J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1171

Published by EMS Press

Bogdan Gheorghe · Daniel C. Isaksen · Achim Krause · Nicolas Ricka

$\mathbb{C}$-motivic modular forms

Received August 8, 2019

Abstract. We construct a topological model for cellular, 2-complete, stable $\mathbb{C}$-motivic homotopy theory that uses no algebro-geometric foundations. We compute the Steenrod algebra in this context, and we construct a "motivic modular forms" spectrum over $\mathbb{C}$.

Keywords. Motivic homotopy theory, motivic modular forms, motivic Steenrod algebra, Adams–Novikov spectral sequence, topological modular forms

1. Introduction

The topological modular forms spectrum *tmf* [4] detects a significant portion of the stable homotopy groups. Classes detected by *tmf* are viewed as well-understood, while classes not detected by *tmf* are viewed as more exotic. Similarly, Adams differentials, or Adams–Novikov differentials, detected by *tmf* are well-understood. From this perspective, the essential property of *tmf* is that its cohomology is the quotient $A/\!/A(2)$ of the Steenrod algebra A by the subalgebra $A(2)$ generated by Sq^1, Sq^2, and Sq^4. This means that the cohomology of $A(2)$ is the E_2-page of the Adams spectral sequence for *tmf*, so this spectral sequence can be computed effectively.

For the purposes of computing motivic stable homotopy groups, it is desirable to have an analogous "motivic modular forms" spectrum *mmf* whose cohomology is $A/\!/A(2)$, in the motivic sense. The existence of such a motivic spectrum over $\mathbb{C}$ immediately resolves the status of many possible differentials in the classical Adams spectral sequence [15]. Such a motivic spectrum was first considered (hypothetically) in [13].

Bogdan Gheorghe: Max-Planck-Institut für Mathematik, 53111 Bonn, Germany; gheorghebg@mpim-bonn.mpg.de

Daniel C. Isaksen: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA; isaksen@wayne.edu

Achim Krause: WWU Münster, Einsteinstr. 62, 48149 Münster, Germany; achim.t.krause@gmail.com

Nicolas Ricka: IRMA, 7 Rue René Descartes, 67000 Strasbourg, France; n.ricka@unistra.fr

Mathematics Subject Classification (2020): Primary 14F42, 55N34, 55S10; Secondary 55Q45, 55T15

One of the main goals of this manuscript is to establish the existence of *mmf* in the $\mathbb{C}$-motivic context. Our techniques heavily depend on an algebraically closed base field of characteristic zero, but our intuition is that motivic modular forms ought to exist in much greater generality.

Conjecture 1.1. *A motivic modular forms spectrum exists over any smooth base.*

One possible approach to constructing *mmf* is to follow the classical construction of *tmf* in the motivic context. This would require a careful understanding of motivic elliptic cohomology theories and moduli spaces of E_∞-structures.

We avoid these technical difficulties with a novel approach. We first construct an ∞-category of $\Gamma_\star S^0$-modules that is entirely topological in nature. Foundationally, it depends only on infinite sequences of classical spectra, with no reference to smooth schemes, affine lines, $\mathbb{A}^1 - 0$, etc.

The ∞-category of $\Gamma_\star S^0$-modules is carefully engineered to mimic the computational properties of $\mathbb{C}$-motivic stable homotopy theory. We define an "Eilenberg–Mac Lane" $\Gamma_\star S^0$-module $\Gamma_\star H\mathbb{F}_2$, and we compute that its homotopy groups are of the form $\mathbb{F}_2[\tau]$, i.e., the same as the motivic cohomology of $\mathbb{C}$. We also directly compute the cooperations for $\Gamma_\star H\mathbb{F}_2$, i.e., the Steenrod algebra in the context of $\Gamma_\star S^0$-modules, and we obtain an answer that is identical to the $\mathbb{C}$-motivic Steenrod algebra. We emphasize that our construction and our computations are independent of the much harder motivic computations of Voevodsky [25]–[27].

The key idea relies on an observation from [12] about the structure of the $\mathbb{C}$-motivic Adams–Novikov spectral sequence (see also [14, Chapter 6]). The weights in this spectral sequence follow a simple pattern. The object $\Gamma_\star S^0$ is defined in such a way that its bigraded homotopy groups are computed by an identical spectral sequence.

Having constructed the ∞-category of $\Gamma_\star S^0$-modules, we study the $\Gamma_\star S^0$-module $\Gamma_\star tmf$, and we show that this object has the desired computational properties, i.e., its cohomology is $A/\!/A(2)$ in the $\Gamma_\star S^0$-module context.

Finally, we prove that the homotopy category of 2-complete $\Gamma_\star S^0$-modules is equivalent to the 2-complete cellular $\mathbb{C}$-motivic stable homotopy category. Thus, the completed $\widehat{\Gamma}_\star tmf$ corresponds to some cellular $\mathbb{C}$-motivic spectrum that deserves to be called *mmf*.

Motivic homotopy theory has been at the center of a recent breakthrough in the computation of stable homotopy groups of spheres [14], [15]. From the perspective of that project, the ∞-category of $\Gamma_\star S^0$-modules makes motivic homotopy theory no longer relevant. In particular, the stable homotopy group computations of [15] no longer logically depend on Voevodsky's computations of the motivic cohomology of a point, nor on his computation of the motivic Steenrod algebra. For this reason, we completely avoid results and constructions from motivic homotopy theory until Section 6.

