The homotopy of *MString* and $MU\langle 6\rangle$ at large primes

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We use Hopf rings to compute the homotopy rings $\pi_*MO\langle 8\rangle$ and $\pi_*MU\langle 6\rangle$ at primes > 3. In this case, the additive structure is well-known, but the ring structure is not polynomial. Instead, these rings are quotients of polynomial rings by infinite regular sequences.

55N22; 57R90, 55R45

Introduction

Recall that a *String manifold* is a Spin manifold M together with a trivialization of the class usually denoted $p_1(M)/2$; this is a characteristic class, defined only for Spin manifolds, so that twice it is the usual first Pontrjagin class. The bordism spectrum of String manifolds is called *MString* or $MO\langle 8\rangle$; it is the Thom spectrum of the map $BO\langle 8\rangle \to BO$ of the 7–connected cover of BO to BO. Similarly, $MU\langle 6\rangle$ is the bordism spectrum associated with the 5–connected cover $BU\langle 6\rangle \to BU$ of BU.

These spectra have received considerable attention because of their close connection with both topological modular forms (see Ando, Hopkins and Rezk [1] and Ando, Hopkins and Strickland [2]), and with string theory (see Witten [13]). In simple terms, the relation with string theory is explained by the fact that the space of strings LM on M, also known as the free loop space of M, is Spin, and so should have a Dirac operator, if and only if M is String.

It is well-known that the only primes p where $\pi_*MString$ and $\pi_*MU\langle 6\rangle$ can have p–torsion are $p=2$ and $p=3$. When $p\geq 5$, $MO\langle 8\rangle_{(p)}$ and $MU\langle 6\rangle_{(p)}$ are coproducts of suspensions of the Brown–Peterson spectrum BP, and so their homotopy groups are completely known. However, the ring structure of $\pi_*MO\langle 8\rangle_{(p)}$ and $\pi_*MU\langle 6\rangle_{(p)}$ is not known when $p\geq 5$. Pengelley and Ravenel worked on this in the 1980's and realized that these rings are NOT polynomial rings, but their work has never appeared.

The object of this paper is to compute these homotopy rings. We show that each of them is a BP_*–polynomial algebra divided by an infinite regular sequence, so they are generalized complete intersection BP_*–algebras.

Published: 20 December 2008 DOI: 10.2140/agt.2008.8.2401

Along the way, we compute the ring structure of $BP_*BO\langle 8\rangle$ and $BP_*BU\langle 6\rangle$ for all primes $p \geq 3$. We hope that this will be useful in a more comprehensive attack on $MO\langle 8\rangle$ at $p = 3$ than was undertaken by the author and Ravenel [6]. The idea for $p = 3$ is that $MO\langle 8\rangle_{(3)}$ and $MU\langle 6\rangle_{(3)}$ should be a coproduct of suspensions of BP and $BtmfP$, where $BtmfP$ is an amalgam of BP and the topological modular forms spectrum tmf, analogous to the spectrum BoP of Pengelley [10]. It is possible, and maybe even likely, that some other summands arise as well, arising from an amalgam of BP and $tmf \wedge C(\alpha)$, where $\alpha \in \pi_3 S$ is a nontrivial 3–torsion element.

We use the Hopf ring $BP_*BP\langle \mathbf{1}\rangle_*$ to compute $BP_*BO\langle 8\rangle$, where $BP\langle 1\rangle$ is the Johnson–Wilson spectrum, closely related to K–theory, whose homotopy is $\mathbb{Z}_{(p)}[v_1]$. However, we do not pursue a complete description of this Hopf ring. This seems like a good topic for further work. There has been much previous work on the Hopf rings $E_*\mathbf{ko}_*$ and $E_*BP\langle \mathbf{1}\rangle_*$ (which are closely related when p is odd) for various E. Dena Cowen Morton computes $H\mathbf{F}_{2*}\mathbf{ko}_*$ in [9]. Boardman, Kramer and Wilson compute $K(1)_*BP\langle \mathbf{1}\rangle_*$, among other things, in [3]. Kitchloo, Laures and Wilson compute $K(n)_*\mathbf{ko}_*$ when $p = 2$, as well as the completed BP–cohomology of these spaces, in [7].

Acknowledgements This paper is a long delayed outcome of my collaboration with Doug Ravenel on [6]. His notes with David Pengelley on $MO\langle 8\rangle$ led to our paper and have now led to this one. But I would never have finished this computation without inspirational conversations with Mike Hill.

Notation Throughout we let p be a prime and we use the usual convention that $q = 2p-2$.

