Equivariant multiplicative closure

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Introduction

This paper concerns a subtle issue that arises when attempting to invert element in the homotopy groups of equivariant commutative (E_∞) rings. It arose for the authors in [**5**], as part of the construction of the spectrum there denoted Ω, and was called to our attention when Justin Noel asked us, "how do you know you can do that?"

To set the stage, let G be a finite group, and R a G-equivariant commutative ring, by which we mean and equivariant E_∞ ring spectrum. We suppose given a subset $S \subset \pi_*^G R$.

QUESTION 1. Can one form an equivariant commutative ring $S^{-1}R$?

EXAMPLE 2. Take $G = \mathbb{Z}/2$, $R = S^0$, and $S = \{2\} \subset \pi_0 S^0$. In this case Noel's question becomes, "is it possible to invert 2 in equivariant stable homotopy theory?"

We will see that the questions is a little misleading as stated. One can in fact always invert 2 (in fact one can invert anything), but in order to know that the outcome is the expected one, there is a non-trivial condition to check.

The authors would like to thank Justin Noel for bringing this issue to our attention. It is underlain by a subtlety in the symmetric monoidal structure on equivariant spectra that seems to have gone unnoticed. It is also intimately connected to the theory of the "norm" in [**5**]. This note describes the main ideas that go into resolving Question 1. Details and a much more expanded discussion will appear elsewhere.

A purely algebraic analogue of some of the structures discussed in this paper were first formulated by Tambara in [**10**], and are now known by the name of "Tambara functors." For a recent general introduction to the algebraic theory the reader is referred to [**9**]. The question of localization in the context of Tambara functors has been investigated by Nakaoka [**8**], in which his Proposition 4.8 is similar to our Proposition 4.9. Our discussion of localization of rings and modules follows, more or less, the presentation of [**3**, Chapter VIII].

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1. Localization of commutative rings

Let's begin with localization in ordinary commutative algebra. Suppose that R is a commutative ring, and $S \subset R$ a subset of elements we intend to invert.

DEFINITION 1.3. The map $R \to S^{-1}R$ is the universal map $f : R \to R'$ having the property that $f(s)$ is a unit in R' for every $s \in S$.

Spelled out more precisely, the condition is that given a map $f : R \to R'$ with the property that $f(s)$ is a unit for every $s \in S$, there is a unique map $S^{-1}R \to R'$ factoring f.

Such a universal arrow clearly exists. One can take $S^{-1}R$ to be the quotient

$$S^{-1}R = R[x_s \mid s \in S]/(s \cdot x_s = 1)$$

of the polynomial algebra over R constructed by adding a variable x_s for each $s \in S$, by the relation indicated. There are advantages and disadvantages to this characterization. On one hand, it is clear that $S^{-1}R$ exists, and that the construction can be generalized. On the other hand, many of the fundamental properties of $S^{-1}R$ are not apparent. For example, it is not clear that $S^{-1}R$ is flat, or even that it is non-zero.

There is a second common description of $S^{-1}R$. Suppose for simplicity that S is finite and let $\{s_1, s_2, \dots\}$ be a sequence of elements of S having the property that each $s \in S$ occurs infinitely often as an s_i. The second description of $S^{-1}R$ is that it is the colimit of the sequence

$$\cdots \xrightarrow{s_i} R \xrightarrow{s_{i+1}} R \xrightarrow{s_{i+2}} \cdots .$$

From this description it is obvious that $S^{-1}R$ is flat, and that it is non-zero as long as there is an element of R which is not annihilated by any product of powers of elements of S. It is not, however, immediately obvious from this description that $S^{-1}R$ is a commutative ring, or even that it is independent of the choice of sequence $\{s_i\}$. These are not difficult things to verify, but the proofs involve explicit use of the elements of R, and so don't generalize well.

In fact, what this second construction actually produces is a universal R-*module* map from R to an R-module M having the property that for each $s \in S$, the map "multiplication by s"

$$s : M \to M$$

is an isomorphism. Since an R-algebra is an R-module there is a canonical map between these two constructions, and our basic question becomes

QUESTION 1.4. When is the canonical map from the localization of R as a module to the localization of R as an algebra and isomorphism?

In raising the issue we have considered only R, first as an R-algebra and then as an R-module. Our formulation will better lend itself to generalization if we think in terms of an arbitrary commutative R-algebra A and an arbitrary commutative R-module M. For a subset $S \subset R$ we let $A \to S_{\text{alg}}^{-1}A$ be the universal map from A to a commutative R-algebra in which each element of S becomes a unit. Similarly, we let $M \to S_{\text{mod}}^{-1}M$ be the universal R-module map to an R-module on which

multiplication by each element of S is an isomorphism. The universal properties give an evident map

$$S_{\mathrm{mod}}^{-1}A \to S_{\mathrm{alg}}^{-1}A$$

and we are interested in the conditions guaranteeing it to be an isomorphism.