On the other hand, the theory of $\Gamma_\star S^0$-modules does not make motivic homotopy theory obsolete. Since $\Gamma_\star S^0$-modules only capture the 2-complete cellular motivic spectra, they miss phenomena of arithmetic interest, including the rational part of motivic homotopy theory. Moreover, we currently cannot construct analogous models for motivic

Tab. 1. Some values of $\widehat{\Gamma}_\star$

X	$\widehat{\Gamma}_\star X$
$H\mathbb{F}_2$	$H\mathbb{F}_2$
$H\mathbb{Z}_2$	$H\mathbb{Z}_2$
KU	$KGL^\wedge$
ku	$kgl^\wedge$
KO	$KQ^\wedge$
ko	$kq^\wedge$
BP	$BPGL^\wedge$
$BP\langle n\rangle$	$BPGL\langle n\rangle^\wedge$
MU	$MGL^\wedge$

homotopy theory over base fields other than $\mathbb{C}$, although it seems plausible that at least $\mathbb{R}$-motivic cellular spectra have a topological model.

Our construction of $\Gamma_\star S^0$-modules can be generalized to other contexts. The complex cobordism spectrum MU is built into the definitions from the very beginning, but one can use other cohomology theories in the same way. Moreover, the basic construction can be iterated to obtain interesting multi-graded homotopy theories. See the work of Pstrągowski [22] in this direction.

The cofiber of τ is a very interesting motivic spectrum with curious properties [6], [8]. It would be interesting to study the cofiber of τ from the perspective of $\Gamma_\star S^0$-modules. We do not carry out this investigation in this manuscript because it is not central to our goal of constructing *mmf*. On the other hand, it is possible that $\Gamma_\star S^0$-modules are useful for understanding exotic motivic periodicities [5], [17], and the cofiber of τ would play a critical role in that pursuit.

The functor $\Gamma_\star$ (or more precisely, its 2-completed version $\widehat{\Gamma}_\star$) provides a new tool for producing motivic spectra from classical spectra. If X is a classical cell complex with cells in only even dimensions, then $\widehat{\Gamma}_\star X$ is a motivic cell complex in which $2k$-dimensional cells of X correspond to $(2k, k)$-dimensional cells of $\widehat{\Gamma}_\star X$. The behavior of odd-dimensional cells under $\Gamma_\star$ is a bit more complicated. Table 1 shows that many of the motivic spectra commonly studied can be constructed with this tool. (Here, $X^\wedge$ refers to the 2-completed version of a motivic spectrum X.) On the other hand, the functor $\Gamma_\star$ does not seem to interact well with η-periodicization; see Remark 3.17 for more discussion.

We mention recent work of Pstrągowski [22] that constructs a topological model for cellular $\mathbb{C}$-motivic stable homotopy theory using different foundational techniques. We discuss in Remark 6.13 how to establish a direct equivalence between $\Gamma_\star S^0$-modules and even MU-based synthetic spectra.

We also mention work of Heine [10] that is more formal.

1.1. Odd primes

In this manuscript, we focus only on the prime 2. We compute homology with $\mathbb{F}_2$-coefficients, and work with the 2-primary Steenrod algebra. We construct a motivic modular forms spectrum *mmf* that has the correct 2-primary properties.

In fact, it is possible to carry out these computations at odd primes as well, relying on results of [24] about the structure of odd primary $\mathbb{C}$-motivic homotopy theory. For example, one can compute odd primary Steenrod algebras, and show that *mmf* has the correct odd primary properties.

Nevertheless, we do not write out the details because the proofs are identical, and the new computations are essentially the same as the classical ones. In particular, the exotic formula

$$\tau_i^2 + \tau\xi_{i+1} = 0$$

from Theorem 4.10 has no odd primary analogue; the classical relation $\tau_i^2 = 0$ still holds. Because the computations are essentially the same as the classical ones at odd primes, we suspect that there are limited potential applications for the odd primary version of our results.

1.2. Organization

We begin in Section 2 with a discussion of a certain diagram ∞-category of spectra. In Section 3, we define the functor $\Gamma_\star$ from ordinary spectra to filtered spectra, and we develop the key properties of this functor that allows for a rich computational theory of $\Gamma_\star S^0$-modules.

In Section 4, we begin our computations by computing the Steenrod algebra for $\Gamma_\star S^0$-modules. We carry these computations further in Section 5 when we study $\Gamma_\star tmf$.

Finally, in Section 6, we show that the homotopy theory of $\Gamma_\star$-modules is equivalent to the 2-complete cellular $\mathbb{C}$-motivic stable homotopy category.

1.3. Notation and conventions

Throughout the article, we adopt the perspective of stable ∞-categories for our homotopy theories. In Section 6, we work entirely in a 2-complete setting.

We use the symbol $\mathbb{Z}$ in two different ways. Sometimes it is an indexing category for a diagram, in which case we are thinking of $\mathbb{Z}$ as a partially ordered set in the standard way. At other times, $\mathbb{Z}$ is used for the coefficients of a computation.

The ∞-category **Sp** is the usual stable ∞-category of spectra (or 2-complete spectra in Section 6).

We will use different gradings in different contexts. For the sake of clarity, we adopt the following conventions:

(1) $*$ denotes an integer grading corresponding to the usual grading on spectra. For example, this grading is used for stable homotopy groups of classical spectra.

(2) $\star$ represents an indexing in $\mathbb{Z}^{\mathrm{op}}$. This symbol is used exclusively for filtered spectra, i.e., ∞-functors from $\mathbb{Z}^{\mathrm{op}}$ to spectra (or 2-complete spectra in Section 6). For example, the notation $X_\star$ indicates an ∞-functor

$$X : \mathbb{Z}^{\mathrm{op}} \to \mathbf{Sp}.$$

(3) $\bullet$ denotes the cosimplicial degree of a cosimplicial object.