1 $BP_*BP\langle 1\rangle_*$

In this section, we compute the ring structure of $BP_*BP\langle \mathbf{1}\rangle_n$ when $n \leq 2p+2$ and n is even. The reason for considering this is that, when p is odd,

$$BO\langle 8\rangle_{(p)} \cong BP\langle \mathbf{1}\rangle_8 \times BP\langle \mathbf{1}\rangle_{12} \times \cdots \times BP\langle \mathbf{1}\rangle_{2p+2},$$

as H–spaces (see Hovey and Ravenel [6, Corollary 1.5]). Similarly,

$$BU\langle 6\rangle_{(p)} \cong BP\langle \mathbf{1}\rangle_6 \times BP\langle \mathbf{1}\rangle_8 \times \cdots \times BP\langle \mathbf{1}\rangle_{2p+2},$$

as H–spaces, no matter what p is.

Since $BP_*BP\langle \mathbf{1}\rangle_n$ is a free BP_*–module for $n \leq 2p+2$ (see the discussion immediately preceding and immediately following Theorem 1.2), we have the following proposition.

Proposition 1.1 *The natural map*

$$BP_*BP\langle \mathbf{1}\rangle_8 \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP\langle \mathbf{1}\rangle_{2p+2} \to BP_*BO\langle 8\rangle,$$

where there is one tensor factor in each dimension divisible by 4 in the indicated range, is an isomorphism for p odd. Similarly, the natural map

$$BP_*BP\langle \mathbf{1}\rangle_6 \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP\langle \mathbf{1}\rangle_{2p+2} \to BP_*BU\langle 6\rangle,$$

where there is one tensor factor in each even dimension in the indicated range, is an isomorphism for all p.

From [12, Section 5], we know that $BP\langle \mathbf{1}\rangle_n$ is a factor of BP_n for $n \le 2p+2$. For $n < 2p+2$, this is true as H–spaces, but not when $n = 2p+2$. From [11], we know that BP_*BP_n is a polynomial algebra over BP_*.

We will need explicit generators for the part of this polynomial algebra that maps nontrivially to $BP_*BP\langle \mathbf{1}\rangle_{2p+2}$. The complex orientation gives a map $\mathbf{C}P^\infty \to BP_2$. The image under this map of (a consistent choice of) the generator in dimension $2i$ will be denoted $b_i \in BP_{2i}BP_2$, as will its image in $BP_*BP\langle \mathbf{1}\rangle_2$. The only indecomposable b_i are the $b_{(i)} = b_{p^i}$. We also have a map $S^0 \to BP_{-q}$ corresponding to the homotopy class v_1. The image of the generator under this map will be denoted $[v_1] \in BP_0BP_{-q}$, as will its image in $BP_0BP\langle \mathbf{1}\rangle_{-q}$. Similarly, we have elements $[v_1^i] \in BP_0BP\langle \mathbf{1}\rangle_{-qi}$.

We can then take circle and star products of these elements in the Hopf rings BP_*BP_* and $BP_*BP\langle \mathbf{1}\rangle_*$. Recall that the circle product corresponds to the ring spectrum structure and defines a map

$$BP_*BP_m \otimes BP_*BP_n \to BP_*BP_{n+m}$$

and similarly for $BP\langle \mathbf{1}\rangle_*$. The star product is just the loop space multiplication in BP_*BP_n. Ravenel and Wilson then show that $BP_*BP\langle \mathbf{1}\rangle_n$ is generated as an algebra over BP_* by elements of the form

$$[v_1^i] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots$$

such that

$$2\sum_k j_k - qi = n.$$

and if $i > 0$, then $j_k < p$ for all k. This element is in degree $2(\sum_k j_k p^k)$.

In particular, suppose $n \le q = 2p-2$. Then $j_k < p$ for all k. Let $\alpha(m)$ denote the sum of the digits in the p–adic expansion of m. Then for every positive dimension $2m$

with $\alpha(m) \equiv n/2 \pmod{p-1}$, there is a unique generator

$$x_{2m} = [v_1^i] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ b_{(k)}^{\circ j_k},$$

where $m = \sum j_i p^i$ is the p–adic expansion of m, and $i = (\alpha(m) - n/2)/(p-1)$. It is useful to note that $m \equiv \alpha(m) \pmod{p-1}$, so actually we have one generator in each positive even dimension that is congruent to $n \pmod{q}$.

When $n = 2p$, we have similar generators x_{2m}, but this time we have the additional condition that $\alpha(m) > 1$. Thus we lose the expected generator x_{2p^k}, but this is replaced by $y_{2p^{k+1}} = b_{(k)}^{\circ p}$. We therefore still have one generator in each positive dimension $2m$ with $2m \equiv 2 \pmod{q}$ except in dimension 2.

We know from [12, Corollary 5.1] that, for $n < 2p+2$, the p–local homology of $BP\langle \mathbf{1} \rangle_n$ is an evenly-graded, torsion-free, polynomial algebra with one generator in each dimension corresponding to $s^n v_1^k$ for $k \geq 0$. Therefore, the Atiyah–Hirzebruch spectral sequence collapses, and the same is true for $BP_*BP\langle \mathbf{1} \rangle_n$. We thus recover the following theorem.

