{Z})$, which is in $H_3^0 M_n$, and $H_3(M_2, \mathbb{Z})$, which is in $H_3^0 M_n$ by Ex. 5.10(d) since the image of $H_3(\Sigma_2, \mathbb{Z})$ is in the image of $H_3([M, M], \mathbb{Z})$ by Ex. 5.12.

The group $M_n = T_n \rtimes \Sigma_n$ contains $M_2 \times \Sigma_{n-2}$ as a subgroup. Let S denote the group $\Sigma_2 \times \Sigma_{n-2}$, and write A for the kernel of the split surjection $H_3(M_2 \times \Sigma_{n-2}, \mathbb{Z}) \to H_3(S, \mathbb{Z})$. By Ex. 5.10(e), $H_3(T_3, \mathbb{Z}) \oplus A$ maps onto $H_3^0 M_n$.

By the Künneth formula for $M_2 \times \Sigma_{n-2}$, A is the direct sum of $H_3^0(M_2, \mathbb{Z})$, $H_2^0(M_2, \mathbb{Z}) \otimes H_1(\Sigma_{n-2}, \mathbb{Z})$ and $F^\times \otimes H_2(\Sigma_{n-2}, \mathbb{Z})$. Suslin proved in [183, 4.2] that the images of the latter two summands in $H_3(M_n)$ are contained in the image of $H_3(T_n, \mathbb{Z})$ for large n. It follows that the image of $H_3^0 M_n$ is the kernel of ψ. □

In Example IV.4.10.1, IV.4.6.1 and Ex. IV.1.27 we saw that the homotopy groups of $K(F^\times\text{-}\mathbf{Sets}_{\mathrm{fin}}) \simeq \mathbb{Z} \times BM^+$ form a graded-commutative ring with $\pi_1(BM^+) \cong F^\times \times \pi_1^s$. By Ex. IV.4.12, $K_*(F^\times\text{-}\mathbf{Sets}_{\mathrm{fin}}) \to K_*(F)$ is a ring homomorphism. By Matsumoto's Theorem III.6.1, it follows that $\pi_2(BM^+) \to K_2(F)$ is onto; in fact, $\pi_2(BM^+) = \pi_2^s \oplus \tilde{\wedge}^2 F^\times$ by Exercise 5.11. Multiplying by $\pi_1(BM^+)$, this implies that $K_3^M(F)$ lies in the image of $\pi_3(BM^+) \to K_3(F)$.

We saw in Theorem 5.7 that the kernel of the map $H_3(GL_2) \to B(F)$ is the image of $H_3(M_2)$. The monomial matrices M_3 in $GL_3(F)$ contain T_3, so Proposition 5.12 shows that the kernel of ψ is contained in the image of $H_3(M_3)$.

VI.5.16 **Theorem 5.16.** *([187, Lemma 5.4]) The cokernel of $\pi_3 BM^+ \to K_3(F)$ is $B(F)/(2c)$, and there is an exact sequence:*

$$\pi_3(BM^+) \longrightarrow K_3(F) \oplus \pi_3^s \xrightarrow{(\psi, -\psi\iota_*)} B(F) \to 0.$$

Proof. Write $H_n G$ for $H_n(G, \mathbb{Z})$, and let P denote the commutator subgroup $[M, M]$; we saw in IV, Ex. 1.27 that P is perfect. By IV.1.19–1.20, π_3^s maps onto $H_3 A_\infty$ and there is a commutative diagram

$$\begin{array}{ccccc}
\pi_2(BP^+) & \xrightarrow{\circ\eta} & \pi_3(BP^+) & \xrightarrow{h} & H_3 P \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
K_2(F) & \xrightarrow{[-1]} & K_3(F) & \xrightarrow{h} & H_3 SL(F) \to 0.
\end{array}$$

Since $\pi_n BP^+ \cong \pi_n BM^+$ for $n \geq 2$ (Ex. IV.1.8), and $\pi_2(BM^+) \to K_2(F)$ is onto (as we noted above), a diagram chase shows that $K_3(F)/\pi_3(BM^+) \cong H_3 SL(F)/H_3 P$.

Let SM denote the kernel of $\det : M \to F^\times$. We saw in Chapter IV, Ex. 1.27 that $BM^+ \simeq BP^+ \times B(F^\times) \times B\Sigma_2$; it follows that $BSM^+ \simeq BP^+ \times B\Sigma_2$. By the Künneth formula, $H_3 SM \cong H_3 P \oplus (H_2 P \otimes H_1 \Sigma_2) \oplus H_3 \Sigma_2$. Under the map $H_3 SM \to H_3 SL(F)$, the final term lands in the image of $H_3 P$ by Ex. 5.12, and the middle term factors through $H_2SL(F)\otimes H_1SL(F)$, which is zero as $SL(F)$ is perfect. Hence $H_3SL(F)/H_3P = H_3SL(F)/H_3SM$.

This explains the top half of the following diagram.

$$\begin{array}{ccccccc}
H_3(P) & \longrightarrow & H_3(SL) & \longrightarrow & K_3(F)/\pi_3(BM^+) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow\cong & & \\
H_3(SM) & \longrightarrow & H_3(SL) & \longrightarrow & H_3(SL)/H_3(SM) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow\cong & & \\
H_3(M) & \longrightarrow & H_3(GL) & \longrightarrow & B(F)/(2c) & \longrightarrow & 0.
\end{array}$$

Similarly, Ex. IV.1.8(a) implies that $H_nGL(F)\cong\oplus_{i+j=n}H_iSL(F)\otimes H_jF^\times$. There is a compatible splitting $H_nM\cong\oplus_{i+j=n}H_iSM\otimes H_jF^\times$ by Ex. IV.1.27. This implies that $H_nSL(F)/H_nSM\to H_nGL(F)/H_nM$ is injective for all n. For $n=3$, the summands $H_3F^\times$ and $H_2SL(F)\otimes F^\times$ of $H_3GL(F)$ are in the image of H_3SM (because $\pi_2BSM^+\to K_2(F)\cong H_2SL(F)$ is onto). Since $H_1SL(F)=0$, we conclude that $H_3SL(F)/H_3SM\to H_3GL(F)/H_3M$ is onto and hence an isomorphism.

Finally, combining Propositions 5.12 and 5.13 with Exercise 5.10 and Lemma 5.14, we see that $H_3SL(F)\to B(F)$ is onto, and the cokernel of $H_3M\to H_3GL(F)$ is $B(F)/(2c)$. Concatenating the isomorphisms yields the first assertion. The second assertion follows from this by the argument in the proof of Proposition 5.15. □

We define $\pi_3^{\text{ind}}(BM^+)$ to be the quotient of $\pi_3(BM^+)$ by all products from $\pi_1(BM^+)\otimes\pi_2(BM^+)$. There is a natural map $\pi_3^{\text{ind}}(BM^+)\to\pi_3^s/(\eta^3)\cong\mathbb{Z}/12$. Since the products map to $K_3^M(F)$ in $K_3(F)$, we have the following reformulation.

VI.5.16.1 **Corollary 5.16.1.** *The sequence of Theorem 5.16 induces an exact sequence*

$$\pi_3^{ind}(BM^+)\to K_3^{ind}(F)\oplus\mathbb{Z}/12\to B(F)\to 0.$$

Thus to prove Theorem 5.2, we need to study $\pi_3^{\text{ind}}(BM^+)$.

VI.5.16.2 **Remark 5.16.2.** The diagram in the proof of Theorem 5.16 shows that the map $\pi_3^{\text{ind}}(BM^+)\to K_3^{\text{ind}}(F)$ is a quotient of the map $H_3(P,\mathbb{Z})\to H_3(SL(R),\mathbb{Z})$.

If E is any homology theory and X any pointed topological space, the Atiyah-Hirzebruch spectral sequence converging to $E_*(X)$ has $E^2_{p,q}=H_p(X,E_q(*))$. For stable homotopy we have $E_*(X)=\pi_*^s(X)$. When $X=BG_+$, the Barratt-Priddy Theorem (IV.4.10.1) states that $\pi_*^s(BG_+)=\pi_*(\mathbb{Z}\times B(G\wr\Sigma_\infty)^+)$.

VI.5.17 **Proposition 5.17.** *There is an exact sequence*

$$0\to\mu_2(F)\to\pi_3^{ind}(BM^+)\xrightarrow{\gamma}\mu(F)\oplus\mathbb{Z}/12\to 0.$$

Proof. ([187, §5]) We analyze the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(F^\times, \pi^s_q) \Rightarrow \pi_{p+q}(\mathbb{Z} \times BM^+)$$

which has a module structure over the stable homotopy ring π^s_*. Note that the y-axis $E^2_{0,*} = \pi^s_*$ is a canonical summand of $\pi_*(\mathbb{Z} \times BM^+)$, so it survives to E^∞. By Ex. 5.11, $\pi_2 BM^+/\pi^s_2 \cong \tilde{\wedge}^2 F^\times$. It is easy to see that the map from $E^2_{1,1} = F^\times/F^{\times 2}$ to $E^\infty_{1,1} = \tilde{\wedge}^2 F^\times$ is injective, as it sends x to $x \otimes x$. It follows that the differential from $E^2_{3,0} = H_3(F^\times, \mathbb{Z})$ to $E^2_{1,1}$ is zero, so $E^\infty_{3,0} = H_3(F^\times, \mathbb{Z})$; see Figure 5.17.1.

π^s_3				
π^s_2	$F^\times/F^{\times 2}$			
π^s_1	$F^\times/F^{\times 2}$	$E^2_{2,1}$		
$\mathbb{Z}$	$F^\times$	$\wedge^2 F^\times$	$H_3(F^\times)$	$H_4(F^\times)$

Figure 5.17.1: The E^2 page converging to $\pi_*(\mathbb{Z} \times BM^+)$ VI.5.17.1

The universal coefficient sequence expresses $E^2_{2,1} = H_2(F^\times, \mathbb{Z}/2)$ as an extension:

$$0 \to (\wedge^2 F^\times)/2 \to H_2(F^\times, \mathbb{Z}/2) \to \mu_2(F) \to 0.$$

The differential $E^2_{4,0} \longrightarrow E^2_{2,1}$ lands in $(\wedge^2 F^\times)/2$ because the composite to $\mu_2(F)$ is zero: by naturality in F, it factors through the divisible group $H_4(\bar{F}^\times, \mathbb{Z})$.

The cokernel of the product maps from $F^\times \otimes \pi^s_2$ and $\wedge^2 F^\times \otimes \pi^s_1 \to H_3(F^\times, \mathbb{Z})$ is therefore the direct sum of $\pi^s_3/(\eta^3)$ and an extension of $H_3(F^\times, \mathbb{Z})$ by $\mu_2(F)$. Modding out by the products from $\wedge^2 F^\times \otimes F^\times$ replaces $H_3(F^\times, \mathbb{Z})$ by $H_3(F^\times, \mathbb{Z})/\wedge^3 F^\times$, which by Ex. 5.9 is isomorphic to $\mu(F)$. □

Since $H_3(P, \mathbb{Z}) \cong \pi_3(BM^+)/\eta \circ \pi_2(BM^+)$ (see IV.1.19), there is a natural surjection $H_3(P, \mathbb{Z}) \to \pi^{\text{ind}}_3(BM^+)$. Consider the homomorphism $\delta : \mu(F) \to P$ sending x to $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$. The composition with the map γ of Proposition 5.17 is:

$$\mu(F) \cong H_3(\mu(F), \mathbb{Z}) \xrightarrow{\delta} H_3(P, \mathbb{Z}) \xrightarrow{\text{onto}} \pi^{\text{ind}}_3(BM^+) \xrightarrow{\gamma} \mu(F).$$

VI.5.18 **Lemma 5.18.** *The composition $\mu(F) \xrightarrow{\delta} \pi^{ind}_3(BM^+) \xrightarrow{\gamma} \mu(F)$ sends x to x^2.*

Proof. Let us write μ for $\mu(F)$ and $H_n G$ for $H_n(G, \mathbb{Z})$. The homomorphism $\mu \xrightarrow{\delta} P$ factors through $D = \mu \rtimes \Sigma_2$ (where $\Sigma_2 \to A_\infty \subset P$ is given by (12)(34)), and the map $\mu \cong H_3(\mu) \to H_3(D)$ lands in the subgroup $H_3(\mu)_{\Sigma_2} \cong \mu/\{\pm 1\}$. By Exercise 5.13, the composition of $\mu \to \mu/\{\pm 1\}$ with $H_3(D) \to H_3(P) \xrightarrow{\gamma} \mu$ sends $x \in \mu$ to x^2. □

Recall that the e-invariant maps $K^{\text{ind}}_3(F)$ to $H^0(F, \mu^{\otimes 2}) \subset \mu(\bar{F})$ (see Definition 2.1). We use it to detect the image of $\mu(F)$ in $\pi^{\text{ind}}_3(BM^+)$.

VI.5.19 **Lemma 5.19.** *The following composition is an injection:*

$$\mu(F) \xrightarrow{\delta} \pi_3^{ind}(BM^+) \to K_3^{ind}(F) \xrightarrow{e} H^0(F, \mu^{\otimes 2}).$$

If F is algebraically closed it is an isomorphism.

Proof. If $F \subset F'$, $\mu(F) \subseteq \mu(F')$. Therefore we may enlarge F to assume that it is algebraically closed. In this case $H^0(F, \mu^{\otimes 2}) = \mu(F)$ and $K_3^M(F)$ is uniquely divisible (III.7.2), so it is a summand of $K_3(F)$ (by V.11.13) and therefore ${}_mK_3(F) \cong {}_mK_3^{\rm ind}(F)$.

Now consider the étale Chern class $c_{2,4} : K_4(F; \mathbb{Z}/m) \to H^0(F, \mu_m^{\otimes 2})$, where $1/m \in F$ and $m \not\equiv 2 \pmod 4$. Since $K_4(F; \mathbb{Z}/m) \cong \mathbb{Z}/m$ on generator β^2 (1.4) and $c_1(\beta) = \zeta$ by V.11.10.1, the product rule yields $c_2(\beta^2) = \zeta^{-1} \otimes \zeta$. Since the Bockstein is an isomorphism: $K_4(F; \mathbb{Z}/m) \cong {}_mK_3(F)$, this implies that the e-invariant is $-c_{2,4}$ on ${}_mK_3(F)$.

The isomorphism $K_4(F; \mathbb{Z}/m) \cong {}_mK_3^{\rm ind}(F)$ factors through the Hurewicz map, the map $H_4(SL(F), \mathbb{Z}/m) \to {}_mH_3(SL(F), \mathbb{Z})$ and the quotient map ${}_mH_3(SL(F), \mathbb{Z}) \to {}_mK_3^{\rm ind}(F)$ of IV.1.20. Therefore we have a commutative diagram:

$$\begin{array}{ccccccc}
H_4(\mu, \mathbb{Z}/m) & \xrightarrow{\delta} & H_4(P, \mathbb{Z}/m) & \longrightarrow & H_4(SL(F), \mathbb{Z}/m) & \xrightarrow{c_{2,4}} & H^0_{\rm et}(F, \mu_m^{\otimes 2}) \\
\cong\downarrow & & \downarrow & & \downarrow & & \| \\
\mu_m & \xrightarrow{\delta} & {}_m\pi_3^{\rm ind}(BM^+) & \longrightarrow & {}_mK_3^{\rm ind}(F) & \xrightarrow{-e} & \mu_m \otimes \mu_m .
\end{array}$$

Since the top composition is an isomorphism by Ex. V.11.5, the result follows. ∎

VI.5.20 **Corollary 5.20.** $\pi_3^{ind}(BM^+) \cong \tilde{\mu}(F) \oplus \mathbb{Z}/12$.

Proof. If $\mathrm{char}(F) = 2$, the result is immediate from Proposition 5.17, so we may suppose that $\mathrm{char}(F) \neq 2$. If $F \subset E$ then $\mu(F) \subseteq \mu(E)$ and therefore, by naturality of Proposition 5.17 in F, the map on $\pi_3(BM^+)$ groups is an injection. We may therefore assume that F is algebraically closed. In this case, it follows from 5.18 and 5.19 that $(\delta, i) : \mu(F) \oplus \mathbb{Z}/12 \to \pi_3(BM^+)$ is an isomorphism, because $\gamma\delta$ is onto and $\delta(-1)$ is a nonzero element of the kernel $\mu_2(F)$ of γ. ∎

Proof of Theorem 5.2. (Suslin) By Corollaries 5.16.1 and 5.20, the kernel of $K_3^{\rm ind}(F) \to B(F)$ is the image of $\tilde{\mu}(F)$. Thus it suffices to show that the summand $\tilde{\mu}(F)$ of $\pi_3(BM^+)$ (given by Corollary 5.20) injects into $K_3^{\rm ind}(F)$. When F is algebraically closed, this is given by Lemma 5.19. Since $\tilde{\mu}(F) \to \tilde{\mu}(E)$ is an injection for all field extensions $F \subset E$, the general case follows. ∎

EXERCISES

EVI.5.1 **5.1.** (Hutchinson) If $|X| = d$, $C_n(X) = 0$ for $n \geq d$ by 5.5. Show that $H_n(C_*(X)) = 0$ for all $n \neq 0, d-1$, and that $H_n(C'_*(X)) = 0$ for all $n \neq d-1$. (C'_* is defined in 5.10.) Conclude that if $|X| \geq 5$ then $H_3(G,\mathbb{Z}) \cong \mathbb{H}_3(G, C_*(X))$ and $H_4(G, C'_*(X)) = 0$.

EVI.5.2 **5.2.** ([187, Lemma 2.3])) In this exercise we prove Lemma 5.8.1. In the hypercohomology spectral sequence (5.8), the differential d^1 from $H_q(G, C_1)$ to $H_q(G, C_0) \cong H_q(B,\mathbb{Z}) \cong H_q(T_2,\mathbb{Z})$ is induced by the map $d^1_0 - d^1_1 : C_1 \to C_0$, where $d^1_0(x_0, x_1) = x_1$ and $d^1_1(x_0, x_1) = x_0$. Let T_2 be the diagonal subgroup, and B the upper triangular subgroup of $GL_2(F)$. We saw after (5.8) that the inclusion of T_2 and B induces isomorphisms $\iota : H_q(T_2,\mathbb{Z}) \xrightarrow{\cong} H_q(G, C_1)$ and $H_q(B,\mathbb{Z}) \xrightarrow{\cong} H_q(G, C_0)$. Show that:
(i) $d^1_0\, \iota = \sigma(d^1_1\, \iota)$, where σ is the involution $\sigma(a, b) = (b, a)$ of F^2;
(ii) The composition $d^1_0\, \iota$ is the natural map $H_q(T_2,\mathbb{Z}) \to H_q(B,\mathbb{Z}) \cong H_q(G, C_0)$;
(iii) The fact that $H_q(T_2,\mathbb{Z}) \to H_q(B,\mathbb{Z})$ is an isomorphism ([184, §3]) implies that $d^1 = 1 - \sigma$.

EVI.5.3 **5.3.** Show that $B(\mathbb{F}_5) \cong \mathbb{Z}/3$ on generator $c = [2] = 2[3]$, with $[-1] = 0$. Then show that $B(\mathbb{F}_7) \cong \mathbb{Z}/4$ on generator $[-1] = 2[3]$ with $c = [4] = 2[-1]$. (In both cases, $[3] \notin B(\mathbb{F})$.) Show that $B(\mathbb{F}_4) \cong \mathbb{Z}/5$.

EVI.5.4 **5.4.** Show that $c = 0$ in $B(\mathbb{F}_q)$ if either: (a) $\mathrm{char}(\mathbb{F}_q) = 2$ and $q \equiv 1 \pmod 3$; (b) $\mathrm{char}(\mathbb{F}_q) = 3$ and $q \equiv 1 \pmod 4$; or (c) $\mathrm{char}(\mathbb{F}_q) > 3$ and $q \equiv 1 \pmod 6$;

EVI.5.5 **5.5.** (Dupont-Sah) Show that $[x^2] = 2\,([x] + [-x] + [-1])$ in $\mathcal{P}(F)$ for all $x \neq \pm 1$. Using this, show that $[x^2] + 2[-x^2] - 2[x-1] - [1/x^2] = 2c$.

EVI.5.6 **5.6.** Given an arbitrary set X, let $\widetilde{C}_n(X)$ be the free abelian group on the set X^{n+1} of all $(n+1)$-tuples $(x_0, \dots, x_n)$ in X, including duplication.

a) Show that $\widetilde{C}_n(X)$ is a simplicial abelian group, whose degeneracy operators σ_i are duplication of the i^{th} entry. Under the Dold-Kan correspondence, the simplicial abelian group $\widetilde{C}_n(X)$ corresponds to the chain complex $C_*(X)$ of Definition 5.5.

b) Show that the rotations $t_n(x_0, \dots, x_n) = (x_n, x_0, \dots, x_{n-1})$ satisfy $t_n^{n+1} = 1$, $\partial_i t_n = t_n \partial_{i-1}$ and $\sigma_i t_n = t_n \sigma_i$ for $i \neq 0$, $\partial_0 t_n = \partial_n$ and $\sigma_0 t_n = t^2_{n+1}\sigma_n$. This shows that $\widetilde{C}_n(X)$ is a cyclic abelian group (Definition 5.9).

EVI.5.7 **5.7.** *Transfer maps.* Let $F \subset E$ be a finite field extension. If the transfer map $K_3(E) \to K_3(F)$ induces a map $N_{E/F} : B(E) \to B(F)$ (via Theorem 5.2), we call $N_{E/F}$ a transfer map. If $N_{E/F}$ exists, the composition $B(F) \to B(E) \to B(F)$ must be multiplication by $[E : F]$.

(a) Conclude that there is no transfer map $N_{E/F} : B(E) \to B(F)$ defined for all $F \subset E$. *Hint:* Consider $\mathbb{F}_5 \subset \mathbb{F}_{25}$ or $\mathbb{R} \subset \mathbb{C}$.

(b) Show that a transfer map $B(E) \to B(F)$ exists if $\mu(F) = \mu(E)$, or more generally if E has an F-basis such that $\mu(E)$ is represented by monomial matrices over F.

EVI.5.8 **5.8.** If F_0 is the field of constants in F, show that $B(F)/B(F_0)$ is the cokernel of $K_3^M(F) \oplus K_3(F_0) \to K_3(F)$. (The Rigidity Conjecture 5.3.1 implies that it is zero.) Conclude that $B(F(t)) \cong B(F)$, using V.6.7.1.

EVI.5.9 **5.9.** If A is an abelian group, $H_*(A, \mathbb{Z})$ is a graded-commutative ring [223, 6.5.14]. Since $H_1(A, \mathbb{Z}) \cong A$, there is a ring map $\wedge^* A \to H_*(A, \mathbb{Z})$. The Künneth formula [223, 6.1.13] for $H_*(A \times A)$ and the diagonal provide a natural map $H_3(A, \mathbb{Z}) \to H_3(A \times A, \mathbb{Z}) \to \mathrm{Tor}(A, A)$, whose image is invariant under the transposition involution τ on $A \times A$.

(a) Show that $\wedge^2 A \to H_2(A, \mathbb{Z})$ is an isomorphism, and that $\wedge^3 A \to H_3(A, \mathbb{Z})$ is an injection whose cokernel $H_3^{\mathrm{ind}}(A)$ is canonically isomorphic to $\mathrm{Tor}(A, A)^\tau$. *Hint:* It is true for cyclic groups; use the Künneth formula to check it for finitely generated A.

(b) If A is a finite cyclic group and σ is an automorphism of A, show that the composite

$$A \cong H_3^{\mathrm{ind}}(A) \xrightarrow{\sigma} H_3^{\mathrm{ind}}(A) \cong A$$

sends $a \in A$ to $\sigma^2(a)$. In particular, σ acts trivially when $\sigma^2 = 1$.

EVI.5.10 **5.10.** In this exercise we analyze H_3 of the group $M_n = T_n \rtimes \Sigma_n$ of monomial matrices in $GL_n(F)$, using the Hochschild-Serre spectral sequence. Here F is a field and $T_n = (F^\times)^n$ denotes the subgroup of diagonal matrices. In this exercise we write $H_i(G)$ for $H_i(G, \mathbb{Z})$.

(a) Show that the images of $H_3(T_3)$ and $H_3(T_n)$ in $H_3(M_n(F))$ are the same.

(b) Show that $H_*(\Sigma_n, T_n) \cong H_*(\Sigma_{n-1}, F^\times)$. (The group Σ_{n-1} is the stabilizer of $T_1 = F^\times$.)

(c) Show that $H_*(\Sigma_n, \wedge^2 T_n) \cong H_*(\Sigma_{n-1}, \wedge^2 F^\times) \oplus H_*(\Sigma_2 \times \Sigma_{n-2}, F^\times \otimes F^\times)$.

(d) When $n = 2$, conclude that $H_3(M_2) \cong H_3(\Sigma_2) \oplus H_3(T_2)_{\Sigma_2} \oplus \tilde{\wedge}^2 F^\times$.

(e) Let A denote the kernel of $H_3(M_2 \times \Sigma_{n-2}, \mathbb{Z}) \to H_3(\Sigma_2 \times \Sigma_{n-2}, \mathbb{Z})$. Conclude that $H_3(T_3) \oplus A \to H_3(M_n) \to H_3(\Sigma_n, \mathbb{Z})$ is exact for $n \geq 5$.