2. Filtered spectra and a colocalization

In this section, we construct a closed symmetric monoidal stable ∞-category of filtered spectra. The objects that we are considering come entirely from the classical stable homotopy category. The monoidal structure arises from Day convolution.

Note that we are using only a few formal properties of the ∞-category $\mathbf{Sp}$ of spectra. Our construction could be generalized to filtered objects in any presentable closed symmetric monoidal stable ∞-category, but we work only with $\mathbf{Sp}$ for the sake of simplicity.

2.1. *Filtered spectra*

Definition 2.1. The stable ∞-category of *filtered spectra* is the ∞-category $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ of ∞-functors from the ∞-nerve of the category $\mathbb{Z}^{\mathrm{op}}$, considered as a poset, to the ∞-category $\mathbf{Sp}$ of spectra.

The objects of $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ are $\mathbb{Z}^{\mathrm{op}}$-diagrams in $\mathbf{Sp}$ up to coherent homotopy, in the following sense. A filtered spectrum $X_\star$ is a sequence $\{X_w \mid w \in \mathbb{Z}^{\mathrm{op}}\}$ of spectra, together with a choice of morphisms $X_w \to X_{w'}$, for all $w \geq w'$, and a choice of coherent homotopies relating compositions of the structure morphisms. We will typically denote a filtered spectrum by a sequence

$$X_\star = \cdots \to X_1 \to X_0 \to X_{-1} \to \cdots,$$

in which the homotopies are implicit. Similarly, we describe a morphism $X_\star \to Y_\star$ in this ∞-category by a sequence $\{X_w \to Y_w \mid w \in \mathbb{Z}^{\mathrm{op}}\}$ of coherently homotopically compatible maps.

In $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$, the weak equivalences are exactly the pointwise weak equivalences, i.e., the maps $f \colon X_\star \to Y_\star$ such that each component $f_w \colon X_w \to Y_w$ induces an isomorphism on homotopy groups. In particular, an object $X_\star$ is contractible precisely when every spectrum X_w is contractible. Moreover, homotopy limits and homotopy colimits are computed pointwise in this diagram ∞-category.

In the ∞-category $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$, we have a bigraded family of sphere objects, as described in Definition 2.2. We will show later in Lemma 2.6 that these spheres generate all filtered spectra in the appropriate homotopical sense.

Definition 2.2. The sphere $S^{s,w}$ of bidegree (s, w) is the object of $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ defined by

$$\cdots \to * \to * \to S^s \xrightarrow{\mathrm{id}} S^s \xrightarrow{\mathrm{id}} \cdots,$$

where $S^{s,w}_v$ is $*$ in if $v > w$ and $S^{s,w}_v$ is S^s if $v \leq w$.

Recall that such ∞-categories of functors canonically possess a closed symmetric monoidal structure, given by the Day convolution product (see [9], [18, Section 2.2.6]), which is induced by the monoidal structure of the source category. Explicitly, in filtered degree w we have

$$(X_\star \otimes Y_\star)_w \simeq \operatorname*{hocolim}_{i+j \geq w} X_i \wedge Y_j. \tag{1}$$

The unit for this product is the sphere $S^{0,0}$ of Definition 2.2. Moreover, the convolution product commutes with homotopy colimits in each variable [9, Lemma 2.13]. We denote by $\Sigma^{s,w}$ the suspension endofunctor

$$S^{s,w} \otimes (-)\colon \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}} \to \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}.$$

More concretely, $\Sigma^{s,w} X_\star$ is the filtered spectrum such that $(\Sigma^{s,w} X_\star)_v$ is $\Sigma^s(X_{v-w})$.

The "formal suspension" in $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ is the same as $\Sigma^{1,0}$, i.e., there is a cofiber sequence

$$X_\star \to * \to \Sigma^{1,0} X_\star$$

for every filtered spectrum $X_\star$.

Definition 2.3. Let $\mathrm{colim}_\star : \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}} \to \mathbf{Sp}$ be the functor that takes a filtered spectrum $X_\star$ to the spectrum $\mathrm{colim}_w X_w$.

We are now ready to show that the bigraded spheres generate the ∞-category of filtered spectra.

Definition 2.4. The stable homotopy group $\pi_{s,w} X_\star$ is the abelian group $[S^{s,w}, X_\star]$.

The direct sum $\pi_{*,\star} X$ of all stable homotopy groups of X is a bigraded abelian group.

Remark 2.5. It follows from a standard adjunction argument that $\pi_{s,w} X_\star$ is equal to the pointwise homotopy group $\pi_s X_w$.

Lemma 2.6. *In the ∞-category $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$, a map is a weak equivalence if and only if it induces an isomorphism on $\pi_{*,\star}$. Equivalently, $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ is generated as a stable ∞-category by the spheres $S^{0,w}$ for $w \in \mathbb{Z}^{\mathrm{op}}$.*

Proof. By definition, a map $f\colon X_\star \to Y_\star \in \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ is a weak equivalence if and only if each $f_w\colon X_w \to Y_w$ is a weak equivalence, and f_w is a weak equivalence if and only if it induces a π_* isomorphism. Finally, Remark 2.5 establishes the first claim.

The second claim follows from a standard argument (see for instance [11, Theorem 1.2.1]), since the first claim implies that $X_\star$ is equivalent to the zero object in $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ if and only if the bigraded abelian group $\pi_{*,\star} X_\star$ is zero. ∎

2.2. t-structure

Definition 2.7. (1) Let $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\geq 0}$ be the full ∞-subcategory of $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ consisting of filtered spectra $X_\star$ such that $\pi_{s,w} X_\star = 0$ for $s < 2w$, i.e., $\pi_s X_w = 0$ for $s < 2w$.

