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ON THE COHOMOLOGY GROUPS OF AN ASSOCIATIVE ALGEBRA

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Introduction.

The cohomology theory of associative algebras is concerned with the m-linear mappings of an algebra $\mathfrak{A}$ into a two-sided $\mathfrak{A}$-module $\mathfrak{P}$. In this theory, the additive group $(\mathfrak{A}^{(m)}:\mathfrak{P})$ of the m-linear mappings of $\mathfrak{A}$ into $\mathfrak{P}$ plays a rôle analogous to that of the group of m-dimensional cochains in combinatorial topology. A linear mapping of $(\mathfrak{A}^{(m)}:\mathfrak{P})$ into $(\mathfrak{A}^{(m+1)}:\mathfrak{P})$ analogous to the coboundary operator of combinatorial topology and leading to the notion of $\mathfrak{P}$-"cohomology group" has been defined by Eilenberg and MacLane[1]. The special cases of dimension one and two (linear and bilinear mappings of $\mathfrak{A}$ into a two-sided $\mathfrak{A}$-module) have appeared before in connection with the so-called first and second lemmas of Whitehead[2].

In a sense, the cohomology theory of associative algebras is degenerate: the 1-dimensional cohomology groups already determine all the others. In fact, if $\mathfrak{P}$ is any two-sided $\mathfrak{A}$-module, one can construct another two-sided $\mathfrak{A}$-module, $(\mathfrak{A}:\mathfrak{P})$, such that (for $m \geqq 2$) the m-dimensional $\mathfrak{P}$-cohomology group of $\mathfrak{A}$ is isomorphic with the $(m - 1)$-dimensional $(\mathfrak{A}:\mathfrak{P})$-cohomology group of $\mathfrak{A}$ (Theorem 3.1).

The present paper is concerned primarily with the connections between the structure of an algebra and the vanishing of its cohomology groups. It is shown that an algebra is separable if and only if all its cohomology groups vanish (Theorem 4.1). This is a generalization of a result obtained previously for the case of a non-modular ground field[3].

The 2-dimensional cohomology groups of an algebra are directly connected with the "extensions" of $\mathfrak{A}$, i.e. algebras $\mathfrak{B}$ of which $\mathfrak{A}$ is a homomorphic image. In particular, the condition that all 2-dimensional cohomology groups of $\mathfrak{A}$ vanish signifies that every extension of $\mathfrak{A}$ has the form $\mathfrak{B} = \mathfrak{A}^* + \mathfrak{K}$, where $\mathfrak{A}^*$ is a subalgebra isomorphic with $\mathfrak{A}$ and $\mathfrak{K}$ is the kernel of the homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ (Theorem 6.1). This is connected with the (generalized) third structure theorem of Wedderburn which may be stated by saying that the 2-dimensional cohomology groups of a separable algebra vanish.

More generally, one would be interested in the structural significance of the condition C_m : "all m-dimensional cohomology groups vanish." Theorem 3.1 implies that C_{m+1} is a consequence of C_m , for $m \geqq 1$. But it is an open question whether or not C_m and C_{m+1} are equivalent.

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[1] Unpublished. The analogous concepts for groups are contained in: S. Eilenberg and S. MacLane, Bull. Am. Math. Soc., vol. 50, (1944), Abstract No. 9.

[2] G. Hochschild, "*Semi-Simple Algebras and Generalized Derivations*," Amer. Jour. Math., vol. 64, (1941) p.p. 677–694.

[3] G. Hochschild, loc. cit., Theorems 3.4, 3.2.

1. $\mathfrak{A}$-modules and Cochains of $\mathfrak{A}$.

Let $\mathfrak{A}$ be an associative algebra over an arbitrary field Φ. We consider linear spaces $\mathfrak{P}$ over Φ which are at the same time $\mathfrak{A}$-left modules and $\mathfrak{A}$-right modules[4]. If $a \in \mathfrak{A}$ we denote the corresponding Φ-endomorphisms[5] of $\mathfrak{P}$ by $u \rightarrow a \cdot u$, $u \rightarrow u \cdot a$ respectively. In addition to the usual requirements on these operations for one-sided modules we demand that, if $a_1, a_2 \in \mathfrak{A}$ and $u \in \mathfrak{P}$, $(a_1 \cdot u) \cdot a_2 = a_1 \cdot (u \cdot a_2)$. Then $\mathfrak{P}$ is called a two-sided $\mathfrak{A}$-module.

We denote the Kronecker product of m linear spaces isomorphic with the linear space underlying $\mathfrak{A}$ by $\mathfrak{A}^{(m)}$. If $a_1, a_2, \cdots, a_n$ is a basis of $\mathfrak{A}$ over Φ, $\mathfrak{A}^{(m)}$ is the n^m-dimensional linear space which is spanned by the elements $a_{i_1} \times a_{i_2} \times \cdots \times a_{i_m}$. By an m-dimensional $\mathfrak{P}$-cochain of $\mathfrak{A}$, briefly an $(m, \mathfrak{P})$-cochain of $\mathfrak{A}$, is meant a linear mapping of $\mathfrak{A}^{(m)}$ into the two-sided $\mathfrak{A}$-module $\mathfrak{P}$. By a $(0, \mathfrak{P})$-cochain of $\mathfrak{A}$ we mean simply an element of $\mathfrak{P}$. The linear space over Φ formed by all the $(m, \mathfrak{P})$-cochains of $\mathfrak{A}$ will be denoted by $(\mathfrak{A}^{(m)} : \mathfrak{P})$.

