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Quasi-categories and Kan complexes

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Abstract

A *quasi-category* X is a simplicial set satisfying the restricted Kan conditions of Boardman and Vogt. It has an associated homotopy category hoX. We show that X is a Kan complex iff hoX is a groupoid. The result plays an important role in the theory of quasi-categories (in preparation). Here we make an application to the theory of initial objects in quasi-categories. We briefly discuss the notions of limits and colimits in quasi-categories.

1. Quasi-categories

We first recall a few basic concepts of the theory of simplicial sets [5]. The simplicial category Δ has for objects the non-empty ordinals $[n] = \{0, \ldots, n\}$ and for arrows the order preserving maps $[m] \to [n]$. It is standard to denote by d_i the injection $[n-1] \to [n]$ omitting $i \in [n]$ and to denote by s_i the surjection $[n] \to [n-1]$ repeating $i \in [n-1]$. A simplicial set is a contravariant functor $X : \Delta \to Sets$; it is standard to denote $X([n])$ by X_n; an element $x \in X_n$ is an *n-simplex* of X. The fundamental simplex $\Delta[n]$ is the representable functor $\Delta(-, [n])$. The category $\mathscr{S}$ of simplicial sets is the category $[\Delta^o, Sets]$ of functors $\Delta^o \to Sets$ and natural transformations. We shall use the Yoneda lemma to identify a simplex $x \in X_n$ with the corresponding map $\Delta[n] \to X$ in $\mathscr{S}$; in particular, we shall identify a map $f : [m] \to [n]$ in Δ with a map $\Delta[m] \to \Delta[n]$ in $\mathscr{S}$; if $x \in X_n$ the simplex $X(f)(x) \in X_m$ will be denoted as the composite $xf : \Delta[m] \to \Delta[n] \to X$. We shall say that a subfunctor $A \subseteq X$ is a *simplicial subset* of X.

Let X be a simplicial set. If $a, b \in X_0$ and $f \in X_1$ we shall often write $f : a \to b$ to indicate that $a = fd_1$ and $b = fd_0$; we shall denote by $X_1(a, b)$ the set of arrows $a \to b$ in X_1; the degenerate arrow as_0 will be denoted as a *unit* $1_a : a \to a$.

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Let ω be the automorphism of Δ which reverses the order relation on each ordinal. If $f:[n]\to[m]$ then the map $\omega(f)=[n]\to[m]$ is given by $\omega(f)(i)=m-f(n-i)$. The *opposite* X^{o} of a simplicial set X is obtained by composing the contravariant functor $X:\Delta\to Sets$ with $\omega:\Delta\to\Delta$. It is convenient to distinguish between the elements of X and X^{o} by writing $x^{\mathrm{o}}\in X^{\mathrm{o}}$ for each element $x\in X$, with the convention that $x^{\mathrm{oo}}=x$. If $f:a\to b$ is an arrow in X then $f^{\mathrm{o}}:b^{\mathrm{o}}\to a^{\mathrm{o}}$ is an arrow in X^{o}.

If $i\in[n]$ and $n>0$ the *face* $\partial_i\Delta[n]\subset\Delta[n]$ is the image of the map $d_i:[n-1]\to[n]$. The *simplicial* $(n-1)$*-sphere* $\partial\Delta[n]\subset\Delta[n]$ is the union the faces $\partial_i\Delta[n]$, with the convention that $\partial\Delta[0]=\emptyset$. If $n>0$ and $k\in[n]$ the *horn* $\Lambda^k[n]\subset\Delta[n]$ is the union of the faces $\partial_i\Delta[n]$ containing the vertex k. We shall say that the horn $\Lambda^k[n]$ is *inner* if $0<k<n$, otherwise we shall say that it is *outer*.

Let X be a simplicial set. If $n>0$ a *simplicial* $(n-1)$*-sphere in* X is a map $x:\partial\Delta[n]\to X$; it is determined by the sequence $(x_0,\dots,x_n)$ of its faces $x_i=xd_i$; a *filler* for x is a map $\Delta[n]\to X$ extending x. If $n>1$ we shall say that $x:\partial\Delta[n]\to X$ *commutes* if it can be filled. A *horn in* X is a map $x:\Lambda^k[n]\to X$ where $n>0$; it is determined by a lacunary sequence of faces $(x_0,\dots,x_{k-1},*,x_{k+1},\dots,x_n)$; a *filler* for x is a map $\Delta[n]\to X$ extending x. The horn $x:\Lambda^k[n]\to X$ is *inner* if $0<k<n$, otherwise it is *outer*.

Let *Cat* be the category of small categories. We have $\Delta\subset Cat$ since an ordinal is a category and since an order preserving map is a functor. The *nerve NC* of a category $C\in Cat$ is the simplicial set $[n]\mapsto Cat([n],C)$. The nerve functor $N:Cat\to\mathscr{S}$ is full and faithful and it has a left adjoint $cat:\mathscr{S}\to Cat$. We shall say that $cat\,X$ is the *fundamental category* of a simplicial set X. Here is a quick description of $cat\,X$. Let FX be the category freely generated by the 1-skeleton of X viewed as a graph with units. Then $cat\,X$ is the quotient $FX/\equiv$ by the congruence generated by the basic relations $gf\equiv h$, one for each commuting 1-sphere (g,h,f) in X. The fundamental groupoid π_1X is obtained by inverting freely every arrow in $cat\,X$.

Recall that a simplicial set X is a *Kan complex* if every horn $\Lambda^k[n]\to X(n>1,\ 0\leqslant k\leqslant n)$ can be filled.

Definition 1.1. We shall say that a simplicial set X is a *quasi-category* (*q-category*) if every inner horn $\Lambda^k[n]\to X(0<k<n)$ can be filled.

The concept of quasi-category was introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [1]. It is sometime called a *weak Kan complex* in the literature [8]. The purpose of our name is to stress the analogy with categories. Most concepts and results of category theory can be extended to quasi-categories [7]. Quasi-categories are special cases of weak ω-categories: exactly those having only invertible cells in dimensions >1. The nerve of a category is a quasi-category. The opposite of a quasi-category is a quasi-category. A *map* $f:X\to Y$ between quasi-categories is a map of simplicial sets. The category of quasi-categories is a full subcategory of the category of simplicial sets. It contains *Cat* as a full subcategory.