(2) Let $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\leq 0}$ be the full ∞-subcategory of $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ consisting of filtered spectra $X_\star$ such that $\pi_{s,w} X_\star = 0$ for $s > 2w$, i.e., $\pi_s X_w = 0$ for $s > 2w$.

Proposition 2.8. *Definition 2.7 equips $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ with a t-structure.*

Proof. Suppose that $X_\star$ and $\Sigma Y_\star$ belong to $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\geq 0}$ and $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\leq 0}$ respectively. We will show that any map $X_\star \to Y_\star$ is trivial. The simplicial mapping space $\mathrm{Map}(X_\star, Y_\star)$ is the homotopy limit

$$\operatorname*{holim}_{i \geq j} \mathrm{Map}(X_i, Y_j).$$

When $i \geq j$, the space $\mathrm{Map}(X_i, Y_j)$ is contractible since $\pi_n X_i = 0$ if $n < 2i$ and $\pi_n Y_j = 0$ if $n \geq 2j$. Therefore, the homotopy limit is contractible as well.

The ∞-subcategory $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\geq 0}$ is closed under suspension since the suspension functor on filtered spectra is defined pointwise. Similarly, the ∞-subcategory $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\leq 0}$ is closed under desuspension.

Finally, for any filtered spectrum $X_\star$, we have the diagram

$$\begin{tikzcd}
{} \arrow[r] & \tau_{\geq 4} X_2 \arrow[r] \arrow[d] & \tau_{\geq 2} X_1 \arrow[r] \arrow[d] & \tau_{\geq 0} X_0 \arrow[r] \arrow[d] & \tau_{\geq -2} X_{-1} \arrow[r] \arrow[d] & \tau_{\geq -4} X_{-2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X_{-1} \arrow[r] \arrow[d] & X_{-2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \tau_{<4} X_2 \arrow[r] & \tau_{<2} X_1 \arrow[r] & \tau_{<0} X_0 \arrow[r] & \tau_{<-2} X_{-1} \arrow[r] & \tau_{<-4} X_{-2} \arrow[r] & {}
\end{tikzcd}$$

where $\tau_{\geq n}$ and $\tau_{<n}$ are the usual connective cover and Postnikov section functors on spectra. The top row of the diagram is a filtered spectrum in $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\geq 0}$, the bottom row is a filtered spectrum in $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}_{\leq -1}$, and the columns are fiber sequences of ordinary spectra. Since fiber sequences of filtered spectra are defined pointwise, this diagram defines a fiber sequence of filtered spectra. ∎

Remark 2.9. The heart of the t-structure of Proposition 2.8 is the category of graded abelian groups. A filtered spectrum $X_\star$ is in the heart if and only if $\pi_{2w} X_w$ is the only non-zero homotopy group of X_w for all w, i.e., $X_\star$ is of the form

$$\cdots \to \Sigma^{2w+2} H\pi_{2w+2} \to \Sigma^{2w} H\pi_{2w} \to \Sigma^{2w-2} H\pi_{2w-2} \to \cdots.$$

Then the structure maps $\Sigma^{2w+2} H\pi_{2w+2} \to \Sigma^{2w} H\pi_{2w}$ are necessarily null-homotopic. This shows that the functor

$$X_\star \mapsto \bigoplus \pi_{2w} X_w$$

from the heart to graded abelian groups is an equivalence.

Lemma 2.10. *The ∞-subcategory $\mathbf{Sp}_{\geq 0}^{\mathbb{Z}^{\mathrm{op}}}$ is closed under Day convolution.*

Proof. Let $X_\star$ and $Y_\star$ belong to $\mathbf{Sp}_{\geq 0}^{\mathbb{Z}^{\mathrm{op}}}$. Then $(X_\star \otimes Y_\star)_w$ is the colimit

$$\operatorname*{colim}_{i+j \geq w} X_i \wedge Y_j .$$

Since X_i is a $2i$-connective spectrum and Y_j is a $2j$-connective spectrum by assumption, the colimit is taken over $2w$-connective spectra, so the result is $2w$-connective. ∎

We write $(\tau_{\geq 0})_\star$ for the right adjoint, in the appropriate ∞-categorical sense, to the inclusion $\mathbf{Sp}_{\geq 0}^{\mathbb{Z}^{\mathrm{op}}} \to \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$. More concretely, $(\tau_{\geq 0})_\star : \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}} \to \mathbf{Sp}_{\geq 0}^{\mathbb{Z}^{\mathrm{op}}}$ is the functor that takes a filtered spectrum $X_\star$ to

$$\cdots \to \tau_{\geq 4} X_2 \to \tau_{\geq 2} X_1 \to \tau_{\geq 0} X_0 \to \tau_{\geq -2} X_{-1} \to \tau_{\geq -4} X_{-2} \to \cdots .$$

Corollary 2.11. *The truncation functor $(\tau_{\geq 0})_\star$ is lax symmetric monoidal.*

Proof. Lemma 2.10 says that the inclusion $\mathbf{Sp}_{\geq 0}^{\mathbb{Z}^{\mathrm{op}}} \to \mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$ is strong symmetric monoidal. Therefore, its right adjoint is lax symmetric monoidal. ∎

3. The functor $\Gamma_\star$

The goal of this section is to define and study a functor $\Gamma_\star$ from $\mathbf{Sp}$ to $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$. The functor is constructed in such a way that it interacts with the Adams–Novikov spectral sequence in an interesting way.

