THE STRUCTURE OF THE HOPF ALGEBRA $H_*(BU)$ OVER A $\boldsymbol{Z}_{(p)}$-ALGEBRA.

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In [5] John Moore gave a description of the Hopf algebra $H_*(BU, R)$ over any ring R using two kinds of universal constructions for Hopf algebras. Over any ring R which is a $\boldsymbol{Q}$-algebra it is well known that $H_*(BU, R)$ decomposes as an infinite tensor product of polynomial Hopf algebras on one generator. Over a $\boldsymbol{Z}_{(p)}$ algebra R (for example the field $\boldsymbol{F}_p$ of p elements), we will show that $H_*(BU, R)$ is an infinite tensor product of Hopf algebras indexed by the integers prime to p. The Hopf algebras in the infinite tensor product are isomorphic up to change of the bottom degree d and have one primitive element and one indecomposible element in exactly each degree of the form $p^i d$. The above discussion applies to $H^*(BU)$, $H_*(BSp)$, and $H^*(BSp)$ and also to $H_*(BO, R)$ and $H^*(BO, R)$ when either R is a $\boldsymbol{Z}_{(p)}$ algebra for p odd or an $\boldsymbol{F}_2$-algebra.

In Sections 1, 2, and 3 we review the universal constructions that are needed. We digress to consider some questions about the universal free commutative algebras $S(M)$ and coalgebras $S'(M)$. For example, in the ungraded case the polynomial algebra $S(M)$ on a free module M is again a free module. In the graded (twisted) commutative case this is not universally true. These constructions are related to ideas in the theory of formal groups, because a smooth formal group can be viewed as a Hopf algebra structure on $S'(M)$ for some free ungraded module M, see [2].

In Section 4 we recount the description of the biuniversal Hopf algebra $B[x, d]$ and give new proofs of some of the results based on methods used in the theory of λ-rings and Witt vectors. In Section 5 we compare $B[x, d]$ and $B[x, rd]$ using the morphism $\theta\colon P(A) \to Q(A)$ which is defined for any Hopf algebra.

In Sections 6, 7 and 8 the biuniversal Hopf algebras $B_{(p)}[x, d]$ are introduced and studied. The decomposition over a $\boldsymbol{Z}_{(p)}$-algebra $B[x, d] = \otimes_{(r,p)=1} B_{(p)}[x, rd]$ is proven. The Hopf algebra $B_{(p)}[x, d]$ is related to $B[x, d]$ in the same manner that the p-Witt vectors are related to the (big) universal Witt vectors, see [6]. In the ungraded case $B_{(p)}[x, d]$ is the analogue of the Hopf algebra (or bialgebra or bigebre) of distributions on

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the p-Witt vectors and then the decomposition comes from the Artin-Hasse exponential series, see [7].

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1. The covariant and invariant tensor spaces $S_n(M)$ and $S_n'(M)$. Let R denote (usually) a commutative ring (with 1), and consider the category $Gr^+(R)$ of positively graded R-modules. In this category we have a tensor product functor $L \otimes M$ where in degree n the module $(L \otimes M)_n$ equals $\coprod_{i+j=n} L_i \otimes M_j$, together with a commutativity morphism

$$T = T(L, M): L \otimes M \to M \otimes L$$

given by the relation $T(x \otimes y) = (-1)^{ij} y \otimes x$ for $x \in L_i$, $y \in M_j$. For discussion of the general setting see [3].

Let $T_n(M)$ denote $M^{n\otimes}$ the tensor product of M with itself n times where $T_0(M) = R$ (concentrated in degree 0 as usual) and $T_1(M) = M$. The symmetric group on n objects acts on $T_n(M)$ by the relation

$$\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}$$

where σ and τ are inverse to each other in the symmetric group and $x_i \in M_{m(i)}$.

Let $S_n'(M) \to T_n(M)$ denote the natural inclusion of the submodule $S_n'(M)$ of invariant tensors under the action of the symmetric group into $T_n(M)$. Let $T_n(M) \to S_n(M)$ denote the natural quotient by the submodule generated by $\sigma(t) - t$ for $t \in t_n(M)$ and σ in the symmetric group. Clearly the functor $M \mapsto T_n(M)$ has $M \mapsto S_n'(M)$ and $M \mapsto S_n(M)$ as a subfunctor and a quotient functor respectively.

If M is projective of finite type, then $T_n(M)$ is also and the dual module $T_n(M)^\vee = T_n(M^\vee)$ where $M^\vee$ is the dual of M. If, in addition, $S_n'(M)$ and $S_n(M)$ are projective of finite type, then the transpose morphism of $S_n'(M) \to T_n(M)$ is $T_n(M^\vee) \to S_n(M^\vee) = S_n'(M)^\vee$ and the transpose morphism of $T_n(M) \to S_n(M)$ is $S_n(M)^\vee = S_n'(M^\vee) \to T_n(M^\vee)$ up to canonical isomorphism. The question of when $S_n(M)$ and $S_n'(M)$ are projective of finite type will be considered in Section 3 using the universal constructions of Section 2.

Recall that a module M is *connected* provided $M_0 = 0$. If M is connected, then $T_n(M)_i = S_n'(M) = 0$ for $i < n$. Let $[x, d]$ denote the (graded) module free on one generator x in degree d. Clearly $[x, d]$ is connected if and only if $d > 0$.

(1.1) *Example.* The module $T_n[x,d]_i$ is 0 if $nd \neq i$ and $Rx^{n\otimes}$, i.e. free of rank 1, if $nd = i$. Also $S_n'[x,d]_i = S_n[x,d]_i = 0$ for $nd \neq i$, and

$$S_n(x,d)_{nd} = \begin{cases} Rx^{n\otimes} & \text{for } n = 0,1 \text{ and all } n \text{ if } d \text{ is even.} \\ R \otimes (\mathbf{Z}/2\mathbf{Z})x^{n\otimes} & \text{for } n > 1 \text{ when } d \text{ is odd.} \end{cases}$$

$$S_n'(x,d)_{nd} = \begin{cases} Rx^{n\otimes} & \text{for } n = 0,1 \text{ and all } n \text{ if } d \text{ is even.} \\ \mathrm{Tor}(R, \mathbf{Z}/2\mathbf{Z})x^{n\otimes} & \text{for } n > 1 \text{ when } d \text{ is odd.} \end{cases}$$

Here $R \otimes (\mathbf{Z}/2\mathbf{Z}) = R/2R$ has a quotient R-module structure and Tor $(R, \mathbf{Z}/2\mathbf{Z}) = {}_2R$, the ideal of all $a \in R$ with $2a = 0$, has an R-submodule structure.

For $R = \mathbf{Z}$ it follows that $S_n'[x,1]_n = 0$ for $n > 1$ and $S_n[x,1]_n = (\mathbf{Z}/2\mathbf{Z}x^{n\otimes}$ for $n > 1$ and thus no simple duality relation holds between $S_n[x,1]$ and $S_n'[x,1]$. The next special cases give the direction for a satisfactory general theory.

(1.2) *Special cases of* (1.1). If R is a $\mathbf{Z}[1/2]$ algebra, then $S_n'[x,d]_{nd} = S_n[x,d]_{nd} = 0$ for $n > 1$ and d odd. If R is an $\boldsymbol{F}_2$-algebra ($\boldsymbol{F}_q$ denotes the finite field of q elements), then $S_n'[x,d]_{nd} = Rx^{n\otimes} = S_n[x,d]_{nd}$ for all n and all d.

2. Universal constructions of algebras, coalgebras and Hopf algebras. In this section we outline those properties of universal constructions needed for analysing the homology Hopf algebras of certain H-spaces. For an introduction to these questions see [5] and for a detailed treatment see [3]. For generalities on Hopf algebras see also [4].

