Hill - MFO - Q+A

Note Title 9/23/2010

Q: What is the slice filtration + why are the subquotients what you say they are?

A: Let G be a finite gp.

Recall the classical Postnikov tower

$$\cdots \leftarrow X_n \leftarrow X_{n+1} \leftarrow X_{n+2}$$

$$\varprojlim X_n = X, \quad \varinjlim X = *$$

X_n = localization where all n-conn spectra are points

Form X_n by killing all htpy gps above dim n.

To do this equivariantly we need subcategories analogous to n-connected spectra

① Choose a collection of rep spheres for G and all of its subgps

$\dim(G_+ \wedge_H S^V) = \dim V$

$\mathcal{T}_n$ = localizing subcat by all spheres of dim $> n$.

The collection should be closed restriction and transfer.

② Define X_n as before

Examples

a) If $V = \mathbb{R}^n$ for any $H \leq G$

$\mathcal{T}_n = \langle G/H_+ \wedge S^m : m > n \rangle$

= subcat of all equivariantly n-connected spectra

$\Longrightarrow$ equiv Postnikov tower

Fibers are EM spectra

b) as above but only for proper subgps H.

$\underrightarrow{\lim}\, X_n = \tilde{E}P \wedge X$ because we are killing all induced cells.

$\varprojlim X_n = X$

Fibers may have the form $\widetilde{EP} \wedge H\underline{A}$

c) $V_H = k\rho_H - \varepsilon$ for $k \in \mathbb{Z}$ and $\varepsilon = 0, 1$

$\dim V_H = k|H| - \varepsilon$.

$\tau_n = \langle G_+ \wedge_H S^{k\rho_H - \varepsilon} : k|H| - \varepsilon > n \rangle$

This gives the slice tower, which refines the classical Postnikov tower. [Closure under colimits means closure under retracts, so we get a spheres of dim $> n$.]

$\tau_{-1} = \langle G/H_+ \mid H \subseteq G \rangle$ = all (-1) connected spectra

$\tau_{-2} = \langle \Sigma^{-1} G/H_+ \mid H \subseteq G \rangle$

so (-1)-slices are the $\Sigma^{-1} H\underline{M}$ for all Mackey functors $\underline{M}$.

$\tau_0 = \langle G_+ \underset{H}{\wedge} S^{\rho_H - 1} \mid H \neq \{e\}$ and $G_+ \wedge S^0 \rangle$

e.g. for $G = C_2$, $\tau_0 = \langle S^\rho, G_+ \wedge S^0 \rangle$

This includes all 0-connected spectra, but S^ρ is not 0-connected.

If $X \in \mathcal{T}_{n-1}$ then X is $\left\lfloor \frac{n}{|G|} \right\rfloor$-connected for $n \geq 0$ because the same is true of each generator.

<u>Cor</u> $\varprojlim X_n = X$ and $\varinjlim X = *$ so slice SS converges.

The set of generators is closed under suspension / desuspension by S^{ρ_G}.

Claim the 0-slice $^{P_0^0 S^0}$ of S^0 is $H\underline{\mathbb{Z}}$. assume G is cyclic p-group.

① $P_0^0 S^0 = H\underline{M}$ for some $\underline{M}$

Let $\underline{A}$ be the Burnside ring Mackey functor

$$[G_+ \wedge_H S^{\rho_H - 1}, H\underline{A}]_G = [S^{\rho_H - 1}, H\underline{A}]_H$$

$$= [[S^{\rho_H - 1}]^1, H\underline{A}]_H$$

1 - skeleton

$\bar{H}$ = index p subgp of H

$$H/\bar{H}_+ \wedge S^0 \longrightarrow S^0 \longrightarrow [\;]^1$$

$$[-, H\underline{A}]: \quad A(\bar{H}) \xleftarrow{\text{res}} A(H) \leftarrow \text{kernel}$$

i.e. $\quad [S^{\rho_H - 1}, H\underline{A}]_H = \ker(\text{restriction } A(H) \longrightarrow A(\overline{H}))$

$$x \in I(G) \lhd A(G)$$

$$c([G/\tilde{G}] - p) + tr(\text{---})$$

$$\Longrightarrow \begin{tikzcd} P_0^0 S^0 & \\ H\underline{A} \arrow[u] \arrow[r] & H\underline{\mathbb{Z}} \arrow[ul, dashed, "\simeq"'] \end{tikzcd}$$

0-slices are $H\underline{M}$ for Mackey functors $\underline{M}$ in which all restrictions are mono

[Are these all modules over $H\underline{\mathbb{Z}}$???

