ODD PRIMARY ANALOGS OF REAL ORIENTATIONS

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Abstract. We define, in C_p-equivariant homotopy theory for $p > 2$, a notion of μ_p-orientation analogous to a C_2-equivariant Real orientation. The definition hinges on a C_p-space $\mathbb{CP}^\infty_{\mu_p}$, which we prove to be homologically even in a sense generalizing recent C_2-equivariant work on conjugation spaces.

We prove that the height $p-1$ Morava E-theory is μ_p-oriented and that $\mathrm{tmf}(2)$ is μ_3-oriented. We explain how a single equivariant map $v_1^{\mu_p} : S^{2\rho} \to \Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$ completely generates the homotopy of E_{p-1} and $\mathrm{tmf}(2)$, expressing a height-shifting phenomenon pervasive in equivariant chromatic homotopy theory.

Contents

1. Introduction

The complex conjugation action on $\mathbb{CP}^\infty$ gives rise to a C_2-equivariant space, $\mathbb{CP}^\infty_{\mathbb{R}}$, with fixed points $\mathbb{RP}^\infty$. The subspace $\mathbb{CP}^1_{\mathbb{R}}$ is invariant and equivalent as a C_2-space to S^ρ, the one-point compactification of the real regular representation of C_2. A C_2-equivariant ring spectrum R is *Real oriented* if it is equipped with a map

$$\Sigma^\infty \mathbb{CP}^\infty_{\mathbb{R}} \to \Sigma^\rho R$$

such that the restriction

$$S^\rho = \Sigma^\infty \mathbb{CP}^1_{\mathbb{R}} \to \Sigma^\infty \mathbb{CP}^\infty_{\mathbb{R}} \to \Sigma^\rho R$$

is the Σ^ρ-suspension of the unit map $S^0 \to R$. Such a Real orientation induces a homotopy ring map

$$\mathrm{MU}_{\mathbb{R}} \to R,$$

with domain the spectrum of Real bordism [AM78, HK01]. These orientations have proved invaluable to the study of 2-local chromatic homotopy theory, leading to an explosion of progress surrounding the Hill–Hopkins–Ravenel solution of the Kervaire invariant one Problem [HHR16, GM17, HM17, KLW17, HLS18, HSWX19, BBHS19, LLQ20, LSWX19, HS20, BHSZ20, MSZ20].

The above papers solve problems, at the prime $p = 2$, that admit clear but often unapproachable analogs for odd primes. To give two examples, the 3 primary Kervaire problem remains unresolved [HHR11], and substantially less precise information is known about odd primary Hopkins–Miller EO-theories [BC20, Conjecture 1.12].

To rectify affairs at $p > 2$, the starting point must be to find a C_p-equivariant space playing the role of $\mathbb{CP}^\infty_{\mathbb{R}}$. This paper began as an attempt of the first two authors to understand a space proposed by the third.

Construction 1.1 (Wilson)**.** For any prime p, let $\mathbb{CP}^\infty_{\mu_p}$ denote the fiber of the C_p-equivariant multiplication map

$$(\mathbb{CP}^\infty)^{\times p} \to \mathbb{CP}^\infty,$$

where the codomain has trivial C_p-action. In other words, a map of spaces $X \to \mathbb{CP}^\infty_{\mu_p}$ consists of the data of:

- A p-tuple of complex line bundles $(\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_p)$ on X.
- A trivialization of the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_p$.

The action on $\mathbb{CP}^\infty_{\mu_p}$ is given by

$$(\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_p) \mapsto (\mathcal{L}_p, \mathcal{L}_1, \cdots, \mathcal{L}_{p-1}).$$

Remark 1.2. There is an equivalence of C_2-spaces $\mathbb{CP}^\infty_{\mu_2} \simeq \mathbb{CP}^\infty_{\mathbb{R}}$. In general, the non-equivariant space underlying $\mathbb{CP}^\infty_{\mu_p}$ is equivalent to $(\mathbb{CP}^\infty)^{\times p-1}$. The fixed points $\left(\mathbb{CP}^\infty_{\mu_p}\right)^{C_p}$ are equivalent to the classifying space BC_p, as can be seen by applying the fixed points functor $(-)^{C_p}$ to the defining fiber sequence for $\mathbb{CP}^\infty_{\mu_p}$. The key point here is that the C_p-fixed points of $(\mathbb{CP}^\infty)^{\times p}$ consist of the diagonal copy of $\mathbb{CP}^\infty$, and BC_p is the fiber of the pth tensor power map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$.

To formulate the notion of Real orientation, it is essential to understand the inclusion of the bottom cell

$$S^\rho = \mathbb{CP}^1_{\mathbb{R}} \to \mathbb{CP}^\infty_{\mathbb{R}}.$$

At an arbitrary prime, the analog of this bottom cell is described as follows:

Notation 1.3. We let $S^{\curlyeqsucc}$ denote the cofiber of the unique non-trivial map of pointed C_p-spaces from $(C_p)_+$ to S^0. This is the *spoke sphere*, and it is a wedge of $(p-1)$ copies of S^1 with action on reduced homology given by the augmentation ideal in the group ring $\mathbb{Z}[C_p]$. We denote the suspension $\Sigma S^{\curlyeqsucc}$ of the spoke sphere by either $S^{1+\curlyeqsucc}$ or $\mathbb{CP}^1_{\mu_p}$, and Remark 1.6 provides a natural inclusion

$$S^{1+\curlyeqsucc} = \mathbb{CP}^1_{\mu_p} \to \mathbb{CP}^\infty_{\mu_p}.$$

We will often also use $S^{1+\curlyeqsucc}$ to denote $\Sigma^\infty S^{1+\curlyeqsucc}$.

With this bottom cell in hand, we propose the following generalization of Real orientation theory:

Definition 1.4. A μ_p-orientation of a C_p-equivariant ring R is a map of spectra

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ} R$$

such that the composite

$$S^{1+\succ} = \Sigma^\infty \mathbb{CP}^1_{\mu_p} \to \Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ} R$$

is the $S^{1+\succ}$-suspension of the unit map $S^0 \to R$.

Remark 1.5. Applying the geometric fixed point functor Φ^{C_p} to a μ_p-orientation we learn that the non-equivariant spectrum $\Phi^{C_p} R$ has $p = 0$ in its homotopy groups.

Remark 1.6. Let $\underline{\mathbb{Z}} \coloneqq \mathrm{H}\underline{\mathbb{Z}}$ denote the C_p-equivariant Eilenberg–MacLane spectrum associated to the constant Mackey functor. Then there is an equivalence of C_p-equivariant spaces

$$\Omega^\infty \Sigma^{1+\succ} \underline{\mathbb{Z}} \simeq \mathbb{CP}^\infty_{\mu_p}.$$

Indeed, suspending and rotating the defining cofiber sequence $(C_p)_+ \to S^0 \to S^{\succ}$ gives rise to a cofiber sequence $S^{1+\succ} \to (C_p)_+ \otimes S^2 \to S^2$. Tensoring with $\underline{\mathbb{Z}}$ and applying Ω^∞ yields the defining fiber sequence for $\mathbb{CP}^\infty_{\mu_p}$.

Under this identification, the natural inclusion $\mathbb{CP}^1_{\mu_p} \to \mathbb{CP}^\infty_{\mu_p}$ is simply adjoint to the $\Sigma^{1+\succ}$-suspension of the unit map $S^0 \to \underline{\mathbb{Z}}$. In particular, the identification $\mathbb{CP}^\infty_{\mu_p} \simeq \Omega^\infty(\Sigma^{1+\succ}\underline{\mathbb{Z}})$ gives a canonical μ_p-orientation of $\underline{\mathbb{Z}}$. In contrast, Bredon cohomology with coefficients in the Burnside Mackey functor cannot be μ_p-oriented, since p is nonzero in the geometric fixed points.

In this paper we explore the interaction between μ_p-orientations and chromatic homotopy theory in the simplest possible case: chromatic height $p-1$. Specifically, we study the following height $p-1$ $\mathbb{E}_\infty$-ring spectra:

Notation 1.7. We let E_{p-1} denote the height $(p-1)$ Lubin–Tate theory associated to the Honda formal group law over $\mathbb{F}_{p^{p-1}}$, with C_p-action given by a choice of order p element in the Morava stabilizer group. At $p = 3$, we let $\mathrm{tmf}(2)$ denote the 3-localized connective ring of topological modular forms with full level 2 structure [Sto12]. The ring $\mathrm{tmf}(2)$ naturally admits an action by $\Sigma_3 \cong \mathrm{SL}_2(\mathbb{F}_2)$, and we restrict along an inclusion $C_3 \subset \Sigma_3$ to view $\mathrm{tmf}(2)$ as a C_3-equivariant ring spectrum.

The underlying homotopy groups of these spectra are given respectively by

$$\pi^e_*(E_{p-1}) \cong \mathbb{W}(\mathbb{F}_{p^{p-1}})[\![u_1, u_2, \cdots, u_{p-2}]\!][u^\pm], \ |u_i| = 0, |u| = -2, \text{ and}$$

$$\pi^e_*(\mathrm{tmf}(2)) \cong \mathbb{Z}_{(3)}[\lambda_1, \lambda_2], \ |\lambda_i| = 4.$$

We will review the C_p-actions on the homotopy groups in Section 5.

Theorem 1.8. *For all primes p, there exists a μ_p-orientation of the C_p-equivariant Morava E-theory E_{p-1}.*

Theorem 1.9. *The (3-localized) C_3-equivariant ring* $\mathrm{tmf}(2)$ *of topological modular forms with full level 2 structure admits a μ_3-orientation.*

Our second main result concerns the fact that, while

$$\pi_* E_{p-1} \cong \mathbb{W}(\mathbb{F}_{p^{p-1}})[\![u_1, u_2, \ldots, u_{p-2}]\!][u^\pm]$$

has $(p-1)$ distinct named generators, the conglomeration of them is generated under the μ_p-orientation by a *single* equivariant map $v_1^{\mu_p}$.

Construction 1.10. In Section 6, we will construct a map of C_p-equivariant spectra

$$v_1^{\mu_p} : S^{2\rho} \to \Sigma^\infty \mathbb{CP}^\infty_{\mu_p}.$$

This map should be viewed as canonical only up to some indeterminacy, just as the classical class v_1 is only well-defined modulo p. As was pointed out to the authors by Mike Hill, one choice of this map is given by *norming* a non-equivariant class in $\pi_2^e \mathbb{CP}^\infty_{\mu_p}$.

Construction 1.11. Suppose a C_p-equivariant ring R is μ_p-oriented via a map

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ} R,$$

so that we may consider the composite

$$S^{2\rho} \xrightarrow{v_1^{\mu_p}} \Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \longrightarrow \Sigma^{1+\succ} R.$$

Using the dualizability of $S^{1+\succ}$, this composite is equivalent to the data of a map

$$S^{2\rho-1-\succ} \to R.$$

The non-equivariant spectrum underlying $S^{2\rho-1-\succ}$ is (non-canonically) equivalent to a direct sum of $p-1$ copies of S^{2p-2}. In particular, by applying π^e_{2p-2} to the map $S^{2\rho-1-\succ} \to R$, one obtains a map from a rank $p-1$ free $\mathbb{Z}_{(p)}$-module to $\pi^e_{2p-2}R$.

Definition 1.12. Given a C_p-equivariant ring R with a μ_p-orientation, the *span of* $v_1^{\mu_p}$ will refer to the subset of $\pi^e_{2p-2}R$ consisting of the image of the rank $p-1$ free $\mathbb{Z}_{(p)}$-module constructed above.

Theorem 1.13. *For any μ_3-orientation of* $\mathrm{tmf}(2)$*, the span of $v_1^{\mu_3}$ in $\pi_4^e\mathrm{tmf}(2)$ is all of* $\pi_4^e\mathrm{tmf}(2)$.

Theorem 1.14. *For any μ_p-orientation of the height $p-1$ Morava E-theory E_{p-1}, the span of $v_1^{\mu_p}$ inside $\pi^e_{2p-2}E_{p-1}$ maps surjectively onto $\pi^e_{2p-2}E_{p-1}/(p, \mathfrak{m}^2)$.*

1.1. **Homological and homotopical evenness**

Non-equivariantly, complex orientation theory is intimately tied to the notion of evenness. A fundamental observation is that, since $\mathbb{CP}^\infty$ has a cell decomposition with only even-dimensional cells, any ring R with $\pi_{2*-1}R \cong 0$ must be complex orientable.

In C_2-equivariant homotopy theory, a ring R is called *even* if $\pi^{C_2}_{*\rho-1}R \cong \pi^e_{2*-1}R \cong 0$, and it is a basic fact that any even ring is Real orientable [HM17, §3.1].

In C_p-equivariant homotopy theory, we propose the appropriate notion of evenness to be captured by the following definition, which we discuss in more detail in Section 3:

Definition 1.15. We say that a C_p-equivariant spectrum E is *homotopically even* if the following conditions hold for all $n \in \mathbb{Z}$:

(1) $\pi^e_{2n-1}E = 0$.
(2) $\pi^{C_p}_{2n\rho-1}E = 0$
(3) $\pi^{C_p}_{2n\rho-2-\succ}E = 0$

Remark 1.16. A C_2-spectrum E is homotopically even, according to our definition above, if and only if it is even in the sense of [HM17, §3.1].