Theorem 1.2 *If n is even and $n < 2p$, then $BP_*BP\langle \mathbf{1} \rangle_n$ is the polynomial algebra on the generators x_{2m} constructed above, where $2m \equiv n \pmod{q}$. If $n = 2p$, then $BP_*BP\langle \mathbf{1} \rangle_n$ is the polynomial algebra on the generators x_{2m}, where $2m \equiv 2 \pmod{q}$ and $\alpha(m) > 1$, together with the generators $y_{2p^{k+1}}$ for $k \geq 0$.*

When $n = 2p+2$, the situation is more complicated. It is still the case that all the generators are in even dimensions $\equiv 4 \pmod{q}$ (and greater than 4). However, there are two generators in some dimensions. In more detail, we have similar generators x_{2m} when $2m \equiv 4 \pmod{q}$, but only when $\alpha(m) > 2$. When $\alpha(m) = 2$, there are generators of the form $t_{i,j} = b_{(i)}^{\circ p} \circ b_{(j)}$ in dimension $2(p^{i+1} + p^j)$. These generators come in distinct varieties. There are the generators $w_{4p^i} = t_{i-1,i}$ for $i \geq 1$, which are the only generators in their dimension. There are the generators

$$y_{2(p^i + p^j)} = t_{j-1,i} = b_{(i)} \circ b_{(j-1)}^{\circ p}$$

for $0 \leq i < j$ and the generators

$$z_{2(p^i + p^j)} = t_{i-1,j} = b_{(i-1)}^{\circ p} \circ b_j$$

when $0 < i < j$. For convenience, we take $z_{2(1+p^j)} = 0$ for $j > 0$.

The fact that there are two generators in degrees $2(p^i + p^j)$ for $0 < i < j$ means that there must be relations between them. Indeed, the p–local cohomology of $BP\langle \mathbf{1} \rangle_{2p+2}$ is again an evenly graded, torsion-free polynomial algebra with one generator in each dimension $2m$ with $\alpha(m)\equiv 2 \pmod{p-1}$ and $m>2$. This means that the Atiyah–Hirzebruch spectral sequence will again collapse, and $BP_*BP\langle\mathbf{1}\rangle_{2p+2}$ will be torsion-free and evenly graded, and in fact a free BP_*–module, but this time there may be multiplicative extensions, because the p–local homology will not be polynomial. However, rationally, it is polynomial on one generator in each degree $2m$ with m satisfying the conditions above. Thus, there must be relations involving the generators $y_{2(p^i+p^j)}$ and $z_{2(p^i+p^j)}$ and p.

In order to find these relations, we work in the Hopf ring $S(*)=BP_*BP\langle\mathbf{1}\rangle_*$. We have the main relation

$$b([p]_{BP}(s))=[p]_{BP\langle 1\rangle}(b(s))$$

of [11]. Here $b(s)=\sum b_i s^i$, but on the right hand side, the sums and products in $[p]_{BP\langle 1\rangle}(s)$ are interpreted as star and circle products respectively. Write the formal group law for $BP\langle 1\rangle$ as

$$F(x,y)=x+y+\sum a_{kl}x^k y^l.$$

Then, using the fact that $[p]_F(x)=\sum^F v_i x^{p^i}$ for a p–typical formal group law (and the Araki generators), we have

$$[p]_{BP\langle 1\rangle}(s)=ps+v_1s^p+\sum a_{kl}(ps)^k(v_1s^p)^l.$$

Thus, the main relation is

$$\sum b_i([p]_{BP}(s))^i=b(s)^{*p}*[v_1]\circ b(s)^{\circ p}*\prod [a_{kl}]\circ(b(s)^{*p})^{\circ k}\circ[v_1^l]\circ b(s)^{\circ pl}.$$

To get anything useful out of such a formula, we must neglect almost all of the terms. To do so, let $I(n)$ be the augmentation ideal of $S(n)=BP_*BP\langle\mathbf{1}\rangle_n$, so that $I(n)$ is the kernel of

$$\epsilon\colon S(n)\to BP_*.$$

Because $\epsilon(x\circ y)=\epsilon(x)\epsilon(y)$, we have $I(n)\circ S(m)\subseteq I(n+m)$. It then follows from the distributive law that $I(n)^{*k}\circ S(m)\subseteq I(n+m)^{*k}$.

Lemma 1.3 *In $BP_*BP\langle\mathbf{1}\rangle_2$, we have*

$$[v_1]\circ b(s)^{\circ p}\equiv[0_2]\pmod{I\cdot I(2)+I(2)^{*p}}.$$

Here $[0_2]$ is the identity for the star product in $BP_*BP\langle\mathbf{1}\rangle_2$, and I denotes the ideal $(p,v_1,v_2,\dots)$ of BP_*.