EVI.5.11 **5.11.** Show that $H_2(P, \mathbb{Z}) \cong \pi_2 BM^+$ equals $\pi_2^s \oplus \tilde{\wedge}^2 F^\times$, where $P = [M, M]$. *Hint:* Recall from IV, Ex. 1.27 that $BM^+ \simeq BP^+ \times B(F^\times) \times B\Sigma_2$. Thus it suffices to compute $H_2(M, \mathbb{Z})$. Now use $M_n = T_n \rtimes \Sigma_n$ and Exercise 5.10(b,c).

EVI.5.12 **5.12.** Show that the linear transformation $\alpha(x_1, x_2, x_3) = (x_2, x_1, -x_3)$ is in SM, inducing a decomposition $SM \cong P \rtimes \Sigma_2$. If $\mathrm{char}(F) \neq 2$, show that α is conjugate in $GL_3(F)$ to the matrix $\mathrm{diag}(-1, -1, +1)$ in P, and hence that the image of $H_*(\Sigma_2, \mathbb{Z}) \to H_*(SM, \mathbb{Z}) \to H_*(GL(F), \mathbb{Z})$ lies in the image of $H_*(P, \mathbb{Z})$. If $\mathrm{char}(F) = 2$, show that α is conjugate to an upper triangular matrix in $GL_3(F)$ and hence that the map $H_*(\Sigma, \mathbb{Z}) \to H_*(GL(F), \mathbb{Z})$ is trivial.

EVI.5.13 **5.13.** Let T' denote the group of diagonal matrices in $SL(F)$.

(a) Show that $P \cong T' \rtimes A_\infty$, and that $H_3(T', \mathbb{Z})_{A_\infty} \cong H_3(F^\times, \mathbb{Z})$.

(b) Use the proof of Proposition 5.17 to show that $\mu(F) \cong H_3(F^\times, \mathbb{Z})/\wedge^3 F^\times$ is the image of $H_3(T', \mathbb{Z})$ in $H_3(P, \mathbb{Z})/\wedge^3 F^\times \cong \pi_3(BM^+)$.

(c) If $D = \mu(F) \rtimes \Sigma_2$, show that the map from $H_3(D, \mathbb{Z})$ to $\pi_3(BM^+)$ sends $\mu(F)/\{\pm 1\}$ to $\mu(F)$.

6 Global fields of finite characteristic

A *global field* of finite characteristic p is a field F which is finitely generated of transcendence degree one over $\mathbb{F}_p$; the algebraic closure of $\mathbb{F}_p$ in F is a finite field $\mathbb{F}_q$ of characteristic p. It is classical (see [85, I.6]) that there is a unique smooth projective curve X over $\mathbb{F}_q$ whose function field is F. If S is any nonempty set of closed points of X, then $X \backslash S$ is affine; we call the coordinate ring R of $X\backslash S$ the ring of *S-integers in F*. In this section, we discuss the K-theory of F, X and the rings of S-integers of F.

Any discussion of the K-theory of F must involve the K-theory of X. For example, $K_1(F)$ is related to the Picard group $\mathrm{Pic}(X)$ by the Units-Pic sequence (I.5.12):

$$1 \to \mathbb{F}_q^\times \to F^\times \to \oplus_{x\in X}\mathbb{Z} \to \mathrm{Pic}(X) \to 0.$$

Recall that $K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$, and that $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus J(X)$, where $J(X)$ is a finite group (see I.5.17).

Since K_2 vanishes on finite fields (III.6.1.1), the localization sequence V.6.12 for X ends in the exact sequence

$$0 \to K_2(X) \to K_2(F) \xrightarrow{\partial} \oplus_{x\in X}\, k(x)^\times \to K_1(X) \to \mathbb{F}_q^\times \to 0.$$

By classical Weil reciprocity V.6.12.1, the cokernel of ∂ is $\mathbb{F}_q^\times$, so $K_1(X) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. By III.2.5.1, if R is the coordinate ring of any affine open in X then $SK_1(R) = 0$. A diagram chase shows that the image of $K_1(X)$ in $K_1(F) = F^\times$ is $\mathbb{F}_q^\times$.

By III.7.2(a) (due to Bass and Tate), the kernel $K_2(X)$ of ∂ is finite of order prime to p. This establishes the low dimensional cases of the following theorem, first proven by Harder [83], using the method pioneered by Borel [28].

VI.6.1 **Harder's Theorem 6.1.** *Let X be a smooth projective curve over a finite field of characteristic p. For $n\geq 1$, each $K_n(X)$ is a finite group of order prime to p.*

Parshin has conjectured that if X is any smooth projective variety over a finite field then $K_n(X)$ is a torsion group for $n \geq 1$. Harder's Theorem 6.1 shows that Parshin's conjecture holds for curves.

Proof. By III.7.2(a), $K_n^M(F) = 0$ for all $n \geq 3$. By Geisser and Levine's Theorem 4.7, the Quillen groups $K_n(F)$ are uniquely p-divisible for $n \geq 3$. For every closed point $x \in X$, the groups $K_n(x)$ are finite of order prime to p ($n > 0$) because $k(x)$ is a finite field extension of $\mathbb{F}_q$. From the localization sequence

$$\oplus_{x\in X} K_n(x) \to K_n(X) \to K_n(F) \to \oplus_{x\in X} K_{n-1}(x),$$

a diagram chase shows that $K_n(X)$ is uniquely p-divisible. By IV.6.9 (due to Quillen), the abelian groups $K_n(X)$ are finitely generated. As any finitely generated p-divisible abelian group A is finite with $p \nmid |A|$, this is true for each $K_n(X)$. $\square$

VI.6.2 **Corollary 6.2.** *If R is the ring of S-integers in $F = \mathbb{F}_q(X)$ (and $S \neq \emptyset$) then:*

a) $K_1(R) \cong R^\times \cong \mathbb{F}_q^\times \times \mathbb{Z}^s$, $|S| = s - 1$;

b) For $n \geq 2$, $K_n(R)$ is a finite group of order prime to p.

Proof. Recall (III.1.1.1) that $K_1(R) = R^\times \oplus SK_1(R)$. We saw in III.2.5.1 that $SK_1(R) = 0$, and the units of R were determined in I.5.17, whence (a). The rest follows from the localization sequence $K_n(X) \to K_n(R) \to \oplus_{x \in S} K_{n-1}(x)$. □

VI.6.3 **Example 6.3** (The e-invariant)**.** The targets of the e-invariant of X and F are the same groups as for $\mathbb{F}_q$, because every root of unity is algebraic over $\mathbb{F}_q$. Hence the inclusions of $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)$ in $K_{2i-1}(X)$ and $K_{2i-1}(F)$ are split by the e-invariant, and this group is the Harris-Segal summand (2.5.1).

The inverse limit of the finite curves $X_\nu = X \times \mathrm{Spec}(\mathbb{F}_{q^\nu})$ is the curve $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ over the algebraic closure $\bar{\mathbb{F}}_q$. To understand $K_n(X)$ for $n > 1$, it is useful to know not only what the groups $K_n(\bar{X})$ are, but how the (geometric) Frobenius $\varphi : x \mapsto x^q$ acts on them.

By II.8.2.1 and I.5.16, $K_0(\bar{X}) = \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X})$, where $J(\bar{X})$ is the group of points on the Jacobian variety over $\bar{\mathbb{F}}_q$; it is a divisible torsion group. If $\ell \neq p$, the ℓ-primary torsion subgroup $J(\bar{X})_\ell$ of $J(\bar{X})$ is isomorphic to the abelian group $(\mathbb{Z}/\ell^\infty)^{2g}$. The group $J(\bar{X})$ may or may not have p-torsion. For example, if X is an elliptic curve then the p-torsion in $J(\bar{X})$ is either 0 or $\mathbb{Z}/p^\infty$, depending on whether X is supersingular (see [85, Ex. IV.4.15]). Note that the localization $J(\bar{X})[1/p]$ is the direct sum over all $\ell \neq p$ of the ℓ-primary groups $J(\bar{X})_\ell$.

Next, recall that the group of units $\bar{\mathbb{F}}_q^\times$ may be identified with the group μ of all roots of unity in $\bar{\mathbb{F}}_q$; its underlying abelian group is isomorphic to $\mathbb{Q}/\mathbb{Z}[1/p]$. Passing to the direct limit of the $K_1(X_\nu)$ yields $K_1(\bar{X}) \cong \mu \oplus \mu$.

For $n \geq 1$, the groups $K_n(\bar{X})$ are all torsion groups, of order prime to p, because this is true of each $K_n(X_\nu)$ by 6.1. We can now determine the abelian group structure of the $K_n(\bar{X})$ as well as the action of the Galois group on them. Recall from Definition 1.7 that $M(i)$ denotes the i^{th} Tate twist of a Galois module M.

VI.6.4 **Theorem 6.4.** *Let X be a smooth projective curve over $\mathbb{F}_q$, and set $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Then for all $n \geq 0$ we have isomorphisms of* $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$*-modules:*

$$K_n(\bar{X}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus J(\bar{X}), & n = 0 \\ \mu(i) \oplus \mu(i), & n = 2i-1 > 0 \\ J(\bar{X})[1/p](i), & n = 2i > 0. \end{cases}$$

For $\ell \neq p$, the ℓ-primary subgroup of $K_{n-1}(\bar{X})$ is isomorphic to $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$, $n > 0$, whose Galois module structure is given by:

$$K_n(\bar{X}; \mathbb{Z}/\ell^\infty) \cong \begin{cases} \mathbb{Z}/\ell^\infty(i) \oplus \mathbb{Z}/\ell^\infty(i), & n = 2i \geq 0 \\ J(\bar{X})_\ell(i-1), & n = 2i-1 > 0. \end{cases}$$

Proof. Since the groups $K_n(\bar{X})$ are torsion for all $n > 0$, the universal coefficient theorem (Ex. IV.2.6) shows that $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$ is isomorphic to the ℓ-primary subgroup of $K_{n-1}(\bar{X})$. Thus we only need to determine the Galois modules $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. For $n = 0, 1, 2$ they may be read off from the above discussion. For $n > 2$ we consider the motivic spectral sequence (4.2); by Theorem 4.1, the terms $E_2^{p,q}$ vanish unless $p = q, q+1, q+2$. There is no room for differentials, so the spectral sequence degenerates at E_2 to yield the groups $K_n(\bar{X}; \mathbb{Z}/\ell^\infty)$. There are no extension issues because the edge maps are the e-invariants $K_{2i}(X; \mathbb{Z}/\ell^\infty) \to H^0_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/\ell^\infty(i)$ of 6.3, and are therefore split surjections. Finally, we note that as Galois modules we have $H^1_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)$, and (by Poincaré Duality [128, V.2]) $H^2_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i+1)) \cong \mathbb{Z}/\ell^\infty(i)$. □

Passing to invariants under the group $G = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, there is a natural map from $K_n(X)$ to $K_n(\bar{X})^G$. For odd n, we see from 6.4 and 2.3 that $K_{2i-1}(\bar{X})^G \cong \mathbb{Z}/(q^i - 1) \oplus \mathbb{Z}/(q^i - 1)$; for even n, we have the less concrete description $K_{2i}(\bar{X})^G \cong J(\bar{X})[1/p](i)^G$. One way of studying this group is to consider the action of the Frobenius on $H^*_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))$ and use the identity $H^*_{\text{et}}(X, \mathbb{Q}_\ell(i)) = H^*_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))^G$, which follows from the spectral sequence $H^p(G, H^*_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))) \Rightarrow H^*_{\text{et}}(X, \mathbb{Q}_\ell(i))$ since $H^p(G, -)$ is torsion for $p > 0$; see [223, 6.11.14].

VI.6.5 **Example 6.5.** φ^* acts trivially on $H^0_{\text{et}}(\bar{X}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^2_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$. It acts as q^{-i} on the twisted groups $H^0_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))$ and $H^2_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i+1))$. Taking G-invariants yields $H^0_{\text{et}}(X, \mathbb{Q}_\ell(i)) = 0$ for $i \neq 0$ and $H^2_{\text{et}}(X, \mathbb{Q}_\ell(i)) = 0$ for $i \neq 1$.

Weil's 1948 proof of the Riemann Hypothesis for Curves implies that the eigenvalues of φ^* acting on $H^1_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))$ have absolute value $q^{1/2-i}$. Taking G-invariants yields $H^1_{\text{et}}(X, \mathbb{Q}_\ell(i)) = 0$ for all i.

For any G-module M, we have an exact sequence [223, 6.1.4]

$$0 \to M^G \to M \xrightarrow{\varphi^* - 1} M \to H^1(G, M) \to 0. \qquad (6.5.1)$$ 6.5.1

The case $i = 1$ of the following result reproduces Weil's theorem that the ℓ-primary torsion part of the Picard group of X is $J(\bar{X})^G_\ell$.

VI.6.6 **Lemma 6.6.** *For a smooth projective curve X over $\mathbb{F}_q$, $\ell \nmid q$ and $i \geq 2$ we have:*

(1) $H^{n+1}_{et}(X, \mathbb{Z}_\ell(i)) \cong H^n_{et}(X, \mathbb{Z}/\ell^\infty(i)) \cong H^n_{et}(\bar{X}, \mathbb{Z}/\ell^\infty(i))^G$ *for all n;*

(2) $H^0_{et}(X, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_i^{(\ell)}(F)$*;*

(3) $H^1_{et}(X, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$*;*

(4) $H^2_{et}(X, \mathbb{Z}/\ell^\infty(i)) \cong \mathbb{Z}/w_{i-1}^{(\ell)}(F)$*; and*

(5) $H^n_{et}(X, \mathbb{Z}/\ell^\infty(i)) = 0$ *for all $n \geq 3$.*

Proof. Since $i \geq 2$, we see from 6.5 that $H^n_{\text{et}}(X, \mathbb{Q}_\ell(i)) = 0$ for all n. Since $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathbb{Z}/\ell^\infty$, this yields $H^n_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i)) \cong H^{n+1}_{\text{et}}(X, \mathbb{Z}_\ell(i))$ for all n.

Since each $H^n = H^n_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i))$ is a quotient of $H^n_{\text{et}}(\bar{X}, \mathbb{Q}_\ell(i))$, $\varphi^* - 1$ is a surjection, *i.e.*, $H^1(G, H^n) = 0$. Since $H^n(G, -) = 0$ for $n > 1$, the Leray spectral sequence $E_2^{p,q} = H^p(G, H^q_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i))) \Rightarrow H^{p+q}_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i))$ for $X \to \operatorname{Spec}(\mathbb{F}_q)$ [127, III.1.18], collapses for $i > 1$ to yield exact sequences

$$0 \to H^n_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i)) \to H^n_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \xrightarrow{\varphi^* - 1} H^n_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i)) \to 0.$$

In particular, $H^n_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i)) = 0$ for $n > 2$. As in the proof of 6.4, $H^1_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i))$ is $J(\bar{X})_\ell(i-1)$, so $H^1_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i)) \cong J(\bar{X})_\ell(i-1)^G$, and $H^2_{\text{et}}(\bar{X}, \mathbb{Z}/\ell^\infty(i))$ is $\mathbb{Z}/\ell^\infty(i-1)$ by duality, so $H^2_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i))$ is $\mathbb{Z}/\ell^\infty(i-1)^G \cong \mathbb{Z}/w^{(\ell)}_{i-1}$. □

Given the calculation of $K_n(\bar{X})^G$ in Theorem 6.4 and the calculation of $H^n_{\text{et}}(X, \mathbb{Z}/\ell^\infty(i))$ in 6.6, we see that the natural map $K_n(X) \to K_n(\bar{X})^G$ is a surjection. Thus the real content of the following theorem is that $K_n(X) \to K_n(\bar{X})^G$ is an isomorphism.

VI.6.7 **Theorem 6.7.** *Let X be the smooth projective curve corresponding to a global field F over $\mathbb{F}_q$. Then $K_0(X) = \mathbb{Z} \oplus \operatorname{Pic}(X)$, and the finite groups $K_n(X)$ for $n > 0$ are given by:*

$$K_n(X) \cong K_n(\bar{X})^G \cong \begin{cases} K_n(\mathbb{F}_q) \oplus K_n(\mathbb{F}_q), & n \text{ odd}, \\ \bigoplus_{\ell \neq p} J(\bar{X})_\ell(i)^G, & n = 2i \text{ even}. \end{cases}$$

Proof. We may assume that $n \neq 0$, so that the groups $K_n(X)$ are finite by 6.1. It suffices to calculate the ℓ-primary part $K_{n+1}(X; \mathbb{Z}/\ell^\infty)$ of $K_n(X)$. But this follows from the motivic spectral sequence (4.2), which degenerates by 6.6. □

VI.6.8 **Theorem 6.8.** *If F is the function field of a smooth projective curve X over $\mathbb{F}_q$, then for all $i \geq 1$: $\mathbb{F}_q \subset F$ induces an isomorphism $K_{2i+1}(\mathbb{F}_q) \cong K_{2i+1}(F)$, and there is an exact reciprocity sequence (generalizing V.6.12.1):*

$$0 \to K_{2i}(X) \to K_{2i}(F) \xrightarrow{\oplus \partial_x} \oplus_{x \in X} K_{2i-1}(\mathbb{F}_q(x)) \xrightarrow{N} K_{2i-1}(\mathbb{F}_q) \to 0.$$

Proof. The calculation of $K_2(F)$ was carried out at the beginning of this section, so we restrict attention to $K_n(F)$ for $n \geq 3$. Because $K_{2i}(\mathbb{F}_q) = 0$, the localization sequence V.6.12 breaks up into exact sequences for $i \geq 2$:

$$0 \to K_{2i}(X) \to K_{2i}(F) \xrightarrow{\oplus \partial_x} \oplus_{x \in X} K_{2i-1}(\mathbb{F}_q(x)) \to K_{2i-1}(X) \to K_{2i-1}(F) \to 0.$$

Let $SK_{2i-1}(X)$ denote the kernel of $K_{2i-1}(X) \to K_{2i-1}(F)$. Since $\mathbb{F}_q \subset F \subset \bar{F}$ induces an injection from the subgroup $K_{2i-1}(\mathbb{F}_q)$ of $K_{2i-1}(\bar{\mathbb{F}}_q)$ into $K_{2i-1}(\bar{F})$ by V.6.7.4, we see from Theorem 6.7 that $|SK_{2i-1}(X)|$ is at most

$|K_{2i-1}(\mathbb{F}_q)|$. As in V.6.12, for each closed point x the composition of the transfer $K_{2i-1}(\mathbb{F}_q(x)) \to SK_{2i-1}(X)$ with the proper transfer $\pi_* : K_{2i-1}(X) \to K_{2i-1}(\mathbb{F}_q)$ is the transfer associated to $\mathbb{F}_q \subset \mathbb{F}_q(x)$, *i.e.,* the transfer map $K_{2i-1}(\mathbb{F}_q(x)) \to K_{2i-1}(\mathbb{F}_q)$; each of these transfer maps are onto by IV.1.13. It follows that $\pi_* : SK_{2i-1}(X) \to K_{2i-1}(\mathbb{F}_q)$ is an isomorphism. The theorem now follows. □

VI.6.9 **Remark 6.9** (The Zeta Function)**.** We can relate the orders of the K-groups of the curve X to values of the zeta function $\zeta_X(s)$. By definition, $\zeta_X(s) = Z(X, q^{-s})$, where

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \, \frac{t^n}{n}\right).$$

Weil proved that $Z(X,t) = P(t)/(1-t)(1-qt)$ for every smooth projective curve X, where $P(t) \in \mathbb{Z}[t]$ is a polynomial of degree $2 \cdot \text{genus}(X)$ with all roots of absolute value $1/\sqrt{q}$. This formula is a restatement of Weil's proof of the Riemann Hypothesis for X (6.5 above), given Grothendieck's formula $P(t) = \det(1 - \varphi^* t)$, where φ^* is regarded as an endomorphism of $H^1_{\text{et}}(\bar{X};\mathbb{Q}_\ell)$. Note that by 6.5 the action of φ^* on $H^0_{\text{et}}(\bar{X};\mathbb{Q}_\ell)$ has $\det(1-\varphi^* t) = (1-t)$, and the action on $H^2_{\text{et}}(\bar{X};\mathbb{Q}_\ell)$ has $\det(1-\varphi^* t) = (1-qt)$.

Here is application of Theorem 6.7, which was conjectured by Lichtenbaum in [113] and proven by Thomason in [199, (4.7)]. For legibility, let $\#A$ denote the order of a finite abelian group A.

VI.6.10 **Proposition 6.10.** *If X is a smooth projective curve over $\mathbb{F}_q$ then for all $i \geq 2$,*

$$\frac{\#K_{2i-2}(X) \cdot \#K_{2i-3}(\mathbb{F}_q)}{\#K_{2i-1}(\mathbb{F}_q) \cdot \#K_{2i-3}(X)} = \prod_\ell \frac{\#H^2_{et}(X;\mathbb{Z}_\ell(i))}{\#H^1_{et}(X;\mathbb{Z}_\ell(i)) \cdot \#H^3_{et}(X;\mathbb{Z}_\ell(i))} = \left|\zeta_X(1-i)\right|.$$

Proof. We have seen that all the groups appearing in this formula are finite. The first equality follows from 6.6 and 6.7. The second equality follows from the formula for $\zeta_X(1-i)$ in 6.9. □

EXERCISES

EVI.6.1 **6.1.** Let X be the projective line $\mathbb{P}^1_{\mathbb{F}_q}$ over $\mathbb{F}_q$. Use Theorem 6.7 to recover the calculation of $K_*(X)$ in V.1.5. Show directly that $Z(X,t) = 1/(1-t)(1-qt)$ and use this to verify the formula in 6.10 for $\zeta_X(i-1)$.

EVI.6.2 **6.2.** Let R be the ring of S-integers in a global field $F = \mathbb{F}_q(X)$ of finite characteristic. Show that $K_n(R) \to K_n(F)$ is an injection for all $n \geq 1$, and that $K_{2i-1}(R) \to K_{2i-1}(F)$ is an isomorphism for all $i > 1$. (This generalizes the Bass-Milnor-Serre Theorem III.2.5, and provides another proof of 6.2(a).)

EVI.6.3 **6.3.** Let $F = \mathbb{F}_q(X)$ be a global field, of degree d over a function field $\mathbb{F}_q(t)$. For $i > 0$, show that the transfer $K_{2i}(F) \to K_{2i}(\mathbb{F}_q(t))$ is onto, and that the transfer $K_{2i-1}(F) \to K_{2i-1}(\mathbb{F}_q(t))$ is multiplication by d (under the identification of both groups with $K_{2i-1}(\mathbb{F}_q)$ in 6.8).

7 Local Fields

A *local field* is a field E which is complete under a discrete valuation v, and whose residue field k_v is finite. The subring V of elements of positive valuation is a complete valuation domain. It is classical that every local field is either a finite extension of the p-adic rationals $\hat{\mathbb{Q}}_p$ or isomorphic to $\mathbb{F}_q((t))$. (See [167].)

We saw in II.2.2 and III.1.4 that $K_0(V) = K_0(E) = \mathbb{Z}$ and $K_1(V) = V^\times$, $K_1(E) = E^\times \cong (V^\times) \times \mathbb{Z}$, where the factor $\mathbb{Z}$ is identified with the powers $\{\pi^m\}$ of a parameter π of V. It is well known that $V^\times \cong \mu(E) \times U_1$, where $\mu(E)$ is the group of roots of unity in E (or V), and $U_1 \subseteq 1+\pi V$ is a torsionfree $\mathbb{Z}_p$-module.

We also saw in Moore's Theorem (Chapter III, Theorem 6.2.4 and Ex. 6.11) that $K_2(E) \cong U_2 \oplus \mathbb{F}_q^\times$, where U_2 is an uncountable, uniquely divisible abelian group. Since $K_2(E) \cong K_2(V) \oplus \mathbb{F}_q^\times$ by V.6.9.2, this implies that $K_2(V) \cong U_2$.