2. Homotopy theoretic commutative algebra

We now pursue the previous considerations in homotopy theory. Fix a finite group G, and let R be a G-equivariant E_∞ ring spectrum. We will just refer to this structure by saying that R is an "equivariant commutative ring." We will also choose a finite subset S of the equivariant homotopy groups of R, and write $|s|$ for the "degree" of an element $s \in S$. We are being deliberately noncommittal about the term "equivariant homotopy groups." When we are led to the eventual formulation of our criteria in §4 we will want to allow s to be an G-equivariant map $S^V \to R$, where V is a virtual real representation of G, and S^V its "one-point compactification." In that case the symbol $|s|$ will refer to V. Nothing will be lost at this point if the reader imagines the set S to consist of G-equivariant maps

$$s : S^n \to R$$

with $n \in \mathbb{Z}$, in which case $|s|$ is the integer n.

It is a little easier to reverse the order of the previous section and begin with the localization of modules.

2.1. Modules. Let R-mod be the category of equivariant left R-modules. We do homotopy theory in R-mod by defining a map to be a weak equivalence if the underlying map of equivariant spectra is, and employ the shorthand notation $[M, N]_{\mathrm{mod}}$ to denote the abelian group of maps from M to N in the homotopy category ho R-mod.

For an equivariant R-module M and $s \in S$ let

(2.1) $$s : S^{|s|} \wedge M \to M$$

be the map representing "multiplication by s." It is given by the composite

$$S^{|s|} \wedge M \xrightarrow{s \wedge 1} R \wedge M \to M.$$

Since any suspension of a module is a module, this means we will be using the same symbol s to denote any suspension of the map (2.1). Similarly, if M is an equivariant *right* R-module we will let $\cdot s$ denote the map "right multiplication by s, given by

$$M \wedge S^{|s|} \xrightarrow{1 \wedge s} M \wedge R \to M.$$

When M is an equivariant R bi-module the map s is a map of right R-modules and $\cdot s$ is a map of equivariant left modules. Of course our assumption that R is commutative means that every left module can be regarded as a bimodule, and the distinction is not actually significant. However since our main concern is a subtlety in the theory of equivariant commutative rings, it seems best to use the commutativity of R sparingly.

DEFINITION 2.2. A left R-module M is *S-local* if for each $s \in S$ the map $s : S^{|s|} \wedge M \to M$ is a weak equivalence. Similarly a right module is S-local if each of the maps $\cdot s$ is a weak equivalence.

2.1.1. *The localization conditions.* The module localization of M is meant to be a universal arrow from M to an S-local R-module N. A little homotopical algebra is needed to state the universal property accurately. Let $R\text{-mod}_{M/}$ be the category of equivariant R-modules equipped with a map from M. We define a map

$$M \to N_1 \to N_2$$

in $R\text{-mod}_{/M}$ to be a *weak equivalence* if the underlying map $N_1 \to N_2$ of equivariant spectra is. We say that an object $M \to N$ is S-*local* if N is. The module localization $M \to S_{\text{mod}}^{-1}M$ is characterized uniquely up to unique isomorphism in ho $R\text{-mod}_{M/}$ by the following conditions:

(i)$_{\text{mod}}$ The module $S_{\text{mod}}^{-1}M$ is S-local;

(ii)$_{\text{mod}}$ If $M \to N$ is S-local, then ho $R\text{-mod}_{M/}(S^{-1}M, N) = \text{pt}$.

In stating these conditions we have abbreviated $M \to S_{\text{mod}}^{-1}M$ and $M \to N$ to just $S_{\text{mod}}^{-1}M$ and N in order to avoid a cumbersome expression for maps in the homotopy category.

2.1.2. *Local modules.* To construct the S-localization of M we first investigate S-local modules. let $\{s_1, s_2, \dots\}$ be a sequence of elements of S having the property that each $s \in S$ occurs infinitely often as an s_i, and let E be the right R-module given by

$$E = \text{ho}\varinjlim \left\{ R \xrightarrow{s_1} S^{-|s_1|} \wedge R \xrightarrow{s_2} S^{-|s_1|-|s_2|} \wedge R \to \cdots \right\}.$$

and E' left R-module

$$E' = \text{ho}\varinjlim \left\{ R \xrightarrow{\cdot s_1} S^{-|s_1|} \wedge R \xrightarrow{\cdot s_2} S^{-|s_1|-|s_2|} \wedge R \to \cdots \right\}.$$

Of course since R is commutative there is a canonical equivalence between E regarded as an equivariant left R-module, and E'. Write

$$\mathcal{Z} = \{S^t \wedge (R \cup_{\cdot s} C(S^{|s|} \wedge R)) \mid s \in S, t \in \mathbb{Z}\}$$

for the set of suspensions mapping cones of the maps s. The elements of $\mathcal{Z}$ are left R-modules.

Though we eventually hope to endow E with the structure of an equivariant commutative R-algebra, it isn't clear at this point whether or not it has a multiplication. It is, however, a right R-module, and it does come equipped with a "unit" $R \to E$.