The fundamental category of a quasi-category has a very simple description due to Boardman and Vogt. Let us write $gf\sim h$ to indicate that the simplicial 1-sphere (g,h,f) in X commutes.

Proposition 1.2 (Boardman and Vogt [1]). *Let X be a quasi-category. If $a,b \in X_0$ and $f,g \in X_1(a,b)$ then the four relations $f1_a \sim g$, $g1_a \sim f$, $1_b f \sim g$ and $1_b g \sim f$ are equivalent. The common relation $f \simeq g$, called the homotopy relation, is an equivalence relation on $X_1(a,b)$. Let us put $ho X(a,b) = X_1(a,b)/\simeq$.* If $f \in X_1(a,b)$, $g \in X_1(b,c)$ *and* $h \in X_1(a,c)$ *then the relation* $gf \sim h$ *induces a map*

$$(ho X)(b,c) \times (ho X)(a,b) \to (ho X)(a,c),$$

which is the composition law of a category $ho X$.

The category $ho X$ is *homotopy category* of X. It is easy to see that there is a canonical isomorphism $ho X = cat X$. If $f \in X_1(a,b)$, $g \in X_1(b,c)$ and $h \in X_1(a,c)$ then the relation $gf = h$ in $ho X$ is equivalent to the relation $gf \sim h$.

We shall say that two arrows $f : a \to b$ and $g : b \to a$ in a quasi-category X are mutually *quasi-inverse* if they are mutually inverse in $ho X$; this means that we have $gf \sim 1_a$ and $fg \sim 1_b$. An arrow having a quasi-inverse is a *quasi-isomorphism*.

The horn $\Lambda^0[n] \subset \Delta[n]$ contains the edge $(0,1) \subset [n]$ if $n > 1$. If X is a quasi-category we shall say that an outer horn $x : \Lambda^0[n] \to X$ (with $n > 1$) is *special* if the arrow $x(0,1)$ is quasi-invertible. Dually, an outer horn $x : \Lambda^n[n] \to X$ is *special* if the arrow $x(n-1,n)$ is quasi-invertible.

Theorem 1.3. *Let X be a quasi-category. Then every special outer horn $x : \Lambda^0[n] \to X$ (resp. $x : \Lambda^n[n] \to X$) can be filled.*

This follows from Theorem 2.2 in the next section. Let us finish this section by examining a few consequences.

Corollary 1.4. *A quasi-category X is a Kan complex iff $ho X$ is a groupoid.*

Proof. The necessity is obvious. Conversely, if $ho X$ is a groupoid let us show that every horn $x : \Lambda^k[n] \to X$ with $n > 1$ can be filled. It is true if $0 < k < n$ since X is a quasi-category; it is true if $k = 0$ or $k = n$ since every outer horn $x : \Lambda^k[n] \to X$ is special when $ho X$ is a groupoid. □

If C is a category let $gr(C)$ be the groupoid of isomorphisms of C; it is the largest subgroupoid of C.

Corollary 1.5. *Every quasi-category X contains a largest Kan complex $k(X) \subseteq X$. A simplex $x : \Delta[n] \to X$ belongs to $k(X)$ iff the arrow $x(i,j)$ is quasi-invertible for every $0 \leqslant i < j \leqslant n$. We have a canonical isomorphism $gr(ho X) = ho(k(X))$.*

Proof. For each $n \geqslant 0$ let S_n be the set of simplicies $x : \Delta[n] \to X$ for which the arrow $x(i,j)$ is quasi-invertible for every $0 \leqslant i < j \leqslant n$. By definition $S_0 = X_0$. It is easy to see that the subsets $S_n \subseteq X_n$ define a simplicial subset $S \subseteq X$ and that S is a quasi-category. But every arrow in S is quasi-invertible in S by definition of S. It follows from Corollary 1.4 that S is a Kan complex. It is clearly the largest Kan complex of X. The formula $gr(ho X) = ho S$ is obvious from the definition of $ho X$. □

Let J be the groupoid generated by one isomorphism $i:0\to 1$ between two distinct objects 0 and 1. The nerve $N(J)$ is an infinite-dimensional sphere S^∞; it is the total space $E(Z_2,1)$ of the classifying space $K(Z_2,1)$. The free action of Z_2 on S^∞ is the antipodal map; it is induced by the automorphism $i\mapsto i^{-1}$ of J. From the inclusion of categories $I\subset J$ we obtain an inclusion of simplicial sets $\Delta[1]\subset S^\infty$.

Corollary 1.6. *Let $f\in X_1$ be an arrow in a quasi-category X. If f is quasi-invertible then the map $f:\Delta[1]\to X$ can be extended to a map $S^\infty\to X$.*

Proof. The map $f:\Delta[1]\to X$ factors through the Kan complex $k(X)\subseteq X$ since f is quasi-invertible. But the inclusion $\Delta[1]\subset S^\infty$ is a weak homotopy equivalence since the simplicial sets $\Delta[1]$ and S^∞ are contractible. It follows that the map $f:\Delta[1]\to k(X)$ can be extended to a map $S^\infty\to k(X)$. □

2. Fibrations

Recall that a map $i:A\to B$ in a category $\mathscr{E}$ is said to be *left orthogonal* to a map $f:X\to Y$, or that f is said to be *right orthogonal* to i, if for any pair of arrows (a,b) making a commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

there is a map $d:B\to X$, called a *diagonal filler*, such that $di=a$ and $fd=b$. We shall denote the orthogonality relation by $i\perp f$.

Remark. In the terminology of Quillen [11] the orthogonality relation $i\perp f$ is expressed by saying that *i has the left lifting property with respect to f*, or that *f has the right lifting property with respect to i*. Our terminology is taken from Freyd and Kelly [4], although these authors ask for uniqueness of the diagonal filler.

Recall that a map of simplicial sets is a *trivial fibration* if it is right orthogonal to the inclusions $\partial\Delta[n]\subset\Delta[n](n\geqslant 0)$. A trivial fibration is right orthogonal to every monomorphism by a classical result [5]. Recall that a map of simplicial sets is a *Kan fibration* if it is right orthogonal to the horn inclusions $\Lambda^k[n]\subset\Delta[n](n>0,\ 0\leqslant k\leqslant n)$. This motivates the following definition:

Definition 2.1. We shall say that a map of simplicial sets $f:X\to Y$ is *mid fibrant* if it is right orthogonal to the horn inclusions $\Lambda^k[n]\subset\Delta[n]$ with $0<k<n$; we shall say that f is *right fibrant* (resp. *left fibrant*) if it is right orthogonal to the horn inclusions $\Lambda^k[n]\subset\Delta[n]$ with $0<k\leqslant n$ (resp. $0\leqslant k<n$).

