ITERATED SPANS AND CLASSICAL TOPOLOGICAL FIELD THEORIES

RUNE HAUGSENG

ABSTRACT. We construct higher categories of iterated spans, possibly equipped with extra structure in the form of higher-categorical local systems, and classify their fully dualizable objects. By the Cobordism Hypothesis, these give rise to framed topological quantum field theories, which are the framed versions of the classical TQFTs considered in the quantization programme of Freed-Hopkins-Lurie-Teleman.

Using this machinery, we also construct an $(\infty, 1)$-category of symplectic derived algebraic stacks and Lagrangian correspondences and show that all its objects are dualizable.

CONTENTS

1. INTRODUCTION

In this paper our main goal is to construct a fundamental family of higher categories, namely the symmetric monoidal (∞, n)-categories $\mathrm{Span}_n(\mathcal{C})$ of iterated spans in an $(\infty, 1)$-category (or ∞-category) $\mathcal{C}$ with finite limits, and to classify the fully dualizable objects in these (∞, n)-categories. Via the Cobordism Hypothesis, these objects correspond to the framed extended topological quantum field theories (TQFTs) valued in the (∞, n)-categories $\mathrm{Span}_n(\mathcal{C})$, which can be interpreted as a simple model for classical topological field theories.

We also construct variants of these (∞, n)-categories where the spans are equipped with extra structure in the form of local systems, and classify their fully dualizable objects. The

Date: November 30, 2018.

corresponding TQFTs are those proposed as classical field theories in the quantization programme outlined by Freed, Hopkins, Lurie, and Teleman [FHLT10].

Finally, we apply our machinery in the context of shifted symplectic structures on derived algebraic stacks, as developed by Pantev, Toën, Vaquié, and Vezzosi [PTVV13], to construct $(\infty, 1)$-categories of n-shifted symplectic derived stacks with the morphisms given by Lagrangian correspondences; we also show that here all objects are dualizable (and thus determine framed 1-dimensional TQFTs).

Before we describe our results in more detail and discuss how they relate to extended TQFTs, we will first look at the analogous classical field theories in the simpler setting of non-extended TQFTs.

1.1. **Topological Quantum Field Theories and Spans.** *Topological quantum field theories* (or TQFTs) originated in physics as a particularly simple class of quantum field theories. They were first formalized mathematically by Atiyah [Ati88] in the late 1980s, and have been the subject of much research over the past two decades. A sketch of Atiyah's definition goes as follows:

Definition 1.1. Let Cob_n be the category with objects closed $(n-1)$-dimensional manifolds and morphisms diffeomorphism classes of n-dimensional cobordisms, i.e. a morphism from M to N is an n-manifold X with an identification of ∂X with $M \amalg N$; composition of morphisms is given by gluing along the boundary, and the disjoint union of manifolds makes this a symmetric monoidal category. If $\mathbf{C}$ is another symmetric monoidal category, a $\mathbf{C}$-valued n-dimensional *topological quantum field theory* is a symmetric monoidal functor $\mathrm{Cob}_n \to \mathbf{C}$.

Reflecting the linearity of quantum mechanics, in examples $\mathbf{C}$ is typically a category of "linear" objects, for example complex vector spaces or chain complexes of these.

Ideas from physics suggest that one should be able to produce interesting examples of TQFTs as quantizations of classical topological field theories — this proposal goes back at least to [Fre94]. In a very simple picture of a classical field theory we assign to a manifold M a collection $\mathcal{F}(M)$ of *fields* on M, which will typically be some form of stack. If K is a cobordism from M to M', we can restrict the fields on K to the boundary, giving a *span*

$$\begin{tikzcd} & \mathcal{F}(K) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathcal{F}(M) & & \mathcal{F}(M'). \end{tikzcd}$$

Moreover, these fields should be *local*: if K is obtained from cobordisms K_1 and K_2 by gluing them along a common boundary N, giving a field on K should be the same as giving fields on K_1 and K_2 that agree on N. In other words, the stack $\mathcal{F}(K)$ should be the pullback $\mathcal{F}(K_1) \times_{\mathcal{F}(N)} \mathcal{F}(K_2)$. In this case the cobordism K is the composite of K_1 and K_2 in Cob_n, so we want $\mathcal{F}$ to be a functor from Cob_n to a category where the morphisms are spans and the composition is given by taking pullbacks.

More precisely, if $\mathbf{C}$ is any category with finite limits, we would like to define a category $\mathrm{Span}(\mathbf{C})$ where the objects are the objects of $\mathbf{C}$ and the morphisms from X to Y are spans

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in $\mathbf{C}$. If $X \leftarrow A \rightarrow Y$ and $Y \leftarrow B \rightarrow Z$ are two spans, their composite should be given by the pullback

$$\begin{tikzcd}
 & & A \times_Y B \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\
X & & Y & & Z.
\end{tikzcd}$$

However, taking pullbacks in this way is not strictly associative, only associative up to canonical isomorphisms, so to get a category we are forced to take morphisms in $\mathrm{Span}(\mathbf{C})$ to be isomorphism classes of spans. The Cartesian product in $\mathbf{C}$ then gives $\mathrm{Span}(\mathbf{C})$ a natural symmetric monoidal structure, and we can think of a classical field theory as a TQFT valued in $\mathrm{Span}(\mathbf{C})$ for some $\mathbf{C}$.

A quantization of such a classical field theory, which is supposed to be analogous to the path integral of quantum field theory, would then assign some algebraic object to the stack $\mathcal{F}(M)$, for example the value of some cohomology theory $E^*\mathcal{F}(M)$, and to the span

$$\mathcal{F}(M) \xleftarrow{s} \mathcal{F}(K) \xrightarrow{t} \mathcal{F}(M')$$

a push-pull composite $t_*s^* \colon E^*\mathcal{F}(M) \to E^*\mathcal{F}(K) \to E^*\mathcal{F}(M')$, where the pushforward t_* is thought of as integrating over the fibres of t.

1.2. **Extended TQFTs and Iterated Spans.** Although relatively easy to define, Atiyah's notion of TQFTs suffers from a number of defects, and recently much work has focused on the more sophisticated notion of *extended* topological quantum field theories. This was first formulated in terms of *n-categories* by Baez and Dolan [BD95], building on earlier work by a number of mathematicians, including Lawrence [Law93] and Freed [Fre94]. Roughly speaking, an n-category is a structure that has objects, morphisms between objects, 2-morphisms between morphisms, and so on up to n-morphisms. For the definition of Baez and Dolan we want to consider an n-category Bord_n whose objects are compact 0-manifolds, with morphisms given by 1-dimensional cobordisms between 0-manifolds, and in general i-morphisms for $i = 1, \ldots, n$ given by i-dimensional cobordisms between manifolds with corners, taking diffeomorphism classes of these for the n-morphisms. Given such a symmetric monoidal n-category Bord_n, with the tensor product again given by taking disjoint unions, an *n-dimensional extended TQFT* valued in a symmetric monoidal n-category $\mathcal{C}$ should be a symmetric monoidal functor $\mathrm{Bord}_n \to \mathcal{C}$.

We would like to define an extended analogue of the classical topological field theories we considered above. For example, if we have an n-manifold K whose boundary ∂K is a manifold with corners, i.e. we have a decomposition of ∂K as $M \cup_{A \amalg B} N$ where M and N are both $(n-1)$-manifolds with boundary the $(n-2)$-manifold $A \amalg B$, then by restricting the fields on K we get a span of spans

$$\begin{tikzcd}
 & & \mathcal{F}(K) \arrow[dl] \arrow[dr] & & \\
 & \mathcal{F}(M) \arrow[dl] \arrow[drrr] & & \mathcal{F}(N) \arrow[dlll] \arrow[dr] & \\
\mathcal{F}(A) & & & & \mathcal{F}(B),
\end{tikzcd}$$

so we want to have such 2-fold spans as 2-morphisms in the target. More precisely, if $\mathbf{C}$ is a category with finite limits we'd like to construct a 2-category where the objects are the objects of $\mathbf{C}$, the 1-morphisms are spans in $\mathbf{C}$, and the 2-morphisms from $X \leftarrow A \rightarrow Y$ to $X \leftarrow B \rightarrow Y$ are given by (isomorphism classes of) 2-fold spans, i.e. diagrams of the form

$$\begin{tikzcd}
 & J \arrow[dl] \arrow[dr] & \\
A \arrow[d] \arrow[drr] & & B \arrow[dll] \arrow[d] \\
X & & Y.
\end{tikzcd}$$

We can think of this 2-fold span as a span in the slice category $\mathbf{C}_{/X\times Y}$ (whose objects are themselves spans from X to Y), which suggests that the general target for a classical extended TQFT should be an n-category $\mathrm{Span}_n(\mathbf{C})$ where an i-morphism between objects X and Y is inductively defined to be an $(i-1)$-morphism in $\mathrm{Span}_{n-1}(\mathbf{C}_{/X\times Y})$. This n-category should also have a symmetric monoidal structure induced by the Cartesian product in $\mathbf{C}$.

To give precise definitions of both of the n-categories Bord_n and $\mathrm{Span}_n(\mathbf{C})$ we need to consider *weak* n-categories: as is typically the case for interesting structures that we want to organize as n-categories, the composition of (higher) morphisms is not strictly associative, only associative up to a coherent choice of (specified) invertible higher morphisms. Unfortunately, although the notion of weak n-categories intuitively makes sense, making it precise becomes increasingly intractable as n increases. For our n-categories we also want symmetric monoidal structures, which introduces additional complications — a complete definition of Bord_2 as a symmetric monoidal 2-category has been given by Schommer-Pries [SP14], but for larger n it seems that an appropriate notion of symmetric monoidal n-category has not even been defined.

We will therefore instead work with *(∞, n)-categories*. In the same way as n-categories these have i-morphisms for $i = 1, \ldots, n$, but in addition they have *invertible* i-morphisms for $i > n$. This might seem an even more complicated notion to rigorously formalize than that of n-categories, but it turns out that by using homotopy theory we can set up notions of (∞, n)-category that are quite easy to work with in practice, such as the iterated complete Segal spaces of Barwick [Bar05] (which is the model we will make use of in this paper) and the complete Θ_n-spaces of Rezk [Rez10]. Heuristically, the advantage of working with (∞, n)-categories is that we can often avoid dealing with coherence issues by not making any choices, e.g. of compositions, and instead only need to check that the space of possible choices is contractible.

Our first main result gives a construction of (∞, n)-categories of iterated spans with the properties we discussed above:

Theorem 1.2. *Suppose $\mathcal{C}$ is an ∞-category (i.e. $(\infty, 1)$-category) with finite limits. Then there exists an (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ such that:*

(i) The objects of $\mathrm{Span}_n(\mathcal{C})$ are the objects of $\mathcal{C}$.
(ii) The 1-morphisms of $\mathrm{Span}_n(\mathcal{C})$ are spans in $\mathcal{C}$.
(iii) For X and $Y \in \mathcal{C}$, the $(\infty, n-1)$-category of maps from X to Y in $\mathrm{Span}_n(\mathcal{C})$ is $\mathrm{Span}_{n-1}(\mathcal{C}_{/X\times Y})$.
(iv) The (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ has a natural symmetric monoidal structure induced by the Cartesian product in $\mathcal{C}$.

We'll construct these (∞, n)-categories (in the form of n-fold Segal spaces) in §5 and prove that they are complete in §8, where we also identify their mapping $(\infty, n-1)$-categories, and we construct the symmetric monoidal structure in §12. The definition we consider generalizes that of Barwick [Bar13a] in the case $n = 1$.

Note that even if $\mathbf{C}$ is an ordinary category, the ∞-category $\mathrm{Span}_1(\mathbf{C})$ will still have invertible 2-morphisms, given by isomorphisms of spans — since composing spans by taking pullbacks is well-defined up to a canonical choice of such an invertible 2-morphism, we do not have to take isomorphism classes at the top level.

We would like to consider these (∞, n)-categories as targets for extended TQFTs, with these also reformulated using the language of (∞, n)-categories. Such a definition was introduced by Lurie, and a sketch of it goes as follows:

Definition 1.3. Let Bord_n be the (∞, n)-category whose objects are closed 0-manifolds and whose i-morphisms for $i \leq n$ are i-dimensional cobordisms between $(i-1)$-manifolds with corners, with $(n+1)$-morphisms being diffeomorphisms of such cobordisms, $(n+2)$-morphisms smooth homotopies of such diffeomorphisms, and so on. The disjoint union of manifolds gives this a symmetric monoidal structure, and if $\mathcal{C}$ is a symmetric monoidal (∞, n)-category, a $\mathcal{C}$-valued n-dimensional *extended topological quantum field theory* is a symmetric monoidal functor $\mathrm{Bord}_n \to \mathcal{C}$.

In [Lur09c] Lurie sketches a definition of the symmetric monoidal (∞, n)-categories Bord_n, and a complete construction has recently been carried out in full detail by Calaque and Scheimbauer [CS15].

Baez and Dolan conjectured that *framed* extended TQFTs (where we consider cobordisms that are equipped with a framing of their tangent bundle) valued in any symmetric monoidal n-category $\mathbf{C}$ are classified by the *fully dualizable* objects in $\mathbf{C}$. (Being fully dualizable is an inductively defined algebraic notion; in the case $n = 1$ it reduces to the usual notion of dualizability for an object in a symmetric monoidal ∞-category. We will review the general definition below in §11.) This conjecture is known as the *Cobordism Hypothesis*. In [Lur09c], Lurie introduced the natural generalization of the Cobordism Hypothesis to (∞, n)-categorical framed extended TQFTs, and gave a detailed sketch of a proof thereof.

Our second main result gives a description of the fully dualizable objects of the (∞, n)-categories $\mathrm{Span}_n(\mathcal{C})$, and thus, via the Cobordism Hypothesis, a classification of the framed extended TQFTs with this target:

Theorem 1.4. *Suppose $\mathcal{C}$ is an ∞-category with finite limits. Then all objects of the (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ are fully dualizable with respect to the natural symmetric monoidal structure induced by the Cartesian product in $\mathcal{C}$.*

We'll show this in section §12. In fact, we'll show that these (∞, n)-categories *have duals* in the sense of [Lur09c], meaning that all the objects are dualizable and all i-morphisms have left and right adjoints for all $i < n$.

1.3. **Iterated Spans with Local Systems.** In [FHLT10], Freed, Hopkins, Lurie, and Teleman discuss extended TQFTs valued in (∞, n)-categories where the higher morphisms are iterated spans of spaces equipped with *local systems*. If $\mathcal{C}$ is an ∞-category (regarded as a complete Segal space) and X is a space, a *$\mathcal{C}$-valued local system* on X is just a functor $X \to \mathcal{C}$, or equivalently a map of spaces $X \to \mathcal{C}_0$, where $\mathcal{C}_0$ is the space of objects in $\mathcal{C}$. If we have a span of spaces $X \leftarrow A \to Y$, we may consider a more elaborate notion of local system on this span, namely a map of spans

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C}_1 \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr] & & \mathcal{C}_0 \\
X \arrow[rr] & & \mathcal{C}_0 &
\end{tikzcd}$$

where $\mathcal{C}_1$ is the space of morphisms in $\mathcal{C}$ and the two maps $\mathcal{C}_1 \to \mathcal{C}_0$ are the source and target projections. Moreover, we can use the composition map in $\mathcal{C}$ to compose such spans: given spans $X \leftarrow A \to Y$ and $Y \leftarrow B \to Z$ over $\mathcal{C}_0 \leftarrow \mathcal{C}_1 \to \mathcal{C}_0$, their composite is given by $X \leftarrow A \times_Y B \to Z$ with the maps from X and Z to $\mathcal{C}_0$ as before, but now equipped with the composite map

$$A \times_Y B \to \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1,$$

where the second map is the composition in $\mathcal{C}$. We can also use the map $\mathcal{C}_0 \to \mathcal{C}_1$ that assigns to objects their identity maps to get identity maps for objects $X \to \mathcal{C}_0$, so from the ∞-category $\mathcal{C}$ we should get a new ∞-category where the objects are spaces with $\mathcal{C}$-valued local systems and the morphisms are given by spans with local systems in this sense. Increasing the category number, from an (∞, n)-category $\mathcal{C}$ we would expect to get an (∞, n)-category where the k-morphisms are k-fold spans of spaces, equipped with a map to the k-fold span obtained from the source and target maps from the space of k-morphisms in $\mathcal{C}$ to the spaces of i-morphisms for all $i < k$. Our third main result is a construction of such (∞, n)-categories:

Theorem 1.5. *Suppose $\mathcal{C}$ is an (∞, n)-category. Then there is an (∞, n)-category $\mathrm{Span}_n(\mathcal{S}; \mathcal{C})$ where the k-morphisms are k-fold spans of spaces with local systems in $\mathcal{C}$.*

We'll construct these (∞, n)-categories in the form of n-fold Segal spaces in §6 and prove that they are complete in §9.

These are the (∞, n)-categories considered as targets for classical topological field theories by Freed, Hopkins, Lurie, and Teleman, who propose that for good choices of $\mathcal{C}$ there should be a symmetric monoidal linearization functor from $\mathrm{Span}_n(\mathcal{S}; \mathcal{C})$, or at least from the subcategory of spans of π-finite spaces, to $\mathcal{C}$. We will not construct any such linearizations here, but in §13 we do describe (via the Cobordism Hypothesis) the framed extended TQFTs with values in these (∞, n)-categories:

Theorem 1.6. *Suppose $\mathcal{C}$ is a symmetric monoidal (∞, n)-category. Then $\mathrm{Span}_n(\mathcal{S}; \mathcal{C})$ inherits a natural symmetric monoidal structure. Moreover, if $\mathcal{C}$ has duals, then so does $\mathrm{Span}_n(\mathcal{S}; \mathcal{C})$.*

We'll also prove analogous results when $\mathcal{S}$ is replaced by an arbitrary ∞-topos $\mathcal{X}$, with $\mathcal{C}$ an (∞, n)-category internal to $\mathcal{X}$.

1.4. **Lagrangian Correspondences.** To get a bit closer to the notion of a (non-extended) classical field theory as considered in physics within the framework of TQFTs, we would like to assign to a closed manifold M a stack of fields $\mathcal{F}(M)$ equipped with a *symplectic structure*. The span $\mathcal{F}(M) \leftarrow \mathcal{F}(X) \to \mathcal{F}(N)$ assigned to a cobordism X from M to N should then be a *Lagrangian correspondence*, i.e. a Lagrangian morphism from $\mathcal{F}(X)$ to $\mathcal{F}(M) \times \overline{\mathcal{F}(N)}$, where $\overline{\mathcal{F}(N)}$ is $\mathcal{F}(N)$ equipped with the negative of its symplectic form. In [PTVV13] Pantev, Toën, Vaquié and Vezzosi introduce a theory of symplectic structures on derived algebraic stacks and Lagrangian morphisms between them. The final main contribution in this paper is to make use of this to construct ∞-categories of symplectic derived stacks and Lagrangian correspondences:

Theorem 1.7. *The n-symplectic derived Artin stacks locally of finite presentation and the Lagrangian correspondences between them determine a subcategory $\mathrm{Lag}^n_{(\infty,1)}$ of $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$, where dSt_k is the ∞-topos of derived stacks over a base field k and $\mathcal{A}^2_{\mathrm{cl}}[n]$ is the derived stack of n-shifted closed 2-forms. Moreover, the ∞-category $\mathrm{Lag}^n_{(\infty,1)}$ inherits a symmetric monoidal structure from $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$ with respect to which all n-symplectic derived Artin stacks are dualizable.*

We will prove this in §14. This result partly generalizes results of Calaque [Cal15] from the level of 1-categories to ∞-categories. Note that the ∞-category $\mathrm{Lag}^n_{(\infty,1)}$ can be viewed as a derived algebro-geometric version of Weinstein's symplectic "category" [Wei81, Wei82].

In a sequel to this paper, joint with Damien Calaque and Claudia Scheimbauer, we will extend this by introducing a definition of *iterated* Lagrangian correspondences and using this to construct (∞, n)-categories of symplectic derived stacks.

1.5. **Related Work.** The classical TQFTs considered in this paper have previously been discussed by a number of authors; particularly inspirational were the accounts of Freed, Hopkins, Lurie, and Teleman [FHLT10] and of Calaque [Cal15].

The construction of the ∞-category of spans in an ∞-category we use is due to Barwick, who has made extensive use of this and variants of it [Bar13a, Bar17, BGS16]. In unpublished work, Barwick has also given an alternative definition of higher categories of iterated spans, in the setting of Rezk's Θ_n-spaces.

In their work [DK12] on 2-Segal spaces, Dyckerhoff and Kapranov introduce an alternative construction of an $(\infty, 2)$-category of spans. I have also been informed that Lurie has given a construction of the $(\infty, 2)$-category of 2-fold spans in the setting of scaled simplicial sets, though this is not currently publicly available.

The idea that the (∞, n)-category of iterated spans could most easily be constructed as that underlying an n-uple ∞-category I gained from the definition sketched by Schreiber in [Sch13, §3.9.14.2]. Schreiber and collaborators have also extensively studied quantization by linearizing iterated spans of stacks, for example in [Sch14a, Sch14b] and [Sch13, §3.9.14]; they consider not necessarily topological quantum field theories valued in iterated spans in a cohesive ∞-topos under the name *local prequantum field theories*. Nuiten [Nui13] has also recently studied the quantization of these.

Analogues of the higher categories we construct here have previously been defined in low dimensions: a weak double category of 2-fold spans in a category was constructed by Morton [Mor09], and a monoidal 3-category of spans in a 2-category was constructed by Hoffnung [Hof13], with the dualizability of its objects subsequently proved by Stay [Sta16]. In the 1-categorical setting, a construction of weak n-fold categories of iterated cospans in a category has been carried out by Grandis [Gra07].

Morton has also studied extended TQFTs valued in 2-fold spans of groupoids [Mor11] and 2-fold spans of groupoids equipped with $U(1)$-valued cocycles [Mor15], and has constructed linearization functors to linear categories (or *2-vector spaces*) in both cases.

Finally, the (∞, n)-categories $\mathrm{Span}_n(\mathcal{S}; \mathcal{C})$ appear in Lurie's work on the cobordism hypothesis [Lur09c], under the name $\mathrm{Fam}_n(\mathcal{C})$, but only a sketch of a definition is given there.

1.6. **Overview.** We begin by reviewing some background on ∞-categories in §2. We then briefly recall Rezk's theory of (complete) Segal spaces in §3 and its relationship to that of ∞-categories, before introducing the model of (∞, n)-categories we will use, namely iterated

Segal spaces, in §4. Then we construct the (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ of iterated spans in an ∞-category $\mathcal{C}$ as an n-fold Segal space in §5, and the (∞, n)-category $\mathrm{Span}_n(\mathcal{X}; \mathcal{D})$ of iterated spans in an ∞-topos $\mathcal{X}$ equipped with local systems in an (∞, n)-category $\mathcal{D}$ internal to $\mathcal{X}$ in §6.

Next, we recall the definition of a *complete* n-fold Segal space (and its generalization to a general ∞-topos) and prove some technical results about these in §7. We then show that the n-fold Segal space $\mathrm{Span}_n(\mathcal{C})$ is complete in §8 and that $\mathrm{Span}_n(\mathcal{X}; \mathcal{D})$ is complete in §9.

In §11 we discuss the notion of (symmetric) monoidal (∞, n)-categories in the form they will appear later (and prove these are equivalent to the definitions found in [Lur17]), before we review the notions of duals and adjoints in (∞, n)-categories in §11, where we also generalize these to (∞, n)-categories internal to an ∞-topos. Then in §12 we prove that $\mathrm{Span}_n(\mathcal{C})$ is symmetric monoidal and that all its objects are fully dualizable, and in §13 we show the same holds for $\mathrm{Span}_n(\mathcal{X}; \mathcal{D})$ provided $\mathcal{D}$ is a symmetric monoidal (∞, n)-category in $\mathcal{X}$ with duals.

Finally, in §14 we construct an ∞-category of symplectic derived algebraic stacks and Lagrangian correspondences, and prove that all of its objects are dualizable.

1.7. **Acknowledgments.** I first learned about ∞-categories of spans from conversations with Clark Barwick back in 2010. The present work was inspired by a number of discussions during my visit to the MSRI programme on algebraic topology in the spring of 2014, in particular with Hiro Tanaka and Owen Gwilliam. I also thank Oren Ben-Bassat, Damien Calaque, Theo Johnson-Freyd, Gregor Schaumann, Claudia Scheimbauer, Chris Schommer-Pries, and Peter Teichner for helpful comments.

2. Background and Notation

In this section we will describe our perspective on ∞-categories and recall some key constructions we'll make use of later on, in the hope that this will make the paper easier to follow for readers who are not intimately familiar with the literature on ∞-categories. (For more background on ∞-categories, we recommend Groth's expository article [Gro15] and Rezk's lecture notes [Rez17].) At the end, we also describe some of our notational conventions.

2.1. **∞-Categories.** As we mentioned above, the basic idea of an *∞-category* is that this should be a structure that has objects and i-morphisms between $(i-1)$-morphisms for $i = 1, 2, \ldots$, where these are all invertible for $i > 1$. Moreover, the composition of morphisms should not be strictly associative, but only associative up to a compatible choice of invertible higher morphisms. If we insist on making explicit choices of composites and of the invertible higher morphisms specifying the associativity data, we get a theory that is essentially intractable, if we can make sense of it at all. A key idea in the homotopical approaches to ∞-categories is that it is better to not make any choices, but instead consider the space of all possible choices, which is contractible.

By far the best-developed implementation of this idea is that of *quasicategories*. A quasicategory is simply a simplicial set satisfying the right lifting property for the *inner horn inclusions* $\Lambda^n_k \hookrightarrow \Delta^n$ $(0 < k < n)$. The quasicategories are the fibrant objects in a model structure on simplicial sets, due to Joyal; we will write Set_Δ^J for this model category. Quasicategories were first introduced by Boardman and Vogt [BV73] under the name *restricted Kan complexes*, and their theory has later been very extensively developed by Joyal [Joy08] and Lurie [Lur09a, Lur17], to the extent that most basic notions in category theory have analogues for quasicategories, generally behaving "as you would expect".

Let us warn the reader that, when working in some quasicategory $\mathfrak{C}$, we will use the same vocabulary for these quasicategorical notions as we would use if $\mathfrak{C}$ were a category. Thus if we speak of a commutative diagram in $\mathfrak{C}$ of shape $\mathfrak{I}$ we mean a functor of quasicategories (i.e. map of simplicial sets) $\mathfrak{I} \to \mathfrak{C}$, even if $\mathfrak{I}$ is (the nerve of) an ordinary category — note that in the latter case such a diagram includes the choice of homotopies in all commuting triangles and so on for higher simplices. Thus if we say we have a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathfrak{C}$, this implicitly includes the data of a homotopy between the two composite maps $A \to D$.

A key advantage of the quasicategorical model is that many important constructions have simple combinatorial incarnations. For example, if $\mathfrak{C}$ is a quasicategory then for any simplicial set the internal Hom of simplicial sets $\mathfrak{C}^K$ is a quasicategory ([Lur09a, Corollary 2.3.2.5]) — this represents the ∞-category of functors from the ∞-category generated by K (given by a fibrant replacement in the Joyal model structure) to $\mathfrak{C}$. To emphasize this, we will usually denote the internal Hom by $\mathrm{Fun}(K, \mathfrak{C})$ when $\mathfrak{C}$ is a quasicategory.

It will be very convenient to apply the quasicategorical viewpoint also to ∞-categories themselves, i.e. we will think of them as living in a quasicategory $\mathfrak{Cat}_\infty$ (which can be obtained as the coherent nerve of a simplicial category of quasicategories) rather than in the model category Set_Δ^J of simplicial sets with the Joyal model structure. This is a precise version of the somewhat vague idea of working with ∞-categories "model-independently". For most of the paper this will allow us to avoid referring explicitly to the implementation of ∞-categories as simplicial sets; this typically lets us make definitions and constructions that are more conceptual, thereby hopefully making it easier for the reader to see what is actually going on.

On a few occasions we will, however, need to make constructions in the model category Set_Δ^J. To avoid confusion we will always be very explicit about this: we will refer to objects of $\mathfrak{Cat}_\infty$ as ∞-*categories* and fibrant objects of Set_Δ^J as quasicategories, and say things like "let $\mathfrak{C} \in \mathrm{Set}_\Delta$ be a quasicategory representing the ∞-category $\mathcal{C}$". Note also that, since ordinary categories form a full subcategory of $\mathfrak{Cat}_\infty$, we will not distinguish notationally between a category $\mathbf{C}$ and its image in $\mathfrak{Cat}_\infty$ — on the other hand, when we think of it as living in Set_Δ^J via its nerve $\mathrm{N}\mathbf{C}$ we will explicitly indicate this.

In fact, though it may seem slightly perverse at first sight, it is pleasant to take this to the next level: we want to be able to work with the ∞-category (rather than the quasicategory) of ∞-categories, so we take Cat_∞ to be the ∞-*category* represented by the *quasicategory* $\mathfrak{Cat}_\infty$. (Here we are implicitly passing to a larger Grothendieck universe of sets.)

If $\mathcal{C}$ is an ∞-category, we write $\mathrm{Map}_{\mathcal{C}}(x, y)$ for the space of maps from x to y in $\mathcal{C}$. Using the quasicategory model there are a variety of (weakly equivalent) ways to describe these mapping spaces as simplicial sets; see [DS11] for an extensive discussion and comparisons.

We will now briefly review some key concepts from the theory of ∞-categories that we will make repeated use of in this paper. We will mainly describe them from our ∞-categorical perspective, but we will also mention how they can be implemented via quasicategories.

2.2. **∞-Groupoids.** Just as a groupoid is a category where all the morphisms are invertible, an *∞-groupoid* is an ∞-category all of whose morphisms are invertible. Grothendieck's *Homotopy Hypothesis* asserts that ∞-groupoids are equivalent to homotopy types. For the homotopical approaches to higher categories that we are concerned with here, this idea is taken as a starting point for the theory; in the case of quasicategories, ∞-groupoids correspond to those quasicategories that are *Kan complexes*,which are of course a well-known model for homotopy types. (Moreover, the weak equivalences in $\mathrm{Set}_{\Delta}^{J}$ restrict to the usual weak equivalences between Kan complexes.) Given this equivalence, we will often refer to ∞-groupoids as just *spaces*.