VI.7.1 **Proposition 7.1.** *Let E be a local field. For $n \ge 3$, $K_n^M(E)$ is an uncountable, uniquely divisible group. The group $K_2(E)$ is the sum of $\mathbb{F}_q^\times$ and an uncountable, uniquely divisible group.*

Proof. The group is uncountable by Ex. III.7.14, and divisibility follows easily from Moore's Theorem (see III, Ex. 7.4). We give a proof that it is uniquely divisible using the isomorphism $K_n^M(E) \cong H^n(E, \mathbb{Z}(n))$. If $\mathrm{char}(E) = p$, $K_n^M(E)$ has no p-torsion by Izhboldin's Theorem III.7.8, so we consider m-torsion when $1/m \in E$. The long exact sequence in motivic cohomology associated to the coefficient sequence $0 \to \mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \to \mathbb{Z}/m(n) \to 0$ yields the exact sequence for m:

$$H^{n-1}_{\mathrm{et}}(E, \mu_m^{\otimes n}) \to K_n^M(E) \xrightarrow{m} K_n^M(E) \to H^n_{\mathrm{et}}(E, \mu_m^{\otimes n}). \qquad (7.1.1)$$ VI.7.1.1

Since $H^n_{\mathrm{et}}(E, -) = 0$ for $n \ge 3$, this immediately implies that $K_n^M(E)$ is uniquely m-divisible for $n > 3$ (and m-divisible for $n = 3$). Moreover, the m-torsion subgroup of $K_3^M(E)$ is a quotient of the group $H^2_{\mathrm{et}}(E, \mu_m^{\otimes 3})$, which by duality is $\mathbb{Z}/(w_2, m)$, where $w_2 = w_2(E)$ is $q^2 - 1$ by 2.3.1. Thus the torsion subgroup of $K_3^M(E)$ is a quotient of w_2. We may therefore assume that w_2 divides m. Now map the sequence (7.1.1) for m^2 to the sequence (7.1.1) for m; the map from $\mathbb{Z}/w_i = H^2_{\mathrm{et}}(E, \mu_{m^2}^{\otimes 3})$ to $\mathbb{Z}/w_i = H^2_{\mathrm{et}}(E, \mu_m^{\otimes 3})$ is the identity but the map from the image ${}_{m^2}K_3^M(E)$ to ${}_mK_3^M(E)$ is multiplication by m and thus zero, as required. □

Equicharacteristic local fields

We first dispose of the equi-characteristic case, where $E = \mathbb{F}_q((t))$, $V \cong \mathbb{F}_q[[\pi]]$ and $\mathrm{char}(E) = p$. In this case, $\mu(E) = \mathbb{F}_q^\times$, and $U_1 = 1 + \pi\mathbb{F}_q[[\pi]]$ is isomorphic to the big Witt vectors of $\mathbb{F}_q$ (II.4.3), which is the product of a countably infinite number of copies of $\mathbb{Z}_p$ (see Ex. 7.2). (In fact, it is a countably infinite product of copies of the ring $\mathbb{Z}_p[\zeta_{q-1}]$ of Witt vectors over $\mathbb{F}_q$.)

Here is a description of the abelian group structure on $K_n(V)$ for $n \ge 2$.

VI.7.2 **Theorem 7.2.** *Let* $V = \mathbb{F}_q[[\pi]]$ *be the ring of integers in the local field* $E = \mathbb{F}_q((\pi))$*. For* $n \geq 2$ *there are uncountable, uniquely divisible abelian groups* U_n *and canonical isomorphisms:*

$$K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q), \quad K_n(V) \cong K_n(\mathbb{F}_q) \oplus U_n.$$

Proof. The splitting $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ was established in V.6.9.2. Now let U_n denote the kernel of the canonical map $K_n(V) \to K_n(\mathbb{F}_q)$. Since $V \to \mathbb{F}_q$ splits, naturality yields $K_n(V) = U_n \oplus K_n(\mathbb{F}_q)$. By Gabber rigidity (IV.2.10), U_n is uniquely ℓ-divisible for all $\ell \neq p$ and $n > 0$. It suffices to show that U_n is uncountable and uniquely p-divisible when $n \geq 2$; this holds for $n = 2$ by 7.1.

Now the Milnor groups $K_n^M(E)$ are uncountable, uniquely divisible abelian groups for $n \geq 3$, by Proposition 7.1. The group $K_n^M(E)$ is a summand of the Quillen K-group $K_n(E)$ by V.11.13, 4.3 or 4.9. On the other hand, the Geisser-Levine theorem 4.7 shows that the complementary summand is uniquely p-divisible. □

p-adic local fields

In the mixed characteristic case, when $\mathrm{char}(E) = 0$, even the structure of $V^\times$ is quite interesting. The torsionfree part U_1 is a free $\mathbb{Z}_p$-module of rank $[E : \mathbb{Q}_p]$; it is contained in $(1 + \pi V)^\times$ and injects into E as a lattice by the convergent power series for $x \mapsto \ln(x)$.

The quotient $V^\times \to \mathbb{F}_q^\times$ splits, and the subgroup of $V^\times$ isomorphic to $\mathbb{F}_q^\times$ is called the group of *Teichmüller units.* Thus $K_1(V) = V^\times$ is a product $U_1 \times \mathbb{F}_q^\times \times \mu_{p^\infty}(E)$, where $\mu_{p^\infty}(E)$ is the finite cyclic group of p-primary roots of unity in V. There seems to be no simple formula for the order of the cyclic p-group $\mu_{p^\infty}(E)$.

To understand the groups $K_n(E)$ for $n \geq 3$, recall from Proposition 7.1 that $K_n^M(E)$ is an uncountable, uniquely divisible abelian group. From Example 4.3, this is a direct summand of $K_n(E)$; since $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ by V.6.9.2, it is also a summand of $K_n(V)$. Thus, as in the equicharacteristic case, $K_n(E)$ will contain an uncountable uniquely divisible summand about which we can say very little.

Before stating our next result, we need an étale calculation. Since E has étale cohomological dimension 2, we may ignore $H^n_{\mathrm{et}}(E, -)$ for $n > 2$. By Tate–Poitou duality [128, I.2.3], $H^2_{\mathrm{et}}(E, \mu_m^{\otimes i+1})$ is isomorphic to $H^0_{\mathrm{et}}(E, \mu_m^{\otimes i})$. We shall assume that $i > 0$ and m is divisible by $w_i(E)$, so that these groups are isomorphic to $\mathbb{Z}/w_i(E)$. Now consider the change of coefficients $\mu_m^{\otimes i} \subset \mu_{m^2}^{\otimes i}$. The induced endomorphisms of $\mathbb{Z}/w_i(E) = H^0_{\mathrm{et}}(E, \mu_m^{\otimes i})$ and $\mathbb{Z}/w_i(E) = H^2_{\mathrm{et}}(E, \mu_m^{\otimes i+1})$ are the identity map and the zero map, respectively. Since $\mu = \cup \mu_m$, passing to the limit over m yields:

$$H^0_{\mathrm{et}}(E, \mu^{\otimes i}) \cong \mathbb{Z}/w_i(E) \quad \text{and} \quad H^2_{\mathrm{et}}(E, \mu^{\otimes i+1}) = 0, \quad i > 0.$$

Recall that an abelian group which is uniquely ℓ-divisible for all $\ell \neq p$ is the same thing as a $\mathbb{Z}_{(p)}$-module.

VI.7.3 **Proposition 7.3.** *For $n > 0$ we have $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$, and the groups $K_n(V)$ are $\mathbb{Z}_{(p)}$-modules.*

When $n = 2i - 1$, $K_n(V) \cong K_n(E)$ is the direct sum of a torsionfree $\mathbb{Z}_{(p)}$-module and the Harris-Segal summand (see 2.5), which is isomorphic to $\mathbb{Z}/w_i(E)$.

When $n = 2i$, $K_n(V)$ is the direct sum of $\mathbb{Z}/w_i^{(p)}(E)$ and a divisible $\mathbb{Z}_{(p)}$-module.

Proof. The decomposition $K_n(E) \cong K_n(V) \oplus K_{n-1}(\mathbb{F}_q)$ was established in V.6.9.2. In particular, $K_{2i-1}(V) \cong K_{2i-1}(E)$.

To see that $K_{2i-1}(E)$ has a cyclic summand of order $w_i(E)$, consider the spectral sequence (4.2) with coefficients $\mathbb{Q}/\mathbb{Z}$. By the above remarks, it degenerates completely to yield $K_{2i}(E;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/w_i(E)$. Since this injects into $K_{2i}(\bar{E};\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ (by Theorem 1.6, $\bar{E}$ being the algebraic closure of E), this implies that the e-invariant is an isomorphism: $K_{2i}(E;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}/w_i(E)$. By Ex. IV.2.6, this implies that $K_{2i-1}(E) \cong T_i \oplus \mathbb{Z}/w_i(E)$ where T_i is torsionfree.

To see that $K_{2i}(E)$ is the sum of a divisible group and $\mathbb{Z}/w_i(E)$, fix i and suppose that $w_i(E)$ divides m. Since $H^n_{\text{et}}(E,-) = 0$ for $n > 2$, the spectral sequence (4.2) with coefficients $\mathbb{Z}/m$ degenerates completely and describes $K_{2i}(E;\mathbb{Z}/m)$ as an extension of $H^0_{\text{et}}(E,\mu_m^{\otimes i}) \cong \mathbb{Z}/w_i(E)$ by $H^2_{\text{et}}(E,\mu_m^{\otimes i+1}) \cong \mathbb{Z}/w_i(E)$. By the previous paragraph, the quotient $\mathbb{Z}/w_i(E)$ is identified with the m-torsion in $K_{2i-1}(E)$, so the kernel $\mathbb{Z}/w_i(E)$ is identified with $K_{2i}(E)/m$. Setting $m = m'w_i(E)$, it follows that the subgroup $D_i = w_i(E)K_{2i}(E)$ of $K_{2i}(E)$ is a divisible group. Thus $K_{2i}(E) \cong D_i \oplus \mathbb{Z}/w_i(E)$, and $K_{2i}(V) \cong D_i \oplus \mathbb{Z}/w_i^{(p)}(E)$.

It remains to show that T_i and D_i are uniquely ℓ-divisible for $\ell \neq p$ and $i > 0$, *i.e.*, that $T_i/\ell T_i = {}_{\ell}(D_i) = 0$. By Universal Coefficients IV.2.5 and the calculations above, $K_{2i-1}(V;\mathbb{Z}/\ell^\nu)$ is isomorphic to $\mathbb{Z}/w_i^{(\ell)}(E) \oplus T_i/\ell^\nu T_i \oplus {}_{\ell^\nu}(D_i)$ for large ν. By Gabber Rigidity IV.2.10, $K_{2i-1}(V;\mathbb{Z}/\ell^\nu)$ and $K_{2i-1}(\mathbb{F}_q;\mathbb{Z}/\ell^\nu) \cong w_i^{(\ell)}(\mathbb{F}_q)$ are isomorphic. Since $w_i^{(\ell)}(E) = w_i^{(\ell)}(\mathbb{F}_q)$ by 2.3.1, we have $T_i/\ell^\nu T_i = {}_{\ell^\nu}(D_i) = 0$, as required. □

VI.7.3.1 **Remark 7.3.1.** Either T_i fails to be p-divisible, or else D_{i-1} has p-torsion. This follows from Corollary 7.4.1 below: $\dim(T_i/pT_i) + \dim({}_pD_{i-1}) = [E:\mathbb{Q}_p]$.

We now consider the p-adic K-groups $K_*(E;\mathbb{Z}_p)$ of E, as in IV.2.9. This result was first proved in [161, 3.7] for $p = 2$, and in [88, thm. A] for $p > 2$.

VI.7.4 **Theorem 7.4.** *Let E be a local field, of degree d over $\mathbb{Q}_p$, with ring of integers V. Then for $n \geq 2$ we have:*

$$K_n(V;\mathbb{Z}_p) \cong K_n(E;\mathbb{Z}_p) \cong \begin{Bmatrix} \mathbb{Z}/w_i^{(p)}(E), & n = 2i, \\ (\mathbb{Z}_p)^d \oplus \mathbb{Z}/w_i^{(p)}(E), & n = 2i-1. \end{Bmatrix}$$

Proof. It is classical that the groups $H^*_{\text{et}}(E;\mathbb{Z}/p^\nu)$ are finitely generated groups, and that the $H^*_{\text{et}}(E;\mathbb{Z}/p)$ are finitely generated $\mathbb{Z}_p$-modules. Since this implies that the groups $K_n(E;\mathbb{Z}/p)$ are finitely generated, this implies (by IV.2.9) that

$K_n(E;\mathbb{Z}_p)$ is an extension of the Tate module of $K_{n-1}(E)$ by the p-adic completion of $K_n(E)$. Since the Tate module of $K_{2i-1}(E)$ is trivial by 7.3, this implies that $K_{2i}(E;\mathbb{Z}_p) \cong \varprojlim K_{2i}(E)/p^\nu \cong \mathbb{Z}/w_i^{(p)}(E)$.

By 7.3 and IV.2.9, $K_n(V;\mathbb{Z}_p) \cong K_n(E;\mathbb{Z}_p)$ for all $n \geq 0$. Hence it suffices to consider the p-adic group $K_{2i-1}(V;\mathbb{Z}_p)$. By 7.3 and IV.2.9 again, $K_{2i-1}(V;\mathbb{Z}_p)$ is the direct sum of the finite p-group $\mathbb{Z}/w_i^{(p)}(E)$ and two finitely generated torsionfree $\mathbb{Z}_p$-modules: the Tate module of D_{i-1} and $T_i \otimes_{\mathbb{Z}} \mathbb{Z}_p$. All that is left is to calculate the rank of $K_{2i-1}(V;\mathbb{Z}_p)$.

Wagoner proved in [213] that the $\mathbb{Q}_p$-vector space $K_n(V;\mathbb{Z}_p) \otimes \mathbb{Q}$ has dimension $[E:\mathbb{Q}_p]$ when n is odd and $n \geq 3$. (Wagoner's continuous K-groups were identified with $K_*(E;\mathbb{Z}_p)$ in [149].) Hence $K_{2i-1}(V;\mathbb{Z}_p)$ has rank $d = [E:\mathbb{Q}_p]$. $\square$

VI.7.4.1 **Corollary 7.4.1.** *For $i > 1$ and all large ν we have*

$$K_{2i-1}(E;\mathbb{Z}/p^\nu) \cong H^1_{et}(E,\mu_{p^\nu}^{\otimes i}) \cong (\mathbb{Z}/p^\nu)^{[E:\mathbb{Q}_p]} \oplus \mathbb{Z}/w_i^{(p)}(E) \oplus \mathbb{Z}/w_{i-1}^{(p)}(E).$$

Proof. This follows from Universal Coefficients and Theorem 7.4. $\square$

VI.7.4.2 **Corollary 7.4.2.** *$K_3(V)$ contains a torsionfree subgroup isomorphic to $\mathbb{Z}^d_{(p)}$, whose p-adic completion is isomorphic to $K_3(V;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$.*

Proof. Combine 7.4 with Moore's theorem III.6.2.4 that D_1 is torsionfree. $\square$

VI.7.4.3 **Remark 7.4.3.** Surprisingly, the cohomology groups $H^1_{\text{et}}(E;\mu_m^{\otimes i})$ (for $m = p^\nu$) were not known before the K-group $K_{2i-1}(E;\mathbb{Z}/m)$ was calculated, circa 2000.

VI.7.5 **Warning 7.5.** Unfortunately, I do not know how to reconstruct the homotopy groups $K_n(V)$ from the information in 7.4. Any of the $\mathbb{Z}_p$'s in $K_{2i-1}(V;\mathbb{Z}_p)$ could come from either a $\mathbb{Z}_{(p)}$ in $K_{2i-1}(V)$ or a $\mathbb{Z}/p^\infty$ in $K_{2i-2}(V)$. Another problem is illustrated by the case $n = 1$, $V = \mathbb{Z}_p$ and $p \neq 2$. The information that $\varprojlim K_1(V)/p^\nu \cong (1+\pi V)^\times \cong V$ is not enough to deduce that $K_1(V) \otimes \mathbb{Z}_{(p)} \cong V$.

Even if we knew that $\varprojlim K_n(V)/p^\nu = \mathbb{Z}_p$, we would still not be able to determine the underlying abelian group $K_n(V)$ exactly. To see why, note that the extension $0 \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z}_{(p)} \to 0$ doesn't split, because there are no p-divisible elements in $\mathbb{Z}_p$, yet $\mathbb{Z}_p/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Q}$ is a uniquely divisible abelian group. For example, I doubt that the extension $0 \to \mathbb{Z}^d_{(p)} \to K_3(V) \to U_3 \to 0$ splits in Corollary 7.4.2.

Here are some more cases when I can show that the $\mathbb{Z}_p$'s in $K_{2i-1}(V;\mathbb{Z}_p)$ come from torsionfree elements in $K_{2i-1}(E)$; I do not know any example where a $\mathbb{Z}/p^\infty$ appears in $K_{2i}(E)$.

VI.7.6 **Example 7.6.** If $k > 0$, then $K_{4k+1}(\mathbb{Z}_2)$ contains a subgroup T isomorphic to $\mathbb{Z}_{(2)}$, and the quotient $K_{4k+1}(\mathbb{Z}_2)/(T \oplus \mathbb{Z}/w_i(\mathbb{Q}_2))$ is uniquely divisible. (By Exercise 2.3, $w_i(\mathbb{Q}_2) = 2(2^{2k+1} - 1)$.) This follows from Rognes' theorem [158, 4.13] that the map from $K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus (\mathbb{Z}/2)$ to $K_{4k+1}(\mathbb{Z}_2; \mathbb{Z}_2)$ is an isomorphism for all $k > 1$. (The information about the torsion subgroups, missing in [158], follows from 7.4 and 7.4.1.) Since this map factors through $K_{4k+1}(\mathbb{Z}_2)$, the assertion follows.

VI.7.7 **Example 7.7.** Let F be a totally imaginary number field of degree $d = 2r_2$ over $\mathbb{Q}$, with s prime ideals over p, and let $E_1, ..., E_s$ be the completions of F at these primes. By Borel's Theorem IV.1.18, there is a subgroup of $K_{2i-1}(F)$ isomorphic to $\mathbb{Z}^{r_2}$; its image in $\oplus K_{2i-1}(E_j)$ is a subgroup of rank at most r_2, while $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$ has rank $2r_2 = \sum [E_j : \mathbb{Q}_p]$. So these subgroups of $K_{2i-1}(E_j)$ can account for at most half of the torsionfree part of $\oplus K_{2i-1}(E_j; \mathbb{Z}_p)$.

VI.7.8 **Example 7.8.** Suppose that F is a totally real number field, of degree $d = r_1$ over $\mathbb{Q}$, and let $E_1, ..., E_s$ be the completions of F at the prime ideals over p. By Borel's Theorem IV.1.18, there is a subgroup of $K_{4k+1}(F)$ isomorphic to $\mathbb{Z}^d$ for all $k > 0$; its image in $\oplus K_{4k+1}(E_j)$ is a subgroup of rank d. Although $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$ also has rank $d = \sum [E_j : \mathbb{Q}_p]$, this does not imply that the p-adic completion $\mathbb{Z}_p^d$ of the subgroup injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p)$.

Implications like this are related to Leopoldt's conjecture, which states that the torsionfree part $\mathbb{Z}_p^{d-1}$ of $(\mathcal{O}_F)^\times \otimes \mathbb{Z}_p$ injects into the torsionfree part $\mathbb{Z}_p^d$ of $\prod_{j=1}^s \mathcal{O}_{E_j}^\times$; see [216, 5.31]. This conjecture has been proven when F is an abelian extension of $\mathbb{Q}$; see [216, 5.32].

When F is a totally real abelian extension of $\mathbb{Q}$, and p is a regular prime, Soulé shows in [173, 3.1, 3.7] that the torsion free part $\mathbb{Z}_p^d$ of $K_{4k+1}(F) \otimes \mathbb{Z}_p$ injects into $\oplus K_{4k+1}(E_j; \mathbb{Z}_p) \cong (\mathbb{Z}_p)^d$, because the cokernel is determined by the Leopoldt p-adic L-function $L_p(F, \omega^{2k}, 2k+1)$, which is a p-adic unit in this favorable scenario. Therefore in this case we also have a summand $\mathbb{Z}_{(p)}^d$ in each of the groups $K_{4k+1}(E_j)$.

We conclude this section with a description of the topological type of the p-adic completions (see IV.2.9) $\hat{\mathbf{K}}(V)_p$ and $\hat{\mathbf{K}}(E)_p$, when p is odd, due to Hesselholt and Madsen [88]. Recall that $F\Psi^k$ denotes the homotopy fiber of $\Psi^k - 1 : \mathbb{Z} \times BU \to BU$. Since $\Psi^k = k^i$ on $\pi_{2i}(BU) = \mathbb{Z}$ when $i > 0$, and $\pi_{2i-1}(BU) = 0$, we see that $\pi_{2i-1} F\Psi^k \cong \mathbb{Z}/(k^i - 1)$, and that all even homotopy groups of $F\Psi^k$ vanish, except for $\pi_0(F\Psi^k) = \mathbb{Z}$.

VI.7.9 **Theorem 7.9.** ([88, thm.D]) *Let E be a local field, of degree d over $\mathbb{Q}_p$, with p odd. Then after p-completion, there is a number k (given below) so that*

$$\hat{\mathbf{K}}(V)_p \simeq SU \times U^{d-1} \times F\Psi^k \times BF\Psi^k, \qquad \hat{\mathbf{K}}(E)_p \simeq U^d \times F\Psi^k \times BF\Psi^k.$$

The number k is defined as follows. As in Proposition 2.2, let p^a be the number of p-primary roots of unity in $E(\mu_p)$ and set $r = [E(\mu_p) : E]$. If γ is a topological generator of $\mathbb{Z}_p^\times$, then $k = \gamma^n$, where $n = p^{a-1}(p-1)/r$. (See Exercise 7.4.)

EXERCISES

EVI.7.1 **7.1.** Show that $M = \prod_{i=1}^{\infty} \mathbb{Z}_p$ is not a free $\mathbb{Z}_p$-module. *Hint:* Consider the submodule S of all $(a_1, ...)$ in M where all but finitely many a_i are divisible by p^ν for all ν. If M were free, S would also be free. Show that S/p is countable, and that $\prod p^i \mathbb{Z}_p$ is an uncountably generated submodule of S. Hence S cannot be free.

EVI.7.2 **7.2.** When $E = \mathbb{F}_q[[\pi]]$, show that the subgroup $W(\mathbb{F}_q) = 1 + \pi\mathbb{F}_q[[\pi]]$ of units is a module over $\mathbb{Z}_p$, by defining $(1+\pi f)^a$ for all power series f and all $a \in \mathbb{Z}_p$. If $\{u_i\}$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, show that every element of $W(\mathbb{F}_q)$ is uniquely the product of terms $(1+u_i t^n)^{a_{ni}}$, where $a_{ni} \in \mathbb{Z}_p$. This shows that $W(\mathbb{F}_q)$ is a countably infinite product of copies of $\mathbb{Z}_p$. Using Ex. 7.1, conclude that $W(\mathbb{F}_q)$ is not a free $\mathbb{Z}_p$-module.

EVI.7.3 **7.3.** Show that the first étale Chern classes $K_{2i-1}(E;\mathbb{Z}/p^\nu) \cong H^1(E, \mu_{p^\nu}^{\otimes i})$ are natural isomorphisms for all i and ν.

EVI.7.4 **7.4.** In Theorem 7.9, check that $\pi_{2i-1}F\Psi^k \cong \mathbb{Z}_p/(k^i-1)$ is $\mathbb{Z}/w_i(E)$ for all i.

8 Number fields at primes where $cd = 2$

In this section we quickly obtain a cohomological description of the odd torsion in the K-groups of a number field, and also the 2-primary torsion in the K-groups of a totally imaginary number field. These are the cases where $cd_\ell(\mathcal{O}_S) = 2$; see [128, 4.10]. This bound forces the motivic spectral sequence (4.2) to degenerate completely, leaving an easily-solved extension problem.

VI.8.1 **Classical Data 8.1.** Let $\mathcal{O}_S$ be a ring of integers in a number field F. By Chapter IV, 1.18 and 6.9, the groups $K_n(F)$ are finite when n is even and nonzero; if n is odd and $n \geq 3$ the groups $K_n(F)$ are the direct sum of a finite group and $\mathbb{Z}^r$, where r is r_2 when $n \equiv 3 \pmod 4$ and $r_1 + r_2$ when $n \equiv 1 \pmod 4$. Here r_1 is the number of real embeddings of F, and r_2 is the number of complex embeddings (up to conjugacy), so that $[F : \mathbb{Q}] = r_1 + 2r_2$. The formulas for $K_0(\mathcal{O}_S) = \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_S)$ and $K_1(\mathcal{O}_S) = \mathcal{O}_S^\times \cong \mathbb{Z}^{r_2+|S|-1} \oplus \mu(F)$ are different; see II.2.6.3 and III.1.3.6.

The Brauer group of $\mathcal{O}_S$ is determined by the sequence

$$0 \to \mathrm{Br}(\mathcal{O}_S) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\substack{v\in S\\ \text{finite}}} (\mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{add}} \mathbb{Q}/\mathbb{Z} \to 0. \qquad (8.1.1)$$ VI.8.1.1

The notation $A_{(\ell)}$ will denote the localization of an abelian group A at the prime ℓ, and the notation ${}_\ell A$ will denote the subgroup $\{a \in A \mid \ell a = 0\}$.

VI.8.2 **Theorem 8.2.** *Let F be a number field, and let $\mathcal{O}_S$ be a ring of integers in F. Fix a prime ℓ; if $\ell = 2$ we suppose F totally imaginary. Then for all $n \geq 2$:*

$$K_n(\mathcal{O}_S)_{(\ell)} \cong \begin{cases} H^2_{et}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i+1)) & \text{for } n = 2i > 0; \\ \mathbb{Z}^{r_2}_{(\ell)} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ even}; \\ \mathbb{Z}^{r_2+r_1}_{(\ell)} \oplus \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i-1,\ i \text{ odd}. \end{cases}$$

Proof. Set $R = \mathcal{O}_S[1/\ell]$. For each prime ideal $\mathfrak{p}$ over ℓ, $K_{n-1}(R/\mathfrak{p})$ has no ℓ-torsion by IV.1.13. By the localization sequence (V, (6.6) or 6.8), $K_n(\mathcal{O}_S)_{(\ell)} = K_n(R)_{(\ell)}$. Thus we may replace $\mathcal{O}_S$ by $R = \mathcal{O}_S[1/\ell]$. Since the rank of $K_n(R)$ is classically known (see 8.1), it suffices by IV.2.9 and Ex. IV.2.6 to determine $K_{2i-1}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}/\ell^\infty)$ and $K_{2i}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}_\ell)$.

