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A COMBINATORIAL DEFINITION OF HOMOTOPY GROUPS

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1. Introduction

The usual definition of the homotopy groups of a simplicial complex involves only its underlying topological space and disregards the simplicial structure. Our main result is a definition of the homotopy groups of a simplicial complex (or more general: c.s.s. complex) *using only the simplicial structure*. Although the simplicial approximation theorem suggests such a definition, the present one is much simpler and resembles the definition of the homology groups. It involves *free groups* where the corresponding definition of the homology groups involves *free abelian groups*.

The results will be stated in terms of the c.s.s. complexes of Eilenberg-Zilber [1]. There are four chapters and an appendix.

In Chapter I, we review several definitions and results on c.s.s. complexes and homotopy groups. Chapter II contains the new definition of the homotopy groups. The main tools used are a construction G which assigns to a connected c.s.s. complex L with base point ϕ a c.s.s. group $G(L; \phi)$ which, roughly speaking, " is of the homotopy type of the loops on L ", and J.C. Moore's definition of the homotopy groups of a c.s.s. group [11]. As is shown in the appendix, the construction G is a generalization of the construction F of J.W. Milnor [10], the abstract analogue of the reduced product construction of I.M. James [3]. In Chapter III, it is shown in what precise sense the homology groups are " *abelianized homotopy groups* ". We reduce the Hurewicz theorem to a purely group theoretical theorem. Chapter IV, contains miscellaneous results on the construction G.

It should be noted that in several definitions either the first or the last face or degeneracy operator plays a special role and that another definition could be obtained by using the other extreme operator. Both definitions, however, are essentially equivalent and it is mainly a matter of convenience which one is chosen.

Most of the results were announced in [5].

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CHAPTER I. C.S.S. COMPLEXES AND HOMOTOPY GROUPS

2. The homotopy groups (special case)

We recall the definitions of c.s.s. complexes and the extension condition and define homotopy groups only for c.s.s. complex which satisfy the extension condition. The general case will be considered in §4.

DEFINITION 2.1. A *c.s.s. complex* K is a collection of elements (called *simplices*), to each of which is attached a *dimension* $n \geqq 0$, such that for every n-simplex $\sigma \in K$ and every integer i with $0 \leqq i \leqq n$ there are defined in K

(a) an $(n-1)$-simplex $\sigma\varepsilon^i$ (called the i^{th} *face* of σ)

(b) an $(n+1)$-simplex $\sigma\eta^i$ (called the i^{th} *degeneracy* of σ)

The operators ε^i and η^i are required to satisfy the following identities

$$
\begin{aligned}
\varepsilon^i \varepsilon^{j-1} &= \varepsilon^j \varepsilon^i && i < j \\
\eta^{j-1} \eta^i &= \eta^i \eta^j && i < j \\
\eta^j \varepsilon^i &= \varepsilon^i \eta^{j-1} && i < j \\
\eta^j \varepsilon^i &= \text{identity} && i = j, j+1 \\
\eta^j \varepsilon^i &= \varepsilon^{i-1} \eta^j && i > j+1
\end{aligned}
$$

The set of the n-simplices of K will be denoted by K_n.

A *c.s.s. map* $f: K \to L$ is a dimension preserving function which commutes with all face and degeneracy operators, i.e. for every n-simplex $\sigma \varepsilon K$

$$
\begin{aligned}
f(\sigma \varepsilon^i) &= (f\sigma)\varepsilon^i && 0 \leqq i \leqq n \\
f(\sigma \eta^i) &= (f\sigma)\eta^i && 0 \leqq i \leqq n .
\end{aligned}
$$

DEFINITION 2.2. A c.s.s. complex K is said to satisfy the *extension condition* if for every pair of integers (k, n) such that $0 \leqq k \leqq n$ and for every n $(n-1)$-simplices $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n \varepsilon K$ such that $\sigma_i \varepsilon^{j-i} = \sigma_j \varepsilon^i$ for $i < j$ and $i \neq k \neq j$, there exists an n-simplex $\sigma \in K$ such that $\sigma\varepsilon^i = \sigma_i$ for $i \neq k$.

The following relation on the simplices of a c.s.s. complex will be needed.

DEFINITION 2.3. Two n-simplices σ and τ of a c.s.s. complex K are called *homotopic* (notation $\sigma \sim \tau$) if

(a) their faces coincide, i.e. $\sigma\varepsilon^i = \tau\varepsilon^i$ for all i

(b) there exists an $(n+1)$-simplex $\rho \varepsilon K$ such that

$$
\begin{aligned}
\rho\varepsilon^n &= \sigma \\
\rho\varepsilon^{n+1} &= \tau \\
\rho\varepsilon^i &= \sigma\varepsilon^i \eta^{n-1} = \tau\varepsilon^i \eta^{n-1} , && 0 \leqq i < n .
\end{aligned}
$$

PROPOSITION 2.4. *Let K satisfy the extension condition. Then the relation $\sim$ is an equivalence on the simplices of K.*

PROOF. For every n-simplex $\sigma \in K$ we have $\sigma\eta^n \varepsilon^n = \sigma\eta^n \varepsilon^{n+1} = \sigma$ and $\sigma\eta^n \varepsilon^i = \sigma\varepsilon^i \eta^{n-1}$ for $0 \leqq i < n$. Hence the relation $\sim$ is reflexive and it thus remains to show that $\sigma \sim \tau$ and $\sigma \sim \psi$ imply $\tau \sim \psi$. Let $\rho_n, \rho_{n+1} \in K_{n+1}$ be such that

$$\rho_n \varepsilon^n = \sigma,\ \rho_n \varepsilon^{n+1} = \tau,\ \rho_n \varepsilon^i = \tau\varepsilon^i \eta^{n-1} \qquad 0 \leqq i < n$$
$$\rho_{n+1} \varepsilon^n = \sigma,\ \rho_{n+1} \varepsilon^{n+1} = \psi,\ \rho_{n+1} \varepsilon^i = \psi\varepsilon^i \eta^{n-1} \qquad 0 \leqq i < n\ .$$

Then the $(n+1)$-simplices $\tau\eta^n \eta^n \varepsilon^0, \cdots, \tau\eta^n \eta^n \varepsilon^{n-1}, \rho_n, \rho_{n+1}$ "match" and application of the extension condition ($k = n+2$) yields an $(n+2)$-simplex $\rho \in K$ such that $\rho\varepsilon^n = \rho_n$, $\rho\varepsilon^{n+1} = \rho_{n+1}$ and $\rho\varepsilon^i = \tau\eta^n \eta^n \varepsilon^i$ for $0 \leqq i < n$. A straightforward computation now yields

$$\rho\varepsilon^{n+2} \varepsilon^n = \tau,\ \rho\varepsilon^{n+2} \varepsilon^{n+1} = \psi,\ \rho\varepsilon^{n+2} \varepsilon^i = \tau\varepsilon^i \eta^{n-1} \qquad 0 \leqq i < n.$$

DEFINITION 2.5. A *complex with base point* is a pair $(K; \phi)$ where K is a c.s.s. complex and $\phi \in K$ is a 0-simplex (the *base point*).

Let $(K; \phi)$ and $(L; \psi)$ be complexes with base point. By a *map* $f: (K; \phi) \to (L; \psi)$ we then mean a c.s.s. map $f: K \to L$ such that $f\phi = \psi$.

DEFINITION 2.6. Let $(K; \phi)$ be a c.s.s. complex with base point, where K satisfies the extension condition. For every integer $n \geqq 0$ we define a set $\pi_n(K; \phi)$ as follows. Let Γ_n be the set consisting of those n-simplices $\sigma \in K$ such that

$$\sigma\varepsilon^i = \phi\eta^0 \cdots \eta^{n-2} \qquad 0 \leqq i \leqq n\ .$$

The equivalence relation $\sim$ divides Γ_n into classes. Then $\pi_n(K; \phi)$ will be the set of these equivalence classes, i.e. $\pi_n(K; \phi) = \Gamma_n/(\sim)$. The class containing an n-simplex σ will be denoted by $\{\sigma\}$. The set $\pi_0(K; \phi)$ is called the *set of components* of K.

For $n > 0$ the set $\pi_n(K; \phi)$ may be converted into a group (the n^{th} *homotopy group* of K rel. ϕ) as follows. Let $\sigma \in a$ and $\tau \in b$ be n-simplices in the classes $a, b \in \pi_n(K; \phi)$. Because K satisfies the extension condition there exists an $(n+1)$-simplex $\rho \in K$ such that

$$\rho\varepsilon^{n-1} = \sigma,\ \rho\varepsilon^{n+1} = \tau,\ \rho\varepsilon^i = \phi\eta^0 \cdots \eta^{n-1} \qquad 0 \leqq i < n-1\ .$$

The product $a \cdot b$ then is defined by

$$a \cdot b = \{\rho\varepsilon^n\}$$

It may be verified (by the method used in the proof of Proposition 2.4) that this multiplication is independent of the choice of σ, τ and ρ. Furthermore

PROPOSITION 2.7. *The multiplication defined above converts $\pi_n(K; \phi)$ into a group.*

A map $f\colon (K;\phi)\to(L;\psi)$ induces for every integer $n \geqq o$ a map $\pi_n(f)\colon \pi_n(K;\phi)\to\pi_n(L;\psi)$. Clearly $\pi_n(f)$ is a homomorphism for $n > o$, while $\pi_0(f)$ is such that $\pi_0(f)\ \{\phi\} = \{\psi\}$.

PROOF OF PROPOSITION 2.7. We must prove left and right divisibility and associativity.

Divisibility. Let $a,\ b \in \pi_n(K;\phi)$ and let $\sigma \in a$ and $\tau \in b$. Using the extension condition ($k = n-1, n+1$) one may find $\rho_0,\ \rho_1 \in K_{n+1}$ such that

$$\rho_0\varepsilon^n = \tau,\ \rho_0\varepsilon^{n+1} = \sigma,\ \rho_0\varepsilon^i = \phi\eta^0\cdots\eta^{n-1} \qquad 0 \leqq i < n-1$$
$$\rho_1\varepsilon^{n-1} = \sigma,\ \rho_1\varepsilon^n = \tau,\ \rho_1\varepsilon^i = \rho\eta^0\cdots\eta^{n-1} \quad 0 \leqq i < n-1\,.$$

Consequently $\{\rho_0\varepsilon^{n-1}\}\cdot\sigma = \tau$ and $\sigma\cdot\{\rho_1\varepsilon^{n+1}\} = \tau$.

Associativity. Let $a, b, c \in \pi_n(K;\phi)$ and let $\sigma \in a$, $\tau \in b$ and $\psi \in c$. Then there exist $\rho_{n-1}, \rho_n, \rho_{n+2}$ (use the extension condition for $k = n$) such that

$$\rho_{n-1}\,\varepsilon^{n-1} = \sigma,\ \rho_{n-1}\,\varepsilon^{n+1} = \tau,\ \rho_{n-1}\,\varepsilon^i = \phi\eta^0\cdots\eta^{n-1} \qquad 0 \leqq i < n-1$$
$$\rho_{n+1}\,\varepsilon^{n-1} = \rho_{n-1}\,\varepsilon^n,\ \rho_{n+1}\,\varepsilon^{n+1} = \psi,\ \rho_{n+1}\,\varepsilon^i = \phi\eta^0\cdots\eta^{n-1} \qquad 0 \leqq i < n-1$$
$$\rho_{n+1}\,\varepsilon^{n-1} = \tau,\ \rho_{n+2}\,\varepsilon^{n+1} = \psi,\ \rho_{n+2}\,\varepsilon^i = \phi\eta^0\cdots\eta^{n-1}\,, \qquad 0 \leqq i < n-1\,.$$

Application of the extension condition one dimension higher ($k = n$) yields a $\rho \in K_{n+2}$ such that $\rho\varepsilon^i = \rho_i$ for $i = n-1, n+1, n+2$ and $\rho\varepsilon^i = \phi\eta^0\cdots\eta^n$ for $0 \leqq i < n-1$. A simple computation yields

$$\rho\varepsilon^n\,\varepsilon^{n-1} = \sigma,\ \rho\varepsilon^n\,\varepsilon^{n+1} = \rho_{n+2}\,\varepsilon^n,\ \rho\varepsilon^n\,\varepsilon^i = \phi\eta^0\cdots\eta^{n-1}\,, \quad 0 \leqq i < n-1\,.$$

Consequently

$$(\sigma\cdot\tau)\cdot\psi = \{\rho_{n-1}\,\varepsilon^n\}\cdot\psi = \{\rho_{n+1}\,\varepsilon^{n-1}\}\cdot\psi = \{\rho_{n+1}\,\varepsilon^n\}$$
$$= \{\rho\varepsilon^n\,\varepsilon^n\} = \sigma\cdot\{\rho_{n+2}\,\varepsilon^n\} = \sigma\cdot(\tau\cdot\psi)\,.$$

3. The homotopy sequence (special case)

In defining the homotopy sequence of a fibre sequence we again restrict ourselves temporarily to c.s.s. complexes which satisfy the extension condition.