We write $\mathcal{S}$ for the ∞-category of ∞-groupoids or spaces. As a quasicategory, this is modelled by the coherent nerve $\mathrm{N}(\mathrm{Set}_{\Delta}^{\circ})$ of the simplicial category $\mathrm{Set}_{\Delta}^{\circ}$ of Kan complexes (which is the full subcategory spanned by the Kan complexes in the simplicial category Set_{Δ} of simplicial sets).

Definition 2.1. The fully faithful inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_{\infty}$ has both a left and a right adjoint. We write ι for the right adjoint, which takes an ∞-category to its underlying ∞-groupoid (obtained by forgetting the non-invertible morphisms) and $\|{-}\|$ for the left adjoint, obtained by inverting all the morphisms in an ∞-category.

Remark 2.2. If $\mathfrak{C}$ is a quasicategory representing an ∞-category $\mathcal{C}$, then the ∞-groupoid $\|\mathcal{C}\|$ is represented by any Kan complex that is a fibrant replacement for the simplicial set $\mathfrak{C}$ in the usual Kan-Quillen model structure on Set_{Δ}.

Definition 2.3. If $\mathcal{C}$ is an ∞-category, we say that $\mathcal{C}$ is *weakly contractible* if the space $\|\mathcal{C}\|$ is contractible.

We call a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ a *presheaf* on $\mathcal{C}$, and write $\mathcal{P}(\mathcal{C})$ for the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves.

2.3. **Joins and Slice ∞-Categories.** We'll frequently make use of slice ∞-categories, which can be defined as follows:

Definition 2.4. If $\mathcal{C}$ is an ∞-category and x is an object of $\mathcal{C}$, then the *overcategory* $\mathcal{C}_{/x}$ is defined by the pullback square

$$\begin{tikzcd}
\mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathrm{Fun}([1], \mathcal{C}) \arrow[d, "\mathrm{ev}_1"] \\
\{x\} \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where ev_1 is the functor given by evaluation at $1 \in [1]$. Similarly, if $p\colon \mathcal{I} \to \mathcal{C}$ is a functor, we define $\mathcal{C}_{/p}$ by the pullback square

$$\begin{tikzcd} \mathcal{C}_{/p} \arrow[r] \arrow[d] & \mathrm{Fun}([1] \times \mathcal{I}, \mathcal{C}) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ \mathcal{C} \times \{p\} \arrow[r, "\mathrm{const} \times *"'] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{I}, \mathcal{C}). \end{tikzcd}$$

Undercategories are of course also defined analogously.

By the universal property of pullbacks, for any ∞-category $\mathcal{D}$ we have a pullback square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D} \star \mathcal{I}, \mathcal{C}) \arrow[d] \\ \{p\} \arrow[r] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}), \end{tikzcd}$$

where the *join* $\mathcal{D} \star \mathcal{I}$ is defined as follows:

Definition 2.5. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, their *join* $\mathcal{C} \star \mathcal{D}$ is defined by the pushout

$$\mathcal{C} \amalg_{\mathcal{C} \times \mathcal{D} \times \{0\}} \mathcal{C} \times \mathcal{D} \times [1] \amalg_{\mathcal{C} \times \mathcal{D} \times \{1\}} \mathcal{D}.$$

in Cat_∞. The *cones* $\mathcal{C}^{\triangleleft}$ and $\mathcal{C}^{\triangleright}$ are then defined as $[0] \star \mathcal{C}$ and $\mathcal{C} \star [0]$, respectively.

If $\mathfrak{C}$ is a quasicategory representing $\mathcal{C}$, the functor $\mathrm{ev}_1\colon \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$ can be represented by $\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \mathfrak{C}) \to \mathfrak{C}$. This is a fibration in the Joyal model structure (combine [Lur09a, Corollary 2.4.7.12] with [Lur09a, Corollary 2.4.6.5]) and so if we define (using the notation of [Lur09a, §4.2.1]) the simplicial set $\mathfrak{C}^{/x}$ by the pullback

$$\begin{tikzcd} \mathfrak{C}^{/x} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathfrak{C}) \arrow[d, "\mathrm{ev}_1"] \\ \{x\} \arrow[r] & \mathfrak{C}, \end{tikzcd}$$

then this is a homotopy pullback square. Thus the quasicategory $\mathfrak{C}^{/x}$ represents $\mathcal{C}_{/x}$. Moreover, this has the universal property that for any simplicial set K, we have a pullback square

$$\begin{tikzcd} \mathrm{Fun}(K, \mathfrak{C}^{/x}) \arrow[r] \arrow[d] & \mathrm{Fun}(K \diamond \Delta^0, \mathfrak{C}) \arrow[d] \\ \{x\} \arrow[r] & \mathfrak{C}, \end{tikzcd}$$

where for simplicial sets K and L, the simplicial set $K \diamond L$ is defined as the pushout

$$K \amalg_{K \times L \times \{0\}} K \times L \times \Delta^1 \amalg_{K \times L \times \{1\}} L.$$

Since this is a homotopy pushout in the Joyal model structure, if $\mathfrak{C}$ and $\mathfrak{D}$ are quasicategories representing ∞-categories $\mathcal{C}$ and $\mathcal{D}$, then the simplicial set $\mathfrak{C} \diamond \mathfrak{D}$ represents $\mathcal{C} \star \mathcal{D}$.

However, the simplicial set $\mathfrak{C} \diamond \mathfrak{D}$ is generally not a quasicategory. It is therefore often convenient to use instead an alternative model for the join by using the *join of simplicial sets* (see [Lur09a, Definition 1.2.8.1]), which we also denote using $\star$: if $\mathfrak{C}$ and $\mathfrak{D}$ are quasicategories, the join $\mathfrak{C} \star \mathfrak{D}$ is a quasicategory by [Lur09a, Proposition 1.2.8.3] and the natural map $\mathfrak{C} \diamond \mathfrak{D} \to \mathfrak{C} \star \mathfrak{D}$ is a weak equivalence by [Lur09a, Proposition 4.2.1.2]. (In other words, $\mathfrak{C} \star \mathfrak{D}$ is a fibrant replacement for $\mathfrak{C} \diamond \mathfrak{D}$ in the Joyal model structure.)

We can use this model for the join to get an alternative quasicategory representing $\mathcal{C}_{/x}$, by replacing $K \diamond \Delta^0$ by $K \star \Delta^0$ in the pullback square above. The resulting quasicategory is denoted $\mathfrak{C}_{/x}$ in [Lur09a]. There is a natural map $\mathfrak{C}^{/x} \to \mathfrak{C}_{/x}$ (induced by the natural inclusion $K \diamond \Delta^0 \to K \star \Delta^0$) and this is a weak equivalence in the Joyal model structure by [Lur09a, Proposition 4.2.1.5]. An analogous discussion applies, of course, to quasicategories $\mathfrak{C}^{/p}$ and $\mathfrak{C}_{/p}$ as well as to the dual notions for undercategories.

2.4. **Cartesian and CoCartesian Fibrations.**

Cartesian fibrations are the ∞-categorical analogue of *Grothendieck fibrations*; in both cases they are characterized by the existence of *Cartesian morphisms*:

Definition 2.6. If $F\colon \mathcal{E} \to \mathcal{B}$ is a functor of ∞-categories, we say a morphism $\bar{f}\colon x \to y$ in $\mathcal{E}$ with image $f\colon a \to b$ in $\mathcal{B}$ is *F-Cartesian* if for every $z \in \mathcal{E}$ over $c \in \mathcal{B}$ the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(z,x) \arrow[r, "\bar{f}_*"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(z,y) \arrow[d] \\ \mathrm{Map}_{\mathcal{B}}(c,a) \arrow[r, "f_*"'] & \mathrm{Map}_{\mathcal{B}}(c,b) \end{tikzcd}$$

is Cartesian. Equivalently (since the mapping space $\mathrm{Map}_{\mathcal{E}}(z,x)$ is the fibre at z of the forgetful functor $\mathcal{C}_{/x} \to \mathcal{C}$) f is F-Cartesian if the commutative square

$$\begin{tikzcd} \mathcal{E}_{/x} \arrow[r] \arrow[d] & \mathcal{E}_{/y} \arrow[d] \\ \mathcal{B}_{/a} \arrow[r] & \mathcal{B}_{/b} \end{tikzcd}$$

is Cartesian. We say F is a *Cartesian fibration* if for every morphism $f\colon a \to b$ in $\mathcal{B}$ and every $y \in \mathcal{E}$ with $F(y) \simeq b$ there exists an F-Cartesian morphism $\bar{f}\colon y \to x$ with $F(\bar{f}) \simeq f$. The notions of *co*Cartesian morphisms and *co*Cartesian fibrations are defined dually, i.e. F is a coCartesian fibration if and only if F^{op} is a Cartesian fibration.

Remark 2.7. It can be shown that if $p\colon \mathfrak{E} \to \mathfrak{B}$ is an inner fibration (meaning p has the right lifting property for the inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$) representing a functor of ∞-categories $F\colon \mathcal{E} \to \mathcal{B}$, then F is a Cartesian fibration in our sense if and only if p is a Cartesian fibration in the sense of [Lur09a, Definition 2.4.2.1]. This is not entirely obvious — see [MG15] for a detailed proof. To avoid confusion we will use the term *(co)Cartesian inner fibration* for a (co)Cartesian fibration of simplicial sets in Lurie's sense.

Grothendieck proved (in [Gro63]) that Grothendieck opfibrations over a category **C** correspond to (pseudo)functors from **C** to the category of categories. Lurie's *straightening equivalence* from [Lur09a, §3.2] establishes an analogous equivalence between Cartesian fibrations over an ∞-category $\mathcal{C}$ and functors from $\mathcal{C}^{\mathrm{op}}$ to the ∞-category Cat_∞ of ∞-categories. If we let $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{Cart}}$ denote the subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ whose objects are the Cartesian fibrations and whose morphisms are the maps that preserve Cartesian morphisms, then this gives the following statement in our language:

Theorem 2.8 (Lurie)**.** *There is an equivalence of ∞-categories* $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{Cart}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$.

This is extremely useful, as it is in practice impossible to "write down" functors to Cat_∞, whereas we can much more easily describe (co)Cartesian fibrations, e.g. by manipulating preexisting fibrations.

Remark 2.9. In ordinary category theory, the Grothendieck fibration associated to a functor is given by the *Grothendieck construction*, which can be identified with the lax colimit (a certain weighted colimit) of the functor. In the ∞-categorical setting, the functor $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ that takes a functor to the source of the associated Cartesian fibration can also be identified with the lax colimit, by [GHN17, Corollary 7.6].

2.5. **Left and Right Fibrations.** An important special case of Cartesian fibrations are those whose fibres are spaces (as opposed to general ∞-categories); these are called *right fibrations*. Dually, coCartesian fibrations whose fibres are spaces are called *left fibrations*. Right fibrations can also be characterized as those functors $\mathcal{E} \to \mathcal{B}$ such that *every* morphism in $\mathcal{E}$ is Cartesian.

Remark 2.10. The terms *left* and *right* fibrations are motivated by the incarnations of these concepts on the level of quasicategories: If $p \colon \mathfrak{E} \to \mathfrak{B}$ is an inner fibration representing a functor of ∞-categories $F \colon \mathcal{E} \to \mathcal{B}$, then F is a right fibration if and only if p satisfies a very simple condition: p must have the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k \leq n$. Similarly, F is a left fibration if and only if p has the right lifting property with respect to the horn inclusions where $0 \leq k < n$.

An easier version of Lurie's straightening equivalence for Cartesian fibrations gives an equivalence of ∞-categories

$$\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{RFib}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

where $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{RFib}}$ is the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the right fibrations.

2.6. **Cofinal and Coinitial Functors.** We will often need to know that objects defined as (co)limits over diagrams of different, but related, shapes are equivalent. Just as in ordinary category theory, the notion of cofinal functors (and the dual notion of coinitial functors) is a very useful tool for proving such statements.

Definition 2.11. A functor $F \colon \mathcal{A} \to \mathcal{B}$ of ∞-categories is *cofinal* if for every diagram $p \colon \mathcal{B} \to \mathcal{C}$, the induced functor $\mathcal{C}_{p/} \to \mathcal{C}_{p\circ F/}$ is an equivalence. Dually, F is *coinitial* if $F^{\mathrm{op}} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is cofinal, i.e. the functor $\mathcal{C}_{/p} \to \mathcal{C}_{/p\circ F}$ is an equivalence for every p.

Since a colimit of p is the same thing as a final object in $\mathcal{C}_{p/}$, we see that if F is cofinal then p has a colimit if and only if $p \circ F$ has a colimit, and these colimits are necessarily given by the same object in $\mathcal{C}$.

The key criterion for cofinality is [Lur09a, Theorem 4.1.3.1]:

Theorem 2.12 ([Lur09a, Theorem 4.1.3.1])**.** *A functor $F \colon \mathcal{A} \to \mathcal{B}$ is cofinal if and only if for every $b \in \mathcal{B}$ the slice ∞-category $\mathcal{A}_{b/} := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{b/}$ is weakly contractible.*

2.7. **∞-Topoi.** Just as a (Grothendieck) topos is a category that abstracts some key properties of the category of sets, an ∞-*topos* is an ∞-category with key properties of the ∞-category $\mathcal{S}$ of spaces. A terse definition is that ∞-topoi are the ∞-categories that arise as left exact accessible localizations (meaning the localization functor preserves finite limits and sufficiently compact objects) of presheaf ∞-categories $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ where $\mathcal{C}$ is a small ∞-category with finite limits; see [Lur09a, Theorem 6.1.0.6, Proposition 6.1.5.3] for several equivalent characterizations.

The key examples of ∞-topoi are ∞-categories of presheaves $\mathcal{P}(\mathcal{C})$ (where $\mathcal{C}$ has finite limits) and ∞-categories of sheaves of spaces on topological spaces, or more generally on sites. The latter are important in the context of derived algebraic geometry.

Remark 2.13. At a number of points below we will prove results for a general ∞-topos $\mathcal{X}$, as the proofs are no more work once certain definitions have been set up. This is motivated by the possibility of applications in derived algebraic geometry, but apart from in §14 we do not make use of this generality in this paper. The reader should therefore feel free to assume that $\mathcal{X}$ is just the ∞-category $\mathcal{S}$ of spaces, and to omit the parts of §7 and §11 where certain results and constructions are generalized from spaces to an arbitrary ∞-topos.

2.8. **Notation.** We generally reuse the notation and terminology used by Lurie in [Lur09a, Lur09c, Lur17]. We note the following conventions, some of which differ slightly from those of Lurie:

- Δ is the simplicial indexing category, with objects the non-empty finite totally ordered sets $[n] := \{0, 1, \ldots, n\}$ and morphisms order-preserving functions between them.
- For $[n] \in \Delta$ we will abbreviate $(\Delta_{/[n]})^{\mathrm{op}}$ to $\Delta^{\mathrm{op}}_{/[n]}$, and for $I \in \Delta^k$ we will abbreviate $(\Delta^k_{/I})^{\mathrm{op}}$ to $\Delta^{k,\mathrm{op}}_{/I}$.
- If $\mathcal{C}$ is an ∞-category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of $\mathcal{C}$, i.e. the largest subspace of $\mathcal{C}$ that is a Kan complex.
- If $f\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $g\colon \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.
- We make use of Grothendieck universes to avoid having to deal with set-theoretical size issues: we fix three nested universes and refer to their elements as small, large, and very large sets, respectively. If we have ∞-categories of small and large versions of the same objects, we will distinguish the large version with a circumflex: thus $\mathcal{S}$ is the (large) ∞-category of small spaces and $\widehat{\mathcal{S}}$ is the (very large) ∞-category of large spaces; similarly Cat_∞ is the ∞-category of small ∞-categories and $\widehat{\mathrm{Cat}}_\infty$ that of large ∞-categories.

3. Segal Spaces

In this section we review the description of ∞-categories as complete Segal spaces. This was introduced by Rezk in [Rez01], though we will discuss it from an ∞-categorical perspective rather than the model-categorical one used by Rezk.

Definition 3.1. Suppose $\mathcal{C}$ is an ∞-category with finite limits. A *category object* in $\mathcal{C}$ is a simplicial object $C_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that the natural maps

$$C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1,$$

induced by the maps $\sigma_i\colon [0] \to [n]$ sending 0 to i and $\rho_i\colon [1] \to [n]$ sending 0 to $i-1$ and 1 to i, are equivalences in $\mathcal{C}$ for all n. We write $\mathrm{Cat}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ spanned by the category objects.

Definition 3.2. A *Segal space* is a category object in the ∞-category $\mathcal{S}$ of spaces. We write $\mathrm{Seg}(\mathcal{S})$ for the ∞-category of Segal spaces.

A category object in Set is the same thing as an ordinary category. Similarly, a Segal space X describes the algebraic structure of an ∞-category: We think of the space X_0 as the space of objects and X_1 as the space of morphisms; the two face maps $X_1 \rightrightarrows X_0$ assign the source and target object to each morphism, and the degeneracy $s_0 \colon X_0 \to X_1$ assigns an identity morphism to every object. Then $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the space of composable sequences of n morphisms, and the face map $d_1 \colon [1] \to [2]$ gives a composition

$$X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1.$$

The remaining data in $X_\bullet$ gives the homotopy-coherent associativity data for this composition and its compatibility with the identity maps.

However, although category objects in Set are categories, isomorphisms in the category Cat(Set) do not give the right notion of equivalence of categories: to describe the correct homotopy theory of categories we must invert the fully faithful and essentially surjective functors. If done in an ∞-categorical (or at least 2-categorical) setting, this produces the (2,1)-category of categories, functors, and natural isomorphisms. An equivalence here, in the 2-categorical sense, is precisely an equivalence of categories. The same phenomenon occurs for ∞-categories: Segal spaces encode the algebraic structure of composition and units in ∞-categories, but the right notion of equivalence between ∞-categories corresponds to the (non-algebraic) notion of fully faithful and essentially surjective morphisms of Segal spaces, in the following sense:

Definition 3.3. Let E^n denote the contractible groupoid with n objects and a unique morphism between any pair of objects; we can regard this as a Segal space by thinking of it as a category object in sets and applying the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$. For $X \in \mathrm{Seg}(\mathcal{S})$ we define a simplicial space by $\iota_\bullet X := \mathrm{Map}_{\mathrm{Seg}(\mathcal{S})}(E^\bullet, X)$; we write ιX for the colimit of this simplicial diagram — this is the *classifying space of equivalences* in X. We say a morphism $f \colon X \to Y$ is *fully faithful and essentially surjective* if

(1) The map $\iota X \to \iota Y$ is an equivalence of spaces.
(2) The diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_0 \times X_0 \arrow[r] & Y_0 \times Y_0, \end{tikzcd}$$

with the vertical maps coming from the two maps $[0] \to [1]$, is a pullback square.

To get the correct ∞-category of ∞-categories we need to localize the ∞-category $\mathrm{Seg}(\mathcal{S})$ at the fully faithful and essentially surjective morphisms. The main result of [Rez01] is that this localization is given by the full subcategory of the *complete* Segal spaces:

Definition 3.4. A Segal space X is *complete* if the natural map $X_0 \to \iota X$ is an equivalence.

Theorem 3.5 (Rezk, [Rez01, Theorem 7.7])**.** *Let* $\mathrm{CSS}(\mathcal{S})$ *denote the full subcategory of* $\mathrm{Seg}(\mathcal{S})$ *spanned by the complete Segal spaces. Then the inclusion* $\mathrm{CSS}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{S})$ *has a left adjoint, which exhibits* $\mathrm{CSS}(\mathcal{S})$ *as the localization of* $\mathrm{Seg}(\mathcal{S})$ *at the fully faithful and essentially surjective morphisms.*

Rezk also proves some alternative characterizations of complete objects, which we recall for use later on:

Theorem 3.6 (Rezk)**.** *Let X be a Segal space. There are two obvious inclusions $[1] \to E^1$ which induce maps $\mathrm{Map}(E^1, X) \to X_1$. We write X_{eq} for the subspace of $\mathcal{C}_1$ consisting of the components in the image of either of these maps (they are the same since E^1 has an autoequivalence that swaps the two). Then the map $\mathrm{Map}(E^1, X) \to X_{\mathrm{eq}}$ is an equivalence, and the following are equivalent:*

(i) X is complete.
(ii) The simplicial space $\iota_\bullet X$ is constant.
(iii) The map $X_0 \to X_{\mathrm{eq}}$ induced by the degeneracy map s_0 is an equivalence.
(iv) The map $X_0 \to \mathrm{Map}(E^1, X)$ induced by composition with either of the maps $[0] \to E^1$ is an equivalence.

Proof. This is Theorem 6.2 and Proposition 6.4 of [Rez01]. □

By analogy with the case of ordinary categories, we would expect that the ∞-category $\mathrm{CSS}(\mathcal{S})$ is equivalent to Cat_∞. This is indeed true, as was proved by Joyal and Tierney:

Theorem 3.7 (Joyal-Tierney [JT07])**.** *The functor $\Delta \to \mathrm{Cat}_\infty$ given by the usual inclusion of ordered sets into categories induces, via the Yoneda embedding, a functor $\mathrm{Cat}_\infty \to \mathcal{P}(\Delta) := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$. This is fully faithful and its essential image consists precisely of the complete Segal spaces. In other words it restricts to an equivalence $\mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{CSS}(\mathcal{S})$.*

It will be useful to introduce a reformulation of the definition of a category object. To give this we must first introduce some notation:

Definition 3.8. A map $\phi\colon [n] \to [m]$ is *inert* if ϕ is the inclusion of a subinterval, i.e. we have $\phi(i) = \phi(0) + i$ for all i. We write Δ_{int} for the subcategory of Δ containing only the inert maps. Let Cell^1 denote the full subcategory of Δ_{int} spanned by the objects $[0]$ and $[1]$, i.e. the category
$$[0] \rightrightarrows [1].$$
For $[n] \in \Delta$ we write $\mathrm{Cell}^1_{/[n]}$ for the category $\mathrm{Cell}^1 \times_{\Delta_{\mathrm{int}}} (\Delta_{\mathrm{int}})_{/[n]}$ of inert maps from $[0]$ and $[1]$ to $[n]$.

Remark 3.9. This is a special case of the general notion of an inert map defined by Barwick [Bar13b] in the context of *operator categories*, which can be viewed as settings for different kinds of algebraic structures.

Remark 3.10. The category $\mathrm{Cell}^1_{/[n]}$ can be depicted as

$$\begin{tikzcd}
\{0\} \arrow[dr] & & \{1\} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & \{n-1\} \arrow[dl] \arrow[dr] & & \{n\} \arrow[dl] \\
& \{0,1\} & & \{1,2\} & \cdots & & \{n-1,n\} &
\end{tikzcd}$$

Lemma 3.11. *Let $\mathcal{C}$ be an ∞-category with finite limits. A simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is a category object if and only if its restriction $X|_{\Delta^{\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{1,\mathrm{op}}$, or in other words, if j denotes the inclusion $\mathrm{Cell}^{1,\mathrm{op}} \to \Delta^{\mathrm{op}}_{\mathrm{int}}$, the unit map*
$$X|_{\Delta^{\mathrm{op}}_{\mathrm{int}}} \to j_* j^* X|_{\Delta^{\mathrm{op}}_{\mathrm{int}}}$$
of the right Kan extension adjunction $j^* \dashv j_*$ is an equivalence.

Proof. The functor $X|_{\Delta^{\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{1,\mathrm{op}}$ if and only if for every object $[n] \in \Delta^{\mathrm{op}}_{\mathrm{int}}$ the natural map

$$X([n]) \to \lim_{([i]\to[n])\in(\mathrm{Cell}^1_{/[n]})^{\mathrm{op}}} X([i])$$

is an equivalence. But from the definition of Cell^1 we see that this is precisely the limit that appears in the definition of a Segal space. □

4. Iterated Segal Spaces

In this section we briefly review the model for (∞, n)-categories we use in this paper: the iterated Segal spaces of Barwick [Bar05]. Following [Lur09b] we'll state the basic definitions using the language of ∞-categories.

Definition 4.1. An *n-fold category object* in an ∞-category $\mathcal{C}$ is inductively defined to be a category object in the ∞-category of $(n-1)$-fold category objects. We write $\mathrm{Cat}^n(\mathcal{C}) := \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathcal{C}))$ for the ∞-category of n-fold category objects in $\mathcal{C}$. We refer to an n-fold category object in $\mathcal{S}$ as an *n-uple Segal space*.

Remark 4.2. The term n-uple Segal space is motivated by the observation that 2-uple (or double) Segal spaces encode the algebraic structure of double ∞-categories, i.e. category objects in Cat_∞. More generally, n-uple Segal spaces can be considered as a model for n-uple ∞-categories, i.e. internal ∞-categories in internal ∞-categories in … in ∞-categories.

Remark 4.3. Unwinding the definition, we see that an n-uple Segal space $\mathcal{D}\colon (\Delta^{\mathrm{op}})^{\times n} \to \mathcal{S}$ consists of the data of:

- a space $\mathcal{D}_{0,\ldots,0}$ of objects
- spaces $\mathcal{D}_{1,0,\ldots,0}, \ldots, \mathcal{D}_{0,\ldots,0,1}$ of n different kinds of 1-morphism, each with a source and target in $\mathcal{D}_{0,\ldots,0}$,
- spaces $\mathcal{D}_{1,1,0,\ldots,0}$, etc., of "commutative squares" between any two kinds of 1-morphism,
- spaces $\mathcal{D}_{1,1,1,0,\ldots,0}$, etc., of "commutative cubes" between any three kinds of 1-morphism,
- …
- a space $\mathcal{D}_{1,1,\ldots,1}$ of "commutative n-cubes",

together with units (from the degeneracies in $(\Delta^{\mathrm{op}})^{\times n}$) and coherently homotopy-associative composition laws (from the face maps) for all these different types of maps.

We can view the algebraic structure of an (∞, n)-category as given by the same kind of data, except that there is only one type of 1-morphism, etc., so we require certain spaces to be "degenerate", i.e. equivalent to the space $\mathcal{D}_{0,\ldots,0}$ via a degeneracy. This leads to Barwick's definition of an n-fold Segal object in an ∞-category:

Definition 4.4. Suppose $\mathcal{C}$ is an ∞-category with finite limits. A *1-fold Segal object* in $\mathcal{C}$ is just a category object in $\mathcal{C}$. For $n > 1$ we inductively define an *n-fold Segal object* in $\mathcal{C}$ to be an n-fold category object $\mathcal{D}$ such that

(i) the $(n-1)$-fold category object $\mathcal{D}_{0,\bullet,\ldots,\bullet}$ is constant,
(ii) the $(n-1)$-fold category object $\mathcal{D}_{k,\bullet,\ldots,\bullet}$ is an $(n-1)$-fold Segal object for all k.

We write $\mathrm{Seg}_n(\mathcal{C})$ for the full subcategory of $\mathrm{Cat}^n(\mathcal{C})$ spanned by the n-fold Segal objects. When $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces, we refer to n-fold Segal objects in $\mathcal{S}$ as *n-fold Segal spaces*.

Remark 4.5. Unwinding the definition, we see that an n-fold Segal space $\mathcal{D}$ consists of

- a space $\mathcal{D}_{0,\ldots,0}$ of objects,
- a space $\mathcal{D}_{1,0,\ldots,0}$ of 1-morphisms,
- a space $\mathcal{D}_{1,1,0,\ldots,0}$ of 2-morphisms,
- ...
- a space $\mathcal{D}_{1,\ldots,1}$ of n-morphisms,

together with units and coherently homotopy-associative composition laws for these morphisms.

Remark 4.6. The notion of n-fold Segal spaces describes precisely the *algebraic* structure we expect from (∞, n)-categories. Just as in the case of Segal spaces, to get the right homotopy theory of (∞, n)-categories we must supplement this algebraic structure with the ("non-algebraic") notion of fully faithful and essentially surjective functors. Thus the ∞-category of (∞, n)-categories is obtained from $\mathrm{Seg}_n(\mathcal{S})$ by inverting the fully faithful and essentially surjective morphisms. As in the case $n = 1$, this localization can be obtained by restricting to a full subcategory of *complete* objects; we will discuss this below in §7.

It will be useful to restate the definition of an n-fold category object non-inductively, via the analogue of Lemma 3.11. To state this we first need some notation:

Definition 4.7. We say an object $I = ([i_1], \ldots, [i_k]) \in \Delta^k$ is a *cell* if $i_j = 0$ or 1 for all i. We write Cell^k for the full subcategory of $\Delta^k_{\mathrm{int}} := (\Delta_{\mathrm{int}})^{\times k}$ spanned by the cells, i.e. $(\mathrm{Cell}^1)^{\times k}$. For $I \in \Delta^k$ we write $\mathrm{Cell}^k_{/I}$ for the pullback $\mathrm{Cell}^k \times_{\Delta^k_{\mathrm{int}}} (\Delta^k_{\mathrm{int}})_{/I}$; if $I = ([i_1], \ldots, [i_k])$ then this category is equivalent to $\mathrm{Cell}^1_{/[i_1]} \times \cdots \times \mathrm{Cell}^1_{/[i_k]}$.

Lemma 4.8. *A functor $\Phi \colon \Delta^{k,\mathrm{op}} \to \mathcal{C}$ is a k-fold category object if and only if the restriction $\Phi|_{\Delta^{k,\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{k,\mathrm{op}}$, or in other words, if j^k denotes the inclusion $j^k \colon \mathrm{Cell}^{k,\mathrm{op}} \hookrightarrow \Delta^{k,\mathrm{op}}_{\mathrm{int}}$, the unit map*

$$\Phi|_{\Delta^{k,\mathrm{op}}_{\mathrm{int}}} \to j^k_* j^{k,*} \Phi|_{\Delta^{k,\mathrm{op}}_{\mathrm{int}}}$$

is an equivalence.