A simplicial set X is a quasi-category iff the map $X\to 1$ is mid fibrant. A map $f:X\to Y$ is left fibrant iff the opposite map $f^{\mathrm{o}}:X^{\mathrm{o}}\to Y^{\mathrm{o}}$ is right fibrant. A map which is both left and right fibrant is a Kan fibration. The class of mid (resp. left, right) fibrations is closed under composition and base change.

Theorem 2.2. *Let $f:X\to Y$ be a mid fibration between quasi-categories and suppose that we have a commutative square*

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{h} & X \\ \downarrow & & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$

where $n>1$. If the arrow $h(0,1)$ is quasi-invertible then the square has a diagonal filler.

It is obvious that Theorem 2.2 implies Theorem 1.3. We shall see in the next section that Theorem 2.2 follows Theorem 3.4. The proof of Theorem 3.4 will be given after Corollary 3.10. The final argument will use the following concepts.

Recall that a functor $p:C\to D$ is said to be *quasi-fibrant* if for every object $b\in C$ and every isomorphism $f\in D$ with target $p(b)$ there is an isomorphism $g\in C$ with target b such that $p(g)=f$.

Definition 2.3. We shall say that a map $p:X\to Y$ between quasi-categories is *quasi-fibrant* if it is mid fibrant and for any vertex $b\in X$ and any quasi-isomorphism $f\in Y$ with target $p(b)$ there exists a quasi-isomorphism $g\in X$ with target b such that $p(g)=f$.

Proposition 2.4. *Let $p:X\to Y$ be a mid-fibration between quasi-categories. Then p is quasi-fibrant iff the functor $ho(p):hoX\to hoY$ is quasi-fibrant.*

Proof. The implication $\Rightarrow$ is clear. Let us prove the implication $\Leftarrow$. Let $b\in X_0$ and let $f\in Y_1$ be a quasi-isomorphism with target $p(b)$. Then there is a quasi-isomorphism $u\in X_1$ with target b such that $p(u)\simeq f$ since the functor $ho(p)$ is quasi-fibrant. There is then a triangle $t\in Y_2$ with boundary $(1_{p(b)}, f, p(u))$. Consider the commutative square

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{h} & Y \\ \downarrow & & \downarrow p \\ \Delta[2] & \xrightarrow{t} & X \end{array}$$

where h is the horn $(1_b,\bigstar,u)$. It has a diagonal filler $s:\Delta[2]\to Y$ since p is mid-fibrant. If $v=sd_1$ then we have $p(v)=f$ and $v\simeq u$. The arrow v is quasi-invertible since $v\simeq u$ and u is quasi-invertible. □

It is easy to see that a functor $p:C\to D$ is quasi-fibrant iff the functor $p^{\mathrm{o}}:C^{\mathrm{o}}\to D^{\mathrm{o}}$ is quasi-fibrant.

Corollary 2.5. *A map $p:X\to Y$ between quasi-categories is quasi-fibrant iff the opposite map $p^{\mathrm{o}}:X^{\mathrm{o}}\to Y^{\mathrm{o}}$ is quasi-fibrant.*

We shall say that a functor $p: C \to D$ is *conservative* if it reflects isomorphisms, that is, if the invertibility of $p(f) \in D$ implies the invertibility of f for every arrow $f \in C$.

Definition 2.6. We shall say that a map $p: X \to Y$ between quasi-categories is *conservative* if the functor $ho(p): ho X \to ho Y$ is conservative.

Proposition 2.7. *Let* $p: X \to Y$ *be a right* (resp. *left*) *fibration between quasi-categories. Then* p *is quasi-fibrant and conservative.*

Proof. By Corollary 2.5 it suffices to consider the case of a right fibration $p: X \to Y$. Let us first verify that p is a conservative functor. Let $f \in X_1(a,b)$ and let us suppose that $p(f)$ is invertible in $ho\, Y$. Then we have $p(f)u = 1_{pb}$ in $ho\, Y$ for some $u: p(b) \to p(a)$. Hence, there exists a triangle $t \in Y_2$ with boundary $(p(f), 1_{pb}, u)$. Consider the commutative square

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{h} & Y \\ \downarrow & & \downarrow p \\ \Delta[2] & \xrightarrow{t} & X \end{array}$$

where h is the horn $(f, 1_b, \bigstar)$. It has a diagonal filler $s: \Delta[2] \to Y$ since p is right fibrant. If $g = sd_2$ then we have $fg = 1_b$ in $ho X$ and this means that g is right inverse to f in $ho X$. To prove that f is invertible in $ho X$ it suffices to show that g has itself a right inverse in $ho X$. But $p(g) = u$ is invertible in $ho X$ and we can repeat the argument with g instead of f, showing that g has a right inverse in $ho X$. It follows that f is invertible in $ho X$. This proves that p is a conservative functor. Let us now see that p is quasi-fibrant. It is mid-fibrant since it is right fibrant. Let $b \in X$ and let $f \in Y$ be a quasi-isomorphism with target $p(b)$. Then there exists an arrow $g \in Y$ with target b such that $p(g) = f$ since p is right fibrant. The arrow g is quasi-invertible by the first part since f is quasi-invertible. □

3. Slice and join

Let Δ_+ be the category of finite ordinals and order preserving maps. We have $\Delta \subset \Delta_+$ and the ordinal $0 = [-1]$ is in Δ_+ but not in Δ. An *augmented simplicial set* is a contravariant functor $X_+ : \Delta_+ \to$ *Sets*; it can be given as a triple $X_+ = (X, \varepsilon, X_{-1})$ where X is a simplicial set, where X_{-1} is a set, and where ε is a map $: X_0 \to X_{-1}$ called the *augmentation* and satisfying $\varepsilon d^0 = \varepsilon d^1$. We shall denote by $\mathscr{S}_+$ the category $[\Delta_+, Sets]$ of augmented simplicial sets. From the inclusion $i: \Delta \subset \Delta_+$ we obtain by restriction a functor $i^*: \mathscr{S}_+ \to \mathscr{S}$; its right adjoint $i^*: \mathscr{S}_+ \to \mathscr{S}$ gives every simplicial set X the trivial augmentation $X_0 \to 1$. We shall view the category $\mathscr{S}$ as a full subcategory of $\mathscr{S}_+$ via the functor i_*. Beware that $i_*(\emptyset) = [-1]$ is the augmented simplicial set $\emptyset \to 1$ which is non-empty. The category Δ_+ has a monoidal structure $\Delta_+ \times \Delta_+ \to \Delta_+$ given by the operation of ordinal sum $(a,b) \mapsto a + b$ with $0 = [-1]$ as the unit. We have