DEFINITION 3.1. A c.s.s. map $p\colon E\to B$ is called a *fibre map* if for every pair of integers (k, n) such that $0 \leqq k \leqq n$, for every n $(n-1)$-simplices $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n \in E$ such that $\sigma_i\,\varepsilon^{j-1} = \sigma_j\,\varepsilon^i$ for $i < j$ and $i \neq k \neq j$ and for every n-simplex $\tau \in B$ such that $\tau\varepsilon^i = p\sigma_i$ for $i \neq k$, there exists an n-simplex $\sigma \in E$ such that $p\sigma = \tau$ and $\sigma\varepsilon^i = \sigma_i$ for all $i \neq k$. The complex E is called the *total complex*, B the *base*. Let $\phi \in B$ be a 0-simplex. Then we mean by the *fibre* of p over ϕ the sub-complex $F \subset E$ such that $F_n = p^{-1}(\phi\eta^0\cdots\eta^{n-1})$ for all n.

PROPOSITION 3.2. *Let $p\colon E\to B$ be a fibre map. Then B satisfies the extension condition if and only if E does so.*

PROOF. Suppose B satisfies the extension condition. Let $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n \in E_{n-1}$ be such that $\sigma_i \varepsilon^{j-1} = \sigma_j \varepsilon^i$ for $i < j$ and $i \neq k \neq j$. Using the extension condition one obtains an n-simplex $\tau \in B$ such that $\tau \varepsilon^i = p\sigma_i$ for $i \neq k$ and because p is a fibre map it follows that there exists an n-simplex $\sigma \in E$ such that $\sigma\varepsilon^i = \sigma_i$ for $i \neq k$. Hence E satisfies the extension condition.

The other half of the proof is similar, although slightly more complicated.

DEFINITION 3.3. A sequence of maps of complexes with base point

$$(F; \phi) \xrightarrow{q} (E; \chi) \xrightarrow{p} (B, \psi)$$

is called a *fibre sequence* if

(a) p is a fibre map onto
(b) q is an isomorphism into,
(c) " the image of q " = " the fibre of p over ψ ".

We call F, E and B respectively the *fibre*, *total complex* and *base* of the fibre sequence.

DEFINITION 3.4. Let

$$(F; \phi) \xrightarrow{q} (E; \chi) \xrightarrow{p} (B; \psi)$$

be a fibre sequence such that B (and hence E) satisfies the extension condition. For every integer $n > 0$ we define a *boundary map* $\partial_n(q, p): \pi_n(B; \psi) \to \pi_{n-1}(F; \phi)$ as follows. Let $a \in \pi_n(B; \psi)$ and let $\tau \in a$ be an n-simplex. Because p is a fibre map there exists an n-simplex $\sigma \in E$ such that $p\sigma = \tau$ and $\sigma\varepsilon^i = \chi\eta^0 \cdots \eta^{n-2}$ for $i \neq n$. We then define $\partial_n(q, p)a \in \pi_{n-1}(F; \phi)$ by

$$\partial_n(q, p)a = \{q^{-1}(\sigma\varepsilon^n)\} \ .$$

It may be verified (as in the proof of Proposition 2.4) that this definition is independent of the choice of τ and σ. For $n > 1$ the map $\partial_n(q, p)$ is a homomorphism, while $\partial_1(q, p)$ is such that $\partial_1(q, p)\{\psi\eta^0\} = \{\phi\}$.

We can now form the *homotopy sequences* of a fibre sequence.

PROPOSITION 3.5. *Let*

$$(F; \phi) \xrightarrow{q} (E; \chi) \xrightarrow{p} (B; \psi)$$

be a fibre sequence such that B (and hence E) satisfies the extension condition. Then the sequence

$$\longrightarrow \pi_n(F;\phi) \xrightarrow{\pi_n(q)} \pi_n(E;\chi) \xrightarrow{\pi_n(p)} \pi_n(B;\psi) \xrightarrow{\partial_n(q,p)} \cdots$$
$$\xrightarrow{\pi_0(p)} \pi_0(B;\psi) \longrightarrow 1$$

is exact (in the sense that "image of each map" = "kernel of the next map").

PROOF. We will show exactness only at $\pi_{n-1}(F;\phi)$. The other parts of the proof are similar.

Let $a \in \pi_{n-1}(F;\phi)$. If $b \in$ *kernel* $\pi_{n-1}(q)$, then there exists a $\rho \in b$ and a $\sigma \in E_n$ such that $q\rho = \sigma\varepsilon^n$ and $\sigma\varepsilon^i = \chi\eta^0 \cdots \eta^{n-2}$ for $i \neq n$. Hence $b = \partial_n(q,p)\{p\sigma\}$. Conversely if $b \in$ *image* $\partial_n(q,p)$ then there exists a $\rho \in b$ and a $\sigma \in E_n$ such that $q\rho = \sigma\varepsilon^n$ and $\sigma\varepsilon^j = \chi\eta^0 \cdots \eta^{n-2}$ for $i \neq n$ and hence $q\rho \sim \chi\eta^0 \cdots \eta^{n-2}$.

5. The general case

We shall now define homotopy groups and boundary maps for all c.s.s. complexes with base point. As use will be made of the definitions given in §§ 2 and 3 we, in order to avoid confusion, denote temporarily the homotopy groups and boundary maps as defined in §§ 2 and 3 by $\bar{\pi}_n$ and $\bar{\partial}_n$.

DEFINITION 4.1. Let $(K;\phi)$ be a complex with base point, let $|K|$ denote its geometrical realization in the sense of J.W. Milnor [9]. ($|K|$ is a CW-complex of which the n-cells are in one-to-one correspondence with the non-degenerate n-simplices of K), let $S|K|$ be the total singular complex of $|K|$ and let $i: K \to S|K|$ denote the natural embedding map (see [9]). For every integer $n \geqq 0$ we then define $\pi_n(K;\phi)$ by

$$\pi_n(K;\phi) = \bar{\pi}_n(S|K|\,;i\phi)$$

Definition 4.1. clearly involves topological spaces and continuous maps. It is however possible to give an equivalent definition which only involves c.s.s. complexes and maps. *Using only c.s.s. complexes and maps*, one may define for every c.s.s. complex K a c.s.s. complex $Ex^\infty K$ and a c.s.s. map $e^\infty(K): K \to Ex^\infty K$ and for every c.s.s. map $f: K \to L$ a c.s.s. map $Ex^\infty f: Ex^\infty K \to Ex^\infty L$ (the exact definitions will be omitted; they can be found in [7]) having the following properties.

4.2a. *For every c.s.s. complex K the complex $Ex^\infty K$ satisfies the extension condition.*

4.2b. *The construction Ex^∞ preserves fibre sequences, i.e., if*

$$(F;\phi) \xrightarrow{q} (E;\chi) \xrightarrow{p} (B;\psi)$$

is a fibre sequence, then so is the sequence

$$(Ex^\infty F\,;\, e^\infty(F)\phi) \xrightarrow{Ex^\infty q} (Ex^\infty E\,;\, e^\infty(E)\chi) \xrightarrow{Ex^\infty p} (Ex^\infty B\,(e^\infty\,;\,(B)\psi)\,.$$

The proof of these properties can also be found in [7].

In view of Property 4.2a we now can define

DEFINITION 4.3. Let $(K; \phi)$ be a complex with base point. For every integer $n \geqq 0$ we then define $\pi_n(K; \phi)$ by

$$\pi_n(K; \phi) = \bar{\pi}_n(Ex^\infty K;\, e^\infty(K)\phi)\,.$$

Let $f\colon (K; \phi) \to (L; \psi)$ be a map. Then for every integer $n \geqq 0$ the induced map $\pi_n(f): \pi_n(K; \phi) \to \pi_n(L; \psi)$ is defined by

$$\pi_n(f) = \bar{\pi}_n(Ex^\infty f)\,.$$

It may be shown that the Definitions 4.1 and 4.3 are equivalent, and that if K satisfies the extension condition, both are equivalent with Definition 2.6. For more details see [7].

In view of Property 4.2b the boundary maps may be defined as follows

DEFINITION 4.4. Let

$$(F; \phi) \xrightarrow{q} (E; \chi) \xrightarrow{p} (B; \psi)$$

be a fibre sequence. For every integer $n > 0$ we then define the *boundary map* $\partial_n(q, p): \pi_n(B; \psi) \to \pi_{n-1}(F; \phi)$ by

$$\partial_n(q, p) = \bar{\partial}_n(Ex^\infty q,\, Ex^\infty p)\,.$$

The following then is an immediate consequence of Proposition 3.5.

PROPOSITION 4.5. *Let*

$$(F; \phi) \xrightarrow{q} (E\,;\, \chi) \xrightarrow{p} (B\,;\, \psi)$$

be a fibre sequence. Then the sequence

$$\longrightarrow \pi_n(F; \phi) \xrightarrow{\pi_n(q)} \pi_n(E; \chi) \xrightarrow{\pi_n(p)} \pi_n(B; \psi) \xrightarrow{\partial_n(q,\,p)} \cdots$$
$$\xrightarrow{\pi_0(p)} \pi_0(B; \psi) \longrightarrow 1$$

is exact (*in the sense that " image of each map " = " kernel of the next map "*).

5. The homotopy groups of c.s.s. groups

DEFINITION 5.1. Let G be a c.s.s. complex. The face and degeneracy operators may be considered as functions $\varepsilon^i: G_n \to G_{n-1}$ and $\eta^i: G_n \to G_{n+1}$, The complex G will be called an (*abelian*) *c.s.s. group* if

(a) G_n is an (abelian) group for all $n \geqq 0$

(b) the functions $\varepsilon^i : G_n \to G_{n-1}$ and $\eta^i : G_n \to G_{n+1}$ are homomorphisms for all $0 \leqq i \leqq n$.

The identity element of G_n will be denoted by e_n.

Let G and H be c.s.s. groups. A c.s.s. map $f : G \to H$ is called a *c.s.s. homomorphism* if the restriction $f|G_n$ is a homomorphism for all $n \geqq 0$.

Let G be a c.s.s. group. *A c.s.s. subgroup* $H \subset G$ is a subcomplex of G such that H_n is a subgroup of G_n for all $n \geqq 0$. By the c.s.s. subgroup *generated* by a subset of $K \subset G$ we mean the smallest c.s.s. subgroup of G containing K.

The following important property of c.s.s. groups was proved by J.C. Moore [11].

PROPOSITION 5.2 *Every c.s.s. group satisfies the extension condition.*

It follows that the homotopy groups of a c.s.s. group G may be defined as in §2 in terms of the c.s.s. structure of G. Following J.C. Moore [11] we shall now give another definition which is completely in terms of the group structure of G

DEFINITION 5.3. Let G be a c.s.s. group. For every integer $n \geqq 0$ let

$$\tilde{G}_n = G_n \cap \textit{kernel}\ \varepsilon^1 \cap \cdots \cap \textit{kernel}\ \varepsilon^n$$

Then $\sigma \in \tilde{G}_{n+1}$ implies $\sigma\varepsilon^0 \in \tilde{G}_n$. Hence we may define homomorphisms $\tilde{\partial}_{n+1} : \tilde{G}_{n+1} \to \tilde{G}_n$ by

$$\tilde{\partial}_{n+1}\sigma = \sigma\varepsilon^0 \qquad \sigma \in \tilde{G}_{n+1} .$$

For each integer $m < 0$ let $\tilde{G}_m$ be trivial and let $\tilde{\partial}_{m+1} : \tilde{G}_{m+1} \to \tilde{G}_m$ be the trivial map. Then it can be shown that *image* $\tilde{\partial}_{n+1}$ is a normal subgroup of *kernel* $\tilde{\partial}_n$ for all n, i.e., $\tilde{G} = \{\tilde{G}_n, \tilde{\partial}_n)$ is a (not necessarily abelian) chain complex. This chain complex will be called the *Moore chain complex* of G.