Proof. We prove this by induction on n — the case $n = 1$ being Lemma 3.11. The functor $\Phi|_{\Delta^{k,\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{k,\mathrm{op}}$ if and only if for every object $I = ([i_1], \ldots, [i_k]) \in \Delta^{k,\mathrm{op}}_{\mathrm{int}}$ the natural map

$$\Phi(I) \to \lim_{(C \to I) \in (\mathrm{Cell}^k_{/I})^{\mathrm{op}}} \Phi(C)$$

is an equivalence. Since $\mathrm{Cell}^k_{/I}$ is the product $\mathrm{Cell}^1_{/[i_1]} \times \mathrm{Cell}^{k-1}_{/I'}$, where $I' = ([i_2], \ldots, [i_k])$, this limit can (e.g. using [Hau16, Corollary 5.7]) be rewritten as the iterated limit

$$\lim_{([j] \to [i_1]) \in (\mathrm{Cell}^1_{/[i_1]})^{\mathrm{op}}} \lim_{(J' \to I') \in (\mathrm{Cell}^{k-1}_{/I'})^{\mathrm{op}}} \Phi([j], J')).$$

Taking $i_1 = 0, 1$ (where $(\mathrm{Cell}^1_{/[i_1]})^{\mathrm{op}}$ has an initial object) we see by the inductive hypothesis that the condition holds in these cases if and only if $\Phi([0], -)$ and $\Phi([1], -)$ are $(k-1)$-fold category objects. Moreover, if the condition holds in these cases we can rewrite the limit for a general I as

$$\lim_{([j] \to [i_1]) \in (\mathrm{Cell}^1_{/[i_1]})^{\mathrm{op}}} \Phi([j], I').$$

Thus we have that $\Phi|_{\Delta^{k,\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{k,\mathrm{op}}$ if and only if $\Phi([0],-)$ and $\Phi([1],-)$ are $(k-1)$-fold category objects, and Φ is a category object in $\mathrm{Fun}(\Delta^{k-1,\mathrm{op}}, \mathcal{C})$. Since $(k-1)$-fold category objects are closed under limits, this is equivalent to Φ being a k-fold category object in $\mathcal{C}$. □

Definition 4.9. We write C_k for the k-cell, i.e. the generic k-morphism, thought of as an n-fold Segal space for any $n > k$. Concretely, it is the representable n-fold simplicial object represented by $([1], \dots, [1], [0], \dots, [0])$ where $[1]$ occurs k times. If $\mathcal{D}$ is an n-fold Segal space, we write $\mathrm{Mor}_k(\mathcal{D})$ for the space $\mathrm{Map}(C_k, \mathcal{D})$ of k-morphisms in $\mathcal{D}$, i.e. $\mathcal{D}_{1,\dots,1,0,\dots,0}$. For $k = 0$ we also write $\mathrm{Ob}(\mathcal{D})$ for $\mathcal{D}_{0,\dots,0}$.

Definition 4.10. Suppose $\mathcal{C}$ is an n-fold Segal space. The two face maps $[1] \to [0]$ induce a map of $(n-1)$-fold Segal spaces from $\mathcal{C}_1$ to the constant $(n-1)$-fold Segal space $\mathcal{C}_0 \times \mathcal{C}_0$. Given two objects X, Y of $\mathcal{C}$, i.e. a point of $\mathcal{C}_{0,\dots,0} \times \mathcal{C}_{0,\dots,0}$, we define the *mapping* $(\infty, n-1)$-*category* $\mathcal{C}(X, Y)$ to be the pullback

$$\begin{tikzcd} \mathcal{C}(X,Y) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\ \{(X,Y)\} \arrow[r] & \mathcal{C}_0^{\times 2}. \end{tikzcd}$$

Since $(n-1)$-fold Segal objects are closed under limits in $(n-1)$-fold simplicial spaces, this is again an $(n-1)$-fold Segal space.

Definition 4.11. Suppose $\mathcal{C}$ is an n-fold Segal object in $\mathcal{X}$. The *underlying k-fold Segal object* $u_{(\infty,k)}\mathcal{C}$ of $\mathcal{C}$ is the k-fold simplicial object obtained by restricting $\mathcal{C}$ along the inclusion $\Delta^{k,\mathrm{op}} \to \Delta^{n,\mathrm{op}}$ that is $[0]$ in the last $n-k$ components.

Our next goal is to prove that there is a canonical way to extract an n-fold Segal space from an n-uple Segal space; in the next section we will apply this to construct an n-fold Segal space of iterated spans from an n-uple Segal space.

Proposition 4.12. *Let $\mathcal{C}$ be an ∞-category with finite limits. The inclusion* $\mathrm{Seg}_n(\mathcal{C}) \hookrightarrow \mathrm{Cat}^n(\mathcal{C})$ *has a right adjoint* $U^n_{\mathrm{Seg}} \colon \mathrm{Cat}^n(\mathcal{C}) \to \mathrm{Seg}_n(\mathcal{C})$. *(We will usually abbreviate U^n_{Seg} to just U_{Seg} and let the integer n be implicitly determined by the context.)*

Remark 4.13. The basic idea, in the case $n = 2$, is that for X a double category object we form $U_{\mathrm{Seg}}X$ by taking pullbacks of simplicial objects

$$\begin{tikzcd} (U_{\mathrm{Seg}}X)_{m,\bullet} \arrow[r] \arrow[d] & X_{m,\bullet} \arrow[d] \\ X_{0,0}^{\times(m+1)} \arrow[r] & X_{0,\bullet}^{\times(m+1)}, \end{tikzcd}$$

where $X_{0,0}$ denotes the constant simplicial space with this value, the right vertical map is induced by the inert maps $\rho_i \colon [1] \to [m]$, and the bottom horizontal map is given by the degeneracies $X_{0,0} \to X_{0,k}$. This pullback extracts, for example, precisely the part of $X_{1,1}$ that is "constant" or degenerate in the second coordinate, i.e. lies over the image of the degeneracy $X_{0,0}^{\times 2} \to X_{0,1}^{\times 2}$. We can also think of this as given by a single pullback of bisimplicial objects : if, given a simplicial object $C_\bullet$, we write i_*C for the bisimplicial object given by $(i_*C)_{n,\bullet} \simeq C_\bullet^{\times(n+1)}$, with the face maps in the first coordinate given by projections and the degeneracies by diagonal maps, then $U_{\mathrm{Seg}}X$ is given by the pullback of bisimplicial objects

$$\begin{tikzcd} U_{\mathrm{Seg}}X \arrow[r] \arrow[d] & X \arrow[d] \\ i_*X_{0,0} \arrow[r] & i_*X_{0,\bullet}, \end{tikzcd}$$

where again $X_{0,0}$ denotes the constant simplicial object with this value. For n-fold category objects we then want to iterate this procedure.

To make this idea rigorous we first prove a sequence of easy lemmas:

Lemma 4.14. *Suppose $\pi\colon \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration and $j\colon \mathcal{C}_0 \to \mathcal{C}$ is a functor with a right adjoint $r\colon \mathcal{C} \to \mathcal{C}_0$. Let*

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, "J"] \arrow[d, "\pi_0"'] & \mathcal{E} \arrow[d, "\pi"] \\ \mathcal{C}_0 \arrow[r, "j"'] & \mathcal{C} \end{tikzcd}$$

be a pullback square. Then the functor J has a right adjoint $R\colon \mathcal{E} \to \mathcal{E}_0$ such that the counit map $JR(X) \to X$ is a π-Cartesian morphism over the counit map $jr\pi(X) \to \pi(X)$.

Proof. By [Lur09a, Proposition 3.1.2.1], the functor of ∞-categories $\mathrm{Fun}(\mathcal{E}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{C})$ induced by composition with π is a Cartesian fibration, and a morphism $\alpha\colon F \to G$ in $\mathrm{Fun}(\mathcal{E}, \mathcal{E})$ is Cartesian if and only if the morphism $\alpha_x\colon F(x) \to G(x)$ is π-Cartesian for every $x \in \mathcal{E}$.

Let $\epsilon\colon \mathcal{C} \times [1] \to \mathcal{C}$ be the counit natural transformation $jr \to \mathrm{id}_{\mathcal{C}}$. Then we may choose a Cartesian morphism $\bar{\epsilon}\colon \mathcal{E} \times \Delta^1 \to \mathcal{E}$ in $\mathrm{Fun}(\mathcal{E}, \mathcal{E})$ over $\epsilon \circ \pi\colon jr\pi \to \pi$ with target $\mathrm{id}_{\mathcal{E}}$. The morphism $\bar{\epsilon}_x$ is π-Cartesian for every $x \in \mathcal{E}$. We let $R' := \bar{\epsilon}|_{\mathcal{E} \times \{0\}}$. By construction we then have a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[dr, dashed, "R"] \arrow[drr, bend left, "R'"] \arrow[ddr, bend right, "r\pi"'] & & \\ & \mathcal{E}_0 \arrow[r, "J"] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "\pi"] \\ & \mathcal{C}_0 \arrow[r, "j"] & \mathcal{C}, \end{tikzcd}$$

and by the universal property of the pullback $\mathcal{E}_0$ there exists a dashed arrow $R\colon \mathcal{E} \to \mathcal{E}_0$. By (the dual of) [Lur09a, Proposition 5.2.2.8], to show that R is right adjoint to J it suffices to show that for all $X \in \mathcal{E}_0$ and $Y \in \mathcal{E}$, the map

$$\mathrm{Map}_{\mathcal{E}_0}(X, RY) \to \mathrm{Map}_{\mathcal{E}}(JX, JRY) \to \mathrm{Map}_{\mathcal{E}}(JX, Y)$$

arising from composition with $\bar{\epsilon}_Y$ is an equivalence. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}_0}(X, RY) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(JX, JRY) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(JX, Y) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}_0}(x, ry) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(jx, jry) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(jx, y). \end{tikzcd}$$

where we write $x := \pi_0 X$, $y := \pi Y$, and use the identifications $\pi_0 RY \simeq ry$, $\pi JX \simeq jx$, and $\pi JRY \simeq jry$. Here the left-hand square is Cartesian since $\mathcal{E}_0$ is a pullback, and the right-hand square is Cartesian since the map $\bar{\epsilon}_Y$ is a π-Cartesian morphism; this means the composite square is also Cartesian. The composite in the bottom row is an equivalence as ϵ is the counit for the adjunction $j \dashv r$, so this means the composite upper horizontal map is also an equivalence, as required. □

Lemma 4.15. *Let $\mathcal{C}$ be an ∞-category with finite limits, and write i for the inclusion $\{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$. Then the functor $i^*\colon \mathrm{Cat}(\mathcal{C}) \to \mathcal{C}$ given by composition with i is a Cartesian fibration.*

Proof. We first observe that the functor i^* has a right adjoint. The right Kan extension functor i_* gives a right adjoint to i^*, considered as a functor from $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ to $\mathcal{C}$. For $C \in \mathcal{C}$ the simplicial object i_*C is given by $(i_*C)_n \simeq C^{\times(n+1)}$, with face maps given by projections and degeneracies by diagonal maps, and so lies in the full subcategory $\mathrm{Cat}(\mathcal{C})$. Thus i_* restricts to a right adjoint to $i^*\colon \mathrm{Cat}(\mathcal{C}) \to \mathcal{C}$.

We now apply the criterion of [Hau17, Corollary 5.28]: Since i^* has a right adjoint, to see that i^* is a Cartesian fibration it is enough to check that, given $X_\bullet \in \mathrm{Cat}(\mathcal{C})$, a map $C \to X_0$, and a pullback square

$$\begin{tikzcd} Y_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ i_*C \arrow[r] & i_*X_0, \end{tikzcd}$$

the induced map $Y_0 \to (i_*C)_0 \to C$ is an equivalence.

Limits in $\mathrm{Cat}(\mathcal{C})$ are computed in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$, and limits in functor ∞-categories are computed objectwise, so this pullback square gives a pullback square when evaluated at every object of Δ^{op}. Thus we in particular have a pullback square

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ (i_*C)_0 \arrow[r] & (i_*X_0)_0 \end{tikzcd}$$

in $\mathcal{C}$. Here the right vertical map is an equivalence, so the left vertical map is also an equivalence. □

Lemma 4.16.

(i) *Suppose $f\colon \mathcal{C} \rightleftarrows \mathcal{D} : g$ is an adjunction. Then there is an induced adjunction $g^*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathrm{Fun}(\mathcal{D}, \mathcal{E}) : f^*$ on functor ∞-categories.*

(ii) Suppose $\mathcal{C}$ is an ∞-category with an initial object $\emptyset$. Then the functor $\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \to \mathcal{E}$ given by evaluation at $\emptyset$ is left adjoint to the constant diagram functor $\mathcal{E} \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

Proof. To prove (i), we just choose unit and counit transformations for $f \dashv g$; composing with these gives unit and counit transformations for an adjunction $g^* \dashv f^*$, since the adjunction identities can be deduced from the adjunction identities for $f \dashv g$. Now (ii) is a special case of (i), applied to the adjunction $\{\emptyset\} \rightleftarrows \mathcal{C}$. $\square$

Lemma 4.17. *Let $\mathcal{C}$ be an ∞-category with finite limits. The inclusion $\mathrm{Seg}_n(\mathcal{C}) \hookrightarrow \mathrm{Cat}(\mathrm{Seg}_{n-1}(\mathcal{C}))$ admits a right adjoint for all n.*

Proof. We prove this by applying Lemma 4.14 to the pullback square

$$\begin{tikzcd} \mathrm{Seg}_n(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Cat}(\mathrm{Seg}_{n-1}(\mathcal{C})) \arrow[d, "i^*"] \\ \mathcal{C} \arrow[r, "c"'] & \mathrm{Seg}_{n-1}(\mathcal{C}), \end{tikzcd}$$

where c is the constant-diagram functor and i denotes the inclusion $\{[0]\} \times (\Delta^{\mathrm{op}})^{\times(n-1)} \hookrightarrow (\Delta^{\mathrm{op}})^{\times n}$.

To see that c has a right adjoint, let e denote the inclusion $\{([0], \ldots, [0])\} \hookrightarrow (\Delta^{\mathrm{op}})^{\times(n-1)}$. By Lemma 4.16 the functor $e^* \colon \mathrm{Fun}(\Delta^{n-1,\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ given by evaluation at $([0], \ldots, [0])$ is right adjoint to the constant-diagram functor $c \colon \mathcal{C} \to \mathrm{Fun}(\Delta^{n-1,\mathrm{op}}, \mathcal{C})$. Since c factors through $\mathrm{Seg}_{n-1}(\mathcal{C})$, this restricts to an adjunction

$$c : \mathcal{C} \rightleftarrows \mathrm{Seg}_{n-1}(\mathcal{C}) : e^*.$$

Moreover, since e^*c is the identity functor on $\mathcal{C}$, the functor c is fully faithful.

It follows from Lemma 4.15 that i^* is a Cartesian fibration, so the hypotheses of Lemma 4.14 are satisfied, which implies the existence of the required right adjoint $\mathrm{Cat}(\mathrm{Seg}_{n-1}(\mathcal{C})) \to \mathrm{Seg}_n(\mathcal{C})$. $\square$

Lemma 4.18. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories with finite limits, and*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is an adjunction such that the left adjoint F preserves finite limits. Then composition with F and G gives an adjunction

$$F_* : \mathrm{Cat}(\mathcal{C}) \rightleftarrows \mathrm{Cat}(\mathcal{D}) : G_*.$$

Proof. Composition with F and G induces an adjunction

$$F_* : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \rightleftarrows \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) : G_*$$

by the same argument as in the proof of Lemma 4.16, and since F preserves finite limits both F_* and G_* preserve the full subcategories of category objects, giving the desired restricted adjunction between these. $\square$

Proof of Proposition 4.12. Since the full subcategory $\mathrm{Seg}_k(\mathcal{C})$ of $\mathrm{Cat}(\mathrm{Seg}_{k-1}(\mathcal{C}))$ is closed under finite limits, by combining Lemma 4.17 with Lemma 4.18 it follows that the inclusion

$$\mathrm{Cat}^i(\mathrm{Seg}_k(\mathcal{C})) \hookrightarrow \mathrm{Cat}^{i+1}(\mathrm{Seg}_{k-1}(\mathcal{C}))$$

has a right adjoint for all i. Composing these adjoints, we see that the composite inclusion

$$\mathrm{Seg}_n(\mathcal{C}) \hookrightarrow \mathrm{Cat}(\mathrm{Seg}_{n-1}(\mathcal{C})) \hookrightarrow \cdots \hookrightarrow \mathrm{Cat}^n(\mathcal{C})$$

also has a right adjoint. □

Definition 4.19. Let $\mathcal{C}$ be an n-uple Segal space. We refer to $U_{\mathrm{Seg}}\mathcal{C}$ as the *underlying n-fold Segal space* of $\mathcal{C}$.

Remark 4.20. In the definition of n-fold Segal space, we privileged one of the spaces of 1-morphisms, etc. By making different choices (i.e. by permuting the coordinates in $\Delta^{n,\mathrm{op}}$) we get $n!$ different n-fold Segal spaces from an n-uple Segal space.

5. The (∞, k)-Category of Iterated Spans

In this section we construct the (∞, k)-category $\mathrm{Span}_k(\mathcal{C})$ of iterated spans in an ∞-category $\mathcal{C}$ with finite limits, in the form of a k-fold Segal space. As we noted in the introduction, we expect the $(\infty, k-1)$-category of morphisms from X to Y in $\mathrm{Span}_k(\mathcal{C})$ to be $\mathrm{Span}_{k-1}(\mathcal{C}_{/X\times Y})$. To avoid dealing with coherence issues our construction will not be immediately recognizable as being of this form, but we'll see below in Proposition 8.3 that the mapping $(\infty, k-1)$-categories do indeed admit this inductive description. Moreover, as the diagrams describing an n-fold span become increasingly complicated as n increases, it turns out to be much easier to define $\mathrm{Span}_k(\mathcal{C})$ as the underlying k-fold Segal space of a k-uple Segal space $\mathrm{SPAN}_k(\mathcal{C})$. This will have all of its k types of 1-morphisms given by spans in $\mathcal{C}$, and the "commutative squares" are given by diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X & A \arrow[l] \arrow[r] & Y \\ C \arrow[u] \arrow[d] & J \arrow[l] \arrow[r] \arrow[u] \arrow[d] & D \arrow[u] \arrow[d] \\ Z & B \arrow[l] \arrow[r] & W, \end{tikzcd}$$

whose shape is given by the *product* of the diagram shape describing spans; similarly, the higher "commutative i-cubes" are described by the i-fold product of this shape. Notice that when we restrict to the underlying k-fold Segal space we consider only those diagrams of the above form where the maps $X \leftarrow C \rightarrow Z$ and $Y \leftarrow D \rightarrow W$ are all identities, in which case this diagram is equivalent to a 2-fold span. In fact, since it is no more work and will be useful in the next section, we'll construct an n-fold category object $\mathrm{SPAN}_k^+(\mathcal{C})$ in Cat_∞ — this may be regarded as a $(k+1)$-uple ∞-category, from which we may extract an $(\infty, k+1)$-category $\mathrm{Span}_k^+(\mathcal{C})$ extending $\mathrm{Span}_k(\mathcal{C})$, with $(k+1)$-morphisms given by natural transformations of k-fold span diagrams.

In the case $k = 1$, the construction we use is due to Barwick [Bar13a], and in general we consider a simple inductive generalization of Barwick's definition. We begin by introducing some notation:

Definition 5.1. Let Σ^n be the partially ordered set with objects pairs (i, j) with $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ if $i \leq i'$ and $j' \leq j$. A map of totally ordered sets $\phi\colon [n] \to [m]$ induces a functor $\Sigma^n \to \Sigma^m$ by sending (i, j) to $(\phi(i), \phi(j))$; we thus get a functor $\Sigma^\bullet\colon \Delta \to \mathrm{Cat}$. We will also write $\Sigma^{m_1,\ldots,m_k}$ for the product $\Sigma^{m_1} \times \cdots \times \Sigma^{m_k}$, which defines a functor $\Sigma^{\bullet,\ldots,\bullet}\colon \Delta^k \to \mathrm{Cat}$.

Definition 5.2. Let Λ^n denote the full subcategory of Σ^n spanned by those pairs (i, j) such that $j - i \leq 1$. These subcategories are not in general preserved by the functors $\Sigma(\phi)$ for ϕ in Δ, but they are preserved by the functors arising from inert maps. We thus get a functor $\Lambda^\bullet \colon \Delta_{\mathrm{int}} \to \mathrm{Cat}$ with a natural transformation $i \colon \Lambda^\bullet \to \Sigma^\bullet|_{\Delta_{\mathrm{int}}}$. We will also write $\Lambda^{m_1,\ldots,m_k}$ for the product $\Lambda^{m_1} \times \cdots \times \Lambda^{m_k}$, which defines a functor $\Lambda^{\bullet,\ldots,\bullet} \colon \Delta^k_{\mathrm{int}} \to \mathrm{Cat}$, with a natural transformation $i \colon \Lambda^{\bullet,\ldots,\bullet} \to \Sigma^{\bullet,\ldots,\bullet}|_{\Delta^k_{\mathrm{int}}}$.

Examples 5.3.

(i) The category $\Sigma^0 = \Lambda^0$ is the trivial one-object category.

(ii) The category $\Sigma^1 = \Lambda^1$ can be depicted as

$$(0,0) \leftarrow (0,1) \rightarrow (1,1).$$

(iii) The category Σ^2 can be depicted as

$$\begin{tikzcd}[column sep=tiny]
 & & (0,2) \arrow[dl] \arrow[dr] & & \\
 & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2),
\end{tikzcd}$$

and the subcategory Λ^2 as

$$\begin{tikzcd}[column sep=tiny]
 & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2).
\end{tikzcd}$$

(iv) The category Σ^3 can be depicted as

$$\begin{tikzcd}[column sep=tiny]
 & & & (0,3) \arrow[dl] \arrow[dr] & & & \\
 & & (0,2) \arrow[dl] \arrow[dr] & & (1,3) \arrow[dl] \arrow[dr] & & \\
 & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2) & & (3,3),
\end{tikzcd}$$

and the subcategory Λ^3 as

$$\begin{tikzcd}[column sep=tiny]
 & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2) & & (3,3).
\end{tikzcd}$$

Remark 5.4. We can think of the object $(i,j) \in \Sigma^n$ as a subinterval $(i, i+1, \ldots, j)$ of the ordered set $[n] \in \Delta$. This gives a functor $\Sigma^n \to \Delta^{\mathrm{op}}_{/[n]}$ that takes (i,j) to the inert map $[j-i] \to [n]$ that sends $t \in [j-i]$ to $i+t$, with a map $(i,j) \to (i',j')$ sent to the unique inert map $[j' - i'] \to [j - i]$ such that the diagram

$$\begin{tikzcd} {[j'-i']} \arrow[rr] \arrow[dr] & & {[j-i]} \arrow[dl] \\ & {[n]} & \end{tikzcd}$$

commutes. This functor identifies Σ^n with the subcategory $\Delta^{\mathrm{op}}_{\mathrm{int},/[n]}$ of inert maps, which will be useful later. Similarly, restricting the functor to Λ^n, this is identified with the full subcategory $\mathrm{Cell}^{1,\mathrm{op}}_{/[n]}$ as defined in Definition 3.8.

Definition 5.5. Suppose $\mathcal{C}$ is an ∞-category with finite limits. A functor $f\colon \Sigma^{n_1,\dots,n_k} \to \mathcal{C}$ is *Cartesian* if f is a right Kan extension of its restriction to $\Lambda^{n_1,\dots,n_k}$ along the inclusion $i = i_{n_1,\dots,n_k}$. Equivalently, f is Cartesian if and only if the unit map

$$f \to i_* i^* f$$

is an equivalence, where $i^*\colon \mathrm{Fun}(\Sigma^{n_1,\dots,n_k}, \mathcal{C}) \to \mathrm{Fun}(\Lambda^{n_1,\dots,n_k}, \mathcal{C})$ is the functor given by composition with i, and i_* denotes its right adjoint, given by right Kan extension. We denote the full subcategory of the ∞-category $\mathrm{Fun}(\Sigma^{n_1,\dots,n_k}, \mathcal{C})$ spanned by the Cartesian functors by $\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\dots,n_k}, \mathcal{C})$, and its underlying space by $\mathrm{Map}^{\mathrm{Cart}}(\Sigma^{n_1,\dots,n_k}, \mathcal{C})$.

We can also formulate this condition inductively:

Lemma 5.6. *Suppose $\mathcal{C}$ is an ∞-category with finite limits. Then the following are equivalent, for a functor $F\colon \Sigma^{n_1,\dots,n_k} \to \mathcal{C}$:*

(1) F is Cartesian.
(2) F is a right Kan extension of its restriction to $\Lambda^{n_1} \times \Sigma^{n_2,\dots,n_k}$, and for every $I \in \Lambda^{n_1}$ the restriction F_I of F to $\{I\} \times \Sigma^{n_2,\dots,n_k}$ is Cartesian.
(3) The functor $\tilde{F}\colon \Sigma^{n_1} \to \mathrm{Fun}(\Sigma^{n_2,\dots,n_k}, \mathcal{C})$ corresponding to F is Cartesian, and for every $X \in \Lambda^{n_1}$ the image $\tilde{F}(X) \in \mathrm{Fun}(\Sigma^{n_2,\dots,n_k}, \mathcal{C})$ is Cartesian.

Proof. Let F' denote the restriction of F to $\Lambda^{n_1} \times \Sigma^{n_2,\dots,n_k}$. We first prove that the following pair of conditions are equivalent:

(a) F' is a right Kan extension of its restriction to $\Lambda^{n_1} \times \Lambda^{n_2,\dots,n_k}$.
(b) For every $I \in \Lambda^{n_1}$, the restriction F'_I of F' to $\{I\} \times \Sigma^{n_2,\dots,n_k}$ is Cartesian.

Condition (a) says that for every $I \in \Lambda^{n_1}$, $J \in \Sigma^{n_2,\dots,n_k}$ the natural map

$$F(I,J) \to \lim_{X \in (\Lambda^{n_1} \times \Lambda^{n_2,\dots,n_k})_{(I,J)/}} F(X)$$

is an equivalence, whereas (b) says that for every $I \in \Lambda^{n_1}$, $J \in \Sigma^{n_2,\dots,n_k}$ the natural map

$$F(I,J) \to \lim_{Y \in \Lambda^{n_2,\dots,n_k}_{J/}} F(I,Y)$$

is an equivalence. But the inclusion $\{I\} \times \Lambda^{n_2,\dots,n_k}_{J/} \to \Lambda^{n_1}_{I/} \times \Lambda^{n_2,\dots,n_k}_{J/}$ is coinitial (as coinitial maps are closed under products by the dual of [Lur09a, Corollary 4.1.1.13]), so these two limits are canonically equivalent, hence (a) and (b) are indeed equivalent.

Thus condition (2) holds if and only if F is the right Kan extension of its restriction F' and F' is the right Kan extension of its restriction to $\Lambda^{n_1,\dots,n_k}$. The transitivity of Kan extensions implies this is the same as condition (1).

To see that (2) is equivalent to (3), we are reduced to showing that the following two conditions are equivalent:

(a′) F is a right Kan extension of its restriction to $\Lambda^{n_1} \times \Sigma^{n_2,\ldots,n_k}$.
(b′) $\tilde{F} \colon \Sigma^{n_1} \to \mathrm{Fun}(\Sigma^{n_2,\ldots,n_k}, \mathcal{C})$ is Cartesian.

Here (a′) says that for every $I \in \Sigma^{n_1}$ and $J \in \Sigma^{n_2,\ldots,n_k}$ the natural map

$$F(I,J) \to \lim_{X \in (\Lambda^{n_1} \times \Sigma^{n_2,\ldots,n_k})_{(I,J)/}} F(X)$$

is an equivalence. By the same argument as above, the inclusion $\Lambda^{n_1}_{I/} \times \{J\} \hookrightarrow (\Lambda^{n_1} \times \Sigma^{n_2,\ldots,n_k})_{(I,J)/}$ is coinitial, so this is equivalent to $F(I,J)$ being $\lim_{Y \in \Lambda^{n_1}_{I/}} F(Y,J)$. As limits in functor ∞-categories are computed objectwise, this is equivalent to (b′), which completes the proof. □

Remark 5.7. Expanding the definition, in the case $k = 1$ we see that a functor $f \colon \Sigma^n \to \mathcal{C}$ is Cartesian if and only if it is obtained by taking iterated pullbacks of the n spans $f_1, \ldots, f_n$ given by restricting f along the inclusions $\Sigma(\rho_i) \colon \Sigma^1 \to \Sigma^n$ coming from the maps $\rho_i \colon [1] \to [n]$ in Δ that send 0 to $i-1$ and 1 to i. In other words, f is Cartesian if and only if it presents the n-fold composite of these spans as 1-morphisms in our desired ∞-category of spans. The preceding Lemma then says that, similarly, a functor $\Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ is Cartesian if it presents the appropriate composite of spans in k different directions.

Definition 5.8. Suppose $\mathcal{C}$ is an ∞-category with finite limits. Let $\overline{\mathrm{SPAN}}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ be a coCartesian fibration associated to the functor $\Delta^{k,\mathrm{op}} \to \mathrm{Cat}_\infty$ given by $\mathrm{Fun}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$. Then we define $\mathrm{SPAN}_k^+(\mathcal{C})$ to be the full subcategory of $\overline{\mathrm{SPAN}}_k^+(\mathcal{C})$ spanned by the Cartesian functors $\Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ for all $n_1, \ldots, n_k$.

Our goal is now to prove that $\mathrm{SPAN}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ is the coCartesian fibration associated to a k-fold category object in Cat_∞. We will first show that this is indeed a coCartesian fibration, which is a consequence of the following result:

Proposition 5.9. *Suppose $\mathcal{C}$ is an ∞-category with finite limits and $F \colon \Sigma^m \to \mathcal{C}$ is a Cartesian functor. Then for any map $\phi \colon [n] \to [m]$ in Δ, the composite functor*

$$\phi^* F \colon \Sigma^n \to \Sigma^m \to \mathcal{C}$$

is again Cartesian.

Before we give the proof, we first prove the key technical observation we need, which requires a bit of notation:

Definition 5.10. Suppose $\phi \colon [n] \to [m]$ is an injective map in Δ. Let $\Lambda^m[\phi]$ denote the full subcategory of Σ^m spanned by the objects $I = (i,j)$ such that either $I \in \Lambda^m$ or $I \geq \phi(J)$ for some $J \in \Lambda^n$.

Lemma 5.11.

(i) *Suppose $\phi \colon [n] \to [m]$ is a surjective map in Δ. Then for every $I \in \Sigma^n$, the induced functor $\Lambda^n_{I/} \to \Lambda^m_{\phi(I)/}$ is coinitial.*
(ii) *Suppose $\phi \colon [n] \to [m]$ is an injective map in Δ. Then for every $I \in \Sigma^n$, the induced functor $\Lambda^n_{I/} \to \Lambda^m[\phi]_{\phi(I)/}$ is coinitial.*

Proof. If $I = (i,j)$ then the category $\Lambda^n_{I/}$ is equivalent to $\Lambda^{n'}$ where $n' = j - i$. If we let ϕ' denote the restriction of ϕ to a map $\phi' \colon [n'] \cong \{i, i+1, \ldots, j\} \to \{\phi(i), \phi(i)+1, \ldots, \phi(j)\} \cong$

$[m']$, where $m' := \phi(j) - \phi(i)$, then we also have $\Lambda^m_{\phi(I)/} \simeq \Lambda^{m'}$ in case (i) and $\Lambda^m[\phi]_{\phi(I)/} \simeq \Lambda^{m'}[\phi']$ in case (ii). Replacing ϕ by ϕ' we may therefore without loss of generality assume that $I = (0, n)$ and $\phi(I) = (0, m)$. Note that for (ii) this implies that $\Lambda^m[\phi]$ consists precisely of those objects I such that $I \geq \phi(J)$ for some $J \in \Lambda^n$.