$[m]+[n]=[m+1+n]$ for every $m,n \geqslant -1$. The *join operation* is the unique functor

$$\star : \mathscr{S}_+ \times \mathscr{S}_+ \to \mathscr{S}_+$$

cocontinuous in each variable extending the ordinal sum along the Yoneda inclusion $\Delta_+ \subset \mathscr{S}_+$. By definition we have $\Delta[m] \star \Delta[n] = \Delta[m+1+n]$ for every $m,n \geqslant -1$. The join operation defines a biclosed monoidal structure on $\mathscr{S}_+$ by a general result [3,9]. The unit object is $\Delta[-1]$. The monoidal structure is not symmetric, but there is a canonical isomorphism $(X \star Y)^{\mathrm{o}} = Y^{\mathrm{o}} \star X^{\mathrm{o}}$.

Proposition 3.1. *For any $X, Y \in \mathscr{S}_+$ and any $n \geqslant -1$ we have*

$$(X \star Y)_n = \bigsqcup_{i+1+j=n} X_i \times Y_j.$$

Proof. For simplicity, let us denote the ordinal $[n-1]$ as n and the set X_{n-1} as $X(n)$. If $X, Y \in \mathscr{S}_+$ then $X \star Y$ is the left Kan extension of the functor $(p,q) \mapsto X(p) \times Y(q)$ along the functor $+ : \Delta_+ \times \Delta_+ \to \Delta_+$. Thus,

$$(X \star Y)(n) = \varinjlim_{n \to p+q} X(p) \times Y(q),$$

where the colimit is taken over the category E_n of elements of the functor $(p,q) \mapsto \Delta_+(n, p+q)$. But every arrow $f : n \to p+q$ is of the form $f = u+v : i+j \to p+q$ for a unique pair of arrows $(u,v) \in \Delta_+ \times \Delta_+$ where $i = f^{-1}(p)$ and $j = f^{-1}(q)$. This means that the set of decompositions $n = i+j$ is initial in the category E_n. Thus,

$$\varinjlim_{n \to p+q} X(p) \times Y(q) = \bigsqcup_{i+j=n} X(i) \times Y(j). \qquad \square$$

In particular, we have $(X \star Y)_{-1} = X_{-1} \times Y_{-1}$. It follows that $X \star Y$ is trivially augmented when X and Y are trivially augmented; there is thus an induced join operation on the subcategory $\mathscr{S} \subset \mathscr{S}_+$. It follows from Proposition 3.1 that for $X, Y \in \mathscr{S}$ we have a natural inclusion $X \sqcup Y \subseteq X \star Y$. Notice that $X \star \emptyset = \emptyset \star X = X$ for $X \in \mathscr{S}$ since $[-1]$ is the unit objects in $\mathscr{S}_+$. It follows that the functor $X \star (-) : \mathscr{S} \to \mathscr{S}$ is not concontinuous. Hence, the join operation does not define a closed monoidal structure on $\mathscr{S}$ even if it does on $\mathscr{S}_+$. The simplicial set $X \star 1$ (resp. $1 \star X$) is the *inductive cone* (resp. the *projective cone*) on X. We have $\Delta[n] \star 1 = \Delta[n+1] = 1 \star \Delta[n]$ for every $n \geqslant 0$. In particular, $1 \star 1$ is the simplicial interval $\Delta[1] = I$.

If T and X are simplicial sets then we have $T = \emptyset \star T \subseteq X \star T$. Fixing T we obtain a functor $(-) \star T : \mathscr{S} \to T \setminus \mathscr{S}$ which is associates to $X \in \mathscr{S}$ the inclusion $T \subseteq X \star T$.

Proposition 3.2. *The functor $(-) \star T : \mathscr{S} \to T \setminus \mathscr{S}$ has a right adjoint.*

Proof. It suffices to show that the functor $F(-) = (-) \star T : \mathscr{S} \to T \setminus \mathscr{S}$ is cocontinuous. It sends the initial object $\emptyset \in \mathscr{S}$ to the initial object $1_T : T \to T$ of $T \setminus \mathscr{S}$. It thus suffices to show that F preserves the colimit of non-empty diagrams. This can be seen directly from Proposition 3.1. Here is another proof of the result. It is obvious that the forgetful functor $U : T \setminus \mathscr{S} \to \mathscr{S}$ and the inclusion functor $J : \mathscr{S} \subset \mathscr{S}_+$ preserve colimits of non-empty diagrams. Observe also that both U and J are conservative functors. We can thus prove that F preserves colimits of non-empty diagrams by showing that the composite functor JUF has the same property. But we have $JUF = HJ$ where $H = (-) \star T : \mathscr{S}_+ \to \mathscr{S}_+$. The result follows since the functor H is cocontinuous by definition of the join operation. $\square$

The right adjoint to the functor $(-) \star T : \mathscr{S} \to T \setminus \mathscr{S}$ takes a map $t : T \to X$ to a simplicial set that we shall denote X/t, or more simply by X/T when the context is clear. A map $S \to X/T$ is equivalent to a map $S \star T \to X$ extending t along the inclusion $T \subseteq S \star T$. In particular, a simplex $\Delta[n] \to X/T$ is the same thing as a map $\Delta[n] \star T \to X$ extending t. If $t : S \star T \to X$ then we have a canonical isomorphism $X/(S \star T) = (X/T)/S$. Dually, the functor $T \star (-) : \mathscr{S} \to T \setminus \mathscr{S}$ has a right adjoint $(t : T \to X) \mapsto t \backslash X = T \backslash X$. A simplex $\Delta[n] \to T \backslash X$ is a map $T \star \Delta[n] \to X$ extending t. By duality we have $(T \backslash X)^{\mathrm{o}} = X^{\mathrm{o}}/T^{\mathrm{o}}$. If $t : S \star T \to X$ then $(S \star T) \backslash X = T \backslash (S \backslash X)$; there is also a simplicial set $S \backslash X/T = (S \backslash X)/T = S \backslash (X/T)$; a simplex $\Delta[n] \to S \backslash X/T$ is the same thing as a map $S \star \Delta[n] \star T \to X$ extending t.