LEMMA 2.3. *Both E and E' are S-local. For each $Z \in \mathcal{Z}$ the G-spectrum $E \underset{R}{\wedge} Z$ is contractible.*

Proof: This result makes use of the commutativity of R. We begin with the assertion that E is S-local. Let $s \in S$ and consider the following commutative ladder

$$
\begin{array}{ccccccc}
R \wedge S^{|s|} & \xrightarrow{s_1} & S^{-|s_1|} \wedge R \wedge S^{|s|} & \xrightarrow{s_s} & S^{-|s_1|-|s_2|} \wedge R \wedge S^{|s|} & \longrightarrow & \cdots \\
\downarrow{\cdot s} & & \downarrow{\cdot s} & & \downarrow{\cdot s} & & \\
R & \xrightarrow{s_1} & S^{-|s_1|} \wedge R & \xrightarrow{s_2} & S^{-|s_1|-|s_2|} \wedge R & \longrightarrow & \cdots
\end{array}
$$

The induced map of homotopy colimits of the rows is right multiplication by s

$$\cdot s : E \wedge S^{|s|} \to E.$$

One easily checks it to be an equivalence by considering the induced map of equivariant homotopy groups, using the homotopy commutativity of R and the condition that each $s \in S$ occurs as an s_i infinitely often. The proof that E' is S-local is similar. The last assertion follows from the cofibration sequence

$$S^t \wedge E \wedge S^{|s|} \xrightarrow{\cdot s} S^t \wedge E \to E \underset{R}{\wedge} Z.$$

$\square$

PROPOSITION 2.4. *For an R-module N the following are equivalent*

i) *For each $s \in S$ the map $s : S^{|s|} \wedge N \to N$ is a weak equivalence;*

ii) *The map $N \to E \underset{R}{\wedge} N$, given by smashing with the unit, is a weak equivalence;*

iii) *For each Z with $E \underset{R}{\wedge} Z \sim *$, the group $[Z, N]_{mod}$ is trivial;*

iv) *For each $Z \in \mathcal{Z}$ the group $[Z, N]_{mod}$ is trivial;*

Proof: The assertion i) $\implies$ ii) follows from the fact that smashing commutes with colimits, and ii) $\implies$ iii) is a consequence of the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & N \\
\downarrow & & \downarrow{\scriptstyle \sim} \\
E \underset{R}{\wedge} Z & \xrightarrow[E \wedge f]{} & E \underset{R}{\wedge} N
\end{array}
$$

which is constructed by smashing an equivariant R-module map $Z \to N$ with the unit map $R \to E$. The implication iii) $\implies$ iv) is immediate from the second assertion in Lemma 2.3. Finally, iv) $\implies$ i) follows from the fact that the map

$$[S^t \wedge R, N]_{\mathrm{mod}} \to [S^t \wedge S^{|s|} \wedge R, N]_{\mathrm{mod}},$$

induced by $s : S^{|s|} \wedge R \to R$, is isomorphic to the map

$$\pi_t N \to \pi_t S^{-|s|} \wedge N$$

induced by s. $\square$

From Lemma 2.3 and the equivalence of the first two parts statements in Proposition 2.4 we have

COROLLARY 2.5. *The map $R \to E'$ extends to a weak equivalence*

$$E \to E \underset{R}{\wedge} E'.$$

$\square$

As mentioned earlier, the commutativity of R allows one to regard left modules as right modules and for us to identify E with E'. It also makes the category of equivariant left modules into a symmetric monoidal category. With this in mind the equivalence in Corollary 2.5 can be written as an equivalence

$$E \to E \underset{R}{\wedge} E,$$

making it appear that the module E *must* be an equivariant commutative R-algebra, and that it must be so in a unique way. As we shall see, something like this is true. But there is more to the story.

2.1.3. *Construction of the module localization.* It follows from part iii) of Proposition 2.4 that an R-module is S-local if and only if it is E-local in the sense of Bousfield [**1**], and that the universal arrow

$$(2.6) \qquad\qquad M \to S^{-1}_{\mathrm{mod}} M$$

we are looking for is the Bousfield localization. This leads to a construction. The Bousfield localization of any R-module M can be constructed as the homotopy colimit of the sequence

$$M_0 \to \cdots \to M_i \to M_{i+1} \to \cdots$$

constructed inductively, starting with $M_0 = M$ and, building M_{i+1} out of M_i as a pushout

$$(2.7) \qquad\qquad \begin{array}{ccc} Z_i & \longrightarrow & CZ_i \\ \downarrow & & \downarrow \\ M_i & \longrightarrow & M_{i+1} \end{array}$$

in which Z_i is the wedge

$$Z_i = \bigvee_{\substack{Z \to M_i \\ Z \in \mathcal{Z}}} Z$$

and the map $Z_i \to M_i$ is the tautological map, whose restriction to each summand is the map indicated by its index.

Let's temporarily write $M \to L_E M$ for the map from M to the homotopy colimit of the sequence above. The fact that the modules $Z \in \mathcal{Z}$ are compact objects implies that $L_E M$ satisfies condition iv) of Proposition 2.4 and hence is S-local. The pushout squares show that for any $M \to N$ in which N is S-local

$$\mathrm{ho}\, R\text{-}\mathrm{mod}_{M/}(M_i, N) = \mathrm{pt}$$

hence

$$\mathrm{ho}\, R\text{-}\mathrm{mod}_{M/}(L_E M, N) = \mathrm{pt}.$$

Thus $M \to L_E M$ satisfies conditions (i)$_{\mathrm{mod}}$ and (ii)$_{\mathrm{mod}}$ of §2.1.1, so can be taken to be the universal arrow $M \to S^{-1}_{\mathrm{mod}} M$ we were seeking.