Using (the dual of) [Lur09a, Theorem 4.1.3.1], to prove (i) we must show that for every $J \in \Lambda^m$ the category $\Lambda^n_{/J}$ is weakly contractible. Since the category Λ^m is a partially ordered set, we can identify this slice category with the full subcategory of Λ^n spanned by the objects X such that $J \geq \phi(X)$. If $J = (j, k)$, set $j' := \max\{x : \phi(x) \leq j\}$ and $k' := \min\{x : \phi(x) \geq k\}$; then $X = (x, y)$ satisfies $J \geq \phi(X)$ if and only if $x \leq j'$ and $k' \leq y$. The integer $k - j$ is either 0 or 1. If $k - j = 1$, then (as ϕ is surjective) $k' = j' + 1$ and so $\Lambda^n_{/J}$ consists of the single object $(j', j' + 1)$. On the other hand, if $k = j$, then $k' \leq j'$ and $\Lambda^n_{/J}$ contains the objects $(k' - 1, k')$, (k', k'), $(k', k' + 1), \ldots, (j', j')$, $(j', j' + 1)$. Thus $\Lambda^n_{/J}$ is an iterated pushout $[1] \amalg_{[0]} [1] \amalg_{[0]} \cdots \amalg_{[0]} [1]$ and hence weakly contractible.

To prove (ii), appealing to [Lur09a, Theorem 4.1.3.1] again we need to show that for every $J \in \Lambda^m[\phi]$ the category $\Lambda^n_{/J}$ is weakly contractible. We can again identify this with the full subcategory of Λ^n spanned by the objects X such that $\phi(X) \leq J$, i.e. those $X = (x, y)$ such that $\phi(x) \leq j \leq k \leq \phi(y)$. Let j' and k' be defined as before, then again this holds if and only if $x \leq j'$ and $y \geq k'$. Since ϕ is injective we have $j' \leq k'$, and by definition of $\Lambda^m[\phi]$ there exists some $Y = (a, b) \in \Lambda^n$ such that $\phi(Y) \leq J$, which forces $k' \leq j' + 1$. Thus (j', k') is an object of Λ^n, and by construction it is then terminal in $\Lambda^n_{/J}$. Since any ∞-category with a terminal object is weakly contractible, this completes the proof. □

Proof of Proposition 5.9. We must show that for all $I \in \Sigma^n$ the natural map

$$(\phi^* F)(I) \to \lim_{X \in \Lambda^n_{/I}} (\phi^* F)(X)$$

is an equivalence.

It suffices to check this in the cases where ϕ is either injective or surjective, since these classes of maps form a factorization system on Δ. In the surjective case this map factors as

$$F(\phi(I)) \to \lim_{Y \in \Lambda^m_{/\phi(I)}} F(Y) \to \lim_{X \in \Lambda^n_{/I}} F(\phi(X)).$$

Here the first map is an equivalence since F is Cartesian, and the second map is an equivalence since the functor $\Lambda^n_{/I} \to \Lambda^m_{/\phi(I)}$ is coinitial by Lemma 5.11(i).

In the injective case, the map factors as

$$F(\phi(I)) \to \lim_{Y \in \Lambda^m[\phi]_{/\phi(I)}} F(Y) \to \lim_{X \in \Lambda^n_{/I}} F(\phi(X)).$$

Since F is Cartesian, it is also the right Kan extension of its restriction to $\Lambda^m[\phi]$, since this full subcategory contains Λ^m. Thus the first map is an equivalence, and the second is an equivalence by Lemma 5.11(ii). □

Corollary 5.12. *The restricted projection* $\mathrm{SPAN}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ *is a coCartesian fibration.*

Proof. Let π denote the projection $\overline{\mathrm{SPAN}}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ and write π' for the restricted projection $\mathrm{SPAN}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$. Since $\mathrm{SPAN}_k^+(\mathcal{C})$ is a full subcategory of $\overline{\mathrm{SPAN}}_k^+(\mathcal{C})$, a π-coCartesian morphism whose source and target are in $\mathrm{SPAN}_k^+(\mathcal{C})$ is necessarily π'-coCartesian. As π is a coCartesian fibration, to see that π' is one it therefore suffices to check that if $\alpha \to \beta$ is a coCartesian morphism in $\overline{\mathrm{SPAN}}_k^+(\mathcal{C})$ such that α is a Cartesian functor, then β is a Cartesian functor. In other words, given a Cartesian functor $\alpha \colon \Sigma^{n_1,\ldots,n_k} \to \mathcal{C}$ and morphisms $\phi_i \colon [n_i] \to [m_i]$ in Δ for $i = 1, \ldots, k$, we must show that the composite functor $(\phi_1, \ldots, \phi_k)^*\alpha \colon \Sigma^{m_1,\ldots,m_k} \to \mathcal{C}$ is also Cartesian. Using Lemma 5.6 we can check this iteratively, which means we only need to prove the case $k = 1$. But this case is precisely Proposition 5.9. □

Proposition 5.13. *The compatible maps $\Lambda^1, \Lambda^0 \to \Lambda^n$ induced by the inert maps $[0], [1] \to [n]$ give a functor $\Lambda^n_{\mathrm{Seg}} := \Lambda^1 \amalg_{\Lambda^0} \cdots \amalg_{\Lambda^0} \Lambda^1 \to \Lambda^n$, where the colimit is formed in Cat_∞. This is an equivalence of ∞-categories.*

Proof. We will describe the ∞-categorical colimit Λ^n_{Seg} as a homotopy colimit in the Joyal model structure. The object Λ^n in Cat_∞ is represented by the nerve $\mathrm{N}\Lambda^n$ in Set_Δ^J, which is a quasicategory. The morphisms $\mathrm{N}\Lambda^0 \to \mathrm{N}\Lambda^1$ induced by the inclusions $[0] \to [1]$ are levelwise injective, i.e. cofibrations in the Joyal model structure. Therefore the iterated (1-categorical) pushout $\mathrm{N}\Lambda^1 \amalg_{\mathrm{N}\Lambda^0} \cdots \amalg_{\mathrm{N}\Lambda^0} \mathrm{N}\Lambda^1$ is a homotopy colimit and a fibrant replacement of it represents Λ^n_{Seg}; it is therefore sufficient to show that the natural map $\mathrm{N}\Lambda^1 \amalg_{\mathrm{N}\Lambda^0} \cdots \amalg_{\mathrm{N}\Lambda^0} \mathrm{N}\Lambda^1 \to \mathrm{N}\Lambda^n$ is a weak equivalence in Set_Δ^J. But this is in fact an *isomorphism* of simplicial sets. □

Proposition 5.14. *The functor associated to the coCartesian fibration $\mathrm{SPAN}_k^+(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ is a k-fold category object.*

Proof. Unwinding the definitions, we must show that for each $([n_1], \ldots, [n_k])$ in Δ^k, the natural map

$$\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C}) \to \lim_{([i_1],\ldots,[i_k]) \in \mathrm{Cell}^{k,\mathrm{op}}_{/([n_1],\ldots,[n_k])}} \mathrm{Fun}(\Sigma^{i_1}, \mathcal{C}) \times \cdots \times \mathrm{Fun}(\Sigma^{i_k}, \mathcal{C})$$

is an equivalence. Now using Proposition 5.13 and the fact that products in Cat_∞ commute with colimits, it follows that the target of this map is equivalent to $\mathrm{Fun}(\Lambda^{n_1,\ldots,n_k}, \mathcal{C})$, and under this equivalence the Segal map corresponds to the map given by composing with the inclusion $\Lambda^{n_1,\ldots,n_k} \hookrightarrow \Sigma^{n_1,\ldots,n_k}$. Since this is fully faithful, and $\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{C})$ is precisely the space of functors that are right Kan extensions along this inclusion, it follows from [Lur09a, Proposition 4.3.2.15] that our map is an equivalence. □

Definition 5.15. Let $\overline{\mathrm{SPAN}}_k(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ and $\mathrm{SPAN}_k(\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ be the underlying left fibrations of the coCartesian fibrations $\overline{\mathrm{SPAN}}_k^+(\mathcal{C})$ and $\mathrm{SPAN}_k^+(\mathcal{C})$, respectively. These correspond to the multisimplicial spaces $\mathrm{Map}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$ and $\mathrm{Map}^{\mathrm{Cart}}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$, which are obtained by composing $\mathrm{Fun}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$ and $\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})$ with the underlying ∞-groupoid functor ι. Since ι preserves limits, being a right adjoint, the latter is a k-uple Segal space, which we also denote $\mathrm{SPAN}_k(\mathcal{C})$.

Definition 5.16. We define the (∞, k)*-category* $\mathrm{Span}_k(\mathcal{C})$ *of iterated spans* in $\mathcal{C}$ to be the k-fold Segal space $U_{\mathrm{Seg}}\mathrm{SPAN}_k(\mathcal{C})$ underlying the k-uple Segal space $\mathrm{SPAN}_k(\mathcal{C})$.

Remark 5.17. Using the complete Segal space model for ∞-categories, we may regard Cat_∞ as a full subcategory of $\mathrm{Seg}(\mathcal{S})$. We may then regard $\mathrm{SPAN}_k^+(\mathcal{C})$ as a k-uple Segal object in $\mathrm{Seg}(\mathcal{S})$, i.e. a $(k+1)$-uple Segal space. We let $\mathrm{Span}_k^+(\mathcal{C})$ be the underlying $(k+1)$-fold Segal space $U_{\mathrm{Seg}}\mathrm{SPAN}^+(\mathcal{C})$; this is an $(\infty, k+1)$-category that extends $\mathrm{Span}_k(\mathcal{C})$ by taking morphisms of k-fold spans as $(k+1)$-morphisms.

Remark 5.18. We may regard $\mathrm{SPAN}_k^+(-)$ as a functor from ∞-categories with finite limits to k-fold category objects in ∞-categories with finite limits. Moreover, as has been proved by David Li-Bland [LB15], this functor preserves limits, which means that we can apply SPAN_k^+ levelwise to get a functor from m-fold category objects in ∞-categories with finite limits to $(k+m)$-fold category objects. We can also iterate the construction SPAN_1^+ to recover SPAN_k^+ as $(\mathrm{SPAN}_1^+)^k$ for $k > 1$.

6. The (∞,k)-Category of Iterated Spans with Local Systems

Our goal in this section is to use the (∞,k)-category $\mathrm{Span}_k(\mathcal{S})$ to make, for every k-fold Segal space $\mathcal{C}$, an (∞,k)-category $\mathrm{Span}_k(\mathcal{S};\mathcal{C})$ of k-fold spans with $\mathcal{C}$-valued local systems. In fact, our construction works more generally: if $\mathcal{X}$ is an ∞-category with finite limits and $\mathcal{C}$ is a k-fold Segal object in $\mathcal{X}$, we will define an (∞,k)-category $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$ of k-fold iterated spans in $\mathcal{X}$ equipped with $\mathcal{C}$-valued local systems. To define this we will first show that any k-fold Segal object in $\mathcal{X}$ determines a section of the projection $\mathrm{SPAN}_k^+(\mathcal{X}) \to \Delta^{k,\mathrm{op}}$, and then use results of Lurie to construct a fibrewise slice category for this section.

Definition 6.1. Let $\widehat{\Sigma} \to \Delta^{\mathrm{op}}$ be the Grothendieck fibration associated to the functor $\Sigma^\bullet\colon \Delta \to \mathrm{Cat}$. Explicitly, $\widehat{\Sigma}$ is the category with objects pairs $([n],(i,j))$ with $[n] \in \Delta$ and $0 \leq i \leq j \leq n$, and morphisms $([n],(i,j)) \to ([m],(i',j'))$ given by a morphism $\phi\colon [m] \to [n]$ in Δ and a morphism $(i,j) \to (\phi(i'),\phi(j'))$ in Σ^n. Then the product $\widehat{\Sigma}^k \to \Delta^{k,\mathrm{op}}$ is the Grothendieck fibration associated to the functor $\Sigma^{\bullet,\ldots,\bullet}\colon \Delta^k \to \mathrm{Cat}$.

By [GHN17, Proposition 7.3], the ∞-category $\overline{\mathrm{SPAN}}_k^+(\mathcal{X})$ has a universal property: for every functor of ∞-categories $\mathcal{C} \to \Delta^{k,\mathrm{op}}$, the ∞-category $\mathrm{Fun}_{\Delta^{k,\mathrm{op}}}(\mathcal{C}, \overline{\mathrm{SPAN}}_k^+(\mathcal{X}))$ is naturally equivalent to $\mathrm{Fun}(\mathcal{C} \times_{\Delta^{k,\mathrm{op}}} \widehat{\Sigma}^k, \mathcal{X})$. In particular, giving a section $\Delta^{k,\mathrm{op}} \to \overline{\mathrm{SPAN}}_k^+(\mathcal{X})$ is equivalent to giving a functor $\widehat{\Sigma}^k \to \mathcal{X}$.

Definition 6.2. Let $\Pi\colon \widehat{\Sigma} \to \Delta^{\mathrm{op}}$ denote the functor that sends $([n],(i,j))$ to $[j-i]$ and a map $(\phi\colon [m] \to [n], (i,j) \to (\phi(i'),\phi(j')))$ to the map $[j-i] \to [j'-i']$ in Δ^{op} corresponding to the map of ordered sets taking $s \in [j'-i']$ to $\phi(i'+s) - i \in [j-i]$. We write $\Pi_n\colon \Sigma^n \to \Delta^{\mathrm{op}}$ for the restriction of Π to the fibre Σ^n — this takes $(i,j) \in \Sigma^n$ to $[j-i]$ and a map $(i,j) \to (i',j')$ to the (inert) inclusion $[j-i] \to [j'-i']$ taking $s \in [j-i]$ to $s + i' - i$.

Thus any map $\Phi\colon \Delta^{k,\mathrm{op}} \to \mathcal{X}$ determines a section $s_\Phi\colon \Delta^{k,\mathrm{op}} \to \overline{\mathrm{SPAN}}_k^+(\mathcal{X})$ via the functor $\Phi \circ \Pi^k\colon \widehat{\Sigma}^k \to \Delta^{k,\mathrm{op}} \to \mathcal{X}$.

Lemma 6.3. *Suppose we have a fully faithful inclusion $\phi\colon \mathcal{I} \to \mathcal{J}$ and a commutative triangle*

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \mathcal{J} \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

such that g is the left Kan extension of f along ϕ. Given $X \in \mathcal{J}$, let $\mathcal{I}_{/X}$ denote the fibre product $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/X}$, let $i\colon \mathcal{I}_{/X} \to \mathcal{I}$ and $j\colon \mathcal{J}_{/X} \to \mathcal{J}$ denote the forgetful functors, and let $\phi_{/X}$ denote the induced map $\mathcal{I}_{/X} \to \mathcal{J}_{/X}$. Then $g \circ j$ is the left Kan extension of $f \circ i$ along $\phi_{/X}$.

Proof. We must show that for any object $\alpha\colon Y \to X$ in $\mathcal{J}$, the object $gj(\alpha)$ is the colimit of the diagram

$$\mathcal{I}_{/X} \times_{\mathcal{J}_{/X}} (\mathcal{J}_{/X})_{/\alpha} \to \mathcal{I}_{/X} \xrightarrow{i} \mathcal{I} \xrightarrow{f} \mathcal{C}.$$

But $(\mathcal{J}_{/X})_{/\alpha}$ is equivalent to $\mathcal{J}_{/Y}$, and so $\mathcal{I}_{/X} \times_{\mathcal{J}_{/X}} (\mathcal{J}_{/X})_{/\alpha}$ is equivalent to $\mathcal{I}_{/Y}$, and the diagram is equivalent to the composite $\mathcal{I}_{/Y} \to \mathcal{I} \xrightarrow{f} \mathcal{C}$. Since g is the left Kan extension of f, we know that $gj(\alpha) \simeq g(Y)$ is the colimit of this diagram, which completes the proof. $\square$

Lemma 6.4. *Suppose $\Phi\colon \mathbb{\Delta}^{k,\mathrm{op}} \to \mathcal{X}$ is a k-fold category object in $\mathcal{X}$. Then the section $s_\Phi\colon \mathbb{\Delta}^{k,\mathrm{op}} \to \overline{\mathrm{SPAN}}_k^+(\mathcal{X})$ factors through $\mathrm{SPAN}_k^+(\mathcal{X})$.*

Proof. From the proof of [GHN17, Proposition 7.3] it follows that the value of the section s_Φ at $I = ([n_1], \ldots, [n_k]) \in \mathbb{\Delta}^{k,\mathrm{op}}$ is the composite

$$\Sigma^{n_1,\ldots,n_k} \xrightarrow{\Pi_{n_1,\ldots,n_k}} \mathbb{\Delta}^{k,\mathrm{op}} \xrightarrow{\Phi} \mathcal{X},$$

where $\Pi_{n_1,\ldots,n_k}$ denotes the product $\Pi_{n_1} \times \cdots \times \Pi_{n_k}$. We thus need to check that each of these functors is Cartesian. But under the identification of $\Sigma^{n_1,\ldots,n_k}$ with $\mathbb{\Delta}^{k,\mathrm{op}}_{\mathrm{int},/([n_1],\ldots,[n_k])}$ of Remark 5.4, the functor $\Pi_{n_1,\ldots,n_k}$ corresponds to the forgetful functor $\mathbb{\Delta}^{k,\mathrm{op}}_{\mathrm{int},/([n_1],\ldots,[n_k])} \to \mathbb{\Delta}^{k,\mathrm{op}}$. By Lemma 4.8, the restriction of Φ to $\mathbb{\Delta}^{k,\mathrm{op}}_{\mathrm{int}}$ is the right Kan extension of its restriction to $\mathrm{Cell}^{k,\mathrm{op}}$, from which it follows by (the dual of) Lemma 6.3 that $\Pi_{n_1,\ldots,n_k}$ is the right Kan extension of its restriction to $\Lambda^{n_1,\ldots,n_k}$, since this corresponds to $\mathrm{Cell}^{k,\mathrm{op}}_{/([n_1],\ldots,[n_k])}$ under this identification. $\square$

Lemma 6.5. *Suppose $f\colon \mathcal{E} \to \mathcal{B}$ is a coCartesian fibration and $s\colon \mathcal{B} \to \mathcal{E}$ is a section. Then there exists an ∞-category $\mathcal{E}_{//s}$ and a coCartesian fibration $\mathcal{E}_{//s} \to \mathcal{B}$ with the universal property that $\mathrm{Fun}_{/\mathcal{B}}(\mathcal{C}, \mathcal{E}_{//s})$ is naturally given by a pullback square*

$$\begin{array}{ccc} \mathrm{Fun}_{/\mathcal{B}}(\mathcal{C}, \mathcal{E}_{//s}) & \longrightarrow & \mathrm{Fun}_{/\mathcal{B}}(\mathcal{C} \times [1] \amalg_{\mathcal{C} \times \{1\}} \mathcal{B}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \{s\} & \longrightarrow & \mathrm{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}). \end{array}$$

Proof. Let $\mathfrak{E} \to \mathfrak{B}$ be a coCartesian inner fibration representing f such that there is a section $s'\colon \mathfrak{B} \to \mathfrak{E}$ representing s. Then, following [Lur09a, Definition 4.2.2.1], we define a simplicial set $\mathfrak{E}_{//s'}$ over $\mathfrak{B}$ by taking $\mathrm{Hom}_{/\mathfrak{B}}(K, \mathfrak{E}_{//s'})$ to be defined by the pullback square

$$\begin{array}{ccc} \mathrm{Hom}_{/\mathfrak{B}}(K, \mathfrak{E}_{//s'}) & \longrightarrow & \mathrm{Hom}_{/\mathfrak{B}}(K \times \Delta^1 \amalg_{K \times \{1\}} \mathfrak{B}, \mathfrak{E}) \\ \downarrow & & \downarrow \\ \{s'\} & \longrightarrow & \mathrm{Hom}_{/\mathfrak{B}}(\mathfrak{B}, \mathfrak{E}). \end{array}$$

Then the projection $\mathfrak{E}_{//s'} \to \mathfrak{B}$ is a coCartesian inner fibration by [Lur09a, Proposition 4.2.2.4]. We define $\mathcal{E}_{//s} \to \mathcal{B}$ to be the morphism in Cat_∞ it represents. The defining property of the simplicial set $\mathrm{Hom}_{/\mathfrak{B}}(K, \mathfrak{E}_{//s'})$ implies that the internal Hom of simplicial sets $\mathrm{Fun}_{/\mathfrak{B}}(K, \mathfrak{E}_{//s'})$ is given by an analogous pullback square

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathfrak{B}}(K, \mathfrak{E}_{//s'}) \arrow[r] \arrow[d] & \mathrm{Fun}_{/\mathfrak{B}}(K \times \Delta^1 \amalg_{K \times \{1\}} \mathfrak{B}, \mathfrak{E}) \arrow[d] \\
\{s'\} \arrow[r] & \mathrm{Fun}_{/\mathfrak{B}}(\mathfrak{B}, \mathfrak{E})
\end{tikzcd}$$

of simplicial sets. Since the inclusion $\mathfrak{B} \hookrightarrow K \times \Delta^1 \amalg_{K \times \{1\}} \mathfrak{B}$ is a cofibration and the Joyal model structure is enriched in itself (see [Joy08, Theorem 5.13]) the right vertical map here is a fibration in the Joyal model structure, hence this is a homotopy pullback square; this gives the pullback squares we want in Cat_∞. □

Definition 6.6. Suppose $\Phi \colon \Delta^{k,\mathrm{op}} \to \mathcal{X}$ is a k-fold category object in $\mathcal{X}$. We then define

$$\mathrm{SPAN}_k^+(\mathcal{X}; \Phi) \to \Delta^{k,\mathrm{op}}$$

to be the coCartesian fibration $\mathrm{SPAN}_k^+(\mathcal{X})_{//s_\Phi} \to \Delta^{k,\mathrm{op}}$ of Lemma 6.5.

Proposition 6.7. *Suppose $\Phi \colon \Delta^{k,\mathrm{op}} \to \mathcal{X}$ is a k-fold category object in $\mathcal{X}$. Then the functor $\Delta^{k,\mathrm{op}} \to \mathrm{Cat}_\infty$ associated to the coCartesian fibration $\mathrm{SPAN}_k^+(\mathcal{X}; \Phi) \to \Delta^{k,\mathrm{op}}$ is a k-fold category object in Cat_∞.*

Proof. Using the defining property of $\mathrm{SPAN}_k^+(\mathcal{X}; \Phi)$ we see that a map from an ∞-category $\mathcal{C}$ to the fibre $\mathrm{SPAN}_k^+(\mathcal{X}; \Phi)_{([n_1],\ldots,[n_k])}$ is naturally equivalent to a map

$$\mathcal{C}^{\triangleright} \to \mathrm{SPAN}_k^+(\mathcal{X})_{([n_1],\ldots,[n_k])}$$

that restricts to $s_\Phi([n_1], \ldots, [n_k])$ at the cone point. In other words, we have a natural equivalence

$$\begin{aligned}
\mathrm{SPAN}_k^+(\mathcal{X}; \Phi)_{([n_1],\ldots,[n_k])} &\simeq \left(\mathrm{SPAN}_k^+(\mathcal{X})_{([n_1],\ldots,[n_k])}\right)_{/s_\Phi([n_1],\ldots,[n_k])} \\
&\simeq \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{X})_{/\Phi \circ \Pi_{n_1,\ldots,n_k}}.
\end{aligned}$$

Now on the one hand we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{X})_{/\Phi \circ \Pi_{n_1,\ldots,n_k}} \arrow[r] \arrow[d] & \mathrm{Fun}([1], \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{X})) \arrow[d] \\
\{\Phi \circ \Pi_{n_1,\ldots,n_k}\} \arrow[r] & \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k}, \mathcal{X}).
\end{tikzcd}$$

On the other hand, since limits commute, we have a pullback square

$$\begin{tikzcd}
\lim \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{i_1,\ldots,i_k}, \mathcal{X})_{/\Phi \circ \Pi_{i_1,\ldots,i_k}} \arrow[r] \arrow[d] & \lim \mathrm{Fun}([1], \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{i_1,\ldots,i_k}, \mathcal{X})) \arrow[d] \\
\lim\{\Phi \circ \Pi_{i_1,\ldots,i_k}\} \arrow[r] & \lim \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{i_1,\ldots,i_k}, \mathcal{X}),
\end{tikzcd}$$

where the limit runs over $([i_1],\ldots,[i_k]) \in \mathrm{Cell}^{k,\mathrm{op}}_{/([n_1],\ldots,[n_k])}$. Since $\Phi \circ \Pi_{n_1,\ldots,n_k}$ restricts to $\Phi \circ \Pi_{i_1,\ldots,i_k}$ under the appropriate inclusion, these two squares are equivalent. But then we get that the natural map

$$\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_k},\mathcal{X})_{/\Phi\circ\Pi_{n_1,\ldots,n_k}} \to \lim_{([i_1],\ldots,[i_k])\in\mathrm{Cell}^{k,\mathrm{op}}_{/([n_1],\ldots,[n_k])}} \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{i_1,\ldots,i_k},\mathcal{X})_{/\Phi\circ\Pi_{i_1,\ldots,i_k}}$$

is an equivalence, and so $\mathrm{SPAN}^+_k(\mathcal{X};\Phi)$ is a k-fold category object. □

Definition 6.8. Suppose $\mathcal{C}$ is a k-fold Segal object in $\mathcal{X}$. We let $\mathrm{SPAN}_k(\mathcal{X};\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ denote the left fibration obtained from the coCartesian fibration $\mathrm{SPAN}^+_k(\mathcal{X};\mathcal{C}) \to \Delta^{k,\mathrm{op}}$ by discarding the non-coCartesian morphisms; this left fibration classifies a k-uple Segal space. The *(∞,k)-category* $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$ *of iterated spans in* $\mathcal{X}$ *with local systems valued in* $\mathcal{C}$ is the underlying k-fold Segal space $U_{\mathrm{Seg}}\mathrm{SPAN}_k(\mathcal{X};\mathcal{C})$ associated to the k-uple Segal space $\mathrm{SPAN}_k(\mathcal{X};\mathcal{C})$.

7. Complete Segal Objects in an ∞-Topos

In §3 we recalled Rezk's result that the localization of the ∞-category of Segal spaces at the fully faithful and essentially surjective morphisms is given by the full subcategory of *complete* Segal spaces. In this section we begin by reviewing the generalization of this to *n-fold* Segal spaces, originally proved by Barwick [Bar05]. We will review the reformulation of the theory due to Lurie [Lur09b], which allows for an inductive construction of complete n-fold Segal spaces. Lurie's version of the theorem also works for n-fold Segal objects in an arbitary ∞-topos, which describe *internal (∞,n)-categories* or more concretely *sheaves of (∞,n)-categories* on an ∞-topos. We will then prove two completeness criteria we'll make use of below: first, we show that completeness of an n-fold Segal object in an ∞-topos $\mathcal{X}$ can be checked on the n-fold Segal spaces of maps from objects of $\mathcal{X}$, and second, we give an inductive criterion for the completeness of an n-fold Segal space using the $(n-1)$-fold Segal spaces of maps.

We begin by reviewing Lurie's notion of a *distributor*. This is a technical definition that encapsulates the properties needed to make sense of complete Segal objects, which hold for both an ∞-topos and the ∞-category of complete Segal objects in an ∞-topos (Theorem 7.7). Using distributors thus allows us to give a single definition of complete Segal objects that can be iterated to give a convenient inductive definition of n-fold complete Segal objects in an ∞-topos.

Definition 7.1. A *distributor* consists of an ∞-category $\mathcal{Y}$ together with a full subcategory $\mathcal{X}$ such that:

(1) The ∞-categories $\mathcal{X}$ and $\mathcal{Y}$ are presentable.
(2) The full subcategory $\mathcal{X}$ is closed under small limits and colimits in $\mathcal{Y}$.
(3) If $Y \to X$ is a morphism in $\mathcal{Y}$ such that $X \in \mathcal{X}$, then the pullback functor $\mathcal{X}_{/X} \to \mathcal{Y}_{/Y}$ preserves colimits.
(4) The functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ that sends $X \in \mathcal{X}$ to $\mathcal{Y}_{/X}$ (and a morphism $X \to X'$ to the pullback functor $\mathcal{Y}_{/X'} \to \mathcal{Y}_{/X}$) preserves small limits.

If $\mathcal{X} \subseteq \mathcal{Y}$ is a distributor, an *$\mathcal{X}$-Segal object* in $\mathcal{Y}$ is a Segal object $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{Y}$ such that $\mathcal{C}_0 \in \mathcal{X}$. We write $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ for the full subcategory of $\mathrm{Seg}(\mathcal{Y})$ spanned by the $\mathcal{X}$-Segal objects.

Remark 7.2. It follows from [Lur09a, Theorem 6.1.3.9] that if $\mathcal{X}$ is an ∞-topos, then the tautological inclusion $\mathcal{X} \subseteq \mathcal{X}$ is a distributor.

Definition 7.3. Write $\mathrm{Gpd}(\mathcal{X})$ for the full subcategory of $\mathrm{Seg}(\mathcal{X})$ spanned by the *groupoid objects*, i.e. the simplicial objects X such that for every partition $[n] = S \cup S'$ where $S \cap S'$ consists of a single element, the diagram

$$\begin{tikzcd} X([n]) \arrow[r] \arrow[d] & X(S) \arrow[d] \\ X(S') \arrow[r] & X(S \cap S') \end{tikzcd}$$

is a pullback square. Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor, and let $\Lambda\colon \mathcal{Y} \to \mathcal{X}$ denote the right adjoint to the inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}$. The inclusion $\mathrm{Gpd}(\mathcal{X}) \hookrightarrow \mathrm{Seg}(\mathcal{X}) \hookrightarrow \mathrm{Seg}(\mathcal{Y})$ admits a right adjoint $\iota\colon \mathrm{Seg}(\mathcal{Y}) \to \mathrm{Gpd}(\mathcal{X})$, which is the composite of the functor $\Lambda\colon \mathrm{Seg}(\mathcal{Y}) \to \mathrm{Seg}(\mathcal{X})$ induced by Λ, and the functor $\iota\colon \mathrm{Seg}(\mathcal{X}) \to \mathrm{Gpd}(\mathcal{X})$ right adjoint to the inclusion, which exists by [Lur09b, Proposition 1.1.14].