The homology groups of $\tilde{G}$ are

$$H_n(\tilde{G}) = \textit{kernel}\ \tilde{\partial}_n / \textit{image}\ \tilde{\partial}_{n+1} .$$

A comparison with the homotopy groups as defined in §2 yields

PROPOSITION 5.4. *Let G be a c.s.s. group. Then for every integer $n > 0$*

$$H_n(\tilde{G}) = \pi_n(G ; e_0) .$$

It follows that $\pi_0(G ; e_0)$ may be converted into a *group*. If $f : G \to H$ is a c.s.s. homomorphism, then clearly $\pi_0(f) : \pi_0(G; e_0) \to \pi_0(H; e_0)$ becomes a *homomorphism*.

6. Principal bundles and loop complexes

In defining the c.s.s. analogue of a principal bundle we roughly follow J.C. Moore [12].

DEFINITION 6.1. A *principal bundle with twisting function* is a triple $((B\,;\psi), F, t)$ where

(i) B is a c.s.s. complex (the *base*) with base point ψ.

(ii) F is a c.s.s. group.

(iii) t is a function $t: B \to F$ (the *twisting function*) which lowers the dimension by one and is such that for every integer $n > 0$ and every simplex $\sigma \in B_{n+1}$

$$\begin{aligned} (t\sigma)\varepsilon^i &= t(\sigma\varepsilon^i) && 0 \leqq i < n \\ (t\sigma)\varepsilon^n &= t(\sigma\varepsilon^n)\cdot(t(\sigma\varepsilon^{n+1}))^{-1} \\ (t\sigma)\eta^i &= t(\sigma\eta^i) && 0 \leqq i \leqq n \\ e_{n+1} &= t(\sigma\eta^{n+1}) \end{aligned}$$

By the *bundle complex* or *total complex* we mean the c.s.s. complex $F \times_t B$ defined as follows. An n-simplex of $F \times_t B$ is any pair (ρ, σ) where $\rho \in F_n$ and $\sigma \in B_n$. Its faces $(\rho, \sigma)\varepsilon^i$ and degeneracies $(\rho, \sigma)\eta^i$ are given by the formulas

$$\begin{aligned} (\rho, \sigma)\varepsilon^i &= (\rho\varepsilon^i, \sigma\varepsilon^i) && 0 \leqq i < n \\ (\rho, \sigma)\varepsilon^n &= (\rho\varepsilon^n \cdot t\sigma, \sigma\varepsilon^n) \\ (\rho, \sigma)\eta^i &= (\rho\eta^i, \sigma\eta^i) && 0 \leqq i \leqq n\,. \end{aligned}$$

REMARK. As principal bundles with twisting function are the only principal bundle we will use, we will frequently omit the mention of the twisting function.

With a principal bundle $((B; \psi), F, t)$ we may associate a *fibre sequence*

$$(F; e_0) \xrightarrow{\ q\ } (F \times_t B; (e_0, \psi)) \xrightarrow{\ p\ } (B; \psi)$$

where the maps p and q are given by

$$\begin{aligned} p(\rho, \sigma) &= \sigma && (\rho, \sigma) \in F x_t B \\ q\rho &= (\rho, \psi\eta^0 \cdots \eta^{n-1}) && \rho \in F_n\,. \end{aligned}$$

It should be noted that for this fibre sequence the boundary map $\partial_1(q, p): \pi_1(B; \psi) \to \pi_0(F; e_0)$ is also a *homomorphism*.

DEFINITION 6.2. Let K be a c.s.s. complex and ϕ a 0-simplex of K. Then K is called *connected* if $\pi_0(K; \phi)$ consists of one element and is called *contractible* if $\pi_n(K; \phi)$ consists of one element for all $n \geqq 0$.

Following J.W. Milnor [10], we now define

DEFINITION 6.3. Let $((K; \phi), G, t)$ be a principal bundle. Then G is called a *loop complex* of K rel. ϕ (under the twisting function t) if the bundle complex $G \times_t K$ is contractible

Combination of Definitions 6.2 and 6.3 with Proposition 4.5 yields

PROPOSITION 6.4. *Let $(K; \phi)$ be a complex with base point and let G be a c.s.s. group. If G is a loop complex of K rel. ϕ (under some twisting function) then K is connected.*

CHAPTER II. THE CONSTRUCTION G AND THE DEFINITION OF THE HOMOTOPY GROUPS

7. The construction G for a reduced complex

Let K be a *reduced complex*, i.e., K is a c.s.s. complex which has only one 0-simplex ϕ. Then we define a *c.s.s. group* GK as follows. The group of the n-simplices, $G_n K$ is a group which has

(i) one generator $\bar{\sigma}$ for every $(n+1)$-simplex $\sigma \in K_{n+1}$

(ii) one relation $\overline{\tau\eta^n} = e_n$ for every n-simplex $\tau \in K_n$.

Clearly the groups $G_n K$ so defined are free. Hence it suffices to define the face and degeneracy homomorphisms $\varepsilon^i: G_n K \to G_{n-1} K$ and $\eta^i: G_n K \to G_{n+1} K$ on the generators of $G_n K$. This is done by the following formulas

$$\begin{aligned} \bar{\sigma}\varepsilon^i &= \overline{\sigma\varepsilon^i} && 0 \leqq i < n \\ \bar{\sigma}\varepsilon^n &= \overline{\sigma\varepsilon^n} \cdot (\overline{\sigma\varepsilon^{n+1}})^{-1} && \\ \bar{\sigma}\eta^i &= \overline{\sigma\eta^i} && 0 \leqq i \leqq n \,. \end{aligned}$$

It is readily verified that GK so defined is a c.s.s. group.

THEOREM 7.1. *Let K be a reduced complex with 0-simplex ϕ. Then the c.s.s. group GK is a loop complex of K rel. ϕ (Definition 6.3) i.e., there exists a principal bundle with K as base, GK as fibre and contractible bundle complex.*

PROOF. Define a function $t: K \to GK$ (which lowers the dimension by one) by

$$t\sigma = \bar{\sigma} \qquad \sigma \in K.$$

Then clearly t is a twisting function and hence $((K; \phi), GK, t)$ is a principal bundle, (Definition 6.1). Let $EK = GK \times_t K$ be the bundle complex. Then Theorem 7.1 is a direct consequence of the following lemma which will be proved in §13.

LEMMA 7.2. *The c.s.s. complex $EK = GK \times_t K$ is contractible.*

8. The homotopy groups of a reduced complex

Let K be a reduced complex. Because K has only one 0-simplex ϕ, the homotopy groups of K do not depend on a choice of base point. Therefore we shall usually write $\pi_n(K)$ instead of $\pi_n(K; \phi)$. Using the c.s.s. group GK we shall now obtain a new definition of $\pi_n(K)$ which is much more straightforward than the one given in §4.

Let K be a reduced complex and consider the fibre sequence

$$(GK; e_0) \xrightarrow{q} (EK; (e_0, \phi)) \xrightarrow{p} (K; \phi)$$

associated with the principal bundle $((K; \phi), GK, t)$ (§6). Combining the exactness of the homotopy sequence of this fibre sequence (Proposition 4.5) with Lemma 7.2, we get that for every integer $n > 0$ the boundary map $\partial_n(q, p)$ is an isomorphism

$$\partial_n(q, p)\colon \pi_n(K) \approx \pi_{n-1}(GK; e_0) .$$

Hence we could define the homotopy groups of K to be those of GK (with a shift in dimension). But GK is a c.s.s. group and the homotopy groups of a c.s.s. group may be expressed in terms of the group structure of GK (§5). Consequently the homotopy groups of a reduced complex K may be defined in the following group theoretical manner.

DEFINITION 8.1. Let K be a reduced complex. Form the c.s.s. group GK(§2). Let $\tilde{G}K$ be the Moore chain complex of GK(Definition 5.3). Then for every integer $n > 0$ we define $\pi_n(K)$, the n^{th} *homotopy group of* K, by

$$\pi_n(K) = H_{n-1}(\tilde{G}K) .$$

9. The construction G for an arbitrary connected c.s.s. complex

We shall extend the definition of G to an arbitrary *connected c.s.s. complex* L *with base point* ψ. The procedure followed, which is a direct generalization of §7 involves the choice of a maximal tree of L. It will however be shown in §12 that the c.s.s. group $G(L; \psi)$ so obtained is independent of the choice of a maximal tree.

Let L be a connected c.s.s. complex with base point ψ. A maximal tree $T \subset L$ is roughly speaking "a connected 1-dimensional subcomplex which contains all vertices of L but no closed loops". We first give a more exact definition. By an *n-loop of* L *of length* k we mean a sequence $(\sigma_1, \cdots, \sigma_{2k})$ of $2k$ $(n+1)$-simplices of L such that

$$\begin{aligned} &\sigma_{2j-1}\, \varepsilon^{n+1} = \sigma_{2j}\, \varepsilon^{n+1} && 1 \leqq j \leqq k \\ &\sigma_{2j}\, \varepsilon^{n} \cdots \varepsilon^{0} = \sigma_{2j+1}\, \varepsilon^{n} \cdots \varepsilon^{0} && 1 \leqq j < k \\ &\sigma_{1}\, \varepsilon^{n} \cdots \varepsilon^{0} = \sigma_{2k}\, \varepsilon^{n} \cdots \varepsilon^{0} = \psi \end{aligned}$$

i.e., two adjacent simplices have alternatingly the same last face or the same last vertex (i.e., the 0-simplex opposite the last face). By a *reduced* n-loop of L of length k we mean an n-loop $(\sigma_1, \cdots, \sigma_{2k})$ such that

$$\sigma_s \neq \sigma_{s+1} \qquad 1 \leqq s < 2k$$

i.e., two adjacent simplices are always distinct. We define a *tree* of L as a connected subcomplex $T \subset L$ with $\psi \in T$, which contains no reduced loops of length > 0. A *maximal tree* $T \subset L$ is a tree which contains all 0-simplices of L, i.e., $T_0 = L_0$.

LEMMA 9.1. *A connected c.s.s. complex L with base point ψ contains a maximal tree.*

PROOF. The proof will only be sketched. The details are left to the reader. Clearly the subcomplex of L consisting only of ψ and its degeneracies is a tree. Let T^1 be a tree which is not maximal. Then because L is connected, there exists a 1-simplex $\sigma \in L$ such that $\sigma\varepsilon^0 \in T^1$ and $\sigma\varepsilon^1 \notin T^1$ (or $\sigma\varepsilon^1 \in T^1$ and $\sigma\varepsilon^0 \notin T^1$). Let $S \subset L$ be the subcomplex generated by σ. Then it is readily verified that $T^1 \cup S$ is also a tree. It follows from the connectedness of L that it is possible to obtain an increasing sequence

$$T^1 \, T^2 \cdots T^n \cdots$$

of trees of L such that every 0-simplex of L is contained in all but a finite number of them. Let $T = \cup_{i=1}^{\infty} T^i$ be their union, then T is a maximal tree

Let L be a connected c.s.s. complex with base point ψ. Choose a maximal tree $T \subset L$. We then define a c.s.s. group $G(L; \psi)$ as follows. The group of the n-simplices, $G_n(L; \psi)$, is a group which has

(i) one generator $\bar{\sigma}$ for every $(n+1)$-simplex $\sigma \in L_{n+1}$,

(ii) one relation $\overline{\tau\eta^n} = e_n$ for every n-simplex $\tau \in L_n$,

(iii) one relation $\bar{\rho} = e_n$ for every $(n+1)$-simplex $\rho \in T_{n+1}$.

Clearly the groups $G_n(L; \psi)$ are free and hence it suffices to define the face and degeneracy homomorphisms $\varepsilon^i : G_n(L; \psi) \to G_{n-1}(L; \psi)$ and $\eta^i : G_n(L; \psi) \to G_{n+1}(L; \psi)$ on the generators of $G_n(L; \psi)$. This is done by the following formulas

$$\bar{\sigma}\varepsilon^i = \overline{\sigma\varepsilon^i} \qquad 0 \leqq i < n$$
$$\bar{\sigma}\varepsilon^n = \overline{\sigma\varepsilon^n} \cdot (\overline{\sigma\varepsilon^{n+1}})^{-1}$$
$$\bar{\sigma}\eta^i = \overline{\sigma\eta^i} \qquad 0 \leqq i \leqq n .$$

Clearly $G(L; \psi)$ so defined is a c.s.s. group.

THEOREM 9.2. *Let L be a connected c.s.s. complex with base point ψ. Then the c.s.s. group $G(L; \psi)$ is a loop complex of L rel. ψ, (Definition 6.3.)*

i.e., there exists a principal bundle with L as base, $G(L;\psi)$ as fibre and contractible bundle complex.