We put together these universal constructions from the modules $T_n(M)$, $S_n'(M)$, and $S_n(M)$ considered in the previous section. Observe that there are canonical isomorphisms $T_{n+m}(M) \to T_n(M) \otimes T_m(M)$ and $T_n(M) \otimes T_m(M) \to T_{n+m}(M)$ inverse to each other inducing respectively the monomorphism $S_{n+m}'(M) \to S_n'(M) \otimes S_m'(M)$ and the epimorphism $S_n(M) \otimes S_m(M) \to S_{n+m}(M)$. These monomorphisms collect to define a natural comultiplication $\Delta: S'(M) \to S'(M) \otimes S'(M)$ where $S'(M) = \coprod_{0 \leqq n} S_n'(M)$, and these epimorphisms collect to define a natural multiplication $\Phi: S(M) \otimes S(M) \to S(M)$ where $S(M) = \coprod_{0 \leqq n} E_n(M)$. In fact, $S(M)$ is a (supplemented) algebra and $S'(M)$ is a (supplemented) coalgebra each of whose properties are stated in the next two paragraphs. They are called respectively the *free commutative algebra* and *coalgebra* on the module M.

(2.1) *The algebra $S(M)$ and the Hopf algebra $S(C)$.*

(a) $S(M)$ is a supplemented algebra with multiplication Φ given above with unit $R \to S(M)$ and augmentation $S(M) \to R$ given by the direct summand $S_0(M) = R$ of $S(M)$. The augmentation ideal is $I(S(M)) = \coprod_{0<n} S_n(M)$.

(b) $S(M)$ is commutative (also called anticommutative), that is, $\Phi T = \Phi : S(M) \otimes S(M) \to S(M)$ where T is the commutative morphism of the tensor product.

(c) The module injection $\beta : M \to S(M)$ has the following universal property: for any morphism $f : M \to I(A)$ of modules where A is a commutative (supplemented) algebra, there exists a unique morphism $g : S(M) \to A$ of (supplemented) algebras such that $I(g)\beta = f$. This defines $S(M)$ as a functor from modules to commutative algebras.

For a supplemented coalgebra C we form the commutative algebra $S(I(C))$. The composite of the restriction $I(\Delta) : I(C) \to I(C) \otimes I(C)$ of the comultiplication Δ on C and $\beta \otimes \beta : I(C) \otimes I(C) \to S(I(C)) \otimes S(I(C))$ defines by (c) a comultiplication Δ on $S(I(C))$ making $S(I(C))$ into a Hopf algebra. By abuse of language we denote this Hopf algebra $S(I(C))$ by simply $S(C)$, and if C is commutative, then $S(C)$ is cocommutative.

(d) The injection $\beta : C \to S(C)$ of coalgebras has the following universal property: for any morphism $f : C \to B$ of coalgebas where B is a commutative Hopf algebra, there exists a unique morphism $g : S(C) \to B$ of Hopf algebras such that $g\beta = f$. This defines S as a functor from coalgebras to commutative Hopf algebras.

(e) For the module $[x, d]$ with d even, the algebra $S[x, d]$ has a basis $(x^i)_{0 \leqq i}$ with multiplication given by $x^i x^j = x^{i+j}$ and degree $(x^i) = id$. For the zero coalgebra $R \oplus [x, d]$ where $\Delta x = x \otimes 1 + 1 \otimes x$ the Hopf algebra $S(R \oplus [x, d])$ is the algebra $S[x, d]$ together with the comultiplication $\Delta(x^m) = \sum_{i+j=m} (i, j) x^i \otimes x^j$ where $(a, b) = (a+b)!/(a!)(b!)$, the binomial coefficient.

(f) If C is a connected coalgebra, i.e. $C_0 = R$, then by multiplication in $S(C)$ the morphism $\beta^{(n)} : C^{n\otimes} \to S(C)$ which is a surjection in degree $\leqq n$.

The above discussion could be carried out with $S(M)$ and $S(C)$ replaced by the tensor algebra $T(M) = \coprod_{0<n} T_n(M)$ and the tensor Hopf algebra $T(C)$ respectively. The universal properties would hold for morphisms into any algebra.

In the next paragraph we restrict ourselves to the connected case, although the discussion could be carried out for cocomplete coalgebras, see [3]

(2.2) *The coalgebra $S'(M)$ and the Hopf algebra $S'(A)$.*

(a) $S'(M)$ is a supplemented coalgebra with comultiplication Δ given above with unit $S'(M) \to R$ and augmentation $R \to S'(M)$ given by the direct summand $S_0'(M) = R$ of $S'(M)$. The augmentation ideal is

$$I(S'(M) = \coprod_{0<n} S_n'(M).$$

(b) $S'(M)$ is commutative, that is, $T\Delta = \Delta\colon S'(M) \to S'(M) \otimes S'(M)$.

(c) For M connected, the module projection $\alpha\colon S'(M) \to M$ has the following universal property: for any morphism $f\colon I(C) \to M$ of modules where C is a commutative connected coalgebra, there exists a uniqne morphism $g\colon C \to S'(M)$ of coalgebras such that $\alpha I(g) = f$. This defines $S'(M)$ as a functor from connected modules to connected coalgebras.

For a connected algebra A we form the commutative coalgebras $S'(I(A))$. The composite of the quotient $I(\Phi)\colon I(A) \otimes I(A) \to I(A)$ of the multiplication Φ on A and $\alpha \otimes \alpha\colon S'(I(A)) \otimes S'(I(A)) \to I(A) \otimes I(A)$ defines by (c) a multiplication Φ on $S'(I(A))$ making $S'(I(A))$ into a Hopf algebra. By abuse of language we denote this Hopf algebra $S'(I(A))$ by simply $S'(A)$, and if A is commutative, then $S'(A)$ is commutative.

(d) The projection $\alpha\colon S'(A) \to A$ of algebras has the following universal property: for any morphism $f\colon B \to A$ of algebras where B is a connected, cocommutative Hopf algebra, there exists a unique morphism, $g\colon B \to S'(A)$ of Hopf algebras such that $\alpha g = f$. This defines S' as a functor from connected algebras to connected cocommutative Hopf algebras.

(e) For the module $[x, d]$ with d even and $0 < d$, the coalgebra $S'[x, d]$ has a basis $(\gamma_i(x))_{0 \leqq i}$ with comultiplication given by $\Delta\gamma_n(x) = \sum_{i+j=m} \gamma_i(x) \otimes \gamma_j(x)$ and degree $(\gamma_i(x)) = \text{id}$. For the zero algebra $R \oplus [x, d]$ where $x^2 = 0$ the Hopf algebra $S'(R \oplus [x, d])$ is the coalgebra $S'[x, d]$ together with the multiplication $\gamma_i(x)\gamma_j(x) = (i, j)\gamma_{i+j}(x)$.

(f) For a connected algebra A, the morphism $\alpha\colon S'(A) \to A$ together with the comultiplication on $S'(A)$ define a morphism $\alpha^{(n)}\colon S'(A) \to A^{n\otimes}$ which is injective in degrees $\leqq n$.

The above discussion could be carried out with $S'(M)$ and $S'(A)$ replaced by the tensor coalgebra $T'(M) = \coprod_{0 \leqq n} T_n(M)$ and tensor Hopf algebra $T'(A)$ respectively. The universal properties would hold for morphisms from any coalgebra.