Definition 7.4. We say an $\mathcal{X}$-Segal object $F\colon \Delta^{\mathrm{op}} \to \mathcal{Y}$ is *complete* if the groupoid object ιF is constant, and write $\mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$ for the full subcategory of $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ spanned by the complete $\mathcal{X}$-Segal objects. The inclusion $\mathrm{CSS}_{\mathcal{X}}(\mathcal{Y}) \hookrightarrow \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ admits a left adjoint by [Lur09a, Lemma 5.5.4.17].

Definition 7.5. Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor, and suppose $f\colon \mathcal{C} \to \mathcal{D}$ is a morphism in $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$. We say that f is *fully faithful and essentially surjective* if:

(1) The map $|\mathrm{Gpd}(\mathcal{C})| \to |\mathrm{Gpd}(\mathcal{D})|$ is an equivalence in the ∞-topos $\mathcal{X}$.
(2) The diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r] \arrow[d] & \mathcal{D}_1 \arrow[d] \\ \mathcal{C}_0 \times \mathcal{C}_0 \arrow[r] & \mathcal{D}_0 \times \mathcal{D}_0 \end{tikzcd}$$

is a pullback square in $\mathcal{Y}$.

Theorem 7.6 ([Lur09b, Theorem 1.2.13])**.** *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. Then the left adjoint*

$$L_{\mathcal{X} \subseteq \mathcal{Y}}\colon \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$$

exhibits $\mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$ as the localization of $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ with respect to the fully faithful and essentially surjective morphisms.

Theorem 7.7 ([Lur09b, Proposition 1.3.2])**.** *Suppose $\mathcal{X} \subseteq \mathcal{Y}$ is a distributor. Then so is $\mathcal{X} \subseteq \mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$, where we regard $\mathcal{X}$ as a full subcategory of $\mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$ via the diagonal embedding $c^*\colon \mathcal{X} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})$.*

We can therefore inductively define distributors $\mathcal{X} \subseteq \mathrm{CSS}^n_{\mathcal{X}}(\mathcal{Y}) := \mathrm{CSS}_{\mathcal{X}}(\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}))$; we refer to the objects of $\mathrm{CSS}^n_{\mathcal{X}}(\mathcal{Y})$ as *complete n-fold $\mathcal{X}$-Segal objects* in $\mathcal{Y}$.

Definition 7.8. Let $\mathcal{X}$ be an ∞-topos. We write $\mathrm{CSS}^n(\mathcal{X})$ for $\mathrm{CSS}^n_{\mathcal{X}}(\mathcal{X})$, which we may regard as a full subcategory of $\mathrm{Seg}_n(\mathcal{X})$. The inclusion $\mathrm{CSS}^n(\mathcal{X}) \hookrightarrow \mathrm{Seg}_n(\mathcal{X})$ has a left adjoint $L_{n,\mathcal{X}}\colon \mathrm{Seg}_n(\mathcal{X}) \to \mathrm{CSS}^n(\mathcal{X})$, obtained inductively as the composite

$$\mathrm{Seg}_n(\mathcal{X}) \xrightarrow{\mathrm{Seg}_{\mathcal{X}}(L_{n-1,\mathcal{X}})} \mathrm{Seg}_{\mathcal{X}}(\mathrm{CSS}^{n-1}(\mathcal{X})) \xrightarrow{L_{\mathcal{X} \subseteq \mathrm{CSS}^{n-1}(\mathcal{X})}} \mathrm{CSS}^n(\mathcal{X}).$$

Remark 7.9. The ∞-category $\mathrm{CSS}^n(\mathcal{X})$ can be identified with the ∞-category of sheaves on $\mathcal{X}$ valued in the ∞-category $\mathrm{CSS}^n(\mathcal{S})$ of complete n-fold Segal spaces (in other words, sheaves of (∞, n)-categories).

We now prove the useful fact that the completion functor preserves certain fibre products:

Lemma 7.10.

(i) *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor. For $X \in \mathcal{X}$, write c^*X for the constant diagram $\Delta^{\mathrm{op}} \to \mathcal{Y}$ with value X; this is an $\mathcal{X}$-Segal object. The localization $L_{\mathcal{X}\subseteq\mathcal{Y}} \colon \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{CSS}_{\mathcal{X}}(\mathcal{Y})$ preserves fibre products over c^*X where $X \in \mathcal{X}$; in particular, $L_{\mathcal{X}\subseteq\mathcal{Y}}$ preserves products.*

(ii) *Let $\mathcal{X}$ be an ∞-topos. Then the localization $L_{n,\mathcal{X}} \colon \mathrm{Seg}_n(\mathcal{X}) \to \mathrm{CSS}^n(\mathcal{X})$ preserves products.*

Proof. Since the inclusion $\mathrm{CSS}_{\mathcal{X}}(\mathcal{Y}) \hookrightarrow \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ is a right adjoint, it preserves limits. Thus we must show that if $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{X}$-Segal objects of $\mathcal{Y}$ over c^*X, then the natural map $L_{\mathcal{X}\subseteq\mathcal{Y}}(\mathcal{C} \times_{c^*X} \mathcal{D}) \to L_{\mathcal{X}\subseteq\mathcal{Y}}(\mathcal{C}) \times_{c^*X} L_{\mathcal{X}\subseteq\mathcal{Y}}(\mathcal{D})$ in $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ is an equivalence. By Theorem 7.6, this is equivalent to proving that the map $\mathcal{C} \times_{c^*X} \mathcal{D} \to L_{\mathcal{X}\subseteq\mathcal{Y}}(\mathcal{C}) \times_{c^*X} L_{\mathcal{X}\subseteq\mathcal{Y}}(\mathcal{D})$ is fully faithful and essentially surjective. Condition (1) in the definition holds since pullbacks over X preserve colimits in the ∞-topos $\mathcal{X}$, and the colimit in question is sifted, and condition (2) holds since limits commute. This proves (i); then (ii) follows inductively as the functor $L_{n,\mathcal{X}}$ is a composite of functors constructed from the functors in (i). □

As a consequence, we have:

Lemma 7.11. *The Cartesian product in $\mathrm{CSS}^n(\mathcal{X})$ preserves colimits separately in each variable.*

Proof. Colimits in $\mathrm{CSS}^n(\mathcal{X})$ are computed by applying the localization L to the colimit of the same diagram in $\mathrm{Seg}_n(\mathcal{X})$. Thus the result follows by combining Lemma 7.10 with the observation that the product preserves colimits in each variable in $\mathrm{Seg}_n(\mathcal{X})$. □

This lets us define internal Homs in complete Segal objects:

Definition 7.12. We denote the internal Hom in $\mathrm{CSS}^n(\mathcal{X})$ of morphisms from $\mathcal{C}$ to $\mathcal{D}$ by $\mathcal{D}^{\mathcal{C}}$. If $X \in \mathcal{X}$ we abbreviate $\mathcal{D}^{c^*X}$ by $\mathcal{D}^X$. We also write $\mathrm{MAP}(\mathcal{C}, \mathcal{D})$ for the object of $\mathcal{X}$ that represents the functor $\mathrm{Map}_{\mathrm{CSS}^n(\mathcal{X})}(\mathcal{C} \times c^*(-), \mathcal{D}) \colon \mathcal{X} \to \mathcal{S}$. Equivalently, this is just $(\mathcal{D}^{\mathcal{C}})_{0,\ldots,0}$.

We now wish to prove a useful criterion for completeness of n-fold Segal objects in an ∞-topos:

Proposition 7.13. *Suppose $\mathcal{X}$ is an ∞-topos and $\mathcal{C}_\bullet$ is a Segal object in $\mathcal{X}$. Then $\mathcal{C}_\bullet$ is complete if and only if the Segal spaces $\mathrm{Map}_{\mathcal{X}}(X, \mathcal{C}_\bullet)$ are complete for all $X \in \mathcal{X}$.*

For the proof it is convenient to first consider functoriality of complete Segal objects in maps of distributors. The usual notion of a map between ∞-topoi is that of a *geometric morphism*: an adjunction where the left adjoint preserves finite limits. The ∞-categories of complete Segal objects are functorial with respect to a slightly more general class of maps:

Definition 7.14. Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi. A *pseudo-geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $f_* \colon \mathcal{X} \to \mathcal{Y}$ such that f_* admits a left adjoint f^* which preserves pullbacks.

Definition 7.15. Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X}' \subseteq \mathcal{Y}'$ be distributors. A *pseudo-geometric morphism* from $\mathcal{Y}$ to $\mathcal{Y}'$ is a functor $G \colon \mathcal{Y} \to \mathcal{Y}'$ such that:

(1) G takes $\mathcal{X}$ to $\mathcal{X}'$.

(2) G has a left adjoint $F\colon \mathcal{Y}' \to \mathcal{Y}$.
(3) F takes $\mathcal{X}'$ to $\mathcal{X}$.
(4) If $\phi\colon \Delta^1 \times \Delta^1 \to \mathcal{Y}'$ is a pullback diagram such that $\phi(1,1) \in \mathcal{X}'$, then $F(\phi)$ is a pullback diagram in $\mathcal{Y}$.

Remark 7.16. It is clear that a pseudo-geometric morphism of ∞-topoi as above is also a pseudo-geometric morphism of distributors.

Proposition 7.17.

(i) Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X}' \subseteq \mathcal{Y}'$ be distributors. Suppose $G\colon \mathcal{Y} \to \mathcal{Y}'$ is a pseudo-geometric morphism of distributors with left adjoint F. Then composition with F and G induces an adjunction

$$LF_* : \mathrm{CSS}_{\mathcal{X}'}(\mathcal{Y}') \rightleftarrows \mathrm{CSS}_{\mathcal{X}}(\mathcal{Y}) : G_*,$$

and this is also a pseudo-geometric morphism.

(ii) Suppose $f^*\colon \mathcal{X}' \rightleftarrows \mathcal{X} : f_*$ is a pseudo-geometric morphism of ∞-topoi. Then the functors given by composition with f^* and f_* induce an adjunction

$$L_{n,\mathcal{X}}(f^*)_* : \mathrm{CSS}^n(\mathcal{X}') \rightleftarrows \mathrm{CSS}^n(\mathcal{X}) : (f_*)_*.$$

Proof. By Lemma 4.16 we have an adjunction

$$F_* : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{Y}') \rightleftarrows \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{Y}) : G_*.$$

It is clear from the definition of a pseudo-geometric morphism that F_* and G_* preserve $\mathcal{X}'$- and $\mathcal{X}$-Segal objects, respectively, so there is an induced adjunction

$$F_* : \mathrm{Seg}_{\mathcal{X}'}(\mathcal{Y}') \rightleftarrows \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}) : G_*.$$

We have a commutative diagram of left adjoints

$$\begin{tikzcd}
\mathrm{Gpd}(\mathcal{X}') \arrow[r, "F_*"] \arrow[d] & \mathrm{Gpd}(\mathcal{X}) \arrow[d] \\
\mathrm{Seg}_{\mathcal{X}'}(\mathcal{Y}') \arrow[r, "F_*"'] & \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}),
\end{tikzcd}$$

where the vertical morphisms denote the obvious inclusions, hence the corresponding diagram of right adjoints also commutes, giving an equivalence $G_*(\mathrm{Gpd}(\mathcal{C})) \simeq \mathrm{Gpd}(G_*\mathcal{C})$. It follows that G_* preserves complete Segal objects, hence there is an induced adjunction

$$L_{\mathcal{X}\subseteq\mathcal{Y}}F_* : \mathrm{CSS}_{\mathcal{X}'}(\mathcal{Y}') \rightleftarrows \mathrm{CSS}_{\mathcal{X}}(\mathcal{Y}) : G_*.$$

To complete the proof of (i), we must show that this is a pseudo-geometric morphism. The functors $L_{\mathcal{X}\subseteq\mathcal{Y}}F_*$ and G_* preserve constant simplicial objects valued in $\mathcal{X}$ and $\mathcal{X}'$, so it remains to show that, given a pullback diagram

$$\begin{tikzcd}
\mathcal{C} \times_X \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{D} \arrow[r] & c^*X
\end{tikzcd}$$

in $\mathrm{CSS}_{\mathcal{X}'}(\mathcal{Y}')$, where c^*X is the constant simplicial object with value $X \in \mathcal{X}'$, its image under $L_{\mathcal{X}\subseteq\mathcal{Y}}F_*$ is also a pullback. Since limits in $\mathrm{CSS}_{\mathcal{X}'}(\mathcal{Y}')$ are computed in $\mathrm{Seg}_{\mathcal{X}'}(\mathcal{Y}')$, and these in turn are computed objectwise, it follows that F_* takes this to a pullback diagram in $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$. Now applying Lemma 7.10 we conclude that the image of this under $L_{\mathcal{X}\subseteq\mathcal{Y}}$ is also a pullback. This completes the proof of (i), and (ii) is just a special case of (i) obtained by induction. $\square$

Proof of Proposition 7.13. Let $r^*\colon \mathcal{S} \to \mathcal{X}$ denote the unique colimit-preserving functor such that $r^*(*)$ is a terminal object of $\mathcal{X}$, and let $r_* := \mathrm{Map}_{\mathcal{X}}(*,-)$ be its right adjoint. By [Lur09a, Proposition 6.3.4.1], the adjunction $r^* \dashv r_*$ is a geometric morphism. It is clear that for any $X \in \mathcal{X}$ the functor $\mathrm{Map}_{\mathcal{X}}(X,-)$ has a left adjoint given by $X \times r^*(-)$, and this preserves pullbacks since r^* preserves finite limits. Thus the adjunction $X \times r^*(-) \dashv \mathrm{Map}_{\mathcal{X}}(X,-)$ is a pseudo-geometric morphism of ∞-topoi, and so by the proof of Proposition 7.17 we have an equivalence $\mathrm{Map}_{\mathcal{X}}(X, \mathrm{Gpd}(\mathcal{C}_\bullet)) \simeq \mathrm{Gpd}(\mathrm{Map}_{\mathcal{X}}(X,\mathcal{C}_\bullet))$. By the Yoneda Lemma the simplicial object $\mathrm{Gpd}(\mathcal{C}_\bullet)$ is constant if and only if $\mathrm{Map}_{\mathcal{X}}(X, \mathrm{Gpd}(\mathcal{C}_\bullet))$ is constant for all $X \in \mathcal{X}$, so $\mathcal{C}_\bullet$ is complete if and only if $\mathrm{Gpd}(\mathrm{Map}_{\mathcal{X}}(X,\mathcal{C}_\bullet))$ is constant for all $X \in \mathcal{X}$, i.e. if and only if $\mathrm{Map}_{\mathcal{X}}(X,\mathcal{C}_\bullet)$ is complete for all $X \in \mathcal{X}$. $\square$

The remainder of this section is devoted to proving the following inductive characterization of completeness for n-fold Segal spaces, which we will make use of in the next section:

Theorem 7.18. *Suppose $\mathcal{C}$ is an n-fold Segal space. Then the following are equivalent:*

(i) $\mathcal{C}$ is complete.
(ii) The Segal space $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete, and the $(n-1)$-fold Segal spaces $\mathcal{C}(x,y)$ are complete for all objects x, y in $\mathcal{C}$.

Remark 7.19. As a consequence of this, we get the expected inductive characterization of the fully faithful and essentially surjective morphisms in $\mathrm{Seg}_n(\mathcal{S})$, i.e. the morphisms that are inverted by the localization to $\mathrm{CSS}_n(\mathcal{S})$: They are precisely the morphisms $f\colon \mathcal{C} \to \mathcal{C}'$ that are *essentially surjective*, in the sense that the underlying morphism of 1-fold Segal spaces is essentially surjective, and *locally* fully faithful and essentially surjective, in the sense that for each pair of objects $x, y \in \mathcal{C}$ the induced morphism $\mathcal{C}(x,y) \to \mathcal{C}'(fx, fy)$ is a fully faithful and essentially surjective functor of $(n-1)$-fold Segal spaces.

For convenience, we make the following inductive definition:

Definition 7.20. Let $\mathcal{C}$ be an n-fold Segal space. We say that $\mathcal{C}$ is *pseudo-complete* if

(1) the Segal space $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete,
(2) the $(n-1)$-fold Segal spaces $\mathcal{C}(X,Y)$ are pseudo-complete for all objects X, Y in $\mathcal{C}$.

Our goal is then to show that an n-fold Segal space is complete if and only if it is pseudo-complete. Before we give the proof we need to make a number of observations:

Lemma 7.21. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be a distributor, and suppose $\mathcal{C} \in \mathrm{Seg}_{\mathcal{X}}(\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}))$. Then $\mathcal{C}$ is in $\mathrm{CSS}^n_{\mathcal{X}}(\mathcal{Y})$ if and only if the Segal object $\mathcal{C}_{\bullet,0,\ldots,0}$ in $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$ is complete.*

Proof. The inclusion functor $\mathrm{Seg}\mathcal{X} \hookrightarrow \mathrm{Seg}_{\mathcal{X}}(\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}))$ factors through the inclusion

$$\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}) \hookrightarrow \mathrm{Seg}_{\mathcal{X}}(\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}))$$

induced by the functor $\mathcal{Y} \to \mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y})$ that sends an object of $\mathcal{Y}$ to the constant $(n-1)$-simplicial object with that value. Thus the right adjoint $\mathrm{Seg}_{\mathcal{X}}\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{Seg}(\mathcal{X})$ is the composite of the right adjoint $\mathrm{Seg}_{\mathcal{X}}\mathrm{CSS}^{n-1}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{Seg}_{\mathcal{X}}(\mathcal{Y})$, which is induced by evaluation at the initial object $([0], \ldots, [0]) \in \Delta^{n-1,\mathrm{op}}$, and the localization $\mathrm{Seg}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{Seg}(\mathcal{X})$. In particular, the groupoid object $\mathrm{Gpd}(\mathcal{C})$ is equivalent to $\mathrm{Gpd}(\mathcal{C}_{\bullet,0,\ldots,0})$ and so $\mathcal{C}$ is complete if and only if $\mathcal{C}_{\bullet,0,\ldots,0}$ is. $\square$

Lemma 7.22. *Let $\mathcal{C}$ be an n-fold Segal space. Then the following are equivalent:*

(i) $\mathcal{C}$ is complete.
(ii) The Segal space $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete, and the $(n-1)$-fold Segal space $\mathcal{C}_{1,\bullet,\ldots,\bullet}$ is complete.

Proof. By Lemma 7.21 we know that $\mathcal{C}$ is complete if and only if $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete and the $(n-1)$-fold Segal spaces $\mathcal{C}_{n,\bullet,\ldots,\bullet}$ are complete for each n. But $\mathcal{C}_{0,\bullet,\ldots,\bullet}$ is constant and so obviously complete, and thus for $n > 1$ the Segal condition implies that $\mathcal{C}_{n,\bullet,\ldots,\bullet}$ is complete if $\mathcal{C}_{1,\bullet,\ldots,\bullet}$ is complete, since complete Segal objects are closed under limits in the ∞-category of presheaves. $\square$

Remark 7.23. Applying Lemma 7.22 inductively, we see that an n-fold Segal space $\mathcal{C}$ is complete if and only if the n Segal spaces $\mathcal{C}_{\bullet,0,\ldots,0}, \mathcal{C}_{1,\bullet,0,\ldots,0}, \ldots, \mathcal{C}_{1,\ldots,1,\bullet}$ are all complete; this is the definition of completeness used in [BSP11]. An alternative proof that this agrees with the definition of completeness we gave above is found in [JFS17, Lemma 2.8].

Lemma 7.24. *Suppose given an n-fold Segal space $\mathcal{C}$ together with a map $\pi\colon \mathcal{C} \to X$ where X is a constant Segal space, and for $x \in X$ let $\mathcal{C}_x$ denote the n-fold Segal space that is the fibre of π at x. Then for any two objects $c, d \in \mathcal{C}$ there is a map $\mathcal{C}(c,d) \to \Omega_{\pi(c),\pi(d)}X$ whose fibres are of the form $\mathcal{C}_{\pi(c)}(c,d')$, where d' is the image of d in $\mathcal{C}_{\pi(c)}$ under the equivalence $\mathcal{C}_{\pi(c)} \simeq \mathcal{C}_{\pi(d)}$ determined by the path from $\pi(c)$ to $\pi(d)$.*

Proof. The map π gives a commutative square

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Ob}(\mathcal{C})^{\times 2} \arrow[r] & X^{\times 2}
\end{tikzcd}$$

so taking fibres over a point $(c,d) \in \mathrm{Ob}(\mathcal{C})^{\times 2}$ we get a map $\mathcal{C}(c,d) \to \Omega_{(\pi c,\pi d)}X$. If πc and πd are not in the same component then both spaces are empty and we are done. Otherwise we want to identify the fibre of this map at a point $p \in \Omega_{(\pi c,\pi d)}X$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(c,d)_p \arrow[r] \arrow[d] & \mathcal{C}(c,d) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \arrow[dr] & \\
* \arrow[r, "p"] & \Omega_{(\pi c,\pi d)}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
 & * \arrow[r] & \mathrm{Ob}(\mathcal{C})^{\times 2} \arrow[r] & X^{\times 2},
\end{tikzcd}$$

where all four squares are pullbacks (and Y is defined as the pullback of $X \to X^{\times 2}$ and $\mathrm{Ob}(\mathcal{C})^{\times 2} \to X^{\times 2}$). Thus we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(c,d)_p \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\
* \arrow[r] \arrow[dr] & Y \arrow[d] \\
& X,
\end{tikzcd}$$

where the top square is a pullback. Taking fibres at the given point of X (which we may identify with πc) we can factor this diagram as

$$\begin{tikzcd}
\mathcal{C}(c,d)_p \arrow[r] \arrow[d] & (\mathcal{C}_{\pi c})_1 \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\
* \arrow[r] & (\mathcal{C}_{\pi c})_0^{\times 2} \arrow[r] \arrow[d] & Y \arrow[d] \\
& * \arrow[r] & X,
\end{tikzcd}$$

where the two right-hand squares are pullbacks. Then as the top composite square is a pullback, so is the top left square. Thus we have identified $\mathcal{C}(c,d)_p$ with a mapping space in $\mathcal{C}_{\pi c}$, as required. □

Lemma 7.25. *Let $\pi\colon \mathcal{C} \to X$ be a map of n-fold Segal spaces, where X is constant. Suppose the fibres $\mathcal{C}_x$ are pseudo-complete for each $x \in X$. Then $\mathcal{C}$ is also pseudo-complete.*

Proof. We prove this by induction on n. The map π induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(E^1, \mathcal{C}_{\bullet,0,\ldots,0}) \arrow[rr] \arrow[dr] & & \mathcal{C}_{0,\ldots,0} \arrow[dl] \\
& X &
\end{tikzcd}$$

where the map on fibres at $x \in X$ is $\mathrm{Map}(E^1, (\mathcal{C}_x)_{\bullet,0,\ldots,0}) \to (\mathcal{C}_x)_{0,\ldots,0}$ as $\mathrm{Map}(E^1,-)$ commutes with limits. This map is an equivalence for all x, since $\mathcal{C}_x$ is pseudo-complete, hence the horizontal map in the triangle is also an equivalence and thus $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete.

Now given objects $c, d \in \mathcal{C}$, by Lemma 7.24 there is a map $\mathcal{C}(c,d) \to \Omega_{\pi(c),\pi(d)}X$ whose fibres are mapping $(n-1)$-fold Segal spaces in $\mathcal{C}_{\pi(c)}$. By assumption these are pseudo-complete, hence by the inductive hypothesis $\mathcal{C}(c,d)$ is pseudo-complete for all c, d. This completes the proof. □

Proof of Theorem 7.18. We will show, by induction on n, that an n-fold Segal space is complete if and only if it is pseudo-complete. For $n = 1$ the two notions coincide, so there is nothing to prove. Suppose we have shown that they agree for $n < k$, and let $\mathcal{C}$ be a k-fold Segal space. By Lemma 7.22 $\mathcal{C}$ is complete if and only if $\mathcal{C}_{\bullet,0,\ldots,0}$ is complete and the $(k-1)$-fold Segal space $\mathcal{C}_1$ is complete. Since by assumption the notions of complete and pseudo-complete $(k-1)$-fold Segal spaces coincide, it remains to show that $\mathcal{C}_1$ is complete if and only if the fibres $\mathcal{C}(c,d)$ at $(c,d) \in \mathrm{Ob}(\mathcal{C})^{\times 2}$ are all complete. One direction follows from applying Lemma 7.25 to the map $\mathcal{C}_1 \to \mathrm{Ob}(\mathcal{C})^{\times 2}$, and the other follows since complete $(k-1)$-fold Segal spaces are closed under limits in $(k-1)$-fold Segal spaces. □

8. Completeness for Iterated Spans

In this section we will prove that the n-fold Segal spaces $\mathrm{Span}_n(\mathcal{C})$ that we constructed above are always complete. We first consider the case $n = 1$, which is due to Barwick:

Proposition 8.1 ([Bar17, Proposition 3.4]). *Let $\mathcal{C}$ be an ∞-category with finite limits. Then the Segal space $\mathrm{Span}_1(\mathcal{C})$ is complete.*

For completeness we include a slightly different proof than that of Barwick, based on the following observation:

Lemma 8.2. *A span $X \xleftarrow{f} A \xrightarrow{g} Y$ in $\mathcal{C}$ is an equivalence in $\mathrm{Span}_1(\mathcal{C})$ if and only if the maps f and g are equivalences.*

Proof. It is clear that a span is an equivalence if both the maps in it are equivalences, so it remains to prove the converse. Suppose $Y \xleftarrow{h} B \xrightarrow{k} X$ is an inverse, then since composing gives the identity we have a diagram

$$\begin{tikzcd}[column sep=small] & & X \arrow[dl, "f'"'] \arrow[dr, "k'"] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & Y & & X, \end{tikzcd}$$

where the composites $ff' \colon X \to A \to X$ and $kk' \colon X \to B \to X$ are equivalent to id_X. Composing in the other order we similarly have maps $h' \colon Y \to B$ and $g' \colon Y \to A$ such that $hh' \colon Y \to B \to Y$ and $gg' \colon Y \to A \to Y$ are equivalent to id_Y. Now taking a double composite in $\mathrm{Span}_1(\mathcal{C})$ we get a commutative diagram

$$\begin{tikzcd}[column sep=small] & & & A \arrow[dl, "f''"'] \arrow[dr, "g''"] & & & \\ & & X \arrow[dl, "f'"'] \arrow[dr, "k'"] & & Y \arrow[dl, "h'"'] \arrow[dr, "g'"] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y & & X & & Y. \end{tikzcd}$$

Since the composite is the original span $X \xleftarrow{f} A \xrightarrow{g} Y$, the composite along the left edge is the original map $f \colon A \to X$. But since the composite $ff' \simeq \mathrm{id}_X$, this means that $f'' \simeq f$, and similarly $g'' \simeq g$. But then since compositions are formed by taking pullbacks, we have a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[r, "f'"] \arrow[d, "k'"] & A \arrow[d, "g"] \\ Y \arrow[r, "h'"] & B \arrow[r, "h"] & Y, \end{tikzcd}$$

where both squares are Cartesian. The composite square is also Cartesian, and so as $hh' \simeq \mathrm{id}_Y$, we have $f'f \simeq \mathrm{id}_A$. Thus f' is a two-sided inverse of f, hence f is an equivalence. Similarly, we get $g'g \simeq \mathrm{id}_A$ and so g is also an equivalence. □

Proof of Proposition 8.1. By Theorem 3.6 the space of equivalences in $\mathrm{Span}_1(\mathcal{C})$ consists of the components of the space $\mathrm{Span}_1(\mathcal{C})_1 \simeq \mathrm{Map}(\Sigma^1, \mathcal{C})$ that correspond to equivalences. But by Lemma 8.2 these are precisely the diagrams that land in the underlying ∞-groupoid $\iota\mathcal{C}$ of $\mathcal{C}$, so the space in question is $\mathrm{Map}(\Sigma^1, \iota\mathcal{C})$. This is equivalent to the space of maps to $\iota\mathcal{C}$ from the ∞-groupoid $\|\Sigma^1\|$ obtained by inverting the morphisms in Σ^1. This is contractible, so the space of equivalences is equivalent to $\iota\mathcal{C} \simeq \mathrm{Span}_1(\mathcal{C})_0$ as required. □

To extend this to iterated spans, we first identify the mapping $(\infty, n-1)$-categories in $\mathrm{Span}_n(\mathcal{C})$:

Proposition 8.3. *Let $\mathcal{C}$ be an ∞-category with finite limits. If X and Y are objects of $\mathcal{C}$, then the $(k-1)$-fold Segal space $\mathrm{Span}_k(\mathcal{C})(X, Y)$ of maps from X to Y in $\mathrm{Span}_k(\mathcal{C})$ is equivalent to $\mathrm{Span}_{k-1}(\mathcal{C}_{/X\times Y})$.*

For the proof we need a simple observation:

Lemma 8.4. *Suppose X and Y are objects of an ∞-category $\mathcal{C}$ that have a product $X \times Y$. Then for any ∞-category $\mathcal{K}$ there is a natural pullback square*

$$\begin{tikzcd} \mathrm{Map}(\mathcal{K}, \mathcal{C}_{/X\times Y}) \arrow[r] \arrow[d] & \mathrm{Map}(\Sigma^1 \times \mathcal{K}, \mathcal{C}) \arrow[d] \\ \{(c_X, c_Y)\} \arrow[r] & \mathrm{Map}(\Sigma^0 \times \mathcal{K}, \mathcal{C})^{\times 2}, \end{tikzcd}$$

where c_X and c_Y denote the functors constant at X and Y.