If F is a number field, the étale ℓ-cohomological dimension of R (and of F) is 2, unless $\ell = 2$ and $r_1 > 0$ (F has a real embedding). Since $H^2_{\mathrm{et}}(R; \mathbb{Z}/\ell^\infty(i)) = 0$ by Ex. 8.1, the motivic spectral sequence (4.2) with coefficients $\mathbb{Z}/\ell^\infty$ has at most one nonzero entry in each total degree at the E_2 page. Thus we may read off:

$$K_n(R; \mathbb{Z}/\ell^\infty) \cong \begin{cases} H^0(R; \mathbb{Z}/\ell^\infty(i)) = \mathbb{Z}/w_i^{(\ell)}(F) & \text{for } n = 2i \geq 2, \\ H^1(R; \mathbb{Z}/\ell^\infty(i)) & \text{for } n = 2i-1 \geq 1. \end{cases}$$

The description of $K_{2i-1}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}/\ell^\infty)$ follows.

The same argument works for ℓ-adic coefficients $\mathbb{Z}_\ell$; for $i > 0$ we have $H^n_{\rm et}(R, \mathbb{Z}_\ell(i)) = 0$ for $n \neq 1, 2$, so the spectral sequence (4.2) degenerates to yield $K_{2i-1}(R; \mathbb{Z}_\ell) \cong H^1_{\rm et}(R, \mathbb{Z}_\ell(i))$ and $K_{2i}(R; \mathbb{Z}_\ell) \cong H^2_{\rm et}(R, \mathbb{Z}_\ell(i+1))$ (which is a finite group by Exercise 8.2). The description of $K_{2i}(R)\{\ell\} = K_{2i}(R; \mathbb{Z}_\ell)$ follows. □

VI.8.3 **Corollary 8.3.** *For all odd ℓ and $i > 0$, $K_{2i}(\mathcal{O}_S)/\ell \cong H^2_{et}(\mathcal{O}_S[1/\ell], \mu_\ell^{\otimes i+1})$.*
The same formula holds for $\ell = 2$ if F is totally imaginary.

Proof. Immediate from 8.2 since $H^2_{\rm et}(R, \mathbb{Z}_\ell(i+1))/\ell \cong H^2_{\rm et}(R, \mu_\ell^{\otimes i+1})$. □

VI.8.3.1 **Example 8.3.1.** Let F be a number field containing a primitive ℓ^{th} root of unity, $\ell \neq 2$, and let S be the set of primes over ℓ in $\mathcal{O}_F$. If t is the rank of $\mathrm{Pic}(\mathcal{O}_S)/\ell$, then $H^2_{\rm et}(\mathcal{O}_S, \mu_\ell)$ has rank $t + |S| - 1$ by (8.1.1). Since $H^2_{\rm et}(\mathcal{O}_S, \mu_\ell^{\otimes i+1}) \cong H^2_{\rm et}(\mathcal{O}_S, \mu_\ell) \otimes \mu_\ell^{\otimes i}$, it follows from Corollary 8.3 that $K_{2i}(\mathcal{O}_S)/\ell$ has rank $t + |S| - 1$. Hence the ℓ-primary subgroup of the finite group $K_{2i}(\mathcal{O}_F)$ has $t + |S| - 1$ nonzero summands for each $i \geq 1$.

VI.8.3.2 **Example 8.3.2.** If $\ell \neq 2$ is a regular prime (see 2.4.1), we claim that $K_{2i}(\mathbb{Z}[\zeta_\ell])$ has no ℓ-torsion. The case K_0 is tautological since $\mathrm{Pic}(\mathcal{O}_F)/\ell = 0$ by definition. Setting $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, we have $|S| = 1$ and $\mathrm{Br}(R) = 0$ by (8.1.1). The case K_2 is known classically; see III.6.9.3. By Theorem 8.2, $K_{2i}(\mathbb{Z}[\zeta_\ell])_{(\ell)} \cong H^2_{\rm et}(R, \mathbb{Z}_\ell(i+1))$. By Example 8.3.1, $H^2_{\rm et}(R, \mathbb{Z}_\ell(i+1)) = 0$ and the claim now follows.

Note that every odd-indexed group $K_{2i-1}(\mathbb{Z}[\zeta_\ell]) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$ has nontrivial ℓ-torsion, because $w_i^{(\ell)}(F) \geq \ell$ for all i by 2.2.

Combining Theorems 8.1 and 8.2, we obtain a description of $K_*(\mathcal{O}_S)$ when F is totally imaginary. This includes exceptional number fields such as $\mathbb{Q}(\sqrt{-7})$.

VI.8.4 **Theorem 8.4.** *Let F be a totally imaginary number field, and let $\mathcal{O}_S$ be the ring of S-integers in F for some set S of finite places. Then:*

$$K_n(\mathcal{O}_S) \cong \begin{cases} \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_S), & \text{for } n = 0; \\ \mathbb{Z}^{r_2+|S|-1} \oplus \mathbb{Z}/w_1(F), & \text{for } n = 1; \\ \oplus_\ell\ H^2_{et}(\mathcal{O}_S[1/\ell]; \mathbb{Z}_\ell(i+1)) & \text{for } n = 2i \geq 2; \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F) & \text{for } n = 2i-1 \geq 3. \end{cases}$$

Proof. The cases $n = 0, 1$ and the ranks of K_n are part of the Classical Data 8.1. Since F is totally imaginary, the torsion comes from Theorem 8.2. □

Similarly, the mod-ℓ spectral sequence (4.2) collapses when ℓ is odd to yield the K-theory of $\mathcal{O}_S$ with coefficients $\mathbb{Z}/\ell$, as our next example illustrates.

VI.8.5 **Example 8.5.** If $\mathcal{O}_S$ contains a primitive ℓ^{th} root of unity and $1/\ell \in \mathcal{O}_S$ then $H^1(\mathcal{O}_S; \mu_\ell^{\otimes i}) \cong \mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell} \oplus {}_\ell\,\mathrm{Pic}(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S; \mu_\ell^{\otimes i}) \cong \mathrm{Pic}(\mathcal{O}_S)/\ell \oplus {}_\ell\mathrm{Br}(\mathcal{O}_S)$ for all i, so $K_0(\mathcal{O}_S; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell \oplus \mathrm{Pic}(\mathcal{O}_S)/\ell$ and

$$K_n(\mathcal{O}_S; \mathbb{Z}/\ell) \cong \begin{cases} \mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell} \oplus {}_\ell\,\mathrm{Pic}(\mathcal{O}_S) & \text{for } n = 2i-1 \geq 1, \\ \mathbb{Z}/\ell \oplus \mathrm{Pic}(\mathcal{O}_S)/\ell \oplus {}_\ell\mathrm{Br}(\mathcal{O}_S) & \text{for } n = 2i \geq 2. \end{cases}$$

The $\mathbb{Z}/\ell$ summands in degrees $2i$ are generated by the powers β^i of the Bott element $\beta \in K_2(\mathcal{O}_S;\mathbb{Z}/\ell)$ (see IV.2.5.2). In fact, $K_*(\mathcal{O}_S;\mathbb{Z}/\ell)$ is free as a graded $\mathbb{Z}[\beta]$-module on $\mathrm{Pic}(\mathcal{O}_S)/\ell$, $K_1(\mathcal{O}_S;\mathbb{Z}/\ell)$ and ${}_\ell\mathrm{Br}(\mathcal{O}_S) \subseteq K_2(\mathcal{O}_S;\mathbb{Z}/\ell)$; this is immediate from the multiplicative properties of (4.2) described in 4.2.1. Taking the direct limit over S, we also have $K_0(F) = \mathbb{Z}/\ell$ and

$$K_n(F;\mathbb{Z}/\ell) \cong \begin{cases} F^\times/F^{\times\ell}, & \text{for } n = 2i-1 \geq 1, \\ \mathbb{Z}/\ell \oplus {}_\ell\mathrm{Br}(F) & \text{for } n = 2i \geq 2. \end{cases}$$

We conclude this section with a comparison to the odd part of $\zeta_F(1-2k)$.

VI.8.6 **Birch-Tate Conjecture 8.6.** If F is a number field, the zeta function $\zeta_F(s)$ has a pole of order r_2 at $s=-1$. Birch and Tate conjectured in 1970 that for totally real number fields ($r_2 = 0$) we have

$$\zeta_F(-1) = (-1)^{r_1}|K_2(\mathcal{O}_F)|/w_2(F).$$

The odd part of this conjecture was proven by Wiles in [229], using Tate's calculation that $K_2(\mathcal{O}_S)/m \cong H^2_{\mathrm{et}}(\mathcal{O}_S,\mu_m^{\otimes 2})$ when $1/m \in \mathcal{O}_S$ (see [198] or III(6.10.4)).

The two-primary part of the Birch-Tate conjecture is still open, but it is known to be a consequence of the 2-adic Main Conjecture of Iwasawa Theory (see Kolster's appendix to [161]). This was proven by Wiles for abelian extensions F of $\mathbb{Q}$ in *loc. cit.*, so the full Birch-Tate Conjecture holds for all abelian extensions of $\mathbb{Q}$. For example, when $F=\mathbb{Q}$ we have $\zeta_\mathbb{Q}(-1) = -1/12$, $|K_2(\mathbb{Z})| = 2$ and $w_2(\mathbb{Q}) = 24$; see the Classical Data 8.1

To generalize the Birch-Tate Conjecture 8.6, we invoke the following deep result of Wiles [229, Thm. 1.6], which is often called the "Main Conjecture" of Iwasawa Theory.

VI.8.7 **Theorem 8.7.** *(Wiles) Let F be a totally real number field. If ℓ is odd and $\mathcal{O}_S = \mathcal{O}_F[1/\ell]$, then for all even integers $2k > 0$ there is a rational number u_k, prime to ℓ, such that:*

$$\zeta_F(1-2k) = u_k\ \frac{|H^2_{et}(\mathcal{O}_S,\mathbb{Z}_\ell(2k)|}{|H^1_{et}(\mathcal{O}_S,\mathbb{Z}_\ell(2k)|}.$$

The numerator and denominator on the right side are finite (Ex. 8.1–8.2). Note that if F is not totally real then $\zeta_F(s)$ has a pole of order r_2 at $s = 1-2k$.

We can now verify a conjecture of Lichtenbaum, made in [112, 2.4–2.6], which was only stated up to powers of 2.

VI.8.8 **Theorem 8.8.** *If F is totally real, and $\mathrm{Gal}(F/\mathbb{Q})$ is abelian, then for all $k \geq 1$:*

$$\zeta_F(1-2k) = (-1)^{kr_1}\, 2^{r_1}\frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|}.$$

Proof. We first show that the left and right sides of 8.8 have the same power of each odd prime ℓ. The group $H^2_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(2k))$ is the ℓ-primary part of $K_{4k-2}(\mathcal{O}_F)$ by Theorem 8.2. The group $H^1_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(2k))$ in the numerator of 8.7 is $K_{4k-1}(\mathcal{O}_F)_{(\ell)} \cong \mathbb{Z}/w^{(\ell)}_{2k}(F)$ by the proof of 8.2; the details of this identification are left to Exercise 8.3.

By the functional equation for ζ_F, the sign of $\zeta_F(1-2k)$ is $(-1)^{kr_1}$. Therefore it remains to check the power of 2 in Theorem 8.8. By Theorem 9.12 in the next section, the power of 2 on the right side equals $|H^2_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_2(2k))|/|H^1_{\text{et}}(\mathcal{O}_F[1/\ell], \mathbb{Z}_2(2k))|$. By the 2-adic Main Conjecture of Iwasawa Theory, which is known for abelian F, this equals the 2-part of $\zeta_F(1-2k)$. □

EXERCISES

EVI.8.1 **8.1.** Suppose that ℓ is odd, or that F is totally imaginary. If R is any ring of integers in F containing $1/\ell$, and $i \ge 2$, show that $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i))$ is a finite group, and conclude that $H^2_{\text{et}}(R, \mathbb{Z}/\ell^\infty(i)) = 0$. *Hint:* Use (4.2) to compare it to $K_{2i-2}(R)$, which is finite by IV.1.18, IV.2.9 and IV.6.9. Then apply Ex. IV.2.6.

EVI.8.2 **8.2.** Supose that F is any number field, and let R be any ring of integers in F containing $1/\ell$. Show that $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i))$ is a finite group for all $i \ge 2$, *Hint:* Let R' be the integral closure of R in $F(\sqrt{-1})$. Use a transfer argument to show that the kernel A of $H^2_{\text{et}}(R, \mathbb{Z}_\ell(i)) \to H^2_{\text{et}}(R', \mathbb{Z}_\ell(i))$ has exponent 2; A is finite because it injects into $H^2_{\text{et}}(R, \mu_2)$. Now apply Exercise 8.1.

EVI.8.3 **8.3.** Let ℓ be an odd prime and F a number field. If $i > 1$, show that for every ring $\mathcal{O}_S$ of integers in F containing $1/\ell$,

$$H^1_{\text{et}}(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \cong H^1_{\text{et}}(F, \mathbb{Z}_\ell(i)) \cong \mathbb{Z}_\ell^r \oplus \mathbb{Z}/w_i^{(\ell)}(F),$$

where r is r_2 for even i and $r_1 + r_2$ for odd i. *Hint:* Compare to $K_{2i-1}(F; \mathbb{Z}_\ell)$, as in the proof of 8.2.

EVI.8.4 **8.4.** It is well known that $\mathbb{Z}[i]$ is a principal ideal domain. Show that the finite group $K_n(\mathbb{Z}[i])$ has odd order for all even $n > 0$. *Hint:* Show that $H^2_{\text{et}}(\mathbb{Z}[\frac{1}{2}, i], \mu_4) = 0$.

EVI.8.5 **8.5.** Show that $K_3(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/24$, $K_7(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/240$ and

$$K_{4k+1}(\mathbb{Z}[i]) \cong \mathbb{Z} \oplus \mathbb{Z}/4$$

for all $k > 0$. Note that the groups $w_i(\mathbb{Q}(\sqrt{-1}))$ are given in 2.1.2.

EVI.8.6 **8.6.** Let F be a number field. Recall (Ex. IV.7.10) that there is a canonical involution on $K_*(F)$, and that it is multiplication by -1 on $K_1(F) = F^\times$. Show that it is multiplication by $(-1)^i$ on $K_{2i-1}(\mathcal{O}_F)$ and $K_{2i-2}(\mathcal{O}_F)$ for $i > 1$. *Hint:* Pick an odd prime ℓ and consider the canonical involution on $K_{2i-1}(F(\zeta_\ell); \mathbb{Z}/\ell)$.

9 Real number fields at the prime 2

Let F be a real number field, i.e., F has $r_1 > 0$ embeddings into $\mathbb{R}$. The calculation of the algebraic K-theory of F at the prime 2 is somewhat different from the calculation at odd primes, for two reasons. One reason is that a real number field has infinite cohomological dimension, which complicates descent methods. A second reason is that the Galois group of a cyclotomic extension need not be cyclic, so that the e-invariant may not split (see Example 2.1.2). A final reason, explained in IV.2.8, is that, while $K_*(F;\mathbb{Z}/2^\nu)$ is a graded ring for $2^\nu = 8$ and a graded-commutative ring for $2^\nu \geq 16$, its graded product may be non-associative and non-commutative for $2^\nu = 4$, and the groups $K_*(F;\mathbb{Z}/2)$ do not have a natural multiplication.

For the real numbers $\mathbb{R}$, the mod 2 motivic spectral sequence has $E_2^{p,q} = \mathbb{Z}/2$ for all p, q in the octant $q \leq p \leq 0$. In order to distinguish between these terms, it is useful to label the nonzero elements of $H^0_{\text{et}}(\mathbb{R}, \mathbb{Z}/2(i))$ as β_i, writing 1 for β_0. Using the multiplicative pairing with the spectral sequence $'E_2^{*,*}$ converging to $K_*(\mathbb{R})$, multiplication by the element η of $'E_2^{0,-1} = H^1(\mathbb{R}, \mathbb{Z}(1))$ allows us to write the nonzero elements in the $-i$th column as $\eta^j\beta_i$. (See Table 9.1.1 below)

From Suslin's calculation of $K_n(\mathbb{R})$ in Theorem 3.1, we know that the groups $K_n(\mathbb{R};\mathbb{Z}/2)$ are cyclic and 8-periodic (for $n \geq 0$) with orders $2, 2, 4, 2, 2, 0, 0, 0$ for $n = 0, 1, \dots, 7$. The unexpected case $K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$ is described in IV.2.5.1.

VI.9.1 **Theorem 9.1.** *In the spectral sequence (4.2) converging to $K_*(\mathbb{R};\mathbb{Z}/2)$, all the d_2 differentials with nonzero source on the lines $p \equiv 1, 2 \pmod 4$ are isomorphisms. Hence the spectral sequence degenerates at E_3. The only extensions are the nontrivial extensions $\mathbb{Z}/4$ in $K_{8a+2}(\mathbb{R};\mathbb{Z}/2)$.*

Proof. Recall from 4.8.1 that the mod 2 spectral sequence has periodicity isomorphisms $E_r^{p,q} \xrightarrow{\simeq} E_r^{p-4,q-4}$, $p \leq 0$. Therefore it suffices to work with the columns $-3 \leq p \leq 0$. These columns are shown in Table 9.1.1.

Because $K_3(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$, the differential closest to the origin, from β_2 to η^3, must be nonzero. Since the pairing with $'E_2$ is multiplicative and $d_2(\eta) = 0$, we must have $d_2(\eta^j\beta_2) = \eta^{j+3}$ for all $j \geq 0$. Thus the column $p = -2$ of E_3 is zero, and every term in the column $p = 0$ of E_3 is zero except for $\{1, \eta, \eta^2\}$.

Similarly, we must have $d_2(\beta_3) = \eta^3\beta_1$ because $K_5(\mathbb{R};\mathbb{Z}/2) = 0$. By multiplicativity, this yields $d_2(\eta^j\beta_3) = \eta^{j+3}\beta_1$ for all $j \geq 0$. Thus the column $p = -3$ of E_3 is zero, and every term in the column $p = -1$ of E_3 is zero except for $\{\beta_1, \eta\beta_1, \eta^2\beta_1\}$. □

VI.9.1.2 **Variant 9.1.2.** The spectral sequence (4.2) with coefficients $\mathbb{Z}/2^\infty$ is very similar, except that when $p > q$, $E_2^{p,q} = H^{p-q}_{\text{et}}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: 0 for p even; $\mathbb{Z}/2$ for p odd. If p is odd, the coefficient map $\mathbb{Z}/2 \to \mathbb{Z}/2^\infty$ induces isomorphisms on the $E_2^{p,q}$ terms, so by Theorem 9.1 all the d_2 differentials with nonzero source in the columns $p \equiv 1 \pmod 4$ are isomorphisms. Again, the spectral sequence converging to $K_*(\mathbb{R};\mathbb{Z}/2^\infty)$ degenerates at $E_3 = E_\infty$. The only extensions are the nontrivial extensions of $\mathbb{Z}/2^\infty$ by $\mathbb{Z}/2$ in $K_{8a+4}(\mathbb{R};\mathbb{Z}/2^\infty) \cong \mathbb{Z}/2^\infty$.

$p=-3$	-2	-1	0
			1
		β_1	η
	β_2	$\eta\beta_1$	η^2
β_3	$\eta\beta_2$	$\eta^2\beta_1$	η^3
$\eta\beta_3$	$\eta^2\beta_2$	$\eta^3\beta_1$	η^4

The first 4 columns of E_2

-3	-2	-1	0
			1
		β_1	η
	0	$\eta\beta_1$	η^2
0	0	$\eta^2\beta_1$	0
0	0	0	0

The same columns of E_3

Table 9.1.1: The mod 2 spectral sequence for $\mathbb{R}$. VI.9.1.1

VI.9.1.3 **Variant 9.1.3.** The analysis of the spectral sequence with 2-adic coefficients is very similar, except that (a) $H^0(\mathbb{R};\mathbb{Z}_2(i))$ is: $\mathbb{Z}_2$ for i even; 0 for i odd and (b) (for $p>q$) $E_2^{p,q}=H_{\rm et}^{p-q}(\mathbb{R};\mathbb{Z}/2^\infty(-q))$ is: $\mathbb{Z}/2$ for p even; 0 for p odd. All differentials with nonzero source in the column $p\equiv 2 \pmod 4$ are onto. Since there are no extensions to worry about, we omit the details.

We now consider the K-theory of the ring $\mathcal{O}_S$ of integers in a number field F with coefficients $\mathbb{Z}/2^\infty$. The E_2 terms in the spectral sequence (4.2) are the (étale) cohomology groups $H^n(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$. Following Tate, the r_1 real embeddings of F define natural maps $\alpha_S^n(i)$:

$$H^n(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \xrightarrow{\alpha_S^n(i)} \bigoplus^{r_1} H^n(\mathbb{R};\mathbb{Z}/2^\infty(i)) \cong \begin{cases}(\mathbb{Z}/2)^{r_1}, & i-n \text{ odd}\\ 0, & i-n \text{ even.}\end{cases} \quad (9.2)$$ VI.9.2

This map is an isomorphism for all $n\geq 3$ by Tate-Poitou duality [128, I(4.20)]. It is also an isomorphism for $n=2$ and $i\geq 2$, as shown in Exercise 9.1.

Write $\widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for the kernel of $\alpha_S^1(i)$.

VI.9.3 **Lemma 9.3.** *For even i, $H^1(F;\mathbb{Z}/2^\infty(i)) \xrightarrow{\alpha^1(i)} (\mathbb{Z}/2)^{r_1}$ is a split surjection. Hence $H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))\cong(\mathbb{Z}/2)^{r_1}\oplus\widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ for sufficiently large S.*

Proof. By the strong approximation theorem for units of F, the left vertical map is a split surjection in the diagram:

$$\begin{array}{ccccc}
F^\times/F^{\times 2} & \xrightarrow{\cong} & H^1(F,\mathbb{Z}/2) & \longrightarrow & H^1(F,\mathbb{Z}/2^\infty(i)) \\
\text{split onto}\downarrow\oplus\sigma & & \downarrow & & \alpha^1(i)\downarrow \\
(\mathbb{Z}/2)^{r_1} = \oplus\,\mathbb{R}^\times/\mathbb{R}^{\times 2} & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2) & \xrightarrow{\cong} & \oplus H^1(\mathbb{R},\mathbb{Z}/2^\infty(i)).
\end{array}$$

Since $F^\times/F^{\times 2}$ is the direct limit (over S) of the groups $\mathcal{O}_S^\times/\mathcal{O}_S^{\times 2}$, we may replace F by $\mathcal{O}_S$ for sufficiently large S. □

The next result is taken from [161, 6.9]. When $n\equiv 5 \pmod 8$, we have an unknown group extension; to express it, we write $A\rtimes B$ for an abelian group extension of B by A.

VI.9.4 **Theorem 9.4.** *Let F be a real number field, and let $\mathcal{O}_S$ be a ring of S-integers in F containing $\mathcal{O}_F[\frac{1}{2}]$. Then $\alpha^1_S(4k)$ is onto ($k>0$), and for all $n \ge 0$:*

$$K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty) \cong \begin{cases} \mathbb{Z}/w_{4k}(F) & \text{for } n=8k,\\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+1)) & \text{for } n=8k+1,\\ \mathbb{Z}/2 & \text{for } n=8k+2,\\ H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+2)) & \text{for } n=8k+3,\\ \mathbb{Z}/2w_{4k+2}\oplus(\mathbb{Z}/2)^{r_1-1} & \text{for } n=8k+4,\\ (\mathbb{Z}/2)^{r_1-1}\rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+3)) & \text{for } n=8k+5,\\ 0 & \text{for } n=8k+6,\\ \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2^\infty(4k+4)) & \text{for } n=8k+7. \end{cases}$$

Proof. Consider the morphism α_S of spectral sequences (VI.4.2 4.2) with coefficients $\mathbb{Z}/2^\infty$, from that for $\mathcal{O}_S$ to the direct sum of r_1 copies of that for $\mathbb{R}$. By naturality, the morphism in the $E_2^{p,q}$ spot is the map $\alpha_S^{p-q}(-q)$ of (VI.9.2 9.2). By Tate-Poitou duality and Ex. EVI.9.1 9.1, this is an isomorphism on $E_2^{p,q}$ except on the diagonal $p=q$, where it is the injection of $\mathbb{Z}/w^{(2)}_{-q}(F)$ into $\mathbb{Z}/2^\infty$, and on the critical diagonal $p=q+1$.

When $p\equiv +1 \pmod 4$, we saw in VI.9.1.2 9.1.2 that $d_2^{p,q}(\mathbb{R})$ is an isomorphism whenever $q\le p<0$. It follows that we may identify $d_2^{p,q}(\mathcal{O}_S)$ with α_S^{p-q}. Therefore $d_2^{p,q}(\mathcal{O}_S)$ is an isomorphism if $p\ge 2+q$, and an injection if $p=q$. As in VI.9.1.2 9.1.2, the spectral sequence degenerates at E_3, yielding $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ as proclaimed, except for two points: (a) when $n=8k+4$, the extension of $\mathbb{Z}/w_{4k+2}$ by $(\mathbb{Z}/2)^{r_1}$ is seen to be nontrivial by comparison with the extension for $\mathbb{R}$, and (b) when $n=8k+6$, it only shows that $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ is the cokernel of $\alpha^1_S(4k+4)$.