(2.3) *The functor Q and $S(M)$.* The composite of $\beta\colon M \to I(S(M))$ and $I(S(M)) \to Q(S(M))$ is an isomorphism where, the *indecomposable element functor*, $Q(A) = \text{coker}(\bar{\Phi}\colon I(A) \otimes I(A) \to I(A))$ is defined on the category of supplemented algebras A. Recall that a morphism $f: A \to A'$ of connected algebras is surjective if and only if $Q(f)$ is an epimorphism, see [4, Prop. 3.8]. This holds under a general completeness hypothesis, see [3].

(2.4) *The functor P and $S'(M)$.* The composition of $P(S'(M)) \to I(S'(M))$ and $\alpha: I(S'(M)) \to M$ is an isomorphism where, the *primitive element functor*, $P(C) = \operatorname{Ker}(\Delta: I(C) \to I(C) \otimes I(C))$ is defined on the category of supplemented coalgebras C. Recall that a morphism $f: C \to C'$ of connected coalgebras is injective if and only if $P(f)$ is a monomorphism, see [4, Prop. 3.9]. This holds under a general cocompleteness hypothesis, see [3].

(2.5) *The morphism $\theta(A): P(A) \to Q(A)$.* For a Hopf algebra A the morphism $\theta(A)$ is the composite of $P(A) \to I(A)$ with $I(A) \to Q(A)$. Suppose that R is an integral domain of characteristic zero and A has no R-torsion. If A is commutative and connected, then $\theta(A)$ is a monomorphism. For R a field see [4, Prop. 4.17], and in general consider the following commutative diagram where F is the field of fractions of R and $PA \to P(A \otimes F)$ is a monomorphism.

$$\begin{array}{ccc} P(A) & \xrightarrow{\theta(A)} & Q(A) \\ \downarrow & & \downarrow \\ P(A \otimes F) & \xrightarrow[\theta(A \otimes F)]{} & Q(A \otimes F) \end{array}$$

Since $\theta(A \otimes F)$ is a monomorphism, $\theta(A)$ is also a monomorphism.

3. Applications of universal properties. Let $\mathcal{A}(R)$, $\mathcal{C}(R)$, and $Gr^c(R)$ denote respectively the categories of commutative connected algebras, commutative connected coalgebras, and connected modules respectively over R. In order to study the structure and duality properties of functors S_n and S_n', we use some of the general properties of these three categories and of the functors S, S', and I which are all developed at length in [3].

By a degreewise construction, $Gr^c(R)$ is a category with (arbitrary) products and coproducts (also called direct sums), and it is also an abelian category.

(3.1) Proposition. *The categories $\mathcal{A}(R)$ and $\mathcal{C}(R)$ both have products and coproducts.*

Proof. Let $A(j)$ $(j \in J)$ be a family of algebras in $\mathcal{A}(R)$, and let

$C(j)$ $(j \in J)$ be a family of coalgebras in $\mathscr{C}(R)$. The product $\prod_{j \in J} A(j)$ as a graded module with projections is given by the relation

$$I(\prod_{j \in J} A(j)) = \prod_{j \in J} I(A(j))$$

in $Gr^c(R)$. The coproduct $\coprod_{j \in J} C(j)$ is constructed using the relation $I(\coprod_{j \in J} C(j)) = \coprod_{j \in J} I(C(j))$ in $Gr^c(R)$. The algebraic structure on $\prod_{j \in J} A(j)$ is determined by the reqnirement that the projections are morphisms and the coalgebra structure on $\coprod_{j \in J} C(j)$ by the requirement that the injections are morphisms. The universal properties are easily verified.

Assume J is finite. Then $\bigotimes_{j \in J} A(j)$ is the coproduct of the $A(j)$ with injections $A(i) \to \bigotimes_{j \in J} A(j)$ given by tensoring $A(i)$ with $\bigotimes_{j \neq i} (R \to A(j))$ and $\bigotimes_{j \in J} C(j)$ is the product of the $C(j)$ with projections $\bigotimes_{j \in J} C(j) \to C(i)$ given by tensoring $C(i)$ with $\bigotimes_{j \neq i} (C(j) \to R)$. To check the universal property in the coalgebra case, consider a family of morphisms $f_j \colon D \to C(j)$. Then $f \colon D \to \bigotimes_{j \in J} C(j)$ which is the composite of the iterated comultiplication $\Delta \colon D \to \bigotimes_{j \in J} D$ and $\bigotimes_{j \in J} f_j \colon \bigotimes_{j \in J} D \to \bigotimes_{j \in J} C(j)$ is the unique morphism which composed with the projection onto $C(i)$ is f_i. Observe Δ is a morphism of coalgebras because D is commutative.

For general J we have

$$\coprod_{j \in J} A(j) = \varinjlim_{J' \text{ finite}, J' \in J} \bigotimes_{j \in J} A(j)$$

and

$$\prod_{j \in J} C(j) = \varprojlim_{J' \text{ finite}, J' \subset J} \bigotimes_{j \in J} C(j)$$

as modules with the evident algebra and coalgebra structures given using

$$\bigotimes_{j \in J'} A(j) \to \bigotimes_{j \in J''} A(j) \to \bigotimes_{j \in J'} A(j)$$

and

$$\bigotimes_{j \in J'} C(j) \to \bigotimes_{j \in J''} C(j) \to \bigotimes_{j \in J'} C(j)$$

for $J' \subset J''$ finite and contained in J. This proves the proposition.

(3.2) PROPOSITION. *Let $M(j)$ $(j \in J)$ be a family of modules in $Gr^c(R)$. Then it follows that*

$$S(\coprod_{j\in J} M(j)) = \coprod_{j\in J} S(M(j)) \qquad S'(\prod_{j\in J} M(j)) = \prod_{j\in J} S'(M(j)).$$

Proof. The universal property (2.1)(c) just says that the adjoint relation $S \dashv I$ holds where $S\colon Gr^c(R) \to \mathcal{A}(R)$, and hence S preserves coproducts by generalities on adjoint functors. The universal property (2.2)(c) says that the adjoint relation $I \dashv S'$ holds where $S'\colon Gr^c(R) \to \mathcal{B}(R)$, and hence S' preserves products by generalities on adjoint functors. The proposition could be also be proved directly.

Thus for J finite we have

$$S(\coprod_{j\in J} M(j)) = \bigotimes_{j\in J} S(M(j)) \text{ and } S'(\coprod_{j\in J} M(j)) = \bigotimes_{j\in J} S'(M(j)).$$

From these preliminaries we can deduce the main theorem on the module structure of $S(M)$ and $S'(M)$. A graded module M is free provided it is a coproduct of $[x, d]$ and projective provided it is a summand of a free module. A graded module M is *even* provided $M_m = 0$ for m odd.

(3.3) THEOREM. *Let M be a graded module over a ring R such that either M is even, R has no 2-torsion, or R is an $\mathbf{F}_2$-algebra, i.e. $2R = 0$. If M is free (resp. projective), then $S'(M)$ and each $S_n'(M)$ are free (resp. projective) modules. If M is free (resp. projective) of finite type, then each $S_n'(M)$ is free (resp. projective) of finite type, and if, in addition, M is connected, then $S'(M)$ is a free (resp. projective) module of finite type.*

Proof. From the relation $S'(M \oplus N) = S'(M) \otimes S'(N)$ and the fact that $S_n'(M)$ is a direct summand of $S'(M)$ it suffices to prove the statements referring to the free module $M = \coprod_{j\in J} [x_j, d_j]$. For J finite $S'(M) = \bigotimes_{j\in J} S'[x_j, d_j]$ by (3.2) and we use (1.1), that is, $S'[x_j, d_j]$ is free. For J infinite, $T_n(M) = \lim_{\to J'} T_n(\coprod_{j\in J'} [x_j, d_j])$, and from the exactness of $\varinjlim$, it follows that $S_n'(M) = \lim_{\to J'} S_n(\coprod_{j\in J'} [x_j, d_j])$ where J' ranges over the finite subsets of J. The theorem now follows easily.