We prove the following theorem in Section 4.

Theorem 1.17. *If a p-local C_p-ring spectrum R is homotopically even, then it is also μ_p-orientable.*

The key point here, as we explain in Section 4, is that $\mathbb{CP}^{\infty}_{\mu_p}$ admits a *slice cell decomposition* with even slice cells. An even more fundamental fact, which turns out to be equivalent to the slice cell decomposition, is a splitting of the *homology* of $\mathbb{CP}^{\infty}_{\mu_p}$:

Definition 1.18. We say that a C_p-spectrum X is *homologically even* if there is a direct sum splitting

$$X \otimes \underline{\mathbb{Z}}_{(p)} \simeq \bigoplus_k A_k \otimes \underline{\mathbb{Z}}_{(p)},$$

where each A_k is equivalent, for some $n \in \mathbb{Z}$, to one of

$$(C_p)_+ \otimes S^{2n},\ S^{2n\rho},\ S^{2n\rho+1+\curlyvee}.$$

Theorem 1.19. *The space* $\mathbb{CP}^{\infty}_{\mu_p}$ *is homologically even.*

Remark 1.20. The notion of homological evenness we propose in this paper restricts, when $p = 2$, to the notion studied by Hill in [Hil19, Definition 3.2]. Notably, our definition differs from Hill's when $p > 2$.

Returning again to the group C_2, work of Pitsch, Ricka, and Scherer relates a version of homological evenness to the study of *conjugation spaces* [PRS19]. An interesting example of a conjugation space, generalized in [HH18] and its in-progress sequel, is $\mathrm{BU}_{\mathbb{R}} = \Omega^\infty\Sigma^\rho \mathrm{BP}\langle 1\rangle_{\mathbb{R}}$. It would be very interesting to develop a C_p-equivariant version of conjugation space theory. Since $\mathrm{tmf}(2)$ is a form of $\mathrm{BP}\langle 1\rangle_{\mu_3}$ (cf. Question 7), we wonder whether there is an interesting slice cell decomposition of $\Omega^\infty\Sigma^{1+\curlyvee}\mathrm{tmf}(2)$.

1.2. A view to the future

The most natural next question, after those tackled in this paper, is the following:

Question 1. Let $n \geqslant 1$, and fix a formal group Γ of height $n(p-1)$ over a perfect field k of characteristic p. When is the associated Lubin–Tate theory $E_{k,\Gamma}$ μ_p-orientable?

We have not fully answered this question even for $n = 1$, since we focus attention on the Honda formal group.

It seems likely that further progress on Question 1, at least for $n \geqslant 2$, must wait for work in progress of Hill–Hopkins–Ravenel, who have a program by which to understand the C_p-action on Lubin–Tate theories. As the authors understand that work in progress, it is to be expected that the height $n(p-1)$ Morava E-theory has homotopy generated by n copies of the reduced regular representation, $v_1^{\mu_p}, v_2^{\mu_p}, \cdots, v_n^{\mu_p}$. One expects to be able to construct μ_p Morava K-theories, generated by a single $v_i^{\mu_p}$, and we expect at least these Morava K-theories to be homotopically even in the sense of this paper.

Question 2. Can one construct homotopically even μ_p Morava K-theories?

In light of the orientation theory of Section 2, it seems useful to know if μ_p Morava K-theories admit *norms*. Indeed, at $p = 2$ the Real Morava K-theories all admit the structure of $\mathbb{E}_\sigma$-algebras. Since the first μ_3 Morava K-theory should be $\mathrm{TMF}(2)/3$, or perhaps $L_{K(2)}\mathrm{TMF}(2)/3$, it seems pertinent to answer the following question first:

Question 3. At the prime $p = 3$, what structure is carried by the C_3-equivariant spectrum $L_{K(2)}\mathrm{TMF}(2)/3$? Is there an analog of the $\mathbb{E}_\sigma$ structure carried by $\mathrm{KU}_{\mathbb{R}}/2$?

In another direction, one might ask about other finite subgroups of Morava stabilizer groups:

Question 4. Is there an analog of the notion of μ_p-orientation related to the Q_8-actions on Lubin–Tate theories at the prime 2?

One may also go beyond finite groups and ask for notions capturing other parts of the Morava stabilizer group, such as the central $\mathbb{Z}_p^\times$ that acts on $\mathbb{CP}^\infty \simeq \mathrm{B}^2\mathbb{Z}_p$ after p-completion.

To make full use of all these ideas, one would like not only an analog of $\mathbb{CP}^\infty_{\mathbb{R}}$, but also an analog of at least one of $\mathrm{MU}_{\mathbb{R}}$ or $\mathrm{BP}_{\mathbb{R}}$. Attempts to construct such analogs have consumed the authors for many years; we consider it one of the most intriguing problems in stable homotopy theory today.

Question 5. (Hill–Hopkins–Ravenel [HHR11]) Does there exist a natural C_p-ring spectrum, BP_{μ_p}, with

- Underlying, non-equivariant spectrum the smash product of $(p-1)$ copies of BP.
- Geometric fixed points $\Phi^{C_p}\mathrm{BP}_{\mu_p} \simeq \mathrm{H}\mathbb{F}_p$.

At $p = 2$, it should be the case that $\mathrm{BP}_{\mu_2} = \mathrm{BP}_{\mathbb{R}}$.

To the above we may add:

Question 6. Does such a natural BP_{μ_p} orient all μ_p-orientable C_p-ring spectra, or at least all those that admit norms in the sense of Section 2?

Most of our attempts to build BP_{μ_p} have proceeded via obstruction theory, while $\mathrm{MU}_{\mathbb{R}}$ is naturally produced via geometry. It would be extremely interesting to see a geometric definition of an object MU_{μ_p}. Alternatively, it would be very clarifying if one could prove that a reasonable BP_{μ_p} does not exist. As some evidence in that direction, the authors doubt any variant of BP_{μ_p} can be homotopically even.

Even if BP_{μ_p} cannot be built, or cannot be built easily, it would be excellent to know whether it is possible to build C_p-ring spectra $\mathrm{BP}\langle 1\rangle_{\mu_p}$.

Question 7. Does there exist, for each prime p, a C_p-ring $\mathrm{BP}\langle 1\rangle_{\mu_p}$ satisfying the following properties:

- $\mathrm{BP}\langle 1\rangle_{\mu_2}$ is the 2-localization of $\mathrm{ku}_{\mathbb{R}}$, and $\mathrm{BP}\langle 1\rangle_{\mu_3}$ is the 3-localization of $\mathrm{tmf}(2)$.
- The homotopy groups are given by
$$\pi_*^e \mathrm{BP}\langle 1\rangle_{\mu_p} \cong \mathbb{Z}_{(p)}[\lambda_1, \lambda_2, \cdots, \lambda_{p-1}],$$
with $|\lambda_i| = 2p-2$. The C_p action on these generators should make $\pi^e_{2p-2}\mathrm{BP}\langle 1\rangle_{\mu_p}$ into a copy of the reduced regular representation.
- There is a C_p-ring map $\mathrm{BP}\langle 1\rangle_{\mu_p} \to E_{p-1}$.
- $\mathrm{BP}\langle 1\rangle_{\mu_p}$ is homotopically even, and in particular μ_p-orientable.
- The underlying spectrum $\left(\mathrm{BP}\langle 1\rangle_{\mu_p}\right)^e$ additively splits into a wedge of suspensions of $\mathrm{BP}\langle p-1\rangle$.
- We have $\Phi^{C_p}\mathrm{BP}\langle 1\rangle_{\mu_p} \simeq \mathbb{F}_p[y]$ for a generator y of degree $2p$.

It is plausible that $\mathrm{BP}\langle 1\rangle_{\mu_p}$ should come in many forms, in the sense of Morava's forms of K-theory [Mor89]. A natural $\mathbb{E}_\infty$ form might be obtained by studying compactifications of the Gorbounov–Hopkins–Mahowald stack [GM00, Hil06] of curves of the form
$$y^{p-1} = x(x-1)(x-a_1)\cdots(x-a_{p-2}).$$
Studying the uncompactified stack, it is possible to construct a C_p-equivariant $\mathbb{E}_\infty$ ring $E(1)_{\mu_p}$ which is a μ_p analog of uncompleted Johnson-Wilson theory.

Remark 1.21. The C_p-action on $\mathbb{CP}^\infty_{\mu_p}$ is naturally the restriction of an action by Σ_p. In fact, most objects in this paper admit actions of Σ_p, or at least of $C_{p-1} \ltimes C_p$, but these are consistently ignored. The reader is encouraged to view this as an indication that the theory remains in flux, and welcomes further refinement.

Remark 1.22. Since work of Quillen [Qui69], the notion of a complex orientation has been intimately tied to the notion of a formal group law. There are hints throughout this paper, particularly in Section 2 and Section 6, that the norm and diagonal maps on $\mathbb{CP}^\infty_{\mu_p}$ lead to equivariant refinements of the p-series of a formal group. It may be interesting to develop the purely algebraic theory underlying these constructions, particularly if algebraically defined $v_i^{\mu_p}$ turn out to be of relevance to higher height Morava E-theories.

1.3. **Notation and Conventions**

- If X is a C_p-space, we use X^e to denote the underlying non-equivariant space, and we use X^{C_p} to denote the fixed point space. If X is a C_p-spectrum, we will use either $\Phi^e X$ or X^e to denote the underlying spectrum, and we use $\Phi^{C_p} X$ to denote the geometric fixed points.
- We fix a prime number p, and throughout the paper all spectra and all (nilpotent) spaces are implicitly p-localized. In the C_p-equivariant setting, this means that we implicitly p-localize both underlying and fixed point spaces and spectra.
- If X is a C_p-space or spectrum, we use $\pi^e_* X$ to denote the homotopy groups of X^e, considered as a graded abelian group *with C_p-action.* If V is a C_p-representation, we use $\pi^{C_p}_V X$ to denote the set of homotopy classes of equivariant maps from S^V to X.
- We let $S^{\succ}$ denote the cofiber of the C_p-equivariant map $(C_p)_+ \to S^0$, and we also use $S^{\succ}$ to refer to the suspension C_p-spectrum of this C_p-space. We let $S^{-\succ}$ denote the Spanier-Whitehead dual of the C_p-spectrum $S^{\succ}$. Given a C_p-representation V and a C_p-spectrum X, we will use $\pi^{C_p}_{V+\succ} X$ and $\pi^{C_p}_{V-\succ}$ to denote the set of homotopy classes of equivariant maps from $S^{V+\succ} := S^V \otimes S^{\succ}$ and $S^{V-\succ} := S^V \otimes S^{-\succ}$ to X.
- We let $\mathbb{CP}^\infty_{\mu_p}$ denote the fiber of the C_p-equivariant multiplication map $(\mathbb{CP}^\infty)^{\times p} \to \mathbb{CP}^\infty$.
- If R is a classical commutative ring, we use $\bar{\rho}_R$ to denote the $R[C_p]$-module given by the augmentation ideal $\ker(R[C_p] \to R)$. This is a rank $p-1$ R-module with generators permuted by the reduced regular representation of C_p. We similarly use $\mathbb{1}_R$ to denote the $R[C_p]$-module that is isomorphic to R with trivial action. We sometimes use ρ_R to denote $R[C_p]$ itself, and write free to denote a sum of copies of ρ_R.

1.4. **Acknowledgments**

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2. Orientation theory

Non-equivariantly, one may study complex orientations of any unital spectrum R. However, if R is further equipped with the a homotopy commutative multiplication, then the theory takes on extra significance: in this case, a complex orientation of R provides an isomorphism $R^*(\mathbb{CP}^\infty) \cong R_*[[x]]$.

In this section, we work out the analogous theory for μ_p-orientations. In particular, we find that the theory of μ_p-orientations takes on special significance for C_p homotopy ring spectra R that are equipped with a *norm* $N_e^{C_p} R \to R$ refining the underlying multiplication. Recall the following definition from the introduction:

Definition 2.1. A μ_p-*orientation* of a unital C_p-spectrum R is a map

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \longrightarrow \Sigma^{1+\succ} R$$

such that the composite

$$S^{1+\succ} \longrightarrow \Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \longrightarrow \Sigma^{1+\succ} R$$

is equivalent to $\Sigma^{1+\succ}$ of the unit.

For any C_p representation sphere S^V, it is traditional to denote by $S^0[S^V]$ the free $\mathbb{E}_1$-ring spectrum

$$S^0[S^V] = S^0 \oplus S^V \oplus S^{2V} \oplus S^{3V} \oplus \cdots$$

Below, we extend this construction to take input not only representation spheres S^V, but spoke spheres as well.

Definition 2.2. For integers n, let

$$S^0[S^{2n\rho-1-\succ}] \coloneqq \mathrm{N}_e^{C_p}(S^0[S^{2np-2}]) \otimes_{S^0[S^{2n\rho-2}]} S^0,$$

where we consider $\mathrm{N}_e^{C_p} S^0[S^{2np-2}]$ as a $S^0[S^{2n\rho-2}]$-bimodule via the E_1-map induced by the composite

$$S^{2n\rho-2} \to (C_p)_+ \otimes S^{2np-2} \to \mathrm{N}_e^{C_p} S^0[S^{2np-2}].$$

In this composite, the first map is adjoint to the identity on S^{2np-2} and the second map is the canonical inclusion. Note that $S^0[S^{2n\rho-1-\succ}]$ is a unital left module over $\mathrm{N}_e^{C_p} S^0[S^{2np-2}]$.