PROOF. Define a function $t: L \to G(L;\psi)$ (which lowers the dimension by one) by

$$t\sigma = \bar{\sigma} \qquad \sigma \in L .$$

Then clearly t is a twisting function and hence $((L;\psi), G(L;\psi), t)$ is a principal bundle, (Definition 6.1). Let $E(L;\psi) = G(L;\psi) \times_t L$ be the bundle complex. Then Theorem 9.2 is a direct consequence of the following lemma.

LEMMA 9.3. *The c.s.s. complex $E(L;\psi) = G(L;\psi) \times_t L$ is contractible,* The proof is similar to that of Lemma 7.2. (See §13).

REMARK. If L has only one 0-simplex, then clearly the only possible maximal tree of L is the subcomplex generated by ψ. The results of this § then reduces to those of §7, i.e., $G(L;\psi) = GL$ and $E(L;\psi) = EL$.

10. The homotopy groups of a connected c.s.s. complex

The definition of homotopy groups of §8 will now be extended to arbitrary connected c.s.s. complexes.

Let L be a connected c.s.s. complex with base point ψ. Consider the fibre sequences.

$$(G(L;\psi);e_0) \xrightarrow{q} (E(L;\psi);(e_0,\psi)) \xrightarrow{p} (L;\psi)$$

associated with the principal bundle $((L;\psi), G(L;\psi), t)$. Combination of the exactness of the homotopy sequence of this fibre sequence with Lemma 9.3 yields that for every integer $n > 0$ the boundary map $\partial_n(q,p)$ is an isomorphism

$$\partial_n(q,p): \pi_n(L;\psi) \approx \pi_{n-1}(G(L;\psi);e_0) .$$

It will be shown in §12 that the above fibre sequence is independent of the choice of the maximal tree $T \subset L$. Hence so are the boundary maps $\partial_n(q,p)$ and as in §8 we may give the following group theoretical definition of the homotopy groups of L.

DEFINITION 10.1. Let L be a connected c.s.s. complex with base point ψ. Choose a maximal tree $T \subset L$. Form the c.s.s. group $G(L;\psi)$ (see §9) and let $\tilde{G}(L;\psi)$ be the Moore chain complex of $G(L;\psi)$ (Definition 5.3). Then for every integer $n > 0$ we define $\pi_n(L;\psi)$, the n^{th} *homotopy group* of L rel. ψ, by

$$\pi_n(L;\psi) = H_{n-1}(\tilde{G}(L;\psi)).$$

REMARK. If L has only one 0-simplex ϕ, then Definition 10.1 reduces to Definition 8.1.

11. The homotopy groups of a simplicial complex

Let M be a connected simplicial complex and let a be one of its vertices. It is well known that the fundamental group $\pi_1(M; a)$ may be defined combinatorially, i.e. in terms of the simplicial structure of M. The usual definitions of the higher homotopy groups, however, involve the underlying topological space of M and disregard completely the simplicial structure. It will now be shown how, using the results of §10, a purely combinatorial definition may be obtained for the homotopy groups of M. For the fundamental group this new definition coincides with the old combinatorial one.

Choose an ordering of the vertices of M. Then we may associate with M a c.s.s. complex L (which depends on the ordering chosen) as follows. An n-simplex of L is an $(n+1)$-tuple $(A_0, \cdots, A_n)$ of vertices of M such that

(a) $A_i \leqq A_{i+1}$ $\qquad 0 \leqq i < n$

(b) $A_0, \cdots, A_n$ lie in a common simplex of M .

Its faces and degeneracies are given by

$$(A_0, \cdots, A_n)\varepsilon^i = (A_0, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n) \qquad 0 \leqq i \leqq n$$
$$(A_0, \cdots, A_n)\eta^i = (A_0, \cdots, A_i, A_i, \cdots, A_n) \qquad 0 \leqq i \leqq n\ .$$

Define a base point $\phi \in L$ by $\phi = (a)$.

Clearly M is homeomorphic with $|L|$, the geometrical realization of L [9], and it follows that the homotopy groups of M are isomorphic with those of L (see §4), which by §10, may be defined in a comparatively simple combinatorial manner. As L was defined in terms of the simplicial structure of M only, it follows that the resulting definition of $\pi_n(M; a)$ also involves only the simplicial structure of M and completely disregards the underlying topological space.

It follows from the results of §18 that for $\pi_1(M; a)$ this combinatorial definition coincides with the old one, in which a maximal tree $S \subset M$ was chosen (i.e. a connected 1-dimensional subcomplex containing all vertices but no closed loops). The fundamental group then was a group with one generator for every 1-simplex of M, not in S, and a relation for every 2-simplex of M.

12. The role of the maximal tree

Let L be a connected c.s.s. complex with base point ϕ and let $T \subset L$ be a maximal tree. In order to show that the c.s.s. group $G(L; \phi)$ does not depend on the choice of T, we construct a c.s.s. group $G'(L; \phi)$, the definition of which does not involve T, together with a (multiplicative) isomorphism

$$g: G'(L; \phi) \approx G(L; \phi)$$

which is completely determined by T. If we identify $G(L; \phi)$ with $G'(L; \phi)$ under this isomorphism, then clearly $G(L; \phi)$ becomes independent of the choice of the maximal tree T. The role played by T then becomes merely that of choosing a basis for the free groups $G_n'(L; \phi)$. A different choice of T induces different bases for these groups.

The proof that the fibre sequence

$$(G(L; \phi); e_0) \xrightarrow{q} (E(L; \phi); (e_0, \phi)) \xrightarrow{p} (L; \phi)$$

is independent of T is similar. This is done by constructing a fibre sequence

$$(G'(L; \phi); e_0) \xrightarrow{q'} (E'(L; \phi); \chi) \xrightarrow{p'} (L; \phi),$$

the definition of which does not involve T, together with an isomorphism

$$e: E'(L; \phi) \approx E(L; \phi)$$

which is completely determined by T and is such that commutativity holds in the diagram

$$\begin{array}{ccccc} G(L; \phi) & \xrightarrow{q} & E(L; \phi) & \xrightarrow{p} & L \\ \downarrow g & & \downarrow e & & \downarrow i \\ G'(L; \phi) & \xrightarrow{q'} & E'(L; \phi) & \xrightarrow{p'} & L \end{array} \tag{12.1}$$

where $i: L \to L$ denotes the identity map.

In order to define the c.s.s. group $G'(L; \phi)$ we need a function $< >$ which assigns to every loop $(\sigma_1, \cdots, \sigma_{2k})$ of L a reduced loop $< \sigma_1, \cdots, \sigma_{2k} >$ as follows If $(\sigma_1, \cdots, \sigma_{2k})$ is a reduced loop, then

$$< \sigma_1, \cdots, \sigma_{2k} > = (\sigma_1, \cdots, \sigma_{2k}) .$$

It $(\sigma_1, \cdots, \sigma_{2k})$ is not reduced, i.e. $\sigma_s = \sigma_{s+1}$ for one or more integers s with $1 \leqq s < 2k$, then choose one of them, s, and define by induction

$$< \sigma_1, \cdots, \sigma_{2k} > = < \sigma_1, \cdots, \sigma_{s-1}, \sigma_{s+2}, \cdots, \sigma_{2k} > .$$

It is readily vertified that this definition is independent of the choice of s.

The c.s.s. group $G'(L; \phi)$ is now defined as follows. An n-simplex of $G'(L; \phi)$ is any reduced n-loop of L. The product of two n-simplices

$(\sigma_1, \cdots, \sigma_{2k})$ and $(\tau_1, \cdots, \tau_{2m})$ is given by

$$(\sigma_1, \cdots, \sigma_{2k}) \cdot (\tau_1, \cdots, \tau_{2m}) = < \sigma_1, \cdots, \sigma_{2k}, \tau_1, \cdots, \tau_{2m} >$$

and it is easily seen that under this multiplication $G_n'(L; \psi)$, the set of the n-simplices of $G'(L; \psi)$, becomes a group. The face and degeneracy homomorphisms $\varepsilon^i: G_n'(L; \psi) \to G_{n-1}'(L; \psi)$ and $\eta^i: G_n'(L; \psi) \to G_{n+1}'(L; \psi)$ are given by the formulas:

$$(\sigma_1, \cdots, \sigma_{2k})\varepsilon^i = < \sigma_1 \varepsilon^i, \cdots, \sigma_{2k}\, \varepsilon^i > \qquad 0 \leqq i \leqq n$$
$$(\sigma_1, \cdots, \sigma_{2k})\eta^i = < \sigma_1 \eta^i, \cdots, \sigma_{2k}\, \eta^i > \qquad 0 \leqq i \leqq n\ .$$

Finally the map $g: G'(L; \psi) \to G(L; \psi)$ is defined by

$$g(\sigma_1, \cdots, \sigma_{2k}) = \sigma_1 \cdot \sigma_2^{-1} \cdot \cdots \cdot \sigma_{2k-1} \cdot \sigma_{2k}^{-1}\ ,$$

and it remains to show that

LEMMA 12.2. *The map $g: G'(L; \psi) \to G(L; \psi)$ is a (multiplicative) isomorphism.*

The proof of Lemma 12.2 will be given in §14.

For the definition of the c.s.s. complex $E'(L; \psi)$ we need the notion of an n-path and a reduced n-path. An *n-path of L of length k ending at the n-simplex* $\sigma \in L$ is a sequence $(\sigma_1, \cdots, \sigma_{2k+1})$ of $2k + 1$ $(n + 1)$-simplices of L such that

$$\sigma_{2j-1}\, \varepsilon^{n+1} = \sigma_{2j}\varepsilon^{n+1} \qquad 1 \leqq j \leqq k$$
$$\sigma_{2k+1}\, \varepsilon^{n+1} = \sigma$$
$$\sigma_1\, \varepsilon^n \cdots \varepsilon^0 = \psi$$
$$\sigma_{2j}\, \varepsilon^n \cdots \varepsilon^0 = \sigma_{2j+1}\, \varepsilon^n \cdots \varepsilon^0 \qquad 1 \leqq j \leqq k$$

i.e., two adjacent simplices have alternatingly the same last face or last vertex. A *reduced* n-path of L is an n-path $(\sigma_1, \cdots, \sigma_{2k+1})$ such that

$$\sigma_s \neq \sigma_{s+1} \qquad 1 \leqq s \leqq 2k$$

i.e., two adjacent simplices are always distinct.

The following function $< \ >$ assigns to every n-path of L a reduced n-path. If $(\sigma_1, \cdots, \sigma_{2k+1})$ is a reduced n-path of L, then $< \sigma_1, \cdots, \sigma_{2k+1} > = (\sigma_1, \cdots, \sigma_{2k+1})$. If an n-path $(\sigma_1, \cdots, \sigma_{2k+1})$ is not reduced, choose an integer s such that $1 \leqq s \leqq 2k$ and $\sigma_s = \sigma_{s+1}$ and define by induction $< \sigma_1, \cdots, \sigma_{2k+1} > = < \sigma_1, \cdots, \sigma_{s-1}, \sigma_{s+2}, \cdots, \sigma_{2k+1} >$. Clearly this definition is independent of the choice of s.

The c.s.s. complex $E'(L; \psi)$ is now defined as follows. An n-simplex of $E'(L; \psi)$ is any reduced n-path of L. The face and degeneracy operators are given by

$$(\sigma_1, \cdots, \sigma_{2k+1})\varepsilon^i = < \sigma_1\, \varepsilon^i, \cdots, \sigma_{2k+1}\, \varepsilon^i > \qquad 0 \leqq i \leqq n$$
$$(\sigma_1, \cdots, \sigma_{2k+1})\eta^i = < \sigma_1\, \eta^i, \cdots, \sigma_{2k+1}\, \eta^i > \qquad 0 \leqq i \leqq n\ .$$

Let $\chi = (\psi\eta^0)$ and define the maps $p'\colon E'(L;\psi) \to L$ and $q'\colon G'(L;\psi) \to E'(L;\psi)$ by the formulas

$$p'(\sigma_1, \cdots, \sigma_{2k+1}) = \sigma_{2k+1}\,\varepsilon^{n+1}$$
$$q'(\sigma_1, \cdots, \sigma_{2k}) = < \sigma_1, \cdots, \sigma_{2k}, \psi\eta^0 \cdots \eta^n > .$$

Finally the map $e\colon E'(L;\psi) \to E(L;\psi)$ is given by

$$e(\sigma_1, \cdots, \sigma_{2k}) = (\gamma, \sigma)$$

where

$$\gamma = \bar{\sigma}_1 \cdot \bar{\sigma}_2^{-1} \cdot \cdots \cdot \bar{\sigma}_{2k-1} \cdot \bar{\sigma}_{2k}^{-1} \cdot \bar{\sigma}_{2k+1} \qquad \in G_n(L;\psi)$$
$$\sigma = \sigma_{2k+1}\,\varepsilon^{n+1} \qquad \in L_n ,$$

and it is readily verified that commutativity holds in Diagram 12.1. It thus remains to show

LEMMA 12.3. *The c.s.s. map $e\colon E'(L;\psi)$ is an isomorphism.*

The proof is similar to that of Lemma 12.2 (see §14).