Proof. Since $X \times Y$ is a product, the ∞-category $\mathcal{C}_{/X\times Y}$ is equivalent to $\mathcal{C}_{/p}$ where p is the diagram $\{0, 1\} \to \mathcal{C}$ sending 0 to X and 1 to Y. The ∞-category $\mathcal{C}_{/p}$ has the universal property that for all ∞-categories $\mathcal{K}$ there are natural pullback squares

$$\begin{tikzcd} \mathrm{Map}(\mathcal{K}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{K} \star \{0,1\}, \mathcal{C}) \arrow[d] \\ \{(X, Y)\} \arrow[r] & \mathrm{Map}(\{0,1\}, \mathcal{C}). \end{tikzcd}$$

There is an evident equivalence between Σ^1 and $\{0,1\}^{\triangleleft}$, i.e. $* \amalg_{\{0,1\}\times\{0\}} \{0,1\} \times [1]$, and since products in Cat_∞ preserve colimits this gives an equivalence

$$\mathcal{K} \times \Sigma^1 \simeq \mathcal{K} \amalg_{\mathcal{K}\times\{0,1\}\times\{0\}} \mathcal{K} \times \{0,1\} \times [1].$$

Moreover, the ∞-category $\mathcal{K} \star \{0,1\}$ is equivalent to the pushout (in Cat_∞)

$$\mathcal{K} \amalg_{\mathcal{K}\times\{0,1\}\times\{0\}} \mathcal{K} \times \{0,1\} \times [1] \amalg_{\mathcal{K}\times\{0,1\}\times\{1\}} \{0,1\},$$

thus we get a pullback square

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{K} \star \{0,1\}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{K} \times \Sigma^1, \mathcal{C}) \arrow[d] \\
\mathrm{Map}(\{0,1\}, \mathcal{C}) \arrow[r] & \mathrm{Map}(\mathcal{K} \times \{0,1\}, \mathcal{C}).
\end{tikzcd}$$

Putting these two pullback squares together then completes the proof. □

Proof of Proposition 8.3. By Lemma 8.4 we have natural pullback squares

$$\begin{tikzcd}
\overline{\mathrm{SPAN}}_{k-1}(\mathcal{C}_{/X\times Y})_{[n_1],\ldots,[n_{k-1}]} \arrow[r] \arrow[d] & \overline{\mathrm{SPAN}}_k(\mathcal{C})_{[1],[n_1],\ldots,[n_{k-1}]} \arrow[d] \\
\{(c_X, c_Y)\} \arrow[r] & \overline{\mathrm{SPAN}}_k(\mathcal{C})^{\times 2}_{[0],[n_1],\ldots,[n_{k-1}]}.
\end{tikzcd}$$

Using Lemma 5.6 we see that this restricts to a pullback square

$$\begin{tikzcd}
\mathrm{SPAN}_{k-1}(\mathcal{C}_{/X\times Y})_{[n_1],\ldots,[n_{k-1}]} \arrow[r] \arrow[d] & \mathrm{SPAN}_k(\mathcal{C})_{[1],[n_1],\ldots,[n_{k-1}]} \arrow[d] \\
\{(c_X, c_Y)\} \arrow[r] & \mathrm{SPAN}_k(\mathcal{C})^{\times 2}_{[0],[n_1],\ldots,[n_{k-1}]},
\end{tikzcd}$$

since a functor to $\mathcal{C}_{/X\times Y}$ is Cartesian if and only if its composite with $\mathcal{C}_{/X\times Y} \to \mathcal{C}$ is Cartesian, as this forgetful functor detects pullbacks. Thus we have a pullback square of $(k-1)$-uple Segal spaces

$$\begin{tikzcd}
\mathrm{SPAN}_{k-1}(\mathcal{C}_{/X\times Y}) \arrow[r] \arrow[d] & \mathrm{SPAN}_k(\mathcal{C})_1 \arrow[d] \\
\{(X, Y)\} \arrow[r] & \mathrm{SPAN}_k(\mathcal{C})_0^{\times 2}.
\end{tikzcd}$$

The functor U^{k-1}_{Seg} is a right adjoint, so applying it we get a pullback square

$$\begin{tikzcd}
\mathrm{Span}_{k-1}(\mathcal{C}_{/X\times Y}) \arrow[r] \arrow[d] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{C})_1) \arrow[d] \\
\{(X, Y)\} \arrow[r] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{C})_0)^{\times 2}.
\end{tikzcd}$$

On the other hand, our constuction of U^k_{Seg} gives us a pullback square

$$\begin{tikzcd}
\mathrm{Span}_k(\mathcal{C})_1 \arrow[r] \arrow[d] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{C})_1) \arrow[d] \\
\mathrm{Span}_k(\mathcal{C})_0^{\times 2} \arrow[r] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{C})_0)^{\times 2}.
\end{tikzcd}$$

The map $\{(X, Y)\} \to U^{k-1}_{\mathrm{Seg}}\mathrm{SPAN}_k(\mathcal{C})_0^{\times 2}$ factors through the constant $(k-1)$-simplicial object $\mathrm{Span}_k(\mathcal{C})_0^{\times 2}$, so we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Span}_{k-1}(\mathcal{C}_{/X\times Y}) \arrow[r] \arrow[d] & \mathrm{Span}_k(\mathcal{C})_1 \arrow[r] \arrow[d] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{C})_1) \arrow[d] \\
\{(X, Y)\} \arrow[r] & \mathrm{Span}_k(\mathcal{C})_0^{\times 2} \arrow[r] & U^{k-1}_{\mathrm{Seg}}\mathrm{SPAN}_k(\mathcal{C})_0^{\times 2}
\end{tikzcd}$$

where the right-hand square and the composite square are both Cartesian. Then the left-hand square is also Cartesian, which completes the proof. □

Corollary 8.5. *Let $\mathcal{C}$ be an ∞-category with finite limits. Then the n-fold Segal space* $\mathrm{Span}_n(\mathcal{C})$ *is complete.*

Proof. We prove this by induction on n. The case $n = 1$ is Proposition 8.1. Suppose the result holds for all $n < k$. By Theorem 7.18 to show that $\mathrm{Span}_k(\mathcal{C})$ is complete it suffices to prove that the Segal space $\mathrm{Span}_k(\mathcal{C})_{\bullet,0,\ldots,0}$ is complete, and the $(n-1)$-fold Segal spaces $\mathrm{Span}_k(\mathcal{C})(X, Y)$ are complete for all X, Y in $\mathcal{C}$. But $\mathrm{Span}_k(\mathcal{C})_{\bullet,0,\ldots,0}$ is equivalent to $\mathrm{Span}_1(\mathcal{C})$, which we know is complete, and by Proposition 8.3 we can identify $\mathrm{Span}_k(\mathcal{C})(X, Y)$ with $\mathrm{Span}_{k-1}(\mathcal{C}_{/X\times Y})$, which is complete by the inductive hypothesis. □

9. Completeness for Iterated Spans with Local Systems

In this section we will show that that the k-fold Segal space $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ is complete, provided $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$. We first consider the case $k = 1$, which follows from the following observation:

Lemma 9.1. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a Segal object in $\mathcal{X}$. Then there is an equivalence*

$$\mathrm{Span}_1(\mathcal{X}; \mathcal{C})_{\mathrm{eq}} \simeq \coprod_{X \in \pi_0 \mathcal{X}} \mathrm{Map}_{\mathcal{X}}(X, \mathcal{C})_{\mathrm{eq}}.$$

Proof. By definition, $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})_{\mathrm{eq}}$ is the subspace of $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})_1 \simeq \iota\mathrm{Fun}(\Sigma^1, \mathcal{C})_{/s_{\mathcal{C}}([1])}$ consisting of those components that correspond to equivalences. The forgetful functor $\mathrm{Span}_1(\mathcal{X}; \mathcal{C}) \to \mathrm{Span}_1(\mathcal{X})$ induces a map $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})_{\mathrm{eq}} \to \mathrm{Span}_1(\mathcal{X})_{\mathrm{eq}}$, so by Proposition 8.1 the underlying span of an equivalence is trivial. We may thus identify $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})_{\mathrm{eq}}$ with a collection of components in $\coprod_{X \in \pi_0(\mathcal{X})} \mathrm{Map}_{\mathcal{X}}(X, \mathcal{C}_1)$. It is moreover immediate from the definition of composition in $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})$ that a map $X \to \mathcal{C}_1$ has an inverse when viewed as a morphism in $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})$ if and only if it has one when viewed as a morphism in $\mathrm{Map}_{\mathcal{X}}(X, \mathcal{C}_\bullet)$, which completes the proof. □

Proposition 9.2. *Suppose $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is a complete Segal object in $\mathcal{X}$. Then* $\mathrm{Span}_1(\mathcal{X}; \mathcal{C})$ *is a complete Segal space.*

Proof. By Theorem 3.6 it suffices to show that the degeneracy map

$$\mathrm{Span}(\mathcal{X};\mathcal{C})_0 \to \mathrm{Span}(\mathcal{X};\mathcal{C})_{\mathrm{eq}}$$

is an equivalence. By Lemma 9.1 we may identify this with the map

$$\coprod_X \mathrm{Map}(X,\mathcal{C}_0) \to \coprod_X \mathrm{Map}(X,\mathcal{C})_{\mathrm{eq}}.$$

But by Proposition 7.13 the Segal spaces $\mathrm{Map}(X,\mathcal{C})$ are complete since $\mathcal{C}$ is complete, and by Theorem 3.6 it follows that for each X the map $\mathrm{Map}(X,\mathcal{C}_0) \to \mathrm{Map}(X,\mathcal{C})_{\mathrm{eq}}$ is an equivalence. □

To extend this to iterated Segal spaces, we first identify the mapping $(\infty,k-1)$-categories of $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$:

Proposition 9.3. *Suppose $\mathcal{C}$ is a k-fold Segal object in $\mathcal{X}$, and that $\xi\colon X \to \mathcal{C}_{0,\ldots,0}$ and $\eta\colon Y \to \mathcal{C}_{0,\ldots,0}$ are objects of $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$. Then the $(k-1)$-fold Segal space $\mathrm{Span}_k(\mathcal{X};\mathcal{C})(\xi,\eta)$ of maps from ξ to η in $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$ is equivalent to $\mathrm{Span}_{k-1}(\mathcal{X};\mathcal{C}_{\xi,\eta})$, where $\mathcal{C}_{\xi,\eta}$ is the $(k-1)$-fold Segal object given by the pullback square*

$$\begin{tikzcd} \mathcal{C}_{\xi,\eta} \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\ X \times Y \arrow[r, "\xi \times \eta"'] & \mathcal{C}_0^{\times 2}. \end{tikzcd}$$

To prove this, we first make the following observations:

Lemma 9.4. *For any ∞-categories $\mathcal{A}$ and $\mathcal{B}$, the natural map*

$$(\mathcal{A}^{\triangleleft} \times \mathcal{B} \amalg_{\mathcal{A}\times\mathcal{B}} \mathcal{A} \times \mathcal{B}^{\triangleleft})^{\triangleleft} \to \mathcal{A}^{\triangleleft} \times \mathcal{B}^{\triangleleft}$$

is an equivalence.

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are quasicategories representing the ∞-categories $\mathcal{A}$ and $\mathcal{B}$. Then the pushout of simplicial sets

$$\mathfrak{A}^{\triangleleft} \times \mathfrak{B} \amalg_{\mathfrak{A}\times\mathfrak{B}} \mathfrak{A} \times \mathfrak{B}^{\triangleleft}$$

is a homotopy pushout in $\mathrm{Set}_{\Delta}^{J}$, since the maps are cofibrations. It therefore suffices to prove that the natural map

$$(\mathfrak{A}^{\triangleleft} \times \mathfrak{B} \amalg_{\mathfrak{A}\times\mathfrak{B}} \mathfrak{A} \times \mathfrak{B}^{\triangleleft})^{\triangleleft} \to \mathfrak{A}^{\triangleleft} \times \mathfrak{B}^{\triangleleft}$$

is a weak equivalence in the Joyal model structure. In fact, we will show that this map is an isomorphism for all simplicial sets $\mathfrak{A}$ and $\mathfrak{B}$. Since both sides preserve colimits in $\mathfrak{A}$ and $\mathfrak{B}$, it suffices to consider the case $\mathfrak{A} = \Delta^n$, $\mathfrak{B} = \Delta^m$. Thus, as $(\Delta^n)^{\triangleleft} \cong \Delta^{n+1}$, we must show that the map

$$(\Delta^{n+1} \times \Delta^m \amalg_{\Delta^n\times\Delta^m} \Delta^n \times \Delta^{m+1})^{\triangleleft} \to \Delta^{n+1} \times \Delta^{m+1}$$

is an isomorphism for all n, m. Now $\Delta^{n+1} \times \Delta^{m+1}$ is the nerve of the category $[n+1] \times [m+1]$, which is the join (of categories) $[0] \star \mathbf{C}$, where $\mathbf{C}$ is the subcategory spanned by all objects except $(0,0)$. The nerve functor takes the join of categories to the join of simplicial sets, so it follows that $\Delta^{n+1} \times \Delta^{m+1} \cong (\mathrm{N}\mathbf{C})^{\triangleleft}$. Moreover, the simplicial set $\mathrm{N}\mathbf{C}$ is the subcomplex of $\Delta^{n+1} \times \Delta^{m+1}$ containing only the simplices that do not have $(0,0)$ as a vertex, which we can identify with $\Delta^{n+1} \times \Delta^m \amalg_{\Delta^n\times\Delta^m} \Delta^n \times \Delta^{m+1}$. □

Lemma 9.5. *Let $\mathcal{A}$ be an ∞-category, and suppose given a functor $\mu\colon \mathcal{A} \times [1] \amalg_{\mathcal{A}\times\{1\}} \mathcal{A}^{\triangleleft} \to \mathcal{C}$ with limit $X \in \mathcal{C}$. Then there is a natural pullback diagram*

$$\begin{tikzcd} \mathcal{C}_{/X} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A}^{\triangleleft}, \mathcal{C})_{/\alpha} \arrow[d] \\ \{\nu\} \arrow[r] & \mathrm{Fun}(\mathcal{A}, \mathcal{C})_{/\beta}, \end{tikzcd}$$

where $\alpha := \mu|_{\mathcal{A}^{\triangleleft}}$, $\beta := \mu|_{\mathcal{A}\times\{1\}}$, and ν is the object corresponding to $\mu|_{\mathcal{A}\times[1]}$.

Proof. Since X is the limit of μ, the ∞-category $\mathcal{C}_{/X}$ is equivalent to $\mathcal{C}_{/\mu}$. And as

$$(\mathcal{A} \times [1] \amalg_{\mathcal{A}} \mathcal{A}^{\triangleleft})^{\triangleleft} \simeq \mathcal{A}^{\triangleleft} \times [1]$$

by Lemma 9.4, the ∞-category $\mathcal{C}_{/\mu}$ fits in a natural pullback square

$$\begin{tikzcd} \mathcal{C}_{/\mu} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A}^{\triangleleft} \times [1], \mathcal{C}) \arrow[d] \\ \{\mu\} \arrow[r] & \mathrm{Fun}(\mathcal{A} \times [1] \amalg_{\mathcal{A}\times\{1\}} \mathcal{A}^{\triangleleft}, \mathcal{C}). \end{tikzcd}$$

Now consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}_{/\mu} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A}^{\triangleleft}, \mathcal{C})_{/\alpha} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A}^{\triangleleft} \times [1], \mathcal{C}) \arrow[d] & \\ \{\nu\} \arrow[r] & \mathrm{Fun}(\mathcal{A}, \mathcal{C})_{/\beta} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A} \times [1] \amalg_{\mathcal{A}\times\{1\}} \mathcal{A}^{\triangleleft}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A} \times [1], \mathcal{C}) \arrow[d] \\ & \{\beta\} \arrow[r] & \mathrm{Fun}(\mathcal{A}^{\triangleleft}, \mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathcal{A}, \mathcal{C}). \end{tikzcd}$$

Here the bottom right square and bottom composite squares are pullbacks, hence so is the bottom left square. Then, as the composite square in the middle column is a pullback, the top right square is a pullback. Finally, as the top composite square is a pullback, this implies the top left square is a pullback, as required. □

Proof of Proposition 9.3. Using Lemma 9.5, we have natural pullback diagrams

$$\begin{tikzcd} \mathrm{Fun}(\Sigma^{n_1,\ldots,n_{k-1}}, \mathcal{X})_{/s_{\mathcal{C}_{\xi,\eta}}(n_1,\ldots,n_{k-1})} \arrow[r] \arrow[d] & \mathrm{Fun}(\Sigma^{1,n_1,\ldots,n_{k-1}}, \mathcal{X})_{/s_{\mathcal{C}}(1,n_1,\ldots,n_{k-1})} \arrow[d] \\ \{(c_\xi, c_\eta)\} \arrow[r] & \mathrm{Fun}(\Sigma^{0,n_1,\ldots,n_{k-1}}, \mathcal{X})^{\times 2}_{/s_{\mathcal{C}}(0,n_1,\ldots,n_{k-1})}. \end{tikzcd}$$

It now follows using Lemma 5.6 that these restrict to pullback diagrams

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{n_1,\ldots,n_{k-1}},\mathcal{X})_{/s_{\mathcal{C}_{\xi,\eta}}(n_1,\ldots,n_{k-1})} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{1,n_1,\ldots,n_{k-1}},\mathcal{X})_{/s_{\mathcal{C}}(1,n_1,\ldots,n_{k-1})} \arrow[d] \\
\{(c_\xi,c_\eta)\} \arrow[r] & \mathrm{Fun}^{\mathrm{Cart}}(\Sigma^{0,n_1,\ldots,n_{k-1}},\mathcal{X})^{\times 2}_{/s_{\mathcal{C}}(0,n_1,\ldots,n_{k-1})}.
\end{tikzcd}$$

The functor ι, which takes the underlying space of an ∞-category, is a right adjoint and hence preserves limits; applying it we therefore get, by naturality, a pullback square of $(k-1)$-uple Segal spaces

$$\begin{tikzcd}
\mathrm{SPAN}_{k-1}(\mathcal{X};\mathcal{C}_{\xi,\eta}) \arrow[r] \arrow[d] & \mathrm{SPAN}_k(\mathcal{X};\mathcal{C})_1 \arrow[d] \\
\{(\xi,\eta)\} \arrow[r] & \mathrm{SPAN}_k(\mathcal{X};\mathcal{C})_0^{\times 2}.
\end{tikzcd}$$

Now applying the right adjoint U^{k-1}_{Seg} we get from this a pullback square

$$\begin{tikzcd}
\mathrm{Span}_{k-1}(\mathcal{X};\mathcal{C}_{\xi,\eta}) \arrow[r] \arrow[d] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{X};\mathcal{C})_1) \arrow[d] \\
\{(\xi,\eta)\} \arrow[r] & U^{k-1}_{\mathrm{Seg}}(\mathrm{SPAN}_k(\mathcal{X};\mathcal{C})_0)^{\times 2}.
\end{tikzcd}$$

As in the proof of Proposition 8.3 we can now combine this with a pullback square we get from the construction of U^k_{Seg}, to get a pullback square

$$\begin{tikzcd}
\mathrm{Span}_{k-1}(\mathcal{X};\mathcal{C}_{\xi,\eta}) \arrow[r] \arrow[d] & \mathrm{Span}_k(\mathcal{X};\mathcal{C})_1 \arrow[d] \\
\{(c_\xi,c_\eta)\} \arrow[r] & \mathrm{Span}_k(\mathcal{X};\mathcal{C})_0^{\times 2},
\end{tikzcd}$$

as required. □

Remark 9.6. In particular, if $X \simeq Y \simeq *$, so that ξ and η are determined by two objects x and y of $\mathcal{C}$, then $\mathrm{Span}_k(\mathcal{X};\mathcal{C})(\xi,\eta) \simeq \mathrm{Span}_{k-1}(\mathcal{X};\mathcal{C}(x,y))$.

Corollary 9.7. *Suppose $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$. Then $\mathrm{Span}_k(\mathcal{X};\mathcal{C})$ is a complete k-fold Segal space.*

Proof. The case $k = 1$ is Proposition 9.2; we will prove the general case by induction on k. Suppose we know the result for k-fold Segal objects for all $k < n$. By Theorem 7.18 to show that $\mathrm{Span}_n(\mathcal{X};\mathcal{C})$ is complete it suffices to prove that the Segal space $\mathrm{Span}_n(\mathcal{X};\mathcal{C})_{\bullet,0,\ldots,0}$ is complete, and the $(n-1)$-fold Segal spaces $\mathrm{Span}_n(\mathcal{X};\mathcal{C})(\xi,\eta)$ are complete for all ξ, η. But $\mathrm{Span}_n(\mathcal{X};\mathcal{C})_{\bullet,0,\ldots,0}$ is equivalent to $\mathrm{Span}_1(\mathcal{X};\mathcal{C}_{\bullet,0,\ldots,0})$, which we know is complete, and by Proposition 9.3 we can identify $\mathrm{Span}_n(\mathcal{X};\mathcal{C})(\xi,\eta)$ with $\mathrm{Span}_{n-1}(\mathcal{X};\mathcal{C}_{\xi,\eta})$. The $(n-1)$-fold Segal object $\mathcal{C}_{\xi,\eta}$ is complete since complete $(n-1)$-fold Segal objects in $\mathcal{X}$ are closed under pullback, so $\mathrm{Span}_{n-1}(\mathcal{X};\mathcal{C}_{\xi,\eta})$ is complete by the inductive hypothesis. □

10. Symmetric Monoidal Structures on Iterated Segal Spaces

In this section we introduce (n-fold) monoids in ∞-categories, with a special case being (n-fold) monoidal structures on iterated Segal spaces. In the limit, we use these to give a definition of ∞-fold monoids and ∞-fold monoidal structures, which is the form in which symmetric monoidal structures will show up below. We then show that n-fold monoids are the same thing as $\mathbb{E}_n$-algebras (i.e. algebras for the ∞-operad corresponding to the little n-disc operad) and ∞-fold monoids are commutative algebras, as defined in [Lur17]; for this we assume the reader has some familiarity with the formalism of ∞-operads, but this discussion can easily be skipped as it is not needed in the rest of the paper.

Definition 10.1. Suppose $\mathcal{C}$ is an ∞-category with finite products. An (*associative*) *monoid* in $\mathcal{C}$ is a simplicial object $A_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that the Segal maps $A_n \to \prod_{i=1}^n A_1$ (induced by the inert maps $\rho_i \colon \{i-1, i\} \to [n]$) are equivalences for all $n = 0, 1, \ldots$. We write $\mathrm{Mon}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ spanned by the monoids. If $\mathcal{C}$ is the ∞-category $\mathrm{Seg}_k(\mathcal{X})$ or $\mathrm{CSS}_k(\mathcal{X})$ we refer to monoids as *monoidal* k-fold (complete) Segal objects.

Definition 10.2. We inductively define an *n-fold monoid* in $\mathcal{C}$ to be an $(n-1)$-fold monoid in $\mathrm{Mon}(\mathcal{C})$. Unwinding this definition, we see that an n-fold monoid is a functor $A \colon \Delta^{n,\mathrm{op}} \to \mathcal{C}$ such that the natural maps

$$A_{t_1,\ldots,t_n} \to \prod_{i_1=1}^{t_1} \cdots \prod_{i_n=1}^{t_n} A_{1,\ldots,1}$$

are equivalences for all $t_1, \ldots, t_n$. We write $\mathrm{Mon}_n(\mathcal{C}) := \mathrm{Mon}_{n-1}(\mathrm{Mon}(\mathcal{C}))$ for the ∞-category of n-fold monoids in $\mathcal{C}$. If $\mathcal{C}$ is the ∞-category $\mathrm{Seg}_k(\mathcal{X})$ or $\mathrm{CSS}_k(\mathcal{X})$ we refer to n-fold monoids as *n-monoidal* k-fold (complete) Segal spaces.

Remark 10.3. Since the localization functor $\mathrm{Seg}_k(\mathcal{X}) \to \mathrm{CSS}_k(\mathcal{X})$ preserves products by Lemma 7.10, the completion of an n-monoidal k-fold Segal space is an n-monoidal k-fold complete Segal space.

Definition 10.4. We define the ∞-category $\mathrm{Mon}_\infty(\mathcal{C})$ of *∞-fold monoids* in $\mathcal{C}$ as the limit of the sequence

$$\cdots \to \mathrm{Mon}_n(\mathcal{C}) \to \mathrm{Mon}_{n-1}(\mathcal{C}) \to \cdots \mathrm{Mon}(\mathcal{C}) \to \mathcal{C},$$

where the functors are given by evaluation at $[1]$ in the first factor of $\Delta^{n,\mathrm{op}}$. Thus an ∞-fold monoid in $\mathcal{C}$ is a sequence $(A^0, A^1, \ldots)$ where A^n is an n-fold monoid, such that $A^{n-1} \simeq A^n([1], –, \ldots, –)$ for each n. If $\mathcal{C}$ is the ∞-category $\mathrm{Seg}_k(\mathcal{X})$ or $\mathrm{CSS}_k(\mathcal{X})$ we refer to ∞-fold monoids as *∞-monoidal* k-fold (complete) Segal objects.

Remark 10.5. An associative monoid in $\mathcal{C}$ is the same thing as a category object X in $\mathcal{C}$ such that $X_0 \simeq *$. Because of this, we can extract a monoid $\Omega_p X$ from a pointed category object, i.e. a category object X equipped with a map $p \colon * \to X$ (or equivalently a map $p \colon * \to X_0$)[1]: Let i denote the inclusion $\{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$; then the functor $i^* \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$, given by evaluation at $[0]$, has a left adjoint $i_!$ and a right adjoint i_*, given by left and right Kan extension along i. From the formula for right Kan extensions we see $i_* C$ is given by $(i_* C)_n \simeq C^{\times(n+1)}$ with face maps being projections and degeneracies diagonal maps. The

[1]This has been changed from the published version, which has an incorrect construction of $\Omega_p X$

map $p\colon * \to X_0$ induces a map $i_*p\colon * \simeq i_** \to i_*X_0$, and so we can take the pullback

$$\begin{tikzcd} \Omega_p X \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & i_*X_0 \end{tikzcd}$$

in category objects. We think of the monoid $\Omega_p X$ as the monoid of endomorphisms of the object p of X.

Remark 10.6. Suppose $\mathcal{X}$ is an ∞-topos. We have a composite functor $\mathrm{Mon}(\mathcal{X}) \to \mathrm{Seg}(\mathcal{X}) \to \mathrm{CSS}(\mathcal{X})$, where the first functor is the natural inclusion and the second is the localization functor. Then the induced functor $\mathrm{Mon}(\mathcal{X}) \to \mathrm{CSS}(\mathcal{X})_*$ to pointed complete Segal objects is fully faithful, and is left adjoint to the functor $\Omega\colon \mathrm{CSS}(\mathcal{X})_* \to \mathrm{Mon}(\mathcal{X})$ we just defined. This can be proved by the same argument as for [GH15, Theorem 6.3.2]; we do not recall this as we will not make any further use of this observation.

Remark 10.7. Similarly, an n-fold monoid is the same thing as an n-fold Segal object X such that $X_{0,\ldots,0} \simeq X_{1,0,\ldots,0} \simeq X_{1,\ldots,1,0} \simeq *$. We can again use this to extract an n-fold monoid from a pointed n-fold Segal object (or n-fold category object) $(X, p\colon * \to X)$.[2] If $X' := X_{\bullet,\ldots,\bullet,0}$ denotes the underlying $(n-1)$-fold Segal object (or n-fold category object), then we get an n-fold monoid $\Omega_p^n X$ by taking the pullback

$$\begin{tikzcd} \Omega_p^n X \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & i_*X', \end{tikzcd}$$

where $*$ denotes the terminal n-fold simplicial object and i_*X' is obtained by taking the right Kan extension along $i\colon \{[0]\} \hookrightarrow \Delta^{\mathrm{op}}$ in the last simplicial coordinate. We think of $\Omega_p^n X$ as the n-fold monoid of endomorphisms of the identity $(n-1)$-morphism of p. As in the case $n = 1$, if $\mathcal{X}$ is an ∞-topos it can be shown that $\Omega^n\colon \mathrm{CSS}_n(\mathcal{X})_* \to \mathrm{Mon}_n(\mathcal{X})$ has a fully faithful left adjoint B^n given by the composite $\mathrm{Mon}_n(\mathcal{X}) \to \mathrm{Seg}_n(\mathcal{X})_* \to \mathrm{CSS}_n(\mathcal{X})_*$ where the second functor is the localization.

Example 10.8. Given an $(n+k)$-fold Segal space X and an object $p \in X_{0,\ldots,0}$, we can extract an n-monoidal k-fold Segal space $\Omega_p^n X$.

Definition 10.9. Given a pointed n-fold Segal object (X, p) in $\mathcal{C}$, we can extract a pointed $(n-1)$-fold Segal object by taking the pullback

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X_1 \arrow[d] \\ * \arrow[r, "p"'] & X_0^{\times 2} \end{tikzcd}$$

[2]This has been changed from the published version, which has an incorrect construction of $\Omega_p^n X$.

with X' pointed via the degeneracy by $s_0 \circ p$. This defines a functor $\phi_n \colon \mathrm{Seg}_n(\mathcal{C})_* \to \mathrm{Seg}_{n-1}(\mathcal{C})_*$, which restricts to the forgetful functor $\mathrm{Mon}_n(\mathcal{C}) \to \mathrm{Mon}_{n-1}(\mathcal{C})$ we used above. The $(n-1)$-fold Segal object $\phi_n(X,p)$ is the mapping object $X(p,p)$. Let us define $\mathrm{Seg}_\infty(\mathcal{C})_*$ to be the limit of the sequence of ∞-categories

$$\cdots \to \mathrm{Seg}_n(\mathcal{C})_* \to \mathrm{Seg}_{n-1}(\mathcal{C})_* \to \cdots \to \mathrm{Seg}(\mathcal{C})_* \to \mathcal{C}_*.$$

The objects of $\mathrm{Seg}_\infty(\mathcal{C})_*$, which we will call *infinite delooping sequences*, can then be described as sequences $((X^0,p^0),(X^1,p^1),\ldots)$ such that (X^n,p^n) is a pointed n-fold Segal object, together with equivalences $(X^{n-1},p^{n-1}) \simeq \phi_n(X^n,p^n)$ for all n. The functors Ω^n we defined above sit in commutative diagrams

$$\begin{tikzcd}
\mathrm{Seg}_n(\mathcal{C})_* \arrow[r, "\Omega^n"] \arrow[d, "\phi_n"'] & \mathrm{Mon}_n(\mathcal{C}) \arrow[d] \\
\mathrm{Seg}_{n-1}(\mathcal{C})_* \arrow[r, "\Omega^{n-1}"'] & \mathrm{Mon}_{n-1}(\mathcal{C}),
\end{tikzcd}$$

and so taking the limit we get a functor $\Omega^\infty \colon \mathrm{Seg}_\infty(\mathcal{C})_* \to \mathrm{Mon}_\infty(\mathcal{C})$ that extracts an ∞-fold monoid from an infinite delooping sequence.