To resolve (b) we must show that the map $\alpha^1_S(4k+4)$ is onto when $k\ge 0$. Set $n=8k+6$. Since $K_n(\mathcal{O}_S)$ is finite, $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)$ must equal the 2-primary subgroup of $K_{n-1}(\mathcal{O}_S)$, which is independent of S by V.6.8 V.6.8. But for sufficiently large S, the map $\alpha^1(4k+4)$ is a surjection by Lemma VI.9.3 9.3, and hence $K_n(\mathcal{O}_S;\mathbb{Z}/2^\infty)=0$. □

VI.9.5 **Theorem 9.5.** *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F. Then for each odd $n\ge 3$, the group $K_n(\mathcal{O}_S)\cong K_n(F)$ is given by:*

(a) If F is totally imaginary, $K_n(F)\cong \mathbb{Z}^{r_2}\oplus\mathbb{Z}/w_i(F)$;

(b) If F has $r_1>0$ real embeddings then, setting $i=(n+1)/2$,

$$K_n(F)\cong\begin{cases} \mathbb{Z}^{r_1+r_2}\oplus\mathbb{Z}/w_i(F), & n\equiv 1 \pmod 8\\ \mathbb{Z}^{r_2}\oplus\mathbb{Z}/2w_i(F)\oplus(\mathbb{Z}/2)^{r_1-1}, & n\equiv 3 \pmod 8\\ \mathbb{Z}^{r_1+r_2}\oplus\mathbb{Z}/\tfrac{1}{2}w_i(F), & n\equiv 5 \pmod 8\\ \mathbb{Z}^{r_2}\oplus\mathbb{Z}/w_i(F), & n\equiv 7 \pmod 8). \end{cases}$$

Note that these groups are determined only by the number r_1, r_2 of real and complex places of F and the integers $w_i(F)$.

Proof. Part (a), when F is totally imaginary, is given by Theorem 8.4. In case (b), since the rank is classically known (see 8.1), and $K_n(\mathcal{O}_S) \cong K_n(F)$ by V.6.8, it suffices to determine the torsion subgroup of $K_n(\mathcal{O}_S)$. The odd torsion is given by Theorem 8.2, so we need only worry about the 2-primary torsion. Since $K_{n+1}(\mathcal{O}_S)$ is finite, it follows from Ex. IV.2.6 that the 2-primary subgroup of $K_n(\mathcal{O}_S)$ is $K_{n+1}(\mathcal{O}_S;\mathbb{Z}/2^\infty)$, which we can read off from Theorem 9.4, recalling from 2.3(b) that $w_i^{(2)}(F) = 2$ for odd i. □

VI.9.5.1 **Example 9.5.1.** $K_n(\mathbb{Q}) \cong \mathbb{Z}$ for all $n \equiv 5 \pmod 8$ as $w_i(\mathbb{Q}) = 2$; see 2.1.2. More generally, if F has a real embedding and $n \equiv 5 \pmod 8$, then $K_n(F)$ has no 2-primary torsion, because $\frac{1}{2}w_i(F)$ is an odd integer when i is odd; see 2.3(b).

The narrow Picard group

To determine the 2-primary torsion in $K_n(\mathcal{O}_S)$ when n is even, we need to introduce the narrow Picard group and the signature defect of the ring $\mathcal{O}_S$. We begin with some terminology.

Each real embedding $\sigma_i : F \to \mathbb{R}$ determines a map $F^\times \to \mathbb{R}^\times \to \mathbb{Z}/2$, detecting the sign of units of F under that embedding. The sum of these is the *sign map* $\sigma : F^\times \longrightarrow (\mathbb{Z}/2)^{r_1}$; it is surjective by the strong approximation theorem for F. The kernel $F_+^\times$ of σ is called the group of *totally positive units* in F, since it consists of units which are positive under every real embedding.

If $R = \mathcal{O}_S$ is a ring of integers in F, we write $R_+^\times$ for $R^\times \cap F_+^\times$, the subgroup of totally positive units in R. Since the sign map σ factors through $F^\times/F^{\times 2} = H^1_{\text{et}}(F,\mathbb{Z}/2)$, the restriction to $R^\times$ also factors through $\alpha^1 : H^1_{\text{et}}(R,\mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$. This map is part of a family of maps

$$\alpha^n : H^n_{\text{et}}(R,\mathbb{Z}/2) \to \oplus_{r_1} H^n_{\text{et}}(\mathbb{R},\mathbb{Z}/2) = (\mathbb{Z}/2)^{r_1} \tag{9.6}$$ VI.9.6

related to the maps $\alpha^n(i)$ in (9.2). By Tate-Poitou duality, α^n is an isomorphism for all $n \geq 3$; it is a surjection for $n = 2$ (see Ex. 9.2). We will be interested in α^1. The following classical definitions are due to Weber; see [42, 5.2.7] or [145, VI.1].

VI.9.6.1 **Definition 9.6.1.** The *signature defect* $j(R)$ of R is defined to be the dimension of the cokernel of α^1. Since the sign of -1 nontrivial, we have $0 \leq j(R) < r_1$. Note that $j(F) = 0$, and that $j(\mathcal{O}_S) \leq j(\mathcal{O}_F)$ for all S.

The *narrow Picard group* $\text{Pic}_+(R)$ is defined to be the cokernel of the restricted divisor map $F_+^\times \to \bigoplus_{\wp \notin S} \mathbb{Z}$ of I.3.5; it is a finite group. Some authors call $\text{Pic}_+(\mathcal{O}_S)$ the *ray class group* and write it as Cl_F^S.

The kernel of the restricted divisor map is clearly $R_+^\times$, and it is easy to see from this that there is an exact sequence

$$0 \to R_+^\times \to R^\times \xrightarrow{\sigma} (\mathbb{Z}/2)^{r_1} \to \text{Pic}_+(R) \to \text{Pic}(R) \to 0.$$

For simplicity we write $H^n(R,\mathbb{Z}/2)$ for $H^n_{\text{et}}(R,\mathbb{Z}/2)$ and, as in (9.2), we define $\widetilde{H}^n(R;\mathbb{Z}/2)$ to be the kernel of α^n. A diagram chase (left to Ex. 9.3 shows that there is an exact sequence

$$0\to\widetilde{H}^1(R;\mathbb{Z}/2)\to H^1(R;\mathbb{Z}/2)\xrightarrow{\alpha^1}(\mathbb{Z}/2)^{r_1}\to\text{Pic}_+(R)/2\to\text{Pic}(R)/2\to 0. \tag{9.6.2}$$ VI.9.6.2

Thus the signature defect $j(R)$ of R is also the dimension of the kernel of $\text{Pic}_+(R)/2\to\text{Pic}(R)/2$. If we let t and u denote the dimensions of $\text{Pic}(R)/2$ and $\text{Pic}_+(R)/2$, respectively, then this means that $u=t+j(R)$. If s denotes the number of finite places of $\mathcal{O}_S$, then $\dim H^1(\mathcal{O}_S;\mathbb{Z}/2)=r_1+r_2+s+t$ and $\dim H^2(\mathcal{O}_S;\mathbb{Z}/2)=r_1+s+t-1$. This follows from 8.1 and (8.1.1), using Kummer theory.

VI.9.6.3 **Lemma 9.6.3.** *Suppose that $\frac{1}{2}\in\mathcal{O}_S$. Then $\dim\widetilde{H}^1(\mathcal{O}_S,\mathbb{Z}/2)=r_2+s+u$, and $\dim\widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}/2)=t+s-1$.*

Proof. The first assertion is immediate from (9.6.2). Since α^2 is onto, the second assertion follows. □

VI.9.7 **Theorem 9.7.** *Let F be a real number field, and $\mathcal{O}_S$ a ring of integers containing $\frac{1}{2}$. If $j=j(\mathcal{O}_S)$ is the signature defect, then the mod 2 algebraic K-groups of $\mathcal{O}_S$ are given (up to extensions) for $n>0$ as follows:*

$$K_n(\mathcal{O}_S;\mathbb{Z}/2)\cong\begin{cases}\widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2)\oplus\mathbb{Z}/2 & \text{for } n=8k,\\ H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+1,\\ H^2(\mathcal{O}_S;\mathbb{Z}/2)\rtimes\mathbb{Z}/2 & \text{for } n=8k+2,\\ (\mathbb{Z}/2)^{r_1-1}\rtimes H^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+3,\\ (\mathbb{Z}/2)^j\rtimes H^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+4,\\ (\mathbb{Z}/2)^{r_1-1}\rtimes\widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+5,\\ (\mathbb{Z}/2)^j\oplus\widetilde{H}^2(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+6,\\ \widetilde{H}^1(\mathcal{O}_S;\mathbb{Z}/2) & \text{for } n=8k+7.\end{cases}$$

Proof. (Cf. [161, 7.8].) As in the proof of Theorem 9.4, we compare the spectral sequence for $R=\mathcal{O}_S$ with the sum of r_1 copies of the spectral sequence for $\mathbb{R}$. For $n\geq 3$ we have $H^n(R;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^{r_1}$. It is not hard to see that we may identify the differentials $d_2: H^n(R,\mathbb{Z}/2)\to H^{n+3}(R,\mathbb{Z}/2)$ with the maps α^n. Since these maps are described in 9.6, we see from periodicity 4.8.1 that the columns $p\leq 0$ of E_3 are 4-periodic, and all nonzero entries are described by Figure 9.7.1.

As in Example 4.6, the E_2 page of the spectral sequence (4.2) has only one nonzero entry for $p>0$, namely $E_3^{+1,-1}=\text{Pic}(R)/2$, and it only affects $K_0(R;\mathbb{Z}/2)$. By inspection, $E_3=E_\infty$, yielding the desired description of the groups $K_n(R,\mathbb{Z}/2)$ in terms of extensions. The proof that the extensions split for $n\equiv 0,6 \pmod 8$ is left to Exercises 9.4 and 9.5. □

$p=-3$	-2	-1	0
			1
		β_1	H^1
	0	H^1	H^2
0	$\widetilde{H}^1$	H^2	$(\mathbb{Z}/2)^{r_1-1}$
$\widetilde{H}^1$	$\widetilde{H}^2$	$(\mathbb{Z}/2)^{r_1-1}$	$(\mathbb{Z}/2)^j$
$\widetilde{H}^2$	0	$(\mathbb{Z}/2)^j$	0
0	0	0	0

The first 4 columns of $E_3 = E_\infty$

Table 9.7.1: The mod 2 spectral sequence for $\mathcal{O}_S$. VI.9.7.1

The case $F = \mathbb{Q}$ has historical importance, because of its connection with the image of J (see 2.1.3 or [155]) and classical number theory. The following result was first established in [224]; the groups are not truly periodic only because the order of $K_{8k-1}(\mathbb{Z})$ depends upon k.

VI.9.8 **Corollary 9.8.** *For $n \geq 0$, the 2-primary subgroups of $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ are essentially periodic, of period eight, and are given by the following table for $n \geq 2$. (When $n \equiv 7 \pmod 8$, we set $k = (n+1)/8$.)*

$n \pmod 8$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})\{2\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/16k$	0

In particular, $K_n(\mathbb{Z})$ and $K_n(\mathbb{Z}[1/2])$ have odd order for all $n \equiv 4, 6, 8 \pmod 8$, and the finite group $K_{8k+2}(\mathbb{Z})$ is the sum of $\mathbb{Z}/2$ and a finite group of odd order. We will say more about the odd torsion in the next section.

Proof. When n is odd, this is Theorem 9.5; $w_{4k}^{(2)}$ is the 2-primary part of $16k$ by 2.3(c). For $R = \mathbb{Z}[1/2]$ we have $s = 1$ and $t = u = j = 0$. By Lemma 9.6.3 we have $\dim \widetilde{H}^1(R;\mathbb{Z}/2) = 1$ and $\widetilde{H}^2(R;\mathbb{Z}/2) = 0$. By 9.7, the groups $K_n(\mathbb{Z}[1/2];\mathbb{Z}/2)$ are periodic of orders 2, 4, 4, 4, 2, 2, 1, 2 for $n \equiv 0, 1, ..., 7$ respectively. The groups $K_n(\mathbb{Z}[1/2])$ for n odd, given in 9.5, together with the $\mathbb{Z}/2$ summand in $K_{8k+2}(\mathbb{Z})$ provided by topology (see 2.1.3), account for all of $K_n(\mathbb{Z}[1/2];\mathbb{Z}/2)$, and hence must contain all of the 2-primary torsion in $K_n(\mathbb{Z}[1/2])$. □

Recall that the 2-*rank* of an abelian group A is just the dimension of $\mathrm{Hom}(\mathbb{Z}/2, A)$. We have already seen (in Theorem 9.5) that for $n \equiv 1, 3, 5, 7 \pmod 8$ the 2-ranks of $K_n(\mathcal{O}_S)$ are: 1, r_1, 0 and 1, respectively.

VI.9.9 **Corollary 9.9.** *For $n \equiv 2, 4, 6, 8 \pmod 8$, $n > 0$, the respective 2-ranks of the finite groups $K_n(\mathcal{O}_S)$ are: $r_1 + s + t - 1$, $j + s + t - 1$, $j + s + t - 1$ and $s + t - 1$.*

Here j is the signature defect of $\mathcal{O}_S$ (9.6.1), s is the number of finite places of $\mathcal{O}_S$ and t is the rank of $\mathrm{Pic}(\mathcal{O}_S)/2$.

Proof. Since $K_n(R;\mathbb{Z}/2)$ is an extension of $\mathrm{Hom}(\mathbb{Z}/2, K_{n-1}R)$ by $K_n(R)/2$, and the dimensions of the odd groups are known, we can read this off from the list given in Theorem 9.7, using Lemma 9.6.3. □

VI.9.9.1 **Example 9.9.1.** Consider $F = \mathbb{Q}(\sqrt{p})$, where p is prime. When $p \equiv 1 \pmod 8$, it is well known that $t = j = 0$ but $s = 2$. It follows that $K_{8k+2}(\mathcal{O}_F)$ has 2-rank 3, while the two-primary summand of $K_n(\mathcal{O}_F)$ is nonzero and cyclic when $n \equiv 4, 6, 8 \pmod 8$.

When $p \equiv 7 \pmod 8$, we have $j = 1$ for both $\mathcal{O}_F$ and $R = \mathcal{O}_F[1/2]$. Since $r_1 = 2$ and $s = 1$, the 2-ranks of the finite groups $K_n(R)$ are: $t+2$, $t+1$, $t+1$ and t for $n \equiv 2, 4, 6, 8 \pmod 8$ by 9.9. For example, if $t = 0$ ($\mathrm{Pic}(R)/2 = 0$) then $K_n(R)$ has odd order for $n \equiv 8 \pmod 8$, but the 2-primary summand of $K_n(R)$ is $(\mathbb{Z}/2)^2$ when $n \equiv 2$ and is cyclic when $n \equiv 4, 6$.

VI.9.9.2 **Example 9.9.2.** (2–regular fields) A number field F is said to be *2–regular* if there is only one prime over 2 and the narrow Picard group $\mathrm{Pic}_+(\mathcal{O}_F[\frac{1}{2}])$ is odd (*i.e.,* $t = u = 0$ and $s = 1$). In this case, we see from 9.9 that $K_{8k+2}(\mathcal{O}_F)$ is the sum of $(\mathbb{Z}/2)^{r_1}$ and a finite odd group, while $K_n(\mathcal{O}_F)$ has odd order for all $n \equiv 4, 6, 8 \pmod 8$ ($n > 0$). In particular, the map $K_4^M(F) \to K_4(F)$ must be zero, since it factors through the odd order group $K_4(\mathcal{O}_F)$, and $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

Browkin and Schinzel [35] and Rognes and Østvær [160] have studied this case. For example, when $F = \mathbb{Q}(\sqrt{m})$ and $m > 0$ ($r_1 = 2$), the field F is 2-regular exactly when $m = 2$, or $m = p$ or $m = 2p$ with $p \equiv 3, 5 \pmod 8$ prime. (See [35].)

A useful example is $F = \mathbb{Q}(\sqrt{2})$. Note that $K_4^M(F) \cong (\mathbb{Z}/2)^2$ is generated by the Steinberg symbols $\{-1,-1,-1,-1\}$ and $\{-1,-1,-1,1+\sqrt{2}\}$. Both symbols must vanish in $K_4(\mathbb{Z}[\sqrt{2}])$, since this group has odd order. This is the case $j = 0$, $r_1 = 2$ of Corollary 9.10.

Let ρ denote the rank of the image of the group $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(F)$.

VI.9.10 **Corollary 9.10.** *Let F be a real number field. Then $j(\mathcal{O}_F[1/2]) \le \rho \le r_1 - 1$. The image $(\mathbb{Z}/2)^\rho$ of $K_4^M(F) \to K_4(F)$ lies in the subgroup $K_4(\mathcal{O}_F)$, and its image in $K_4(\mathcal{O}_S)/2$ has rank $j(\mathcal{O}_S)$ whenever S contains all primes over 2.*

In particular, the image $(\mathbb{Z}/2)^\rho$ of $K_4^M(F) \to K_4(F)$ lies in $2 \cdot K_4(F)$.

Proof. By Ex. IV.1.12(d), $\{-1,-1,-1,-1\}$ is nonzero in $K_4^M(F)$ but zero in $K_4(F)$. Since $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ by III.7.2(d), we have $\rho < r_1$. The assertion that $K_4^M(F) \to K_4(F)$ factors through $K_4(\mathcal{O}_F)$ follows from the fact that $K_3(\mathcal{O}_F) = K_3(F)$ (see V.6.8), by multiplying $K_3^M(F)$ and $K_3(\mathcal{O}_F) \cong K_3(F)$ by $[-1] \in K_1(\mathbb{Z})$. We saw in 4.3 that the edge map $H^n(F, \mathbb{Z}(n)) \to K_n(F)$ in the motivic spectral sequence agrees with the usual map $K_n^M(F) \to K_n(F)$. By Theorem 4.1 (due to Voevodsky), $K_n^M(F)/2^\nu \cong H^n(F, \mathbb{Z}(n))/2^\nu \cong H^n(F, \mathbb{Z}/2^\nu(n))$. For $n = 4$, the image of the edge map from $H^4(\mathcal{O}_S, \mathbb{Z}/2^\nu(4)) \cong H^4(F, \mathbb{Z}/2^\nu(4)) \to K_4(\mathcal{O}_S; \mathbb{Z}/2)$ has rank j by Table 9.7.1; this implies the assertion that the image in $K_4(\mathcal{O}_S)/2 \subset K_4(\mathcal{O}_S;\mathbb{Z}/2)$ has rank $j(\mathcal{O}_S)$. Finally, taking $\mathcal{O}_S = \mathcal{O}_F[1/2]$ yields the inequality $j(\mathcal{O}_S) \le \rho$. □

VI.9.10.1 **Example 9.10.1.** ($\rho = 1$) Consider $F = \mathbb{Q}(\sqrt{7})$, $\mathcal{O}_F = \mathbb{Z}[\sqrt{7}]$ and $R = \mathcal{O}_F[1/2]$; here $s = 1$, $t = 0$ and $j(R) = \rho = 1$ (the fundamental unit $u = 8+3\sqrt{7}$ is totally positive). Hence the image of $K_4^M(F) \cong (\mathbb{Z}/2)^2$ in $K_4(\mathbb{Z}[\sqrt{7}])$ is $\mathbb{Z}/2$ on the symbol $\sigma = \{-1,-1,-1,\sqrt{7}\}$, and this is all of the 2-primary torsion in $K_4(\mathbb{Z}[\sqrt{7}])$ by 9.9.

On the other hand, $\mathcal{O}_S = \mathbb{Z}[\sqrt{7}, 1/7]$ still has $\rho = 1$, but now $j = 0$, and the 2-rank of $K_4(\mathcal{O}_S)$ is still one by 9.9. Hence the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/48 \to 0$ of V.6.8 cannot be split, implying that the 2-primary subgroup of $K_4(\mathcal{O}_S)$ must then be $\mathbb{Z}/32$.

In fact, the nonzero element σ is *divisible* in $K_4(F)$. This follows from the fact that if $p \equiv 3 \pmod{28}$ then there is an irreducible $q = a+b\sqrt{7}$ whose norm is $-p = q\bar{q}$. Hence $R' = \mathbb{Z}[\sqrt{7}, 1/2q]$ has $j(R') = 0$ but $\rho = 1$, and the extension $0 \to K_4(\mathcal{O}_F) \to K_4(\mathcal{O}_S) \to \mathbb{Z}/(p^2-1) \to 0$ of V.6.8 is not split. If in addition $p \equiv -1 \pmod{2^\nu}$ — there are infinitely many such p for each ν — then there is an element v of $K_4(R')$ such that $2^{\nu+1}v = \sigma$ See [226] for details.

VI.9.10.2 **Question 9.10.2.** Can ρ be less than the minimum of $r_1 - 1$ and $j+s+t-1$?

As in (9.2), when i is even we define $\widetilde{H}^2(R;\mathbb{Z}_2(i))$ to be the kernel of $\alpha^2(i) : H^2(R;\mathbb{Z}_2(i)) \to H^2(\mathbb{R};\mathbb{Z}_2(i))^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. By Lemma 9.6.3, $\widetilde{H}^2(R;\mathbb{Z}_2(i))$ has 2-rank $s+t-1$. The following result is taken from [161, 0.6].

VI.9.11 **Theorem 9.11.** *Let F be a number field with at least one real embedding, and let $R = \mathcal{O}_S$ denote a ring of integers in F containing $1/2$. Let j be the signature defect of R, and write w_i for $w_i^{(2)}(F)$.*

Then there is an integer ρ, $j \le \rho < r_1$, such that, for all $n \ge 2$, the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of $K_n(\mathcal{O}_S)$ is isomorphic to:

$$K_n(\mathcal{O}_S)\{2\} \cong \begin{cases} H^2_{et}(R;\mathbb{Z}_2(4k+1)) & \text{for } n = 8k, \\ \mathbb{Z}/2 & \text{for } n = 8k+1, \\ H^2_{et}(R;\mathbb{Z}_2(4k+2)) & \text{for } n = 8k+2, \\ (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}/2w_{4k+2} & \text{for } n = 8k+3, \\ (\mathbb{Z}/2)^{\rho} \rtimes H^2_{et}(R;\mathbb{Z}_2(4k+3)) & \text{for } n = 8k+4, \\ 0 & \text{for } n = 8k+5, \\ \widetilde{H}^2_{et}(R;\mathbb{Z}_2(4k+4)) & \text{for } n = 8k+6, \\ \mathbb{Z}/w_{4k+4} & \text{for } n = 8k+7. \end{cases}$$

Proof. When $n = 2i-1$ is odd, this is Theorem 9.5, since $w_i^{(2)}(F) = 2$ when $n \equiv 1 \pmod 4$ by 2.3(b). When $n = 2$ it is III.6.9.3. To determine the two-primary subgroup $K_n(\mathcal{O}_S)\{2\}$ of the finite group $K_{2i+2}(\mathcal{O}_S)$ when $n = 2i+2$, we use the universal coefficient sequence

$$0 \to (\mathbb{Z}/2^\infty)^r \to K_{2i+3}(\mathcal{O}_S;\mathbb{Z}/2^\infty) \to K_{2i+2}(\mathcal{O}_S)\{2\} \to 0,$$

where r is the rank of $K_{2i+3}(\mathcal{O}_S)$ and is given by 8.1 ($r = r_1 + r_2$ or r_2). To compare this with Theorem 9.4, we note that $H^1(\mathcal{O}_S, \mathbb{Z}/2^\infty(i))$ is the direct sum of $(\mathbb{Z}/2^\infty)^r$ and a finite group, which must be $H^2(\mathcal{O}_S, \mathbb{Z}_2(i))$ by universal coefficients; see [161, 2.4(b)]. Since $\alpha_S^1(i) : H^1(R;\mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$

must vanish on the divisible group $(\mathbb{Z}/2^\infty)^r$, it induces the natural map $\alpha_S^2(i) : H^2_{\rm et}(\mathcal{O}_S;\mathbb{Z}_2(i)) \to (\mathbb{Z}/2)^{r_1}$ and

$$\widetilde{H}^1(\mathcal{O}_S,\mathbb{Z}/2^\infty(i)) \cong (\mathbb{Z}/2^\infty)^r \oplus \widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}_2(i)).$$

This proves all of the theorem, except for the description of $K_n(\mathcal{O}_S)$, $n = 8k+4$. By mod 2 periodicity 4.8.1, the integer ρ of 9.10 equals the rank of the image of $H^4(\mathcal{O}_S,\mathbb{Z}/2(4)) \cong H^4(\mathcal{O}_S,\mathbb{Z}/2(4k+4)) \cong (\mathbb{Z}/2)^{r_1}$ in $\mathrm{Hom}(\mathbb{Z}/2, K_n(\mathcal{O}_S))$, considered as a quotient of $K_{n+1}(\mathcal{O}_S;\mathbb{Z}/2)$. □

We can combine the 2-primary information in 9.11 with the odd torsion information in 8.2 and 8.8 to relate the orders of K-groups to the orders of étale cohomology groups. Up to a factor of 2^{r_1}, they were conjectured by Lichtenbaum in [112]. Let $|A|$ denote the order of a finite abelian group A.