(3.4) THEOREM. *Let M be a graded module over a ring R such that either M is even, R is a $\mathbf{Z}[1/2]$-algebra, or R is $\mathbf{F}_2$-algebra. If M is free (resp. projective), then $S(M)$ and each $S_n(M)$ are free (resp. projective) modules. If M is free (resp. projective) of finite type, then each $S_n(M)$ is free (resp. projective) of finite type, and if, in addition, M is connected, then $S(M)$ is a free (resp. projective) module of finite type.*

Proof. From the relation $S(M \oplus N) = S(M) \otimes S(N)$ and the fact that

$S_n(M)$ is a direct summand of $S(M)$ it suffices to prove the statements referring to the free module $M = \coprod_{j \in J} [x_j, d_j]$. By (3.2) we have $S(M) = \bigotimes_{j \in J} S[x_j, d_j]$, and we use that $S[x_j, d_j]$ is free, see (1.1). The general tensor product, which is just $\varinjlim$ of the finite tensor products, of free modules is again a free module. The theorem follows now easily.

(3.5) THEOREM. *Let M be a graded module over a ring R such that either M is even, R is a $\mathbf{Z}[1/2]$-algebra, or R is a $\mathbf{F}_2$-algebra. If M is projective of finite type, then transpose of $T_n(M) \to S_n(M)$ is isomorphic to $S_n'(M^\vee) \to S_n(M^\vee)$ and the transpose of $S_n'(M) \to T_n(M)$ is isomorphic to $T_n(M^\vee) \to S_n(M^\vee)$. If M is connected and projective of finite type, then the dual coalgebra of the algebra $S(M)$ is $S'(M^\vee)$ and the dual algebra of the coalgebra $S'(M)$ is $S(M^\vee)$.*

Proof. The second statement is a straight forward consequence of the first one. We have two exact sequences

$$\coprod_{\sigma \in S(n)} T_n(M) \xrightarrow{\beta_n} T_n(M) \to S_n(M) \to 0$$

and

$$0 \to S'(M) \to T_n(M) \xrightarrow{\alpha_n} \prod_{\sigma \in S(n)} T_n(M)$$

defining $S_n(M)$ and $S_n'(M)$ where $S(n)$ is the symmetric group on n letters and the restriction of β_n or the projection of α_n to the factor σ is $\sigma - 1 \colon T_n(M) \to T_n(M)$. For $\sigma \in S(n)$ acting on $T_n(M)$ the transpose is σ^{-1} acting on $T_n(M^\vee) = T_n(M)^\vee$. Thus under our hypothesis of projective of finite type the two exact sequences are dual to each other. This proves the theorem.

(3.6) *Remark*. Let $R \to R'$ be a morphism of rings, i.e. view R' as an R-algebra. The change of coefficient (or change of base) functor $M \to R' \otimes_R M$ is defined $Gr^+(R) \to Gr^+(R')$. It induces functors $\mathcal{A}(R) \to \mathcal{A}(R')$, $\mathcal{C}(R) \to \mathcal{C}(R')$ and from Hopf algebras over R to Hopf algebras over R'. From the canonical isomorphisms

$$R' \otimes_R T_n(M) = T_n(R' \otimes_R M) \text{ and } R' \otimes_R T(M) = T(R' \otimes_R M)$$

we have the commutativity of the following diagram by passing to quotients

$$\begin{array}{ccc} Gr^c(R) & \longrightarrow & Gr^c(R') \\ S \downarrow & & \downarrow S \\ \mathcal{A}(R) & \longrightarrow & \mathcal{A}(R'). \end{array}$$

On the other hand such a general commutativity assertion does not hold for S'. If $R \to R'$ is a morphism of either $\boldsymbol{Z}[1/2]$-algebras or $\boldsymbol{F}_2$-algebras, then the following diagrams of functors are commutative up to natural isomorphism

$$\begin{array}{ccc} Gr^c(R) & \longrightarrow & Gr^c(R') \\ S' \downarrow & & \downarrow S' \\ \mathcal{B}(R) & \longrightarrow & \mathcal{B}(R'). \end{array}$$

The horizontal arrows in both diagrams are defined by change of coefficients.

(3.7) *Remarks.* The algebra $S(M)$ has a natural Hopf algebra structure induced by viewing M as the zero coalgebra, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in M$ or $P(M) = M$. Then $\theta\colon PS(M) \to QS(M) = M$ is an epimorphism, see also (2.1)(e) for M of rank 1. Similarly the coalgebra $S'(M)$ has a natural Hopf algebra structure induced by viewing M as the zero algebra, i.e. $xy = 0$ for all $x, y \in M$ or $Q(M) = M$. Then $\theta\colon M = PS'(M) \to QS'(M)$ is a monomorphism, see also (2.2)(e) for M of rank 1. When M has rank 1 we see that $S'(M)$ is an algebra with divided power operations $\gamma_m(x)$. From the tensor product theorem [1, exposé 7, théorème 2, p. 7-04] and (3.2) we see that $S'(M)$ has divided power operations for a free module M. In fact for M a free module, $S'(M)$ is just the universal algebra $U(M)$ with divided powers, see [1, pp. from 8-06 to 8-09].

4. The Hopf algebra $\boldsymbol{B[x, d]}$. Recall from [5; pp. 10-05, 10-06] the universal bicommutative Hopf algebras $S(S'(M))$ defined over the coalgebra $S'(M)$ and $S'(S(M))$ defined over the algebra $S(M)$ are related by a unique morphism of Hopf algebras $\lambda\colon S(S'(M)) \to S'(S(M))$ such that the following diagram is commutative.

$$\begin{array}{ccc} S'(M) & \xrightarrow{\;S'(\beta(M))\;} & S'(S(M)) \\ \beta(S'(M)) \downarrow & \nearrow \lambda & \downarrow \alpha(S(M)) \\ S(S'(M)) & \xrightarrow{\;S(\alpha(M))\;} & S(M). \end{array}$$

The morphism λ is constructed from either the universal property of $\beta(S'(M))$ or of $\alpha(S(M))$.

Let M be a graded module over a ring R such that either M is even, R is a $\boldsymbol{Z}[1/2]$-algebra, or R is a $\boldsymbol{F}_2$-algebra. Then the transpose of $\lambda: S(S'(M)) \to S'(S(M))$ is

$$S'(S(M))^{\vee} = S(S'(M^{\vee})) \xrightarrow{\lambda} S'(S(M^{\vee})) = S(S'(M))^{\vee}$$

by Theorem (3.5).

(4.1) Theorem. *Let $d \in \boldsymbol{Z}$ with $1 \leqq d$ such that either d is even, R is a $\boldsymbol{Z}[1/2]$-algebra, or R is a $\boldsymbol{F}_2$-algebra. Then $\lambda: S(S'[x,d]) \to S'(S[x,d])$ is an isomorphism.*

Proof. The case where d is even is contained in [5; Theorem 1]. The case where d is odd and R is an $\boldsymbol{F}_2$-algebra follows by the same argument as for d even. When d is odd and R a $\boldsymbol{Z}[1/2]$-algebra reduces to the observation that $S(S'[x,d])$ and $S'(S[x,d])$ are each the exterior Hopf algebras on one generator x with $x^2 = 0$ and x primitive together with the fact that $\beta(S'(M))$, $S'(\beta(M))$, $\alpha(S(M))$, and $S(\alpha(M))$ are all isomorphisms. This proves the theorem.