Bousfield's construction leads to a further property of localization that is special to *modules*. Applying $E \underset{R}{\wedge} (-)$ to the pushout square (2.7) one finds that the map

$$E \underset{R}{\wedge} M_i \to E \underset{R}{\wedge} M_{i+1}$$

is a weak equivalence. It follows that the map

$$E \underset{R}{\wedge} M \to E \underset{R}{\wedge} L_E M$$

is a weak equivalence. This leads to another characterization of the module localization $M \to S^{-1}_{\mathrm{mod}} M$ (which, in Bousfield's formulation [**1**] is taken as the definition).

PROPOSITION 2.8. *Let $M \to N$ be a map of equivariant left R-modules in which N is S-local. If*

$$E \underset{R}{\wedge} M \to E \underset{R}{\wedge} N$$

is a weak equivalence then the unique map $S^{-1}M \to N$ making

(2.9)
$$
\begin{array}{ccc}
M & \longrightarrow & S^{-1}M \\
 & \searrow & \downarrow \\
 & & N
\end{array}
$$

commute is a weak equivalence.

Proof: Smashing (2.9) with E gives a diagram

$$
\begin{array}{ccc}
E \underset{R}{\wedge} M & \longrightarrow & E \underset{R}{\wedge} S^{-1}M \\
 & \searrow & \downarrow \\
 & & E \underset{R}{\wedge} N
\end{array}
$$

in which the top arrow is a weak equivalence by the above discussion and the diagonal arrow is a weak equivalence by assumption. This means that the vertical is also a weak equivalence. Now consider the diagram

$$
\begin{array}{ccc}
S^{-1}M & \longrightarrow & N \\
\downarrow & & \downarrow \\
E \underset{R}{\wedge} S^{-1}M & \longrightarrow & E \underset{R}{\wedge} N.
\end{array}
$$

We have just shown that the bottom horizontal arrow to be a weak equivalence. The vertical arrows are weak equivalences by part ii) of Proposition 2.4. It follows that the top arrow is a weak equivalence. □

COROLLARY 2.10. *The unit map $R \to E'$ is the S-localization of R, and the R-modules E and E' are independent of the choice of sequence $\{s_1, s_2, \dots\}$.*

Proof: By Lemma 2.3 and Corollary 2.5 the map $R \to E'$ satisfies the conditions of Proposition 2.8. □

We have now seen that the module localization of R, characterized by the properties described in §2.1.1, has the homotopy type we expect both the algebra and module the localizations to have, namely that of

$$
E' = \mathrm{ho}\varinjlim \left\{ R \xrightarrow{\cdot s_1} S^{-|s_1|} \wedge R \xrightarrow{\cdot s_2} S^{-|s_1|-|s_2|} \wedge R \to \cdots \right\}.
$$

Our next aim is to investigate what actually happens with the algebra localization.

2.2. Algebras. We now turn to the localization in algebras. Let R-alg be the category of G-equivariant E_∞ algebras over R. We do homotopy theory in R-alg by defining a map to be a *weak equivalence* if the underlying map of equivariant spectra is. We will say that an R-algebra is *S-local* if it is so when regarded as a left R-module. As with modules, in order to state the universal property of algebra localization we will need to consider the category of equivariant commutative R-algebras B equipped with an R-algebra map $A \to B$. But such a B is just an A-algebra, so we will denote this category A-alg, rather than using the analogue

R-alg$_{A/}$ of our notation for modules. We do homotopy theory in A-alg by defining a map $A \to B_1 \to B_2$ to be a weak equivalence if the underlying map of G-spectra $B_1 \to B_2$ is.

Given an R-algebra A we wish to construct an R-algebra map $A \to S_{\mathrm{alg}}^{-1}A$ which is universal for maps from A to R-algebras B which are S-local. As in the discussion of modules, the universal property consists of the conditions

(i)$_{\mathrm{alg}}$ The R-algebra $S_{\mathrm{alg}}^{-1}A$ is S-local;

(ii)$_{\mathrm{alg}}$ If $A \to B$ is S-local, then the set ho A-alg$(S_{\mathrm{alg}}^{-1}A, B)$ consists of just one point.

Imitating the discussion in §1, one can certainly construct $S_{\mathrm{alg}}^{-1}A$ as the pushout of

$$\begin{array}{ccc} \mathrm{Sym}_A[r_s \mid s \in S] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Sym}_A[x_s \mid s \in S] & \longrightarrow & S_{\mathrm{alg}}^{-1}A \end{array}$$

in which $|x_s| = -|s|$, $|r_s| = 0$, the left vertical arrow sends r_s to $s \cdot x_s$ and the top arrow sends r_s to 1. This construction produces an algebra which is clearly S-local, however it requires a small computation in topological Andre-Quillen cohomology to show that it satisfies property (ii)$_{\mathrm{alg}}$.

