GREENLEES E HT II

Note Title 10/26/2010

This is the most elementary talk this week.

I. Isotropy separation

G = finite gp

$E\mathcal{F}$ = classifying space for family of subgps $\mathcal{F}$.

$\mathcal{F}$ is closed under subgps + conjugacy

$$E\mathcal{F}^H = \begin{cases} * & \text{if } H \in \mathcal{F} \\ \phi & \text{else} \end{cases}$$

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow S^0 * E\mathcal{F} =: \tilde{E}\mathcal{F}$$

Let $\mathcal{S}\mathcal{J}(X) = \{H : X^H \not\simeq *\}$

$\Omega\tilde{\mathcal{J}}(E\mathcal{F}_+) = \mathcal{F}$ = family

$\Omega\tilde{\mathcal{J}}(\tilde{E}\mathcal{F}) = \mathcal{F}^c$ = cofamily (closed under supergp + conj)

<u>Lemma</u> Given families $\mathcal{F}$ and $\mathcal{F}'$, cofamilies $\mathcal{C}$ and $\mathcal{C}'$, if $\mathcal{F}\cap\mathcal{C} = \mathcal{F}'\cap\mathcal{C}' =: \mathcal{N}$

then $E\mathcal{F}_+ \wedge \tilde{E}\mathcal{C}^c \simeq E\mathcal{F}'_+ \wedge \tilde{E}\mathcal{C}'^c$

Let $X[\mathcal{N}] = X \wedge E\mathcal{F}_+ \wedge \tilde{E}\mathcal{C}^c$

$$\Omega\tilde{\mathcal{J}}(X[\mathcal{N}]) = \Omega\tilde{\mathcal{J}}(X) \cap \mathcal{N}$$

<u>Idea</u> Analyse X by separating isotropy

e.g. (1) X nonequiv $\rightsquigarrow$ isotropy $\{1\}$

(2) X^N $(N \trianglelefteq G, N \neq 1) \rightsquigarrow [\supseteq N]$

Examples

$$\begin{array}{c} C_8 \\ \uparrow \\ C_4 \\ \uparrow \\ C_2 \\ \uparrow \\ 1 = C_1 \end{array}$$

$[\supseteq C_2]$ = family of nontrivial subgps

$\{1\}$ = trivial subgp

λ = rotation by $\pi/4$

$$S^{\infty\lambda} = \bigcup_k S^{k\lambda} \quad (= \tilde{E}[\not\supseteq C_2])$$

$$EG_+ = S(\infty\lambda)_+ = \bigcup_{k\geq 0} S(k\lambda)$$

C_8, C_4: $S^{\infty\lambda_2}$

C_2, C_1: $S(\infty\lambda_2)$

Note $\tilde{\mathcal{F}}(S^{\infty V}) = \{H \mid V^H = 0\}$

$$X \wedge S^{\infty V} = \varinjlim \{X \wedge S^0 \xrightarrow{X\wedge a_V} X \wedge S^V \longrightarrow X \wedge S^{2V} \longrightarrow \cdots\}$$

colimit along $S^0 \xrightarrow{a_V} S^V$

so $\pi_\star^G X \wedge S^{\infty V} = a_V^{-1} \pi_\star^G X$

<u>Geometric fixed points</u> We want

(1) $\Phi^N(\Sigma^\infty X) \simeq \Sigma^\infty(X^N)$ for a based G-space X

(2) $\Phi^N(X \wedge Y) \simeq (\Phi^N X) \wedge (\Phi^N Y)$ for G-spectra X, Y

(3) Φ^N commutes with htpy colimits.

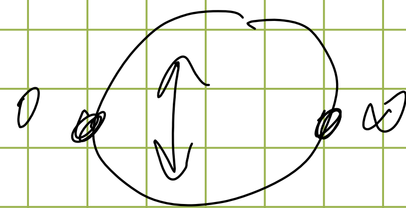
For $X \simeq \underset{\overrightarrow{k}}{\mathrm{holim}}\ S^{-k\rho} \wedge X_{k\rho}$

$\Phi^N X \simeq \underset{\overrightarrow{k}}{\mathrm{holim}}\ S^{-k\rho^N} \wedge X_{k\rho}^N$

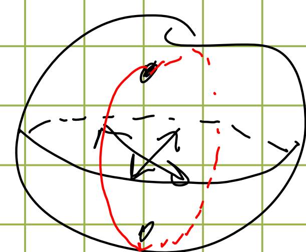
e.g. $\Phi^N X = (X[\geq N])^N = (X \wedge \tilde{E}[\geq N])^N$