Now we study $\lambda: S(S'[x,d]) \to S'(S[x,d])$ in the case where d is even or R is an $\boldsymbol{F}_2$-algebra and denote this Hopf algebra by $B[x,d]$. We have a morphism $\beta: S'[x,d] \to B[x,d]$ of coalgebras and $\alpha: B[x,d] \to S[x,d]$ of algebras with universal properties given in (2.1)(d) and (2.2)(d) respectively.

In terms of the element x there is a unique basis of $S'[x,d]$ consisting of $c_i(x) \in S'[x,d]_{id}$ such that $c_1(x) = x$ and $\Delta(c_m(x)) = \sum_{i+j=m} c_i(x) \otimes c_j(x)$. For $S[x,d]$ the powers $x^i \in S[x,d]_{id}$ form a base. Thus the image $\beta(c_i(x))$ in $B[x,d]$, also denoted $c_i(x)$, form a family of elements whose image in $QB[x,d]$ is a base for $1 \leqq i$ and there is a family of primitive elements $b_i(x) \in PB[x,d]_{id}$ such that $\alpha(b_i(x)) = x^i$ which form a basis of $PB[x,d]$. In [5; p. 10-06] we find a relation between $c_i(x)$ and $b_i(x)$ (with different notation). This we state again and give a new proof the result which is motivated by the definition of the Adams operations in K-theory.

(4.2) Proposition. *The natural basis $(b_i(x))$ of the primitive elements $PB[x,d]$ is related to the natural basis $(c_i(x))$ of the indecomposable elements $QB[x,d]$ by the Newton relations:*

$$b_n(x) = \sum_{0 \leqq j < n} (-1)^{j+1} c_j(x) b_{n-j}(x) + (-1)^{n+1} n c_n(x) \qquad (n > 0).$$

Proof. Introduce an indeterminant t of degree $-d$ and consider the algebras of formula series $B[x,d][[t]]$ and $(B[x,d]\otimes B[x,d])[[t]]$ and the morphism induced by Δ also denoted

$$\Delta\colon B[x,d][[t]]\to(B[x,d]\otimes B[x,d])[[t]].$$

For the element $c_t(x)=\sum_{0\leqq i}c_i(x)t^i$ with $c_0(x)=1$ we have the relation $\Delta c_t(x)=c_t(x)\otimes c_t(x)$. Now we introduce the element $b_t(x)=\sum_{1\leqq i}b^*{}_i(x)t^i$ by the requirement that

$$b^*{}_{-t}(x)=-t(\frac{d}{dt}c_t(x))c_t(x)^{-1}$$

Since $b^*{}_{-t}(x)$ is a logarithmic derivative, it follows that

$$\Delta b_t(x)=b^*{}_t(x)\otimes 1+1\otimes b^*{}_t(t)$$

by calculus and the coefficients $b^*{}_i(x)$ of $b^*{}_t(x)$ are primitive of degree id.

From the relation

$$(\sum_{1\leqq i}b^*{}_i(x)(-t)^i)(\sum_{0\leqq j}c_j(x)t^j)=-\sum_{1\leqq n}nc_n(x)t^n$$

we derive the Newton relations for $b^*{}_i(x)$ by equating coefficients of t^n in the form

$$(-1)^n b^*{}_n(x)+\sum_{0<j<n}(-1)^{n-j}b^*{}_{n-j}(x)c_j(x)=-nc_n(x).$$

Since $\alpha\colon B[x,d]\to S[x,d]$ is a morphism of algebras with $\alpha(c_m(x))=0$ for $m>1$ and $\alpha(c_1(x))=\alpha(x)=x$, it follows from the Newton relations that $\alpha(b^*{}_m(x))=x\alpha(b^*{}_{m-1}(x))$. Since $b^*{}_1(x)=x$, we have $\alpha(b^*{}_m(x))=x^m=\alpha(b_m(x))$ by induction on m and the definition of $b_m(x)$. But $\alpha\mid PB[x,d]$ is injective. Thus $b_m(x)=b^*{}_m(x)$ and the Newton relations hold for $b_m(x)$. This proves the proposition.

The following theorem, which is the basis for our analysis, is an immediate consequence of (4.2) since $b_n(x)$ is a free generator of $PB[x,d]_{nd}$, $c_n(x)$ or $QB[x,d]_{nd}$, and $\theta(b_n(x))=(-1)^{n+1}nc_n(x)$ in $QB[x,d]_{nd}$.

(4.3) THEOREM. *For d even or R an $\mathbf{F}_2$-algebra the following sequence is exact.*

$$0\to\operatorname{Tor}(R,\mathbf{Z}/n\mathbf{Z})\to PB[x,d]_{nd}\xrightarrow{\theta}QB[x,d]_{nd}\to R\otimes\mathbf{Z}/n\mathbf{Z}\to 0$$

(4.4) PROPOSITION. *The Hopf algebra $B[x,d]_{nd}$ is isomorphic to its dual.*

Proof. The dual of the isomorphism $\lambda\colon S(S'[x,d]) \to S'(S[x,d])$ is again the same isomorphism for the module $[x,d]$.

(4.5) *Remark.* Since $B[x,d]$ is a polynomial algebra on one generator in each degree md, it follows that the Euler-Poincaré series for $B[x,d]$ is $\prod_{i\leqq m}(1-t^{md})^{-1}$.

(4.6) *Examples.* Over any ring R of coefficients $H_*(BU;R)$ (and $H^*(BU;R)) = B[x,2]$ where x is the first universal chern class, $H_*(BS_p;R)$ (and $H^*(BS_p;R)) = B[x,4]$ where x is the first universal symplectic Pontrjagin class. Over a $\boldsymbol{Z}[1/2]$-algebra R of coefficients $H_*(BO;R)$ (and $H^*(BO;R)) = B[x,4]$ where x is the first universal Pontrjagin class. Over an $\boldsymbol{F}_2$-algebra R of coefficients $H_*(BO;R)) = B[x,1]$ where x is the first universal Stiefel-Whitney class.

5. Comparison of $B[x,d]$ and $B[x,rd]$. In this section, d is even or or R is an $\boldsymbol{F}_2$-algebra, and r is a natural number with $1 \leqq r$.

There exists a unique morphism of algebras $\bar{f}_r\colon S[y,rd] \to S[x,d]$ such that $\bar{f}_r(y^i) = x^{ir}$ and a unique morphism of coalgebras $\bar{g}_r\colon S'[x,d] \to S'[y,rd]$ such that $\bar{g}_r(\gamma_{ir}(x)) = \gamma_i(y)$ and $\bar{g}_r(\gamma_j(x)) = 0$ if r does not divide j.

(5.1) PROPOSITION. *There exist unique morphisms of Hopf algebras $f_r\colon B[y,rd] \to B[x,d]$ and $g_r\colon B[x,d] \to B[y,rd]$ such that the following diagrams are commutative*

$$\begin{array}{ccc} B[y,rd] & \xrightarrow{f_r} & B[x,d] \\ \alpha\downarrow & & \downarrow\alpha \\ S[y,rd] & \xrightarrow{\bar{f}_r} & S[x,d] \end{array} \qquad \begin{array}{ccc} S'[x,d] & \xrightarrow{\bar{g}_r} & S'[y,rd] \\ \beta\downarrow & & \downarrow\beta \\ B[x,d] & \xrightarrow{g_r} & B[y,rd] \end{array}$$

in the categories of algebras and coalgebras respectively. Moreover, $P(f_r)$ is a monomorphism with $f_r(b_i(y)) \to b_{ir}(x)$, thus an isomorphism in degrees ird, and $Q(g_r)$ is an epimorphism which is an isomorphism in degrees ird.

Proof. See also [5; Prop. 10]. Observe the existence and uniqueness of f_r and g_r follow from the biuniversal character of $B[x,d]$. The other assertions are either immediate or follow from $\alpha(b_i(y)) = y^i$.