Furthermore, given a C_p-equivariant spectrum R, we set

$$R[S^{2n\rho-1-\succ}] \coloneqq R \otimes S^0[S^{2n\rho-1-\succ}].$$

Construction 2.3. Suppose that R is a homotopy ring in C_p-spectra, further equipped with a genuine norm map

$$\mathrm{N}_e^{C_p} R \to R$$

which is unital and restricts on underlying spectra to the composite

$$(\Phi^e R)^{\otimes p} \xrightarrow{\mathrm{id} \otimes \gamma \otimes \cdots \otimes \gamma^{p-1}} (\Phi^e R)^{\otimes p} \xrightarrow{m} \Phi^e R,$$

where $\gamma \in C_p$ is the generator and m is the p-fold multiplication map.

If R is μ_p-oriented by a map

$$S^{-1-\succ} \to R^{\mathbb{CP}^\infty_{\mu_p +}}$$

then we may produce a map

$$R[S^{-1-\succ}] \to R^{\mathbb{C}P^\infty_{\mu_p +}}$$

as follows. First, the composite

$$S^{-2} \xrightarrow{e} \Phi^e(C_{p+} \wedge S^{-2}) \to \Phi^e(S^{-1-\succ}) \to \Phi^e(R^{\mathbb{CP}^\infty_{\mu_p +}}),$$

where the map e is the inclusion of the factor of S^{-2} corresponding to the identity in C_p, extends to a map

$$S^0[S^{-2}] \to \Phi^e(R^{\mathbb{CP}^\infty_{\mu_p +}})$$

since the target is a homotopy ring. Norming up, and combining the norm on R with the diagonal map $\mathbb{CP}^\infty_{\mu_p} \to \mathrm{Map}(C_p, \mathbb{CP}^\infty_{\mu_p})$, we get a map

$$\mathrm{N}_e^{C_p}(S^0[S^{-2}]) \to \mathrm{N}_e^{C_p}(R^{\mathbb{CP}^\infty_{\mu_p+}}) \to R^{\mathbb{CP}^\infty_{\mu_p+}}.$$

Finally, the extension of $C_{p+} \wedge S^{-2} \to R^{\mathbb{CP}^\infty_{\mu_p+}}$ over $S^{-1-\succ}$ provides a nullhomotopy of the composite

$$S^{-2} \to (C_p)_+ \otimes S^{-2} \to R^{\mathbb{CP}^\infty_{\mu_p+}},$$

producing a map

$$S^0[S^{-1-\succ}] \to R^{\mathbb{CP}^\infty_{\mu_p+}}.$$

We finish by extending scalars to R.

Construction 2.4. If R is μ_p-oriented then so too is the Postnikov truncation $R_{\leqslant n}$. The construction above is natural, and so we may form a map

$$R[\![S^{-1-\succ}]\!] := \varprojlim R_{\leqslant n}[S^{-1-\succ}] \to \varprojlim (R_{\leqslant n})^{\mathbb{CP}^\infty_{\mu_p+}} \simeq R^{\mathbb{CP}^\infty_{\mu_p+}}.$$

Theorem 2.5. *Suppose R is a μ_p-oriented homotopy C_p ring, further equipped with a unital homotopy $N_e^{C_p}R$-module structure such that the unit*

$$N_e^{C_p}R \to R$$

respects the underlying multiplication in the sense of Construction 2.3. Then, with notation as above, the map

$$R[\![S^{-1-\succ}]\!] \longrightarrow R^{\mathbb{CP}^\infty_{\mu_p+}}$$

is an equivalence.

Proof. By construction, it suffices to prove that the map

$$R_{\leqslant n}[S^{-1-\succ}] \longrightarrow (R_{\leqslant n})^{\mathbb{CP}^\infty_{\mu_p+}}$$

is an equivalence for each $n \geqslant 0$. This is clear on underlying spectra. On geometric fixed points we can factor this map as

$$(\Phi^{C_p}R_{\leqslant n})[S^{-1}] \to (\Phi^{C_p}R_{\leqslant n})^{\mathrm{B}C_{p+}} \to \Phi^{C_p}\left((R_{\leqslant n})^{\mathbb{CP}^\infty_{\mu_p+}}\right),$$

being careful to interpret the source as a module (this is not a map of rings). Specifically, the above composite is one of unital $\Phi^{C_p}N_e^{C_p}S^0[S^{-2}] \simeq S^0[S^{-2}]$-modules and, separately, one of $\Phi^{C_p}R_{\leqslant n}$-modules.

The second map is an equivalence by Lemma 2.6 below, so we need only prove the first map is an equivalence. Since $p = 0$ in $\Phi^{C_p}R$, the Atiyah-Hirzebruch spectral sequence computing $\pi_*(\Phi^{C_p}R_{\leqslant n})^{BC_{p+}}$ has E_2-page given by

$$\pi_*(\Phi^{C_p}R_{\leqslant n}) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(x) \otimes_{\mathbb{F}_p} \mathbb{F}_p[y]$$

The class x is realized by applying geometric fixed point to the μ_p-orientation. The powers of y are obtained from the unit of the unital $S^0[S^{-2}]$-module structure. Using the $\Phi^{C_p}R_{\leqslant n}$-module structure, this implies that the spectral sequence degenerates and moreover that the first map is an equivalence. □

Lemma 2.6. *If R is bounded above, and X is a C_p-space of finite type, then the map*

$$(\Phi^{C_p}R)^{X_+^{C_p}} \to \Phi^{C_p}(R^{X_+})$$

is an equivalence.

Proof. Write $X = \operatorname{colim} X_n$ where the X_n are skeleta for a C_p-CW-structure on X with each X_n finite. Then the fiber of

$$\Phi^{C_p}(R^{X_+}) \to \Phi^{C_p}(R^{X_{n+}})$$

becomes increasingly coconnective, and hence the map

$$\Phi^{C_p}(R^{X_+}) \to \varprojlim \Phi^{C_p}(R^{X_{n+}})$$

is an equivalence. We are thus reduced to the case $X = X_n$ finite, where the result follows since $\Phi^{C_p}(-)$ is exact. □

Since $\underline{\mathbb{Z}}$ is μ_p-oriented by Remark 1.6 and truncated, we have the following corollary of Theorem 2.5:

Corollary 2.7. *There is a natural equivalence*

$$\underline{\mathbb{Z}}[S^{-1-\lambda}] \simeq \underline{\mathbb{Z}}^{\mathbb{CP}^{\infty}_{\mu_p +}}.$$

3. Evenness

In this section, we will introduce a notion of *evenness* in C_p-equivariant homotopy theory. This is a generalization of the notion of evenness in non-equivariant homotopy theory. Evenness comes in two forms: *homological* evenness and *homotopical* evenness. Homological evenness is a C_p-equivariant version of the condition that a spectrum have homology concentrated in even degrees, and homotopical evenness corresponds to the condition that a spectrum have homotopy concentrated in even degrees.

The main results in this section are Proposition 3.9, which shows that, under certain conditions, a bounded below homologically even spectrum admits a cell decomposition into *even slice spheres* (defined below), and Proposition 3.16, which shows that there are no obstructions to mapping in a bounded below homologically even spectrum to a homotopically even spectrum.

3.1. Homological Evenness

We begin our discussion of evenness with the definition of an even slice sphere.

Definition 3.1. We say that a C_p-equivariant spectrum is an *even slice sphere* if it is equivalent to one of the following for some $n \in \mathbb{Z}$:

$$(C_p)_+ \otimes S^{2n},\ S^{2n\rho},\ S^{2n\rho+1+\lambda}.$$

A *dual even slice sphere* is the dual of an even slice sphere. The *dimension* of a (dual) even slice sphere is the dimension of its underlying spectrum.

Remark 3.2. The phrase *slice sphere* is taken from [Wil17b, Definition 2.3], where a G-equivariant slice sphere is defined to be a compact G-equivariant spectrum, each of whose geometric fixed point spectra is a finite direct sum of spheres of a given dimension.

It is easy to check that the (dual) even slice spheres of Definition 3.1 are slice spheres in this sense.

Remark 3.3. In the case $p = 2$, the even slice spheres are precisely those of the form

$$(C_2)_+ \otimes S^{2n} \text{ or } S^{n\rho}$$

for some $n \in \mathbb{Z}$.

Definition 3.4. We say that a C_p-equivariant spectrum X is *homologically even* if there is an equivalence of $\underline{\mathbb{Z}}_{(p)}$-modules

$$X \otimes \underline{\mathbb{Z}}_{(p)} \simeq \bigoplus_n S_n \otimes \underline{\mathbb{Z}}_{(p)},$$

where S_n is a direct sum of even slice spheres of dimension $2n$.

Remark 3.5. When $p = 2$, this recovers the notion of homological purity given in [Hil19, Definition 3.2]. However, when p is odd, our definition of homological evenness differs from Hill's definition of homological purity. The most important difference is that we allow the spoke spheres $S^{2n\rho+1+\succ}$ to appear in our definition. This is necessary for $\mathbb{CP}^\infty_{\mu_p}$ to be homologically even.

As in the non-equivariant case, homological evenness for a bounded below spectrum is equivalent to the existence of an even cell structure. To prove this, we need to recall the following definition:

Definition 3.6. We say that a C_p-equivariant spectrum X is *regular slice n-connective* if:

(1) X^e is n-connective, and
(2) $\Phi^{C_p} X$ is $\lceil \frac{n}{p} \rceil$-connective.

Furthermore, we say that X is *bounded below* if it is regular slice n-connective for some integer n.

Lemma 3.7. *Let X be a bounded below C_p-spectrum with the property that $\Phi^{C_p} X$ is of finite type. Then X is regular slice n-connective if and only if $X \otimes \underline{\mathbb{Z}}_{(p)}$ is regular slice n-connective.*

Proof. For the underlying spectrum, the follows from the fact that $\mathbb{Z}_{(p)}$ detects connectivity of bounded below p-local spectra. For the geometric fixed points, we use the fact that $\Phi^{C_p} \underline{\mathbb{Z}}_{(p)} = \mathbb{F}_p[y]$, $|y| = 2$, detects connectivity of bounded below p-local spectra which are of finite type, since a finitely generated $\mathbb{Z}_{(p)}$-module is trivial if and only if it is trivial after tensoring with $\mathbb{F}_p$. □

Lemma 3.8. *Let W denote an even slice sphere of dimension n, and suppose that X is regular slice n-connective. Then we have $[W, \Sigma X] = 0$.*

Proof. If W is of dimension n, then its underlying spectrum W^e is a direct sum of n-spheres and $\Phi^{C_p} W$ is a $\lceil \frac{n}{p} \rceil$-sphere. It therefore follows that W is a regular slice n-sphere in the sense of [Wil17b, §2.1], so the conclusion follows from [Wil17b, Proposition 2.22]. □

Proposition 3.9. *Suppose that X is a bounded below, homologically even C_p-equivariant spectrum with the property that $\Phi^{C_p} X$ is of finite type, so that there exists a splitting*

$$X \otimes \underline{\mathbb{Z}}_{(p)} \simeq \bigoplus_{k \geqslant n} S_k \otimes \underline{\mathbb{Z}}_{(p)},$$

where S_k is a direct sum of $2k$-dimensional even slice spheres. Then X admits a filtration $\{X_k\}_{k \geqslant n}$ such that $X_k / X_{k-1} \simeq S_k$ for each $k \geqslant n$.

Proof. By assumption, we are given a splitting

$$X \otimes \underline{\mathbb{Z}}_{(p)} \simeq \bigoplus_{k \geqslant n} S_k \otimes \underline{\mathbb{Z}}_{(p)},$$

where S_k is a direct sum of $2k$-dimensional even slice spheres. By induction on n, it will suffice to show that the dashed lifting exists in the diagram

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ S_n & \longrightarrow & \bigoplus_{k \geqslant n} S_k \otimes \underline{\mathbb{Z}}_{(p)} \simeq X \otimes \underline{\mathbb{Z}}_{(p)}, \end{array}$$

since the cofiber of any such lift is a bounded below homologically even C_p-spectrum with $\Phi^{C_p} X$ of finite type and whose $\underline{\mathbb{Z}}_{(p)}$-homology is $\bigoplus_{k \geqslant n+1} S_k \otimes \underline{\mathbb{Z}}_{(p)}$.

Note that Lemma 3.7 implies that X is regular slice $2n$-connected. Let F be the fiber of the Hurewicz map $S^0 \to \underline{\mathbb{Z}}_{(p)}$. Then F is easily seen to be regular slice 0-connective, so that $F \otimes X$ is regular slice $2n$-connective. This implies that $[S_n, \Sigma F \otimes X] = 0$ by Lemma 3.8. The result now follows from the cofiber sequence

$$X \to \underline{\mathbb{Z}}_{(p)} \otimes X \to \Sigma F \otimes X. \qquad \square$$

Remark 3.10. It will follow from Example 3.15 and Proposition 3.17 that the following converse of Proposition 3.9 holds: if X is bounded below and admits an even slice cell structure, then X is homologically even.