A better approach is to imitate the construction of the Bousfield localization described in §2.1 and construct $S_{\mathrm{alg}}^{-1}A$ as the homotopy colimit of the sequence

$$\cdots \to A_i \to A_{i+1} \to \cdots$$

constructed inductively, starting with $A_0 = A$, and constructing A_{i+1} from A_i as the (algebra) pushout of

(2.11)
$$\begin{array}{ccc} \mathrm{Sym}_R(Z_i) & \longrightarrow & \mathrm{Sym}_R(CZ_i) \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & A_{i+1}, \end{array}$$

in which Z_i is the wedge of R-modules

$$Z_i = \bigvee_{\substack{Z \to A_i \\ Z \in \mathcal{Z}}} Z$$

and the map $\mathrm{Sym}(Z_i) \to A_i$ is the algebra extension of the tautological map whose restriction to each summand is indicated by its index. From this description the universal property is more obvious. That $S_{\mathrm{alg}}^{-1}A$ is S-local follows, as before, from the compactness of the $Z \in \mathcal{Z}$. The Meyer-Vietoris sequence associated to each pushout square shows that if B is an S-local A-algebra then for each i the restriction map

$$\mathrm{ho}\, A\text{-alg}(A_{i+1}, B) \to \mathrm{ho}\, A\text{-alg}(A_i, B)$$

is a bijection. Since $A_0 = A$ each of these sets consists of just one point, and hence so does

$$\mathrm{ho}\, A\text{-alg}(S_{\mathrm{alg}}^{-1}A, B).$$

Thus Bousfield's construction produces an A-algebra which is easily verified to have the two properties characterizing the localization of A at S.

Since $S_{\mathrm{alg}}^{-1}A$ is S-local as an R-module, there is a unique a module map

(2.12)
$$S_{\mathrm{mod}}^{-1}A \to S_{\mathrm{alg}}^{-1}A,$$

and our main question becomes

QUESTION 2.13. Under what conditions on S is (2.12) a weak equivalence?

By Proposition 2.8, this is equivalent to

QUESTION 2.14. Under what conditions on S is the map

$$E \underset{R}{\wedge} A \to E \underset{R}{\wedge} S_{\mathrm{alg}}^{-1}A$$

a weak equivalence?

An analysis of the pushout squares (2.11) shows that this, in turn, is equivalent to

QUESTION 2.15. Under what conditions on S is the map

$$E \underset{R}{\wedge} \mathrm{Sym}_R(Z) \to E \underset{R}{\wedge} R$$

induced by $Z \to *$ a weak equivalence for all $Z \in \mathcal{Z}$?

Finally, this is equivalent to

QUESTION 2.16. Under what conditions on S is

$$E \underset{R}{\wedge} \mathrm{Sym}^n Z$$

contractible, for all $n \geq 1$ and all $Z \in \mathcal{Z}$?

We will address this question in the next section.

3. Extended powers and indexed monoidal products

3.1. The homotopy type of symmetric powers. We have a spectrum Z which is E-acyclic in the sense that $E \underset{R}{\wedge} Z$ is contractible, and we wish to know whether or not $\mathrm{Sym}^n Z$ is E-acyclic. To define $\mathrm{Sym}^n Z$ requires that R be commutative so that the category R-mod acquires a symmetric monoidal structure. The definitions are arranged (see for example [**3**,**6**,**7**]) so that $\mathrm{Sym}^n Z$ has the homotopy type of

$$(E_G\Sigma_n)_+ \underset{\Sigma^n}{\wedge} Z^{(n)},$$

in which

$$Z^{(n)} = Z \underset{R}{\wedge} \ldots \underset{R}{\wedge} Z,$$

is the iterated smash product over R, and $E_G\Sigma_n$ is the G-equivariant analogue of the free contractible Σ_n-space $E\Sigma_n$. We first consider the case of the trivial group. When G is trivial one has

$$\mathrm{Sym}^n Z \sim (E\Sigma_n)_+ \underset{\Sigma^n}{\wedge} Z^{(n)}.$$

Since $E\Sigma_n$ is a homotopy colimit of free Σ_n-sets, it suffices to show that if T is a free Σ_n-set then

$$T_+ \underset{\Sigma_n}{\wedge} Z^{(n)}$$

is E-acyclic. This reduces to the case in which $T = \Sigma_n$, in which case

$$T_+ \underset{\Sigma_n}{\wedge} Z^{(n)} = (\Sigma_n)_+ \underset{\Sigma_n}{\wedge} Z^{(n)} = Z^{(n)}.$$

But if Z is E-acyclic, then so is $Z^{(n)}$ by the associativity of the smash product. We therefore have

PROPOSITION 3.1. *When G is the trivial group, the map $S_{mod}^{-1}A \to S_{alg}^{-1}A$ is a weak equivalence for any equivariant commutative R-algebra A.*   □

How does this change if G is non-trivial? The main difference is that the Σ_n-space $E\Sigma_n$ must be replaced by the $E_G\Sigma_n$. As mentioned above, $E_G\Sigma_n$ is the G-analogue of $E\Sigma_n$. It is a space with a $G \times \Sigma_n$-action and can be defined to be the total space of the universal G-equivariant principal Σ_n-bundle. It is characterized up to $G \times \Sigma_n$-equivariant weak equivalence by the property that its fixed point space for $H \subset G \times \Sigma_n$ is empty if $H \cap \Sigma_n$ is non-zero, and contractible otherwise. This characterization shows that the $E_G\Sigma_n$ is a homotopy colimit of transitive $G \times \Sigma_n$-sets T which are Σ_n-free. Thus to show that

$$\mathrm{Sym}^n Z = (E_G\Sigma_n)_+ \underset{\Sigma^n}{\wedge} Z^{(n)}$$

is E-acyclic when Z is, it suffices to show that for such T, if Z is E-acyclic, so is the result of forming

$$T_+ \underset{\Sigma_n}{\wedge} Z^{(n)}.$$

We have therefore reduced Question 2.13 to

QUESTION 3.2. What conditions on S guarantee that

$$T_+ \underset{\Sigma_n}{\wedge} Z^{(n)}$$

is E-acyclic when Z is E-acyclic, where T is a Σ_n-free finite $G \times \Sigma_n$-set?

