Note Title 10/18/2009

HILL – Equivariance and the Kervaire invariant

[Will discuss the slice SS.]

Recall $\tilde{\Omega} = D^{-1} MU_{\mathbb{R}}^{(4)}$

1. If $\exists\, \theta_j$ then its image in $\pi_*(\tilde{\Omega}^{hC_8})$ is $\neq 0$
2. $\pi_*(\tilde{\Omega}^{hC_8})$ is 256-periodic
3. $\pi_{-2}(\tilde{\Omega}^{C_8}) = 0$
4. $\tilde{\Omega}^{hC_8} \simeq \tilde{\Omega}^{C_8}$

Hope to discuss 2 briefly, 3 explicitly, and a toy version of 4. $\quad G$ = finite gp

Two flavors of homotopy: X is a G-spectrum

1 $\quad [G_+ \wedge_H S^n, X]_G \qquad H \subseteq G$

$$\| $$

$$[S^n, X]_H = \pi_n(X^H)$$

$$G_+ \wedge_H S^n = (G/H)_+ \wedge S^n$$

$$G/H \longrightarrow G/K \quad \text{for} \quad H \subset K$$

$$\Rightarrow \pi_n(X^K) \longrightarrow \pi_n(X^H) \quad (\text{restriction})$$

We also have a transfer map $tr : \pi_n(X^H) \to \pi_n(X^K)$

Let $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$

This forms a Mackey functor.

Mackey functors are to equiv stable htpy
as
abelian grps are to stable htpy

We have an Eilenberg-MacLane object $H\underline{M}$ for any Mackey functor $\underline{M}$.

(2) If $\alpha \in RO(G)$ is a virtual rep, we have

$$\pi_\alpha(X) = [S^\alpha, X]_G = [S^0, X \wedge S^{-\alpha}]_G$$

This is an $RO(G)$ graded Mackey functor

$$\underline{\pi}_n(X \wedge S^{\alpha})$$

The slice SS is one of $RO(G)$-graded Mackey functors.

Thm There is an equiv filtration of $MU_{\mathbb{R}}^{((4))}$ (for $G = C_8$) s.t. the associated graded

$$GM = \bigvee_{\text{indexing}} H\underline{\mathbb{Z}} \wedge (C_{8+} \wedge_H S^{k\rho_H})$$

Will discuss the indexing set and how to compute htpy.

Indexing

$\pi_\star (MU_{\mathbb{R}}^{(4)}) (C_8/C_2)$ = underlying htpy gp

$= \mathbb{Z}[r_1, \gamma(r_1), \gamma^2(r_1), \gamma^3(r_1), r_2, \dots]$ $|r_i| = 2i$.

$C_8 \ni \gamma$ sends $\gamma^3(r_i)$ to $-r_i$

G permutes mod 2 monomials

Given a monomial p mod 2

① $C_2 \leq \mathrm{Stab}^H(p) \leq C_8 = G$

To the orbit of p we associate

$H\underline{\mathbb{Z}} \wedge (G_+ \wedge_H S^{|p|\rho_H/|H|})$

Example:
$$\begin{tikzcd} M_1 \arrow[r] & \gamma^1(M_1) \arrow[r] & \gamma^2(M_1) \arrow[r] & \gamma^3(M_1) \arrow[lll, bend left, "\sim"] \end{tikzcd}$$

so $\mathrm{Stab}(M_1) = C_2$ and we get

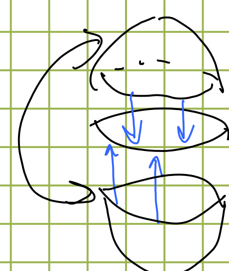
$$H\underline{\mathbb{Z}} \wedge (C_{8+} \wedge_{C_2} S^{\rho_{C_2}})$$

Now we compute

Let $G = C_2$ and we want $\pi_*(H\underline{\mathbb{Z}} \wedge S^{\rho_{C_2}})$

$\rho_{C_2} = 1 \oplus \sigma$ where σ = sign rep

$(S^{\rho_{C_2}})^{C_2} = S^1$



Equivariant cell complex with two 2-cells.