VI.9.12 **Theorem 9.12.** *Let F be a totally real number field, with r_1 real embeddings, and let $\mathcal{O}_S$ be a ring of integers in F. Then for all even $i > 0$*

$$2^{r_1}\cdot\frac{|K_{2i-2}(\mathcal{O}_S)|}{|K_{2i-1}(\mathcal{O}_S)|} = \frac{\prod_\ell |H^2_{et}(\mathcal{O}_S[1/\ell];\mathbb{Z}_\ell(i))|}{\prod_\ell |H^1_{et}(\mathcal{O}_S[1/\ell];\mathbb{Z}_\ell(i))|}.$$

Proof. Since $2i-1 \equiv 3 \pmod 4$, all groups involved are finite (see 8.1, Ex. 8.2 and Ex. 8.3.) Write $h^{n,i}(\ell)$ for the order of $H^n_{\rm et}(\mathcal{O}_S[1/\ell];\mathbb{Z}_\ell(i))$. By Ex. 8.3, $h^{1,i}(\ell) = w_i^{(\ell)}(F)$. By 9.5, the ℓ-primary subgroup of $K_{2i-1}(\mathcal{O}_S)$ has order $h^{1,i}(\ell)$ for all odd ℓ and all even $i > 0$, and also for $\ell = 2$ with the exception that when $2i-1 \equiv 3 \pmod 8$ then the order is $2^{r_1}h^{1,i}(2)$.

By Theorems 8.2 and 9.11, the ℓ-primary subgroup of $K_{2i-2}(\mathcal{O}_S)$ has order $h^{2,i}(\ell)$ for all ℓ, except when $\ell = 2$ and $2i-2 \equiv 6 \pmod 8$ when it is $h^{1,i}(2)/2^{r_1}$. Combining these cases yields the formula asserted by the theorem. □

Theorem 9.12 was used in the previous section (Theorem 8.8) to equate the ratio of orders of the finite groups $K_{4k-2}(\mathcal{O}_F)$ and $K_{4k-1}(\mathcal{O}_F)$ with $|\zeta_F(1-2k)|/2^{r_1}$.

EXERCISES

EVI.9.1 **9.1.** Suppose that F has $r_1 > 0$ embeddings into $\mathbb{R}$. Show that

$$H^2_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \cong H^2_{\rm et}(F;\mathbb{Z}/2^\infty(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1}, & i \ge 3 \text{ odd} \\ 0, & i \ge 2 \text{ even.}\end{cases}$$

Using (8.1.1), determine $H^2_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2^\infty(1))$. *Hint:* Compare F with $F(\sqrt{-1})$, and use Exercise 8.1 to see that $H^2_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2^\infty(i))$ has exponent 2. Hence the Kummer sequence is:

$$0 \to H^2_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \to H^3_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2) \to H^3_{\rm et}(\mathcal{O}_S;\mathbb{Z}/2^\infty(i)) \to 0.$$

Now plug in the values of the H^3 groups, which are known by (9.2).

EVI.9.2 **9.2.** Show that α^2 is onto. *Hint:* Use Ex. 9.1 and the coefficient sequence for $\mathbb{Z}/2 \subset \mathbb{Z}/2^\infty(4)$ to show that the map $H^2_{\mathrm{et}}(R;\mathbb{Z}/2) \to H^2_{\mathrm{et}}(R;\mathbb{Z}/2^\infty(4))$ is onto.

EVI.9.3 **9.3.** Establish the exact sequence (9.6.2). (This is taken from [161, 9.6].)

EVI.9.4 **9.4.** The stable homotopy group $\pi_{8k}(QS^0;\mathbb{Z}/2)$ contains an element β_{8k} of exponent 2 which maps onto the generator of $K_{8k}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$; see [161, 5.1]. Use it to show the extension $K_{8k}(\mathcal{O}_S;\mathbb{Z}/2)$ of $\mathbb{Z}/2$ by $\widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}/2)$ splits in Theorem 9.7.

EVI.9.5 **9.5.** Show that the extension $K_{8k+6}(\mathcal{O}_S;\mathbb{Z}/2)$ splits in Thm. 9.7. Conclude that $K_{8k+6}(\mathcal{O}_S)/2 \cong \widetilde{H}^2(\mathcal{O}_S,\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^j$. *Hint:* use Example 9.5.1.

EVI.9.6 **9.6.** Let $R = \mathcal{O}_F[1/2]$, where F is a real number field. Show that $K_{8k+4}(R;\mathbb{Z}/2)$ is an extension of ${}_2\mathrm{Br}(R)$ by $\mathrm{Pic}_+(R)/2$.

Let $\mathrm{Br}_+(R)$ denote the kernel of the canonical map $\mathrm{Br}(R) \to (\mathbb{Z}/2)^{r_1}$ induced by (8.1.1). Show that $K_{8k+6}(R;\mathbb{Z}/2) \cong \mathrm{Pic}_+(R)/2 \oplus {}_2\mathrm{Br}_+(R)$. (See [161, 7.8].)

10 The K-theory of $\mathbb{Z}$

The determination of the groups $K_n(\mathbb{Z})$ has been a driving force in the development of K-theory. We saw in Chapters II and III that the groups $K_0(\mathbb{Z})$, $K_1(\mathbb{Z})$ and $K_2(\mathbb{Z})$ are related to very classical mathematics. In the 1970's, homological methods led to the calculation of the rank of $K_n(\mathbb{Z})$ by Borel (see 8.1) and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ by Lee and Szczarba (see Example 2.1.2 or [108]).

In order to describe the groups $K_n(\mathbb{Z})$, we use the Bernoulli numbers B_k. We let c_k denote the numerator of $B_k/4k$; c_k is a product of irregular primes (see 2.4.1). We saw in Lemma 2.4 that the denominator of $B_k/4k$ is w_{2k}, so $B_k/4k = c_k/w_{2k}$.

Theorem 10.1. *For $n \not\equiv 0 \pmod 4$ and $n > 1$, we have:*

(1) If $n = 8k+1$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$;

(2) If $n = 8k+2$, $|K_n(\mathbb{Z})| = 2c_{2k+1}$;

(3) If $n = 8k+3$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/2w_{4k+2}$;

(5) If $n = 8k+5$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}$;

(6) If $n = 8k+6$, $|K_n(\mathbb{Z})| = c_{2k+1}$;

(7) If $n = 8k+7$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/w_{4k+4}$.

As a consequence, for $k \ge 1$ we have: $\dfrac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|} = \dfrac{B_k}{4k} = \dfrac{(-1)^k}{2}\zeta(1-2k)$.

Proof. The equality $B_k/4k = (-1)^k\zeta(1-2k)/2$ comes from 2.4.2. The equality of this with $|K_{4k-2}(\mathbb{Z})|/|K_{4k-1}(\mathbb{Z})|$ comes from Theorem 8.8 (using 9.12). This gives the displayed formula.

When n is odd, the groups $K_n(\mathbb{Z})$ were determined in Theorem 9.5, and $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q})$ by V.6.8. Thus we may suppose that $n = 4k-2$. Since the 2-primary torsion in $K_n(\mathbb{Z})$ was determined in Corollary 9.8, we can ignore factors of 2. But up to a factor of 2, $|K_{4k-1}(\mathbb{Z})| = w_{2k}(\mathbb{Q})$ so the displayed formula yields $|K_{4k-2}(\mathbb{Z})|/w_{2k} = B_k/4k$ and hence $|K_{4k-2}(\mathbb{Z})| = c_k$. □

The groups $K_n(\mathbb{Z})$ are much harder to determine when $n \equiv 0 \pmod 4$. The group $K_4(\mathbb{Z})$ was proven to be zero in the late 1990's by Soulé and Rognes (see Remark 10.1.3 or [159]). If $n = 4i \ge 8$, the orders of the groups $K_{4i}(\mathbb{Z})$ are known to be products of irregular primes ℓ, with $\ell > 10^8$, and are conjectured to be zero; this conjecture follows from, and implies, Vandiver's conjecture (stated in 10.8 below).

In Table 10.1.1, we have summarized what we know for $n < 20{,}000$; conjecturally the same pattern holds for all n (see Theorem 10.2).

$K_0(\mathbb{Z}) = \mathbb{Z}$	$K_8(\mathbb{Z}) = (0?)$	$K_{16}(\mathbb{Z}) = (0?)$	$K_{8k}(\mathbb{Z}) = (0?),\ k \geq 1$
$K_1(\mathbb{Z}) = \mathbb{Z}/2$	$K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{8k+1}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
$K_2(\mathbb{Z}) = \mathbb{Z}/2$	$K_{10}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{18}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{8k+2}(\mathbb{Z}) = \mathbb{Z}/2c_{2k+1}$
$K_3(\mathbb{Z}) = \mathbb{Z}/48$	$K_{11}(\mathbb{Z}) = \mathbb{Z}/1008$	$K_{19}(\mathbb{Z}) = \mathbb{Z}/528$	$K_{8k+3}(\mathbb{Z}) = \mathbb{Z}/2w_{4k+2}$
$K_4(\mathbb{Z}) = 0$	$K_{12}(\mathbb{Z}) = (0?)$	$K_{20}(\mathbb{Z}) = (0?)$	$K_{8k+4}(\mathbb{Z}) = (0?)$
$K_5(\mathbb{Z}) = \mathbb{Z}$	$K_{13}(\mathbb{Z}) = \mathbb{Z}$	$K_{21}(\mathbb{Z}) = \mathbb{Z}$	$K_{8k+5}(\mathbb{Z}) = \mathbb{Z}$
$K_6(\mathbb{Z}) = 0$	$K_{14}(\mathbb{Z}) = 0$	$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$	$K_{8k+6}(\mathbb{Z}) = \mathbb{Z}/c_{2k+2}$
$K_7(\mathbb{Z}) = \mathbb{Z}/240$	$K_{15}(\mathbb{Z}) = \mathbb{Z}/480$	$K_{23}(\mathbb{Z}) = \mathbb{Z}/65520$	$K_{8k+7}(\mathbb{Z}) = \mathbb{Z}/w_{4k+4}$.

Table 10.1.1: The groups $K_n(\mathbb{Z})$, $n < 20,000$.
The notation '(0?)' refers to a finite group, conjecturally zero,
whose order is a product of irregular primes $> 10^8$. VI.10.1.1

VI.10.1.2 **Example 10.1.2** (Relation to π_n^s)**.** Using homotopy-theoretic techniques, the torsion subgroups of $K_n(\mathbb{Z})$ had been detected by the late 1970's, due to the work of Quillen [155], Harris-Segal [84], Soulé [171] and others.

As pointed out in Remark 2.1.3, the image of the natural maps $\pi_n^s \to K_n(\mathbb{Z})$ capture most of the Harris-Segal summands 2.5.1. When n is $8k+1$ or $8k+2$, there is a $\mathbb{Z}/2$-summand in $K_n(\mathbb{Z})$, generated by the image of Adams' element μ_n. (It is the 2-torsion subgroup by 9.8.) Since $w_{4k+1}(\mathbb{Q}) = 2$, we may view it as the Harris-Segal summand when $n = 8k+1$. When $n = 8k+5$, the Harris-Segal summand is zero by Example 9.5.1. When $n = 8k+7$ the Harris-Segal summand of $K_n(\mathbb{Z})$ is isomorphic to the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+4}(\mathbb{Q})$ of π_n^s.

When $n = 8k+3$, the subgroup $J(\pi_n O) \cong \mathbb{Z}/w_{4k+2}(\mathbb{Q})$ of π_n^s is contained in the Harris-Segal summand $\mathbb{Z}/(2w_i)$ of $K_n(\mathbb{Z})$; the injectivity was proven by Quillen in [155], and Browder showed that the order of the summand was $2w_i(\mathbb{Q})$.

The remaining calculations of $K_*(\mathbb{Z})$ depend upon the development of motivic cohomology, via the tools described in Section 4, and date to the period 1997–2007. The 2-primary torsion was resolved in 1997 using [211] (see Section 9), while the order of the odd torsion (conjectured by Lichtenbaum) was only determined using the Norm Residue Theorem 4.1 of Rost and Voevodsky.

VI.10.1.3 **Remark 10.1.3** (Homological methods)**.** Lee-Szczarba [108] and Soulé [172] used homological methods in the 1970s to show that $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ and that there is no p-torsion in $K_4(\mathbb{Z})$ or $K_5(\mathbb{Z})$ for $p > 3$. Much later, Rognes [159] and Elbaz-Vincent–Gangl–Soulé [53] refined this to show that $K_4(\mathbb{Z}) = 0$, $K_5(\mathbb{Z}) = \mathbb{Z}$, and that $K_6(\mathbb{Z})$ has at most 3-torsion. This used the calculation in [161] (using [211]) that there is no 2-torsion in $K_4(\mathbb{Z})$, $K_5(\mathbb{Z})$ or $K_6(\mathbb{Z})$.

Our general description of $K_*(\mathbb{Z})$ is completed by the following assertion, which follows immediately from Theorems 10.1, 10.9 and 10.10 below, It was observed independently by Kurihara [105] and Mitchell [137].

VI.10.2 **Theorem 10.2.** *If Vandiver's conjecture holds, then the groups $K_n(\mathbb{Z})$ are given by Table 10.2.1, for all $n \geq 2$. Here k is the integer part of $1 + \frac{n}{4}$.*

$n \pmod 8$	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

Table 10.2.1: The K-theory of $\mathbb{Z}$, assuming Vandiver's Conjecture VI.10.2.1

When n is at most 20,000 and $n \equiv 2 \pmod 4$, we show that the finite groups $K_n(\mathbb{Z})$ are cyclic in Examples 10.3 and 10.3.2. (The order is c_k or $2c_k$, where $k = (n+2)/4$, by Theorem VI.10.1.)

VI.10.3 **Examples 10.3.** For n at most 450, the group $K_n(\mathbb{Z})$ is cyclic because its order is squarefree. For $n \le 30$ we need only consult 2.4 to see that the groups $K_2(\mathbb{Z})$, $K_{10}(\mathbb{Z})$, $K_{18}(\mathbb{Z})$ and $K_{26}(\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2$, while $K_6(\mathbb{Z}) = K_{14}(\mathbb{Z}) = 0$. Since $c_6 = 691$, $c_8 = 3617$, $c_9 = 43867$ and $c_{13} = 657931$ are all prime, we have

$$K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691, \qquad K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617,$$

$K_{34}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/43867$ and $K_{50} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/657931$.

The next hundred values of c_k are also squarefree: $c_{10} = 283\cdot 617$, $c_{11} = 131\cdot 593$, $c_{12} = 103\cdot 2294797$, $c_{14} = 9349\cdot 362903$ and $c_{15} = 1721\cdot 1001259881$ are all products of two primes, while $c_{16} = 37\cdot 683\cdot 305065927$ is a product of 3 primes. Hence $K_{38}(\mathbb{Z}) = \mathbb{Z}/c_{10}$, $K_{42}(\mathbb{Z}) = \mathbb{Z}/2c_{11}$, $K_{46} = \mathbb{Z}/c_{12}$, $K_{54}(\mathbb{Z}) = \mathbb{Z}/c_{14}$, $K_{58}(\mathbb{Z}) = \mathbb{Z}/2c_{15}$ and

$$K_{62}(\mathbb{Z}) = \mathbb{Z}/c_{16} = \mathbb{Z}/37 \oplus \mathbb{Z}/683 \oplus \mathbb{Z}/305065927.$$

Thus the first occurrence of the smallest irregular prime (37) is in $K_{62}(\mathbb{Z})$; it also appears as a $\mathbb{Z}/37$ summand in $K_{134}(\mathbb{Z})$, $K_{206}(\mathbb{Z})$, $K_{494}(\mathbb{Z})$. In fact, there is 37-torsion in every group $K_{72a+62}(\mathbb{Z})$ (see Ex. 10.2). This direct method fails for $K_{454}(\mathbb{Z})$, because its order $2c_{114}$ is divisible by 103^2.

To go further, we need to consider the torsion in the groups $K_{4k-2}(\mathbb{Z})$ on a prime-by-prime basis. Since the 2-torsion has order at most 2 by 9.8, we may suppose that ℓ is an odd prime. Our method is to consider the cyclotomic extension $\mathbb{Z}[\zeta_\ell]$ of $\mathbb{Z}$, $\zeta_\ell = e^{2\pi i/\ell}$. Because $K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[1/\ell])$ is an isomorphism on ℓ-torsion (by the Localization Sequence V.6.6), and similarly for $K_n(\mathbb{Z}[\zeta_\ell]) \to K_n(\mathbb{Z}[\zeta_\ell, 1/\ell])$, it suffices to work with $\mathbb{Z}[1/\ell]$ and $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$.

VI.10.3.1 **The usual transfer argument 10.3.1.** The ring extension $\mathbb{Z}[1/\ell] \subset R$ is Galois and its Galois group $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is cyclic of order $\ell - 1$. The map $i^* : K_*(\mathbb{Z}) \to K_*(R)$ identifies $K_n(\mathbb{Z}[1/\ell])_{(\ell)}$ with $K_n(R)^G_{(\ell)}$ for all n, because $i_* i^*$ is multiplication by $|G|$ on $K_*(\mathbb{Z})$ and $i^* i_*$ is multiplication by $|G|$ on $K_n(R)^G$ (see Ex. IV.6.13). This style of argument is called the *usual transfer argument.*

VI.10.3.2 **Example 10.3.2.** The group $K_{4k-2}(\mathbb{Z})$ is cyclic (of order c_k or $2c_k$) for all $k \le 5000$. To see this, we observe that $K_{4k-2}(\mathbb{Z})_{(\ell)}$ is cyclic if ℓ^2 does not divide c_k, and in this range only seven of the c_k are not square-free; see [170, A090943].

The numerator c_k is divisible by ℓ^2 only for the following pairs (k, ℓ): $(114, 103)$, $(142, 37)$, $(457, 59)$, $(717, 271)$, $(1646, 67)$, $(2884, 101)$ and $(3151, 157)$. In each of these cases, we note that $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell = \mathrm{Pic}(R)/\ell \cong \mathbb{Z}/\ell$. By Example 8.3.1, $K_{4k-2}(R)/\ell \cong \mathrm{Pic}(R)/\ell \cong \mathbb{Z}/\ell$. The usual transfer argument (10.3.1) now shows that $K_{4k-2}(\mathbb{Z})/\ell$ is either 0 or $\mathbb{Z}/\ell$ for all k. Since c_k is divisible by ℓ^2 but not ℓ^3, $K_{4k-2}(\mathbb{Z})_{(\ell)} \cong \mathbb{Z}/\ell^2$.

VI.10.4 **Representations of G over $\mathbb{Z}/\ell$ 10.4.** When G is the cyclic group of order $\ell-1$, a $\mathbb{Z}/\ell[G]$-module is just a $\mathbb{Z}/\ell$-vector space on which G acts linearly. By Maschke's theorem, $\mathbb{Z}/\ell[G] \cong \prod_{i=0}^{\ell-2} \mathbb{Z}/\ell$ is a simple ring, so every $\mathbb{Z}/\ell[G]$-module has a unique decomposition as a sum of its $\ell-1$ irreducible modules. Since μ_ℓ is an irreducible G-module, it is easy to see that the irreducible G-modules are $\mu_\ell^{\otimes i}$, $i = 0, 1, ..., \ell-2$. The "trivial" G-module is $\mu_\ell^{\otimes \ell-1} = \mu_\ell^{\otimes 0} = \mathbb{Z}/\ell$. By convention, $\mu_\ell^{\otimes i} = \mu_\ell^{\otimes i + a(\ell-1)}$ for all integers a.

For example, the G-submodule $\langle \beta^i \rangle$ of $K_{2i}(\mathbb{Z}[\zeta]; \mathbb{Z}/\ell)$ generated by β^i is isomorphic to $\mu_\ell^{\otimes i}$. It is a trivial G-module only when $(\ell-1)|i$.

If A is any $\mathbb{Z}/\ell[G]$-module, it is traditional to decompose $A = \oplus A^{[i]}$, where $A^{[i]}$ denotes the sum of all G-submodules of A isomorphic to $\mu_\ell^{\otimes i}$.

VI.10.4.1 **Example 10.4.1.** Set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$. It is known that the torsionfree part $R^\times/\mu_\ell \cong \mathbb{Z}^{\frac{\ell-1}{2}}$ of the units of R is isomorphic as a G-module to $\mathbb{Z}[G] \otimes_{\mathbb{Z}[c]} \mathbb{Z}$, where c is complex conjugation. (This is sometimes included as part of Dirichlet's theorem on units.) It follows that as a G-module,

$$R^\times/R^{\times\ell} \cong \mu_\ell \oplus (\mathbb{Z}/\ell) \oplus \mu_\ell^{\otimes 2} \oplus \cdots \oplus \mu_\ell^{\otimes \ell-3}.$$

The first two terms μ_ℓ and $\mathbb{Z}/\ell$ are generated by the root of unity ζ_ℓ and the class of the unit ℓ of R. It will be convenient to choose units $x_0 = \ell$, $x_1, \dots, x_{(\ell-3)/2}$ of R such that x_i generates the summand μ_ℓ^{-2i} of $R^\times/R^{\times\ell}$; the notation is set up so that $x_i \otimes \zeta_\ell^{\otimes 2i}$ is a G-invariant element of $R^\times \otimes \mu_\ell^{\otimes 2i}$.

VI.10.4.2 **Example 10.4.2.** The G-module decomposition of $M = R^\times \otimes \mu_\ell^{\otimes i-1}$ is obtained from Example 10.4.1 by tensoring with $\mu_\ell^{\otimes i-1}$. If i is even, $\mathbb{Z}/\ell$ occurs only when $i \equiv 0 \pmod{\ell-1}$, corresponding to $\zeta^{\otimes i}$. If i is odd, exactly one term of M is $\mathbb{Z}/\ell$; M^G is $\mathbb{Z}/\ell$ on the generator $x_j \otimes \zeta_\ell^{i-1}$, where $i \equiv 1 + 2j \pmod{\ell-1}$.

Torsion for odd regular primes

Suppose that ℓ is an odd regular prime. By definition, $\mathrm{Pic}(\mathbb{Z}[\zeta])$ has no ℓ-torsion, and $K_1(R)/\ell \cong R^\times/R^{\times\ell}$ by III.1.3.6. Kummer showed that ℓ cannot divide the order of any numerator c_k of B_k/k (see 2.4.1). Therefore the case $2i = 4k-2$ of the following result follows from Theorem 10.1.

VI.10.5 **Proposition 10.5.** *When ℓ is an odd regular prime, the group $K_{2i}(\mathbb{Z})$ has no ℓ-torsion. Thus the only ℓ-torsion subgroups of $K_*(\mathbb{Z})$ are the Harris-Segal subgroups $\mathbb{Z}/w_i^{(\ell)}(\mathbb{Q})$ of $K_{2i-1}(\mathbb{Z})$ when $i \equiv 0 \pmod{\ell-1}$.*

Proof. Since ℓ is regular, we saw in Example 8.3.2 that the group $K_{2i}(\mathbb{Z}[\zeta])$ has no ℓ-torsion. Hence the same is true for its G-invariant subgroup, $K_{2i}(\mathbb{Z})$. The restriction on i comes from Example 2.2.2. □

We can also describe the algebra structure of $K_*(\mathbb{Z};\mathbb{Z}/\ell)$. For this we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$ and $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, noting that $K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell) \cong K_*(R;\mathbb{Z}/\ell)^G$ by the usual transfer argument (10.3.1). Recall from Example 8.5 that $K_*(R;\mathbb{Z}/\ell)$ is a free graded $\mathbb{Z}/\ell[\beta]$-module on $\frac{\ell+1}{2}$ generators: the x_i of $R^\times/R^{\times\ell} = K_1(R;\mathbb{Z}/\ell)$, together with $1 \in K_0(R;\mathbb{Z}/\ell)$.

Thus $K_{2i}(R;\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ is generated by β^i, and is isomorphic to $\mu_\ell^{\otimes i}$ as a G-module. It follows that $K_{2i}(R;\mathbb{Z}/\ell)^G$ is zero unless $i = a(\ell-1)$, when it is $\mathbb{Z}/\ell$ on the generator β^i. By abuse of notation, we shall write $\beta^{\ell-1}$ for the element of $K_{2(\ell-1)}(\mathbb{Z};\mathbb{Z}/\ell)$ corresponding to $\beta^{\ell-1}$; if $i = a(\ell-1)$ we shall write β^i for the element $(\beta^{\ell-1})^a$ of $K_{2i}(\mathbb{Z};\mathbb{Z}/\ell)$ corresponding to $\beta^i \in K_{2i}(R;\mathbb{Z}/\ell)^G$.

By Example 8.5, $K_{2i-1}(R;\mathbb{Z}/\ell)$ is just $R^\times \otimes \mu_\ell^{\otimes i-1}$ when ℓ is regular. The G-module structure was determined in Example 10.4.2: if i is even, exactly one term is $\mathbb{Z}/\ell$; if i is odd, $\mathbb{Z}/\ell$ occurs only when $i \equiv 1 \pmod{\ell-1}$.

Multiplying $[\zeta] \in K_1(R;\mathbb{Z}/\ell)$ by $\beta^{\ell-2}$ yields the G-invariant element $v = [\zeta]\,\beta^{\ell-2}$ of $K_{2\ell-3}(R;\mathbb{Z}/\ell)$. Again by abuse of notation, we write v for the corresponding element of $K_{2\ell-3}(\mathbb{Z};\mathbb{Z}/\ell)$.