The linear space $(\mathfrak{A}^{(1)} : \mathfrak{P})$ formed by the $(1, \mathfrak{P})$-cochains of $\mathfrak{A}$ will play an important part in what follows. We make $(\mathfrak{A}^{(1)} : \mathfrak{P})$ into a two-sided $\mathfrak{A}$-module in a way specially adapted to our purpose: If $f \in (\mathfrak{A}^{(1)} : \mathfrak{P})$ and $a \in \mathfrak{A}$ we define an element $a*f$ of $(\mathfrak{A}^{(1)} : \mathfrak{P})$ as follows: $(a*f)\{a'\} = a \cdot f\{a'\}$, for every $a' \in \mathfrak{A}$. Clearly, this makes $(\mathfrak{A}^{(1)} : \mathfrak{P})$ into an $\mathfrak{A}$-left module. Furthermore, we define an element $f*a$ of $(\mathfrak{A}^{(1)} : \mathfrak{P})$ by setting $(f*a)\{a'\} = f\{aa'\} - f\{a\} \cdot a'$, for every $a' \in \mathfrak{A}$. It can be verified directly that the operation $f \rightarrow f*a$ makes $(\mathfrak{A}^{(1)} : \mathfrak{P})$ into an $\mathfrak{A}$-right module and that $(\mathfrak{A}^{(1)} : \mathfrak{P})$ has in fact become a two-sided $\mathfrak{A}$-module. We state these facts in the following proposition:

PROPOSITION 1.1. *The linear space $(\mathfrak{A}^{(1)} : \mathfrak{P})$ of the $(1, \mathfrak{P})$-cochains of $\mathfrak{A}$ is a two-sided $\mathfrak{A}$-module with the operations $f \rightarrow a*f$ and $f \rightarrow f*a$, where*

$$(a*f)\{a'\} = a \cdot f\{a'\} \quad \textit{and}$$
$$(f*a)\{a'\} = f\{aa'\} - f\{a\} \cdot a'.$$

When $(\mathfrak{A}^{(1)} : \mathfrak{P})$ is regarded as a two-sided $\mathfrak{A}$-module in this particular fashion we denote it by $(\mathfrak{A} : \mathfrak{P})$.

2. Coboundary, Cocycle, Cohomology Group.

We define a "coboundary operator," δ, operating on the set of all cochains of $\mathfrak{A}$ as follows:

DEFINITION 2.1. *δ maps $(\mathfrak{A}^{(m)} : \mathfrak{P})$ linearly into $(\mathfrak{A}^{(m+1)} : \mathfrak{P})$. If $f \in (\mathfrak{A}^{(m)} : \mathfrak{P})$,*

$$\delta f\{a_1, \cdots, a_{m+1}\} = a_1 \cdot f\{a_2, \cdots, a_{m+1}\} + \sum_{i=1}^{m} (-1)^i f\{a_1, \cdots, a_i a_{i+1}, \cdots, a_{m+1}\} + (-1)^{m+1} f\{a_1, \cdots, a_m\} \cdot a_{m+1}.$$

[4] In the conventional sense. See, for instance, N. Jacobson, "*The Theory of Rings*", Math. Surv. No. II, Amer. Math. Soc. (1943), p.p. 14–16.

[5] Cf. N. Jacobson, loc. cit., pp. 4–5.

We wish to prove that $\delta\delta f = 0$, for every cochain f. We shall do this by making an induction on the dimension m of f: For $m = 0$, $f \epsilon \mathfrak{P}$, and $\delta f\{a\} = a\cdot f - f\cdot a$,

$$\begin{aligned}\delta\delta f\{a_1, a_2\} &= a_1\cdot\delta f\{a_2\} - \delta f\{a_1a_2\} + \delta f\{a_1\}\cdot a_2 \\ &= a_1\cdot(a_2\cdot f - f\cdot a_2) - (a_1a_2\cdot f - f\cdot a_1a_2) + (a_1\cdot f - f\cdot a_1)\cdot a_2 \\ &= 0.\end{aligned}$$

Corresponding to any element f of $(\mathfrak{A}^{(m)}:\mathfrak{P})$, $m \geqq 1$, we define an element $\bar{f} \epsilon (\mathfrak{A}^{(m-1)}:(\mathfrak{A}:\mathfrak{P}))$ by the relation $\bar{f}\{a_1, \cdots, a_{m-1}\}\{a_m\} = f\{a_1, \cdots, a_m\}$. (We identify $(\mathfrak{A}^{(0)}:(\mathfrak{A}:\mathfrak{P}))$ with $(\mathfrak{A}:\mathfrak{P})$). Then we can verify directly from Definition 2.1 that, if $f \epsilon (\mathfrak{A}^{(m)}:\mathfrak{P})$, $m \geqq 1$,

$$\begin{aligned}\delta f\{a_1, \cdots, a_{m+1}\} &= (a_1 * \bar{f}\{a_2, \cdots, a_m\})\{a_{m+1}\} \\ &\quad + \sum_{i=1}^{m-1}(-1)^i\bar{f}\{a_1, \cdots, a_ia_{i+1}, \cdots, a_m\}\{a_{m+1}\} \\ &\quad + (-1)^m(\bar{f}\{a_1, \cdots, a_{m-1}\}*a_m)\{a_{m+1}\} \\ &= \delta\bar{f}\{a_1, \cdots, a_m\}\{a_{m+1}\}, \quad \text{whence} \quad \overline{\delta f} = \delta\bar{f}\end{aligned}$$

Thus, $\overline{\delta\delta f} = \delta\overline{\delta f} = \delta\delta\bar{f}$.
Since dim $(\bar{f})$ = dim $(f) - 1$, and since $\bar{f} = 0$ only if $f = 0$, we can now apply induction to prove our result.

We now make the customary definitions:

DEFINITION 2.2. *A cochain f is called a cocycle if $\delta f = 0$. f is said to be a coboundary if there exists a cochain g such that $f = \delta g$.*

Clearly, the set of $(m, \mathfrak{P})$-cocycles of $\mathfrak{A}$ constitutes a subgroup of the additive group of the $(m, \mathfrak{P})$-cochains of $\mathfrak{A}$. By the last result, the set of coboundaries constitutes a subgroup of the corresponding group of cocycles.

DEFINITION 2.3. *For $m \geqq 1$, the m-dimensional cohomology group of $\mathfrak{A}$ for the module $\mathfrak{P}$, denoted by $H^m(\mathfrak{A}, \mathfrak{P})$, is the group of the $(m, \mathfrak{P})$-cocycles of $\mathfrak{A}$, modulo the subgroup of coboundaries.*[6]

3. A Fundamental Isomorphism.

It is obvious that the mapping $f \rightarrow \bar{f}$, defined in section 2, maps $(\mathfrak{A}^{(m)}:\mathfrak{P})$ isomorphically onto $(\mathfrak{A}^{(m-1)}:(\mathfrak{A}:\mathfrak{P}))$, for $m \geqq 1$. The relation $\overline{\delta f} = \delta\bar{f}$ shows furthermore that this isomorphism maps the group of $(m, \mathfrak{P})$-cocycles onto the group of $(m - 1, (\mathfrak{A}:\mathfrak{P}))$-cocycles. Finally, if dim $(f) \geqq 2$, $\bar{f} = \delta g$ if and only if $f = \delta g$. Hence the mapping $f \rightarrow \bar{f}$ induces an isomorphism between $H^m(\mathfrak{A}, \mathfrak{P})$ and $H^{m-1}(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$:

THEOREM 3.1. *If $m \geqq 2$, $H^m(\mathfrak{A}, \mathfrak{P}) \cong H^{m-1}(\mathfrak{A}, (\mathfrak{A}:\mathfrak{P}))$.*

4. Separability and the 1-dimensional Cohomology Groups.

In this section, we wish to prove the fundamental result that an algebra is separable if and only if all its 1-dimensional cohomology groups vanish. We

[6] $H^0(\mathfrak{A}, \mathfrak{P})$ is of no interest. We may define it as the additive subgroup of $\mathfrak{P}$ consisting of all $u \epsilon \mathfrak{P}$ such that $a.u = u.a$, for all $a \epsilon \mathfrak{A}$.

recall that an algebra $\mathfrak{A}$ over the field Φ is said to be separable if $\mathfrak{A}_\Gamma$ is semi-simple for every extension field Γ of Φ. Here, $\mathfrak{A}_\Gamma$ denotes the algebra over Γ which consists of all linear combinations $\sum_{i=1}^n \gamma_i a_i$, where $\gamma_i \in \Gamma$ and where $a_1, \cdots, a_n$ constitute a basis of $\mathfrak{A}$ over Φ, multiplication being derived from the multiplication of the a_i in $\mathfrak{A}$ over Φ.[7]

Now, let $\mathfrak{A}$ be a separable algebra over Φ, and let $\mathfrak{P}$ be any two-sided $\mathfrak{A}$-module. $\mathfrak{A}$ has an identity element: e, say. For any element $u \in \mathfrak{P}$ we may write $u = e \cdot u + (1 - e) \cdot u$, where 1 stands for the identity operator on $\mathfrak{P}$. This gives a direct decomposition of $\mathfrak{P}$ into the $\mathfrak{A}$-invariant linear subspaces $e \cdot \mathfrak{P}$ and $(1 - e) \cdot \mathfrak{P}$. If f is any $(1, \mathfrak{P})$-cocycle of $\mathfrak{A}$ we have

$$(1 - e) \cdot f\{a\} = (1 - e) \cdot f\{ea\} = (1 - e) \cdot f\{e\} \cdot a.$$

If we set $u = (1 - e) \cdot f\{e\}$ we have therefore $f\{a\} = e \cdot f\{a\} - (a \cdot u - u \cdot a)$. It follows that f cobounds if and only if the $(1, e \cdot \mathfrak{P})$-cocycle $\hat{f}$, where $\hat{f}\{a\} = e \cdot f\{a\}$, cobounds. Thus, in proving that every 1-dimensional cocycle of $\mathfrak{A}$ cobounds, we may confine our attention to those two-sided $\mathfrak{A}$-modules $\mathfrak{P}$ for which the identity e of $\mathfrak{A}$ is the left identity operator.

Since $\mathfrak{A}$ is separable there exists a finite algebraic extension Γ of Φ such that $\mathfrak{A}_\Gamma$ is the direct sum of full matrix algebras over Γ; say $\mathfrak{A}_\Gamma = \Gamma_{n_1} + \cdots + \Gamma_{n_m}$. If f is any $(1, \mathfrak{P})$-cocycle of $\mathfrak{A}$ over Φ we define a $(1, \mathfrak{P}_\Gamma)$-cocycle f^* of $\mathfrak{A}_\Gamma$ by setting $f^*\{\sum_{i=1}^n \gamma_i a_i\} = \sum_{i=1}^n \gamma_i f\{a_i\}$. Let $e_{i,j}^q$ $(i, j = 1, \cdots, n_q)$ denote the matrix units of Γ_{n_q}. Then $e_{i,j}^q e_{k,p}^r = \delta_{q,r}\delta_{j,k} e_{i,p}^r$, and $\sum_{q=1}^m \sum_{i=1}^{n_q} e_{i,i}^q = e$. Consider now the element $u^* = \sum_{q=1}^m \sum_{i=1}^{n_q} e_{i,1}^q \cdot f^*\{e_{1,i}^q\}$ of $\mathfrak{P}_\Gamma$: We have

$$\begin{aligned} e_{h,t}^r \cdot u^* - u^* \cdot e_{h,t}^r &= \sum_{q=1}^m \sum_{i=1}^{n_q} e_{h,t}^r e_{i,1}^q \cdot f^*\{e_{1,i}^q\} - \sum_{q=1}^m \sum_{i=1}^{n_q} e_{i,1}^q \cdot (f^*\{e_{1,i}^q e_{h,t}^r\} - e_{1,i}^q \cdot f^*\{e_{h,t}^r\}) \\ &= \sum_{i=1}^{n_r} e_{h,t}^r e_{i,1}^r \cdot f^*\{e_{1,i}^r\} - \sum_{i=1}^{n_r} e_{i,1}^r \cdot f^*\{e_{1,i}^r e_{h,t}^r\} + e \cdot f^*\{e_{h,t}^r\} \\ &= f^*\{e_{h,t}^r\}. \end{aligned}$$

By linearity, it follows that $f^*\{a^*\} = a^* \cdot u^* - u^* \cdot a^*$, for every $a^* \in \mathfrak{A}_\Gamma$. In particular, $f\{a\} = a \cdot u^* - u^* \cdot a$, for every $a \in \mathfrak{A}$. The question of whether there exists an element $u \in \mathfrak{P}$ such that $f\{a\} = a \cdot u - u \cdot a$, for every $a \in \mathfrak{A}$ can be reduced to the question of whether a certain system of linear equations with coefficients in Φ possesses a solution in Φ. The last equation means that there does exist a solution in Γ. Hence, there must already be a solution in Φ, and we have proved that every 1-dimensional cocycle of $\mathfrak{A}$ is a coboundary.

Conversely, suppose that $\mathfrak{A}$ is an algebra over Φ such that $H^1(\mathfrak{A}, \mathfrak{P}) = \{0\}$ for every two-sided $\mathfrak{A}$-module $\mathfrak{P}$. It is known that under these circumstances $\mathfrak{A}$ is semi-simple[8]. We shall, however, reproduce the proof here since it will throw some light on the significance of the 1-dimensional cohomology groups.

[7] From here on we shall make use of the classical results in the theory of separable algebras without giving specific references. The results in question can be found in any of the standard expositions, such as: Deuring, "*Algebren*"; Van der Waerden, "*Moderne Algebra*"; or Jacobson, loc. cit.