Proof In view of the main relation, it suffices to show that

$$b([p]_{BP}(s)) \equiv b_0 = [0_2] \pmod{I \cdot I(2)},\ b(s)^{*p} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}},$$

and $$[a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}}.$$

Now $$[p]_{BP}(s) = ps +_F v_1 s^p +_F v_2 s^{p^2} +_F \cdots$$

is clearly in I, and each b_i except $b_0 = [0_2]$ is in $I(2)$, so

$$b([p]_{BP}(s)) \equiv b_0 = [0_2] \pmod{I \cdot I(2)}.$$

On the other hand,

$$\begin{aligned} b(s)^{*p} &= \Big([0_2] + \sum_{i>0} b_i s^i\Big)^{*p} \\ &= [0_2] + p\Big(\sum_{i>0} b_i s^i\Big) + \binom{p}{2}\Big(\sum_{i>0} b_i s^i\Big)^{*2} + \cdots + \Big(\sum_{i>0} b_i s^i\Big)^{*p}. \end{aligned}$$

Each b_i for $i > 0$ is in $I(2)$, and $p \in I$, so

$$b(s)^{*p} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}}.$$

Now taking the circle product preserves multiplication by I and the star product, and moves $I(2)$ to $I(k)$ as needed. We also have $[0_k] \circ y = \epsilon(y)[0_{k+l}]$ for any $y \in S(l)$. Putting all this together gives

$$[a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}},$$

as required. □

Theorem 1.4 *In $BP_*BP\langle 1\rangle_2$, we have*

$$\begin{aligned} & b_{(0)} \sum v_i s^{p^i} \\ &\equiv p \sum b_{(i)} s^{p^i} + \sum b_{(i)}^{*p} s^{p^{i+1}} + [v_1] \circ \Big(\sum b_{(i)} s^{p^i}\Big)^{\circ p} + [v_1] \circ \sum_{j \neq p^i} b_j^{\circ p} s^{pj} \\ &\qquad \pmod{I^2 \cdot I(2) + I \cdot I(2)^{*2} + I(2)^{*p+1}}. \end{aligned}$$

Proof Note that

$$[p]_{BP}(s) \equiv ps + v_1 s^p + v_2 s^{p^2} + \cdots \pmod{I^2},$$

since all the cross-terms in the formal sum will be in I^2. It follows that

$$b([p]_{BP}(s)) \equiv [0_2] + b_{(0)}(ps + v_1 s^p + v_2 s^{p^2} + \cdots) \pmod{I^2 \cdot I(2)}.$$

We want to perform a similar computation for the right side of the main relation. We begin with the larger terms. Using the Hopf ring distributive law as in the proof of Lemma 1.7 of [6], we find that

$$[a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} = ([a_{kl}] \circ b(s)^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl})^{*p^k}.$$

We know from Lemma 1.3 that

$$[v_1] \circ b(s)^{\circ p} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}},$$

and so

$$[a_{kl}] \circ b(s)^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}}$$

also. Then an easy computation shows that if $f \equiv [0_2] \pmod{I \cdot I(2) + I(2)^{*p}}$ then

$$f^{*p} \equiv [0_2] \pmod{I^2 \cdot I(2) + I \cdot I(2)^{*2} + I(2)^{*p+1}},$$

and this is not even the smallest possible ideal we could use.

The term $b(s)^{*p}$ is easily dealt with, since

$$\begin{aligned} b(s)^{*p} &= \Big([0_2] + \sum_{i>0} b_i s^i\Big)^{*p} \\ &= [0_2] + p\Big(\sum_{i>0} b_i s^i\Big) + \binom{p}{2}\Big(\sum_{i>0} b_i s^i\Big)^{*2} + \cdots + \Big(\sum_{i>0} b_i s^i\Big)^{*p}, \end{aligned}$$

and each b_i is in $I(2)^{*2}$ except the $b_{(i)}$. Hence

$$b(s)^{*p} \equiv [0_2] + p\sum b_{(i)} s^{p^i} + \sum b_{(i)}^{*p} s^{p^{i+1}} \pmod{I \cdot I(2)^{*2} + I(2)^{*p+1}}.$$

Now consider the term $[v_1] \circ b(s)^{\circ p}$. Writing $b(s) = [0_2] + \sum_{i>0} b_i s^i$, and raising to the p–th circle power, we get

$$b(s)^{\circ p} = [0_{2p}] + \Big(\sum_{i>0} b_i s^i\Big)^{\circ p}.$$

The cross-terms go away because we are circling with a $[0_k]$. Now in $(\sum_{i>0} b_i s^i)^{\circ p}$ we will have terms like $b_j^{\circ p} s^{pj}$ and terms like

$$pcb_{i_1} \circ \cdots \circ b_{i_p} s^{i_1 + \cdots + i_p},$$

where c depends on which of the i_j are equal to each other (pc is a multinomial coefficient with p as the top number, and we have already taken care of the one case where such a multinomial coefficient is not divisible by p, when all the i_j are equal). If any of the i_j is not a power of p, this latter term is in $I \cdot I(2)^{*2}$ so we can ignore it.