Now in degrees mrd we have the following two morphisms of exact sequences:

$$\begin{array}{ccccccccc}
0 \to \mathrm{Tor}(R, \mathbf{Z}/m\mathbf{Z}) & \to & PB[y, rd]_{mrd} & \xrightarrow{\theta} & QB[y, r]_{mrd} & \to & R \otimes \mathbf{Z}/m\mathbf{Z} \to 0 \\
\downarrow & & \wr \downarrow P(f_r) & & \downarrow Q(f_r) & & \downarrow \\
0 \to \mathrm{Tor}(R, \mathbf{Z}/rm\mathbf{Z}) & \to & PB[x, d]_{mrd} & \xrightarrow{\theta} & QB[x, d]_{mrd} & \to & R \otimes \mathbf{Z}/rm\mathbf{Z} \to 0
\end{array}$$

where the first and fourth vertical arrows are induced by the inclusion $\mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/mr\mathbf{Z}$ and

$$\begin{array}{ccccccccc}
0 \to \mathrm{Tor}(R, \mathbf{Z}/mr\mathbf{Z}) & \to & PB[x, d]_{mrd} & \xrightarrow{\theta} & QB[x, d]_{mrd} & \to & R \otimes \mathbf{Z}/mr\mathbf{Z} \to 0 \\
\downarrow & & \downarrow P(g_r) & & \wr \downarrow Q(g_r) & & \downarrow \\
0 \to \mathrm{Tor}(R, \mathbf{Z}/m\mathbf{Z}) & \to & PB[y, rd]_{mrd} & \xrightarrow{\theta} & QB[y, r]_{mrd} & \to & R \otimes \mathbf{Z}/m\mathbf{Z} \to 0
\end{array}$$

where the first and fourth vertical arrows are induced by the quotient $\mathbf{Z}/mr\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z}$. To calculate $P(g_r)$ and $Q(f_r)$, we observe that over $\mathbf{Z}$ it is multiplication by r and this result holds over a ring R since $B[x, d]_R$ over R is $B[x, d] \otimes_{\mathbf{Z}} R$.

(5.2) Proposition. *In degrees mrd for $m \geqq 1$, it follows that $P(f_r)$ and $Q(g_r)$ are isomorphisms and $P(g_r)$ and $Q(f_r)$ are multiplications by r. If R is a $\mathbf{Z}[1/r]$-algebra, then the composite of $f_r\colon B[y, rd] \to B[x, d]$ and $g_r\colon B[x, d] \to B[y, rd]$ is the identity on $B[y, rd]$.*

Proof. The first statement follows from the above remarks, and for the second, observe that $P(g_r f_r)$ and $Q(g_r f_r)$ are isomorphisms and by (2.3) and (2.4) it follows that $g_r f_r$ is an isomorphism. Since $g_r f_r(y) = y$, one induces that $g_r f_r$ is the identity on each $c_i(y)$, and hence, on $B[y, rd]$.

(5.3) *Remark.* If R is a $\mathbf{Z}[1/r]$-algebra, then the following diagram is commutative for a unique morphism $h\colon \ker(g_r) \to \mathrm{coker}(f_r)$, and in addition, h is an isomorphism

$$\begin{array}{ccccc}
\ker(g_r) & & \xrightarrow{\quad h \quad} & & \mathrm{coker}(f_r) \\
& \searrow & & \nearrow & \\
& & B[x, d] & & \\
& f_r \nearrow & & \searrow g_r & \\
B[y, rd] & & \xrightarrow[\quad 1 \quad]{} & & B[y, rd].
\end{array}$$

These morphisms define isomorphisms $B[y, rd] \otimes \ker(g_r) \to B[x, d]$ and $B[x, d] \to B[y, rd] \otimes \operatorname{coker}(f_r)$ which split $B[x, d]$. The Euler-Poincaré series for $\ker(g_r) \cong \operatorname{coker}(f_r)$ is given by $\prod_{1 \leqq i, r \nmid i} (1 - t^{id})^{-1}$.

6. Decomposition of $B[x, d]$ over $Z_{(p)}$-algebras. Let $Z_{(p)}$ denote the ring Z localized at the prime ideal $(p) = Zp$ for a prime number p. In this section R will denote $Z_{(p)}$-algebra, that is, a ring such that all n prime to p are invertible in R.

(6.1) PROPOSITION. *In degree m where $pd \nmid m$ the morphism*

$$\theta \colon PB[x, d]_m \to QB[x, d]_m$$

is an isomorphism and for $m = np^i d$ where $p \nmid n$ the morphism θ leads to the following exact sequence.

$$0 \to \operatorname{Tor}(R, Z/p^i Z) \to PB[x, d]_m \xrightarrow{\theta} QB[x, d]_m \to R \otimes (Z/p^i Z) \to 0$$

Proof. This proposition follows from (4.3) and the fact that for a $Z_{(p)}$-algebra R the quotient morphism $Z/np^i Z \to Z/p^i Z$ induces isomorphisms $R \otimes (Z/np^i Z) \to R \otimes (Z/p^i Z)$ and $\operatorname{Tor}(R, Z/np^i Z) \to \operatorname{Tor}(R, Z/p^i Z)$.

Now consider the following diagram where $fq_l = f_l$ and $p_l g = g_l$ are defined in (5.1) and $w = vu$.

$$\begin{array}{ccccc} \bigotimes_{\substack{p \neq l \\ p \text{ prime}}} B[y_l, ld] & \longrightarrow & B[x, d] & \longrightarrow & \bigotimes_{\substack{p \neq l \\ p \text{ prime}}} B[y_l, ld] \\ & u \nearrow & & \searrow v & \\ & \ker(g) & \xrightarrow{w} & \operatorname{coker}(f) & \end{array}$$

(6.2) PROPOSITION. *With the above notations for $m = p^i d$ the morphisms*

$$P(u)_m \colon P \ker(g)_m \to PB[x, d]_m, \quad P(v)_m \colon PB[x, d]_m \to P \operatorname{coker}(f)_m$$

$$Q(u)_m \colon Q \ker(g_m) \to QB[x, d]_m, \quad Q(v)_m \colon QB[x, d]_m \to Q \operatorname{coker}(f)_m$$

are isomorphisms and if m does not have the form $p^i d$ for some $i \geqq 0$, then $P \ker(g)_m = P \operatorname{coker}(f)_m = Q \ker(g)_m = Q \operatorname{coker}(f)_m = 0$. The morphism w is an isomorphism of Hopf algebras.

Proof. We have the following exact sequences since P preserves kernels and Q preserves cokernels, see [4; 3.12 and 3.11]:

$$0 \to P \ker(g)_m \to PB[x, d]_m \to P\Big(\bigotimes_{l \neq p} B[y_l, ld]\Big) = \coprod_{l \neq p} PB[y_l, ld]$$

and

$$\coprod_{l \neq p} QB[y_l, ld] = Q(\bigotimes_{l \neq p} B[y_l, ld]) \to QB[x, d]_m \to Q\,\mathrm{coker}(f)_m \to 0$$

from (5.2) it follows that the projections $PB[x,d]_{nd} \to PB[y_l, ld]_{nd}$ and $QB[y_l, ld]_{nd} \to QB[x,d]_{nd}$ are isomorphisms. Thus $P\ker(g)_m = 0$ and $Q\,\mathrm{coker}(f)_m = 0$ except for $m = p^i d$. In the case $m = p^i d$ the sum terms are 0 and it follows that $P\ker(g)_m \to PB[x,d]_m$ and $QB[x,d]_m \to Q\,\mathrm{coker}(f)_m$ are isomorphisms.