Example 10.10. Given a sequence (X^n,p^n) where (X^n,p^n) is a pointed $(k+n)$-fold Segal space, such that $(X^{n-1},p^{n-1}) \simeq (X^n(p^n,p^n),\mathrm{id}_{p^n})$, we can extract an ∞-monoidal k-fold Segal space as $\Omega^\infty_{p^\bullet} X^\bullet$.

We now wish to compare our notions of n-fold monoids to $\mathbb{E}_n$-algebras, where $\mathbb{E}_n$ is the ∞-operad associated to the little n-disc operad. This is a straightforward consequence of results proved in [Lur17]:

Proposition 10.11. *Let $\mathcal{C}$ be an ∞-category with finite products, and let $\mathcal{C}^\times$ denote the associated Cartesian symmetric monoidal ∞-category (see* [Lur17, §2.4.1]*). Then:*

(i) There is a natural equivalence $\mathrm{Mon}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}^\times)$.

(ii) For every integer n there is a natural equivalence $\mathrm{Mon}_n(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}^\times)$*; under this equivalence the forgetful map* $\mathrm{Mon}_n(\mathcal{C}) \to \mathrm{Mon}_{n-1}(\mathcal{C})$ *corresponds to the map induced by a map of ∞-operads $\mathbb{E}_{n-1} \to \mathbb{E}_n$.*

(iii) There is a natural equivalence $\mathrm{Mon}_\infty(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^\times)$*, where $\mathbb{E}_\infty$ denotes the commutative ∞-operad.*

Proof. (i) follows by combining [Lur17, Proposition 4.1.2.10], [Lur17, Proposition 2.4.2.5] and [Lur17, Example 5.1.0.7]. Now we prove (ii) by induction: if it holds for $n-1$ we have an equivalence

$$\mathrm{Mon}_n(\mathcal{C}) \simeq \mathrm{Mon}(\mathrm{Mon}_{n-1}(\mathcal{C})) \simeq \mathrm{Mon}(\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}^\times)) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}^\times)^\times).$$

The universal property of the Boardman-Vogt tensor product (see [Lur17, §2.2.5]) implies that this is naturally equivalent to $\mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{E}_{n-1}}(\mathcal{C})$. By the Additivity Theorem, [Lur17, Theorem 5.1.2.2] the tensor product $\mathbb{E}_1 \otimes \mathbb{E}_{n-1}$ is equivalent to $\mathbb{E}_n$, so we obtain a natural equivalence $\mathrm{Mon}_n(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}^\times)$. Moreover, under this equivalence the forgetful functor $\mathrm{Mon}_n(\mathcal{C}) \to \mathrm{Mon}_{n-1}(\mathcal{C})$ corresponds to the map induced by

$$\mathbb{E}_{n-1} \simeq \mathrm{Triv} \otimes \mathbb{E}_{n-1} \to \mathbb{E}_1 \otimes \mathbb{E}_{n-1} \simeq \mathbb{E}_n,$$

where $\mathrm{Triv} \to \mathbb{E}_1$ is the obvious map from the trivial operad corresponding to the forgetful functor from $\mathbb{E}_1$-algebras in $\mathcal{C}$ to $\mathcal{C}$.

Taking the limit of the equivalences in (iii) as n goes to ∞, we get an equivalence $\mathrm{Mon}_\infty(\mathcal{C}) \simeq \lim_{n\to\infty} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$. Since the functors in the diagram come from maps of ∞-operads, we can identify the right-hand side with $\mathrm{Alg}_{\mathrm{colim}_{n\to\infty}\mathbb{E}_n}(\mathcal{C})$. But by [Lur17, Corollary 5.1.1.5] the colimit $\mathrm{colim}_{n\to\infty}\mathbb{E}_n$ is the commutative ∞-operad $\mathbb{E}_\infty$. Putting these equivalences together now gives a natural equivalence $\mathrm{Mon}_\infty(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^\times)$. □

Given these natural equivalences, we will allow ourselves to refer to n- and ∞-monoidal (complete) k-fold Segal spaces as $\mathbb{E}_n$*-monoidal* and *symmetric monoidal* (complete) k-fold Segal spaces.

11. Adjoints and Duals in Iterated Segal Spaces

In this section we first review the notions of (∞, k)-categories with adjoints and (symmetric) monoidal (∞, k)-categories with duals from [Lur09c], and then extend these notions to (∞, k)-categories internal to an ∞-topos. We begin by recalling some key facts about adjunctions in $(\infty, 2)$-categories due to Riehl and Verity:

Definition 11.1. Let Adj denote the generic adjunction, i.e. the universal 2-category containing an adjunction between two 1-morphisms. An explicit description of Adj can be found in [RV16, §4]. We will think of Adj as a 2-fold Segal space via the nerve functor from 2-categories to 2-fold Segal spaces. An *adjunction* in a (complete) 2-fold Segal space $\mathcal{C}$ is then a map of 2-fold Segal spaces $\mathrm{Adj} \to \mathcal{C}$. If $\mathcal{C}$ is a complete 2-fold Segal space, we write $\mathrm{Adj}(\mathcal{C}) := \mathrm{Map}(\mathrm{Adj}, \mathcal{C})$ for the space of adjunctions in $\mathcal{C}$.

Theorem 11.2 ([RV16, Theorem 5.3.9])**.** *Every adjunction in the homotopy 2-category of an* $(\infty, 2)$*-category extends to an adjunction in the* $(\infty, 2)$*-category. In particular, a 1-morphism in an* $(\infty, 2)$*-category has a (left or right) adjoint if and only if it has one in the homotopy 2-category.*

Definition 11.3. More or less keeping the notation of [RV16], among the data defining the $(\infty, 2)$-category Adj we have:

- two objects $+$ and $-$,
- 1-morphisms $\mathfrak{f}\colon - \to +$ (the left adjoint) and $\mathfrak{g}\colon + \to -$ (the right adjoint),
- 2-morphisms $\mathfrak{u}\colon \mathrm{id}_+ \to \mathfrak{g}\mathfrak{f}$ (the unit) and $\mathfrak{c}\colon \mathfrak{f}\mathfrak{g} \to \mathrm{id}_-$ (the counit), satisfying the triangle identities.

Theorem 11.4 ([RV16, Theorem 5.4.22])**.** *Suppose* $\mathcal{C}$ *is a complete 2-fold Segal space. The maps* $\mathfrak{f}^*$ *and* $\mathfrak{g}^*\colon \mathrm{Adj}(\mathcal{C}) \to \mathrm{Mor}_1(\mathcal{C})$ *sending an adjunction in* $\mathcal{C}$ *to the left and right adjoint, respectively, are (-1)-connected, i.e. their fibres are either empty or contractible.*

Remark 11.5. The results of Riehl and Verity are proved in the context of categories strictly enriched in simplicial sets equipped with the Joyal model structure. Reformulating these theorems in terms of complete 2-fold Segal spaces is justified because these two models of $(\infty, 2)$-categories are equivalent by the unicity theorem of Barwick and Schommer-Pries [BSP11]. (An explicit equivalence can also be obtained by combining Theorem 5.9 and Corollary 7.21 of [Hau15].)

Now we recall what it means for an (∞, n)-category to *have adjoints*:

Definition 11.6. Suppose $\mathcal{C}$ is a (complete) n-fold Segal space with $n > 1$. We say that $\mathcal{C}$ *has adjoints for 1-morphisms* if every 1-morphism in the homotopy 2-category of $\mathcal{C}$ has a left and a right adjoint. Equivalently, $\mathcal{C}$ has adjoints for 1-morphims if the maps

$$\mathfrak{f}^*, \mathfrak{g}^* \colon \mathrm{Adj}(u_{(\infty,2)}\mathcal{C}) \to \mathrm{Mor}_1(u_{(\infty,2)}\mathcal{C})$$

are both equivalences.

Definition 11.7. Suppose $\mathcal{C}$ is a (complete) n-fold Segal space with $n > 1$. For $1 < k < n$ we say that $\mathcal{C}$ *has adjoints for k-morphisms* if for all objects X, Y of $\mathcal{C}$ the $(n-1)$-fold Segal space $\mathcal{C}(X, Y)$ has adjoints for $(k-1)$-morphims. We say that $\mathcal{C}$ *has adjoints* if it has adjoints for k-morphims for all $k = 1, \dots, n-1$.

Remark 11.8. To see that a not necessarily complete n-fold Segal space $\mathcal{C}$ has adjoints, it is not necessary to complete it: Whether $\mathcal{C}$ has adjoints for 1-morphisms only depends on the homotopy 2-category, which is easy to describe without completing $\mathcal{C}$. Moreover, by [Hau17, Lemma 5.50] the mapping $(n-1)$-fold Segal spaces in the completion of $\mathcal{C}$ are the completions of the mapping $(n-1)$-fold Segal spaces of $\mathcal{C}$, so by induction we do not need to complete to see that $\mathcal{C}$ has adjoints for k-morphisms also for $k > 1$.

Definition 11.9. We say that a monoidal n-fold Segal space $\mathcal{C}^{\otimes}$ *has duals* if $\mathcal{C}$ has adjoints when regarded as an $(n+1)$-fold Segal space. We also say a symmetric monoidal or $\mathbb{E}_k$-monoidal n-fold Segal space *has duals* if the underlying monoidal n-fold Segal space has duals.

Lemma 11.10. *We may regard a space X as an n-fold Segal space for any n by taking the constant functor with value X. If X is an associative monoid in $\mathcal{S}$ (or in other words an A_∞-space), then X has duals if and only if X is grouplike, i.e. under the induced multiplication the monoid $\pi_0 X$ is a group.*

Proof. It suffices to check that the homotopy 1-category of X, equipped with the induced monoidal structure, has duals. But this is just the fundamental 1-groupoid of X, and an object of a monoidal groupoid has a dual if and only if it has an inverse. □

We now prove a characterization of n-fold Segal spaces with adjoints that will be useful later:

Proposition 11.11. *Suppose $\mathcal{C}$ is a complete n-fold Segal space. Then $\mathcal{C}$ has adjoints for k-morphisms for any $2 < k < n$ if and only if for every map $\phi \colon X \to \mathrm{Ob}(\mathcal{C})^{\times 2}$ in $\mathcal{S}$, the complete $(n-1)$-fold Segal space $\mathcal{C}_\phi$, defined by the pullback square*

$$\begin{tikzcd}
\mathcal{C}_\phi \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\
X \arrow[r] & \mathcal{C}_0^{\times}2
\end{tikzcd}$$

in $(n-1)$-fold Segal spaces, has adjoints for $(k-1)$-morphisms.

The proof depends on the following observation:

Lemma 11.12. *Suppose given a morphism of n-fold Segal spaces $\mathcal{C} \to X$, where X is constant. If all the fibres $\mathcal{C}_x$ for $x \in X$ have adjoints for k-morphisms, then so does $\mathcal{C}$.*

Proof. To prove this, we induct on n. For $n = 1$, there is nothing to prove, so we may suppose that the statement is true for $(n-1)$-fold Segal spaces for all $k = 1, \dots, n-1$.

We first consider the case $k = 1$. Since $\mathrm{Adj}(X) \simeq \mathrm{Mor}_1(X) \simeq X$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Adj}(\mathcal{C}) \arrow[rr] \arrow[dr] & & \mathrm{Mor}_1(\mathcal{C}) \arrow[dl] \\ & X. & \end{tikzcd}$$

Since the functors $\mathrm{Adj}(-)$ and $\mathrm{Mor}_1(-)$ preserve limits, the induced map on fibres over $p \in X$ can be identified with $\mathrm{Adj}(\mathcal{C}_p) \to \mathrm{Mor}_1(\mathcal{C}_p)$. By assumption this is an equivalence for all $p \in X$, and so $\mathrm{Adj}(\mathcal{C}) \to \mathrm{Mor}_1(\mathcal{C})$ is also an equivalence, i.e. $\mathcal{C}$ has adjoints for 1-morphisms.

For $k > 2$, we must show that $\mathcal{C}(c, d)$ has adjoints for $(k-1)$-morphisms for all $c, d \in \mathrm{Ob}(\mathcal{C})$. But by Lemma 7.24 there is a map $\mathcal{C}(c, d) \to \Omega_{\pi(c),\pi(d)}X$ whose fibres are mapping $(n-1)$-fold Segal spaces in the fibres of π, and so have adjoints for $(k-1)$-morphisms. The result therefore holds by the inductive hypothesis. □

Proof of Proposition 11.11. One direction is obvious: If $\mathcal{C}_\phi$ has adjoints for $(k-1)$-morphisms for every map ϕ, then in particular this is true for the $(n-1)$-fold Segal spaces $\mathcal{C}(x, y)$ for all objects x, y, so $\mathcal{C}$ has adjoints for k-morphisms.

For the other direction we must show that if $\mathcal{C}$ has adjoints for k-morphisms then $\mathcal{C}_\phi$ has adjoints for $(k-1)$-morphisms for all ϕ. By Lemma 11.12, to see this it suffices to show that the fibres of the map $\mathcal{C}_\phi \to X$ have adjoints for $(k-1)$-morphisms. But the fibre of this map at $p \in X$ is $\mathcal{C}(a, b)$ where $\phi(p) \simeq (a, b)$, which by assumption has adjoints for k-morphisms. □

An advantage of the characterization of Proposition 11.11 is that this has a straightforward generalization to other ∞-topoi. We introduce this after some preliminary discussion:

Definition 11.13. Suppose $\mathcal{X}$ is an ∞-topos, and let

$$r^* : \mathcal{S} \rightleftarrows \mathcal{X} : r_*$$

denote the unique geometric morphism from the ∞-category of spaces. By Proposition 7.17, this induces an adjunction

$$L_n(r^*)_* : \mathrm{CSS}^n(\mathcal{S}) \rightleftarrows \mathrm{CSS}^n(\mathcal{X}) : (r_*)_*$$

If $\mathcal{C}$ is a complete 2-fold Segal object in $\mathcal{X}$, then an *adjunction* in $\mathcal{C}$ is a functor $(r^*)_*\mathrm{Adj} \to \mathcal{C}$. We write $\mathrm{Adj}(\mathcal{C}) \in \mathcal{X}$ for the mapping object $\mathrm{MAP}((r^*)_*\mathrm{Adj}, \mathcal{C})$ in $\mathcal{X}$, defined in Definition 7.12. Similarly, if $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$, we write $\mathrm{Ob}(\mathcal{C}) := \mathrm{MAP}((r^*)_*C_0, \mathcal{C}) \simeq \mathcal{C}_{0,\ldots,0}$ and $\mathrm{Mor}_n(\mathcal{C}) := \mathrm{MAP}((r^*)_*C_n, \mathcal{C})$ for $n = 1, \ldots, k$.

Lemma 11.14. *Let $\mathcal{C}$ be a complete 2-fold Segal object in an ∞-topos $\mathcal{X}$. Then the morphisms $\mathfrak{f}^*$ and $\mathfrak{g}^* \colon \mathrm{Adj}(\mathcal{C}) \to \mathrm{Mor}_1(\mathcal{C})$ are (-1)-truncated.*

Proof. We must show that for any $X \in \mathcal{X}$, the map $\mathrm{Map}_{\mathcal{X}}(X, \mathrm{Adj}(\mathcal{C})) \to \mathrm{Map}_{\mathcal{X}}(X, \mathrm{Mor}_1(\mathcal{C}))$ is (-1)-truncated. But there is a natural equivalence

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}}(X, \mathrm{Adj}(\mathcal{C})) &\simeq \mathrm{Map}_{\mathrm{CSS}^2(\mathcal{X})}(X \times (r^*)_*\mathrm{Adj}, \mathcal{C}) \simeq \mathrm{Map}_{\mathrm{CSS}^2(\mathcal{X})}((r^*)_*\mathrm{Adj}, \mathcal{C}^X) \\ &\simeq \mathrm{Map}_{\mathrm{CSS}^2(\mathcal{S})}(\mathrm{Adj}, (r_*)_*\mathcal{C}^X) \simeq \mathrm{Adj}((r_*)_*\mathcal{C}^X), \end{aligned}$$

and similarly $\mathrm{Map}_{\mathcal{X}}(X, \mathrm{Mor}_1(\mathcal{C})) \simeq \mathrm{Mor}_1((r_*)_*\mathcal{C}^X)$. Thus this follows by applying Theorem 11.4 to the complete 2-fold Segal spaces $(r_*)_*\mathcal{C}^X$ for all $X \in \mathcal{X}$. □

Definition 11.15. Suppose $\mathcal{C}$ is a complete n-fold Segal object in $\mathcal{X}$ with $n > 1$. We say that $\mathcal{C}$ *has adjoints for 1-morphisms* if the maps $\mathfrak{f}^*, \mathfrak{g}^*\colon \mathrm{Adj}(u_{(\infty,2)}\mathcal{C}) \to \mathrm{Mor}_1(u_{(\infty,2)}\mathcal{C})$ are both equivalences.

Definition 11.16. Suppose $\mathcal{C}$ is a complete n-fold Segal object in $\mathcal{X}$ with $n > 1$. For $1 < k < n$ we say that $\mathcal{C}$ *has adjoints for k-morphisms* if for all maps $\phi\colon X \to \mathrm{Ob}(\mathcal{C})^{\times 2}$ in $\mathcal{X}$, the complete $(n-1)$-fold Segal object $\mathcal{C}_\phi$, defined by the pullback square

$$\begin{tikzcd} \mathcal{C}_\phi \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\ X \arrow[r] & \mathcal{C}_0^{\times}2 \end{tikzcd}$$

in $(k-1)$-fold Segal objects, has adjoints for $(k-1)$-morphisms. We say that $\mathcal{C}$ *has adjoints* if it has adjoints for k-morphims for all $k = 1, \ldots, n-1$.

Definition 11.17. If $\mathcal{C}$ is a (not necessarily complete) n-fold Segal object in $\mathcal{X}$, we say that $\mathcal{C}$ *has adjoints* (for k-morphisms) if this is true of the completion $L\mathcal{C}$.

Definition 11.18. We say that a monoidal complete n-fold Segal space $\mathcal{C}$ *has duals* if it has adjoints when regarded as an $(n+1)$-fold Segal space. We also say a symmetric monoidal or $\mathbb{E}_k$-monoidal complete n-fold Segal object *has duals* if the underlying monoidal complete n-fold Segal object has duals.

Proposition 11.19. *We may regard an object $X \in \mathcal{X}$ as a complete n-fold Segal object for any n by taking the constant functor with value X. If X is an associative monoid in $\mathcal{X}$ then X has duals if and only if X is grouplike, i.e. it is a groupoid object in the sense of Definition 7.3.*

Proof. Write $\mathcal{C}$ for the associative monoid corresponding to X, regarded as an $(n+1)$-fold Segal object in $\mathcal{X}$. Then by Lemma 11.14, the $(n+1)$-fold Segal object $\mathcal{C}$ has adjoints for 1-morphisms if and only if for every $Y \in \mathcal{X}$ the $(n+1)$-fold Segal space $(r_*)_*\mathcal{C}^Y$ has adjoints for 1-morphisms. Similarly, $\mathcal{C}$ is a groupoid object if and only if $(r_*)_*\mathcal{C}^Y$ is a groupoid object for all $Y \in \mathcal{X}$. The result therefore follows from Lemma 11.10. □

12. Full Dualizability for Iterated Spans

In this section we will show that $\mathrm{Span}_k(\mathcal{C})$ is symmetric monoidal, and that all its objects are fully dualizable — in fact, we will show that $\mathrm{Span}_k(\mathcal{C})$ has duals.

Proposition 12.1. *Suppose $\mathcal{C}$ is an ∞-category with finite limits. Then the (∞,k)-category $\mathrm{Span}_k(\mathcal{C})$ is symmetric monoidal.*

Proof. By Proposition 8.3 we can identify $\mathrm{Span}_k(\mathcal{C})$ with the (∞,k)-category $\mathrm{Span}_{k+1}(\mathcal{C})(*,*)$ of endomorphisms of $*$ in $\mathrm{Span}_{k+1}(\mathcal{C})$. The sequence $(\mathrm{Span}_k(\mathcal{C}), *)$ of pointed k-fold Segal spaces therefore defines an infinite delooping sequence $\mathrm{Span}_{\infty+k}(\mathcal{C})$ in k-fold complete Segal spaces. From this we can extract an ∞-fold monoid $\Omega^\infty\mathrm{Span}_{\infty+k}(\mathcal{C})$ in complete k-fold Segal spaces. By Proposition 10.11 this is equivalent to a symmetric monoidal structure on $\mathrm{Span}_k(\mathcal{C})$. □

Remark 12.2. In the case $k = 1$, an explicit construction of this symmetric monoidal structure can also be found in [BGS16].

Lemma 12.3. *Let $\mathcal{C}$ be an ∞-category with finite limits. For all $k \geq 2$, the 1-morphisms in $\mathrm{Span}_k(\mathcal{C})$ have adjoints.*

Proof. It suffices to check this in the homotopy 2-category of $\mathrm{Span}_k(\mathcal{C})$. A 1-morphism $\phi\colon A \to B$ in $\mathrm{Span}_k(\mathcal{C})$ is a span

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A & & B. \end{tikzcd}$$

We will show that the reversed span $\bar{\phi}$ given by

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ B & & A \end{tikzcd}$$

is a right adjoint to this, with unit $\eta\colon \mathrm{id}_A \to \bar{\phi}\phi$ given by the span

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ A & & X \times_B X \end{tikzcd}$$

over $A \times A$, and counit $\epsilon\colon \phi\bar{\phi} \to \mathrm{id}_B$ given by

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times_A X & & B \end{tikzcd}$$

over $B \times B$, where Δ denotes the relevant diagonal maps. To see this it suffices to check that the triangle equations hold up to homotopy. The 2-morphism $\phi\eta\colon \phi \to \phi\bar{\phi}\phi$ is given by the span

$$\begin{tikzcd} & X \times_B X \arrow[dl, "\pi_1"'] \arrow[dr, "\mathrm{id} \times \Delta"] & \\ X & & X \times_B X \times_A X, \end{tikzcd}$$

and $\epsilon\phi$ is given by

$$\begin{tikzcd} & X \times_A X \arrow[dl, "\Delta \times \mathrm{id}"'] \arrow[dr, "\pi_2"] & \\ X \times_B X \times_A X & & X. \end{tikzcd}$$

The composite $\phi \to \phi$ of these two maps is therefore given by the pullback

$$(X \times_B X) \times_{(X \times_B X \times_A X)} (X \times_A X).$$

We claim that this pullback is equivalent to the limit of the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
X \arrow[dr] & & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \\
 & A & & B. &
\end{tikzcd}$$

To see this, take the right Kan extension of this diagram along the inclusion into the category with shape

$$\begin{tikzcd}
 & \circ \arrow[d] \arrow[dr] & & \circ \arrow[dl] \arrow[d] & \\
 & \bullet \arrow[dl] \arrow[dr] & \circ \arrow[dll] \arrow[d] \arrow[drr] & \bullet \arrow[dl] \arrow[dr] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \\
 & \bullet & & \bullet, &
\end{tikzcd}$$

where $\circ$ denotes the new objects. This produces a diagram

$$\begin{tikzcd}
 & X \times_B X \arrow[d] \arrow[dr] & & X \times_A X \arrow[dl] \arrow[d] & \\
 & X \arrow[dl] \arrow[dr] & X \times_A X \times_B X \arrow[dll] \arrow[d] \arrow[drr] & X \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \\
 & A & & B, &
\end{tikzcd}$$

whose limit can be identified with $(X \times_B X) \times_{(X \times_B X \times_A X)} (X \times_A X)$ by a simple cofinality argument. On the other hand, since right Kan extensions are transitive, this must agree with the limit of the first diagram, which can be identified with X (again by an easy cofinality argument). Thus $(\epsilon\phi) \circ (\phi\eta) \simeq \mathrm{id}_\phi$, and the other triangle equivalence, $(\bar{\phi}\epsilon) \circ (\eta\bar{\phi}) \simeq \mathrm{id}_{\bar{\phi}}$, is proved similarly. $\square$

Theorem 12.4. *The (∞, k)-category $\mathrm{Span}_k(\mathcal{C})$ has adjoints.*

Proof. We prove this by induction on k. For $k = 1$, there is nothing to prove. Suppose we have shown that for all $\mathcal{C}$ the $(\infty, k-1)$-category $\mathrm{Span}_{k-1}(\mathcal{C})$ has adjoints. We saw in Lemma 12.3 that $\mathrm{Span}_k(\mathcal{C})$ has adjoints for 1-morphisms, and for every pair X, Y of objects in $\mathcal{C}$ the $(\infty, k-1)$-category $\mathrm{Span}_k(\mathcal{C})(X, Y)$ can be identified with $\mathrm{Span}_{k-1}(\mathcal{C}_{/X \times Y})$ by Proposition 8.3, and so has adjoints by the inductive hypothesis. Thus $\mathrm{Span}_k(\mathcal{C})$ also has adjoints. $\square$

Corollary 12.5. *The (symmetric) monoidal (∞, k)-category $\mathrm{Span}_k(\mathcal{C})$ has duals.*

Proof. As a monoidal (∞, k)-category, we may identify $\mathrm{Span}_k(\mathcal{C})$ with the endomorphism (∞, k)-category $\mathrm{Span}_{k+1}(\mathcal{C})(*, *)$. Since $\mathrm{Span}_{k+1}(\mathcal{C})$ has adjoints, it follows that $\mathrm{Span}_k(\mathcal{C})$ has duals. □

Invoking the cobordism hypothesis in the form [Lur09c, Theorem 2.4.6], we get:

Corollary 12.6. *Suppose $\mathcal{C}$ is an ∞-category with finite limits. Then every object C of $\mathcal{C}$ defines a framed k-dimensional TQFT $\mathcal{Z}^k_C \colon \mathrm{Bord}^{\mathrm{fr}}_k \to \mathrm{Span}_k(\mathcal{C})$, where $\mathrm{Bord}^{\mathrm{fr}}_k$ denotes the (∞, k)-category of framed cobordisms.*

Remark 12.7. For $\mathcal{D}$ an ∞-category with finite colimits, we write $\mathrm{Cospan}_k(\mathcal{D})$ for the (∞, k)-category $\mathrm{Span}_k(\mathcal{D}^{\mathrm{op}})$. If $\mathrm{Bord}^{\mathrm{un}}_k$ denotes the unoriented cobordism (∞, k)-category, it is reasonable to expect that there is a symmetric monoidal "forgetful functor" $\mathrm{Bord}^{\mathrm{un}}_k \to \mathrm{Cospan}_k(\mathcal{S}^{\mathrm{fin}})$, where $\mathcal{S}^{\mathrm{fin}}$ is the ∞-category of finite CW-complexes. This would send a cobordism between manifolds with corners to the iterated cospan given by the inclusions of the underlying homotopy types of the incoming and outgoing boundaries and corners. If we assume this, we can give an explicit construction of the framed field theory $\mathcal{Z}^k_C$ valued in $\mathrm{Span}_k(\mathcal{C})$ associated to an object $C \in \mathcal{C}$:

(1) If $\mathcal{C}$ is an ∞-category with finite limits, then $\mathcal{C}$ is cotensored over $\mathcal{S}^{\mathrm{fin}}$. Thus given $C \in \mathcal{C}$ there is a functor $C^{(-)} \colon (\mathcal{S}^{\mathrm{fin}})^{\mathrm{op}} \to \mathcal{C}$. Since $\mathrm{Span}_k(-)$ is natural in limit-preserving functors, this induces a functor $C^{(-)} \colon \mathrm{Cospan}_k(\mathcal{S}^{\mathrm{fin}}) \to \mathrm{Span}_k(\mathcal{C})$ for all k.
(2) Identifying $\mathrm{Span}_k(\mathcal{C})$ as an endomorphism (∞, k)-category in $\mathrm{Span}_{k+n}(\mathcal{C})$ for all n, we conclude that the functor $C^{(-)}$ is $\mathbb{E}_n$-monoidal for all n, hence symmetric monoidal.
(3) Composing, we get a symmetric monoidal functor $\widehat{\mathcal{Z}}^k_C \colon \mathrm{Bord}^{\mathrm{un}}_k \to \mathrm{Span}_k(\mathcal{C})$ that sends a cobordism X to C^X and the iterated span coming from the iterated boundary of X.
(4) By the cobordism hypothesis, the composite of $\widehat{\mathcal{Z}}^k_C$ with the forgetful functor $\mathrm{Bord}^{\mathrm{fr}}_k \to \mathrm{Bord}^{\mathrm{un}}_k$ is the unique symmetric monoidal functor $\mathrm{Bord}^{\mathrm{fr}}_k \to \mathrm{Span}_k(\mathcal{C})$ that sends the point to C, hence it must be equivalent to $\mathcal{Z}^k_C$.

This construction would also allow us to understand the $O(k)$-action on the space $\iota\mathcal{C}$ of objects of $\mathrm{Span}_k(\mathcal{C})$: This action is given on the space of TQFTs by acting by $O(k)$ on the framings in $\mathrm{Bord}^{\mathrm{fr}}_k$. Since we know all the framed TQFTs factor through the forgetful functor $\mathrm{Bord}^{\mathrm{fr}}_k \to \mathrm{Cospan}_k(\mathcal{S})$, which is $O(k)$-equivariant with respect to the trivial action on the target, we see that the action on $\iota\mathcal{C}$ is trivial. As a consequence, we can also classify other kinds of TQFTs in $\mathrm{Span}_k(\mathcal{C})$, since by [Lur09c, Theorem 2.4.18] these are determined by $O(k)$-equivariant maps to $\iota\mathcal{C}$. For example, the space of unoriented field theories is equivalent to $\mathrm{Map}(BO(n), \mathcal{C})$ — when $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces, this is precisely the ∞-groupoid of spaces equipped with an n-dimensional vector bundle.

Actually constructing such a forgetful functor from cobordisms to cospans would, however, depend on the details of a construction of $\mathrm{Bord}^{\mathrm{un}}_k$, such as that of Calaque and Scheimbauer [CS15], and we will not attempt to do so here.

13. Full Dualizability for Iterated Spans with Local Systems

In this section we consider dualizability for iterated spans with local systems. We'll prove that if $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$, then a symmetric monoidal structure on $\mathcal{C}$ induces one on $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$. Moreover, we will show that if $\mathcal{C}$ has duals then the same is true of $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$.