13. Proof of Lemma 7.2

Clearly the standard 0-simplex $\Delta[0]$ ([7], §2), is contractible. Also $H_n(\Delta[0]) = 0$ for $n > 0$. Let $f\colon EK \to \Delta(0)$ be the only such c.s.s. map. Then it follows from the c.s.s. version of a theorem of J.H.C. Whitehead [13] that EK is contractible if and only if EK is simply connected and $H_n(EK) = 0$ for $n > 0$. Hence it remains to prove the following two lemmas

LEMMA 13.1. *EK is simply connected.*

LEMMA 13.2. *$H_n(EK) = 0$ for $n > 0$.*

PROOF OF LEMMA 13.1. The proof given here is analogous to one given by J.W. Milnor in case K is a suspension [10]. Throughout this proof we shall write E instead of EK

It must be shown that $\pi_1(E) = 0$, i.e. that $\pi_1(|E|) = 0$ where $|E|$ is the geometrical realization of E (by a CW-complex of which the n cells are in one to one correspondence with the non degenerate n-simplices of E). Choose a maximal tree in $|E|$. Then $\pi_1(|E|)$ can be considered as a group with one generator corresponding to each 1-cell of $|E|$, not in the tree, and one relation corresponding to each 2-cell.

As maximal tree in $|E|$ take the union of all 1-cells which correspond to 1-simplices of the form $(\gamma\eta^0, \sigma)$ where $\gamma \in G_0K$ and $\sigma \in K_1$. Then as generators of $\pi_1(|E|)$ we have all elements $(\gamma, \sigma) \in E_1$ such that γ is non-degenerate. A 2-simplex of the form $(\gamma\eta^0, \sigma\eta^1)$ yields a relation

$$(\gamma\eta^0, \sigma\eta^1)\varepsilon^1 = (\gamma\eta^0, \sigma\eta^1)\varepsilon^0 \cdot (\gamma\eta^0, \sigma\eta^1)\varepsilon^2 ,$$

or, because $(\gamma\eta^0, \sigma\eta^1)\varepsilon^2 = (\gamma\varepsilon^1\eta^0 \cdot \overline{\sigma\eta^1}, \sigma)$ is in the tree,

(13.3a.) $$(\gamma, \sigma) = (\gamma, \phi\eta^0) .$$

Similarly a 2-simplex of the form $(\gamma\eta^0, \tau)$ yield a relation

(13.3b.) $$(\gamma, \tau\varepsilon^1) = (\gamma \cdot \bar{\tau}, \tau\varepsilon^2) .$$

It now follows easily from the relations (13.3a) and (13.b) that $\pi_1 (|E|) = 0$.

PROOF OF LEMMA 13.2. Let $C = \{C_n, \partial_n\}$ be the chain complex of EK, i.e. C_n is the free abelian group generated by the n-simplices of EK and the homomorphism $\partial_n: C_n \to C_{n-1}$ is defined by $\partial_n\tau = \sum_{i=0}^{n}(-1)^i \tau\varepsilon^i$. It then must be shown that there exist homomorphisms

$$D_n : C_n \to C_{n+1}$$

such that

$$\partial_{n+1} D_n + D_{n-1}\partial_n = \text{identity} \qquad n > 0 .$$

In the definitions of those homomorphisms D_n use will be made of the results of §§9 and 12.

Let $(\gamma, \sigma) \in E_n K$ and let λ be the reduced n-path of K such that $e\lambda = (\gamma, \sigma)$. Choose an n-path $\mu = (\sigma_1, \cdots, \sigma_{2k+1})$ such that $< \mu > = \lambda$ and define for every pair of integers (i,s) such that $0 \leqq i \leqq n$ and $1 \leqq s \leqq 2k + 1$ an $(n + 1)$-path $\mu(i, s)$ of K by

$$\mu(i, s) = (\sigma_1 \eta^i, \cdots, \sigma_{s-1} \eta^i, \sigma_s \varepsilon^n \cdots \varepsilon^{i+1} \eta^{i+1} \cdots \eta^{n+1}) \qquad s \text{ odd}$$
$$\mu(i, s) = (\sigma_1 \eta^i, \cdots, \sigma_s \eta^i, \sigma_s \varepsilon^n \cdots \varepsilon^{i+1} \eta^{i+1} \cdots \eta^{n+1} \qquad s \text{ even} .$$

Let $\bar{\mu}(i, s) = e < \mu(i, s) >$ and define

$$D_n(\gamma, \sigma) = \sum_{s=1}^{2k+1} \sum_{i=0}^{n} (-1)^{i+s} \bar{\mu}(i,s) .$$

Then it is readily verified that this definition is independent of the choice of μ, and it thus remains to show that the homomorphisms D_n so defined have the desired property. Let

$$\mu^m = (\sigma_1 \varepsilon^m, \cdots, \sigma_{2k+1} \varepsilon^m) \qquad 0 \leqq m \leqq n .$$

Then clearly

$$e < \mu^m > = (\gamma, \sigma)\varepsilon^m .$$

Hence

$$D_{n-1} ((\gamma, \sigma)\varepsilon^m) = \sum_{s=1}^{2k+1} \sum_{i=0}^{n-1} (-1)^{i+s} \bar{\mu}^m(i, s)$$

A straightforward computation yields

$$\bar{\mu}(i, s)\varepsilon^m = \bar{\mu}^m(i + 1, s) \qquad m < i$$
$$\bar{\mu}(i, s)\varepsilon^m = \bar{\mu}^{m-1}(i, s) \qquad m > i + 1$$
$$\bar{\mu}(i, s)\varepsilon^i = \bar{\mu}(i - 1, s)\varepsilon^i$$
$$\bar{\mu}(0, s)\varepsilon^0 = \bar{\mu}(0, s - 1)\varepsilon^0 \qquad s \text{ odd}$$
$$\bar{\mu}(n, s)\varepsilon^{n+1} = \bar{\mu}(n, s - 1)\varepsilon^{n+1} \qquad s \text{ even} .$$

Consequently

$$\begin{aligned}\partial_{n+1} D_n(\gamma, \sigma) &+ D_{n-1}\partial_n(\gamma, \sigma) \\ &= \sum_{m=0}^{n+1}\sum_{s=1}^{2k+1}\sum_{i=0}^{n}(-1)^{i+s+m}\bar{\mu}(i, s)_i^m \\ &\quad + \sum_{m=0}^{n}\sum_{s=1}^{2k+1}\sum_{i=0}^{n-1}(-1)^{i+s+m}\bar{\mu}^m(i, s) = (\gamma, \sigma)\ .\end{aligned}$$

14. Proof of Lemma 12.2

It is readily verified that the map $g: G'(L; \psi) \to G(L; \psi)$ is a c.s.s. homomorphism. It thus remains to show that g is onto and has trivial kernel.

In order to prove that g is onto, it suffices to show that for every generator $\bar{\sigma} \in G_n(L; \psi)$ there exists a reduced n-loop $(\sigma_1, \cdots, \sigma_{2k})$ such that $g(\sigma_1, \cdots, \sigma_{2k}) = \bar{\sigma}$. Let $\alpha = \sigma\varepsilon^n \cdots \varepsilon^0$ be the $(n+1)$-vertex of σ and $\beta = \sigma\varepsilon^{n+1}\varepsilon^{n-1}\cdots\varepsilon^0$ the n-vertex. Choose in the maximal tree $T \subset L$ two sequences $(\alpha_1, \cdots, \alpha_{2p})$, $(\beta_1, \cdots, \beta_{2q})$ of 1-simplices such that

$$\begin{array}{lll} \alpha_1\varepsilon^0 = \psi & \beta_1\varepsilon^0 = \psi & \\ \alpha_j\varepsilon^0 = \alpha_{j+1}\varepsilon^0 & \beta_j\varepsilon^0 = \beta_{j+1}\varepsilon^0 & j \text{ even} \\ \alpha_{2p}\varepsilon^0 = \alpha & \beta_{2q}\varepsilon^0 = \beta & \\ \alpha_j\varepsilon^1 = \alpha_{j+1}\varepsilon^1 & \beta_j\varepsilon^1 = \beta_{j+1}\varepsilon^1 & j \text{ odd}\ , \end{array}$$

and let

$$\alpha_j' = \alpha_j\eta^0 \cdots \eta^{n-1} \quad \beta_j' = \beta_j\eta^0 \cdots \eta^{n-1}\ .$$

That it is easily verified that

$$\mu = < \alpha_1', \cdots, \alpha_{2p}', \sigma, \sigma\varepsilon^{n+1}\eta^n, \beta_{2q}', \cdots, \beta_1' >$$

is a reduced n-loop of L such that $g\mu = \bar{\sigma}$. Thus g is onto.

We now show that the kernel of g is trivial. Let $\mu = (\sigma_1, \cdots, \sigma_{2k})$ be a reduced n-loop of L such that $g\mu = e_n$

Suppose first that $\bar{\sigma}_i = e_n$ for all $1 \leqq i \leqq 2k$. If $\sigma_j \in T$ for some j, then $\sigma_j = \tau\eta^n$ for some $\tau \in L_n$. Let $i = j-1$ if j is even, and $i = j+1$ if j is odd. Then $\sigma_i\varepsilon^{n+1} = \sigma_j\varepsilon^{n+1} = \tau \in T$. Hence $\sigma^i \in T$ and consequently $\sigma^i = \tau\eta^n = \sigma_j$. But this is impossible as μ is reduced. Hence $\sigma^i \in T$ for all $1 \leqq i \leqq 2k$ and because T is a tree and μ is reduced it follows that $k=0$ and hence μ is the identity.

Now suppose that $\bar{\sigma}_j \neq e_n$ for some j. Then because $g\mu = e_n$ there is an integer i such that $\sigma_i = \sigma_j$ and because μ is reduced $|i-j| > 1$. Choose a pair (i, j) such that $\sigma_i = \sigma_j$, $\bar{\sigma}_j \neq e_n$, $i - j > 1$ and $\bar{\sigma}_t = e_n$ for all $j < t < i$. It then follows by the same kind of argument that there exists an integer s with $j < s < i-1$ such that $\sigma_s = \sigma_{s+1}$. This is in contradiction with the fact that μ is reduced. Hence it is impossible that $\bar{\sigma}_j \neq e_n$ for some j. This completes the proof that the kernel of g is trivial.

CHAPTER III: THE HUREWICZ HOMOMORPHISMS

15. The homology groups

Let K be a reduced complex (this restriction is not essential). It is well known that the *first homology group* of K is isomorphic with the *fundamental group* of K *made abelian*. This statement will be generalized; it will be shown that the higher homology groups may also be regarded as a kind of " abelianized " homotopy groups.

First we abelianize the c.s.s. group GK, i.e., we form the abelian c.s.s. group

$$AK = GK/[GK, GK]$$

By this we mean that for every integer $n \geqq 0$ the group A_nK is " G_nK made abelian ", i.e., A_nK ia an abelian group with one generator $A(\sigma)$ for every $(n+1)$-simplex $\sigma \in K_{n+1}$ and one relation $A(\tau\eta^n) = 0$ for every n-simplex $\tau \in K_n$. Clearly A_nK is free abelian. The face and degeneracy homomorphisms $\varepsilon^i: A_nK \to A_{n-1}K$ and $\eta^i: A_nK \to A_{n+1}K$ are those induced by the corresponding homomorphisms of GK, i.e. they are given by

$$\begin{aligned} A(\sigma)\varepsilon^i &= A(\sigma\varepsilon^i) & 0 \leqq i < n \\ A(\sigma)\varepsilon^n &= A(\sigma\varepsilon^n) - A(\sigma\varepsilon^{n+1}) & \\ A(\sigma)\eta^i &= A(\sigma\eta^i) & 0 \leqq i \leqq n \end{aligned}$$

The homotopy groups of the abelian c.s.s. group AK are closely related to the homology groups of K. In fact

THEOREM 15.1. *There exists* (*in a natural manner*) *an isomorphism*

$$\alpha_n: H_n(K) \approx \pi_{n-1}(AK; e_0) \qquad n > 0 .$$

We are now able to formulate in what sense the homology groups are " *abelianized* " homotopy groups. In §8 it was shown that $\pi_n(K)$ is isomorphic with $\pi_{n-1}(GK; e_0) = H_{n-1}(\tilde{G}K)$. In view of this fact a new definition of the homotopy groups of K was given, using GK. However if we " *abelianize* " this definition, i.e., if we replace GK by AK then it follows from Theorem 15.1 and Proposition 6.4 that we get the following definition of the homology groups of K.