3.2. Indexed monoidal products. To get a feeling for the construction

$$T_+ \underset{\Sigma_n}{\wedge} Z^{(n)}$$

let's focus on a simple example. Take $n = 2$ and $G = \mathbb{Z}/2$ and write $g \in G$ for the generator. We consider two examples of Σ_2-free, transitive $G \times \Sigma_2$-sets

$$T_\Delta = G \times \Sigma_2/\text{diagonal} \quad \text{and}$$
$$T_0 = G \times \Sigma_2/G.$$

For a G-spectrum Z one easily checks that

$$(T_0)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z$$

is just $Z \wedge Z$ with the diagonal G-action

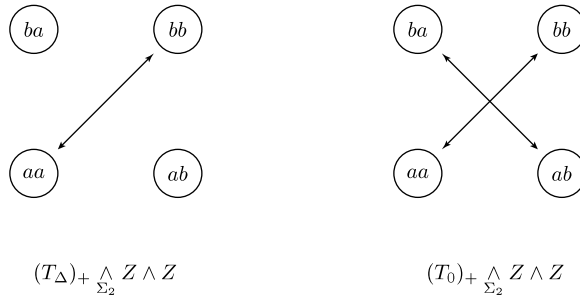$$x \wedge y \mapsto g(x) \wedge g(y)$$

and

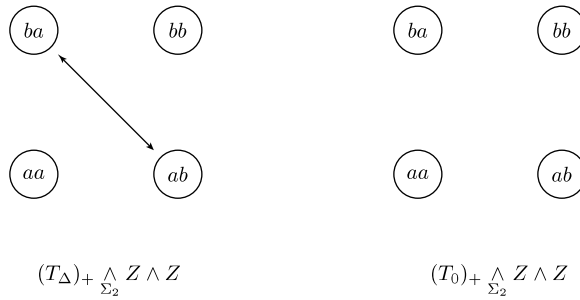$$(T_\Delta)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z$$

is $Z \wedge Z$ with G-action

$$x \wedge y \mapsto g(y) \wedge g(x).$$

For instance, let Let $Z = G_+$, i.e. $S^0 \vee S^0$ with the "flip" action of G. Then the two constructions work out as indicated below:

$$(T_\Delta)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z \qquad\qquad (T_0)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z$$

and when Z is the suspension spectrum of $\{a, b\}_+$ with the trivial action of G one gets

$$(T_\Delta)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z \qquad\qquad (T_0)_+ \underset{\Sigma_2}{\wedge} Z \wedge Z$$

In the general case $T_+ \underset{\Sigma_n}{\wedge} Z^{(n)}$ is constructed as a combination of the following processes. First, starting with a subgroup $H \subseteq G$, an H-spectrum X and a finite H-set U one forms

$$W = \bigwedge_{u \in U} X.$$

This is a spectrum with an H-action. Next one induces the result up to a G-spectrum by forming

$$G_+ \underset{H}{\wedge} W = \bigvee_{G/H} g \cdot W.$$

These processes are studied in much more detail in [**5**] where they are called "indexed monoidal products." Their similarity is more apparent with a slightly more general setup. Let U be finite H-set and $u \mapsto X_u$ a functor from the action category $U /\!/ H$ to spectra. Thus X_u is equivariant for the isotropy group $H_u \subset G$ of u, and there are maps $g_* : X_u \to X_{gu}$ whose formation is associative, etc. Out of this functor one can construct H-spectra by forming the "indexed sum"

$$\bigvee_{u \in U} X_u,$$

the "indexed product"

$$\prod_{u \in U} X_u,$$

and the "indexed smash product"

$$\bigwedge_{u \in U} X_u.$$

Pursuing this, what emerges is the principle that in equivariant homotopy theory it isn't enough to just study ordinary coproducts, products and smash products. One must also systematically include the situation in which the group G is acting on the indexing set. The benefits from doing this impact even the most basic notions of the field. For example, the additivity of non-equivariant stable homotopy theory is expressed by the assertion that for any X, Y, the map

$$X \vee Y \to X \times Y$$

is a weak equivalence. Equivariant stable homotopy theory possesses an enhanced *equivariant additivity*: the map

$$\bigvee_{u \in U} X_u \to \prod_{u \in U} X_u$$

is a weak equivalence. In the presence of stability (the fact that fibration sequences are weakly equivalent to cofibration sequences), this property is equivalent to the dualizability of finite G CW-spectra and to the invertibility of representation spheres S^V. The invertibility of representation spheres is arguably *the* fundamental characteristic of equivariant stable homotopy theory. It is often referred to as *equivariant stability*, and is the only aspect of equivariant stable homotopy theory that is not easily described in algebraic terms. But (for finite G) once one thinks in terms of indexed coproducts, equivariant stability becomes simply the enhancement of ordinary stability one finds by requiring *equivariant additivity*. And that is a fundamentally algebraic notion.