3.2. **Homotopical Evenness**

We now introduce the homotopical version of evenness.

Definition 3.11. We say that a C_p-equivariant spectrum E is *homotopically even* if the following conditions hold for all $n \in \mathbb{Z}$:

(1) $\pi^{e}_{2n-1} E = 0$.
(2) $\pi^{C_p}_{2n\rho-1} E = 0$.
(3) $\pi^{C_p}_{2n\rho-2-\succ} E = 0$.

Remark 3.12. All of the examples of homotopically even C_p-spectra that we will encouter will satisfy the following condition for all $n \in \mathbb{Z}$:

(4) $\pi^{C_p}_{2n\rho+\succ} E = 0$.

We will say that a homotopically even C_p-spectrum *satisfies condition (4)* if this holds.

In fact, the examples which we study satisfy even stronger evenness properties. We have chosen the weakest possible set of properties for which our theorems hold.

Remark 3.13. If we assume condition (1), then we may rewrite conditions (3) and (4) as follows:

(3′) the transfer maps $\pi^{e}_{2np-2} E \to \pi^{C_p}_{2n\rho-2} E$ are surjective for all $n \in \mathbb{Z}$.
(4′) the restrction maps $\pi^{C_p}_{2n\rho} E \to \pi^{e}_{2np} E$ are injective for all $n \in \mathbb{Z}$.

This follows directly from the cofiber sequences defining $S^{-\succ}$ and $S^{\succ}$:

$$S^{-\succ} \to S^0 \xrightarrow{\text{tr}} (C_p)_+ \otimes S^0$$

$$(C_p)_+ \otimes S^0 \xrightarrow{\text{res}} S^0 \to S^{\succ}.$$

Remark 3.14. If $p = 2$, Definition 3.11 reduces to the requirement that, for all $n \in \mathbb{Z}$:

(1) $\pi^{e}_{2n-1} E = 0$.
(2) $\pi^{C_2}_{n\rho-1} E = 0$.

A C_2-equivariant spectrum is therefore homotopically even if and only if it is *even* in the sense of [HM17, Definition 3.1]. Moreover, condition (4) is redundant in the C_2-equivariant setting.

Example 3.15. The Eilenberg–Maclane spectra $\underline{\mathbb{F}}_p$ and $\underline{\mathbb{Z}}_{(p)}$ are examples of homotopically even C_p-spectra which satisfy condition (4). To verify this, we refer to the reader to the appendix of third author's thesis [Wil17a, §A], where one may find a computation of the slice graded homotopy groups of $\underline{\mathbb{F}}_p$ and $\underline{\mathbb{Z}}_{(p)}$.

At the prime $p = 2$, there are many examples of homotopically even C_2-spectra in the literature, such as $\mathrm{MU}_{\mathbb{R}}, \mathrm{BP}_{\mathbb{R}}, \mathrm{BP}\langle n\rangle_{\mathbb{R}}, E(n)_{\mathbb{R}}$, $K(n)_{\mathbb{R}}$ and E_n, where E_n is equipped with the Goerss-Hopkins C_2-action [HM17, HS20].

The main result of Section 5 is that the C_p-spectra E_{p-1} and the C_3-spectrum tmf(2) are homotopically even and satisfy condition (4).

When trying to map a bounded below homologically even C_p-spectrum into a homotopically even C_p-spectrum, there are no obstructions:

Proposition 3.16. *Let E be a homotopically even C_p-spectrum, and suppose that X is a C_p-spectrum equipped with a bounded below filtration $\{X_k\}_{k\geqslant n}$ such that each $S_k := X_k/X_{k-1}$ is a direct sum of $2k$-dimensional even slice spheres.*

Then, for any $k \geqslant n$, every C_p-equivariant map $X_k \to E$ extends to an equivariant map $X \to E$.

Proof. It suffices to prove by induction that any map $X_k \to E$ extends to a map $X_{k+1} \to E$. Using the cofiber sequence

$$\Sigma^{-1} S_{k+1} \to X_k \to X_{k+1},$$

we just need to know that any map from the desuspension of an even slice sphere into E is nullhomotopic. This follows precisely from the definition of homotopical evenness. ☐

If E further satisfies condition (4), we have the stronger result:

Proposition 3.17. *Let E be a homotopically even C_p-ring spectrum which satisfies condition (4), and suppose that X is a C_p-spectrum equipped with a bounded below filtration $\{X_k\}_{k\geqslant n}$ such that each $S_k := X_k/X_{k-1}$ is a direct sum of $2k$-dimensional even slice spheres.*

Then there is a splitting of the induced filtration on $X \otimes E$ by E-modules:

$$X \otimes E \simeq \bigoplus_{k\geqslant n} S_k \otimes E.$$

Proof. We need to show that the filtration $\{X_k\}_{k\geqslant n}$ splits upon smashing with E. Working by induction, we see that it suffices to show that all maps

$$S_k \to \Sigma S_m \otimes E,$$

where $k > m$, are automatically null. Enumerating through all of the possible even slice spheres that can appear in S_k and S_m, and making use of the (non-canonical) equivalence

$$S^{\succ} \otimes S^{-\succ} \simeq S^0 \oplus \bigoplus_{p-2} \left((C_p)_+ \otimes S^0\right),$$

we find that this follows precisely from the hypothesis that E is homotopically even and satisfies condition (4). ☐

4. The homological evenness of $\mathbb{CP}^{\infty}_{\mu_p}$

The main goal of this section is to prove the following theorem:

Theorem 4.1. *The C_p-spectrum $\Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$ is homologically even.*

Noting that $\Phi^{C_p}\Sigma^\infty\mathbb{CP}^\infty_{\mu_p} = \Sigma^\infty BC_p$ is of finite type, we may apply Proposition 3.9 and so deduce the following corollary:

Corollary 4.2. *There is a filtration $\{\Sigma^\infty \mathbb{CP}^n_{\mu_p}\}_{n\geqslant 0}$ of $\Sigma^\infty\mathbb{CP}^\infty_{\mu_p}$ with subquotients as follows*

$$\Sigma^\infty\mathbb{CP}^n_{\mu_p}/\Sigma^\infty \mathbb{CP}^{n-1}_{\mu_p} \simeq \begin{cases} S^{2m\rho} \oplus \bigoplus((C_p)_+ \otimes S^{2n}), & \text{if } n = mp \\ S^{2m\rho+1+\curlyvee} \oplus \bigoplus((C_p)_+ \otimes S^{2n}), & \text{if } n = mp+1 \\ \bigoplus((C_p)_+ \otimes S^{2n}), & \text{otherwise.} \end{cases}$$

Warning 4.3. We believe that there is a filtration $\{\mathbb{CP}^n_{\mu_p}\}_{n\geqslant 0}$ of the space $\mathbb{CP}^\infty_{\mu_p}$ that recovers $\{\Sigma^\infty \mathbb{CP}^n_{\mu_p}\}_{n\geqslant 0}$ upon applying Σ^∞, but we do not prove this here. As such, our name $\Sigma^\infty\mathbb{CP}^n_{\mu_p}$ must be regarded as an abuse of notation: we do not prove that $\Sigma^\infty\mathbb{CP}^n_{\mu_p}$ is Σ^∞ of a C_p-space $\mathbb{CP}^n_{\mu_p}$. In light of the Dold-Thom theorem, it seems likely that the space $\mathbb{CP}^n_{\mu_p}$ could be defined as the nth symmetric power of $S^{1+\curlyvee}$.

Remark 4.4. The identification of the particular even slice spheres appearing in this decomposition is determined by the cohomology of $\mathbb{CP}^\infty_{\mu_p}$ as a C_p-representation, and in particular from the combination of Corollary 2.7, Lemma 4.9 and Proposition 4.10.

As an application, we obtain the following analog of the fact that any ring spectrum with homotopy groups concentrated in even degrees admits a complex orientation:

Corollary 4.5. *Let E be a homotopically even C_p-ring spectrum. Then E is μ_p-orientable.*

Proof. We wish to show that that the $(1+\curlyvee)$-suspension of the unit map factors as

$$S^{1+\curlyvee} \to \Sigma^\infty\mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\curlyvee}E.$$

This is an immediate consequence of Corollary 4.2 and Proposition 3.16. □

We devote the remainder of the section to the proof of Theorem 4.1. By Corollary 2.7, there is an equivalence

$$\underline{\mathbb{Z}}[S^{-1-\curlyvee}] \simeq \underline{\mathbb{Z}}^{\mathbb{CP}^\infty_{\mu_p +}}.$$

This is of finite type, so to prove Theorem 4.1 it will suffice to prove the following theorem and dualize:

Theorem 4.6. *As a C_p-equivariant spectrum, $S^0[S^{2n\rho-1-\curlyvee}]$ is a direct sum of dual even slice spheres for all $n \in \mathbb{Z}$.*

To prove this, we will construct a map in from a wedge of dual even slice spheres which is an equivalence on underlying spectra and geometric fixed points.

Construction 4.7. *The composition*

$$S^{2n\rho-2} \to (C_p)_+ \otimes S^{2np-2} \to \mathrm{N}_e^{C_p} S^0[S^{2np-2}] \to S^0[S^{2n\rho-1-\curlyvee}]$$

is canonically null, and hence induces a map

$$\widetilde{x} : S^{2n\rho-1-\curlyvee} \to S^0[S^{2n\rho-1-\curlyvee}].$$

On the other hand, letting

$$x : S^{2np-2} \to S^0[S^{2np-2}]$$

denote the canonical inclusion, there is the norm map

$$\mathrm{Nm}(x) : S^{(2np-2)\rho} \to \mathrm{N}_e^{C_p} S^0[S^{2np-2}] \to S^0[S^{2n\rho-1-\curlyvee}].$$

Since $S^0[S^{2n\rho-1-\curlyvee}]$ is a module over $\mathrm{N}_e^{C_p} S^0[S^{2np-2}]$, this implies the existence of maps

$$\mathrm{Nm}(x)^k \cdot \widetilde{x}^\varepsilon : S^{k(2np-2)\rho+\varepsilon(2n\rho-1-\curlyvee)} \to S^0[S^{2n\rho-1-\curlyvee}]$$

for $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$.

We first show that the sum of these maps induces an equivalence on geometric fixed points:

Proposition 4.8. *Let*

$$\Psi : \bigoplus_{\substack{k\geqslant 0\\ \varepsilon\in\{0,1\}}} S^{k(2np-2)\rho+\varepsilon(2n\rho-1-\succ)} \to S^0[S^{2n\rho-1-\succ}]$$

denote the direct sum of the maps $\mathrm{Nm}(x)^k \cdot \widetilde{x}^{\varepsilon}$. *Then* $\Phi^{C_p}(\Psi)$ *is an equivalence.*

Proof. We have an identification

$$\Phi^{C_p} S^0[S^{2n\rho-1-\succ}] \simeq S^0[S^{2np-2}] \otimes_{S^0[S^{2n-2}]} S^0 \simeq S^0[S^{2np-2}] \otimes (S^0 \otimes_{S^0[S^{2n-2}]} S^0).$$

Under this identification, the map

$$\Phi^{C_p}(\mathrm{Nm}(x)) : S^{2np-2} \to \Phi^{C_p} S^0[S^{2n\rho-1-\succ}]$$

corresponds to the inclusion of S^{2np-2} into the left factor.

Moreover, there is an isomorphism

$$\mathrm{H}^e_*(S^0 \otimes_{S^0[S^{2n-2}]} S^0; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_{2n-1}),$$

and the map

$$\Phi^{C_p}(\widetilde{x}) : S^{2n-1} \to \Phi^{C_p} S^0[S^{2n\rho-1-\succ}]$$

sends the fundamental class of S^{2n-1} to x_{2n-1}.

It follows that $\Phi^{C_p}(\Psi)$ induces an isomorphism on homology, so is an equivalence. □

Our next task is to extend Ψ to a map that also induces an equivalence on underlying spectra. We will see that this can be accomplished by taking the direct sum with maps from induced even spheres, which are easy to produce. The main input is a computation of the homology of the underlying spectrum of $S^0[S^{2n\rho-1-\succ}]$ as a C_p-representation.

Lemma 4.9. *There is a* C_p*-equivariant isomorphism*

$$\mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{Z}) \cong \mathrm{Sym}^*_{\mathbb{Z}}(\overline{\rho}),$$

where $\overline{\rho}$ *lies in degree* 2.