DEFINITION 15.2 Let K be a reduced complex. Form the abelian c.s.s. group $AK = GK/[GK, GK]$. Let $\tilde{A}K$ be the Moore chain complex of AK (Definition 5.3). Then for every integer $n > 0$ we define $H_n(K)$, the n^{th} *homology group of* K, by

$$H_n(K) = H_{n-1}(\tilde{A}K)$$

PROOF OF THEOREM 15.1. Let $\bar{A} = \{A_nK, \partial_n\}$ be the chain complex where the homomorphism $\partial_n: A_nK \to A_{n-1}K$ is given by

$$\partial_n \rho = \sum_{i=0}^{n} (-1)^i \rho \varepsilon^i \qquad \rho \in A_nK$$

Let C^NK be the chain complex of K normalized in the last direction (i.e., C_n^NK is the free abelian group with a generator $C^N(\sigma)$ for every $\sigma \in K_n$ and a relation $C^N(\tau\eta^n) = 0$ for every $\tau \in K_{n-1}$). It is readily verified that the function $\alpha: C^NK \to \bar{A}$ defined by

$$\alpha C^N(\sigma) = A(\sigma) \qquad \sigma \in K$$

lowers dimensions by one, is an isomorphism in every dimension and commutes with the boundary homomorphisms. Hence it induces isomorphisms

$$\alpha_*: H_n(K) \approx H_{n-1}(\bar{A}) .$$

Let CK be the chain complex of K normalized in all directions (i.e. C_nK is the free abelian group with a generator $C(\sigma)$ for every $\sigma \in K_n$ and relation $C(\tau\eta^i) = 0$ for every $\tau \in K_{n-1}$), and let $p: C^NK \to CK$ be the projection. Denote by $j: \tilde{A}K \to \bar{A}$ the inclusion map. Then it may be verified by a straightforward although rather long computation that the composite map

$$\tilde{A}K \xrightarrow{j} \bar{A} \xrightarrow{\alpha^{-1}} C^NK \xrightarrow{p} CK$$

raises dimensions by one, is an isomorphism in every dimension and commutes with the boundary homomorphisms. Hence it induces an isomorphism $H_{n-1}(\tilde{A}K) \approx H_n(K)$. Application of Proposition 5.4 now yields an isomorphism

$$\alpha_n: H_n(K) \approx \pi_{n-1}(\tilde{A}K; e_0) .$$

16. The Hurewicz homomorphisms

Let K be a reduced complex (again this restriction is not essential). Let

$$k: GK \to AK$$

be the *projection*, i.e., k maps an n-simplex of GK on the coset of $[G_nK, G_nK]$ containing it. Clearly k is a c.s.s. homomorphism. It induces chain map $\tilde{k}: \tilde{G}K \to \tilde{A}K$ of the Moore chain complexes and hence induces homomorphisms

$$\tilde{k}_*: H_{n-1}(\tilde{G}K) \to H_{n-1}(\tilde{A}K) .$$

In view of the isomorphisms (§§8 and 15)

$$\partial_n(q, p): \pi_n(K) \approx H_{n-1}(\tilde{G}K)$$
$$\alpha_n: H_n(K) \approx H_{n-1}(\tilde{A}K)$$

the homomorphisms $\tilde{k}_*$ induce homomorphisms of the homotopy groups of K into the homology groups. It will be shown that these homomorphisms are exactly the *Hurewicz homomorphisms* [2]

$$h_*: \pi_n(K) \to H_n(K) .$$

In fact we have

THEOREM 16.1. *Commutativity holds in the diagram*

$$\begin{array}{ccc} \pi_n(K) & \xrightarrow{\quad h_* \quad} & H_n(K) \\ \partial_n(q, p) \Big\downarrow \approx & & \alpha_n \Big\downarrow \approx \\ H_{n-1}(\tilde{G}K) & \xrightarrow{\quad \tilde{k}_* \quad} & H_{n-1}(\tilde{A}K) \end{array} \qquad n > 0$$

It follows from Theorem 16.1 that if we use for the homotopy and homology groups the Definitions 8.1 and 15.2, then we may give the following corresponding definition for the Hurewicz homomorphisms.

DEFINITION 16.2 Let K be a reduced complex. Form GK and $AK = GK/[GK, GK]$. Let $k: GK \to AK$ be the projection and let $\tilde{k}: \tilde{G}K \to \tilde{A}K$ be the induced chain map on the Moore chain complexes. Then for every integer $n > 0$ we define the *Hurewicz homomorphism* $h_*: \pi_n(K) \to H_n(K)$ by

$$h_* = \tilde{k}_*.$$

PROOF OF THEOREM 16.1. It follows readily from the naturality of $\partial_n(q, p)$, α_n, h_* and $\tilde{k}_*$ that it suffices to prove the theorem for the case that K is an n-sphere.

Let S^n be a reduced complex with an n-simplex ζ as its only non-degenerate simplex in dimension > 0. Embed S^n in $S|S^n|$, the total singular complex of its geometrical realization, under the map $i: S^n \to S|S^n|$ [9]. Clearly $\{\zeta\}$ generates $\pi_n(S^n)$, $\{C^N(\zeta)\}$ generates $H_n(S^n)$ and $h_*\{\zeta\} = \{C^N(\zeta)\}$. Let $\partial_n(q, p): \pi_n(S^n) \to \pi_{n-1}(GS^n; e_0)$ be the boundary map, then $\partial_n(q, p)\{\zeta\} = \{\bar{\zeta}\}$. Furthermore $k(\zeta) = A(\zeta)$ and $\alpha C^N(\zeta) = A(\zeta)$. Hence

$$\alpha_n h_*\{\zeta\} = \tilde{k}_* \partial_n(q, p)\{\zeta\} .$$

17. The Hurewicz Theorem

It will be shown that the Hurewicz Theorem may be regarded as a special case of a purely group theoretical theorem.

We first formulate both halves of the Hurewicz Theorem [2] in Theorems 17.1 and 17.2 below. Again we restrict ourselves to reduced complexes.

THEOREM 17.1. *Let K be a reduced complex. Then the Hurewicz homomorphism* $h_* : \pi_1(K) \to H_1(K)$ *is onto and has* $[\pi_1(K), \pi_1(K)]$ *as kernel.*

THEOREM 17.2. *Let K be a reduced complex and let* $\pi_i(K) = 0$ *for* $0 < i \leqq n$. *Then the Hurewicz homomorphism* $h_*: \pi_{n+1}(K) \to H_{n+1}(K)$ *is an isomorphism and* $h_*: \pi_{n+2}(K) \to H_{n+2}(K)$ *is onto.*

As the definitions 8.1, 15.2 and 16.2 for homotopy groups, homology groups and Hurewicz homomorphisms are equivalent with the usual ones, it follows that Theorem 17.1 and 17.2 are equivalent with the following theorems.

THEOREM 17.3. *Let K be a reduced complex. Let* $k: GK \to AK$ *be the projection and let* $\tilde{k}: \tilde{G}K \to \tilde{A}K$ *be the induced chain map. Then* $\tilde{k}_*: H_0(\tilde{G}K) \to H_0(\tilde{A}K)$ *is onto and has* $[H_0(\tilde{G}K), H_0(\tilde{G}K)]$ *as kernel.*

THEOREM 17.4. *Let K be a reduced complex and let* $\pi_i(K) = 0$ *for* $0 < i \leqq n$. *Let* $k: GK \to AK$ *be the projection and let* $\tilde{k}: \tilde{G}K \to \tilde{A}K$ *be the induced chain map. Then* $\tilde{k}_*: H_n(\tilde{G}K) \to H_n(\tilde{A}K)$ *is an isomorphism and* $\tilde{k}_*: H_{n+1}(\tilde{G}K) \to H_{n+1}(\tilde{A}K)$ *is onto.*

If one tries to prove Theorems 17.3 and 17.4 group theoretically, then it appears that the complex K plays no role at all; the only facts used in the proof are

(a) G_nK is free for all n,
(b) AK is " GK made abelian "
(c) $k: GK \to AK$ is the projection.

Hence Theorems 17.3 and 17.4 and, therefore, the Hurewicz Theorem may be considered as a special case of the following group theoretical theorems.

THEOREM 17.5. *Let F be a c.s.s. group such that F_n is free for all n. Let* $B = F/[F, F]$ *and let* $\tilde{l}: \tilde{F} \to \tilde{B}$ *be the chain map induced by the projection* $l: F \to B$. *Then* $\tilde{l}_*: H_0(\tilde{F}) \to H_0(\tilde{B})$ is *onto and has* $[H_0(\tilde{F}), H_0(\tilde{F})]$ *as kernel.*

THEOREM 17.6. *Let F be a c.s.s. group such that F_n is free for all n and let* $H_i(\tilde{F}) = 0$ *for* $0 \leqq i < n$. *Let* $B = F/[F, F]$ *and let* $\tilde{l}: \tilde{F} \to \tilde{B}$ *be the chain map induced by the projection* $l: F \to B$. *Then* $\tilde{l}_*: H_n(\tilde{F}) \to H_n(\tilde{B})$ *is an isomorphism and* $\tilde{l}_*: H_{n+1}(\tilde{F}) \to H_{n+1}(\tilde{B})$ *is onto.*

A proof of Theorems 17.5 and 17.6 is given in [6].

CHAPTER IV. THE GROUPS $\tilde{G}_n(L; \phi)$.

18. Construction of a free basis

Let L be a connected c.s.s. complex with base point ϕ. Then $G_n(L; \phi)$ is a free group for every n. By the Nielsen-Schreier theorem [8] a subgroup of a free group is free. Hence the groups $\tilde{G}_n(L; \phi)$ are free for all n.

It should be noted that, even in case L has only a finite number of simplices in every dimension, and hence the groups $G_n(L; \phi)$ are finitely generated, in general the groups $\tilde{G}_n(L; \phi)$ have no finite basis.

It will be shown how a (in general infinite) free basis may be obtained for the groups $\tilde{G}_n(L; \phi)$.

We first recall the notion of a Schreier system of coset representatives and the Kurosch-Schreier theorem [8].

Let A be a free group freely generated by elements α_j (j runs through some set J) and let $B \subset A$ be a subgroup. Denote the right cosets of B in A by B_k(k runs through the set of right cosets). In each coset B_k select a representative $|B_k|$ such that $|B| = 1$. Such a system of *coset representatives* is called a *Schreier system* if it satisfies the following condition:

Let $\alpha_{j_1}{}^{\varepsilon_1} \cdots \alpha_{j_n}{}^{\varepsilon_n}$ be a reduced word, i.e. $\varepsilon_i = \pm 1$ and $\varepsilon_i = \varepsilon_{i+1}$ whenever $j_i = j_{i+1}$. If $\alpha_{j_1}{}^{\varepsilon_1} \cdots \alpha_{j_n}{}^{\varepsilon_n}$ is a coset representative, then so is $\alpha_{j_1}{}^{\varepsilon_1} \cdots \alpha_{j_i}{}^{\varepsilon_i}$ for every $i < n$.

The Kurosch-Schreier theorem may be stated as follows.

THEOREM 18.1. *Let A be a free group freely generated by elements α_j ($j \in J$) and let $B \subset A$ be a subgroup. Let $|B_k|$ be a Schreier system of representatives for the right cosets B_k of B in A. Then B is freely generated by the elements $|B_k| \alpha_j |B_k \alpha_j|^{-1}$ for those pairs (k, j) for which $|B_k| \alpha_j \neq |B_k \alpha_j|$.*

We now define the notion of a *free* c.s.s. group and show how, if G is a free c.s.s. group, a free basis may be obtained for the groups $\tilde{G}_n$. This result then will be applied to the c.s.s. group $G(L; \phi)$.