So we have

$$b(s)^{\circ p} \equiv [0_{2p}] + \left(\sum b_{(i)} s^{p^i}\right)^{\circ p} + \sum_{j \neq p^i} b_j^{\circ p} s^{pj} \qquad (\mathrm{mod}\ I^2 \cdot I(2) + I \cdot I(2)^{*2} + I(2)^{*p+1}).$$

Putting it all together and cancelling the $[0_2]$, we get the desired result. □

Corollary 1.5 *In $BP_*BP\langle \mathbf{1}\rangle_2$,*

$$[v_1] \circ b_{(i-1)}^{\circ p} \equiv v_i b_{(0)} - p b_{(i)} - b_{(i-1)}^{*p} \quad (\mathrm{mod}\ I^2 \cdot I(2) + I \cdot I(2)^{*2} + I(2)^{*p+1})$$

for all $i > 0$.

Proof Look at the coefficient of s^{p^i} in the above theorem. □

Corollary 1.6 *In $BP_*BP\langle \mathbf{1}\rangle_{2p+2}$ we have the relations*

$$v_i y_{2(1+p^j)} - v_j y_{2(1+p^i)} + p(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) + z^p_{2(p^{i-1}+p^{j-1})} - y^p_{2(p^{i-1}+p^{j-1})} \in I^2 \cdot I(2p+2) + I \cdot I(2p+2)^{*2} + I(2p+2)^{*p+1}$$

for $0 < i < j$.

Of course, when $i = 1$, we have to remember that $z_{2(1+p^{j-1})} = 0$.

Proof Let $J(2p+2)$ denote the ideal $I^2 \cdot I(2p+2) + I \cdot I(2p+2)^{*2} + I(2p+2)^{*p+1}$. Take $0 < i < j$, and apply the corollary above to $([v_1] \circ b_{(i-1)}^{\circ p}) \circ b_{(j-1)}^{\circ p}$ and to $([v_1] \circ b_{(j-1)}^{\circ p}) \circ b_{(i-1)}^{\circ p}$. We get

$$v_i b_{(0)} \circ b_{(j-1)}^{\circ p} - p b_{(i)} \circ b_{(j-1)}^{\circ p} - b_{(i-1)}^{*p} \circ b_{(j-1)}^{\circ p} \equiv v_j b_{(0)} \circ b_{(i-1)}^{\circ p} - p b_{(j)} \circ b_{(i-1)}^{\circ p} - b_{(j-1)}^{*p} \circ b_{(i-1)}^{\circ p} \quad (\mathrm{mod}\ J(2p+2)).$$

Looking back at the definition of the generators, this means

$$v_i y_{2(1+p^j)} - v_j y_{2(1+p^i)} + p(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) + b_{(j-1)}^{*p} \circ b_{(i-1)}^{\circ p} - b_{(i-1)}^{*p} \circ b_{(j-1)}^{\circ p} \in J(2p+2).$$

Now, recall the consequence

$$a^{*p^k} \circ b_{(i)} = (a \circ b_{(i-k)})^{*p^k}$$

of the Hopf ring distributive law, derived just above [6, Lemma 1.7], where $a^{*p^k} \circ b_{(i)} = 0$ if $i < k$. Applying this gives us

$$b_{(j-1)}^{*p} \circ b_{(i-1)}^{\circ p} = (b_{(j-1)} \circ b_{(i-2)}^{\circ p})^{*p} = z_{2(p^{i-1}+p^{j-1})}^{p}.$$

Note that this is still true if $i = 1$ because we have defined $z_{2(1+p^{j-1})} = 0$. Similarly,

$$b_{(i-1)}^{*p} \circ b_{(j-1)}^{\circ p} = (b_{i-1} \circ b_{j-2}^{\circ p})^{*p} = y_{2(p^{i-1}+p^{j-1})}^{p}. \qquad \square$$

This corollary gives us relations r_{ij} for $0 < i < j$ in R, the polynomial algebra over BP_* on the x_{2m} for $\alpha(m) \equiv 2 \pmod{p-1}$ and $\alpha(m) > 2$, the w_{4p^i} for $i > 0$, the $y_{2(p^i+p^j)}$ for $0 \le i < j$, and the $z_{2(p^i+p^j)}$ for $0 < i < j$, which must be satisfied in $BP_*BP\langle \mathbf{1}\rangle_{2p+2}$. Let $\mathfrak{b}$ denote the ideal of R generated by the r_{ij}. Then we have a surjection $f\colon R/\mathfrak{b} \to BP_*BP\langle \mathbf{1}\rangle_{2p+2}$. The generators of $\mathfrak{b}$ are in the right dimensions for f to be an isomorphism, so the following theorem comes as no surprise.

Theorem 1.7 *The map above*

$$R/\mathfrak{b} \xrightarrow{f} BP_*BP\langle \mathbf{1}\rangle_{2p+2}$$

is an isomorphism of BP_*–algebras.