2. <u>G-cells</u>

$G = C_2$



$= S^\sigma = S^0 \cup (G_+ \wedge e^1)$



$= S^{2\sigma} = S^0 \cup (G_+ \wedge e^1) \cup (G_+ \wedge e^2)$

This differs from the smash product description

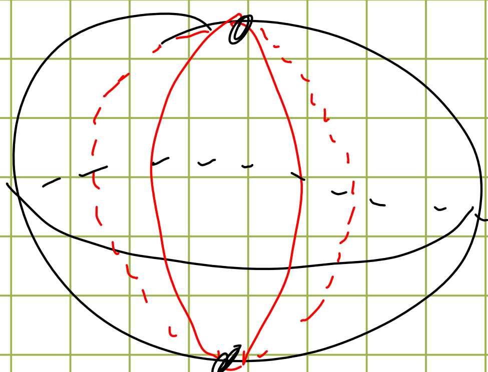
$\left(S^\sigma \to S^{2\sigma} \to S^\sigma \wedge (S^\sigma/S^0) \simeq S^1 \wedge (S^\sigma/S^0)\right)$

Orbit category $\mathcal{O}_G$ has 2 objects

$G/G \xleftarrow{\pi} G/1 \circlearrowleft$

Example 1:

$G = C_4$, reps are $\{1, \sigma, \lambda = \pi/2\text{-turn}\}$



$= S^\lambda = S^0 \cup (G_+ \wedge e^1) \cup (G_+ \wedge e^2)$

$S^\sigma = (S^{\lambda+\sigma})^{C_2}$

(to be discussed by J_G) $S^{\lambda+\sigma} = \underbrace{S^0 \cup (G/C_{2+} \wedge e^1)}_{S^\sigma} \cup (G_+ \wedge e^2) \cup (G_+ \wedge e^3)$

$\mathcal{O}_G$

$$\begin{array}{ll} & C_4/C_4 \\ & \uparrow \pi_2^4 \\ G \circlearrowright & C_4/C_2 \quad C_4/C_2 \\ & \uparrow \pi_1^2 \\ G \circlearrowright & C_4/1 \quad C_4 \end{array}$$

3. Ordinary (Bredon) cohomology

Satisfies E-S axioms, including dim axiom

Value determined by values on orbits G/H_+

i.e. by functor $M: \mathcal{O}_G^{op} \to \mathbb{Z}\text{-mod}$

$$\begin{tikzcd} M(G/G) \arrow[r, "\mathrm{Res}^G_H"] & M(G/H) \arrow[r, "\mathrm{Res}^H_1"] \arrow[loop below, "G/H"] & M(G/1) \arrow[loop below, "G"] \end{tikzcd}$$

This lets us calculate (for $G = C_2$ with gen γ)

$$S^\lambda = S^0 \cup (G_+ \wedge e^1) \cup (G_+ \wedge e^2)$$

$$H^*\left(M(G/G) \xrightarrow{\mathrm{Res}^G_1} M(G/1) \xrightarrow{1-\gamma} M(G/1) \right)$$

$$= H^*(S^\lambda; M)$$

If X is a G-space X define

$$\underline{C}^G_*(X) : \mathcal{O}_G^{op} \longrightarrow Ch$$

$$G/H \longmapsto C_*(X^H) = C_*(G\text{-map}(G/H, X))$$

Bredons Def

$$H^*_G(X;M) = H^*(\mathrm{Hom}_{coeff}(\underline{C}^G_* X, M))$$

Chains can be singular, cellular or simplicial.

To see as before $\mathrm{Hom}(\underline{C}^G_*(G/H), M) = M(G/H)$

e.g. $G = C_4$ $\qquad G/G \quad 0$

$H = C_2$ $\qquad G/H \quad \mathbb{Z}G/H$

$\qquad\qquad\quad G/1 \quad \mathbb{Z}G/H$

Examples (1) Given abelian gp A $\quad (G = C_n)$

constant coeff system

$$\begin{array}{c} A \\ \downarrow \\ A \\ \downarrow \\ A \end{array}$$

hence $H^*_G(X; \underline{A}) = H^*(X/G; A)$

(2) $\begin{array}{c} i_! A \\ 0 \end{array}$

$$\boxed{\begin{array}{cc} A & G/G \\ \downarrow & \\ 0 & G/H \\ \downarrow & \\ 0 & G/H \end{array}}$$