Proof. There are equivariant isomorphisms

$$\mathrm{H}^e_*(S^0[S^{2n\rho-2}]; \mathbb{Z}) \cong \mathrm{Sym}^*_{\mathbb{Z}}(x)$$

and

$$\mathrm{H}^e_*(\mathrm{N}_e^{C_p} S^0[S^{2np-2}]; \mathbb{Z}) \cong \mathrm{Sym}^*_{\mathbb{Z}}(\rho),$$

where x and ρ both lie in degree 2. Since $S^0[S^{2n\rho-1-\succ}]$ is a unital $\mathrm{N}_e^{C_p} S^0[S^{2np-2}]$-module, we obtain a map

$$\mathrm{Sym}^*_{\mathbb{Z}}(\rho) \to \mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{Z})$$

of $\mathrm{Sym}^*_{\mathbb{Z}}(\rho)$-modules. Since x goes to zero in $\mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{Z})$, it follows that this factors through a map

$$\mathrm{Sym}^*_{\mathbb{Z}}(\overline{\rho}) \cong \mathrm{Sym}^*_{\mathbb{Z}}(\rho) \otimes_{\mathrm{Sym}^*_{\mathbb{Z}}(x)} \mathbb{Z} \to \mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{Z}).$$

Examining the Künneth spectral sequence, we see that this map must be an isomorphism. □

The following theorem in pure algebra determines the structure of the mod p reduction $\mathrm{Sym}^*_{\mathbb{F}_p}(\overline{\rho})$ as a C_p-representation:

Proposition 4.10 ([AF78, Propositions III.3.4-III.3.6]). *Let $\overline{\rho}$ denote the reduced regular representation of C_p over $\mathbb{F}_p$, and let $e_1, \dots e_p \in \overline{\rho}$ denote generators which are cyclically permuted by C_p and satisfy $e_1 + \cdots + e_p = 0$. We set $\mathrm{Nm} = e_1 \cdots e_p \in \mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho})$. Then the symmetric powers of $\overline{\rho}$ decompose as follows:*

$$\mathrm{Sym}^k_{\mathbb{F}_p}(\overline{\rho}) \cong \begin{cases} \mathbb{1}\{\mathrm{Nm}^\ell\} \oplus \mathrm{free} & \text{if } k = \ell \cdot p \\ \overline{\rho}\{\mathrm{Nm}^\ell e_1, \dots, \mathrm{Nm}^\ell e_p\} \oplus \mathrm{free} & \text{if } k = \ell \cdot p + 1 \\ \mathrm{free} & \text{otherwise.} \end{cases}$$

Proof of Theorem 4.6. Let Ψ be as in Proposition 4.8. It follows from Lemma 4.9 and Proposition 4.10 that the mod p homology of $\Phi^e(S^0[S^{2n\rho-1-\succ}])$ splits as $\mathrm{im}(\mathrm{H}^e_*(\Psi)) \oplus \mathrm{free}$. Moreover, Ψ is an equivalence on geometric fixed points by Proposition 4.8.

It therefore suffices to show that, given any summand of $\mathrm{H}^e_{2k}(S^0[S^{2n\rho-1-\succ}]; \mathbb{F}_p)$ isomorphic to ρ, there is a map $(C_p)_+ \otimes S^{2k} \to S^0[S^{2n\rho-1-\succ}]$ whose image is that summand. Taking the direct sum of Ψ with an appropriate collection of such maps, we obtain an $\mathbb{F}_p$-homology equivalence. Since both sides have finitely-generated free $\mathbb{Z}$-homology, this must in fact be a p-local equivalence, as desired.

To prove the remaining claim, it suffices to show that the mod p Hurewicz map

$$\pi^e_*(S^0[S^{2n\rho-1-\succ}]) \to \mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{F}_p)$$

is surjective in every degree. This follows from the following square

$$\begin{array}{ccc} \pi^e_*(\mathrm{N}^{C_p}_e S^0[S^{2np-2}]) & \twoheadrightarrow & \mathrm{H}^e_*(\mathrm{N}^{C_p}_e S^0[S^{2np-2}]; \mathbb{F}_p) \\ \downarrow & & \downarrow\!\!\downarrow \\ \pi^e_*(S^0[S^{2n\rho-1-\succ}]) & \longrightarrow & \mathrm{H}^e_*(S^0[S^{2n\rho-1-\succ}]; \mathbb{F}_p), \end{array}$$

where the top horizontal arrow is a surjection because $\mathrm{N}^{C_p}_e S^0[S^{2np-2}]$ is a non-equivariant direct sum of spheres, and the right vertical arrow is a surjection by the proof of Lemma 4.9. □

5. Examples of homotopical evenness

In this section, we introduce our principal examples of homotopically even C_p-ring spectra. By Corollary 4.5, they are also μ_p-orientable.

Our first examples are the the Morava E-theories E_{p-1} associated to the height $p-1$ Honda formal group. As we will recall in Section 5.1, E_{p-1} admits an essentially unique C_p-action by $\mathbb{E}_\infty$-automorphisms. We use this action to view E_{p-1} as a Borel C_p-equivariant $\mathbb{E}_\infty$-ring.

Our second example is the connective $\mathbb{E}_\infty$-ring $\mathrm{tmf}(2)$ of topological modular forms with full level 2 structure. The group $\mathrm{GL}_2(\mathbb{Z}/2\,\mathbb{Z}) \cong \Sigma_3$ acts on $\mathrm{tmf}(2)$ via modification of the level 2 structure, and we view $\mathrm{tmf}(2)$ as a C_3-equivariant $\mathbb{E}_\infty$-ring via the inclusion $C_3 \subset \Sigma_3$. We will discuss this example in Section 5.2.

The main result of this section is the homotopical evenness of the above C_p-ring spectra:

Theorem 5.1. *The Borel C_p-equivariant height $p-1$ Morava E-theories E_{p-1} associated to the Honda formal group over $\mathbb{F}_{p^{p-1}}$ are homotopically even and satisfy condition (4).*

Theorem 5.2. *The C_3-ring spectrum $\mathrm{tmf}(2)$ of connective topological modular forms with full level 2 structure is homotopically even and satisfies condition (4).*

Applying Corollary 4.5, we obtain the following corollary:

Corollary 5.3. *The C_p-ring spectra E_{p-1} and* $\mathrm{tmf}(2)$ *are μ_p-orientable.*

5.1. **Height $p-1$ Morava E-theory**

Given a pair $(k, \mathbb{G})$, where k is a perfect field of characterstic $p > 0$ and $\mathbb{G}$ is a formal group $\mathbb{G}$ over k of finite height h, we may functorially associate an $\mathbb{E}_\infty$-ring $E(k, \mathbb{G})$, the Lubin-Tate spectrum or Morava E-theory spectrum of $(k, \mathbb{G})$ [GH04, Lur18]. There is a non-canonical isomorphism

$$\pi_* E(k, \mathbb{G}) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{h-1}]\!][u^{\pm 1}],$$

where $|u_i| = 0$ and $|u| = -2$.

Given a prime p and finite height h, a formal group particularly well-studied in homotopy theory is the Honda formal group. The Honda formal group $\mathbb{G}_h^{\mathrm{Honda}}$ is defined over $\mathbb{F}_p$, so the Frobenius isogney may be viewed as a endomorphism

$$F : \mathbb{G}_h^{\mathrm{Honda}} \to \mathbb{G}_h^{\mathrm{Honda}}.$$

The Honda formal group is uniquely determined by the condition that $F^h = p$ in $\mathrm{End}(\mathbb{G}_h^{\mathrm{Honda}})$.

The endomorphism ring of the base change of $\mathbb{G}_h^{\mathrm{Honda}}$ to $\mathbb{F}_{p^h}$ is the maximal order $\mathcal{O}_h$ in the division algebra $\mathbb{D}_h$ of Hasse invariant $1/h$ and center $\mathbb{Q}_p$. By the functoriality of the Lubin-Tate theory construction, the automorphism group $\mathbb{S}_h = \mathcal{O}_h^\times$ of $\mathbb{G}_h^{\mathrm{Honda}}$ over $\mathbb{F}_{p^h}$ acts on $E(\mathbb{F}_{p^h}, \mathbb{G}_h^{\mathrm{Honda}})$. To keep our notation from becoming too burdensome, we set

$$E_{p-1} := E(\mathbb{F}_{p^{p-1}}, \mathbb{G}_{p-1}^{\mathrm{Honda}}).$$

There is a subgroup $C_p \subset \mathbb{S}_{p-1}$, which is unique up to conjugation. Indeed, such subgroups correspond to embeddings $\mathbb{Q}_p(\zeta_p) \subset \mathbb{D}_{p-1}$. Since $\mathbb{Q}_p(\zeta_p)$ is of degree $p-1$ over $\mathbb{Q}_p$, it follows from a general fact about division algebras over local fields that such a subfield exists and is unique up to conjugation (cf. [Ser67, Application on pg. 138]). Using any such C_p, we may view E_{p-1} as a Borel C_p-equivariant $\mathbb{E}_\infty$-ring spectrum.

Homotopical evenness of E_{p-1} will follow from the computation of the homotopy fixed point spectral sequence for $E_{p-1}^{hC_p}$, which was first carried out by Hopkins and Miller and has been written down in [Nav10] and again reviewed in [HMS17]. We recall this computation below. The homotopy fixed point spectral sequence takes the form

$$\mathrm{H}^s(C_p, \pi_t E_{p-1}) \Rightarrow \pi_{t-s} E_{p-1}^{hC_p},$$

so the first step is to compute the action of C_p on $\pi_* E_{p-1}$.

This action may be determined as follows. Abusing notation, let $v_1 \in \pi_{2p-2} E_{p-1}$ denote a lift of the canonically defined element $v_1 \in \pi_{2p-2} E_{p-1}/p$. The element v_1 is fixed modulo p by the $\mathbb{S}_{p-1}$ and in particular the C_p-action on E_{p-1}, so if we fix a generator $\gamma \in C_p$ we find that the element $v_1 - \gamma v_1$ is divisible by p. Set $v = \frac{v_1 - \gamma v_1}{p}$. Then the two key properties of v are that:

1. $v + \gamma v + \cdots + \gamma^{p-1} v = 0$.
2. v is a unit in $\pi_* E_{p-1}$. As a consequence, $\mathrm{Nm}(v) = v \cdot \gamma v \cdots \gamma^{p-1} v$ is a unit in $\pi_* E_{p-1}$ which is fixed by the C_p-action [Nav10, pg. 498].

The existence of an element v satisfying the above two conditions completely determines the action of C_p on $\pi_* E_{p-1}$, as follows. First, let $\widetilde{w} \in \pi_{-2} E_{p-1}$ denote any unit, and set $w = v \cdot \mathrm{Nm}(\widetilde{w}) \in \pi_{-2} E_{p-1}$. Then w continues to satisfy (1) and (2) above and determines a map of C_p-representations

$$\overline{\rho}_{\mathbb{W}(\mathbb{F}_{p^{p-1}})} \to \pi_{-2} E_{p-1}.$$

This determines a C_p-equivariant map

$$\mathrm{Sym}^*_{\mathbb{W}(\mathbb{F}_{p^{p-1}})}(\overline{\rho})[\mathrm{Nm}(w)^{-1}] \to \pi_* E_{p-1},$$

which identifies $\pi_* E_{p-1}$ with the graded completion of $\mathrm{Sym}^*_{\mathbb{W}(\mathbb{F}_{p^{p-1}})}(\overline{\rho})[\mathrm{Nm}(w)^{-1}]$ at the graded ideal generated by the kernel of the essentially unique nonzero map of $\mathbb{W}(\mathbb{F}_{p^{p-1}})[C_p]$-modules $\overline{\rho}_{\mathbb{W}(\mathbb{F}_{p^{p-1}})} \to \mathbb{1}_{\mathbb{F}_{p^{p-1}}}$.

Remark 5.4. In Section 7, we will see that the element v is intimately related to the μ_p-orientability of E_{p-1}. For later use, we note that it follows from the above analysis that the map $\overline{\rho}_{\mathbb{F}_{p^{p-1}}} \to \pi_{2p-2} E_{p-1}/(p, \mathfrak{m}^2)$ induced by v is an isomorphism.

Using the above determination of the C_p-action on $\pi_* E_{p-1}$, as well as Proposition 4.10, one may obtain with some work the following description of $\mathrm{H}^s(C_p, \pi_t E_{p-1})$:

Proposition 5.5 (Hopkins–Mahowald, cf. [HMS17, Proposition 2.6])**.** *There is an exact sequence*

$$\pi_* E_{p-1} \xrightarrow{tr} \mathrm{H}^*(C_p, \pi_* E_{p-1}) \to \mathbb{F}_{p^{p-1}}[\alpha, \beta, \delta^{\pm 1}]/(\alpha^2) \to 0, \tag{1}$$

where $|\alpha| = (1, 2p-2), |\beta| = (2, 2p^2 - 2p)$, *and* $|\delta| = (0, 2p)$.

Finally, we must recall the differentials in the homotopy fixed point spectral sequence. We let $\doteq$ denote equality up to multiplication by an element of $\mathbb{W}(\mathbb{F}_{p^{p-1}})^\times$. Then, as explained in [HMS17, §2.4], the spectral sequence is determined multiplicatively by the following differentials:

$$d_{2(p-1)+1}(\delta) \doteq \alpha\beta^{p-1}\delta^{1-(p-1)^2} \quad \text{and} \quad d_{2(p-1)^2+1}(\delta^{(p-1)^3}\alpha) \doteq \beta^{(p-1)^2+1},$$

along with the fact that all differentials vanish on the image of the transfer map.

In particular, on the E_∞-page of the homotopy fixed point spectral sequence there are no elements in positive filtration in total degrees 0, -1 or -2. Indeed, there are no elements at all in the (-1)-stem.

We now have enough information to establish the homotopical evenness of E_{p-1}.

Proof of Theorem 5.1. Let $u \in \pi^e_2 E_{p-1}$ denote the periodicity element. Then $\mathrm{Nm}(u) \in \pi^{C_p}_{2\rho} E_{p-1}$ is also invertible, so the $RO(C_p)$-graded equivariant homotopy of E_{p-1} is 2ρ-periodic.