Remark. An augmented simplicial set X is canonically the coproduct $X = \bigsqcup_{i \in I} X(i)$ in $\mathscr{S}_+$ of a family $(X(i)\colon i \in I)$ of simplicial sets. If $Y = \bigsqcup_{j \in J} Y(j)$ is the canonical decomposition of another augmented simplicial set then

$$X \star Y = \bigsqcup_{(i,j) \in I \times J} X(i) \star Y(j).$$

A map $X \to Y$ is given by a pair (ϕ, f) where $\phi : I \to J$ and where $f = (f_i\colon i \in I)$ is a family of maps $f_i : X(i) \to Y(\phi i)$. Let $[X, -]$ be the right adjoint to the functor $X \star (-) : \mathscr{S}_+ \to \mathscr{S}_+$. The canonical decomposition of the augmented simplicial set $[X, Y]$ is given by

$$[X, Y] = \bigsqcup_{(\phi, f) : X \to Y} \prod_{i \in I} X(i) \setminus Y(\phi i),$$

where (ϕ, f) is a pair as above and where $X(i) \setminus Y(\phi i) = f_i \setminus Y(\phi i)$. In particular, if X and Y are simplicial sets then we have

$$[X, Y] = \bigsqcup_{t : X \to Y} t \setminus Y.$$

The formula illustrates the fact that $[X, Y]$ is augmented by the set of maps $X \to Y$.

Lemma 3.3. *We have*

$$(\partial\Delta[m] \star \Delta[n]) \cup (\Delta[m] \star \partial\Delta[n]) = \partial\Delta[m + 1 + n],$$

$$(\Lambda^k[m] \star \Delta[n]) \cup (\Delta[m] \star \partial\Delta[n]) = \Lambda^k[m + 1 + n],$$

$$(\partial\Delta[m] \star \Delta[n]) \cup (\Delta[m] \star \Lambda^k[n]) = \Lambda^{m+1+k}[m + 1 + n].$$

Proof. The face $\partial_i \Delta[m+n+1]$ is contained in $\partial\Delta[m] \bigstar \Delta[n]$ if $0 \leqslant i \leqslant m$ and in $\Delta[m] \bigstar \partial\Delta[n]$ if $m+1 \leqslant i \leqslant m+1+n$. In the first case we have $\partial_i \Delta[m+n+1] = \partial_i \Delta[m] \bigstar \Delta[n]$ and in the second case we have $\partial_i \Delta[m+n+1] = \Delta[m] \bigstar \partial_{i-m-1}\Delta[n]$. The first formula follows. The other formulas are proved similarly. □

Theorem 3.4. *Let $f: X \to Y$ be a mid-fibration between quasi-categories and suppose that we have a commutative square*

$$\begin{array}{ccc} (\{0\} \bigstar T) \cup (\Delta[1] \bigstar S) & \xrightarrow{h} & X \\ \downarrow & & \downarrow f \\ \Delta[1] \bigstar T & \longrightarrow & Y. \end{array}$$

If the arrow $h(0,1)$ is quasi-invertible then the square has a diagonal filler.

The result will be proved after Corollary 3.10. Let us now see that it implies Theorem 2.2. By the second formula of Lemma 3.3 we have

$$(\Lambda^0[1] \bigstar \Delta[m]) \cup (\Delta[1] \bigstar \partial\Delta[m]) = \Lambda^0[m+2]$$

since $\{0\} = \Lambda^0[1]$. It follows that the commutative square of Theorem 2.2 can written as the following commutative square:

$$\begin{array}{ccc} (\Delta[1] \bigstar S) \cup (\{0\} \bigstar T) & \xrightarrow{h} & X \\ \downarrow & & \downarrow f \\ \Delta[1] \bigstar T & \longrightarrow & Y. \end{array}$$

where $n = m+2$, $S = \partial\Delta[m]$ and $T = \Delta[m]$. The arrow $h(0,1)$ is quasi-invertible by the hypothesis of Theorem 2.2. It follows from Theorem 3.4 that the square has a diagonal filler.

Theorem 3.4 will be proved after Corollary 3.10. The proof is based on a few intermediate results. The first of which is of a very general in nature.

If $\mathscr{E}$ is a category let us denote by $\mathscr{E}^I$ the category of arrows in $\mathscr{E}$. Recall that an object u of $\mathscr{E}^I$ is a map $u: A_0 \to A_1$ in the category $\mathscr{E}$, and that an arrow $u \to v$ from $u: A_0 \to A_1$ to $v: B_0 \to B_1$ is a commutative square

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A_1 & \xrightarrow{f_1} & B_1 \end{array}$$

in the category $\mathscr{E}$. Suppose now that we have two categories $\mathscr{D}$ and $\mathscr{E}$, two pairs of adjoint functors

$$F_i: \mathscr{D} \leftrightarrow \mathscr{E}: G_i \quad (i = 0, 1)$$

together with a pair of adjoint natural transformations $\alpha: F_0 \to F_1$ and $\beta: G_1 \to G_0$. Suppose, moreover, that the category $\mathscr{E}$ admits pushouts and that $\mathscr{D}$ admits pullbacks.