Definition 3.1. Let $MU^{\bullet+1}$ be the cosimplicial spectrum whose nth term is the $(n+1)$-fold smash product of MU with itself, and whose faces and degeneracies are induced from the multiplication and unit of the ring structure on MU.

The cosimplicial spectrum $MU^{\bullet+1}$ is the usual tool for constructing the MU-based Adams spectral sequence. More precisely, the Bousfield–Kan spectral sequence that computes the totalization $\mathrm{Tot}(X \wedge MU^{\bullet+1})$ is the Adams–Novikov spectral sequence that converges to the homotopy of X. Convergence of the Adams–Novikov spectral sequence just means that the totalization $\mathrm{Tot}(X \wedge MU^{\bullet+1})$ is equivalent to the (appropriately completed) spectrum X.

We work with the MU-based Adams spectral sequence, rather than the BP-based Adams spectral sequence, because MU has better formal multiplicative properties than BP. Computationally, there is no difference between the two perspectives.

Definition 3.2. Let X be a spectrum.

(1) Let $\tau_{\geq 2w}(X \wedge MU^{\bullet+1})$ be the cosimplicial spectrum formed by taking the $(2w-1)$-connected cover of each term of the cosimplicial spectrum $X \wedge MU^{\bullet+1}$.

(2) Define the filtered spectrum $\Gamma_\star X$ such that each $\Gamma_w X$ is the totalization $\mathrm{Tot}\, \tau_{\geq 2w}(X \wedge MU^{\bullet+1})$.

We now turn to a particular spectral sequence computing the homotopy groups of $\Gamma_\star X$. We grade Ext in the form (s, f), where f is the homological degree and $s + f$ is the total degree. In other words, s is the stem, and f is the Adams–Novikov filtration. While inconsistent with historical notation, this choice aligns well with the usual graphical representations of the Adams–Novikov spectral sequence.

Proposition 3.3. *Let X be a spectrum, and let $w \in \mathbb{Z}^{\mathrm{op}}$. There is a spectral sequence*

$$E_2^{s,f,w}(X) = \begin{cases} \mathrm{Ext}^{s,f}_{MU_*MU}(MU_*, MU_*X) & \text{if } s+f \geq 2w, \\ 0 & \text{otherwise} \end{cases}$$

that converges to $\pi_{s,w}\Gamma_\star X$. These spectral sequences are compatible with the filtered spectrum structure on $\Gamma_\star X$, in the sense that the morphism $\Gamma_w X \to \Gamma_{w-1} X$ induces the evident inclusion $E_2^{s,f,w}(X) \to E_2^{s,f,w-1}(X)$.

Proof. The spectral sequence is the Bousfield–Kan spectral sequence associated to the cosimplicial spectrum $\tau_{\geq 2w}(X \wedge MU^{\bullet+1})$. The spectral sequence converges to $\pi_{*,\star}\Gamma_\star X$ by definition of $\Gamma_\star X$. Our goal is to identify the E_2-page of this Bousfield–Kan spectral sequence.

The E_1-page of the Bousfield–Kan spectral sequence is

$$E_1^{s,f,w} = \pi_{s+f}(\tau_{\geq 2w} MU^{\wedge f+1}),$$

and the d_1 differential is the map

$$\pi_{s+f}(\tau_{\geq 2w}(X \wedge MU^{\wedge f+1})) \to \pi_{s+f}(\tau_{\geq 2w}(X \wedge MU^{\wedge f+2}))$$

induced by the alternating sum of cofaces. When $s + f < 2w$, the E_1-page is zero. When $s + f \geq 2w$, the E_1-page is isomorphic to

$$E_1^{s,f,w}(X) = \pi_{s+f}(X \wedge MU^{\wedge f+1}),$$

and the d_1 differential is the map

$$\pi_{s+f}(X \wedge MU^{\wedge f+1}) \to \pi_{s+f}(X \wedge MU^{\wedge f+2}).$$

This is identical to the E_1-page and d_1 differential for the classical Adams–Novikov spectral sequence for X. Therefore, $E_2^{s,f,w}(X)$ is equal to the Adams–Novikov E_2-page for X when $s + f \geq 2w$. ∎

Remark 3.4. For fixed w, the spectral sequence $E_2^{s,f,w}(X)$ is a truncated version of the classical Adams–Novikov spectral sequence for X, as shown in Figure 1. Above the line $s + f = 2w$ of slope -1, $E_2^{s,f,w}(X)$ equals the classical Adams–Novikov spectral sequence. Below this line, $E_2^{s,f,w}(X)$ is zero. If X is n-connected and $2w \leq n$, then the truncation is trivial, and $E_2^{s,f,w}(X)$ is equal to the classical Adams–Novikov spectral sequence.

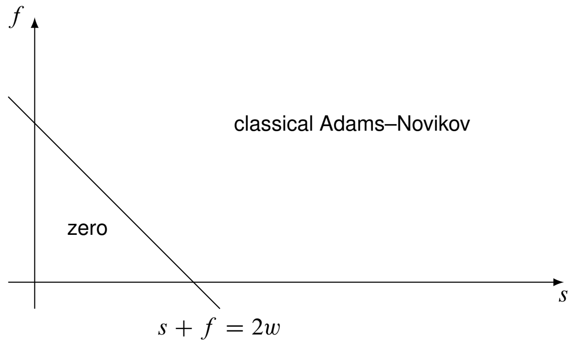

Fig. 1. The spectral sequence of Proposition 3.3. The region below the diagonal line is zero. The region above the diagonal line is isomorphic to the classical Adams–Novikov spectral sequence in that range.