A c.s.s. group G will be called *free* if

(a) G_n is a free group with a *given* basis, for all n,

(b) the bases of the groups G_n are stable under all degeneracy homomorphisms, i.e. for every generator $\sigma \in G_n$ and integer i with $0 \leqq i \leqq n$, the element $\sigma\eta^i$ is a generator of G_{n+1}.

Let G be a free c.s.s. group. Define a c.s.s. group $'G$ as follows. For every integer $n \geqq 0$, $'G_n$ is the subgroup of G_{n+1} given by

$$'G_n = kernel\ \varepsilon^{n+1}\ .$$

The face and degeneracy homomorphisms $\varepsilon^i: 'G_n \to 'G_{n-1}$ and $\eta^i: 'G_n \to 'G_{n+1}$ are the restrictions

$$\varepsilon^i = \varepsilon^i \,|\, 'G_n \qquad 0 \leqq i \leqq n$$
$$\eta^i = \eta^i \,|\, 'G_n \qquad 0 \leqq i \leqq n\ .$$

Then we have

THEOREM 18.2. *The c.s.s. group $'G$ may be converted into a free c.s.s. group by taking as a basis for $'G_n$ the elements $\gamma\eta^n \cdot \sigma \cdot \sigma^{-1}\varepsilon^{n+1}\eta^n \cdot \gamma^{-1}\eta^n$ for which $\gamma \in G_n$ and $\sigma \in G_{n+1}$ is a generator such that $\sigma \neq \sigma\varepsilon^{n+1}\eta^n$.*

Let ${}^{(0)}G = G$ and define ${}^{(n)}G$ by ${}^{(n)}G = {}'{}^{(n-1)}G$. Then clearly

$$\tilde{G}_n = {}^{(n)}G_0$$

Hence n-fold application of Theorem 18.2 yields a free basis for the group $\tilde{G}_n$.

We now apply this result to the c.s.s. group $G(L; \psi)$. Choose a maximal tree $T \subset L$. Then $G_n(L; \psi)$ is a free group freely generated by the elements $\bar{\sigma}$ where σ is an $(n + 1)$-simplex of L, not in T, and which is not of the form $\sigma = \tau\eta^n$. If $\bar{\sigma}$ is a generator of $G_n(L; \psi)$, then clearly $\bar{\sigma}\eta^i = \overline{\sigma\eta^i}$ is a generator of $G_{n+1}(L; \psi)$ for $0 \leqq i \leqq n$. Hence $G(L; \psi)$ is a free c.s.s. group and n-fold application of Theorem 18.2 yields a free basis for the group $\tilde{G}_n(L; \psi)$.

For the proof of Theorem 18.2 we need the following lemma.

LEMMA 18.3. *The elements $\gamma\eta^n$, where $\gamma \in G_n$, form a Schreier system of representatives for the right cosets of $'G_n$ in G_{n+1}.*

PROOF OF THEOREM 18.2. It follows from Lemma 18.3 and Theorem 18.1 that the elements $\gamma\eta^n \cdot \sigma \cdot \sigma^{-1}\varepsilon^{n+1}\eta^n \cdot \gamma^{-1}\eta^n$ for which $\gamma\eta^n \cdot \sigma \neq \gamma\eta^n \cdot \sigma\varepsilon^{n+1}\eta^n$ form a free basis for $'G_n$. As clearly $\gamma\eta^0 \cdot \sigma = \gamma\eta^n \cdot \sigma\varepsilon^{n+1}\eta^n$ if and only if $\sigma = \sigma\varepsilon^{n-1}\eta^n$ it thus remains to show that these bases are stable under all degeneracy homomorphisms.

Let $\gamma \in G_n$ and $\sigma \in G_{n+1}$. Then for every integer i with $0 \leqq i \leqq n$ $(\gamma\eta^n \cdot \sigma \cdot \sigma^{-1}\varepsilon^{n+1}\eta^n \cdot \gamma^{-1}\eta^n)\eta^i = \gamma\eta^i\eta^{n+1} \cdot \sigma\eta^i \cdot (\sigma\eta^i)^{-1}\varepsilon^{n+2}\eta^{n+1} \cdot (\gamma\eta^i)^{-1}\eta^{n+1}$. If σ is a generator and $\sigma \neq \sigma\varepsilon^{n+1}\eta^n$, then $\sigma\eta^i$ is a generator of G_{n+1} (because G is free) and it is readily verified that $\sigma\eta^i \neq \sigma\eta^i\varepsilon^{n+2}\eta^{n+1}$. Hence $\gamma\eta^i\eta^{n+1} \cdot \sigma\eta^i \cdot (\sigma\eta^i)^{-1}\varepsilon^{n+2}\eta^{n+1} \cdot (\gamma\eta^i)^{-1}\eta^{n+1}$ is a generator of $'G_{n+1}$. This completes the proof.

PROOF OF LEMMA 18.3. Because the composite map

$$G_n \xrightarrow{\ \eta^n\ } G_{n+1} \xrightarrow{\ \varepsilon^{n+1}\ } G_n$$

is the identity and $'G_n = kernel\ \varepsilon^{n+1}$, it follows that the elements $\gamma\eta^n(\gamma \in G_n)$ form a system of representatives for the right cosets of $'G_n$ in G_{n+1}. That they even form a Schreier system can now easily be derived from the fact that the given basis of G_n is stable under the degeneracy homomorphism $\eta^n\colon G_n \to G_{n+1}$.

19. The fundamental group

Let L be a connected c.s.s. complex with base point ϕ and let $T \subset L$ be a maximal tree. We shall show, using the results of §18, that $\pi_1(L;\phi)$ is a group with a generator for every 1-simplex of L, not in T, and a relation for every non-degenerate 2-simplex of L. In particular this is the case when L is a c.s.s. complex associated with a simplicial complex (see §11).

By definition

$$\pi_1(L;\phi) = kernel\ \tilde{\partial}_0/image\ \tilde{\partial}_1\,.$$

As $kernel\ \tilde{\partial}_0 = G_0(L;\phi)$ it follows that $kernel\ \tilde{\partial}_0$ is the free group generated by the elements $\bar{\tau}$ where $\tau \in (L_1 - T_1)$. By Theorem 18.2 $\tilde{G}_1(L;\phi)$ is the free group generated by the elements $\gamma\eta^0 \cdot \bar{\sigma} \cdot \bar{\sigma}^{-1}\varepsilon^1\eta^0 \cdot \gamma^{-1}\eta^0$ where $\gamma \in G^0(L;\phi)$ and $\sigma \in (L_2 - T_2)$ is such that $\sigma \neq \tau\eta^1$ for some $\tau \in L_1$ and $\bar{\sigma} \neq \bar{\sigma}\varepsilon^1\eta^0$, i.e. where $\gamma \in G_0(L;\phi)$ and σ is a non-degenerate 2-simplex of L. Thus $\pi_1(L;\phi)$ is a group with one generator $\bar{\tau}$ for every 1-simplex $\tau \in (L_1 - T_1)$ and a relation $\tilde{\partial}_1(\gamma\eta^0 \cdot \bar{\sigma} \cdot \bar{\sigma}^{-1}\varepsilon^1\eta^0 \cdot \gamma^{-1}\eta^0) = 1$ for every $\gamma \in G_0(L;\phi)$ and non-degenerate $\sigma \in L_2$. However for fixed non-degenerate $\sigma \in L_2$ and any $\gamma \in G_0(L;\phi)$

$$\tilde{\partial}_1(\gamma\eta^0 \cdot \bar{\sigma} \cdot \bar{\sigma}^{-1}\varepsilon^1\eta^0 \cdot \gamma^{-1}\eta^0) = \gamma \cdot \overline{\sigma\varepsilon^0} \cdot \overline{\sigma\varepsilon^2} \cdot (\overline{\sigma\varepsilon^1})^{-1} \cdot \gamma^{-1}$$

and hence the relation $\tilde{\partial}_1(\gamma\eta^0 \cdot \bar{\sigma} \cdot \bar{\sigma}^{-1}\varepsilon^1\eta^0 \cdot \gamma^{-1}\eta^0) = 1$ is a consequence of the relation

$$\overline{\sigma\varepsilon^0} \cdot \overline{\sigma\varepsilon^2} = \overline{\sigma\varepsilon^1}$$

Thus $\pi_1(L;\phi)$ is the group with

(a) one generator $\bar{\tau}$ for every 1-simplex $\tau \in (L_1 - T_1)$

(b) one relation $\overline{\sigma\varepsilon^0} \cdot \overline{\sigma\varepsilon^2} = \overline{\sigma\varepsilon^1}$ for every non-degenerate 2-simplex $\sigma \in L_2$.

20. The van Kampen Theorem for G

Let K be a reduced complex and let A and B be subcomplexes such that $A \cup B = K$. An immediate consequence of the result of §19 then, is the *van Kampen theorem* [4] which asserts that $\pi_1(K)$ only depends on $\pi_1(A)$, $\pi_1(B)$, $\pi_1(A \cap B)$ and the homomorphisms $a_*\colon \pi_1(A \cap B) \to \pi_1(A)$ and

$b_*: \pi_1(A \cap B) \to \pi_1(B)$ induced by the inclusion maps $a: A \cap B \to A$ and $b: A \cap B \to B$.

An analogous theorem for GK will be given below.

We recall the definitions of free product and free product with amalgamated subgroup [8] only for the special case needed here.

Let x, y and z be disjoint sets and denote by $F(x)$, etc. the free group freely generated by the elements of x, etc. The *free product* of the groups $F(x)$ and $F(y)$ then is the group $F(x \cup y)$. It is denoted by $F(x) * F(y)$. Clearly the group $F(z)$ is a subgroup of $F(x \cup z)$ and of $F(y \cup z)$. The *free product* of $F(x \cup z)$ and $F(y \cup z)$ with *amalgamation* of the common subgroup $F(z)$ then is the group $F(x \cup y \cup z)$. It is denoted by $(F(x \cup z) * F(y \cup z))_{F(z)}$.

The notions of free product and free product with an amalgamated subgroup carry over to c.s.s. groups by applying them dimensionwise.

The van Kampen Theorem for G now asserts.

THEOREM 20.1. *Let K be a reduced complex and let A and B be subcomplexes of K such that $A \cup B = K$. Then GK is the free product of GA and GB with amalgamated c.s.s. subgroup $G(A \cap B)$, i.e.,*

$$GK = (GA * GB)_{G(A \cap B)}$$

COROLLARY 20.2. *Let A and B be reduced complexes with only the vertex (and its degeneracies) in common and let $K = A \cup B$. Then GK is the free product of GA and GB, i.e.,*

$$GK = GA * GB .$$

PROOF OF THEOREM 20.1. For every integer $n \geqq 0$ the group G_nK is freely generated by the elements $\bar{\sigma}$ where σ is an $(n+1)$-simplex of K which is not of the form $\sigma = \tau\eta^n$, i.e. $\sigma \in (K_{n+1} - K_n\eta^n)$. The theorem now follows immediately from the fact that

$$K_{n+1} - K_n\eta^n = (A_{n+1} - A_n\eta^n) \cup (B_{n+1} - B_n\eta^n)$$
$$(A \cap B)_{n+1} - (A \cap B)_n\eta^n = (A_{n+1} - A_n\eta^n) \cap (B_{n+1} - B_n\eta^n) .$$

21. Computation of $\pi_3(S^2)$

Let S^2 be a reduced complex with a 2-simplex ζ as its only non-degenerate simplex in dimension > 0. We shall compute $\pi_3(S^2)$ with the methods of § 18.

It follows from the definition of S^2 that

(a) $G_1(S^2)$ is infinite cyclic with $\bar{\zeta}$ as generator,

(b) $G_2(S^2)$ is freely generated by $\bar{\zeta}\eta^0$ and $\bar{\zeta}\eta^1$,

(c) $G_3(S^2)$ is freely generated by $\bar{\zeta}\eta^0\eta^1$, $\bar{\zeta}\eta^0\eta^2$ and $\bar{\zeta}\eta^1\eta^2$.