For $f:A\to B$ in $\mathscr{D}$ and $g:X\to Y$ in $\mathscr{E}$ there are commutative squares

$$\begin{array}{ccc} F_0(A) & \xrightarrow{\alpha_A} & F_1(A) \\ {\scriptstyle F_0(f)}\downarrow & & \downarrow{\scriptstyle F_1(f)} \\ F_0(B) & \xrightarrow{\alpha_B} & F_1(B) \end{array} \qquad \begin{array}{ccc} G_1(X) & \xrightarrow{\beta_X} & G_0(X) \\ {\scriptstyle G_1(g)}\downarrow & & \downarrow{\scriptstyle G_0(g)} \\ G_1(Y) & \xrightarrow{\beta_Y} & G_0(Y) \end{array}$$

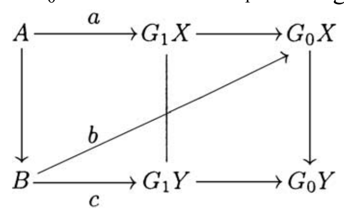
and hence maps

$$F(f):F_0(B)\sqcup_{F_0(A)}F_1(A)\to F_1(B) \quad \text{and} \quad G(g):G_1(X)\to G_1(Y)\times_{G_0(Y)}G_0(X).$$

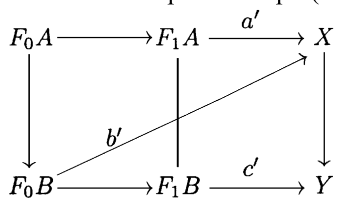
This defines a pair of functors $F:\mathscr{D}^I\to\mathscr{E}^I$ and $G:\mathscr{E}^I\to\mathscr{D}^I$.

Lemma 3.5. *The functor F is left adjoint to the functor G. The orthogonality relation $F(f)\perp g$, for $f\in\mathscr{D}$ and $g\in\mathscr{E}$, is equivalent to the orthogonality relation $f\perp G(g)$.*

Proof. A map $f\to G(g)$ is equivalent to a triple (a,b,c) of arrows $a:A\to G_1X$, $b:B\to G_0X$ and $c:B\to G_1Y$ fitting in a commutative diagram.



By the adjointness, the map $a:A\to G_1X$ corresponds to a map $a':F_1A\to X$, the map $b:B\to G_0X$ to a map $b':F_0B\to X$ and the map $c:B\to G_1Y$ to a map $c':F_1B\to Y$. The triple of maps (a',b',c') fits into a commutative diagram



and this defines a map $F(f)\to g$. The adjointness $F\dashv G$ follows. A diagonal filler for the square $f\to G(g)$ is the same thing as a map $d:B\to G_1X$ fitting commutatively into the first diagram. By adjointness it corresponds to a map $d':F_1B\to X$ fitting commutatively into the second diagram, but this is the same thing as a diagonal filler for the square $F(f)\to g$. $\square$

It is easy to see from Proposition 3.1 that the functor $X\star(-)$ preserve monomorphisms, and moreover that if $A\subseteq B$ and $S\subseteq T$ then we have $(A\star T)\cap(B\star S)=A\star S$ where the intersection is taken in $B\star T$. It follows that the square

$$\begin{array}{ccc} A\star S & \longrightarrow & A\star T \\ \downarrow & & \downarrow \\ B\star S & \longrightarrow & (A\star T)\cup(B\star S) \end{array}$$

is a pushout where the union is taken in $B\star T$.

From an inclusion $j:S \subseteq T$ and maps $t:T \to X$ and $f:X \to Y$ we obtain a commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & Y/T \\ \big\downarrow & & \big\downarrow \\ X/S & \longrightarrow & Y/S \end{array}$$

and hence a map $p:X/T \to X/S \times_{Y/S} Y/T$ where $X/S = X/tj$, $Y/T = Y/ft$ and $Y/S = Y/ftj$. We shall call p the *projection*.

Lemma 3.6. *For $i:A \subseteq B$ and $j:S \subseteq T$ consider the inclusion u: $(A \star T) \cup (B \star S) \subseteq B \star T$. If $f:X \to Y$ then there is a canonical bijection between the following commutative squares*:

$$\begin{array}{ccc} (A \star T) \cup (B \star S) & \longrightarrow & X \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B \star T & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & X/T \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y/T \times_{Y/S} X/S \end{array}$$

where the first square is in $\mathscr{S}$ and the second in $T/\mathscr{S}$; the structure map $T \to X$ by which the object X/T in the second square is obtained is the composite $T \subseteq (A \star T) \cup (B \star S) \to X$. If one of the squares has a diagonal filler then so has the other.

Proof. We shall use Lemma 3.5 with the functors $F_0, F_1 : \mathscr{S} \to T/\mathscr{S}$ given by $F_1(A) = A \star T$, $F_0(A) = (A \star S) \cup T \subseteq A \star T$, and the natural transformation $\alpha: F_0 \to F_1$ obtained from the inclusion $(A \star S) \cup T \subseteq A \star T$. If $i:A \subseteq B$ then $F(i)$ is the inclusion $u:(A \star T) \cup (B \star S) \subseteq B \star T$. We have $F_i \dashv G_i$ for $(i = 0, 1)$ where the functors $G_i : T \backslash \mathscr{S} \to \mathscr{S}$ are given by $G_1(X) = X/T$ and $G_0(X) = X/S$ for $t:T \to X$. The natural transformation $\beta: G_1 \to G_0$ is the map $X/T \to X/S$ obtained from the inclusion $S \subseteq T$. If $f:X \to Y$ then $G(f)$ is the projection $p:X/T \to X/S \times_{Y/S} Y/T$. If we fix $t:T \to X$ then Lemma 3.5 shows that there is a canonical bijection between the maps $u \to f$ in $(T/\mathscr{S})^I$ and the maps $i \to p$ in $\mathscr{S}^I$. The result follows. $\square$

Remark. Lemma 3.6 can be proved by working in $\mathscr{S}_+$ and by using techniques developed for studying monoidal closed Quillen structures as in [6].

Lemma 3.7. *Let $f:X \to Y$ be a mid-fibration. If $t:\Delta[n] \to X$ and $0 \leqslant k < n$ then the projection*

$$p : X/\Delta[n] \to X/\Lambda^k[n] \times_{Y/\Lambda^k[n]} Y/\Delta[n]$$

is a trivial fibration.

Proof. We have to show that p is right orthogonal to every inclusion $\partial\Delta[m] \subset \Delta[m]$. By Lemma 3.6 it suffices to show that every commutative square

$$\begin{array}{ccc} (\Delta[m] \star \Lambda^k[n]) \cup (\partial\Delta[m] \star \Delta[n]) & \longrightarrow & X \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \Delta[m] \star \Delta[n] & \longrightarrow & Y \end{array}$$

has a diagonal filler. But u is the inclusion $\Lambda^{k+m+1}[m+n+1] \subset \Delta[m+n+1]$ by Lemma 3.3. Hence, the square has a diagonal filler since f is mid fibrant and $0 < m+k+1 < m+n+1$. □

Theorem 3.8. *Let $f:X \to Y$ be a mid-fibration. If $S \subseteq T$ and $t:T \to X$ then the projection*

$$p:X/T \to X/S \times_{Y/S} Y/T$$

is right fibrant.