[8] G. Hochschild, loc. cit., Th. 2.3.

As is well known, a necessary and sufficient condition for an algebra to be semi-simple is that every representation be completely decomposable into irreducible parts. This in turn is equivalent to the following condition: If $\mathfrak{P}$ is any left $\mathfrak{A}$-module, $\mathfrak{Q}$ an $\mathfrak{A}$-invariant subspace of $\mathfrak{P}$, then there exists a (complementary) $\mathfrak{A}$-invariant subspace $\mathfrak{R}$ such that $\mathfrak{P}$ is the direct sum of $\mathfrak{Q}$ and $\mathfrak{R}$.

We shall show that this condition is satisfied if all the 1-dimensional cohomology groups of $\mathfrak{A}$ are $\{0\}$:

Given the left $\mathfrak{A}$-module $\mathfrak{P}$ and an invariant subspace $\mathfrak{Q}$, let us first choose any linear subspace $\mathfrak{S}$ of $\mathfrak{P}$ such that $\mathfrak{P}$, as a linear space, is the direct sum of $\mathfrak{Q}$ and $\mathfrak{S}$. For any $s \in \mathfrak{S}$ and $a \in \mathfrak{A}$ we may write $a \cdot s = S_a\{s\} + Q_a\{s\}$, where $S_a\{s\}$ is a uniquely determined element of $\mathfrak{S}$ and $Q_a\{s\}$ a uniquely determined element of $\mathfrak{Q}$. Clearly, S_a and Q_a are linear mappings of $\mathfrak{S}$ into $\mathfrak{S}$ and $\mathfrak{Q}$ respectively From the relation $a_1 \cdot (a_2 \cdot s) = a_1a_2 \cdot s$ and the fact that the sum $\mathfrak{Q} + \mathfrak{S}$ is direct we obtain the relations

$$S_{a_1}S_{a_2} = S_{a_1a_2} \quad \text{and} \quad Q_{a_1}S_{a_2} + a_1 * Q_{a_2} = Q_{a_1a_2},$$

where $a*Q$ is defined by $(a*Q)\{s\} = a \cdot Q\{s\}$.

Now, let $\mathfrak{M}$ denote the linear space of all linear mappings of $\mathfrak{S}$ into $\mathfrak{Q}$. We make $\mathfrak{M}$ a two-sided $\mathfrak{A}$-module by the definitions: $(a*M)\{s\} = a \cdot M\{s\}$, $M*a = MS_a$, for $M \in \mathfrak{M}$, $a \in \mathfrak{A}$, $s \in \mathfrak{S}$. It is easy to see from the above relations that the mapping $a \to Q_a$ is a $(1, \mathfrak{M})$-cocycle of $\mathfrak{A}$. By our hypothesis, there exists an element $M_0 \in \mathfrak{M}$ such that $Q_a = a*M_0 - M_0S_a$, i.e. $Q_a\{s\} = a \cdot M_0\{s\} - M_0\{S_a\{s\}\}$. Now, consider the mapping $s \to s - M_0\{s\}$ of $\mathfrak{S}$ into $\mathfrak{P}$. This maps $\mathfrak{S}$ onto some linear subspace $\mathfrak{R}$ of $\mathfrak{P}$. We have $a \cdot (s - M_0\{s\}) = S_a\{s\} + Q_a\{s\} - a \cdot M_0\{s\} = S_a\{s\} - M_0\{S_a\{s\}\}$, which shows that $\mathfrak{R}$ is $\mathfrak{A}$-invariant. It is obvious that the mapping $s \to s - M_0\{s\}$ is a linear isomorphism, whence $\dim(\mathfrak{R}) = \dim(\mathfrak{S})$. Furthermore, it is easy to see that $\mathfrak{R} \cap \mathfrak{Q} = \{0\}$. Hence $\mathfrak{P}$ is the direct sum of $\mathfrak{Q}$ and $\mathfrak{R}$. This completes the proof that $\mathfrak{A}$ is semi-simple.

Hence, what we have to show now is that if $\mathfrak{A}$ is semi-simple and inseparable there exist a two-sided $\mathfrak{A}$-module $\mathfrak{P}$ and a non-cobounding $(1, \mathfrak{P})$-cocycle of $\mathfrak{A}$. Since $\mathfrak{A}$ must have an inseparable simple component and since any non-cobounding 1-cocycle of a simple component of $\mathfrak{A}$ can be trivially extended to give a non-cobounding 1-cocycle of $\mathfrak{A}$ we may suppose from now on that $\mathfrak{A}$ is simple. Then—as is well known—the center $\mathfrak{C}$ of $\mathfrak{A}$ is a field which is algebraic and inseparable over Φ. In the case where $\mathfrak{A}$ itself is an inseparable extension field of Φ, the result in question follows immediately from the known theory of derivations—i.e. $(1, \mathfrak{C})$-cocycles of $\mathfrak{C}$—in fields of characteristic p.[9] Of the results of this theory we shall require only the following lemma:

Lemma 4.1. *Let Φ be a field of characteristic p ($\neq 0$), $\mathfrak{C}$ an inseparable algebraic extension of Φ. Then there exists a non-zero derivation of $\mathfrak{C}$ (i.e. a non-cobounding $(1, \mathfrak{C})$-cocycle of $\mathfrak{C}$) over Φ.*

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[9] F. K. Schmidt, "*Zusatz bei der Korrektur*", Crelle's Journal, Vol. 177, pp. 223–237. R. Baer, "*Algebraische Theorie der differentiierbaren Funktionen-koerper I,*", Sitzungsberichte, Heidelberger Akademie, 1927, p.p. 15–32. N. Jacobson, "*Abstract Derivations and Lie Algebras,*" Trans. Amer. Math. Soc., vol. 42 (1937), p.p. 206–224.

Proof. It is easy to see that there exists an intermediate field $\mathfrak{S}$ over Φ such that $\mathfrak{C} = \mathfrak{S}(c)$, where $c^p \in \mathfrak{S}$. We define a linear mapping D of $\mathfrak{C}$ into itself by setting $D\{s\} = 0$, for every $s \in \mathfrak{S}$, and $D(sc^q) = sqc^{q-1}c_0$, where c_0 is an arbitrary fixed element ($\neq 0$) of $\mathfrak{C}$. It is easy to verify that D is a non-zero derivation of $\mathfrak{C}$ over $\mathfrak{S}$, and hence also over Φ.