For $m = p^i d$ consider the following commutative diagram

$$\begin{array}{ccccc} P(\ker(g))_m & \xrightarrow[\sim]{P(u)_m} & PB[x,d]_m & \xrightarrow{P(v)_m} & P(\mathrm{coker}(f))_m \\ \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\ Q(\ker(g))_m & \xrightarrow{Q(u)_m} & QB[x,d]_m & \xrightarrow[\sim]{Q(v)_m} & Q(\mathrm{coker}(f))_m. \end{array}$$

Over $\boldsymbol{Z}_{(p)}$ both $\theta\colon PB[x,d]_m \to QB[x,d]_m$ and $Q(v)\theta P(u)$ are multiplication by p^i. Thus $P(v)$ and $Q(u)$ must be isomorphism. Since $P(w) = P(v)P(u)$ and $Q(w) = Q(v)Q(u)$ are isomorphisms, it follows by the remarks (2.3) and (2.4) that w is an isomorphism over $\boldsymbol{Z}_{(p)}$. Since the general case of a $\boldsymbol{Z}_{(p)}$-algebra R results by tensoring with R over $\boldsymbol{Z}_{(p)}$, this proves the proposition.

(6.3) *Notation.* For a $\boldsymbol{Z}_{(p)}$-algebra R we denote by $B_{(p)}[x,d]$ the Hopf algebra $\ker(g)$. Then we have the inclusion morphism $u\colon B_{(p)}[x,d] \to B[x,d]$ and the projection $w^{-1}v$, which we denote v, $B[x,d] \to B_{(p)}[x,d]$ such that vu is the identity on $B_{(p)}[x,d]$.

Observe that $(b_{p^i}(x))_{0 \leqq i}$ is a basis for $PB_{(p)}[x,d]$ with $b_{p^i}(x)$ in degree $p^i d$ and the images of $c_{p^i}(x)$ in $QB_{(p)}[x,d]$ form a basis for $0 \leqq i$.

(6.4) *Remark.* Putting together the morphisms considered in (6.3) and in (5.1), we have

$$B_{(p)}[x, rd] \xrightarrow{f_r u} B[x,d] \xrightarrow{vg_r} B_{(p)}[x, rd]$$

such that $(vg_r)(f_r p) = 1$. These are used in the next theorem for the splitting of $B[x,d]$.

(6.5) THEOREM. *Consider the morphisms* Φ *and* ψ *defined*

$$\bigotimes B_{(p)}[x_r, rd] \xrightarrow{\phi} B[x,d] \xrightarrow{\psi} \bigotimes_{1 \leqq r, p \nmid r} B_{(p)}[x_r, d]$$

such that ϕ *restricted to the factor* $B_{(p)}[x_r, d]$ *is* $f_r u$ *and* ψ *projected to the factor* $B_{(p)}[x_r, rd]$ *is* vg_r. *Then* ϕ *and* ψ *are isomorphisms inverse to each other.*

Proof. Since $(vg_r)(f_s u)$ is the identity for $r = s$ and zero (except in degree 0 on the unit), it follows that $\psi\phi =$ identity. By (6.2) and (5.2) it follows that $P(\phi)$ and $Q(\phi)$ are isomorphisms, and thus ϕ is an isomorphism which proves the theorem.

7. Properties of the algebras $\boldsymbol{B}_{(p)}[\boldsymbol{x}, \boldsymbol{d}]$. As in the previous section, R will denote a $\boldsymbol{Z}_{(p)}$-algebra and $d \geqq 1$ an integer which is even except possibly when R is an $\boldsymbol{F}_2$-algebra.

(7.1) PROPOSITION. *As an algebra* $B_{(p)}[x, d]$ *is a projective in the category* $\mathfrak{A}(R)$, *that is a graded polynomial algebra, with one indecomposable element in each degree of the form* $p^i d$. *As a coalgebra* $B_{(p)}[x, d]$ *is an injective in the category* $\mathfrak{C}(R)$ *with one primitive element in each degree of the form* $p^i d$.

Proof. The statements are true for $B[x, d]$ and $B[x, d] = B_{(p)}[x, d] \otimes B$ as Hopf algebras. Thus they hold for $B_{(p)}[x, d]$ with the structure of $PB_{(p)}[x, d]$ and $QB_{(p)}[x, d]$ given in (6.2).

(7.2) PROPOSITION. *The Hopf algebra* $B_{(p)}[x, d]$ *is isomorphic to its dual.*

Proof. The statement is true for $B[x, d]$ by (4.5). The dual of the diagram preceding (6.2) and defining $B_{(p)}[x, d]$ is isomorphic to the same diagram which defines $B_{(p)}[x, d]$.

(7.3) *Remark.* Since $B_{(p)}[x, d]$ is a polynomial algebra on one generator in each degree $p^i d$, it follows that the Euler-Poincaré series for $B_{(p)}[x, d]$ is $\prod_{0 \leqq i} (1 - t^{p^i d})^{-1}$.

(7.4) *Remarks.* By composing $\beta \colon S'[x, d] \to B[x, d]$ with the projection $B[x, d] \to B_{(p)}[x, d]$, we have a morphism of coalgebras $\beta_{(p)} \colon S'[x, d] \to B_{(p)}[x, d]$ such that for each morphism $f \colon S'[x, d] \to A$ of coalgebras where A is bicommutative Hopf algebra over R, there exists a unique morphism of Hopf algebras $g \colon B_{(p)}[x, d] \to A$ such that $g\beta_{(p)} = f$. The universal factorization property follows from (2.1)(d). Similarly, by composing $\alpha \colon B[x, d] \to S[x, d]$ with the injection $B_{(p)}[x, d] \to B[x, d]$, we have a morphism of algebras $\alpha_{(p)} \colon B_{(p)}[x, d] \to S[x, d]$ such that for each morphism $f \colon A \to S[x, d]$ of algebras where A is a bicommutative Hopf algebra over R, there exists a unique morphism of Hopf algebras $g\colon A \to B_{(p)}[x, d]$ such that $\alpha_{(p)} g = f$.

(7.5) *Remarks.* The morphisms $f_r\colon B[y, rd] \to B[x, d]$ and $g_r\colon B[x, d] \to B[y, rd]$ induce similar morphisms for $r = p^j$ such that the following diagram is commutative.

$$\begin{array}{ccccc}
B_{(p)}[y, p^j d] & \xrightarrow{u} & B[y, p^j d] & \xrightarrow{v} & B_{(p)}[y, p^j d] \\
\downarrow f_{(j)} & & \downarrow g_{p^j} & & \downarrow f_{(j)} \\
B_{(p)}[x, d] & \xrightarrow{u} & B[x, d] & \xrightarrow{v} & B_{(p)}[x, d] \\
\downarrow g_{(j)} & & \downarrow f_{p^j} & & \downarrow g_{(j)} \\
B_{(p)}[y, p^j d] & \xrightarrow{u} & B[y, p^j d] & \xrightarrow{v} & B_{(p)}[y, p^j d].
\end{array}$$

This can be seen immediately from the definitions of u and v in the paragraph before (6.2). Moreover by (5.2) and (6.2), it follows that

$$P(f_{(j)m}\colon PB_{(p)}[y, p^j d]_m \to PB_{(p)}[x, d]_m$$

is an isomorphism for $m = p^{j+a} d$ where $a \geqq 0$ and zero on the zero module otherwise, and similarly

$$Q(g_{(j)})_m\colon QB_{(p)}[x, d]_m \to QB_{(p)}[x, p^j d]_m$$

is an isomorphism for $m = p^{j+a}$ where $a \geqq 0$ and zero into the zero module otherwise.