Proof Let K denote the kernel of f, so we have a short exact sequence of BP_*–modules

$$0 \to K \to R/\mathfrak{b} \xrightarrow{f} BP_*BP\langle \mathbf{1}\rangle_{2p+2} \to 0.$$

We want to show that K is 0. Note that R is a finitely generated (p–local) abelian group in each degree, so K will be as well. Since K is also bounded below, it will suffice to show that K has no generators. That is, it will suffice to show that

$$K/IK = K \otimes_{BP_*} \mathbf{F}_p = 0.$$

Indeed, if n is the smallest degree in which $K_n \neq 0$, then $(K/IK)_n = K_n/pK_n$, so if this is 0 then K_n must also be.

Since $BP_*BP\langle \mathbf{1}\rangle_{2p+2}$ is a free BP_*–module, the short exact sequence above splits. Thus it remains exact upon tensoring with $\mathbf{F}_p$. Also, $BP_*BP\langle \mathbf{1}\rangle_{2p+2} \otimes_{BP_*} \mathbf{F}_p \cong H_*BP\langle \mathbf{1}\rangle_{2p+2}$, again because $BP_*BP\langle \mathbf{1}\rangle_{2p+2}$ is free. We are then reduced to showing that the surjection

$$R/\mathfrak{b} \otimes_{BP_*} \mathbf{F}_p \xrightarrow{\bar{f}} H_*BP\langle \mathbf{1}\rangle_{2p+2}$$

is an isomorphism.

Now $R/\mathfrak{b}\otimes_{BP_*}\mathbf{F}_p \cong \overline{R}/(\overline{r_{ij}})$ where $\overline{R}$ is the polynomial algebra over $\mathbf{F}_p$ on the same generators as R, and if $0<i<j$, then

$$\overline{r_{ij}} \equiv z^p_{2(p^{i-1}+p^{j-1})} - y^p_{2(p^{i-1}+p^{j-1})} \pmod{J^{p+1}}.$$

Here J denotes the augmentation ideal of $\overline{R}$. Replace the generator $y_{2(p^i+p^j)}$ by $y'_{2(p^i+p^j)} = z_{2(p^i+p^j)} - y_{2(p^i+p^j)}$. Then $\overline{R}/(\overline{r_{ij}})$ is spanned by all monomials in the generators such that the exponent of each $y'_{2(p^i+p^j)}$ is less than p. This has the same Poincaré series as the polynomial algebra over $\mathbf{F}_p$ on the generators x_{2m}, w_{4p^i}, $z_{2(p^i+p^j)}$ for $0<i<j$, and generators $a_{2(1+p^j)}$ for $j>0$. That is, $\overline{R}/\overline{r_{ij}}$ has the same Poincaré series as a polynomial algebra on one generator in each dimension $2m$ with $\alpha(m)\equiv 2 \pmod{p-1}$. This is the same Poincaré series as that of $H^*BP\langle\mathbf{1}\rangle_{2p+2}$ given in [12] just after Corollary 5.1. Thus our surjection must be an isomorphism. □

Theorem 1.8 *The sequence (r_{ij}) is a regular sequence in any order, and hence $BP_*BP\langle\mathbf{1}\rangle_{2p+2}$ is a (non-Noetherian) complete intersection ring.*

We remind the reader that a complete intersection ring is a regular local ring divided by a regular sequence. The word "regular" usually implies Noetherian, but in fact a general commutative ring is called regular if every finitely generated ideal has finite projective dimension [5]. The ring R of Theorem 1.7 is then a regular coherent local ring, and $BP_*BP\langle\mathbf{1}\rangle_{2p+2}$ is the quotient of R by an infinitely long regular sequence.

Proof Fix n. Let A_n be the polynomial algebra over $\mathbb{Z}_{(p)}[v_1,\dots,v_n]$ on all the generators of $BP_*BP\langle\mathbf{1}\rangle_{2p+2}$ of dimension $\leq 2(p^{n-1}+p^n)$. This will include one generator in each dimension with $\alpha(m)\equiv 2 \pmod{p-1}$ and $m>2$, plus an extra generator in each dimension $2(p^i+p^j)$ with $0<i<j\leq n$. Now we consider $A_n/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the $\binom{n}{2}$ relations r_{ij} for $0<i<j\leq n$. We will prove these r_{ij} are a regular sequence in A_n in any order. Since n is arbitrary, this will complete the proof.

By Theorem 17.4 of [8], it suffices to show that the Krull dimension of $A_n/\mathfrak{a}$ is the Krull dimension of A minus $\binom{n}{2}$, since A_n is a Cohen–Macaulay local ring. This also proves that the order of the r_{ij} is irrelevant. Note that if we invert p, we can use the relation r_{ij} to solve for $y_{2(p^i+p^j)}$ in terms of $z_{2(p^i+p^j)}$ and lower degree terms, because of the form of r_{ij}. (Note that the only terms in r_{ij} that can possibly involve $y_{2(p^i+p^j)}$ are of the form $p^k y_{2(p^i+p^j)}$ for $k\geq 1$). Hence $p^{-1}(A_n/\mathfrak{a})$ is a polynomial ring over $\mathbb{Q}$ on all of the generators of A_n except the $y_{2(p^i+p^j)}$, and therefore has Krull dimension $\binom{n}{2}+1$ less than the Krull dimension of A (we also lost the prime ideal (p), hence the additional 1).