Probably since all spectra are modules

over S^0 and $P_0^0 S^0 = H\underline{\mathbb{Z}}$]

For $G = C_2$ we understand all slices since we know (-1)- and (0)-slices.

<u>MU-slices</u>!

$$\pi_* MU = \mathbb{Z}[x_1, x_2 \ldots]$$

x_i is refined by $\overline{x}_i : S^{i\rho_2} \longrightarrow MU_{\mathbb{R}}$

<u>Thm</u> $MU_{\mathbb{R}} / (\overline{x}_1, \overline{x}_2 \ldots) = H\underline{\mathbb{Z}}$ and is $P_0^0 MU_{\mathbb{R}}$

This is a hard theorem.

The sequence $(\overline{x}_1, \overline{x}_2 \ldots)$ is not regular in $\pi_\star MU_{\mathbb{R}}$

Conj: $MU/(\bar{x}_i, \bar{x}_{i+1}, \ldots)$ is a ring.

We have an A_∞-map $\forall i > 0$

$$W_i = \bigvee_{j \geq 0} S^{ij\rho_2} \xrightarrow{\bar{x}_i^j} MU_{\mathbb{R}}$$

$$\begin{tikzcd}
\bigwedge_{i>0} W_i = \bigvee_{\substack{\text{monomials} \\ P \text{ in } MU_*}} S^{\frac{|P|}{2}\rho_2} \arrow[r] \arrow[d] & MU_{\mathbb{R}} \\
S^0 &
\end{tikzcd}$$

$\bigwedge_{i>0} W_i = A$

$MU_{\mathbb{R}} \wedge_A S^0$ is defined and is $H\mathbb{Z}$.

A has filtration by monomial degree

$\{F_n\}$ = sub-bimodule of degree $\geq n$
dim $\geq 2n$

and $\{MU_{\mathbb{R}} \wedge_A F_n\}$ is a filtration of $MU_{\mathbb{R}}$

n th subquotient is

$$(MU_{\mathbb{R}} \wedge_A S^0) \wedge F^n/F^{n+1}$$
$$\parallel$$
$$H\underline{\mathbb{Z}} \wedge F^n/F^{n+1} = H\underline{\mathbb{Z}} \wedge (\text{wedge of } S^{n\rho_2})$$

There is an easy lemma that says any tower (such as this one) that looks like the slice tower <u>is</u> the slice tower.

Notation $\rho_{|G|} = \rho_G$ $\qquad |G| = g$

A as above

We have C_2 map $A \longrightarrow MU^{((2^n))}$ using the generators r_i instead of x_i. Apply N_2^g and get

$$\begin{array}{ccc} N_2^g\, A & \longrightarrow & N_2^G\, \mathrm{res}^g_2\, MU^{((2^n))} \\ \| \quad \cup & & \downarrow \\ \quad B & & N_2^G\, MU_{\mathbb{R}} \end{array}$$

wedge that appears in slice theorem.

$$MU^{((2^n))} \bigwedge_B S^0$$

Reduction Thm $MU^{((\Sigma_n))} \underset{B}{\wedge} S^0 = H\underline{\mathbb{Z}}$.

Proof requires Φ^G of both sides which are the same. The left side is not a ring because B is not commutative.

Slice Theorem follows for the same formal reasons

Homework: $\left(G_+ \underset{H}{\wedge} S^{k\rho_H - \varepsilon}\right) \wedge H\underline{\mathbb{Z}}$ is a $(k|H|-\varepsilon)$-slice.

For the slices of $MU^{((G))}$ we need attaching maps from $\Sigma^{-1}\left(G_+ \underset{H}{\wedge} S^{k\rho_H}\right)$