We can now begin the construction of a non-cobounding 1-cocycle of the simple inseparable algebra $\mathfrak{A}$. If m is the dimension of $\mathfrak{A}$ over its center $\mathfrak{C}$ the regular representation of $\mathfrak{A}$ over $\mathfrak{C}$ provides us with an isomorphic mapping $a \to M_a$ of $\mathfrak{A}$ into the matrix algebra $\mathfrak{C}_m$ (m-rowed square matrices with elements in $\mathfrak{C}$). It follows that the definitions $a \cdot M = M_a M$, $M \cdot a = MM_a$ make $\mathfrak{C}_m$ (regarded as a linear space over Φ) a two-sided $\mathfrak{A}$-module.

Now let D be a derivation of $\mathfrak{C}$ over Φ which does not map all of $\mathfrak{C}$ into zero (such a D exists, by Lemma 4.1). If $M = (c_{ij})$ is any element of $\mathfrak{C}_m$ we define $D\{M\} = (D\{c_{ij}\})$ and verify that this is a derivation of $\mathfrak{C}_m$ over Φ. If we now set $f\{a\} = D\{M_a\}$, it is clear that f is a $(1, \mathfrak{C}_m)$-cocycle of $\mathfrak{A}$. If $c \in \mathfrak{C}$, we must obviously have $M_c = cI_m$, where I_m denotes the identity matrix in $\mathfrak{C}_m$. Hence, $f\{c\} = D\{c\}I_m$, which is different from zero for some $c \in \mathfrak{C}$. On the other hand, if f were a coboundary, we should have a matrix $M_0 \in \mathfrak{C}_m$ such that $f\{a\} = M_a M_0 - M_0 M_a$, for every $a \in \mathfrak{A}$. But this would imply that $f\{c\} = 0$, for every $c \in \mathfrak{C}$. Hence, f cannot cobound. Thus, we have proved the following theorem:

Theorem 4.1. *A necessary and sufficient condition for an algebra $\mathfrak{A}$ to be separable is that $H^1(\mathfrak{A}, \mathfrak{P}) = \{0\}$ for every two-sided $\mathfrak{A}$-module $\mathfrak{P}$.*

5. A Natural Homomorphism.

If $\mathfrak{B}$ is an ideal of the algebra $\mathfrak{A}$ and $\mathfrak{P}$ a two-sided $\mathfrak{A}$-module, then $\mathfrak{P}$ is also a two-sided $\mathfrak{B}$-module. A $\mathfrak{P}$-cochain f of $\mathfrak{A}$, when restricted to $\mathfrak{B}$, determines a $\mathfrak{P}$-cochain $\hat{f}$ of $\mathfrak{B}$. The mapping $f \to \hat{f}$ is clearly linear and maps cocycles into cocycles, and coboundaries into coboundaries. Therefore, it defines a homomorphism of $H^m(\mathfrak{A}, \mathfrak{P})$ into $H^m(\mathfrak{B}, \mathfrak{P})$ which we shall call the natural homomorphism.

We shall confine our attention to the 1-dimensional cohomology groups and prove the following theorem:

Theorem 5.1. *Let $\mathfrak{Q}$ be the subspace of $\mathfrak{P}$ which is annihilated by $\mathfrak{B}$. Then the kernel of the natural homomorphism of $H^1(\mathfrak{A}, \mathfrak{P})$ into $H^1(\mathfrak{B}, \mathfrak{P})$ is isomorphic with $H^1(\mathfrak{A}/\mathfrak{B}, \mathfrak{Q})$, where $\mathfrak{Q}$ is regarded as a two-sided $\mathfrak{A}/\mathfrak{B}$-module.*

Proof. If f is a 1-dimensional $\mathfrak{P}$-cocycle of $\mathfrak{A}$ which cobounds in $\mathfrak{B}$ there exists an element $u \in \mathfrak{P}$ such that $f\{b\} = b \cdot u - u \cdot b$, for every $b \in \mathfrak{B}$. Consider the $(1, \mathfrak{P})$-cocycle f' of $\mathfrak{A}$ defined by: $f'\{a\} = f\{a\} - a \cdot u + u \cdot a$. Then $f'\{b\} = 0$, for every $b \in \mathfrak{B}$, i.e. $\hat{f}' = 0$. Furthermore, $b \cdot f'\{a\} - f'\{ba\} + f'\{b\} \cdot a = 0$, since f' is a cocycle. This reduces to $b \cdot f'\{a\} = 0$, and similarly, $f'\{a\} \cdot b = 0$, for all $b \in \mathfrak{B}$ and all $a \in \mathfrak{A}$. Thus, f' maps $\mathfrak{A}$ into $\mathfrak{Q}$. Clearly, f' may be regarded as a $(1, \mathfrak{Q})$-cocycle f^* of $\mathfrak{A}/\mathfrak{B}$. If f^* cobounds there exists an element $v \in \mathfrak{Q}$ such that $f^*\{a\} = a \cdot v - v \cdot a$. But then $f\{a\} = a \cdot (u + v) - (u + v) \cdot a$, i.e. f cobounds. Hence the mapping $f \to f^*$, which is determined to within a coboundary, defines an isomorphism of the kernel of our homomorphism $H^1(\mathfrak{A}, \mathfrak{p}) \to H^1(\mathfrak{B}, \mathfrak{p})$ into $H^1(\mathfrak{A}/\mathfrak{B}, \mathfrak{Q})$.

Finally, if h^* is any $(1, \mathfrak{Q})$-cocycle of $\mathfrak{A}/\mathfrak{B}$ we can define a $(1, \mathfrak{P})$-cochain h of $\mathfrak{A}$ by the relation $h\{a\} = h^*\{\bar{a}\}$, where $\bar{a}$ is the coset of a mod $\mathfrak{B}$. It is easily seen that h is actually a cocycle which vanishes on $\mathfrak{B}$. Obviously, h is mapped into h^* by the above isomorphism, which is therefore an isomorphism *onto* $H^1(\mathfrak{A}/\mathfrak{B}, \mathfrak{Q})$. This proves Theorem 5.1.