In the next section we will turn to the structures that arise from systematically considering equivariant multiplicativity.

4. Equivariant multiplicative closure

The indexed monoidal smash product allows one to express an enhanced commutativity property possessed by equivariant commutative rings. For a non-equivariant commutative ring, the multiplication map

$$R \wedge R \to R$$

is a ring homomorphism. This is, of course, related to the fact that the coproduct of E_∞ rings is weakly equivalent to the smash product. One might naturally guess that the indexed coproduct of equivariant commutative rings is weakly equivalent to the smash product

$$\coprod_{u \in U} R_u \approx \bigwedge_{u \in U} R_u,$$

and that there is canonical equivariant ring homomorphism

(4.1) $$\bigwedge_{u \in U} R_u \to R.$$

This is in fact the case. The particular proof depends on the particular model for equivariant stable homotopy theory you are using, but ultimately it comes down to the fact that the indexed product of the universal $G_u \times \Sigma_n$-spaces

$$\prod_{u \in U} E_{G_u} \Sigma_n$$

is a model for the universal $G \times \Sigma_n$-space $E_G \Sigma_n$. For more details see of [**5**, §2, App. A, B]. The same discussion applies to an equivariant commutative R-algebra A, and with the smash product replaced with the smash product over R.

These considerations lead to two related notions of *equivariant multiplicative closure*.

DEFINITION 4.2. A collection $\mathcal{C}$ of pairs (Z, H) with H a subgroup of G and Z and H-equivariant R-module is *equivariantly multiplicatively closed* if it is stable under the formation of indexed smash products. It is an *equivariant ideal* if for every finite H-set U and every functor Z from $U/\!/H$ to spectra, if for some u, (Z_u, H_u) is in $\mathcal{C}$ then so is the indexed smash product

$$\bigwedge_{u \in U} Z_u.$$

EXAMPLE 4.3. If A is an equivariant commutative R-algebra, the set

$$\{(Z, H) \mid A \underset{R}{\wedge} Z \underset{H}{\sim} *\}$$

of *A-acyclic R-modules* is an equivariant ideal. (The symbol $\sim_H$ indicates an H-equivariant weak equivalence.) Indeed if for some u

$$A \underset{R}{\wedge} Z_u$$

is weakly H_u−contractible, then the indexed smash product

$$\bigwedge_{u \in U} (A \wedge Z_u)$$

is H-contractible. The indexed smash product is a symmetric monoidal functor, so this implies that

$$\left(\bigwedge_{u \in U} A \right) \wedge \left(\bigwedge_{u \in U} Z_u \right)$$

is H-contractible. But then so is

$$(A) \wedge \left(\bigwedge_{u \in U} Z_u \right)$$

since there is an equivariant ring map

$$\bigwedge_{u \in U} A \to A.$$

Example 4.3 bears directly on the situation we found ourselves considering in the previous section. In fact it tells us that the map

$$S_{\mathrm{mod}}^{-1} A \to S_{\mathrm{alg}}^{-1} A$$

is a weak equivalence for every A if and only if it is so for R, and that this holds if the set of E-acyclic modules forms an ideal. By the above discussion, this condition is also necessary, since after all, what we're aiming to show in the first place is that E is an equivariant commutative ring.

We can address this issue entirely in terms of S by considering a second notion of equivariant multiplicative closure. Let V be a G-equivariant vector bundle over

U and S^V its one-point compactification. Writing V_u for the fiber over u we may think of S^V as the indexed wedge

$$\bigvee_{u \in U} S^{V_u}.$$

A G-equivariant map

$$\bigvee_{u \in U} S^{V_u} \to R$$

corresponds to a collection of compatible G_u-equivariant maps

$$\alpha_u : S^{V_u} \to R.$$

Forming the indexed smash product and composing with (4.1) gives a map

$$\bigwedge_{u \in U} S^{V_u} \to \bigwedge_{u \in U} R \to R$$

which we call the *indexed product* of the α_u. We denote this indexed product by

$$\prod_{u \in U} \alpha_u : S^W \to R$$

in which

$$W = \bigoplus_{u \in U} V_u$$

is the space of global sections of V.

EXAMPLE 4.4. Consider the case $R = S^0$. Then $\pi_0^G R$ is the Burnside ring $A(G)$ of finite G-sets. A functor $T(u) = T_u$ from $U/\!/G$ to the category of finite sets determines a G-equivariant homotopy class of maps

$$\bigvee_{u \in U} S^0 \to S^0.$$

The indexed product in this case corresponds to the G-set given by the indexed product

$$\prod_{u \in U} T_u.$$

The examples at the end of §3.2 illustrate this situation in the case $G = \mathbb{Z}/2$.

EXAMPLE 4.5. Consider a general finite G and a transitive G-set $U = G/H$ for some proper subgroup $H \subset G$. The indexed product of set of G_u-equivariant maps

$$\alpha_u : S^1 \to R$$

will be a G-equivariant map

$$\prod_{u \in U} \alpha_u : S^W \to R$$

where W is the real G-representation induced from the trivial representation of H.