Let s denote the Krull dimension of $p^{-1}(A_n/\mathfrak{a})$. We must show that the Krull dimension of $p^{-1}A_n$ is $s+1$. Of course, the primes of $p^{-1}(A_n/\mathfrak{a})$ are in one-to-one correspondence with the primes of $A_n/\mathfrak{a}$ that do not contain p. We therefore have a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s$ of prime ideals in $A_n/\mathfrak{a}$ which do not contain p. Since $p^{-1}(A_n/\mathfrak{a})$ is local, there is a unique prime ideal $\mathfrak{p}_s$ maximal among those which do not contain p. In fact, $\mathfrak{p}_s$ is the ideal generated by the positive degree elements of A_n, and $A_n/\mathfrak{p}_s = \mathbb{Z}_{(p)}$. Letting $\mathfrak{p}_{s+1}$ be the maximal ideal, we get a saturated chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{s+1}$. In a general Noetherian ring B, it is quite possible for two saturated chains of prime ideals to have different lengths, but this cannot happen in a finitely generated algebra over $\mathbb{Z}_{(p)}$, because $\mathbb{Z}_{(p)}$, and all Cohen–Macaulay rings, are *universally catenary* [4, Corollary 18.10]. □

2 $MO\langle 8\rangle$ and $MU\langle 6\rangle$ at large primes

In this section, we compute the homotopy rings $\pi_* MO\langle 8\rangle_{(p)}$ and $\pi_* MU\langle 6\rangle_{(p)}$ for $p \geq 5$.

If p is odd, there is a natural map

$$f\colon MO\langle 8\rangle \to MSO \to BP$$

of ring spectra. Similarly, for all p, there is a natural map

$$f\colon MU\langle 6\rangle \to MU \to BP$$

of ring spectra.

Lemma 2.1 *If $p \geq 5$, the induced map $BP_* MO\langle 8\rangle \to BP_* BP$ is surjective, and similarly for $MU\langle 6\rangle$.*

Proof Both sides are locally finite free BP_*–modules, and if we mod out by the maximal ideal I we get the map

$$H\mathbf{F}_{p*} MO\langle 8\rangle \to H\mathbf{F}_{p*} BP$$

which is onto by Rosen's theorem [6, Theorem 1.1]. We now use the standard technique to prove that f_* is surjective, by induction on the degree. It is certainly surjective in degree 0, so suppose it is surjective in all degrees $<k$, and x is in $BP_k BP$. Then we can find a y in $BP_k MO\langle 8\rangle$ such that $f_* y \equiv x \pmod{I}$, and then we can modify y using the fact that f_* is onto in lower degrees to find a z such that $f_* z \equiv x \pmod{p}$. We then have a map of finitely generated free $\mathbb{Z}_{(p)}$–modules that is surjective after we mod out by p. Such a map is easily seen to be surjective. □

Now, for each i, choose a generator u_i in $BP_{2(p^i-1)}MO\langle 8\rangle$ mapping to the generator t_i of BP_*BP. Note that all the tensor factors $BP_*BP\langle\mathbf{1}\rangle_n$ of $BP_*MO\langle 8\rangle$ must map to 0 in BP_*BP except $BP_*BP\langle\mathbf{1}\rangle_q$ for dimensional reasons (and since $p \geq 5$, $BP_*BP\langle\mathbf{1}\rangle_q$ is a tensor factor of $BP_*MO\langle 8\rangle$). Therefore, u_i lies in the tensor factor $BP_*BP\langle\mathbf{1}\rangle_q$, where, since it is indecomposable, it is congruent to the generator $x_{2(p^i-1)}$ modulo decomposables. Similar considerations apply to $MU\langle 6\rangle$.

Proposition 2.2 *For $p \geq 5$, the map*

$$g\colon H\mathbf{F}_{p*}MO\langle 8\rangle \xrightarrow{\psi} P_* \otimes H\mathbf{F}_{p*}MO\langle 8\rangle \to P_* \otimes H\mathbf{F}_{p*}MO\langle 8\rangle/(u_1, u_2, \dots)$$

is an isomorphism of comodule algebras, where ψ denotes the coaction map, and the coaction on the right is all in the P_* tensor factor. There is a similar isomorphism for $MU\langle 6\rangle$.

Of course, $u_i \in H\mathbf{F}_{p*}MO\langle 8\rangle$ is just the image of $u_i \in BP_*MO\langle 8\rangle$.