We may also start with any two-sided $\mathfrak{A}/\mathfrak{B}$-module $\mathfrak{Q}$ and make it a two-sided $\mathfrak{A}$-module by the definitions: $a \cdot v = \bar{a} \cdot v$, $v \cdot a = v \cdot \bar{a}$, where $v \in \mathfrak{Q}$ and $\bar{a}$ is the coset of a mod $\mathfrak{B}$. Theorem 5.1 then shows that the kernel of the natural homomorphism of $H^1(\mathfrak{A}, \mathfrak{Q})$ into $H^1(\mathfrak{B}, \mathfrak{Q})$ is isomorphic with $H^1(\mathfrak{A}/\mathfrak{B}, \mathfrak{Q})$. Combining these results with Theorem 4.1 we obtain the following corollary:

Corollary. *A necessary and sufficient condition that $\mathfrak{A}/\mathfrak{B}$ be separable is that the natural homomorphism of $H^1(\mathfrak{A}, \mathfrak{P})$ into $H^1(\mathfrak{B}, \mathfrak{P})$ be an isomorphism, for every two-sided $\mathfrak{A}$-module $\mathfrak{P}$.*

6. Two-dimensional Cohomology and Extensions.

Let $\mathfrak{A}$ be an algebra over Φ. By an extension of $\mathfrak{A}$ we shall mean a pair $\{\mathfrak{B}, \sigma\}$, where $\mathfrak{B}$ is an algebra over Φ and σ a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. The extension $\{\mathfrak{B}, \sigma\}$ is called singular if the kernel $\mathfrak{K}$ of σ satisfies the condition $\mathfrak{K}^2 = \{0\}$. $\mathfrak{A}$ is said to be segregated in the extension $\{\mathfrak{B}, \sigma\}$ if $\mathfrak{B}$ is the direct sum of $\mathfrak{K}$ and a subalgebra $\mathfrak{A}^*$ isomorphic with $\mathfrak{A}$.

We wish to investigate those algebras which have the property of being segregated in every extension. It will be shown that this property is equivalent to that of having all 2-dimensional cohomology groups vanish. It turns out that, in this connection, only the singular extensions are of importance. In fact, we have the following proposition:

Proposition 6.1. *If $\mathfrak{A}$ is segregated in every singular extension then $\mathfrak{A}$ is segregated in every extension.*

Proof. We shall show first that we can reduce the problem to the case where the kernel $\mathfrak{K}$ of the extension is nilpotent. Specifically, we suppose that $\mathfrak{A}$ is segregated in every extension with nilpotent kernel and prove that then $\mathfrak{A}$ is segregated in every extension.

Let $\{\mathfrak{B}, \sigma\}$ be any extension of $\mathfrak{A}$, let $\mathfrak{K}$ denote the kernel of the homomorphism σ, and let $\mathfrak{R}$ denote the radical of $\mathfrak{B}$. Consider $\mathfrak{B}/\mathfrak{R}$: clearly, $(\mathfrak{K} + \mathfrak{R})/\mathfrak{R}$ is an ideal of $\mathfrak{B}/\mathfrak{R}$ (for $\mathfrak{K}$ is an ideal of $\mathfrak{B}$). Since $\mathfrak{B}/\mathfrak{R}$ is semi-simple (we need not consider the case where $\mathfrak{B}$, and hence also $\mathfrak{K}$, is nilpotent), so is $(\mathfrak{K} + \mathfrak{R})/\mathfrak{R}$. Further, $(\mathfrak{K} + \mathfrak{R})/\mathfrak{R}$ is isomorphic with $\mathfrak{K}/\mathfrak{N}$, where $\mathfrak{N} = \mathfrak{K} \cap \mathfrak{R}$. Obviously, $\mathfrak{N}$ is a nilpotent ideal of $\mathfrak{B}$, $\mathfrak{K}/\mathfrak{N}$ is a semi-simple ideal of $\mathfrak{B}/\mathfrak{N}$, and we have the isomorphisms $\mathfrak{A} \cong \mathfrak{B}/\mathfrak{K} \cong (\mathfrak{B}/\mathfrak{N})/(\mathfrak{K}/\mathfrak{N})$.

Now, if $\mathfrak{C}$ is any algebra and $\mathfrak{S}$ a semi-simple ideal of $\mathfrak{C}$ then $\mathfrak{C}$ is the direct sum of $\mathfrak{S}$ and another ideal $\mathfrak{T}$. For, if $\mathfrak{H}$ is the radical of $\mathfrak{C}$ we have $\mathfrak{S} \cap \mathfrak{H} = \{0\}$; $\mathfrak{C}/\mathfrak{H}$ is semi-simple and contains the semi-simple ideal $(\mathfrak{S} + \mathfrak{H})/\mathfrak{H} \cong \mathfrak{S}$. Hence $\mathfrak{C}/\mathfrak{H}$ is the direct sum of $(\mathfrak{S} + \mathfrak{H})/\mathfrak{H}$ and another ideal $\mathfrak{U}$. Clearly, there exists an ideal $\mathfrak{T}$ ($\supset \mathfrak{H}$) of $\mathfrak{C}$ such that $\mathfrak{U} = \mathfrak{T}/\mathfrak{H}$. Since $\mathfrak{U} \cap (\mathfrak{S} + \mathfrak{H})/\mathfrak{H} = \{0\}$ we have $\mathfrak{T} \cap \mathfrak{S} \subset \mathfrak{H}$, and since $\mathfrak{S} \cap \mathfrak{H} = \{0\}$ this implies that $\mathfrak{T} \cap \mathfrak{S} = \{0\}$. Thus, $\mathfrak{C}$ is the direct sum of $\mathfrak{S}$ and $\mathfrak{T}$.

Applying this to the above we conclude that there exists an ideal $\bar{\mathfrak{D}}$ of $\mathfrak{B}/\mathfrak{N}$ such that $\mathfrak{A} \cong \bar{\mathfrak{D}}$. But $\bar{\mathfrak{D}}$ is of the form $\mathfrak{D}/\mathfrak{N}$, where $\mathfrak{D}$ is an ideal of $\mathfrak{B}$, and there is a homomorphism of $\mathfrak{D}$ onto $\mathfrak{A}$ with the nilpotent kernel $\mathfrak{N}$. By hypothesis, there exists a subalgebra $\mathfrak{A}^*$ of $\mathfrak{D}$ such that $\mathfrak{A}^* \cong \mathfrak{A}$ and $\mathfrak{D}$ is the direct sum of $\mathfrak{N}$ and $\mathfrak{A}^*$. Obviously, $\mathfrak{A}^*$is also a subalgebra of $\mathfrak{B}$, and $\mathfrak{B}$ is the direct sum of $\mathfrak{K}$ and $\mathfrak{A}^*$, i.e. $\mathfrak{A}$ is segregated in $\{\mathfrak{B}, \sigma\}$.