Therefore, using Remark 3.13, we see that it suffices to show that:

1. $\pi^e_{-1} E_{p-1} = 0$.
2. $\pi_{-1} E^{hC_p}_{p-1} = 0$.
3. The transfer map $\pi^e_{-2} E_{p-1} \to \pi_{-2} E^{hC_p}_{p-1}$ is a surjection.
4. The restriction map $\pi_0 E^{hC_p}_{p-1} \to \pi^e_0 E_{p-1}$ is an injection.

Condition (1) is immediate from the fact that E_{p-1} is even periodic. Condition (2) is a direct consequence of the above computation of the homotopy fixed point spectral sequence. Condition (3) follows from the following two facts:

- The short exact sequence (1) implies that $\mathrm{H}^0(C_p, \pi_{-2} E_{p-1})$ is spanned by the image of the transfer.
- On the E_∞-page of the homotopy fixed point spectral sequence, there are no positive filtration elements in stem -2.

Condition (4) follows from the fact that on the E_∞-page of the homotopy fixed point spectral sequence, there are no positive filtration elements in the zero stem. ☐

5.2. The spectrum $\mathrm{tmf}(2)$ as a form of $\mathrm{BP}\langle 1\rangle_{\mu_3}$

Recall from [Sto12] or [HL16] the spectrum $\mathrm{tmf}(2)$ of connective topological modular forms with full level 2 structure.[1] In this section we will consider $\mathrm{tmf}(2)$ as implictly 3-localized. It is a genuine Σ_3-equivariant $\mathbb{E}_\infty$-ring spectrum with Σ_3-fixed points $\mathrm{tmf}(2)^{\Sigma_3} = \mathrm{tmf}$, the (3-localized) spectrum of connective topological modular forms. We view $\mathrm{tmf}(2)$ as a C_3-spectrum via restriction along an inclusion $C_3 \subset \Sigma_3$.

This spectrum has been well-studied by Stojanoska [Sto12]. In particular, Stojanoska computes $\pi_*^e\mathrm{tmf}(2) = \mathbb{Z}_{(3)}[\lambda_1, \lambda_2]$, where $|\lambda_i| = 4$ and a generator γ of C_3 acts by $\lambda_1 \mapsto \lambda_2 - \lambda_1$ and $\lambda_2 \mapsto -\lambda_1$. It follows that λ_1 and λ_2 span a copy of $\overline{\rho}$, so that $\pi_*\mathrm{tmf}(2) \cong \mathrm{Sym}^*_{\mathbb{Z}_{(3)}}(\overline{\rho})$. The corresponding family of elliptic curves is cut out by the explicit equation

$$y^2 = x(x-\lambda_1)(x-\lambda_2).$$

For later use, we note down some facts about the associated formal group law.

Proposition 5.6. *The 3-series of the formal group law associated to* $\mathrm{tmf}(2)$ *is given by the following formula:*

$$\begin{aligned}[3](x) = 3x &+ 8(\lambda_1+\lambda_2)x^3 + 24(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2)x^5 + 72(\lambda_1^3 - \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 + \lambda_2^3)x^7 \\ &+ 8(27\lambda_1^4 - 76\lambda_1^3\lambda_2 + 98\lambda_1^2\lambda_2^2 - 76\lambda_1\lambda_2^3 + 27\lambda_2^4)x^9 + O(x^{10})\end{aligned}$$

It follows that we have the following formulas for v_1 and v_2:

$$v_1 \equiv -\lambda_1 - \lambda_2 \mod 3$$

and

$$v_2 \equiv \lambda_1^4 \equiv \lambda_2^4 \mod (3, v_1).$$

Proof. This is an elementary computation using the method of [Sil09, §IV.1]. ☐

Remark 5.7. Let $v = -\lambda_1 - \lambda_2$, so that $v \equiv v_1 \mod p$. Then we have

$$\gamma v - v = ((\lambda_1 - \lambda_2) + \lambda_1) + \lambda_1 + \lambda_2 = 3\lambda_1,$$

so that

$$\frac{\gamma v - v}{3} = \lambda_1.$$

Note that this element generates $\pi_*\mathrm{tmf}(2)$ as a $\mathbb{Z}_{(3)}$-algebra with C_3-action. In Section 7, we will relate this element to the μ_3-orientation of $\mathrm{tmf}(2)$.

In his thesis, the third author has computed the slices of $\mathrm{tmf}(2)$ (cf. [HHR16, §4]):

Proposition 5.8 ([Wil17a, Corollary 3.2.1.10])**.** *Given a C_p-equivariant spectrum X, let $P_n^n X$ denote the nth slice of X. The slices of* $\mathrm{tmf}(2)$ *are of the form:*

$$\bigoplus_n P_n^n \mathrm{tmf}(2) \simeq \underline{\mathbb{Z}}_{(3)}[S^{2\rho - 1 - \succ}].$$

We now turn to the proof of Theorem 5.2. Given the computation of the slices of $\mathrm{tmf}(2)$ in Proposition 5.8, this will follow from Theorem 4.6 and the following proposition:

Proposition 5.9. *Let X be a C_p-spectrum whose slices are of the form $P_n^n X \simeq S_n \otimes \underline{\mathbb{Z}}_{(p)}$, where S_n is a direct sum of dual even slice n-spheres. Then X is homotopically even and satisfies condition (4).*

[1] The spectrum $\mathrm{tmf}(2)$ is obtained from the spectrum $\mathrm{Tmf}(2)$ discussed in the references by taking the Σ_3-equivariant connective cover.

Using the slice spectral sequence, the proof of Proposition 5.9 reduces to the following lemma:

Lemma 5.10. *Let S denote a dual even slice sphere. Then $S \otimes \underline{\mathbb{Z}}_{(p)}$ is homotopically even and satisfies condition (4).*

Proof. If $S \simeq S^{2n} \otimes (C_p)_+$, then this follows from the fact that $\pi_{2n-1}\mathbb{Z}_{(p)} = 0$ for all $n \in \mathbb{Z}$.

If $S \simeq S^{2n\rho}$, then this follows from the fact that $\underline{\mathbb{Z}}_{(p)}$ is homotopically even, since the definition of homotopically even is invariant under 2ρ-suspension.

If $S \simeq S^{2n\rho-1-\succ}$, then condition (1) of Definition 3.11 is clearly satisfied, and conditions (2)-(4) follow from the following statements for all $n \in \mathbb{Z}$, which may be read off from [Wil17a, §A.2]:

- $\pi^{C_p}_{2n\rho+\succ}\underline{\mathbb{Z}}_{(p)} = 0$,
- $\pi^{C_p}_{2n\rho-1}\underline{\mathbb{Z}}_{(p)} = 0$,
- $\pi^{C_p}_{2n\rho+1+\lambda}\underline{\mathbb{Z}}_{(p)} = 0$,

where in the proofs of (3) and (4) we have implicitly used the existence of equivalences

$$S^{\succ} \otimes S^{-\succ} \simeq S^0 \oplus \bigoplus_{p-2} (C_p)_+ \otimes S^0$$

and

$$S^{\succ} \otimes S^{\succ} \simeq S^{\lambda} \oplus \bigoplus_{p-2} (C_p)_+ \otimes S^2. \qquad \square$$

6. $v_1^{\mu_p}$ and a formula for its span

In this section, given a μ_p-oriented C_p-ring spectrum R, we will define a class

$$v_1^{\mu_p} \in \pi^{C_p}_{2\rho}(\Sigma^{1+\succ}R) \cong \pi^{C_p}_{2\rho-1-\succ}R.$$

When $p = 2$, our construction agrees with the class $v_1^{\mathbb{R}} \in \pi^{C_2}_{\rho}R$ in the homotopy of a Real oriented C_2-ring spectrum. Just as v_1 is well-defined modulo p, we will see that $v_1^{\mu_p}$ is well-defined modulo the transfer. We will also give a formula for the image of $v_1^{\mu_p}$ in the the underlying homotopy of R in terms of the classical element v_1 and the C_p-action.

To define $v_1^{\mu_p}$, we first construct a class $v_1^{\mu_p} \in \pi^{C_p}_{2\rho}\Sigma^{\infty}\mathbb{CP}^{\infty}_{\mu_p}$, and then we take its image along the μ_p-orientation $\Sigma^{\infty}\mathbb{CP}^{\infty}_{\mu_p} \to \Sigma^{1+\succ}R$. To begin, we recall an analogous construction of the classical element v_1.

6.1. The non-equivariant v_1 as a pth power

We recall some classical, non-equivariant theory that we will generalize to the equivariant setting in the next section.

Notation 6.1. We let $\beta : S^2 \simeq \Sigma^{\infty}\mathbb{CP}^1 \to \Sigma^{\infty}\mathbb{CP}^{\infty}$ denote a generator of the stable homotopy group $\pi_2(\Sigma^{\infty}\mathbb{CP}^{\infty})$.

Since $\mathbb{CP}^{\infty} \simeq \Omega^{\infty}\Sigma^2\mathbb{Z}$ is an infinite loop space, its suspension spectrum $\Sigma^{\infty}\mathbb{CP}^{\infty}$ is a non-unital ring spectrum. This allows us to make sense of the following definition.

Definition 6.2. We define the class $v_1 \in \pi_{2p}\Sigma^{\infty}\mathbb{CP}^{\infty}$ to be β^p, the pth power of the degree 2 generator.

There are at least two justifications for naming this class v_1, which might more commonly be defined as the coefficient of x^p in the p-series of a complex-oriented ring. The relationship is expressed in the following proposition:

Proposition 6.3. *Let R denote a (non-equivariant) homotopy ring spectrum, equipped with a complex orientation*

$$\Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty} \to R,$$

which can be viewed as a class $x \in R^2(\mathbb{CP}^{\infty})$. Then the composite

$$S^{2p-2} \xrightarrow{v_1} \Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty} \to R$$

records, up to addition of a multiple of p, the coefficient of x^p in the p-series $[p]_F(x)$.

Proof. Consider the p-fold multiplication map of infinite loop spaces

$$(\mathbb{CP}^{\infty})^{\times p} \xrightarrow{m} \mathbb{CP}^{\infty}$$

Applying R^* to the above, we obtain a map

$$R_*[\![x]\!] \to R_*[\![x_1, x_2, \cdots, x_p]\!].$$

By the definition of the formal group law $- +_F -$ associated to the complex orientation, the class $x \in R^2(\mathbb{CP}^{\infty})$ is sent to the formal sum

$$f(x_1, x_2, \cdots, x_p) = x_1 +_F x_2 +_F \cdots +_F x_p.$$

The commutativity of the formal group law ensures that this power series is invariant under cyclic permutation of the x_i. The composite in $\pi_{2p-2}R$ that we must compute is the coefficient of the product $x_1 x_2 \cdots x_p$ in $f(x_1, x_2, \cdots, x_p)$. We can consider the power series in a single variable $[p](x) = f(x, x, \cdots, x)$. Since the only degree p monomial in $x_1, ..., x_p$ that is invariant under cyclic permutation of the x_i is the product $x_1 \cdots x_p$, the coefficient of x^p in $[p](x)$ will be equal to the coefficient of $x_1 x_2 \cdots x_p$ in $f(x_1, x_2, \ldots, x_p)$ up to addition of a multiple of p. □

Remark 6.4. The integral homology $H_*(\mathbb{CP}^{\infty}; \mathbb{Z}_{(p)})$ is a divided power ring on the Hurewicz image of β. In particular, the Hurewicz image of $v_1 = \beta^p$ is a multiple of p times a generator of $H_{2p}(\mathbb{CP}^{\infty}; \mathbb{Z}_{(p)})$.

Consider the ring spectrum MU together with its canonical complex orientation

$$\Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty} \to MU.$$

The integral homology $H_*(MU; \mathbb{Z})$ is the symmetric algebra on the image, under this map, of $\widetilde{H}_*(\mathbb{CP}^{\infty}; \mathbb{Z})$. In particular, the Hurewicz image of v_1 in $H_{2p}(\Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty}; \mathbb{Z}_{(p)})$ is sent to p times an indecomposable generator of $H_{2p-2}(MU; \mathbb{Z}_{(p)})$. By [Mil60], this provides another justification for the name v_1.

Remark 6.5. One might ask whether higher v_i, with $i > 1$, can be defined in $\pi_*(\Sigma^{\infty}\mathbb{CP}^{\infty})$. A classical argument with topological K-theory [Mos68] shows that the Hurewicz image of $\pi_*(\Sigma^{\infty}\mathbb{CP}^{\infty})$ inside of $H_*(\Sigma^{\infty}\mathbb{CP}^{\infty}; \mathbb{Z}_{(p)})$ is generated as a $\mathbb{Z}_{(p)}$-module by powers of β. For i larger than 1, β^{p^i} is not simply p times a generator of $H_{2p^i}(\mathbb{CP}^{\infty}; \mathbb{Z}_{(p)})$, so it is impossible to lift the corresponding indecomposable generators of $\pi_*(MU)$ to $\pi_*(\Sigma^{-2}\Sigma^{\infty}\mathbb{CP}^{\infty})$. However, it may be possible to lift multiples of such generators.