Proof. We have to show that p is right orthogonal to every inclusion $\Lambda^k[n] \subset \Delta[n]$ with $0 < k \leqslant n$. By Lemma 3.6 it suffices to show that every commutative square

$$\begin{array}{ccc} (\Lambda^k[n] \star T) \cup (\Delta[n] \star S) & \longrightarrow & X \\ {\scriptstyle u}\downarrow & & \downarrow {\scriptstyle f} \\ (\Delta[n] \star T) & \longrightarrow & Y \end{array}$$

has a diagonal filler. By a dual of Lemma 3.6 this is equivalent to showing that every commutative square

$$\begin{array}{ccc} S & \longrightarrow & \Delta[n] \setminus X \\ \downarrow & & \downarrow {\scriptstyle q} \\ T & \longrightarrow & \Lambda^k[n] \setminus X \times_{\Lambda^k[n]\setminus Y} \Delta[n] \setminus Y \end{array}$$

has a diagonal filler. But q is a trivial fibration by the dual of Lemma 3.7 since $0 < k \leqslant n$. The result follows since a trivial fibration is right orthogonal to every monomorphism. □

Corollary 3.9. *If X is a quasi-category then so is X/T for any map $T \to X$. Moreover, the projection $X/T \to X/S$ is a right fibration for any inclusion of simplicial sets $S \subseteq T$.*

Proof. The projection $X/T \to X/S$ is right fibrant by Theorem 3.8 applied to the case $Y = 1$. This proves the second statement. In particular, it shows that the projection $X/T \to X$ is right fibrant if we put $S = \emptyset$; it is thus mid fibrant and it follows that X/T is a quasi-category since X is a quasi-category. □

Corollary 3.10. *Let $f:X \to Y$ be a map between quasi-categories. Then the simplicial set $X/S \times_{Y/S} Y/T$ is a quasi-category and the projection $p_1:X/S \times_{Y/S} Y/T \to X/S$ is a right fibration for any inclusion of simplicial sets $S \subseteq T$.*

Proof. Consider the pullback square

$$\begin{array}{ccc} X/S \times_{Y/S} Y/T & \longrightarrow & Y/T \\ {\scriptstyle p_1}\downarrow & & \downarrow {\scriptstyle q} \\ X/S & \longrightarrow & Y/S. \end{array}$$

The map q is a right fibration by Corollary 3.9. It follows that p_1 is a right fibration. This proves the second statement. But X/S is a quasi-category by Corollary 1. Thus, $X/S \times_{Y/S} Y/T$ is a quasi-category since p_1 is mid fibrant. This proves the first statement. □

We can now prove Theorem 3.4. By Lemma 3.6 we obtain an adjoint square

$$\begin{array}{ccc} \{0\} & \longrightarrow & X/T \\ \downarrow & & \downarrow p \\ \Delta[1] & \xrightarrow{i} & X/S \times_{Y/S} Y/T. \end{array}$$

It suffices to show that this adjoint square has a diagonal filler. The result will follows from Corollary 2.5 if we show that p is a quasi-fibration between quasi-categories and moreover that $i(0,1)$ is quasi-invertible. The domain and codomain of p are quasi-categories by Corollaries 3.9 and 3.10. But p is a right fibration by Theorem 3.8. It follows from Proposition 2.7 that p is quasi-fibrant. It remains to show that the arrow $i(0,1)$ is quasi-invertible. For this it suffices to show that the map $kp_1 : X/S \times_{Y/S} Y/T \to X$ is conservative since the arrow $kp_1 i(0,1) = h(0,1)$ is quasi-invertible by hypothesis. By Proposition 2.7 it suffices to show that kp_1 is a right fibration. But the projection $k : X/S \to X$ is a right fibration by Corollary 3.9; hence, so is the composite $kp_1 : X/S \times_{Y/S} Y/T \to X$ since p_1 is a right fibration by Corollary 3.10. □

4. Application to the theory of initial objects

Definition 4.1. Let X be a quasi-category. We shall say that a vertex $a \in X$ is *initial* iff every simplicial sphere $x : \partial\Delta[n] \to X (n > 0)$ with $a = x(0)$ can be filled.

Dually, a vertex $a \in X$ is *terminal* if the vertex $a^{o} \in X^{o}$ is initial.

Proposition 4.2. *Let X be a quasi-category and let $a \in X$ be a vertex. Then the following conditions are equivalent*:
(a) *a is initial*;
(b) *the projection $a \backslash X \to X$ is a trivial fibration*;
(c) *the projection $a \backslash X \to X$ has a sections $s : X \to a \backslash X$ such that $s(a) = 1_a$*; *and*
(d) *the inclusion $X \subset 1 \star X$ has a retraction $r : 1 \star X \to X$ such that $r(1 \star a) = 1_a$*.

Proof. (a) ⇔ (b) Let us use the decomposition $\partial\Delta[n+1] = (1 \star \partial\Delta[n]) \cup (\emptyset \star \Delta[n])$. By Lemma 3.6 there is a bijection between the maps $x : \partial\Delta[n+1] \to X$ with $x(0) = a$ and the commutative squares

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & a \backslash X \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & X. \end{array}$$

The square has a diagonal filler iff x can be filled. (b $\Rightarrow$ c) The commutative square

$$\begin{array}{ccc} 1 & \xrightarrow{1_a} & a\setminus X \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \xrightarrow{1_X} & X \end{array}$$

has a diagonal filler $s:X\to a\setminus X$ since the projection $a\setminus X\to X$ is a trivial fibration. (c $\Rightarrow$ d) By adjointness the map $s:X\to a\setminus X$ corresponds to a map $r:1\star X\to X$ such that $r(1)=a$. Moreover, r is a retraction since s is a section and we have $r(1\star a)=1_a$ since we have $s(a)=1_a$. (d $\Rightarrow$ a) Let $x:\partial\Delta[n]\to X$ be a simplicial sphere such that $x(0)=a$. The map $y=r(1\star x):1\star\partial\Delta[n]\to 1\star X\to X$ extends x and we have $y(1\star 0)=r(1\star a)=1_a$. But $1\star\partial\Delta[n]=\Lambda^0[n+1]$ and the arrow $y(0,1)=y(1\star 0)=1_a$ is quasi-invertible since it is a unit. It follows by Theorem 1.3 that y has a filler $z:\Delta[n+1]\to X$. The face zd_0 then fills x. □