It remains to show that if $\mathfrak{A}$ is segregated in every singular extension then $\mathfrak{A}$ is segregated in every extension whose kernel is nilpotent. This we prove by an induction on the dimension of the kernel. If the kernel is of dimension 0, $\mathfrak{B}$ is isomorphic with $\mathfrak{A}$ and there is nothing to prove. Suppose now that the dimension of the nilpotent kernel is n, and that our result has already been established for all extensions with a nilpotent kernel of dimension less than n. The homomorphism σ of $\mathfrak{B}$ onto $\mathfrak{A}$ induces a homomorphism $\bar{\sigma}$ of $\mathfrak{B}/\mathfrak{K}^2$ onto $\mathfrak{A}$ whose kernel is $\mathfrak{K}/\mathfrak{K}^2$, so that $\{\mathfrak{B}/\mathfrak{K}^2, \bar{\sigma}\}$ is a singular extension of $\mathfrak{A}$. Hence, there exists a subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{B}/\mathfrak{K}^2$ such that $\bar{\mathfrak{A}} \cong \mathfrak{A}$ and $\mathfrak{B}/\mathfrak{K}^2$ is the direct sum of $\mathfrak{K}/\mathfrak{K}^2$ and $\bar{\mathfrak{A}}$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}$ such that $\mathfrak{C}/\mathfrak{K}^2 = \bar{\mathfrak{A}}$. Then there exists a homomorphism τ of $\mathfrak{C}$ onto $\mathfrak{A}$ whose kernel is $\mathfrak{K}^2$. Since $\mathfrak{K}$ is nilpotent, dim $(\mathfrak{K}^2) < n$, and it follows from our inductive hypothesis that there exists a subalgebra $\mathfrak{A}^*$ of $\mathfrak{C}$ such that $\mathfrak{A}^* \cong \mathfrak{A}$ and $\mathfrak{C}$ is the direct sum of $\mathfrak{K}^2$ and $\mathfrak{A}^*$. Clearly, $\mathfrak{A}^*$ is also a subalgebra of $\mathfrak{B}$ and $\mathfrak{B}$ is the direct sum of $\mathfrak{K}$ and $\mathfrak{A}^*$. This completes the proof of Proposition 6.1.

In view of Proposition 6.1 we may confine our attention to singular extensions. We shall say that two extensions $\{\mathfrak{B}, \sigma\}$ and $\{\mathfrak{B}^*, \sigma^*\}$ are isomorphic if there exists an isomorphism I of $\mathfrak{B}$ onto $\mathfrak{B}^*$ such that $\sigma^* I = \sigma$. We wish to show that there is a one to one correspondence between the classes of isomorphic singular extensions of $\mathfrak{A}$ and its 2-dimensional cohomology classes.

Let $\{\mathfrak{B}, \sigma\}$ be a singular extension of $\mathfrak{A}$ with kernel $\mathfrak{K}$. Let ρ be any linear mapping of $\mathfrak{A}$ into $\mathfrak{B}$ such that

$$\sigma\{\rho\{a\}\} = a, \quad \text{for every} \quad a \in \mathfrak{A}, \quad \text{or—which is equivalent—}$$

$$\rho\{\sigma\{b\}\} \equiv b \bmod \mathfrak{K}, \quad \text{for every} \quad b \in \mathfrak{B}. \tag{1}$$

We can make $\mathfrak{K}$ a two-sided $\mathfrak{A}$-module by the definitions $a \cdot k = \rho\{a\}k$, $k \cdot a = k\rho\{a\}$, where $k \in \mathfrak{K}$, $a \in \mathfrak{A}$. Since the difference of any two mappings ρ satisfying (1) is a mapping of $\mathfrak{A}$ into $\mathfrak{K}$ and since $\mathfrak{K}^2 = \{0\}$ it is clear that $k \cdot a$ and $a \cdot k$ are independent of the particular mapping ρ we have selected. Thus the two-sided $\mathfrak{A}$-module $\mathfrak{K}$ is uniquely determined by the extension $\{\mathfrak{B}, \sigma\}$. Furthermore, if $\{\mathfrak{B}^*, \sigma^*\}$ is an isomorphic extension, we have:

$$\sigma^*\{\rho^*\{a\}\} = a = \sigma\{\rho\{a\}\} = \sigma^* I\{\rho\{a\}\}, \quad \text{i.e.} \quad \rho^*\{a\} \equiv I\{\rho\{a\}\} \bmod \mathfrak{K}^*.$$

Hence,

$$a \cdot I\{k\} = \rho^*\{a\} I\{k\} = I\{\rho\{a\}\} I\{k\} = I\{a \cdot k\},$$

and, similarly,

$$I\{k\}\cdot a = I\{k\cdot a\}.$$

This means that I is an operator isomorphism between the two-sided $\mathfrak{A}$-modules $\mathfrak{K}$ and $\mathfrak{K}^*$, and we have shown that isomorphic extensions lead to operator-isomorphic $\mathfrak{A}$-modules.

Consider now the mapping $a_1 \times a_2 \to f_\rho\{a_1, a_2\} = \rho\{a_1\}\rho\{a_2\} - \rho\{a_1a_2\}$ of $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{K}$. It is easy to verify that f_ρ is a $(2, \mathfrak{K})$-cocycle of $\mathfrak{A}$.

If ρ' is a second linear mapping of $\mathfrak{A}$ into $\mathfrak{B}$ satisfying (1), we have:

$$\begin{aligned} f_\rho\{a_1, a_2\} - f_{\rho'}\{a_1, a_2\} &= \rho\{a_1\}[\rho\{a_2\} - \rho'\{a_2\}] \\ &\qquad - [\rho\{a_1a_2\} - \rho'\{a_1a_2\}] + [\rho\{a_1\} - \rho'\{a_1\}]\rho'\{a_2\} \\ &= a_1\cdot[\rho\{a_2\} - \rho'\{a_2\}] - [\rho\{a_1a_2\} - \rho'\{a_1a_2\}] \\ &\qquad + [\rho\{a_1\} - \rho'\{a_1\}]\cdot a_2, \end{aligned}$$

i.e. $f_\rho - f_{\rho'} = \delta(\rho - \rho')$. Hence the cohomology class $\bar{f}$ of f_ρ is uniquely determined by the extension $\{\mathfrak{B}, \sigma\}$.