$H^*_G(X; i_! A) = H^*(X^G; A)$

4. Ordinary homology (Bredon)

Satisfies dimension axiom and is determined by values on orbits O_G

i.e. by functor $N : O_G \to \mathbb{Z}\text{-mod}$

dual coeff system

$$\begin{tikzcd}
N(G/G) \\
N(G/H) \arrow[u] \arrow[r, phantom, "\circlearrowleft"] & G/H \\
N(G/1) \arrow[u] \arrow[r, phantom, "\circlearrowleft"] & G\text{-mod}
\end{tikzcd}$$

Def $H^G_*(X;N) = H_*\left(\underline{C}^G_*(X) \otimes_{\mathcal{O}} N\right)$

(Note: Need to know $\underline{C}_*(G/H) \otimes_{\mathcal{O}} N = N(G/H)$)

Examples (1) Constant dual coeff system $\overline{A}$

$$\begin{array}{c} A \\ {\scriptstyle 1}\uparrow \\ A \\ {\scriptstyle 1}\uparrow \\ A \end{array} \qquad H^G_*(X;\overline{A}) = H_*(X/G; A)$$

(2) A = abe gp

$G = C_4$

$$\operatorname{ind}\mathbb{Z} = \boxed{\begin{array}{c} \mathbb{Z} \\ \uparrow{\scriptstyle 2} \\ \mathbb{Z} \\ \uparrow{\scriptstyle 2} \\ \mathbb{Z} \end{array}}$$

in general

$\operatorname{ind}(\mathbb{Z})(G/H) = \mathbb{Z}$

$K \leq H \quad G/K \longrightarrow G/H$

$\mathbb{Z} \xrightarrow{[H:K]} \mathbb{Z}$

Lemma: $H_*^G(X; \underline{\mathbb{Z}}) = H_*\left(C_*(X)^G\right)$

Proof

$$\begin{array}{ccc} (\mathbb{Z}G/K)^G & \xrightarrow{(\pi_K^H)^G} & (\mathbb{Z}\, G/H)^G \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{|H:K|} & \mathbb{Z} \end{array}$$

QED

Example $G = C_4$, $S^\lambda = S^0 \cup (G_+ \wedge e^1) \cup (G_+ \wedge e^2)$

$$C_*(S^\lambda): \quad \mathbb{Z} \xleftarrow{\pi} \mathbb{Z}G \xleftarrow{1-\gamma} \mathbb{Z}G$$

$$\qquad\qquad\quad 0 \qquad\quad 1 \qquad\quad 2$$

$$C_*(S^\lambda)^G: \quad \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$$H_*^G(S^\lambda; \underline{\mathbb{Z}}): \quad \mathbb{Z}/4 \qquad 0 \qquad \mathbb{Z}$$

Def S^V is <u>orientable</u> ($d = \dim V$) if G acts trivially on $H_d(S^V)$

If so orient top cells so G preserves

$$\begin{array}{ccc} \theta_d & \in & \ker\left(C_d(S^V) \longrightarrow C_{d-1}(S^V)\right) \\ \uparrow & & \\ \mu_V & \in & \ker\left(C_d(S^V)^G \longrightarrow C_{d-1}(S^V)^G\right) \end{array}$$

5. <u>Mackey functors</u>

If $H_G^*(\ ; M)$ is to be defined on spectra, we must have $M : (\mathcal{SO}_G)^{op} \longrightarrow \mathbb{Z}\text{-mod}$

(where $\mathcal{SO}_G(G/H_+, G/K_+) = \varinjlim_{V} [S^V \wedge G/H_+, S^V \wedge G/K_+]^G$)

called a <u>Mackey functor</u>

e.g. for a G-spectrum X

$$\underline{\pi}_0^G(X) : \mathcal{SO}_G \longrightarrow \mathbb{Z}\text{-mod}$$

$$G/H \longmapsto [G/H_+ \wedge X]^G = \pi_0(X^H)$$

Example: The constant coeff system $\underline{\mathbb{Z}}$ extends to a Mackey functor

The dual coeff system of this is still $\underline{\mathbb{Z}}$

$$H^*_G(X; M) = [X, HM]^*_G \cong H^*_G(X; cM)$$

$$H_*^G(X; M) = [S^0, HM \wedge X]_*^G = H_*^G(X; dM)$$

underlying coeff system

underlying dual coeff system