Proposition 4.3. *The arrow 1_a is an initial vertex of $a\setminus X$.*

Proof. Let $x:\partial\Delta[n]\to a\setminus X$ be a simplicial sphere such that $x(0)=1_a$ where $n>0$. The map x corresponds by adjointness to a map $y:1\star\partial\Delta[n]\to X$. But we have $1\star\partial\Delta[n]=\Lambda^0[n+1]$ and the arrow $y(0,1)=x(0)=1_a$ is quasi-invertible since it is a unit. By Theorem 1.3 y has a filler $z:\Delta[n+1]\to X$. The simplex $w:\Delta[n]\to a\setminus X$ corresponding to z by adjointness fills x. □

We shall say that a simplicial subset $U\subseteq X$ is *full* if every simplex $\Delta[n]\to X$ having all its vertices in U belongs to U. For every subset $S\subseteq X_0$ there is a unique full simplicial subset $U\subseteq X$ such that $U_0=S$; we shall say that U is *spanned* by S.

Proposition 4.4. *The full simplicial subset spanned by the initial vertices of a quasi-category X is a contractible Kan complex if non-empty. Every vertex $a\in X$ which is quasi-isomorphic to an initial vertex is initial. If a vertex $a\in X_0$ is initial in X then it is initial in hoX; the converse is true if X admits at least one initial vertex.*

Proof. The first statement is obvious since every simplicial sphere $x:\partial\Delta[n]\to X$ with $x(0)$ an initial vertex can be filled. Let us prove the third statement. If $a\in X_0$ is initial in X let us show that it is initial in hoX. We have $X_1(a,b)\neq\emptyset$ for every $b\in X_0$ since the 0-sphere $(b,a):\partial\Delta[1]\to X$ can be filled by the initiality of a. Moreover, if $f,g:a\to b$ then we have $f\simeq g$ since the 1-sphere $(f,g,1_a):\partial\Delta[2]\to X$ can be filled for the same reason. Thus, $(hoX)(a,b)=1$ for every $b\in X_0$ and this shows that a is initial in hoX. Let us prove the fourth statement. Suppose that X has an initial vertex $a\in X_0$ and let us show that every $b\in X_0$ which is initial in hoX is also initial in X. Observe that every arrow $a\to b$ is quasi-invertible since a is initial in hoX by what we just proved. Let $x:\partial\Delta[n]\to X$ be a simplicial sphere with $x(0)=b$. It follows from Proposition 4.2(d) that the map $x'=r(1\star x):1\star\partial\Delta[n]\to X$ extends x and that $x'(1)=a$. But $1\star\partial\Delta[n]=\Lambda^0[n+1]$ and the arrow $x'(0,1):a\to b$ is quasi-invertible by the observation above. It follows by Theorem 1.3 that $x':\Lambda^0[n+1]\to X$ has a filler $z:\Delta[n+1]\to X$. The face $zd_0:\Delta[n]\to X$ then fills x. This proves the fourth statement.

The second statement follows since every vertex $a \in X$ which is quasi-isomorphic to an initial vertex $b \in X$ must be initial in $ho X$ since b is initial in $ho X$ by the first part of the proof. □

We conclude this paper by discussing the notions of limits and colimits in quasi-categories. The classical notions are based on the concepts of initial and terminal object [10]. The same is true in the context of quasi-categories. The notions introduced here are equivalent to the concepts of homotopy limits and colimits of Bousfield and Kan [2]. See [7] for a complete theory.

A *diagram* in a quasi-category X is just a map $d : T \to X$ where T is a simplicial set. A *projective cone on* d is a map $d' : 1 \star T \to X$ extending d; by adjointness d' is equivalent to a vertex $1 \to X/T$; we shall say that d' is *exact* if the corresponding vertex $1 \to X/T$ is terminal in X/T.

Definition 4.5. We shall say that a diagram $d : T \to X$ *has a limit* if the quasi-category X/T has at least one initial vertex, in which case a *limit* $\varprojlim_T d$ is the vertex $d'(1)$ of an exact projective cone $d' : 1 \star T \to X$ extending d. The concept of *colimit* $\varinjlim_T d$ is defined dually by using the initial vertices of $T \setminus X$ and coexact inductive cones instead.

Remark. It would be more precise to define a limit of $d : T \to X$ as an exact projective cone $d' : 1 \star T \to X$ extending d. We are conforming to the common usage of exhibiting the vertex $d'(1)$ while leaving the cone d' in the shadow. A particular choice of an exact cone $d' : 1 \star T \to X$ always stands behind a given choice of $\varprojlim_T d$.

Let $d : T \to X$ be a diagram in a quasi-category X. Let $K \subseteq X/T$ be the full simplicial subset of X/T spanned by the terminal objects of X/T. The simplicial set K parametrises all exact projective cones with base d. By composing the inclusion $K \subseteq X/T$ with the projection $X/T \to X$ we obtain a map $K \to X$; it associates to a cone $d' \in K$ its vertex $d'(1)$. Let us put $K = Proj(d)$. A diagram d has a limit iff $Proj(d) \neq \emptyset$.

Proposition 4.6. *Let $d : T \to X$ be a diagram in a quasi-category X. If d has a limit then $Proj(d)$ is a contractible Kan complex and the map $Proj(d) \to X$ is a quasi-fibration.*

Proof. The simplicial subset $Proj(d) = K \subseteq X/T$ is a contractible Kan complex by Proposition 4.4. The inclusion $K \subseteq X/T$ is mid fibrant since it is full; it is also quasi-fibrant since every vertex of X/T which is quasi-isomorphic to a vertex in K belongs to K by Proposition 4.4. But the projection $X/T \to X$ is quasi-fibrant by Proposition 2.7 and Corollary 3.9. It follows by composing $K \subseteq X/T \to X$ that the map $Proj(d) \to X$ is quasi-fibrant. □

The simplicial set $Proj(d)$ parametrises all exact projective cone based on d. The fact that it is a contractible Kan complex if non-empty means that the limit of d is *homotopy unique* when it exists. The fact that the map $Proj(d) \to X$ is a quasi-fibrant implies that if $a \in X$ is quasi-isomorphic to a limit of d then it is also a limit of d.

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