Proof Coassociativity implies that g is a map of comodule algebras. Both sides have the same Poincaré series, as follows from the fact that the u_i are polynomial generators of $H\mathbf{F}_{p*}MO\langle 8\rangle$. So it suffices to show that the given map is surjective, for which it is sufficient to prove it is surjective on indecomposables. There is a basis of the indecomposables of the right-hand side consisting of the $\zeta_i \otimes 1$ and the $1 \otimes x$, where ζ_i is the conjugate of the usual generator ξ_i and x runs through a basis for the indecomposables of $H\mathbf{F}_{p*}MO\langle 8\rangle$ that are not multiples of one of the u_i (each u_i is the only indecomposable in its dimension, up to $\mathbf{F}_p$ multiples). Now, for any $x \in H\mathbf{F}_{p*}MO\langle 8\rangle$, examination of the commutative diagram

$$\begin{array}{ccc} H\mathbf{F}_{p*}MO\langle 8\rangle & \xrightarrow{\psi} & P_* \otimes H\mathbf{F}_{p*}MO\langle 8\rangle \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*\otimes 1} \\ H\mathbf{F}_{p*}BP = P_* & \xrightarrow[\psi]{} & P_* \otimes P_* \end{array}$$

shows that $g(x) = f_*(x) \otimes 1 + 1 \otimes \overline{x}$ modulo decomposables in $P_* \otimes H\mathbf{F}_{p*}MO\langle 8\rangle$, where $\overline{x}$ is the image of x in $H\mathbf{F}_{p*}MO\langle 8\rangle/(u_1, u_2, \dots)$.

In particular, $g(u_i) = f_*(u_i) \otimes 1 = \zeta_i \otimes 1$ modulo decomposables, since the reduction of t_i in $H\mathbf{F}_{p*}BP = P_*$ is ζ_i. On the other hand, if x is an indecomposable that is not a multiple of the u_i, then f_*x is decomposable, since x must be in a dimension where there are no indecomposables in P_*, and so $g(x) = 1 \otimes \overline{x}$. Thus g is surjective on indecomposables, as required. □

This proposition then allows us to prove a similar result for BP–homology.

Theorem 2.3 *For $p \geq 5$, the map*

$$g\colon BP_*MO\langle 8\rangle \xrightarrow{\psi} BP_*BP \otimes_{BP_*} BP_*MO\langle 8\rangle \to BP_*BP \otimes_{BP_*} BP_*MO\langle 8\rangle/(u_1, u_2, \dots)$$

is an isomorphism of comodule algebras, where ψ denotes the coaction map and the coaction on the right side is all in the BP_*BP tensor factor. The analogous result holds for $MU\langle 6\rangle$.

Proof Again, coassociativity implies that g is a map of comodule algebras. In each dimension, both sides are finitely generated free $\mathbb{Z}_{(p)}$–modules. It therefore suffices to show that the map in question is surjective. Using an argument similar to that of Lemma 2.1, it suffices to prove that this map is surjective after taking the quotient of both sides by the maximal ideal I. On the left hand side, this quotient is $H\mathbf{F}_{p*}MO\langle 8\rangle$. On the right-hand side, we have to use the fact that I is an invariant ideal. Let $J = (u_1, u_2, \dots)$ for convenience of notation. This gives

$$\begin{aligned}
(BP_*BP\otimes_{BP_*}&BP_*MO\langle 8\rangle/J)/I \\
&\cong BP_*/I \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*MO\langle 8\rangle/J \\
&\cong BP_*/I \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*/I \otimes_{BP_*} BP_*MO\langle 8\rangle/I \\
&\cong H\mathbf{F}_{p*}BP \otimes BP_*MO\langle 8\rangle/(I, J) \cong P_* \otimes H\mathbf{F}_{p*}MO\langle 8\rangle/J.
\end{aligned}$$

The preceding proposition now shows that the map g/I is surjective. □

The Adams–Novikov spectral sequence then gives us the following theorem.

Theorem 2.4 *For $p \geq 5$, there is an isomorphism of rings*

$$\begin{aligned}
\pi_*MO\langle 8\rangle_{(p)} &\cong BP_*MO\langle 8\rangle/(u_1, u_2, \dots) \\
&\cong BP_*BP\langle \mathbf{1}\rangle_8 \otimes_{BP_*} BP_*BP\langle \mathbf{1}\rangle_{12} \\
&\qquad \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP\langle \mathbf{1}\rangle_q/(u_1, \dots) \otimes_{BP_*} BP_*BP\langle \mathbf{1}\rangle_{q+4}.
\end{aligned}$$

A similar theorem holds for $\pi_*MU\langle 6\rangle_{(p)}$, except there are tensor factors in every even degree.

Proof The preceding proposition tells us that the E_2 term of the Adams–Novikov spectral sequence converging to $\pi_*MO\langle 8\rangle_{(p)}$ is $BP_*MO\langle 8\rangle/(u_1, u_2, \dots)$, concentrated in filtration 0. So the spectral sequence collapses with no possible extensions, either additive or multiplicative. □

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Received: 15 August 2008 Revised: 24 November 2008