Write ζ_0, ζ_1 and ζ_{01} instead of $\bar{\zeta}\eta^0$, $\bar{\zeta}\eta^1$ and $\bar{\zeta}\eta^0\eta^1$. Then it follows from (a), (b) and (c) and Theorem 18.2 that

(d) ${}^{(1)}G_1(S^2)$ is freely generated by the elements $\beta_p = \zeta_1^p \cdot \zeta_0 \cdot \zeta_1^{-p}$,

(e) ${}^{(2)}G_0(S^2) = \tilde{G}_2(S^2)$ is freely generated by the elements

$$\gamma_{q,p} = \zeta_0^q \cdot \zeta_1^p \cdot \zeta_0 \cdot \zeta_1^{-p} \cdot \zeta_0^{-q-1}$$

which are $\neq 1$(i. e. $p \neq 0$),

(f) ${}^{(1)}G_2(S^2)$ is freely generated by the elements $\alpha\eta^2 \cdot \zeta_{01} \cdot \alpha^{-1}\eta^{-2}$ where $\alpha \in G_2(S^2)$

(g) ${}^{(2)}G_1(S^2)$ is freely generated by the elements

$$\beta\eta^1 \cdot \alpha\eta^2 \cdot \zeta_{01} \cdot \alpha^{-1}\eta^2 \cdot \alpha\eta^1 \cdot \zeta_{01}^{-1} \cdot \alpha^{-1}\eta^1 \cdot \beta^{-1}\eta^1$$

which are $\neq 1$, where $\alpha \in G_2(S^2)$ and $\beta \in {}^{(1)}G_1(S^2)$

(h) ${}^{(3)}G_0(S^2) = \tilde{G}_3(S^2)$ is generated by the elements $g(\alpha, \beta, \gamma) = \gamma\eta^0 \cdot \beta\eta^1 \cdot \alpha\eta^2 \cdot \zeta_{01} \cdot \alpha^{-1}\eta^2 \cdot \alpha\eta^1 \cdot \zeta_{01}^{-1} \cdot \alpha^{-1}\eta^1 \cdot \beta^{-1}\eta^1 \cdot \beta\eta^0 \cdot \alpha\eta^0 \cdot \zeta_{01} \cdot \alpha^{-1}\eta^0 \cdot \alpha\varepsilon^1\eta^0\eta^2 \cdot \zeta_{01}^{-1} \cdot \alpha^{-1}\varepsilon^1\eta^0\eta^2 \cdot \beta^{-1}\eta^0 \cdot \gamma^{-1}\eta^0$, where $\alpha \in G_2(S^2)$, $\beta \in {}^{(1)}G_1(S^2)$ and $\gamma \in {}^{(2)}G_0(S^2) = \tilde{G}_2(S^2)$.

As $\gamma_{q,p}\varepsilon^0 = 1$ for every generator $\gamma_{q,p} \in \tilde{G}_2(S^2)$ it follows that *kernel* $\tilde{\partial}_2 = \tilde{G}_2(S^2)$. By definition

$$\pi_3(S^2) = \textit{kernel}\ \tilde{\partial}_2/\textit{image}\ \tilde{\partial}_3$$

Hence $\pi_3(S^2)$ is the group obtained from $\tilde{G}_2(S^2)$ by addition of the relations $\tilde{\partial}_3 g(\alpha, \beta, \gamma) = 1$ for every element $\alpha \in G_2(S^2)$, $\beta \in {}^{(1)}G_1(S^2)$ and $\gamma \in \tilde{G}_2(S^2)$.

We now compute

$$\tilde{\partial}_3 g(\alpha, \beta, \gamma) = \gamma \cdot \beta\varepsilon^0\eta^0 \cdot \alpha\varepsilon^0\eta^1 \cdot \zeta_0 \cdot \alpha^{-1}\varepsilon^0\eta^1 \cdot \alpha\varepsilon^0\eta^0 \cdot \zeta_0^{-1} \cdot \alpha^{-1}\varepsilon^0\eta^0 \cdot \beta^{-1}\varepsilon^0\eta^0 \cdot$$
$$\cdot \beta \cdot \alpha \cdot \zeta_0 \cdot \alpha^{-1} \cdot \alpha\varepsilon^1\eta^1 \cdot \zeta_0^{-1} \cdot \alpha^{-1}\varepsilon^1\eta^1 \cdot \beta^{-1} \cdot \gamma^{-1}$$

in terms of the generators $\gamma_{q,p}$. It is readily verified that

(j) there exists a unique element $\gamma_\beta \in \tilde{G}_2(S^2)$ and a unique integer q such that $\beta = \gamma_\beta \cdot \zeta_0^q$

(k) $\beta\varepsilon^0\eta^0 = \zeta_0^q$

(l) there exists a unique element $\beta_\alpha \in {}^{(1)}G_1(S^2)$ and a unique integer r such that $\alpha = \beta_\alpha \cdot \zeta_1^r$

(m) there exist a unique integer p, an integer $n \geqq 0$ and integers

$$\varepsilon_1, \cdots, \varepsilon_n, q_1, \cdots, q_n, p_1, \cdots, p_n$$

such that $\alpha = \gamma_{q_1,p_1}{}^{\varepsilon_1} \cdots \gamma_{q_n,p_n}{}^{\varepsilon_n} \cdot \zeta_0^p \cdot \zeta_1^r$

(n) $\alpha\varepsilon^0\eta^0 = \zeta_0^p$

(o) $\alpha\varepsilon^0\eta^1 = \zeta_1^p$

(p) $\alpha\varepsilon^1\eta^1 = \zeta_1^{p+r}$

Consequently

$$\beta\varepsilon^0\eta^0 \cdot \alpha\varepsilon^0\eta^1 \cdot \zeta_0 \cdot \alpha^{-1}\varepsilon^0\eta^1 \cdot \alpha\varepsilon^0\eta^0 \cdot \zeta_0^{-1} \cdot \alpha^{-1}\varepsilon^0\eta^0 \cdot \beta^{-1}\varepsilon^0\eta^0 = \gamma_{q,p}$$

Furthermore

$$\zeta_0^{\ q} \cdot \gamma_{q_1,p_1}{}^{\varepsilon_1} \cdot \cdots \cdot \gamma_{q_n,p_n}{}^{\varepsilon_n} = \gamma_{q_1+q,p_1}{}^{\varepsilon_1} \cdot \cdots \cdot \gamma_{q_n+q,p_n}{}^{\varepsilon_n} \cdot \zeta_0^q$$

Hence the relation $\tilde{\partial}_3 g(\alpha, \beta, \gamma) = 1$ may be written

$$\gamma \cdot \gamma_{q,p} \cdot \gamma_\beta \cdot \gamma_{q_1+q,p_1}{}^{\varepsilon_1} \cdot \cdots \cdot \gamma_{q_n+q,\, p_n}{}^{\varepsilon_n} \cdot \gamma_{p+q,r} \cdot$$
$$\cdot\, \gamma_{q+q+1,p_n}{}^{-\varepsilon_n} \cdot \cdots \cdot \gamma_{q_1+q+1,p_1}{}^{-\varepsilon_1} \cdot \gamma_{q,p+r}{}^{-1} \cdot \gamma_\beta^{-1} \cdot \gamma^{-1} = 1$$

Given two integers s and t with $s \neq 0$ it is clearly possible to choose α, β and γ in such a manner that $p_1 = s$, $q_1 = t$, $p = q = r = p_i = q_i = 0$ for $i > 1$, $\varepsilon_1 = 1$ and $\gamma = \gamma_\beta = 1$. This yields the relation

$$\gamma_{t,s} \cdot \gamma_{t+1,s}{}^{-1} = 1 \tag{21.1}$$

Given three integers s, t and u with $s \neq 0$, $u \neq 0$, it is also possible to choose α, β and γ such that $p = s$, $q = t$, $r = u$, $p_i = q_i = 1$ for all i and $\gamma = \gamma_\beta = 1$. This yields the relation

$$\gamma_{t,s} \cdot \gamma_{s+t,u} \cdot \gamma_{t,s+u}{}^{-1} = 1\ ,$$

and combination of this relation with relation 21.1 yields the

$$\gamma_{t,s} \cdot \gamma_{t,u} \cdot \gamma_{t,s+u}{}^{-1} = 1\ . \tag{21.2}$$

It is readily verified that all relations $\tilde{\partial}_3 g(\alpha, \beta, \gamma) = 1$ are consequences of the relations 21.1 and 21.2 and it follows that $\pi_3(S^2)$ is an infinite cyclic group generated by the coset of *image* $\tilde{\partial}_3$ in $\tilde{G}_2(S^2)$ which contains the element

$$\gamma_{0,1} = \zeta_1 \cdot \zeta_0 \cdot \zeta_1^{-1} \cdot \zeta_0^{-1} = [\bar{\zeta}\eta^1, \bar{\zeta}\eta^0]\ .$$

APPENDIX

22. The construction F of J. W. Milnor

Let L be a c. s. s. complex with base point ψ. J. W. Milnor [10] defined a c.s.s. group FL which is a loop complex for the suspension of L. Let SL denote the reduced suspension of L (see below), then it will be shown that FL is, in a natural manner, isomorphic with GSL.

We recall the definition of FL. The group of the n-simplices, F_nL, is a free group with one generator $\sigma^\cdot$ for every n-simplex $\sigma \in L_n$ and one relation $\psi_n^\cdot = e_n$, where $\psi_n = \psi\eta^0 \cdots \eta^{n-1}$. The face and degeneracy homomorphisms are those induced by the corresponding face and degeneracy operators of L, i. e. they are given by

$$\sigma^\cdot \varepsilon^i = (\sigma\varepsilon^i)^\cdot \qquad 0 \leqq i \leqq n$$
$$\sigma^\cdot \eta^i = (\sigma\eta^i)^\cdot \qquad 0 \leqq i \leqq n\ .$$

The *reduced suspension* of L is the c.s.s. complex SL defined as follows. For every integer $n > 0$ the n-simplices of SL are the pairs (k, s), where $k \geqq 0$ is an integer and $\sigma \in L_{n-k}$ is a simplex, identifying (k, ψ_{n-1}) with (n, ψ), and $(0, \sigma)$ with $(\dim \sigma, \psi)$. The face and degeneracy operators are determined by the formulas

$$(k, \sigma)\varepsilon^i = (k, \sigma\varepsilon^i) \qquad (k, \sigma)\eta^i = (k, \sigma\eta^i) \qquad i \leqq \dim \sigma$$
$$(k+1, \sigma)\varepsilon^i = (k, \sigma) \qquad (k, \sigma)\eta^i = (k+1, \sigma) \qquad i > \dim \sigma .$$

Clearly SL is a reduced complex, its only 0-simplex being $(0, \psi)$.

We can now formulate

THEOREM 22.1. *Let L be a c.s.s. complex with base point. Then the c.s.s. homomorphism $\lambda : FL \to GSL$ defined by*

$$\lambda(\sigma^{\cdot}) = \overline{(1, \sigma)} \qquad \sigma \in L$$

is an isomorphism.

The proof is straightforward.

23. The construction G^+

A c.s.s. complex H^+ is called a c.s.s. *monoid* if

(a) H^+_n is a monoid (i. e. associative semi-group with unit) for all n,

(b) all face and degeneracy operators are homomorphisms.

Let K be a reduced complex. A subcomplex $G^+K \subset GK$ will be defined such that

(i) G^+K is a c.s.s. monoid

(ii) if K satisfies the extension condition, then $G^+K = GK$

(iii) if $K = SL$, then G^+K is, in a natural way, isomorphic with F^+L, where $F^+L \subset FL$ is the c.s.s. monoid defined by J. W. Milnor [10]

Let K be a reduced complex. Then we define G^+K as the smallest subcomplex of GK such that

(a) $\bar{\sigma} \in G^+K$ for every simplex $\sigma \in K$

(b) G^+K is a c.s.s. monoid

THEOREM 23.1. *Let K be a reduced complex which satisfies the extension condition. Then $G^+K = GK$.*

PROOF. It suffices to show that $\bar{\sigma}^{-1} \in G^+K$ for every simplex $\sigma \in K$.

Let $\sigma \in K_{n+1}$. Then there exists an $(n+2)$-simplex $\rho \in K$ such that $\rho\varepsilon^{n+2} = \sigma$ and $\rho\varepsilon^{n+1} = \sigma\varepsilon^{n+1}\eta^n$. But $\bar{\rho} \in G^+K$ and hence

$$\bar{\rho}\varepsilon^{n+1} = \overline{\rho\varepsilon^{n+1}} \cdot (\overline{\rho\varepsilon^{n+2}})^{-1} = \bar{\sigma}^{-1} \in G^+K .$$

q. e. d.

Let L be a c.s.s. complex with base point. Then F^+L may be defined [10] as the smallest subcomplex of FL such that

(a) $\sigma^{\cdot} \in F^+L$ for every simplex $\sigma \in L$

(b) F^+L is a c.s.s. monoid.

Hence an immediate consequence of Theorem 22.1 is

THEOREM 23.2. *Let L be a* c.s.s. *complex with base point. Then the restriction* $\lambda \mid F^+L : F^+L \to G^+SL$ *is an isomorphism.*

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