Similarly, multiplying $x_k \in R^\times = K_1(R)$ by $\beta^{2k} \in K_{4k}(R;\mathbb{Z}/\ell)$ gives a G-invariant element $y_k = x_k\beta^{2k}$ of $K_{4k+1}(R;\mathbb{Z}/\ell)$ with $y_0 = [\ell]$ in $K_1(R;\mathbb{Z}/\ell)$. Again by abuse of notation, we write y_k for the corresponding element of $K_{4k+1}(\mathbb{Z};\mathbb{Z}/\ell)$.

VI.10.6 **Theorem 10.6.** *If ℓ is an odd regular prime then $K_* = K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is a free graded module over the polynomial ring $\mathbb{Z}/\ell[\beta^{\ell-1}]$. It has $(\ell+3)/2$ generators: $1 \in K_0$, $v \in K_{2\ell-3}$, and the elements $y_k \in K_{4k+1}$ ($k = 0, ..., \frac{\ell-3}{2}$) described above.*

Similarly, $K_*(\mathbb{Z};\mathbb{Z}/\ell)$ is a free graded module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$; a generating set is obtained from the generators of K_* by replacing y_0 by $y_0\beta^{\ell-1}$.

The $\mathbb{Z}/\ell[\beta^{\ell-1}]$-submodule generated by v and $\beta^{\ell-1}$ comes from the Harris-Segal summands of $K_{2i-1}(\mathbb{Z})$. The submodule generated by the y's comes from the $\mathbb{Z}$ summands in $K_n(\mathbb{Z})$, $n \equiv 1 \pmod 4$.

Proof. $K_*(\mathbb{Z}[1/\ell];\mathbb{Z}/\ell)$ is the G-invariant subalgebra of $K_*(R;\mathbb{Z}/\ell)$. Given 10.5, it is not very hard to check that this is just the subalgebra described in the theorem. Since $\ell-1$ is even, the elements $y_k\beta^{a(\ell-1)}$ are in $K_n(\mathbb{Z};\mathbb{Z}/\ell)$ for some $n \equiv 1 \pmod 4$. Since $K_{n-1}(\mathbb{Z})$ has no ℓ-torsion by Proposition 10.5, $K_n(\mathbb{Z};\mathbb{Z}/\ell) = K_n(\mathbb{Z})/\ell$. Since $1 \le 4k+1 \le 2\ell-4$, we have $n \equiv 4k+1 \not\equiv 0 \pmod{2\ell-2}$ and hence $K_n(\mathbb{Z})$ has no ℓ-torsion (combine 10.1 with 2.1.2). Hence the element $y_k\beta^{a(\ell-1)}$ must come from the $\mathbb{Z}$-summand of $K_n(\mathbb{Z}[1/\ell])$. □

VI.10.6.1 **Examples 10.6.1.** When $\ell = 3$, the groups $K_n = K_n(\mathbb{Z}[1/3];\mathbb{Z}/3)$ are 4-periodic of ranks $1, 1, 0, 1$, generated by an appropriate power of $\beta^2 \in K_4$ times one of $\{1, [3], v\}$. Here $v \in K_3$.

When $\ell = 5$, the groups $K_n = K_n(\mathbb{Z}[1/5]; \mathbb{Z}/5)$ are 8-periodic, with respective ranks $1, 1, 0, 0, 0, 1, 0, 1$ ($n = 0, ..., 7$), generated by an appropriate power of $\beta^4 \in K_8$ times one of $\{1, [5], y_1, v\}$. Here $y_1 \in K_5$ (x_1 is the golden mean) and $v \in K_7$.

Torsion for irregular primes

Now suppose that ℓ is an irregular prime, so that $\mathrm{Pic}(R)$ has ℓ-torsion for $R = \mathbb{Z}[\zeta, 1/\ell]$. Then $H^1_{\mathrm{et}}(R, \mu_\ell)$ is $R^\times/\ell \oplus {}_\ell\mathrm{Pic}(R)$ and $H^2_{\mathrm{et}}(R, \mu_\ell) \cong \mathrm{Pic}(R)/\ell$ by Kummer theory and (8.1.1). This yields $K_*(R; \mathbb{Z}/\ell)$ by Example 8.5.

Set $P = \mathrm{Pic}(R)/\ell$. When ℓ is irregular, the G-module structure of P is not fully understood; see Vandiver's conjecture 10.8 below.

VI.10.7 **Lemma 10.7.** *For $i = 0, -1, -2, -3$, $P = \mathrm{Pic}(R)/\ell$ contains no summands isomorphic to $\mu_\ell^{\otimes i}$, i.e., $P^{[i]} = 0$.*

Proof. The usual transfer argument shows that $P^G \cong \mathrm{Pic}(\mathbb{Z}[1/\ell])/\ell = 0$. Hence P contains no summands isomorphic to $\mathbb{Z}/\ell$. By III.6.9.3, there is a G-module isomorphism $(P \otimes \mu_\ell) \cong K_2(R)/\ell$. Since $K_2(R)/\ell^G \cong K_2(\mathbb{Z}[1/\ell])/\ell = 0$, $(P \otimes \mu_\ell)$ has no $\mathbb{Z}/\ell$ summands — and hence P contains no summands isomorphic to $\mu_\ell^{\otimes -1}$.

Finally, we have $(P \otimes \mu_\ell^{\otimes 2}) \cong K_4(R)/\ell$ and $(P \otimes \mu_\ell^{\otimes 3}) \cong K_6(R)/\ell$ by 8.3. Again, the transfer argument shows that $K_n(R)/\ell^G \cong K_n(\mathbb{Z}[1/\ell])/\ell$ for $n = 4, 6$. The groups $K_4(\mathbb{Z})$ and $K_6(\mathbb{Z})$ are known to be zero by [159] and [53]; see 10.1.3. It follows that P contains no summands isomorphic to $\mu_\ell^{\otimes -2}$ or $\mu_\ell^{\otimes -3}$. □

VI.10.8 **Vandiver's Conjecture 10.8.** If ℓ is an irregular prime then $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}])$ has no ℓ-torsion. Equivalently, the natural representation of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ on $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$ is a sum of G-modules $\mu_\ell^{\otimes i}$ with i odd.

This means that complex conjugation c acts as multiplication by -1 on the ℓ-torsion subgroup of $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell])/\ell$, because c is the unique element of G of order 2.

As partial evidence for this conjecture, we mention that Vandiver's conjecture has been verified for all primes up to 163 million; see [38]. We also known from Lemma 10.7 that $\mu_\ell^{\otimes i}$ does not occur as a summand of $\mathrm{Pic}(R)/\ell$ for $i = 0, -2$.

VI.10.8.1 **Remark 10.8.1.** The Herbrand-Ribet theorem [216, 6.17–18] states that $\ell | B_k$ if and only if $(\mathrm{Pic}\ R/\ell)^{[\ell - 2k]} \neq 0$. Among irregular primes < 4000, this happens for at most 3 values of k. For example, $37 | c_{16}$ (see 10.3), so $(\mathrm{Pic}\ R/\ell)^{[5]} = \mathbb{Z}/37$ and $(\mathrm{Pic}\ R/\ell)^{[k]} = 0$ for $k \neq 5$.

VI.10.8.2 **Historical Remark 10.8.2.** What we now call "Vandiver's conjecture" was actually discussed by Kummer and Kronecker in 1849–1853; Harry Vandiver was not born until 1882 and only started using this assumption circa 1920 (e.g., in [202] and [203]), but only retroactively claimed to have conjectured it "about 25 years ago" in the 1946 paper [204, p. 576].

In 1849, Kronecker asked if Kummer conjectured that a certain lemma ([216, 5.36]) held for all p, and that therefore p never divided h^+ (*i.e.,* Vandiver's conjecture holds). Kummer's reply [104, pp.114–115] pointed out that the Lemma could not hold for irregular p, and then referred to the assertion [Vandiver's conjecture] as the *noch zu beweisenden Satz* (theorem still to be proven). Kummer also pointed out some of its consequences. In an 1853 letter (see [104, p.123]), Kummer wrote to Kronecker that in spite of months of effort, the assertion [now called Vandiver's conjecture] was still unproven.

For the rest of this chapter, we set $R = \mathbb{Z}[\zeta_\ell, 1/\ell]$, and $P = \mathrm{Pic}(R)/\ell$.

VI.10.9 **Theorem 10.9.** *(Kurihara [105]) Let ℓ be an irregular prime number. Then the following are equivalent for every integer k between 1 and $\frac{\ell-1}{2}$:*

(1) $\mathrm{Pic}(\mathbb{Z}[\zeta])/\ell^{[-2k]} = 0$.

(2) $K_{4k}(\mathbb{Z})$ has no ℓ-torsion;

(3) $K_{2a(\ell-1)+4k}(\mathbb{Z})$ has no ℓ-torsion for all $a \geq 0$;

(4) $H^2_{et}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) = 0$.

In particular, Vandiver's conjecture for ℓ is equivalent to the assertion that $K_{4k}(\mathbb{Z})$ has no ℓ-torsion for all $k < \frac{\ell-1}{2}$, and implies that $K_{4k}(\mathbb{Z})$ has no ℓ-torsion for all k.

Proof. By Kummer theory and (8.1.1), $P \cong H^2_{\mathrm{et}}(R, \mu_\ell)$. Hence $P \otimes \mu_\ell^{\otimes 2k} \cong H^2_{\mathrm{et}}(R, \mu_\ell^{\otimes 2k+1})$ as G-modules. Taking G-invariant subgroups shows that

$$H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1}) \cong (P \otimes \mu_\ell^{\otimes 2k})^G \cong P^{[-2k]}.$$

Hence (1) and (4) are equivalent. By 8.3, $K_{4k}(\mathbb{Z})/\ell \cong H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k+1})$ for all $k > 0$. Since $\mu_\ell^{\otimes b} = \mu_\ell^{\otimes a(\ell-1)+b}$ for all a and b, this shows that (2) and (3) are separately equivalent to (4). □

VI.10.10 **Theorem 10.10.** *If Vandiver's conjecture holds for ℓ then the ℓ-primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all k.*

If Vandiver's conjecture holds for all ℓ, the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all k.

(We know that the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all $k < 5000$, by 10.3.2.)

Proof. Vandiver's conjecture also implies that each of the "odd" summands $P^{[1-2k]} = P^{[\ell-2k]}$ of P is cyclic; see [216, 10.15]. Taking the G-invariant subgroups of $\mathrm{Pic}(R) \otimes \mu_\ell^{\otimes 2k-1} \cong H^2_{\mathrm{et}}(R, \mu_\ell^{\otimes 2k})$, this implies that the group $P^{[1-2k]} \cong H^2_{\mathrm{et}}(\mathbb{Z}[1/\ell], \mu_\ell^{\otimes 2k})$ is cyclic. By Corollary 8.3, this group is the ℓ-torsion in $K_{4k-2}(\mathbb{Z}[1/\ell])/\ell$. □

VI.10.11 **Remark 10.11.** The elements of $K_{2i}(\mathbb{Z})$ of odd order become divisible in the larger group $K_{2i}(\mathbb{Q})$. (The assertion that an element a is divisible in A means that for every m there is an element b so that $a = mb$.) This was proven by Banaszak and Kolster for i odd (see [12, Thm. 2]), and for i even by Banaszak and Gajda [13, Proof of Prop. 8]. It is an open question whether there are any divisible elements of even order.

For example, recall from 10.3 that $K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ and $K_{30}(\mathbb{Z}) \cong \mathbb{Z}/3617$. Banaszak observed [12] that these groups are divisible in $K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Q})$, *i.e.,* that the inclusions $K_{22}(\mathbb{Z}) \subset K_{22}(\mathbb{Q})$ and $K_{30}(\mathbb{Z}) \subset K_{30}(\mathbb{Q})$ do not split.

Let t_j (resp., s_j) be generators of the summand of $\mathrm{Pic}(R)/\ell$ (resp. $K_1(R;\mathbb{Z}/\ell)$) isomorphic to $\mu_\ell^{\otimes -j}$. The following result follows easily from Examples 8.5 and 10.4.1, using the proofs of 10.6, 10.9 and 10.10. It was originally proven in Mitchell [137].

VI.10.12 **Theorem 10.12.** *If ℓ is an irregular prime for which Vandiver's conjecture holds, then $K_* = K_*(\mathbb{Z};\mathbb{Z}/\ell)$ is a free module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$ on 1, $v \in K_{2\ell-3}$, the $(\ell-3)/2$ generators $y_k \in K_{4k+1}$ described in Theorem 10.6, together with the generators $t_j\beta^j \in K_{2j}$ and $s_j\beta^j \in K_{2j+1}$ $(j = 3, 5, ..., (\ell-8))$.*

EXERCISES

EVI.10.1 **10.1.** Let ℓ be an irregular prime and suppose that $K_n(\mathbb{Z})$ has no ℓ-torsion for some positive $n \equiv 0 \pmod 4$. Show that $K_{4k}(\mathbb{Z})$ has no ℓ-torsion for every k satisfying $n \equiv 4k \pmod{2\ell-2}$.

EVI.10.2 **10.2.** Show that $K_n(\mathbb{Z})$ has nonzero 37-torsion for all positive $n \equiv 62 \pmod{72}$, and that $K_n(\mathbb{Z})$ has nonzero 103-torsion for all positive $n \equiv 46 \pmod{204}$.

EVI.10.3 **10.3.** Give a careful proof of Theorem 10.12, by using Examples 8.5 and 10.4.1 for $\mathbb{Z}[\zeta_\ell, 1/\ell]$ to modify the proof of Theorem 10.6.

EVI.10.4 **10.4.** The Bockstein operation $b : K_n(R;\mathbb{Z}/\ell) \to K_{n+1}(R;\mathbb{Z}/\ell)$ is the boundary map in the long exact sequence associated to the coefficient sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$. Show that when $R = \mathbb{Z}$ the Bockstein sends v to $\beta^{\ell-1}$, t_j to s_j and $t_j\beta^j$ to $s_j\beta^j$ in Theorems 10.6 and 10.12.

Nomenclature

ΩR algebraic loop ring of ring R, page 355
$\mathcal{P}(F)$ scissors congruence group, page 485
$\mathbf{P}(R)$ category of projective modules, page 8
$\pi_1(BC)$ fundamental group of a category, page 288
$\pi_3^{\text{ind}}(BM^+)$ indecomposables of $\pi_3(BM^+)$, page 495
$\pi_n(X;\mathbb{Z}/\ell)$ homotopy with coefficients, page 277
$\mathbf{Pic}(R)$ Picard category (line bundles), page 105
$\text{Pic}(R)$ Picard group of R, page 18
$\text{Pic}(X)$ Picard group of X, page 50
$\text{Pic}_+(R)$ narrow Picard group, page 518
$\mathbb{P}^n$ projective n-space, page 50
$Q\mathcal{A}$ Quillen's Q-construction, page 317
$\mathbf{Quad}^\epsilon(A)$ category of quadratic modules, page 299
$\mathbf{Quad}(F)$ category of quadratic spaces, page 110
ρ rank of $K_4^M(F) \to K_4(F)$, page 521
$R(G)$ Representation ring of G, page 66
r_1, r_2 number of real (complex) embeddings, page 270
$R[\Delta^\bullet]$ simplicial ring of standard simplices, page 351
$\mathbf{Rep}_{\mathbb{C}}(G)$ category of complex representations of G, page 105
$\mathcal{R}_f(X)$ finite spaces over X, page 159
$\mathcal{R}_{\text{fd}}(X)$ finitely dominated spaces over X, page 169
R^n free R-module of rank n, page 1
$\sigma(M)$ shift automorphism on graded modules, page 126
Σ_n symmetric group of permutations, page 262
$S^{-1}S$ group completion category, page 299
$Seq(F,R)$ sequence for contracted functors, page 209
$\mathbf{Sets}_{\text{fin}}$ category of finite sets, page 105
$SK_0(R)$ ideal of $K_0(R)$, page 75
$SK_0(X)$ ideal of $K_0(X)$, page 145
$SK_1(A)$ subgroup of $K_1(A)$, A semisimple, page 180
$SK_1(R)$ subgroup of $K_1(R)$, page 179
$SL_n(R)$ special linear group of a ring, page 179
(S_n) stable range condition, page 4
$S_n\mathcal{C}$ category of n-fold extensions, page 333
$sr(R)$ stable range, page 4
$St(R)$ Steinberg group, page 216
$\bigstar$ star operation on $St(R)$, page 223
$\langle S,X\rangle$, $S^{-1}X$ localization categories, page 303
$U(R)$ group of units in R, page 2
$Unip(R)$ group of unipotent matrices, page 350
v_1^4 generator of $\pi^s(S^8;\mathbb{Z}/16)$, page 483
$\mathbf{VB}(X)$ category of algebraic vector bundles, page 46
$\mathbf{VB}_{\mathbb{C}}(X)$ category of complex vector bundles, page 31
$\mathbf{VB}_{\mathbb{R}}(X)$ category of real vector bundles, page 31
$\mathbf{VB}_n(X)$ vector bundles of rank n, page 38
$w_{ij}(r)$ special element of $St(R)$, page 224

Index

Bibliography

Adams [1] J.F. Adams, *Infinite Loop Spaces*, Annals Studies in Math.l90, Princeton University Press, Princeton, 1978.

And72 [2] D. W. Anderson, Relationship among K-theories, pp. 57–72 in *Lecture Notes in Math.* 341, Springer-Verlag, 1973.

AT65 [3] S. Araki and H. Toda, Multiplicative structures in mod q cohomology theories I, *Osaka J. Math.* 2 (1965), 71–115.

Art [4] V. Artamonov, Projective Modules over Group Rings of Nilpotent Groups, *Algebra, Moskov. Gos. Univ.* (1982), 7–23 (Russian). English transl. in *AMS Translations* 154 (1992), 11–24.

Atiy56 [5] M. Atiyah, On the Krull–Schmidt theorem with application to sheaves, *Bull. Soc. Math. France* 84 (1956), 307–317.

Atiy57 [6] M. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.* 7 (1957), 414–452.

Atiyah [7] M. Atiyah, *K-theory*, Benjamin, New York, 1967.

AM [8] M. Atiyah and I. G. McDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

AT [9] M. Atiyah and D. Tall, Group Representations, λ-rings and the J-homomorphism, *Topology* 8 (1969), 253–297.

BA [10] N. Jacobson, *Basic Algebra II*, Freeman & Co., 1980.

Bak [11] A. Bak, *K-theory of Forms*, Annals Studies in Math. 98, Princeton University Press, Princeton, 1981.

Ban [12] G. Banaszak, Algebraic K-theory of number fields and rings of integers and the Stickelberger ideal, *Annals of Math.* 135 (1992), 325–360.

BaGj [13] G. Banaszak and W. Gajda, Euler systems for higher K-theory of number fields, *J. Number Theory* 58 (1996), 213–252.

BP71 [14] M. Barratt and S. Priddy, On the homology of non-connected monoids and their associated groups, pp.31–36 in *Proc. Symp. Pure Math.* 22, 1971.

Bass [15] H. Bass, *Algebraic K-theory*, Benjamin, 1968.

Bass76 [16] H. Bass, Euler characteristics and characters of discrete groups, *Inv. Math.* 35 (1976), 155–196.

B72 [17] H. Bass, Unitary algebraic K-theory, in *Lecture Notes in Math.* 342, Springer-Verlag, 1973.

BHS [18] H. Bass, A. Heller and R. Swan, The Whitehead group of a polynomial extension, *Publ. Math.* IHES 22 (1964), 61–79.

BMS [19] H. Bass, J. Milnor and J.-P. Serre, Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$, *Publ. Math. IHES* 33 (1967), 59–137.

BM [20] H. Bass and M.P. Murthy, Grothendieck groups and Picard groups of abelian group rings, *Annals Math.* 86 (1967), 16–73.

BT [21] H. Bass and J. Tate, The Milnor ring of a global field, pp. 349–446 in *Lecture Notes in Math.* 342, Springer-Verlag, 1973.

BBD [22] A. Beilinson, J. Berstein, and P. Deligne, Faisceaux pervers, *Asterisque* 100, 1982.

Berrick [23] J. Berrick, *An Approach to Algebraic K-theory*, Pitman, 1982.

Bl74 [24] S. Bloch, K_2 and algebraic cycles, *Annals of Math.* 99 (1974), 349–379.

Bl86 [25] S. Bloch, Algebraic cycles and higher K-theory, *Adv. in Math.* 61 (1986), 267–304.

BK [26] S. Bloch and K. Kato, p-adic etale cohomology. *Publ. IHES* 63 (1986), 107–152.

BL [27] S. Bloch and S. Lichtenbaum, A spectral sequence for motivic cohomology, unpublished 1995 preprint, available at www.math.uiuc.edu/K-theory/0062

Bor [28] A. Borel, Cohomologie réelle stable de groupes S-arithmétiques classiques, *C. R. Acad. Sci.* (Paris) Sér. A-B (1972), A1700–A1702. Cohomology of Arithmetic Groups, pp. 435–442 in *Proc. 1974 ICM*, Vancouver, 1975.

Bor1 [29] A. Borel, Stable real cohomology of arithmetic groups, *Ann. scient. Éc. Norm. Sup.* 7 (1974), 235–272.

BoSe [30] A. Borel and J.-P. Serre, Le Théorème de Riemann-Roch (d'après Grothendieck), *Bull. Soc. Math. de France* 86 (1958), 97–136.

BSh [31] Z. Borevich and I. Shafarevich, *Number Theory*, Benjamin, New York, 1967.

B-AC [32] N. Bourbaki, *Algèbre commutative*, Chap.1-Chap.9, Hermann, Paris, 1961-83.

BF [33] A. K. Bousfield and E. M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, pp. 80–130 in *Lecture Notes in Math.* 658, Springer-Verlag, 1978.

Br [34] W. Browder, Algebraic K-theory with coefficients $\mathbb{Z}/p$, *Lecture Notes in Math.* 657, Springer-Verlag, 1978.

BrwS [35] J. Browkin and A. Schinzel, On Sylow 2-subgroups of $K_2(\mathcal{O}_F)$ for quadratic number fields F, *J. Reine Angew. Math.* 331 (1982), 104–113.

BG [36] K. Brown and S. Gersten, Algebraic K-theory as generalized sheaf cohomology, pp. 266–292 in *Lecture Notes in Math.* 341, Springer-Verlag (1973

Bu12 [37] J. Buhler, R. Crandall, R. Ernvall, T. Metsänkylä and M. Shokrollahi, Irregular primes and cyclotomic invariants to 12 million, *J. Symbolic Comput.* 31 (2001), 89–96.

BH [38] J. Buhler and D. Harvey, Irregular primes to 163 million, Preprint,(2008), arXiv:0912.2121.

Carter [39] D. Carter, Lower K-theory of finite groups, *Comm. in Alg.* 8 (1980), 1927–1937.

CCMT [40] J. Caruso, F. Cohen, J. P. May and L. Taylor, James maps, Segal maps, and the Kahn-Priddy theorem, *Trans. Amer. Math. Soc.* 281 (1984), 243–283.

CLL [41] K. Choo, T. Lam and E. Luft, On free products of rings and the coherence property, pp. 135–143 in *Lecture Notes in Math.* 342, Springer-Verlag, 1973.

Co [42] H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics 138, Springer-Verlag, 1993.

Cohen [43] M. Cohen, *A Course in Simple–Homotopy Theory* Springer-Verlag, 1973.

Cohn65 [44] P.M. Cohn, *Universal Algebra*, Harper&Row, 1965.

Cohn66 [45] P.M. Cohn, Some remarks on the invariant basis property, *Topology* 5 (1966), 215–228.

CHSW [46] G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel, Cyclic homology, *cdh*-cohomology and negative K-theory, *Annals of Math.* 167 (2008), 549-563.

CHWW [47] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, Bass' NK groups and *cdh*-fibrant Hochschild homology, *Inventiones Math.* 181 (2010), 421-448.

DS [48] R.K. Dennis and M. Stein, K_2 of radical ideals and semilocal rings revisited, *Lecture Notes in Math.* 342, Springer-Verlag, 1973.

Drin [49] V. Drinfeld Infinite-dimensional vector bundles in algebraic geometry: an introduction, pp. 263–304 in *Progress in Math.* 244, Birkhäuser, Boston, 2006.

Dror [50] E. Dror, Acyclic spaces, *Topology* 11 (1972), 339–348.

DSah [51] J. Dupont and C.-H. Sah, Scissors Congruences II, *J. Pure Applied Algebra* 25 (1982), 159–195.

DM1 [52] W. Dwyer and S. Mitchell, On the K-theory spectrum of a smooth curve over a finite field, *Topology* 36 (1997), 899–929.

EGS [53] P. Elbaz-Vincent, H. Gangl and C. Soulé, Quelques calculs de la cohomologie de $GL_N(\mathbb{Z})$ et de la K-théorie de $\mathbb{Z}$, *C. R. Acad. Sci.* (Paris) 335 (2002), 321–324.

FJ [54] T. Farrell and L. E. Jones, Isomorphism conjectures in algebraic K-theory, *J. Amer. Math. Soc.* 6 (1993), 249–297.

Foss [55] R. Fossum Vector bundles over spheres are algebraic, *Inv. Math.* 8 (1969), 222–225.