Proposition 13.1. *Suppose $\mathcal{C}$ is a symmetric monoidal complete k-fold Segal object in an ∞-topos $\mathcal{X}$. Then the (∞, k)-category $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ is symmetric monoidal.*

Proof. Since $\mathcal{C}$ is symmetric monoidal, we can choose a sequence of "deloopings" $(\mathcal{C}_i, c_i)$ such that $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_i \simeq \mathcal{C}_{i+1}(c_{i+1}, c_{i+1})$, i.e. an infinite delooping sequence in complete k-fold Segal objects. By Proposition 9.3, we can then identify $\mathrm{Span}_{k+i}(\mathcal{X}; \mathcal{C}_i)$ with the mapping $(\infty, k+i)$-category

$$\mathrm{Span}_{k+i+1}(\mathcal{X}; \mathcal{C}_{i+1})(x_{i+1}, x_{i+1})$$

in $\mathrm{Span}_{k+i+1}(\mathcal{X}; \mathcal{C}_{i+1})$, where the object x_{i+1} is the map $* \to \mathrm{Ob}(\mathcal{C}_{i+1})$ corresponding to the object c_{i+1}. Thus we get an infinite delooping sequence $\mathrm{Span}_{\infty+k}(\mathcal{X}; \mathcal{C})$, from which we can extract an ∞-fold monoid $\Omega^\infty \mathrm{Span}_{\infty+k}(\mathcal{X}; \mathcal{C})$ in complete k-fold Segal spaces. By Proposition 10.11 this is equivalent to a symmetric monoidal structure on $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$. □

Proposition 13.2. *Suppose $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$ that has adjoints for 1-morphisms. Then $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ has adjoints for 1-morphisms.*

Proof. Suppose given a 1-morphism in $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$, i.e. a span $A \leftarrow X \to B$ in $\mathcal{X}$ equipped with a map to the span $\mathrm{Ob}(\mathcal{C}) \leftarrow \mathrm{Mor}_1(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$. We will show that a right adjoint to this morphism is given by $B \leftarrow X \to A$, now with X equipped with the map

$$X \to \mathrm{Mor}_1(\mathcal{C}) \xrightarrow{(\mathfrak{f}^*)^{-1}} \mathrm{Adj}(\mathcal{C}) \xrightarrow{\mathfrak{g}^*} \mathrm{Mor}_1(\mathcal{C}),$$

which interchanges the source and target of 1-morphisms in $\mathcal{C}$.

The unit for the adjunction is given by the span $A \leftarrow X \to X \times_B X$ over $A \times A$, where the map $X \to \mathrm{Mor}_2(\mathcal{C})$ is the composite

$$X \to \mathrm{Mor}_1(\mathcal{C}) \xrightarrow{(\mathfrak{f}^*)^{-1}} \mathrm{Adj}(\mathcal{C}) \xrightarrow{\mathfrak{u}^*} \mathrm{Mor}_2(\mathcal{C})$$

and the counit by $B \leftarrow X \to X \times_A X$, where X is now equipped with

$$X \to \mathrm{Mor}_1(\mathcal{C}) \xrightarrow{(\mathfrak{f}^*)^{-1}} \mathrm{Adj}(\mathcal{C}) \xrightarrow{\mathfrak{c}^*} \mathrm{Mor}_2(\mathcal{C}).$$

The triangle identities for the adjunction then follow by combining the proof of Lemma 12.3 with the homotopies coming from the triangle identities for the generic adjunction. Thus all 1-morphisms in $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ have right adjoints. To see that they also have left adjoints, we simply interchange the roles of the morphisms $\mathfrak{f}^*$ and $\mathfrak{g}^*$ above. □

Theorem 13.3. *Suppose $\mathcal{C}$ is a complete k-fold Segal object in $\mathcal{X}$ that has adjoints. Then $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ has adjoints.*

Proof. We will show that if $\mathcal{C}$ has adjoints for i-morphisms then $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ also has adjoints for i-morphisms. The case $i = 1$ was proved in Proposition 13.2. Suppose $i > 1$, then we must show that $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})(\xi, \eta)$ has adjoints for $(i-1)$-morphisms for all $\xi, \eta \in \mathrm{Span}_k(\mathcal{X}; \mathcal{C})$. By Proposition 9.3, this $(k-1)$-fold Segal space can be identified with $\mathrm{Span}_{k-1}(\mathcal{X}; \mathcal{C}_{\xi,\eta})$, and by definition (or by Proposition 11.11 in the case of spaces) $\mathcal{C}_{\xi,\eta}$ has adjoints for $(i-1)$-morphisms if $\mathcal{C}$ has adjoints for i-morphisms. Thus by induction we see that $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ has adjoints for i-morphisms. □

Corollary 13.4. *Suppose $\mathcal{C}$ is a (symmetric) monoidal complete k-fold Segal object in $\mathcal{X}$ that has duals. Then the (symmetric) monoidal (∞, k)-category $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ also has duals.*

Proof. Since $\mathcal{C}$ is monoidal, by Definition 11.18 there is a pointed complete $(n+1)$-fold Segal object $(\mathcal{C}^{\otimes}, *)$ with adjoints such that $\mathcal{C}$ is the endomorphism n-fold Segal object $\mathcal{C}^{\otimes}(*, *)$. Then $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ is the endomorphism (∞, n)-category $\mathrm{Span}_k(\mathcal{X}; \mathcal{C}^{\otimes})(x, x)$, where x is the object $* \to \mathrm{Ob}(\mathcal{C}^{\otimes})$ corresponding to the base point. By Theorem 13.3 the $(\infty, n+1)$-category $\mathrm{Span}_k(\mathcal{X}; \mathcal{C}^{\otimes})$ has adjoints, hence the monoidal (∞, n)-category $\mathrm{Span}_k(\mathcal{X}; \mathcal{C})$ has duals. $\square$

Invoking the cobordism hypothesis, we get:

Corollary 13.5. *Suppose $\mathcal{C}$ is a symmetric monoidal complete k-fold Segal object in $\mathcal{X}$ that has duals. Every morphism $\phi\colon X \to \mathrm{Ob}(\mathcal{C})$ in $\mathcal{X}$ defines a framed k-dimensional TQFT $\mathcal{Z}^k_\phi\colon \mathrm{Bord}^{\mathrm{fr}}_k \to \mathrm{Span}_k(\mathcal{X}; \mathcal{C})$, where $\mathrm{Bord}^{\mathrm{fr}}_k$ denotes the (∞, k)-category of framed bordisms.*

Example 13.6. Suppose A is a grouplike E_∞-algebra in an ∞-topos $\mathcal{X}$. Then by Proposition 11.19 we may regard A as a symmetric monoidal n-fold complete Segal object with duals in $\mathcal{X}$ for any n, and so we get for every n a symmetric monoidal (∞, n)-category $\mathrm{Span}_n(\mathcal{X}; A)$. The underlying (∞, n)-category of this is just $\mathrm{Span}_n(\mathcal{X}_{/A})$, but the symmetric monoidal structure is not that coming from the Cartesian product in $\mathcal{X}_{/A}$ (i.e. the fibre product over A); instead, the tensor product of two maps $X, Y \to A$ is the product $X \times Y$ equipped with the composite map $X \times Y \to A \times A \to A$ where the second map is the multiplication in A. Similarly, the unit for the symmetric monoidal structure is the unit map $* \to A$, and the dual of an object $X \to A$ is the composite $X \to A \to A$ where the second map is the inverse mapping for A.

Remark 13.7. According to [Lur09c, Proposition 3.2.7], the ∞-category $\mathrm{Span}_k(\mathcal{S}; \mathcal{C})$, where $\mathcal{C}$ is a complete k-fold Segal space, should have a universal property. Namely, if $\mathcal{B}$ is a symmetric monoidal (∞, k)-category equipped with a symmetric monoidal functor to $\mathrm{Span}_k(\mathcal{S})$, then the space of symmetric monoidal functors $\mathcal{B} \to \mathrm{Span}_k(\mathcal{S}; \mathcal{C})$ over $\mathrm{Span}_k(\mathcal{S})$ should be naturally equivalent to the space of symmetric monoidal functors from the pullback $\mathcal{B} \times_{\mathrm{Span}_k(\mathcal{S})} \mathrm{Span}_k(\mathcal{S}_*)$ to $\mathcal{C}$. Moreover, [Lur09c, Proposition 3.2.6] gives a description of the general cobordism (∞, k)-category $\mathrm{Bord}^{(X,\zeta)}_k$ (where ζ is a map of spaces $X \to BO(k)$) as a pullback of this form, namely

$$
\begin{tikzcd}
\mathrm{Bord}^{(X,\zeta)}_k \arrow[r] \arrow[d] & \mathrm{Span}_k(\mathcal{S}_*) \arrow[d] \\
\mathrm{Bord}^{\mathrm{un}}_k \arrow[r] & \mathrm{Span}_k(\mathcal{S})
\end{tikzcd}
$$

where the bottom map is the unoriented TQFT determined by ζ. If we assume this, as well as the consequences of a hypothetical forgetful functor from cobordisms to cospans discussed in Remark 12.7, we would get much more information about field theories valued in $\mathrm{Span}_k(\mathcal{S}; \mathcal{C})$. For example, we would be able to describe the $O(k)$-action on the space of objects: the forgetful functor $\mathrm{Span}_k(\mathcal{S}; \mathcal{C}) \to \mathrm{Span}_k(\mathcal{S})$ induces an $O(k)$-equivariant map from framed field theories valued in $\mathrm{Span}_k(\mathcal{S}; \mathcal{C})$ to those valued in $\mathrm{Span}_k(\mathcal{S})$. Since the $O(k)$-action on the target is trivial, this would mean that in the source $O(k)$ can only act on the fibres of this map. The fibre at $X \in \mathcal{S}$ can be identified with the space of (X, ζ)-field theories valued in $\mathcal{C}$, where ζ is the map $X \to * \to BO(k)$; invoking the cobordism hypothesis it would follow that the fibre is $\mathrm{Map}(X, \mathrm{Ob}(\mathcal{C}))$, with the obvious $O(k)$-action induced from that on $\mathrm{Ob}(\mathcal{C})$.

14. The $(\infty,1)$-Category of Lagrangian Correspondences

In this section we will use the theory of symplectic derived stacks and Lagrangian morphisms developed by Pantev, Toën, Vaquié, and Vezzosi [PTVV13] to construct an ∞-category $\mathrm{Lag}^n_{(\infty,1)}$ of n-symplectic derived stacks, with morphisms given by Lagrangian correspondences between them. This construction is based on ideas of Calaque, who describes the underlying homotopy category in [Cal15]. We begin by briefly recalling the setup for derived stacks and (closed) p-forms; for full details we refer to [TV08] and [PTVV13].

Definition 14.1. Let k be a field of characteristic 0. We write $\mathrm{cdga}_k^{\leq 0}$ for the category of non-positively graded commutative algebras in cochains over k, equipped with the usual model structure, and $\mathrm{dAff}_k^{\mathrm{op}}$ for the associated ∞-category. We may equip this ∞-category with an *étale topology*, as described in [TV08], and we write $\mathrm{dSt}_k := \mathrm{Sh}_{\text{ét}}(\mathrm{dAff}_k)$ for the associated (very large) ∞-topos of étale sheaves of (large) spaces.

Remark 14.2. It is not necessary to take k to be a field of characteristic zero. However, for more general rings commutative differential graded algebras are often not the most appropriate notion of "derived rings", the more useful notions being simplicial commutative algebras and (connective) E_∞-algebras. For a field of characteristic zero, however, all three notions coincide.

Remark 14.3. For many purposes we do not want to consider arbitrary objects of dSt_k, but only some subclass of "geometric" objects. The notion of *derived Artin stacks* provides a good definition of such geometric stacks; roughly speaking they are derived stacks obtained as iterated realizations of smooth groupoids — see [TV08] for details. In particular, derived Artin stacks always have cotangent complexes. As in [PTVV13] we will implicitly add the assumption that all derived Artin stacks considered are locally of finite presentation, so that their cotangent complexes are dualizable quasicoherent sheaves. We write $\mathrm{dSt}_k^{\mathrm{Art}}$ for the full subcategory of dSt_k spanned by the derived Artin stacks locally of finite presentation — by [TV08, Corollary 1.3.3.5] this is closed under finite limits in dSt_k.

Definition 14.4. In [PTVV13], functors Ω^p and Ω^p_{cl} from $\mathrm{dAff}_k^{\mathrm{op}}$ to the ∞-category of cochain complexes that take a derived affine scheme to its complex of p-forms and closed p-forms, respectively, are constructed and shown to be étale sheaves. We let $\mathcal{A}^p[n]$ and $\mathcal{A}^p_{\mathrm{cl}}[n]$ be the derived stacks (i.e. sheaves of spaces on dSt_k) obtained from the shifts $\Omega^p[n]$ and $\Omega^p_{\mathrm{cl}}[n]$ via the Dold-Kan construction. If X is a derived stack, we refer to $\mathcal{A}^p[n](X)$ as the space of *n-shifted p-forms* on X. There is a "forgetful" map $\Omega^p_{\mathrm{cl}} \to \Omega^p$, which induces natural transformations $\mathcal{A}^p_{\mathrm{cl}}[n] \to \mathcal{A}^p[n]$, but the components are not in general monomorphisms (i.e. inclusions of a subset of the connected components).

Remark 14.5. Since $\mathcal{A}^p_{\mathrm{cl}}[n]$ comes from a sheaf of cochain complexes on dSt_k, and hence a sheaf of spectra, we may regard it as a sheaf of grouplike E_∞-spaces, i.e. a grouplike E_∞-monoid in dSt_k. Thus by Example 13.6 there is a symmetric monoidal (∞,k)-category $\mathrm{Span}_k(\mathrm{dSt}_k; \mathcal{A}^p_{\mathrm{cl}}[n])$ with duals for all p, n.

Definition 14.6. If X is a derived Artin stack, an n-shifted 2-form $\omega \in \mathcal{A}^2[n](X)$ corresponds to a morphism $\Lambda^2\mathbb{T}_X \to \mathcal{O}_X[n]$ of quasi-coherent sheaves on X, where the tangent complex $\mathbb{T}_X$ is the dual of the cotangent complex $\mathbb{L}_X$. We say that ω is *non-degenerate* if the induced morphism $\mathbb{T}_X \to \mathbb{L}_X[n]$ is an equivalence, and write $\mathcal{A}^2_{\mathrm{nd}}[n](X)$ for the collection of components of $\mathcal{A}^2[n](X)$ corresponding to the non-degenerate n-shifted 2-forms.

Definition 14.7. An *n-shifted symplectic form* on a derived Artin stack X is a non-degenerate closed 2-form, i.e. an element of the pullback

$$\mathrm{Sympl}_n(X) := \mathcal{A}^2_{\mathrm{cl}}[n](X) \times_{\mathcal{A}^2[n](X)} \mathcal{A}^2_{\mathrm{nd}}[n](X),$$

which is a subset of the connected components of $\mathcal{A}^2_{\mathrm{cl}}[n](X)$. An *n-symplectic derived Artin stack* (X, ω) is a derived Artin stack X equipped with an n-shifted symplectic form ω.

Definition 14.8. Suppose X is a derived stack, and ω is an n-shifted closed 2-form on X. If $f\colon L \to X$ is a morphism of derived stacks, then an *ω-isotropic structure* on f is a commutative square

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d] & X \arrow[d, "\omega"] \\ * \arrow[r, "0"'] & \mathcal{A}^2_{\mathrm{cl}}[n]. \end{tikzcd}$$

Equivalently, it is a path from 0 to the composite closed n-shifted 2-form $f^*\omega$ in $\mathcal{A}^2_{\mathrm{cl}}[n](L)$.

Definition 14.9. Suppose (X, ω) is an n-symplectic derived Artin stack, and $f\colon L \to X$ is a morphism of derived Artin stacks. Then an isotropic structure on f induces a commutative square (see [PTVV13, §2.2] for the details)

$$\begin{tikzcd} \mathbb{T}_L \arrow[r] \arrow[d] & f^*\mathbb{T}_X \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_L[n] \end{tikzcd}$$

of quasi-coherent sheaves on L. We say that the isotropic structure is *Lagrangian* if this square is Cartesian.

Definition 14.10. Suppose (X, ω_X) and (Y, ω_Y) are n-shifted symplectic derived Artin stacks. A span

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

in $(\mathrm{dSt}_k^{\mathrm{Art}})_{/\mathcal{A}^2_{\mathrm{cl}}[n]}$ induces a commutative square

$$\begin{tikzcd} \mathbb{T}_L \arrow[r] \arrow[d] & f^*\mathbb{T}_X \arrow[d] \\ g^*\mathbb{T}_Y \arrow[r] & \mathbb{L}_L[n] \end{tikzcd}$$

of quasi-coherent sheaves on L. We say the span is a *Lagrangian correspondence* if this square is Cartesian.

Remark 14.11. If we write $\overline{Y}$ for the derived stack Y equipped with the *negative* of the symplectic form of Y, which is also a symplectic form, we may identify spans of the form $X \xleftarrow{f} L \xrightarrow{g} Y$ over $\mathcal{A}^2_{\mathrm{cl}}[n]$ with isotropic morphisms $L \to X \times \overline{Y}$: Making the maps f and g into a span over $\mathcal{A}^2_{\mathrm{cl}}[n]$ precisely corresponds to choosing a path from $f^*\omega_X$ to $g^*\omega_Y$ in the space of closed 2-forms on L, which is the same as giving a path from the zero form to $f^*\omega_X - g^*\omega_Y$. Under this equivalence, Lagrangian correspondences correspond to Lagrangian morphisms to $X \times \overline{Y}$: since quasicoherent sheaves on L form a stable ∞-category, producing a pullback square

$$\begin{tikzcd} \mathbb{T}_L \arrow[r] \arrow[d] & f^*\mathbb{T}_X \arrow[d] \\ g^*\mathbb{T}_Y \arrow[r] & \mathbb{L}_L[n] \end{tikzcd}$$

is the same thing as producing a fibre sequence

$$\begin{tikzcd} \mathbb{T}_L \arrow[r] \arrow[d] & f^*\mathbb{T}_X \oplus g^*\mathbb{T}_Y \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_L[n], \end{tikzcd}$$

where the right vertical map is the *difference* of the two maps to $\mathbb{L}_L[n]$ in the first square.

Proposition 14.12 ([Cal15, Theorem 4.4])**.** *Suppose X, Y, and Z are n-symplectic derived Artin stacks, and $X \xleftarrow{f} K \xrightarrow{g} Y$ and $Y \xleftarrow{h} L \xrightarrow{k} Z$ are Lagrangian correspondences. Then the composite span*

$$\begin{tikzcd} & K \times_Y L \arrow[dl] \arrow[dr] & \\ X & & Z \end{tikzcd}$$

is also a Lagrangian correspondence.

Proof. A Cartesian square

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[d, "\psi"'] & K \arrow[d, "g"] \\ L \arrow[r, "h"'] & Y \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathbb{T}_N \arrow[r] \arrow[d] & \phi^*\mathbb{T}_K \arrow[r] \arrow[d] & \phi^* f^*\mathbb{T}_X \arrow[d] \\
\psi^*\mathbb{T}_L \arrow[r] \arrow[d] & \phi^* g^*\mathbb{T}_Y \arrow[r] \arrow[d] & \phi^*\mathbb{L}_K[n] \arrow[d] \\
\psi^* k^*\mathbb{T}_Z \arrow[r] & \psi^*\mathbb{L}_L[n] \arrow[r] & \mathbb{L}_N[n].
\end{tikzcd}$$

Here the upper right square is Cartesian since $X \leftarrow K \rightarrow Y$ is a Lagrangian correspondence, and the bottom left square is Cartesian since $Y \leftarrow L \rightarrow Z$ is a Lagrangian correspondence. The top left square is Cartesian since N is the fibre product of K and L over Y. Finally, since Y is symplectic we have an equivalence $\phi^* g^*\mathbb{T}_Y \simeq \phi^* g^*\mathbb{L}_Y[n]$, and so we can identify the bottom right square with a shift of the dual of the top left square. The bottom right square is therefore coCartesian, but we're in a stable ∞-category so coCartesian and Cartesian squares coincide. Thus the boundary square in the diagram is also Cartesian, which by definition means that $X \leftarrow N \rightarrow Z$ is a Lagrangian correspondence. □

Definition 14.13. By Proposition 14.12, Lagrangian correspondences are closed under composition in the homotopy category of $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$. We can therefore define the ∞-category $\mathrm{Lag}^n_{(\infty,1)}$ to be the subcategory of $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$ whose objects are the n-symplectic derived Artin stacks and whose 1-morphisms are the Lagrangian correspondences between these.

Remark 14.14. The idea of considering symplectic derived stacks and Lagrangian correspondences as a subcategory of $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^p_{\mathrm{cl}}[n])$ is taken from [Sch14b].

Lemma 14.15. *The symmetric monoidal structure on* $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$ *induces a symmetric monoidal structure on* $\mathrm{Lag}^n_{(\infty,1)}$.

Proof. To show that $\mathrm{Lag}^n_{(\infty,1)}$ inherits a symmetric monoidal structure, it suffices to prove that it contains the unit of the symmetric monoidal structure on $\mathrm{Span}_1(\mathrm{dSt}_k; \mathcal{A}^2_{\mathrm{cl}}[n])$, and that its objects and morphisms are closed under this. The unit is the map $* \rightarrow \mathcal{A}^2_{\mathrm{cl}}[n]$ corresponding to 0, which is obviously symplectic. If X and Y are n-symplectic derived Artin stacks, then their tensor product is the Cartesian product $X \times Y$ equipped with the sum of the symplectic forms on X and Y, which is again symplectic. Finally, the tensor product of two Lagrangian correspondences is again their Cartesian product, which is Lagrangian with respect to the sum symplectic structures (since we just get the direct sum of the two Cartesian squares of quasi-coherent sheaves, which is again Cartesian). □

Proposition 14.16. *With respect to the induced symmetric monoidal structure, all n-symplectic derived Artin stacks are dualizable in* $\mathrm{Lag}^n_{(\infty,1)}$.

Proof. Let (X, ω) be an n-symplectic derived Artin stack. We must show that the dual of X is also an n-symplectic derived Artin stack, and the evaluation and coevaluation maps, as described in the proof of Proposition 13.2, are Lagrangian correspondences. By Example 13.6 the dual of X is $\bar{X}$, meaning X equipped with the negative $-\omega$ of its symplectic form. This is again symplectic, as the morphism $\mathbb{T}_X \rightarrow \mathbb{L}_X[n]$ induced by $-\omega$ is simply the negative of that induced by ω, and so is also an equivalence. The coevaluation map is given by the span $* \leftarrow X \xrightarrow{\Delta} \bar{X} \times X$, where Δ is the diagonal and $\bar{X} \times X$ is equipped with the sum symplectic structure $(-\omega, \omega)$. The induced diagram of quasi-coherent sheaves on X is

$$\begin{tikzcd} \mathbb{T}_X \arrow[r] \arrow[d] & \mathbb{T}_X \oplus \mathbb{T}_X \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_X[n], \end{tikzcd}$$

where the top horizontal map is $(-\mathrm{id}, \mathrm{id})$. This is Cartesian if and only if the square

$$\begin{tikzcd} \mathbb{T}_X \arrow[r] \arrow[d] & \mathbb{T}_X \arrow[d] \\ \mathbb{T}_X \arrow[r] & \mathbb{L}_X[n] \end{tikzcd}$$

is Cartesian, where the top horizontal and left vertical maps are both the identity, but this is true since X is symplectic as this means the other two maps, which are also identical, are equivalences. Thus this span is a Lagrangian correspondence. The evaluation map $\bar{X} \times X \xleftarrow{\Delta} X \to *$ is likewise a Lagrangian correspondence by a similar argument. □

Corollary 14.17. *Every n-sympectic derived Artin stack X determines a framed 1-dimensional TQFT*

$$\mathcal{Z}_X \colon \mathrm{Bord}_1^{\mathrm{fr}} \to \mathrm{Lag}^n_{(\infty,1)}.$$

REFERENCES

[Ati88] Michael Atiyah, *Topological quantum field theories*, Inst. Hautes Études Sci. Publ. Math. **68** (1988), 175–186 (1989).

[BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105.

[Bar05] Clark Barwick, *(∞, n)-Cat as a closed model category*, 2005. Thesis (Ph.D.)–University of Pennsylvania.

[Bar13a] ———, *On the Q-construction for exact ∞-categories* (2013), available at arXiv:1301.4725.

[Bar13b] ———, *From operator categories to topological operads* (2013), available at arXiv:1302.5756.

[Bar17] ———, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727, available at arXiv:1404.0108.

[BGS16] Clark Barwick, Saul Glasman, and Jay Shah, *Spectral Mackey functors and equivariant algebraic K-theory (II)* (2016), available at arXiv:1505.03098.

[BSP11] Clark Barwick and Christopher Schommer-Pries, *On the unicity of the homotopy theory of higher categories* (2011), available at arXiv:1112.0040.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR0420609

[Cal15] Damien Calaque, *Lagrangian structures on mapping stacks and semi-classical TFTs*, Stacks and categories in geometry, topology, and algebra, Contemp. Math., vol. 643, Amer. Math. Soc., Providence, RI, 2015, pp. 1–23, available at arXiv:1306.3235.

[CS15] Damien Calaque and Claudia Scheimbauer, *A note on the (∞, n)-category of cobordisms* (2015), available at arXiv:1509.08906.

[DS11] Daniel Dugger and David I. Spivak, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 263–325.

[DK12] Tobias Dyckerhoff and Mikhail Kapranov, *Higher Segal spaces I* (2012), available at arXiv:1212.3563.

[Fre94] Daniel S. Freed, *Higher algebraic structures and quantization*, Comm. Math. Phys. **159** (1994), no. 2, 343–398.

[FHLT10] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman, *Topological quantum field theories from compact Lie groups*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403.

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716, available at arXiv:1312.3178.

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), 1225–1266, available at arXiv:1501.02161.

[Gra07] Marco Grandis, *Higher cospans and weak cubical categories (Cospans in algebraic topology, I)*, Theory Appl. Categ. **18** (2007), No. 12, 321–347.

[Gro15] Moritz Groth, *A short course on ∞-categories* (2015), available at arXiv:1007.2925.

[Gro63] Alexander Grothendieck, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique, vol. 1960/61, Institut des Hautes Études Scientifiques, Paris, 1963.

[Hau15] Rune Haugseng, *Rectifying enriched ∞-categories*, Algebr. Geom. Topol. **15** (2015), 1931–1982, available at arXiv:1312.3178.

[Hau16] ———, *Bimodules and natural transformations for enriched ∞-categories*, Homology Homotopy Appl. **18** (2016), 71–98, available at arXiv:1506.07341.

[Hau17] ———, *The higher Morita category of E_n-algebras*, Geom. Topol. **21** (2017), 1631–1730, available at arXiv:1412.8459.

[Hof13] Alexander E. Hoffnung, *Spans in 2-categories: a monoidal tricategory* (2013), available at arXiv:1112.0560.

[JFS17] Theo Johnson-Freyd and Claudia Scheimbauer, *(Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories*, Adv. Math. **307** (2017), 147–223, available at arXiv:1502.06526.

[JT07] André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326, available at arXiv:math/0607820.

[Joy08] André Joyal, *The theory of quasi-categories and its applications*, Advanced course on simplicial methods in higher categories, CRM Quaderns, vol. 45-2, 2008, available at http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[Law93] R. J. Lawrence, *Triangulations, categories and extended topological field theories*, Quantum topology, Ser. Knots Everything, vol. 3, World Sci. Publ., River Edge, NJ, 1993, pp. 191–208.

[LB15] David Li-Bland, *The stack of higher internal categories and stacks of iterated spans* (2015), available at arXiv:1506.08870.

[Lur09a] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available at http://math.harvard.edu/~lurie/papers/highertopoi.pdf.

[Lur09b] ———, *(∞,2)-Categories and the Goodwillie Calculus I* (2009), available at http://math.harvard.edu/~lurie/papers/GoodwillieI.pdf.

[Lur09c] ———, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280, available at http://math.harvard.edu/~lurie/papers/cobordism.pdf.

[Lur17] ———, *Higher Algebra*, 2017. Available at http://math.harvard.edu/~lurie/.

[MG15] Aaron Mazel-Gee, *A user's guide to co/cartesian fibrations* (2015), available at arXiv:1510.02402.

[Mor09] Jeffrey Colin Morton, *Double bicategories and double cospans*, J. Homotopy Relat. Struct. **4** (2009), no. 1, 389–428, available at arXiv:math/0611930.

[Mor11] ———, *Two-vector spaces and groupoids*, Appl. Categ. Structures **19** (2011), no. 4, 659–707.

[Mor15] ———, *Cohomological Twisting of 2-Linearization and Extended TQFT*, J. Homotopy Relat. Struct. **10** (2015), no. 2, 127–187, available at arXiv:1003.5603.

[Nui13] Joost Nuiten, *Cohomological quantization of local prequantum boundary field theory* (2013), available at http://ncatlab.org/schreiber/show/master+thesis+Nuiten.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328.

[Rez01] Charles Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic).

[Rez10] ———, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571.

[Rez17] ———, *Stuff about quasicategories* (2017), available at http://www.math.uiuc.edu/~rezk/595-fal16/quasicats.

[RV16] Emily Riehl and Dominic Verity, *Homotopy coherent adjunctions and the formal theory of monads*, Adv. Math. **286** (2016), 802–888, available at arXiv:1310.8279.

[SP14] Christopher J. Schommer-Pries, *The Classification of Two-Dimensional Extended Topological Field Theories* (2014), available at arXiv:1112.1000.

[Sch13] Urs Schreiber, *Differential cohomology in a cohesive ∞-topos* (2013), available at arXiv:1310.7930.

[Sch14a] ———, *Quantization via linear homotopy types* (2014), available at arXiv:1402.7041.

[Sch14b] ———, *Classical field theory via cohesive homotopy types* (2014), available at http://ncatlab.org/schreiber/show/Classical+field+theory+via+Cohesive+homotopy+types.

[Sta16] Michael Stay, *Compact closed bicategories*, Theory Appl. Categ. **31** (2016), 755–798, available at arXiv:1301.1053.

[TV08] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry II: geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, available at arXiv:math/0404373.

[Wei81] Alan Weinstein, *Symplectic geometry*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 1, 1–13.

[Wei82] ———, *The symplectic "category"*, Differential geometric methods in mathematical physics (Clausthal, 1980), Lecture Notes in Math., vol. 905, Springer, Berlin-New York, 1982, pp. 45–51.

UNIVERSITY OF COPENHAGEN, COPENHAGEN, DENMARK
E-mail address: haugseng@math.ku.dk
URL: http://sites.google.com/site/runehaugseng/