Example 4.5 shows that even if one were only interested in $\pi_n^G R$ for $n \in \mathbb{Z}$, the presence of indexed products for equivariant commutative rings would necessitate consideration of the groups $\pi_V^G R$ for certain virtual representations V of G. In fact by the time everything is all laid out, one is actually led to consider all of the homotopy groups $\pi_V^H R$ with $H \subset G$ a subgroup, and V a virtual representation of H. In order not to overly complicate the discussion we have restricted our focus in

this groups $\pi_V^G R$. In view of this, we henceforth take the expression "equivariant stable homotopy groups" to mean the sum of the groups π_V^G, and view it as graded over the real representation ring of G. Elements of the equivariant stable homotopy groups will always be assumed to be homogeneous.

The enhancement of equivariant multiplication given by the formation of "indexed products" goes back to the work of Greenlees and May [**4**], and is related to concept of Tambara functors in the work of Brun [**2**]. It leads to our second notion of *equivariant multiplicative closure*.

DEFINITION 4.6. Let R be an equivariant commutative ring. A subset of the equivariant stable homotopy groups of R is *multiplicatively closed* if it is closed under the formation of indexed products.

EXAMPLE 4.7. Let A be an equivariant commutative ring. The set of $\alpha \in \pi_V^G R$ which are *units* is multiplicatively closed.

Example 4.7 tells us that if we are going to force all the elements of some set S to become units in an equivariant commutative ring, we are also going to force all indexed products of those elements to become units as well. This is the subtle difference between the equivariant and non-equivariant case.

DEFINITION 4.8. Let R be an equivariant commutative ring, and S a subset of the equivariant homotopy groups of R. The *equivariant multiplicative closure* $\hat{S}$ of S is the smallest multiplicatively closed subset of the equivariant homotopy groups of R containing S.

We then have

PROPOSITION 4.9. *Let R be an equivariant commutative ring and A an equivariant commutative R-algebra. For every subset S of the equivariant homotopy groups of R, the maps*

$$S_{alg}^{-1}A \to \hat{S}_{alg}^{-1}A$$

and

$$\hat{S}_{alg}^{-1}A \to \hat{S}_{mod}^{-1}A$$

are weak equivalences. □

PROPOSITION 4.10. *With the above notation, the map*

$$S_{mod}^{-1}A \to \hat{S}_{mod}^{-1}A$$

is a weak equivalence if and only if every indexed product of elements of S divides an ordinary product of elements of S.

Combining Propositions 4.9 and 4.10, and specializing to the case of interest, we conclude

COROLLARY 4.11. *The map*

$$S_{mod}^{-1}R \to S_{alg}^{-1}R$$

is a weak equivalence if and only if every indexed product of elements of S divides an ordinary product of elements of S. □

The following example will be discussed in much greater detail in a later paper.

EXAMPLE 4.12. Consider the case $R = S^0$ and $G = \mathbb{Z}/2$. Let σ be the 1-dimensional sign representation and $s \in \pi_{-\sigma} S^0$ the map obtained by desuspending the fixed point inclusion $S^0 \to S^\sigma$. Set $S = \{s\}$. Then $S_{\mathrm{mod}}^{-1} R$ is the suspension spectrum of the reduced suspension of $E\mathbb{Z}/2$, and is not (equivariantly) contractible. On the other hand the indexed product

$$(T_\Delta)_+ \underset{\Sigma_2}{\wedge} S^{-\sigma} \wedge S^{-\sigma} \to R$$

depends only on the non-equivariant map underlying s, which is null. It follows that the equivariant multiplicative closure of S contains 0 and so $S_{\mathrm{alg}}^{-1} R \sim *$. Thus in this case the map $S_{\mathrm{mod}}^{-1} R \to S_{\mathrm{alg}}^{-1} R$ is not a weak equivalence.

4.1. Inverting 2. We can now turn to our specific example of Noel's question. Take $G = \mathbb{Z}/2$ and $R = S^0$ and $S = \{2\}$. As mentioned in Example 4.4, $\pi_0^G S^0$ is the Burnside ring $A(G)$. When $G = \mathbb{Z}/2$ it is one has

$$A(G) \approx \mathbb{Z}[\rho]/\rho^2 - 2\rho$$

where $\rho = G$ is the free, transitive G-set. Under the map $A(G) \to \pi_0^G S^0$, the element 2 corresponds to the G-set with 2 elements and trivial G-action. We calculated the non-trivial indexed product of this set with itself at the end of §3.2. A glance at the illustrations gives the formula

$$\prod_{u \in \mathbb{Z}/2} 2 = 2 + \rho.$$

Noel's question is thus reduced to

QUESTION 4.13. Does $2 + \rho$ divide a power of 2 in $Z[\rho]/(\rho^2 - 2\rho)$?

If the answer to the above question is "yes" then inverting 2 in an equivariant commutative ring has the expected effect. If it doesn't then there is something else going on. Fortunately the answer to this question is "yes"

$$(2 + \rho)(4 - \rho) = 8.$$

In fact, for any finite group G inverting any integer n has the expected effect. There is, however, a condition to check.

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