Recall the notations for a basis of $PB_{(p)}[x, d]$; that is, $(b_{p^i}(x))_{0 \leqq i}$ where $b_{(i)}(x) = b_{p^i}(x)$ is the generator of $PB_{(p)}[x, d]$ in degree $p^i d$ such that $\alpha(b_{(i)}(x)) = x^{p^i}$ for $\alpha\colon B_{(p)}[x, d] \to S[x, d]$. Let $a_{(i)}(x) = \beta(c_{p^i}(x))$ for $\beta\colon S'[x, d] \to B_{(p)}[x, d]$. Then the image of $a_{(i)}(x)$ in $QB_{(p)}[x, d]_{p^i d}$ is a free generator. From (4.3) we have the following theorem.

(7.6) THEOREM. *For d even or R an $\mathbf{F}_2$-algebra, and, in addition a $\mathbf{Z}_{(p)}$-algebra structure on R, the following sequence is exact.*

$$0 \to \operatorname{Tor}(R, \mathbf{Z}_{(p)}/p^j \mathbf{Z}_{(p)}) \to PB_{(p)}[x, d]_{p^j d} \xrightarrow{\theta} QB_{(p)}[x, d]_{p^j d} \to R \otimes \mathbf{Z}_{(p)}/p^j \mathbf{Z}_{(p)} \to 0.$$

In the next theorem we relate the elements $a_{(i)}(x)$ and $b_{(i)}(x)$ just as in (4.2) the elements $c_n(x)$ and $b_n(x)$ were related. Observe that $a_{(i)}(x)$

and $c_{p^i}(x)$ are different for $i \geqq 1$ even though they determine the same element in $QB[x,d] \xrightarrow{\sim} QB_{(p)}[x,d]_m$ for $m = p^i d$.

8. Alternative parametrization of $B[x,d]$ and $B_{(p)}[x,d]$. Let $W[x,d]$ be the graded algebra over R with one polynomial generator $a_m(x)$ in each degree md. We define a morphism $\xi\colon B[x,d] \to W[x,d]$ by the requirement that in $W[x,d][[t]]$ the following relation holds

$$1 + \sum_{i \leqq m} \xi(c_m(x))t^m = \prod_{i \leqq m} (1 - a_m(x)t^m)$$

(8.1) PROPOSITION. *With the notations $\xi\colon B[x,d] \to W[x,d]$ is an isomorphism of algebras.*

Proof. Observe that

$$\xi(c_m(x)) = -a_m(x) + \text{a polynomial in } a_1(x), \cdots, a_{m-1}(x).$$

Thus $a_m(x) = -\xi(c_m(x)) +$ a polynomial in $c_1(x), \cdots, c_{m-1}(x)$ and ξ has an inverse.

With the isomorphism ξ, we transfer the coalgebra structure on $B[x,d]$ to $W[x,d]$ making $W[x,d]$ into a Hopf algebra. Recall that the primitive elements $PB[x,d]$ have a free base of $(b_m(x))_{1 \leqq m}$ where

$$\sum_{1 \leqq m} b_m(x)(-t)^m = -t\frac{d}{dt}\log(1 + \sum_{1 \leqq m} c_m(x)t^m).$$

(8.2) PROPOSITION. *The primitive elements $PW[x,d]$ have as a free base*

$$(-1)^m \xi(b_m(x)) = \sum_{d/m} da_d(x)^{m/d}.$$

Proof. Apply ξ to the above formula coefficientwise yielding in $W[x,d][[t]]$

$$\sum_{i \leqq n} (-1)^n \xi(b_n(x))t^n = -t \sum_{1 \leqq d} \frac{-da_d(x)t^{d-1}}{1 - a_d(x)t^d}$$
$$= \sum_{1 \leqq d} d[a_d(x)t^d + a_d(x)^2 t^{2d} + \cdots + a_d(x)^m t^{md} + \cdots]$$
$$= \sum_{1 \leqq n} (\sum_{i \leqq d} da_d(x)^{n/d})t^n.$$

This proves the formula.

Now assume again that R is a $\boldsymbol{Z}_{(p)}$-algebra and $d \geqq 1$ is an integer which is even except possibly when R is an $\boldsymbol{F}_2$-algebra. Thus the splitting

$$B_{(p)}[x,d] \to B[x,d] \to B_{(p)}[x,d]$$

is defined. Let $W_{(p)}[x, d]$ be the subalgebra of $W[x, d]$ with generators $a_m(x)$ for $m = p^i$, $0 \leqq i$ which is defined for all R.

(8.3) PROPOSITION. *For R a $\mathbf{Z}_{(p)}$-algebra $W_{(p)}[x, d]$ is a sub Hopf algebra of $W[x, d]$ and $\xi: B[x, d] \to W[x, d]$ restricts to an isomorphism $\xi: B_{(p)}[x, d] \to W_{(p)}[x, d]$ of Hopf algebras.*

Proof. The primitive element given by (8.2)

$$(-1)^{p^i}\xi(b_{p^i}(x)) = x^{p^i} + pa_p(x)^{p^{i-1}} + \cdots + p^i a_{p^i}(x)$$

is a member of $W_{(p)}[x, d]$, and hence $\xi(PB_{(p)}[x, d]) \subset W_{(p)}[x, d]$. Since over $\mathbf{Q} = \mathbf{Z}_{(p)}[1/p]$ the primitive elements $PB_{(p)}[x, d]$ generate $B_{(p)}[x, d]$, it follows that $\xi(B_{(p)}[x, d]) \subset W_{(p)}[x, d]$ over $\mathbf{Q}$. Since $B_{(p)}[x, d]$ and $W_{(p)}[x, d]$ are direct summands, $\xi(B_{(p)}[x, d]) \subset W_{(p)}[x, d]$ over any $\mathbf{Z}_{(p)}$-algebra R.

Now $Q(\xi): QB_{(p)}[x, d] \to QW_{(p)}[x, d]$ is an isomorphism, so that restriction $\xi: B_{(p)}[x, d] \to W_{(p)}[x, d]$ is an isomorphism of algebras. Since $\xi: B[x, d] \to W[x, d]$ is an isomorphism of Hopf algebras, $W_{(p)}[x, d]$ must be a subHopf algebra and the restriction of ξ to $B_{(p)}[x, d] \to W_{(p)}[x, d]$ is also an isomorphism of Hopf algebras.

(8.4) *Remark.* By transferring elements from $W_{(p)}[x, d]$ to $B_{(p)}[x, d]$, we observe that we have two families of element in $B_{(p)}[x, d]$ namely $(\alpha_{(i)}(x))$ and $(\beta_{(i)}(x))$ where $\beta_{(i)}(x) = (-1)^{p^i} b_{p^i}(x)$ and $\alpha_{(i)}(x) = \xi^{-1}(a_{p^i}(x))$ such that $(\beta_{(i)}(x))$ is a base for $PB_{(p)}[x, d]$ and the classes of $\alpha_{(i)}(x)$ are a base for $QB_{(p)}[x, d]$. Moreover, the primitive elements $\beta_{(i)}(x)$ are the "ghost" Witt vectors of $\alpha_{(i)}(x)$, i.e.

$$\beta_{(i)}(x) = w_{(i)}(\alpha_{(0)}(x), \cdots, \alpha_{(i)}(x))$$

where $w_{(i)}(x_0, \cdots, x_i) = x_0^{p^i} + px_1^{p^{i-1}} + \cdots + p^{i-1}x_{i-1}^p + p^i x_i$. Also the relations $x = \alpha_{(0)}(x) = \beta_{(0)}(x)$ hold. Finally observe that in the case where R is an $\mathbf{F}_p$-algebra, the primitive element $\beta_{(i)}(x) = x^{p^i}$. This can be seen also directly.

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