Classical Adams–Novikov differentials whose source is above the line of slope -1 occur in $E_r^{s,f,w}(X)$ as well. However, classical Adams–Novikov differentials whose source is below the line of slope -1 do not occur in $E_r^{s,f,w}(X)$. Of particular interest are classical differentials that cross the line $s + f = 2w$. Consequently, there are non-zero permanent cycles in $E_\infty^{s,f,w}(X)$ that are zero in the classical Adams–Novikov E_∞-page.

Remark 3.5. Not coincidentally, when $X = S^0$, the spectral sequence of Proposition 3.3 is identical to the motivic Adams–Novikov spectral sequence for computing the stable homotopy groups of the motivic sphere spectrum [12], [14]. Consequently, the bigraded homotopy groups of the filtered spectrum $\Gamma_\star S^0$ are the same as the bigraded motivic homotopy groups of the motivic sphere spectrum.

Later when we study specific examples of $\Gamma_\star X$, we will use specific computational information about X to deduce analogous computational information about $\Gamma_\star X$. Proposition 3.6 is the precise tool for transporting such information.

Proposition 3.6. *Let X be a bounded-below spectrum. Then* $\operatorname{colim}_\star \Gamma_\star X$ *is equivalent to X.*

Proof. By Definition 3.2, $\operatorname{colim}_\star \Gamma_\star X$ equals

$$\operatorname*{colim}_w \operatorname{Tot}(\tau_{\geq 2w}(X \wedge MU^{\bullet+1})).$$

Since X is bounded-below, we can write the Tot as an inverse limit of Tot^n, where the maps $\operatorname{Tot}^n \to \operatorname{Tot}^{n-1}$ increase in connectivity. Since filtered colimits commute with the finite limits Tot^n, and preserve connectivity, one sees that the filtered colimit commutes with totalization in the present situation. So we have

$$\operatorname{Tot}\Big(\operatorname*{colim}_w \tau_{\geq 2w}(X \wedge MU^{\bullet+1})\Big) = \operatorname{Tot}(X \wedge MU^{\bullet+1}).$$

The latter object is the same as X, precisely because the Adams–Novikov spectral sequence converges for connective X. ∎

Remark 3.7. The functor $\Gamma_\star$ admits a different perspective. Recall that on filtered spectra, there is the *Beilinson t-structure* (not to be confused with the t-structure of Subsection 2.2). The Beilinson t-structure is characterized by the property that a connective object is a filtered spectrum $X_\star$ whose associated "chain complex"

$$\cdots \to \Sigma^n X_n/X_{n+1} \to \Sigma^{n+1} X_{n+1}/X_{n+2} \to \cdots$$

consists of connective spectra. This chain complex in the homotopy category of spectra induces the E_1-page of the spectral sequence associated to the filtered spectrum.

The following observation was privately communicated to us by Ben Antieau. As far as we are aware, it is not documented in this form in the literature yet. For a general discussion of the Beilinson t-structure, we refer the reader to [2].

Starting with a filtered spectrum $X_\star$, we can first take the Whitehead tower $\tau^{\mathrm{Beil}}_{\geq w} X_\star$ in the Beilinson t-structure (which is now a bi-filtered spectrum), and then eliminate the original filtration by passing to $X'_w = \operatorname{colim}_\star \tau^{\mathrm{Beil}}_{\geq w} X_\star$. The resulting new filtered spectrum $X'_\star$ has the same colimit, meaning we can think of it as a filtration on the same "underlying spectrum" as $X_\star$. Furthermore, its associated spectral sequence agrees with the one of $X_\star$ up to a page shift: $E_r(X'_\star) = E_{r+1}(X_\star)$. This construction is a higher algebra version of Deligne's *décalage* functor, which in turn is a more structured version of deriving an exact couple, directly on the level of filtered chain complexes.

The pointwise Postnikov filtration on the cosimplicial Adams–Novikov resolution of a spectrum X leads to the Whitehead filtration in the Beilinson t-structure on the associated filtered spectrum $\mathrm{Tot}_\star(X \wedge MU^{\bullet+1})$. Thus, in the case where X has even MU-homology (for example, for the sphere spectrum), $\Gamma_\star X$ agrees, up to a reindexing to account for the "double speed" of our filtration, with Antieau's spectral décalage functor applied to the Adams–Novikov tower of X. In particular, the associated graded object of $\Gamma_\star X$ has homotopy groups given by the Adams–Novikov E_2-page of X, which one can directly see from the Bousfield–Kan spectral sequence used in Proposition 3.3. Under the correspondence to $\mathbb{C}$-motivic homotopy theory discussed in Section 6, this corresponds to the observation that the motivic homotopy groups of $S^{0,0}/\tau$ coincide with the classical Adams–Novikov E_2-page.

3.1. The ring $\Gamma_\star S^0$

Our next goal is to show that $\Gamma_\star S^0$ is a commutative algebra object in the ∞-category $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$. This will allow us to consider $\Gamma_\star S^0$-modules in $\mathbf{Sp}^{\mathbb{Z}^{\mathrm{op}}}$.

We will need to work in the ∞-category of cosimplicial filtered spectra, or equivalently in filtered cosimplicial spectra. Such objects are $(\mathbb{Z}^{\mathrm{op}} \times \Delta)$-shaped diagrams of spectra.

In accordance with our general grading conventions, $X_\star^\bullet$ denotes a cosimplicial filtered spectrum, where $\star$ refers to the filtered degree while $\bullet$ refers to the cosimplicial degree. Thus every $X_\star^s$ is a filtered spectrum, and every $X_w^\bullet$ is a cosimplicial spectrum.