Now let $\{\mathfrak{B}^*, \sigma^*\}$ be a singular extension of $\mathfrak{A}$ which is isomorphic (by the isomorphism I) with $\{\mathfrak{B}, \sigma\}$. Then $I\rho$ is clearly a linear mapping of $\mathfrak{A}$ into $\mathfrak{B}^*$ which satisfies (1). Hence $f_{I\rho}$ is a $(2, \mathfrak{K}^*)$-cocycle of $\mathfrak{A}$ which lies in the cohomology class determined by the extension $\{\mathfrak{B}^*, \sigma^*\}$. Since $f_{I\rho} = If_\rho$, the cohomology class determined by the extension $\{\mathfrak{B}^*, \sigma^*\}$ is the isomorphic image by I of the cohomology class determined by $\{\mathfrak{B}, \sigma\}$.

Conversely, suppose that we are given a two-sided $\mathfrak{A}$-module $\mathfrak{P}$ and a $(2, \mathfrak{P})$-cocycle f of $\mathfrak{A}$. Let $\mathfrak{B}$ be the direct sum of the linear space underlying $\mathfrak{A}$ and the linear space $\mathfrak{P}$. Let (a_1, u_1), (a_2, u_2) $(a_i \in \mathfrak{A}, u_i \in \mathfrak{P})$ be any two elements of $\mathfrak{B}$. We define their product by the equation $(a_1, u_1)(a_2, u_2) = (a_1a_2, a_1\cdot u_2 + u_1\cdot a_2 + f\{a_1, a_2\})$. Using the fact that f is a cocycle we verify easily that this multiplication is associative. Hence $\mathfrak{B}$ has been given the structure of an algebra, $\mathfrak{B}_f$, say. The mapping $(a, u) \to \sigma\{(a, u)\} = a$ is obviously a homomorphism of $\mathfrak{B}_f$ onto $\mathfrak{A}$. The kernel of σ is the ideal $\mathfrak{K} = (0, \mathfrak{P})$ of $\mathfrak{B}_f$ and, clearly, $\mathfrak{K}^2 = \{0\}$. Thus $\{\mathfrak{B}_f, \sigma\}$ is a singular extension of $\mathfrak{A}$. Moreover, if ρ is the mapping $a \to (a, 0)$ of $\mathfrak{A}$ into $\mathfrak{B}_f$, ρ satisfies condition (1) and the $(2, \mathfrak{K})$-cocycle f_ρ derived from the extension $\{\mathfrak{B}_f, \sigma\}$ and ρ is (to within a trivial isomorphism) identical with f. It is obvious that if f and f^* are corresponding cocycles over operator-isomorphic $\mathfrak{A}$-modules the corresponding extensions of $\mathfrak{A}$ are isomorphic.

Finally, let f' be another $(2, \mathfrak{P})$-cocycle of $\mathfrak{A}$ which is cohomologous to f. Then there exists a linear mapping τ of $\mathfrak{A}$ into $\mathfrak{P}$ such that $f - f' = \delta\tau$, i.e.

$$f\{a_1, a_2\} = f'\{a_1, a_2\} + a_1\cdot\tau\{a_2\} - \tau\{a_1a_2\} + \tau\{a_1\}\cdot a_2.$$

Consider the mapping $(a, u) \to I\{(a, u)\} = (a, u + \tau\{a\})$ of $\mathfrak{B}_f$ onto $\mathfrak{B}_{f'}$. We have:

$$\begin{aligned} I\{(a_1, u_1)(a_2, u_2)\} &= (a_1a_2, a_1\cdot u_2 + u_1\cdot a_2 + f\{a_1, a_2\} + \tau\{a_1a_2\}) \\ &= (a_1a_2, a_1\cdot u_2 + u_1\cdot a_2 + a_1\cdot\tau\{a_2\} + \tau\{a_1\}\cdot a_2 + f'\{a_1, a_2\}) \\ &= (a_1, u_1 + \tau\{a_1\})(a_2, u_2 + \tau\{a_2\}) = I\{(a_1, u_1)\}I\{(a_2, u_2)\}, \end{aligned}$$

i.e. I is an isomorphism of $\mathfrak{B}_f$ onto $\mathfrak{B}_{f'}$. Moreover,

$$\sigma'\{I\{(a, u)\}\} = \sigma'\{(a, u + \tau\{a\})\} = a = \sigma\{(a, u)\}.$$

Hence the extensions $\{\mathfrak{B}_f, \sigma\}$ and $\{\mathfrak{B}_{f'}, \sigma'\}$ are isomorphic. We state the foregoing results in the following proposition:

Proposition 6.2. *There is a one to one correspondence between the classes of isomorphic singular extensions, and the* 2*-dimensional cohomology classes of* $\mathfrak{A}$. (*We identify corresponding cohomology classes over operator-isomorphic* $\mathfrak{A}$*-modules.*)

Next we prove:

Proposition 6.3. $\mathfrak{A}$ *is segregated in a singular extension if and only if the corresponding cohomology class is zero.*

Proof: We note that if $\mathfrak{A}$ is segregated in the extension $\{\mathfrak{B}^*, \sigma^*\}$, and if $\{\mathfrak{B}, \sigma\}$ is isomorphic with $\{\mathfrak{B}^*, \sigma^*\}$, then $\mathfrak{A}$ is segregated also in $\{\mathfrak{B}, \sigma\}$. Now let $\{\mathfrak{B}, \sigma\}$ be an extension which determines a zero cohomology class. Then it is obvious that $\mathfrak{A}$ is segregated in the extension $\{\mathfrak{B}_0, \sigma^*\}$ constructed from the representative $f = 0$ of this class. Since $\{\mathfrak{B}, \sigma\}$ is isomorphic with $\{\mathfrak{B}_0, \sigma^*\}$, $\mathfrak{A}$ is segregated in $\{\mathfrak{B}, \sigma\}$. The necessity of the condition is obvious.

From Propositions 6.1, 6.2, and 6.3 follows immediately the main result of this section:

Theorem 6.1. *A necessary and sufficient condition for* $\mathfrak{A}$ *to be segregated in every extension is that* $H^2(\mathfrak{A}, \mathfrak{P}) = \{0\}$, *for every two-sided* $\mathfrak{A}$*-module* $\mathfrak{P}$.

This, with Theorems 4.1 and 3.1 implies a result which is essentially the classical generalization (to arbitrary ground fields) of Wedderburn's third structure theorem:

Corollary: *A separable algebra is segregated in every extension.*

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