The resulting chain complex has the form

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z}^2 = \mathbb{Z}[C_2] \arrow[d, "{[1\ 1]}"'] \\ 1 & \mathbb{Z} = \mathbb{Z}[C_2/C_2] \end{tikzcd}$$

We know $H_*(\text{complex}) = \tilde{H}_*(S^2)$

Passing to fixed points gives

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d, "2"] & \mathbb{Z}[C_2] \arrow[d, "{[1\ 1]}"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

so $\pi_*\left(\left(H\mathbb{Z} \wedge S^{\rho_{C_2}}\right)^{C_2}\right) = \begin{cases} \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$

$G = C_4$ $\qquad \rho_{C_4} = 1 \oplus \sigma \oplus \lambda$ $\qquad \lambda$ = rotation by $\pi/2$

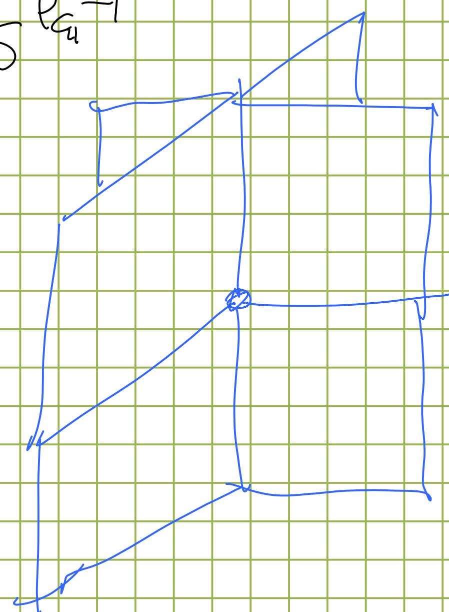
$C_4 = \mu_4 \hookrightarrow S^1$

Will compute $S^{\rho_{C_4}-1}$

$$(S^{\rho_G-1})^G = S^0$$

$$(S^{\rho_G-1})^{C_2} = S^1$$

There are four 2-cells and four 3-cells.



$$\begin{array}{ll}
3 & \mathbb{Z}^4 = \mathbb{Z}[C_4] \\
 & \downarrow \\
2 & \mathbb{Z}^4 = \mathbb{Z}[C_4] \\
 & \downarrow \\
1 & \mathbb{Z}^2 = \mathbb{Z}[C_4/C_2] \\
 & \downarrow {\scriptstyle [1\ 1]} \\
0 & \mathbb{Z}
\end{array}$$

General case $\quad H\underline{\mathbb{Z}} \wedge S^{kp_G} \qquad G = C_8$

we get 1 cell in dim k

2 cells in dims $k+1$ to $2k$

4 cells $\quad 2k+1$ to $4k$

8 cells $\quad 4k+1$ to $8k$

Lemma $\quad \pi_{-2}\left(\left(H\underline{\mathbb{Z}} \wedge \left(C_{8+} \underset{H}{\wedge} S^{kp_H}\right)\right)^{C_8}\right)$ for $|H| \geq 2$

This is obvious for $k \geq 0$

For $k < 0$ compute via the dual complex to that for S^{-kp_G}

For $k < -2$ there are no cells in dim -2

Will consider $S^{-\rho_{C_2}}$ and $S^{-\rho_{C_{2}}-1}$

fixed pts

$$\begin{array}{ccccc}
\mathbb{Z}^2 & & \mathbb{Z} & & \mathbb{Z} \\
\downarrow {\scriptstyle [1\ 1]} & & \downarrow {\scriptstyle \begin{bmatrix}1\\1\end{bmatrix}} & & \downarrow {\scriptstyle 2} \\
\mathbb{Z} & & \mathbb{Z}^2 & & \mathbb{Z} \\
 & & & & \\
S^{\rho_{C_2}} & & S^{-\rho_{C_2}} & &
\end{array}$$

so $\pi_{-2}(H\mathbb{Z} \wedge S^{-\rho_{C_2}}) = 0$

The same happens for S^{kH} for $k = -1, -2$ and all nontrivial subgps H.

Inverting D — D is carried by $S^{k\rho_{C_8}}$ for some k.

Cor $\pi_{-2}\left(\Sigma^{-k\rho_8} MU_{\mathbb{R}}^{((4))}\right)^{C_8} = 0 \Rightarrow \pi_{-2}\,\tilde{\Omega}^{C_8} = 0$

What is the slice SS, studied earlier by Dugger, Hu-Kriz, Hopkins-Morel, Voevodsky. It is a type of Postnikov tower.

We kill off space of maps from spheres of the form

$$G_+ \wedge_H S^{k\rho_H} \text{ and } G_+ \wedge_H S^{k\rho_H - 1}$$

The filtration is the dim of the underlying spheres

<u>Example</u> KR as a C_2-spectrum (Dugger)

Slice assoc. graded of KR is

$$\bigvee_{n \in \mathbb{Z}} H\mathbb{Z} \wedge S^{n\rho_{C_2}}$$

Picture of slice S