The ∞-category of cosimplicial filtered spectra is symmetric monoidal with respect to Day convolution applied pointwise in the cosimplicial direction. More concretely, if $X_\star^\bullet$ and $Y_\star^\bullet$ are cosimplicial filtered spectra, then $X_\star^\bullet \otimes Y_\star^\bullet$ is the cosimplicial filtered spectrum such that

$$(X_\star^\bullet \otimes Y_\star^\bullet)_\star^s = X_\star^s \otimes Y_\star^s,$$

where the latter product is the Day convolution product of filtered spectra as in (1). We refer to this product in the proof of Proposition 3.8 when we study lax symmetric monoidal functors taking values in cosimplicial filtered spectra.

Proposition 3.8. *The functor $\Gamma_\star$ is lax symmetric monoidal.*

Proof. We will show that $\Gamma_\star$ is a composition of three lax symmetric monoidal functors.

First, consider the functor $\mathbf{Sp} \to \mathbf{Sp}^\Delta$ that takes a spectrum X to the cosimplicial spectrum $X \wedge MU^{\bullet+1}$. This functor is lax symmetric monoidal, i.e., there are natural maps

$$(X \wedge MU^{\bullet+1}) \wedge (Y \wedge MU^{\bullet+1}) \to (X \wedge Y) \wedge MU^{\bullet+1},$$

because $MU^{\bullet+1}$ is a commutative ring object in the ∞-category of cosimplicial spectra.

Second, recall the truncation functor $(\tau_{\geq 0})_\star$ on filtered spectra defined at the end of Section 2.2. Let $(\tau_{\geq 0})_\star^\bullet$ be the functor from cosimplicial filtered spectra to cosimplicial filtered spectra that applies $(\tau_{\geq 0})_\star$ pointwise in the cosimplicial direction. This functor is also lax symmetric monoidal because $(\tau_{\geq 0})_\star$ is lax symmetric monoidal by Corollary 2.11.

Finally, we have the totalization functor from cosimplicial filtered spectra to filtered spectra that applies Tot pointwise in the filtered direction. This functor is lax symmetric monoidal because it is a homotopy limit in the ∞-category of filtered spectra. ■

Remark 3.9. The functor $\Gamma_\star$ is not strong monoidal. We will see in Examples 3.14 and 3.16 that $\Gamma_\star S^2$ equals $\Sigma^{2,1}\Gamma_\star S^0$, while $\Gamma_\star S^1$ equals $\Sigma^{1,0}\Gamma_\star S^0$. Therefore, the map

$$\Gamma_\star S^1 \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star S^1 \to \Gamma_\star(S^1 \wedge S^1)$$

is the map $\tau : \Sigma^{2,0}\Gamma_\star S^0 \to \Sigma^{2,1}\Gamma_\star S^0$, which is not an equivalence of filtered spectra.

Theorem 3.10. *The filtered spectrum $\Gamma_\star S^0$ is an E_∞-ring object in the ∞-category of filtered spectra.*

Proof. This follows immediately from Proposition 3.8 because lax symmetric monoidal functors preserve E_∞-ring objects. ■

We can now define the ∞-category in which we are primarily interested.

Definition 3.11. Let $\mathrm{Mod}_{\Gamma_\star S^0}$ be the ∞-category of left $\Gamma_\star S^0$-modules in the ∞-category of filtered spectra.

We showed in Proposition 3.8 that the functor $\Gamma_\star$ is lax symmetric monoidal. Therefore, $\Gamma_\star X$ is a $\Gamma_\star S^0$-module for every spectrum X.

Equivalences in $\mathrm{Mod}_{\Gamma_\star S^0}$ are defined to be equivalences on the underlying filtered spectra. For any two $\Gamma_\star S^0$-modules X and Y, let $[X, Y]_{\Gamma_\star S^0}$ be the set of homotopy classes of $\Gamma_\star S^0$-module maps from X to Y.

Proposition 3.12. *In the ∞-category $\mathrm{Mod}_{\Gamma_\star S^0}$, a map is a weak equivalence if and only if it induces an isomorphism on $[\Sigma^{p,q}\Gamma_\star S^0, -]_{\Gamma_\star S^0}$ for all p and q. Equivalently, $\mathrm{Mod}_{\Gamma_\star S^0}$ is generated under homotopy colimits by the objects $\Sigma^{p,q}\Gamma_\star S^0$ for all p and q.*

Proof. This follows from Lemma 2.6, together with the adjunction

$$[\Sigma^{p,q}\Gamma_\star S^0, X]_{\Gamma_\star S^0} \cong \pi_{p,q} X. \qquad \blacksquare$$

3.2. *Exactness properties of $\Gamma_\star$*

In general, the functor $\Gamma_\star$ is not exact, in the sense that it does not preserve all cofiber sequences. However, we shall show that $\Gamma_\star$ preserves certain types of cofiber sequences. These results are essential for computations later in Sections 4 and 5.

Lemma 3.13. *For any spectrum X, the filtered spectrum $\Gamma_\star(\Sigma^{2k} X)$ is equivalent to $\Sigma^{2k,k}\Gamma_\star X$.*

Proof. We have that $\Gamma_w(\Sigma^{2k} X)$ is equal to

$$\mathrm{Tot}(\tau_{\geq 2w}(S^{2k} \wedge X \wedge MU^{\bullet+1})),$$

which is equivalent to

$$\mathrm{Tot}(S^{2k} \wedge \tau_{\geq 2(w-k)}(X \wedge MU^{\bullet+1})).$$

The functor Tot commutes (up to homotopy) with suspension since homotopy limits commute with desuspension. We conclude that $\Gamma_w(\Sigma^{2k} X)$ is equal to $\Sigma^{2k}\Gamma_{w-k} X$. ■

Example 3.14. When X is S^0, Lemma 3.13 implies that $\Gamma_\star S^{2k}$ equals $\Sigma^{2k,k}\Gamma_\star S^0$.