Finally, we record the following proposition for later use:

Proposition 6.6. *Let A denote a (non-equivariant) homotopy ring spectrum, equipped with a map*

$$f : \Sigma^{\infty}\mathbb{CP}^{\infty} \to \Sigma^2 A$$

that induces the zero *homomorphism on π_2 (in particular, f is not a complex orientation). Then the image of v_1 in $\pi_{2p-2}A$ is a multiple of p.*

Proof. Let $C\alpha_1$ denote the cofiber of $\alpha_1 : S^{2p-3} \to S^0$.

We recall first that, p-locally, the spectrum

$$\Sigma^\infty \mathbb{CP}^p$$

admits a splitting as $\Sigma^2 C\alpha_1 \oplus \bigoplus_{k=2}^{p-2} S^{2k}$. Indeed, since α_1 is the lowest positive degree element in the p-local stable stems, most of the attaching maps in the standard cell structure for $\mathbb{CP}^p$ are automatically p-locally trivial. The only possibly non-trivial attaching map is between the $(2p)$th cell and the bottom cell, and this attaching map is detected by the P^1 action on $H^*(\mathbb{CP}^\infty; \mathbb{F}_p)$.

By cellular approximation, $v_1 : S^{2p} \to \Sigma^\infty \mathbb{CP}^\infty$ must factor through $\Sigma^\infty \mathbb{CP}^p$, and again the lack of elements in the p-local stable stems ensures a further factorization of v_1 through $\Sigma^2 C\alpha_1$. Thus, to determine the image of v_1 in $\pi_{2p}(\Sigma^2 A)$, it suffices to consider the composite

$$\tilde{f} : \Sigma^2 C\alpha_1 \to \Sigma^\infty \mathbb{CP}^p \to \Sigma^\infty \mathbb{CP}^\infty \to \Sigma^2 A.$$

There is by definition a cofiber sequence $S^2 \to \Sigma^2 C\alpha_1 \to S^{2p}$. By the assumption that f is trivial on π_2, $\tilde{f}$ must factor as a composite

$$\Sigma^2 C\alpha_1 \to S^{2p} \to \Sigma^2 A.$$

We now finish by noting that the composite $v_1 : S^{2p} \to \Sigma^2 C\alpha_1 \to S^{2p}$ must be a multiple of p, because otherwise $C\alpha_1$ would split as $S^{2p} \oplus S^2$. □

6.2. The equivariant $v_1^{\mu_p}$ as a norm

As we defined the non-equivariant $v_1 \in \pi_{2p}\Sigma^\infty \mathbb{CP}^\infty$ to be the pth power of a degree 2 class, we similarly define an equivariant $v_1^{\mu_p} \in \pi_{2\rho}^{C_p}\Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$ to be the *norm* of a degree 2 class. We thank Mike Hill for suggesting this conceptual way of constructing $v_1^{\mu_p}$. To see that $\Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$ is equipped with norms, we will make use of the following proposition:

Proposition 6.7. *There is an equivalence of C_p-equivariant spaces*

$$\Omega^\infty \Sigma^{1+\lambda} \underline{\mathbb{Z}} \simeq \mathbb{CP}^\infty_{\mu_p},$$

where $\underline{\mathbb{Z}}$ denotes the C_p-equivariant Eilenberg–Maclane spectrum associated to the constant Mackey functor.

Proof. This is Remark 1.6. □

Construction 6.8. The above proposition equips the space $\mathbb{CP}^\infty_{\mu_p}$ with a natural *norm*, meaning a map

$$N_e^{C_p}((\mathbb{CP}^\infty_{\mu_p})^e) \to \mathbb{CP}^\infty_{\mu_p}.$$

Indeed, any C_p-equivariant infinite loop space $\Omega^\infty Y$, like $\Omega^\infty S^{1+\lambda}\underline{\mathbb{Z}}$, is equipped with a norm

$$N_e^{C_p}(\Omega^\infty Y)^e \to \Omega^\infty Y.$$

This norm is Ω^∞ applied to the C_p-spectrum map

$$(C_p)_+ \otimes Y \to Y$$

that is induced from the identity on Y^e.

Convention 6.9. *For the remainder of this section we fix a (non-canonical) equivalence*

$$(\mathbb{CP}^\infty_{\mu_p})^e \simeq (\mathbb{CP}^\infty)^{\times p-1}.$$

The natural map of C_p-spaces

$$S^{1+\lambda} = \mathbb{CP}^1_{\mu_p} \to \mathbb{CP}^\infty_{\mu_p}$$

then induces an (again, non-canonical) equivalence

$$(S^{1+\succ})^e \simeq \bigvee_{p-1} S^2,$$

giving $p-1$ *classes*

$$\beta_1, \beta_2, \ldots, \beta_{p-1} \in \pi_2^e(\mathbb{CP}^\infty_{\mu_p}).$$

Choosing our non-canonical equivalence appropriately, we may suppose that the C_p*-action on* $\pi_2^e(\mathbb{CP}^\infty_{\mu_p}; \mathbb{Z}_{(p)})$ *is given by the rules*

(1) $\gamma(\beta_i) = \beta_{i+1}$*, if* $1 \leqslant i \leqslant p-2$
(2) $\gamma(\beta_{p-1}) = -\beta_1 - \beta_2 - \cdots - \beta_{p-1}$.

Definition 6.10. We let

$$v_1^{\mu_p} : S^{2\rho} \to \Sigma^\infty \mathbb{CP}^\infty_{\mu_p}.$$

denote the *norm* of β_1. Explicitly, norming the non-equivariant β_1 map yields a map

$$S^{2\rho} \simeq N_e^{C_p} S^2 \to N_e^{C_p}(\Phi^e(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p})),$$

and we may compose this with the norm map of Construction 6.8 to make the class

$$v_1^{\mu_p} \in \pi_{2\rho}^{C_p}(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p}).$$

Remark 6.11. Of course, the choice of the class β_1 above is not canonical. We view this as a mild indeterminancy in the definition of $v_1^{\mu_p}$, related to the fact that the classical v_1 should only be well-defined modulo p. As we will see later, many formulas we write for $v_1^{\mu_p}$ will similarly be well-defined only modulo transfers.

6.3. A formula for $v_1^{\mu_p}$ in terms of v_1

Our next aim will be to give an explicit formula for the *image* of $v_1^{\mu_p}$ in the underlying homotopy of a μ_p-oriented cohomology theory. Our formula is stated as Theorem 6.20. To begin its derivation, our first order of business is to give a different formula for $v_1^{\mu_p}$ modulo transfers:

Proposition 6.12. *In* $\pi_{2p}^e(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p})$*, the class* $pv_1^{\mu_p}$ *and the class* $\mathrm{Tr}(\beta_1^p)$ *differ by* p *times a transferred class. In particular,* $\mathrm{Tr}(\beta_1^p)$ *is divisible by* p*, and the class* $\frac{\mathrm{Tr}(\beta_1^p)}{p}$ *is the restriction of a class in* $\pi_{2\rho}^{C_p} \Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$*.*

Proof. Identifying $\pi_2^e(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p})$ with $\overline{\rho}_{\mathbb{Z}_{(p)}}$ and using the nonunital $\mathbb{E}_\infty$-ring structure on $\Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$, we obtain a map

$$\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}}) \to \pi_{2p}^e(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p})$$

under which the norm class Nm maps to the image of $v_1^{\mu_p}$. The conclusion of the proposition then follows from Lemma 6.13 below. □

Lemma 6.13. *Let* $\overline{\rho}_{\mathbb{Z}_{(p)}}$ *denote the reduced regular representation of* C_p *over* $\mathbb{Z}_{(p)}$*, and let* $e_1, \ldots, e_p \in \overline{\rho}_{\mathbb{Z}_{(p)}}$ *denote generators which are cyclically permuted by* C_p *and satisfy* $e_1 + \cdots + e_p = 0$*. We set* $\mathrm{Nm} = e_1 \cdots e_p \in \mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})$*.*

Then $\mathrm{Tr}(e_1^p)$ *is divisible by* p*, and* Nm *and* $\frac{\mathrm{Tr}(e_1^p)}{p}$ *differ by a transferred class in* $\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})$*.*

Proof. To see that $\mathrm{Tr}(e_1^p)$ is divisible by p, we expand it out in terms of the basis $e_1, \ldots, e_{p-1}$ of $\overline{\rho}_{\mathbb{Z}_{(p)}}$:

$$\mathrm{Tr}(e_1^p) = e_1^p + \cdots + e_{p-1}^p + (-e_1 - e_2 - \cdots - e_{p-1})^p.$$

It is clear from linearity of the Frobenius modulo p that $\mathrm{Tr}(e_1^p)$ is divisible by p. Our next goal is to show that $\mathrm{Nm} - \frac{\mathrm{Tr}(e_1^p)}{p}$ is a transferred class. It is clearly fixed by the C_p-action, so we wish to show that its image in

$$\frac{\left(\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})\right)^{C_p}}{\mathrm{Tr}\left(\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})\right)}$$

is zero. Since p times any fixed point of C_p is the transfer of an element, there is an isomorphism

$$\frac{\left(\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})\right)^{C_p}}{\mathrm{Tr}\left(\mathrm{Sym}^p_{\mathbb{Z}_{(p)}}(\overline{\rho}_{\mathbb{Z}_{(p)}})\right)} \cong \frac{\left(\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p})\right)^{C_p}}{\mathrm{Tr}\left(\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p})\right)}.$$

By Proposition 4.10, there is an isomorphism of C_p-representations

$$\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p}) \cong \mathbb{1}_{\mathbb{F}_p}\{\mathrm{Nm}\} \oplus \text{free},$$

so that any choice of C_p-equivariant map $\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p}) \to \mathbb{1}_{\mathbb{F}_p}$ which is nonzero on Nm restricts to an isomorphism

$$\frac{\left(\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p})\right)^{C_p}}{\mathrm{Tr}\left(\mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p})\right)} \cong \mathbb{1}_{\mathbb{F}_p}.$$

A choice of such a map may be made as follows. First, let $f : \overline{\rho}_{\mathbb{F}_p} \to \mathbb{1}_{\mathbb{F}_p}$ denote the equivariant map sending each e_i to 1. This induces a map $\mathrm{Sym}^p_{\mathbb{F}_p}(f) : \mathrm{Sym}^p_{\mathbb{F}_p}(\overline{\rho}_{\mathbb{F}_p}) \to \mathrm{Sym}^p_{\mathbb{F}_p}(\mathbb{1}_{\mathbb{F}_p}) \cong \mathbb{1}_{\mathbb{F}_p}$ which sends Nm to 1. We now need to show that the image of $\frac{\mathrm{Tr}(e_1^p)}{p}$ under $\mathrm{Sym}^p_{\mathbb{F}_p}(f)$ is also equal to 1. Writing

$$\frac{\mathrm{Tr}(e_1^p)}{p} = \frac{e_1^p + \cdots + e_{p-1}^p + (-e_1 - e_2 - \cdots - e_{p-1})^p}{p},$$

we find that its image of $\mathrm{Sym}^p_{\mathbb{F}_p}(f)$ is equal to

$$\frac{p-1-(p-1)^p}{p} = \frac{p-1-(-1+O(p^2))}{p} \equiv 1 \mod p,$$

as desired. ☐

Proposition 6.12 can be read as the statement that $\frac{\mathrm{Tr}(\beta_1^p)}{p}$ is a formula for $v_1^{\mu_p} \in \pi_{2\rho}^{C_p}\Sigma^\infty\mathbb{CP}^\infty_{\mu_p}$, if one is only interested in $v_1^{\mu_p}$ modulo transfers. We often find this formula for $v_1^{\mu_p}$ to be more useful in computational contexts.

Convention 6.14. *For the remainder of this section, we fix a C_p-ring R together with a μ_p-orientation*

$$\Sigma^\infty\mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\curlyvee}R.$$

Definition 6.15. The μ_p-orientation of R gives rise to a map

$$(\Sigma^\infty \mathbb{CP}^\infty_{\mu_p})^e \to (\Sigma^{1+\succ} R)^e,$$

which under our fixed identification of $(\mathbb{CP}^\infty_{\mu_p})^e$ is given by a map

$$\Sigma^\infty (\mathbb{CP}^\infty)^{\times p-1} \to \bigoplus_{p-1} \Sigma^2 R.$$

By mapping in the first of the $(p-1)$ copies of $\mathbb{CP}^\infty$, and then projecting to the first of the $(p-1)$ copies of R, we obtain *the underlying complex orientation of* R.

Warning 6.16. While it is convenient to give formulas in terms of the underlying complex orientation of Definition 6.15, we stress once again that this is non-canonical, depending on Convention 6.9. There is no *canonical* classical complex orientation associated to a μ_p-oriented C_p-ring.

Notation 6.17. Using Definition 6.2, the underlying complex orientation of R gives rise to a class $v_1 = \beta_1^p \in \pi^e_{2p-2} R$.

Notation 6.18. Recall our fixed non-canonical identification $(S^{1+\succ})^e \simeq \bigoplus_{p-1} S^2$. Let $y_i \in \pi^e_2 S^{1+\succ}$ correspond to the ith copy of S^2, so that we have

(1) $\gamma(y_i) = y_{i+1}$ if $1 \leqslant i \leqslant p-2$, and
(2) $\gamma(y_{p-1}) = -y_1 - \cdots - y_{p-1}$.