FS [56] E. Friedlander and A. Suslin, The spectral sequence relating algebraic K-theory to motivic cohomology, *Ann. Sci. École Norm. Sup.* 35 (2002), 773–875.

Fuchs [57] L. Fuchs, *Infinite Abelian Groups II*, Academic Press, 1973.

Fulton [58] W. Fulton, *Intersection Theory*, Ergebnisse Math. 2, Springer-Verlag, 1984.

FL [59] W. Fulton and S. Lang, *Riemann-Roch Algebra*, Springer-Verlag, 1985.

Gabber [60] O. Gabber, K-theory of Henselian local rings and Henselian pairs, pp. 59–70 in *Contemp. Math.* 126 (1992); Letter to Karoubi, May 1983.

Gabriel [61] P. Gabriel, Des catégories abeliennes, *Bull. Soc. Math. France* 90 (1962), 323–448.

GeiL [62] T. Geisser and M. Levine, The K-theory of fields in characteristic p, *Invent. Math.* 139 (2000), 459–493.

GL01 [63] T. Geisser and M. Levine, The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky, *J. Reine Angew. Math.* 530 (2001), 55–103.

Ger72 [64] S. Gersten, Higher K-theory of rings, pp. 3–42 in *Lecture Notes in Math.* 341, Springer-Verlag, 1973.

Ger74 [65] S. Gersten, K-theory of free rings, *Comm. Alg.* 1 (1974), 39–64.

GSz [66] P. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge Univ. Press, 2006.

Gillet [67] H. Gillet, Riemann-Roch theorems for higher algebraic K-theory, *Adv. in Math.* 40 (1981), 203–289.

GG [68] H. Gillet and D. Grayson, The loop space of the Q-construction, *Ill. J. Math.* 11 (1987), 574–597.

GiL [69] H. Gillet and M. Levine, The relative form of Gersten's conjecture over a discrete valuation ring: the smooth case, *J. Pure Applied Algebra* 46 (1987), 59–71.

GiS [70] H. Gillet and C. Soulé, Filtrations on higher K-theory, pp. 89–148 in *Proc. Symp. Pure Math.* 67, 1999.

Gdearl [71] K. Goodearl, *Von Neumann regular rings*, Pitman, 1979.

Gdrl1 [72] K. Goodearl, Torsion in K_0 of unit-regular rings, *Proc. Edinburgh Math. Soc.* 38 (1995), 331-341.

GR [73] H. Grauert and R. Remmert, *Theory of Stein Spaces*, Springer-Verlag, 1977.

GQ [74] D. Grayson, Higher algebraic K-theory: II (after D. Quillen), *Lecture Notes in Math.* 551, Springer-Verlag, 1976.

GQ82 [75] D. Grayson. Finite generation of K-groups of a curve over a finite field (after D. Quillen), pp. 69–90 in *Lecture Notes in Math.* 966, Springer-Verlag, 1982.

Gra87 [76] D. Grayson, Exact Sequences in algebraic K-theory, *Illinois J. Math.* 31 (1987), 598–617.

Gra88 [77] D. Grayson, The K-theory of semilinear endomorphisms, *J. Algebra* 113 (1988), 358–372.

Gra95 [78] D. Grayson, Weight filtrations via commuting automorphisms, *K-Theory* 9 (1995), 139–172.

GA [79] P. Griffiths and J. Adams, *Topics in Algebraic and Analytic Geometry*, Princeton U. Press, 1974.

Groth57 [80] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphere de Riemann, *Amer. J. Math.* 79 (1957), 121–138.

GDix [81] A. Grothendieck, Classes de Chern et représentations linéaires des groupes discrets, pp. 215–305 in *Dix Exposés sur la Cohomologie des Schémas*, North-Holland, 1968.

HW [82] C. Haesemeyer and C. Weibel, *The Norm Residue Theorem in Motivic Cohomology*, book in preparation.

Har [83] G. Harder, Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern, *Invent. Math.* 42 (1977), 135–175.

HS [84] B. Harris and G. Segal, K_i groups of rings of algebraic integers, *Annals of Math.* 101 (1975), 20–33.

Hart [85] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, 1977.

H20 [86] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, 1966.

HH [87] J.-C. Hausmann and D. Husemoller, Acyclic maps, *Enseign. Math.* 25 (1979), 53–75.

HM [88] L. Hesselholt and I. Madsen, On the K-theory of local fields, *Annals of Math.* 158 (2003), 1-113.

Hiller [89] H. Hiller, λ-rings and algebraic K-theory, *J. Pure Applied Algebra* 20 (1981), 241–266.

Hovey [90] M. Hovey, *Model categories*, Math. Surveys Monographs 63, AMS, 1999.

HSS [91] M. Hovey, B. Shipley and J. Smith, Symmetric spectra, *J. AMS* 13 (2000), 149–208.

Hu [92] A. Huber, Realization of Voevodsky's motives *J. Algebraic Geometry* 9 (2000), 755-799.

Huse [93] D. Husemoller, *Fibre Bundles* (Second Edition), Springer-Verlag, 1974.

Igusa [94] K. Igusa, What happens to Hatcher and Wagoner's formulas for $\pi_0 C(M)$ when the first Postnikov invariant of M is nontrivial?, pp. 104–172 in *Lecture Notes in Math.* 1046, Springer-Verlag, 1984.

Izh [95] O. Izhboldin, On the torsion subgroup of Milnor's K-groups, *Soviet Math. Doklady* 35 (1987), 493–495; On p-torsion in K_*^M for fields of characteristic p, pp. 129–144 in *Algebraic K-theory*, Adv. Soviet Math. 4, Amer. Math. Soc., 1991.

J37 [96] N. Jacobson, p-alghebras of exponent p, *Bull. AMS* 43 (1937), 667–670.

J [97] J. F. Jardine, *Generalized étale cohomology theories*, Birkhäuser, 1997.

Kap58 [98] I. Kaplansky, Projective modules , *Annals of Math.* 68 (1958), 372–377.

Kar63 [99] M. Karoubi, Les isomorphismes de Chern et de Thom-Gysin en K-théorie, Exposé 16 in *Seminaire H. Cartan* (1963/64), Secrétariat mathématique, Paris, 1965.

KV71 [100] M. Karoubi and O. Villamayor, K-théorie algébrique et K-théorie topologique, *Math. Scand.* 28 (1971), 265–307.

Ka82 [101] K. Kato, Symmetric bilinear forms, Quadratic forms and Milnor K-theory in characteristic two, *Invent. Math.* 66 (1982), 493–510.

Ke90 [102] B. Keller Chain complexes and stable categories, *Manuscripta Math.* 67 (1990), 379–417.

Keu78 [103] F. Keune, The relativization of K_2, *J. Algebra* 54 (1978), 159–177.

Kum [104] E. Kummer, *Collected Papers I*, ed. A. Weil, Springer Verlag, 1975.

Kur [105] M. Kurihara, Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$, *Compositio Math.* 81 (1992), 223–236.

Lam [106] T.Y. Lam, *Serre's Conjecture*, Lecture Notes in Math. 635, Springer-Verlag, 1978.

Landsbg [107] S. Landsburg, Patching theorems for projective modules, *J. Pure Appl. Alg.* 21 (1981), 261–277.

LSz [108] R. Lee and R. Szczarba, The group $K_3(\mathbb{Z})$ is cyclic of order forty-eight, *Annals Math.* 104 (1976), 31–60.

Le01 [109] M. Levine, Techniques of localization in the theory of algebraic cycles, *J. Alg. Geom.* 10 (2001), 299–363.

Le08 [110] M. Levine The homotopy coniveau tower, *J. Topology* 1 (2008), 217–267.

Li [111] S. Lichtenbaum, On the values of zeta and L-functions: I, *Annals of Math.* 96 (1972), 338–360.

Li2 [112] S. Lichtenbaum, Values of zeta functions, étale cohomology, and algebraic K-theory, pp. 489–501 in *Lecture Notes in Math.* 342, Springer Verlag, 1973.

Li3 [113] S. Lichtenbaum, Values of zeta functions at non-negative integers, pp. 127–138 in *Lecture Notes in Math.* 1068, Springer Verlag, 1984.

Lo76 [114] J.-L. Loday, K-théorie algébrique et représentations de groupes, *Annales Sci. ENS* 9 (1976), 309–377.

Lo78 [115] J.-L. Loday, Cohomologie et groupe de Steinberg relatifs, *J. Algebra* 54 (1978), 178–202.

Mac [116] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

Matsu [117] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, 1986.

May [118] J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, 1967.

May74 [119] J.P. May, E_∞ ring spaces, group completions and permutative categories, pp. 61–93 in *London Math. Soc. Lecture Notes* 11, Cambridge Univ. Press, 1974.

May80 [120] J.P. May, Pairings of Categories and Spectra, *J. Pure Applied Algebra* (1980), 299-346.

MTh [121] J.P. May and R. Thomason, The uniqueness of infinite loop space machines, *Topology* 17 (1978), 205–224.

MVW [122] C. Mazza, V. Voevodsky and C. Weibel, *Lecture Notes in Motivic Cohomology*, Clay Math. Monographs 4, AMS, 2006.

MM82 [123] P. Menal and J. Moncasi, On regular rings with stable range 2, *J. Pure Applied Algebra* 22 (1982), 25–40.

Merk [124] A. Merkurjev, On the torsion in K_2 of local fields, *Annals of Math.* 118 (1983), 375–381.

MS [125] A. Merkurjev and A. Suslin, K-cohomology of Severi-Brauer Varieties and the Norm Residue Homomorphism, *Math. USSR Izvestiya* 21 (1983), 307–340.

MS2 [126] A. Merkurjev and A. Suslin, On the norm residue homomorphism of degree three, *Izv. Akad. Nauk SSSR* Ser. Mat. 54 (1990), 339–356.

Milne [127] J. Milne, *Étale Cohomology*, Princeton Univ. Press, 1980.

Milne2 [128] J. Milne, *Arithmetic Duality Theorems*, Academic Press, 1986.

M-Axiom [129] J. Milnor, On axiomatic homology theory, *Pacific J. Math.* 12 (1962), 337–341.

M-QF [130] J. Milnor, Algebraic K-theory and Quadratic Forms, *Invent. Math.* 9 (1970), 318–344.

Milnor [131] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math. Study 72, Princeton Univ. Press, 1971.

M-Lie [132] J. Milnor, On the homology of Lie groups made discrete, *Comment. Math. Helv.* 58 (1983), 72–85.

M-SBF [133] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer-Verlag, 1973.

MM [134] J. Milnor and J. Moore, On the structure of Hopf algebras, Annals of Math. 81 (1965), 211–264.

MSt [135] J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Math. Study 76, Princeton Univ. Press, 1974.

MimTo [136] M. Mimura and H. Toda, *Topology of Lie Groups*, AMS Translations of Math. Monographs 91, 1991.

Mit [137] S. Mitchell, Hypercohomology spectra and Thomason's descent theorem, pp. 221–277 in *Algebraic K-theory (Toronto, ON, 1996)*, Fields Inst. Commun. vol. 16, Amer. Math. Soc., Providence, 1997.

MV [138] F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, *Publ. IHES* 90 (2001), 45–143.

Mum [139] D. Mumford, *Lectures on curves on an algebraic surface*, Annals Studies in Math. 59, Princeton University Press, Princeton, 1966.

MmAB [140] D. Mumford, *Abelian Varieties*, Oxford U. Press, 1970.

Nak [141] M. Nakaoka, Decomposition Theorem for Homology Groups of Symmetric Groups, *Annals of Math.* 71 (1960), 16–42.

N [142] J. Neisendorfer, *Primary homotopy theory*, Memoirs AMS 232, 1980.

Nen [143] A. Nenashev, K_1 by generators and relations, *J. Pure Appl. Alg.* 131 (1998), 195–212.

NS [144] Y. Nesterenko and A. Suslin, Homology of the general linear group over a local ring, and Milnor's K-theory, *Izv. Akad. Nauk SSSR* Ser. Mat. 53 (1989), 121–146.

Neu [145] J. Neukirch, *Algebraic number theory* Springer-Verlag, 1999. Translated from the 1992 German original.

Oliver [146] R. Oliver, *Whitehead Groups of Finite Groups*, Cambridge Univ. Press, 1988.

OMeara [147] T. O'Meara, *Introduction to quadratic forms*, Springer, 1963. Reprinted in 1971 and 2000.

OVV [148] D. Orlov, A. Vishik and V. Voevodsky, An exact sequence for $K_*^M/2$ with applications to quadratic forms, *Annals Math.* 165 (2007), 1–13.

Pa [149] I. Panin, On a theorem of Hurewicz and K-theory of complete discrete valuation rings, *Math. USSR Izvestiya* 29 (1987), 119–131.

Puppe [150] D. Puppe, A theorem on semi-simplicial monoid complexes, *Annals Math.* 70 (1959), 379–394.

Q66 [151] D. Quillen, Spectral sequences of a double semi-simplicial group, *Topology* 5 (1966), 155–157.

Q70 [152] D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, *Annals of Math.* 96 (1972), 552–586. Revised version of 1970 preprint "the K-theory associated to a finite field."

Q341 [153] D. Quillen, Higher algebraic K-theory: I, pp. 85–147 in *Lecture Notes in Math.* 341), Springer-Verlag, 1973.

Q73 [154] D. Quillen, Finite generation of the groups K_i of rings of algebraic integers, pp. 195–214 in *Lecture Notes in Math.* 341, Springer-Verlag, 1973.

Qlet [155] D. Quillen, Letter from Quillen to Milnor on $\Im(\pi_i O \to \pi_i^S \to K_i\mathbb{Z})$, pp. 182–188 in *Lecture Notes in Math.* 551 Springer Verlag, 1976.

Q76 [156] D. Quillen, Characteristic classes of representations, pp. 189–216 in *Lecture Notes in Math.* 551), Springer-Verlag, 1976

Reid [157] L. Reid, N-dimensional rings with an isolated singular point having nonzero K_{-N}, *K-Theory* 1 (1987), 197–205.

RognesZ2 [158] J. Rognes, Algebraic K-theory of the two-adic integers, *J. Pure Appl. Algebra* 134 (1999), 287–326.

Rognes [159] J. Rognes, $K_4(\mathbb{Z})$ is the trivial group, *Topology* 39 (2000), 267–281.

ROst [160] J. Rognes and P. A. Østvaer, Two-primary algebraic K-theory of two-regular number fields, *Math. Zeit.* 233 (2000), 251–263.

RW [161] J. Rognes and C. Weibel. Two-primary algebraic K-theory of rings of integers in number fields, *Jour. AMS* 13 (2000), 1–54.

Ro96 [162] J. Rosenberg, The algebraic K-theory of operator algebras, *K-theory* 12 (1997), 75–99.

Rosenberg [163] J. Rosenberg, *Algebraic K-theory and its applications*, Graduate Texts in Mathematics, 147, Springer-Verlag, 1994.

Schl [164] M. Schlichting, Delooping the K-theory of exact categories, *Topology* 43 (2004), 189–1103.

Segal [165] G. Segal, Categories and cohomology theories, *Topology* 13 (1974), 293–312.

S-FAC [166] J.-P. Serre, Faisceaux algébriques cohérents, *Annals of Math.* 61 (1955), 197–278.

S-LF [167] J.-P. Serre, *Local Fields*, Springer-Verlag, 1979.

Shatz [168] S. Shatz, *Profinite grups, arithmetic, and geometry*, Annals of Math. Study 67, Princeton Univ. Press, 1972.

Siv [169] I. Sivitskii, Torsion in Milnor K-groups for a local field, *Mat. Sb.* 126 (168) (1985), 576–583.

OEIS [170] N. Sloan, ed., *On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/ñjas/sequences/`, 2003.

Sou [171] C. Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, *Invent. Math.* 55 (1979), 251–295.

So78 [172] C. Soulé, Addendum to the article "On the torsion in $K_*(\mathbb{Z})$" by R. Lee and R. Szczarba, *Duke Math. J.* 45 (1978), 131–132.

Sou81 [173] C. Soulé, On higher p-adic regulators, pp. 372–401 in *Lecture Notes in Math.* 854, Springer Verlag, 1981.

Sou85 [174] C. Soulé, Opérations en K-théorie algébrique, *Canad. J. Math.* 37 (1985), 488–550.

Staf [175] R. Staffeldt, On Fundamental Theorems of Algebraic K-theory, *K-theory* 1 (1989), 511–532.

St82 [176] J. Stienstra, Operations in the higher K-theory of endomorphisms, pp. 59–115 in *CMS Conf. Series* 2, AMS, 1982.

St85 [177] J. Stienstra, Correction to Cartier-Dieudonné theory for Chow groups, *J. Reine Angew. Math.* 365 (1985), 218–220.

Su76 [178] A. Suslin, On a theorem of Cohn, *Zap. Nauk LOMI* (1976), 127–130. English transl. in *J. Soviet Math.* 17 (1981), 1801–1803.

Su77 [179] A. Suslin, On the structure of the special linear group over polynomial rings, *Math. USSR Izv.* 11 (1977), 221–238.

Su81 [180] A. Suslin, On the equivalence of K-theories, *Comm. in Alg.* 9 (1981), 1559–1565.

Su82 [181] A. Suslin, Stability in algebraic K-theory, pp. 304–333 in *Lecture Notes in Math.* 966, Springer-Verlag, 1982.

Su83 [182] A. Suslin, On the K-theory of algebraically closed fields, *Invent. Math.* 73 (1983), 241–245.

Su-KM [183] A. Suslin Homology of GL_n, characteristic classes and Milnor K-theory, pp. 357–375 in *Lecture Notes in Math.* 1046, Springer Verlag, 1984.

Su84 [184] A. Suslin, On the K-theory of local fields, *J. Pure Appl. Alg.* 34 (1984), 319–330.

Su86 [185] A. Suslin, Algebraic K-theory of fields, pp. 222–244 in *Proc. Berkeley ICM* Vol. 1, Amer. Math. Soc., Providence, 1987.

Su87 [186] A. Suslin, Torsion in K_2 of fields, *K-theory* 1 (1987), 5–29.

Su91 [187] A. Suslin, K_3 of a field, and the Bloch group (Russian), *Trudy Mat. Inst. Steklov* 183 (1990), 180–199. English transl. in *Proc. Steklov Inst. Math.* (1991), 217–239.

Su95 [188] A. Suslin, Excision in integer algebraic K-theory (Russian), *Trudy Mat. Inst. Steklov* 208 (1995), *Teor. Chisel, Algebra i Algebr. Geom.*, 290–317.

Su03 [189] A. Suslin, On the Grayson spectral sequence, *Trudy Mat. Inst. Steklova* 241 (2003), 218–253. English transl. in *Proc. Steklov Inst. Math.* 241 (2003), 202–237.

SV00 [190] A. Suslin and V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients, pp. 117–189 in *NATO Sci. Ser. C Math. Phys. Sci.* 548, Kluwer, 2000.

SuW [191] A. Suslin and M. Wodzicki, Excision in algebraic K-theory, *Annals of Math.* 136 (1992), 51–122.

Suz [192] M. Suzuki, *Group Theory I* Springer-Verlag, 1982.

Swan [193] R. Swan, *Algebraic K-theory*, Lecture Notes in Math. 76, Springer, 1968.

Swan70 [194] R. Swan, A Splitting Principle in Algebraic K-theory, pp.155–159 in *Proc. Sympos. Pure Math.* XXI, AMS, 1971.

Swan71 [195] R. Swan, Excision in algebraic K-theory, *J. Pure Applied Algebra* 1 (1971), 221–252.

Swan72 [196] R. Swan, Some relations between higher K-functors, *J. Algebra* 21 (1972), 113–136.

Swan80 [197] R. Swan, On Seminormality, *J. Algebra* 67 (1980), 210–229.

Tate [198] J. Tate, On the torsion in K_2 of fields, *Algebraic Number Theory (1976 Kyoto Internat. Sympos.)*, Japan Soc. Promotion Sci., Tokyo, 1977.

Th [199] R. W. Thomason, Algebraic K-theory and Étale Cohomology, *Ann. Sci. Éc. Norm. Sup.* (Paris) 118 (1985), 437–552.

TT [200] R. W. Thomason and T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, pp. 247–435 in *The Grothendieck Festschrift III* Progress in Math. 88, Birkhäuser. 1990.

Totaro [201] B. Totaro, Milnor K-theory is the simplest part of algebraic K-theory, *K-Theory* 6 (1992), 177–189.

Van29 [202] H. Vandiver, On Fermat's Last Theorem, *Trans. AMS* 31 (1929), 613–642.

Van34 [203] H. Vandiver, Fermat's Last Theorem and the second factor in the cyclotomic class number, *Bull. AMS* 40 (1934), 118–126.

Van46 [204] H. Vandiver, Fermat's Last Theorem: its history and the nature of the known results concerning it, *Amer. Math. Monthly* 53 (1946), 555–578.

V69 [205] L. Vaserstein, On the stabilization of the general linear group over a ring, *Math USSR Sb.* 8 (1969), 383–400.

V76 [206] L. Vaserstein, Foundations of algebraic K-theory (Russian), *Uspehi Mat. Nauk* 31:4 (1976), 87–149. English transl. in *Russian Math. Surveys* 31:4 (1976), 89–156.

V04 [207] L. Vaserstein, On the Whitehead determinant for semi-local rings, *J. Algebra* 283 (2005), 690–699.

Verd [208] J.-L. Verdier, Catégories dérivées, pp. 262–311 in *Cohomologie étale (SGA4-1/2)*, Lecture Notes in Math. 569, Springer-Verlag, 1977.

VV02a [209] V. Voevodsky, Open problems in the motivic stable homotopy theory I, pp. 3–34 in *Motives, polylogarithms and Hodge Theory I*, Int. Press, Somerville, 2002.

VV02b [210] V. Voevodsky, A possible new approach to the motivic spectral sequence for algebraic K-theory, pp. 371–379 in *Contemp. Math.* 293, AMS, 2002.

V-MC [211] V. Voevodsky, Motivic cohomology with $\mathbb{Z}/2$-coefficients, *Publ. Math. IHES* 98 (2003), 59–104.

Vo [212] A. Vorst, Localization of the K-theory of Polynomial Extensions, *Math. Ann.* 244 (1979), 33–53.

Wag [213] J. B. Wagoner, Continuous cohomology and p-adic K-theory, pp. 241–248 in *Lecture Notes in Math.* 551, Springer Verlag, 1976.

Wa78 [214] F. Waldhausen, Algebraic K-theory of generalized free products, *Annals of Math.* 108 (1978), 135–256.

W1126 [215] F. Waldhausen, Algebraic K-theory of spaces, *Lecture Notes in Math.* 1126, Springer-Verlag, 1985.

Wash [216] L. Washington, *Introduction to Cyclotomic Fields*, Springer, 1982.

We80 [217] C. Weibel, K-theory and Analytic Isomorphisms, *Invent. Math.* 61(1980), 177–197.

We81 [218] C. Weibel, A survey of products in algebraic K-theory, pp. 494–517 in *Lecture Notes in Math.* 854, Springer-Verlag, 1981.

We-Az [219] C. Weibel, K-theory of Azumaya algebras, *Proc. AMS* 81 (1981), 1–7.

We89 [220] C. Weibel, Homotopy algebraic K-theory, pp. 461–488 in *Contemporary Math.* 83, AMS, 1989.

We91 [221] C. Weibel, Pic is a contracted functor, *Invent. Math.* 103 (1991), 351–377.

We93 [222] C. Weibel, Étale Chern classes at the prime 2, pp. 249–286 in *NATO Sci. Ser. C Math. Phys. Sci.* 407, Kluwer, 1993.

WHomo [223] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, 1994.

We97 [224] C. Weibel, The 2-torsion in the K-theory of the integers, *C. R. Acad. Sci. Paris* 324 (1997), 615–620.

We06 [225] C. Weibel, Higher wild kernels and divisibility in the K-theory of number fields, *J. Pure Appl. Alg.* 206 (2006), 222–244.

WeTK [226] C. Weibel, K_4 of number fields, in preparation.

Wh50 [227] J.H.C. Whitehead, Simple Homotopy Types, *Amer. J. Math* 72 (1950), 1–57.

Wh [228] G.W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, 1978.

Wi [229] A. Wiles, The Iwasawa conjecture for totally real fields, *Annals of Math.* 131 (1990), 493–540.

EGA [EGA] A. Grothendieck and J. Dieudonné, Eléments de Géométrie Algébrique, *Publ. Math. IHES* I 4 (1960); II 8 (1961); III 11 (1961), 17 (1963); IV 20 (1964), 24 (1965), 28 (1966), 32 (1967). Part I (revised), Springer-Verlag, 1971.

GAGA [GAGA] J.-P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* 6 (1956), 1–42.

SGA1 [SGA1] A. Grothendieck, Revêtements et groups fondamental, Séminaire de Géometrie Algébrique, 1960/61. Lecture Notes in Math. 224, Springer, 1971.

SGA4 [SGA4] M. Artin, A. Grothendieck and J.-L. Verdier, *Théorie des topos et cohomologie étale des schemas*, Séminaire de Géométrie Algébrique, 1963/64. Lecture Notes in Math. 269, 270 and 305, Springer, 1972–73.

SGA6 [SGA6] P. Berthelot, A. Grothendieck and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique, 1966/67. Lecture Notes in Math. 225, Springer, 1971.