Then a generic class

$$r \in \pi^e_{2p}(\Sigma^{1+\succ} R) \cong \pi^e_2 S^{1+\succ} \otimes \pi^e_{2p-2} R$$

may be written as

$$r = y_1 \otimes r_1 + y_2 \otimes r_2 + \cdots + y_{p-1} \otimes r_{p-1},$$

where $r_i \in \pi^e_{2p-2} R$.

The key relationship between the equivariant $v_1^{\mu_p}$ and non-equivariant v_1 is expressed in the following lemma:

Lemma 6.19. *The class $v_1 = \beta_1^p \in \pi^e_{2p} \Sigma^\infty \mathbb{CP}^\infty_{\mu_p}$ maps to $y_1 \otimes v_1$ plus a multiple of p in $\pi^e_{2p}(\Sigma^{1+\succ} R)$.*

Proof. The class β_1^p maps to $y_1 \otimes r_1 + y_2 \otimes r_2 + \cdots + y_{p-1} \otimes r_{p-1}$ for some collection of elements $r_1, r_2, ..., r_{p-1} \in \pi^e_{2p-2} R$.

By Definition 6.2, $r_1 = v_1$, so it suffices to show that each of $r_2, ..., r_{p-1}$ is divisible by p. These statements in turn each follow by application of Proposition 6.6. □

At last, we are ready to state the main result of this section:

Theorem 6.20. *Suppose that the underlying homotopy groups $\pi^e_* R$ are torsion-free. Then the class $v_1^{\mu_p} \in \pi^e_{2p}(\Sigma^{1+\succ} R)$ is given, modulo transfers, by the class*

$$y_1 \otimes \frac{v_1 - \gamma^{p-1} v_1}{p} + y_2 \otimes \frac{\gamma v_1 - v_1}{p} + \cdots + y_{p-1} \otimes \frac{\gamma^{p-2} v_1 - \gamma^{p-3} v_1}{p}.$$

Proof. By Proposition 6.12, it is equivalent to show the above formula determines $\mathrm{Tr}(\beta_1^p)/p \in \pi^e_{2p}(\Sigma^{1+\succ} R)$ modulo transfers. But this may be computed directly from Lemma 6.19. □

Remark 6.21. Consider the class

$$y_1 \otimes \frac{v_1 - \gamma^{p-1}v_1}{p} + y_2 \otimes \frac{\gamma v_1 - v_1}{p} + \cdots + y_{p-1} \otimes \frac{\gamma^{p-2}v_1 - \gamma^{p-3}v_1}{p}.$$

of Theorem 6.20. If in this formula we replace v_1 by $v_1' = v_1 + px$, for an arbitrary class $x \in \pi^e_{2p-2}R$, the resulting expression differs from the original by

$$y_1 \otimes (x - \gamma^{p-1}x) + y_2 \otimes (\gamma x - x) + \cdots + y_{p-1} \otimes (\gamma^{p-2}x - \gamma^{p-3}x).$$

This is exactly the transfer, in $\pi^e_{2p}(\Sigma^{1+\succ}R)$, of $y_1 \otimes x$. Thus, altering v_1 by a multiple of p does not change the class $v_1^{\mu_p}$ modulo transfers.

7. The span of $v_1^{\mu_p}$ in height $p-1$ theories

In this section, we use the formula of Theorem 6.20 to compute the span of $v_1^{\mu_p}$ in the height $p-1$ theories E_{p-1} and $\mathrm{tmf}(2)$, which we verified were μ_p-orientable in Section 5. Our main result, stated in Theorems 7.3 and 7.4, proves that the span of $v_1^{\mu_p}$ generates the homotopy of these theories in a suitable sense. This demonstrates a height-shifting phenomenon in equivariant homotopy theory: though these theories are height $p-1$ classically, the fact that their homotopy is generated by $v_1^{\mu_p}$ indicates that they should be regarded as height 1 objects in C_p-equivariant homotopy theory.

Notation 7.1. Let R denote a C_p-ring spectrum, equipped with a μ_p-orientation

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ}R.$$

Precomposition with $v_1^{\mu_p}$ then yields a map

$$S^{2\rho} \to \Sigma^{1+\succ}R,$$

which by the dualizability of $S^{1+\succ}$ is equivalent to a map of C_p-spectra

$$S^{2\rho-1-\succ} \to R.$$

Engaging in a slight abuse of notation, we will throughout this section denote this map by

$$v_1^{\mu_p} : S^{2\rho-1-\succ} \to R.$$

Definition 7.2. Given a μ_p-oriented C_p-ring R, applying π^e_{2p-2} gives a homomorphism of $\mathbb{Z}_{(p)}[C_p]$-modules

$$\pi^e_{2p-2}v_1^{\mu_p} : \pi^e_{2p-2}S^{2\rho-1-\succ} \to \pi^e_{2p-2}R.$$

The main theorems of this section are as follows:

Theorem 7.3. *Suppose that*

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ}\mathrm{tmf}(2)$$

is any μ_3-orientation of $\mathrm{tmf}(2)$*. Then the map $\pi^e_4 v_1^{\mu_p} : \pi^e_4 S^{2\rho-1-\succ} \to \pi^e_4\mathrm{tmf}(2)$ is an isomorphism of $\mathbb{Z}_{(3)}$-modules, and thus also of $\mathbb{Z}_{(3)}[C_3]$-modules.*

Theorem 7.4. *Suppose that*

$$\Sigma^\infty \mathbb{CP}^\infty_{\mu_p} \to \Sigma^{1+\succ}E_{p-1}$$

is any μ_p-orientation of E_{p-1}. Then the image of $\pi^e_{2p-2}v_1^{\mu_p}$ in $\pi^e_{2p-2}E_{p-1}$ maps surjectively onto the degree $2p-2$ component of $\pi_*(E_{p-1})/(p, \mathfrak{m}^2)$.

Remark 7.5. Note that the map $\pi_4^e S^{2\rho-1-\succ} \to \pi_4^e \mathrm{tmf}(2)$ of Theorem 7.3 is a map of rank 2 free $\mathbb{Z}_{(3)}$-modules. Thus, it is an isomorphism if and only if its mod 3 reduction is, which is a map of rank 2 vector spaces over $\mathbb{F}_3$.

Similarly, the degree $2p-2$ component $\pi_*(E_{p-1})/(p,\mathfrak{m}^2)$ is a rank $p-1$ vector space over $\mathbb{F}_p$, generated by $u^{p-1}, u_1u^{p-1}, u_2u^{p-1}, \cdots, u_{p-2}u^{p-1}$. The map $\pi_{2p-2}^e S^{2\rho-1-\succ} \to \pi_{2p-2}(E_{p-1}/(p,\mathfrak{m}^2))$ of Theorem 7.4 factors through the mod p reduction of its domain, after which it becomes a map of rank $p-1$ vector spaces over $\mathbb{F}_p$.

Both Theorems 7.3 and 7.4 thus reduce to a question of whether maps of rank $p-1$ vector spaces over $\mathbb{F}_p$ are isomorphisms. These maps are furthermore equivariant, or maps of $\mathbb{F}_p[C_p]$-modules, with the actions of C_p given by reduced regular representations. We will therefore find Lemma 7.7 below particularly useful. First, we recall some basic facts from representation theory.

Recollection 7.6. Given two $\mathbb{F}_p[C_p]$-modules V and W, the space $\mathrm{Hom}_{\mathbb{F}_p}(V,W)$ inherits the structure of a C_p-module via conjugation, where $\gamma \in C_p$ sends $F : V \to W$ to $\gamma \circ F \circ \gamma^{-1}$. Then there is an identification

$$\mathrm{Hom}_{\mathbb{F}_p}(V,W)^{C_p} = \mathrm{Hom}_{\mathbb{F}_p[C_p]}(V,W),$$

so that the transfer determines a linear map

$$\mathrm{Tr} : \mathrm{Hom}_{\mathbb{F}_p}(V,W) \to \mathrm{Hom}_{\mathbb{F}_p[C_p]}(V,W).$$

Lemma 7.7. *Let $\bar{\rho}$ denote the $\mathbb{F}_p[C_p]$-module corresponding to the reduced regular representation of C_p. Then a homomorphism*

$$\phi \in \mathrm{Hom}_{\mathbb{F}_p[C_p]}(\bar{\rho},\bar{\rho})$$

is an isomorphism if and only if $\phi + \mathrm{Tr}(\psi)$ is for any transferred homomorphism $\mathrm{Tr}(\psi)$. More precisely, $\mathrm{Hom}_{\mathbb{F}_p[C_p]}(\bar{\rho},\bar{\rho})$ is a local $\mathbb{F}_p[C_p]$-algebra, with maximal ideal the ideal of transferred homomorphisms.

Proof. Note that $\overline{\rho}$ is a uniserial $\mathbb{F}_p[C_p]$-module, i.e. its submodules are totally ordered by inclusion. Since the endomorphism ring of a uniserial module over a Noetherian ring is local [Lam01, Proposition 20.20], the ring $\mathrm{Hom}_{\mathbb{F}_p[C_p]}(\overline{\rho},\overline{\rho})$ is local.

There is an identification $\overline{\rho}^{C_p} = \mathbb{1}$, so we obtain a ring homomorphism

$$\mathrm{Hom}_{\mathbb{F}_p[C_p]}(\overline{\rho},\overline{\rho}) \to \mathrm{Hom}_{\mathbb{F}_p[C_p]}(\overline{\rho}^{C_p},\overline{\rho}^{C_p}) = \mathrm{Hom}_{\mathbb{F}_p[C_p]}(\mathbb{1},\mathbb{1}) = \mathbb{F}_p\,.$$

Since this homomorphism is clearly surjective, we learn that its kernel must be equal to the maximal ideal of $\mathrm{Hom}_{\mathbb{F}_p[C_p]}(\overline{\rho},\overline{\rho})$.

On the other hand, for any $x \in \overline{\rho}^{C_p}$ and $\psi \in \mathrm{Hom}_{\mathbb{F}_p}(\overline{\rho},\overline{\rho})$, we have

$$\mathrm{Tr}(\psi)(x) = \sum_{i=0}^{p-1} \gamma^i \psi(\gamma^{-i}x) = \sum_{i=0}^{p-1} \gamma^i \psi(x) = \mathrm{Tr}(\psi(x)) = 0,$$

where the last equality follows from the fact that the transfer is zero on $\overline{\rho}$. It follows that $\mathrm{Tr}(\psi)$ lies in the maximal ideal of $\mathrm{Hom}_{\mathbb{F}_p[C_p]}(\overline{\rho},\overline{\rho})$.

Finally, the equivalence

$$\mathrm{Hom}_{\mathbb{F}_p}(\overline{\rho},\overline{\rho}) \cong \mathbb{1}\{\mathrm{id}_{\overline{\rho}}\} \oplus \text{free},$$

shows that the maximal ideal is equal to the image of Tr for dimension reasons. □

Proof of Theorem 7.3. Recall that $\pi_4^e\mathrm{tmf}(2)$ is a free $\mathbb{Z}_{(3)}$-module with basis λ_1 and λ_2. In light of Remark 7.5, it suffices to analyze the image of $v_1^{\mu_3}$ in its mod 3 reduction, which is a free $\mathbb{F}_3$-module generated by the reductions of λ_1 and λ_2. By combining Lemma 7.7 with Theorem 6.20, it suffices to show that a basis for this rank 2 $\mathbb{F}_3$-module is given by the mod 3 reduction of classes

$$\frac{v_1 - \gamma^2 v_1}{3}, \frac{\gamma v_1 - v_1}{3} \in \pi_4^e\mathrm{tmf}(2).$$

Here, $v_1 \in \pi_4^e\mathrm{tmf}(2)$ refers to the class of Notation 6.17, which depends on the chosen μ_3-orientation. By combining Remark 6.21 and Proposition 5.6, we may as well set v_1 to be $-\lambda_1 - \lambda_2$. Using the formulas of [Sto12, Lemma 7.3] (cf. Remark 5.7), we calculate

$$\frac{v_1 - \gamma^2 v_1}{3} \equiv -\lambda_2 \mod 3, \text{ and}$$

$$\frac{\gamma v_1 - v_1}{3} \equiv \lambda_1 \mod 3.$$

These clearly generate all of $\pi_4^e\mathrm{tmf}(2)$ modulo 3, as desired. □

Proof of Theorem 7.4. By arguments analogous to those in the previous proof, it suffices to check that

$$\frac{v_1 - \gamma^{p-1} v_1}{p}, \frac{\gamma v_1 - v_1}{p}, \cdots, \frac{\gamma^{p-2} v_1 - \gamma^{p-3} v_1}{p} \in \pi_{2p-2}^e E_{p-1}$$

reduce to generators of the degree $2p-2$ component of $\pi_*(E_{p-1})/(p, \mathfrak{m}^2)$. By Remark 6.21, we may assume that $\frac{\gamma v_1 - v_1}{p}$ in $\pi_{2p-2}^e E_{p-1}$ is the element v defined in Section 5.1. Under this assumption, the $p-1$ classes of interest become v and its translates under the C_p action on $\pi_{2p-2}^e E_{p-1}$. As noted in Remark 5.4, these span $\pi_{2p-2}^e E_{p-1}/(p, \mathfrak{m}^2)$. □

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