Lemma 3.15. *Let X be a spectrum such that $MU_* X$ is concentrated in even degrees. Then $\Gamma_\star(\Sigma^{2k+1} X)$ equals $\Sigma^{2k+1,k}\Gamma_\star X$.*

Proof. We see that $\Gamma_w(\Sigma^{2k+1} X)$ is equal to $\mathrm{Tot}(\tau_{\geq 2w}(S^{2k+1} \wedge X \wedge MU^{\bullet+1}))$, which is equivalent to $\mathrm{Tot}(S^{2k+1} \wedge \tau_{\geq 2(w-k)}(X \wedge MU^{\bullet+1}))$ because each $X \wedge MU^{\bullet+1}$ has homotopy groups concentrated in even degrees. Similarly to the proof of Lemma 3.13, we conclude that $\Gamma_w(\Sigma^{2k+1} X)$ is equal to $\Sigma^{2k+1}\Gamma_{w-k} X$. ■

Example 3.16. When X is S^0, Lemma 3.15 implies that the filtered spectrum $\Gamma_\star S^{2k+1}$ equals $\Sigma^{2k+1,k}\Gamma_\star S^0$.

Remark 3.17. The functor $\Gamma_\star$ does not commute with suspensions. For example, consider the first Hopf map $\eta : S^1 \to S^0$. Then $\Gamma_\star \eta$ is a map $\Sigma^{1,0}\Gamma_\star S^0 \to \Gamma_\star S^0$, of relative degree $(1,0)$. On the other hand, consider $\Sigma\eta : S^2 \to S^1$. Then $\Gamma_\star(\Sigma\eta)$ is a map $\Sigma^{2,1}\Gamma_\star S^0 \to \Sigma^{1,0}\Gamma_\star S^0$, of relative degree $(1,1)$.

Proposition 3.18. *Let $X \to Y \to Z$ be a cofiber sequence such that $MU_{2w-1}X \to MU_{2w-1}Y$ is injective for all w. Then*

$$\Gamma_\star X \to \Gamma_\star Y \to \Gamma_\star Z$$

is a cofiber sequence of filtered spectra.

Proof. The given condition implies that

$$\pi_{2w-1}(X \wedge MU^{n+1}) \to \pi_{2w-1}(Y \wedge MU^{n+1})$$

is injective for all w and all n. Since the composite

$$\tau_{\geq 2w}(X \wedge MU^{n+1}) \to \tau_{\geq 2w}(Y \wedge MU^{n+1}) \to \tau_{\geq 2w}(Z \wedge MU^{n+1}) \tag{2}$$

is null-homotopic, we get a map

$$\mathrm{cofib}(\tau_{\geq 2w}(X \wedge MU^{n+1}) \to \tau_{\geq 2w}(Y \wedge MU^{n+1})) \to \tau_{\geq 2w}(Z \wedge MU^{n+1}).$$

A diagram chase in homotopy groups shows that this is an equivalence, so (2) is a cofiber sequence for all w and all n. The functor Tot preserves cofiber sequences because fiber sequences are the same as cofiber sequences, and Tot is a homotopy limit. Therefore,

$$\Gamma_w X \to \Gamma_w Y \to \Gamma_w Z$$

is a cofiber sequence for all w. ∎

Proposition 3.18 includes a technical condition about odd MU-homology. We would like to show that this condition holds for a large class of spectra. With that goal in mind, Definition 3.19 encapsulates some standard notions in a convenient form.

Definition 3.19. A spectrum X is a *bounded-below, finite type, even-cell complex* if it is a finite complex with cells only in even dimensions, or if it is the homotopy colimit of a sequence

$$* = X^{(0)} \to X^{(1)} \to X^{(2)} \to \cdots,$$

where there are cofiber sequences

$$X^{(n-1)} \to X^{(n)} \to S^{2k_n}$$

for some integers k_n that tend to ∞ as $n \to \infty$. Analogously, X is a *p-local* (resp., *complete*) *bounded-below, finite type, even-cell complex* if it admits the same structure with p-localized (resp., completed) spheres as cofibers.

Remark 3.20. A spectrum X satisfies Definition 3.19 if and only if it is bounded below, and $H_*(X;\mathbb{Z})$ is concentrated in even degrees, is torsion-free, and is degreewise finitely generated. Specific examples include MU, as well as BP and $BP\langle n\rangle$ in the p-local sense.

In the p-complete context, one can use $H_*(X;\mathbb{F}_p)$ instead of integral homology to detect bounded-below, finite type, even-cell complexes.

Note that if X and Y are bounded-below, finite type, even-cell complexes, then so is $X \wedge Y$. This follows from the standard fact that $X \wedge Y$ has a cell structure in which the n-cells correspond to pairs of i-cells in X and j-cells in Y such that $i + j = n$. This remains true for the p-local variant, and true for the p-complete variant if we use the completed smash product.

Lemma 3.21. *Let X be a (p-local) bounded-below, finite type, even-cell complex. Then $MU \wedge X$ splits as an MU-module into a wedge $\bigvee \Sigma^{2k_n} MU$ of even shifts of MU (even shifts of $MU_{(p)}$ in the p-local variant). In particular, if Y is a spectrum whose MU-homology is concentrated in even degrees, then $MU_*(X \wedge Y)$ is concentrated in even degrees.*

Proof. Let X be the homotopy colimit of the diagram

$$X^{(0)} \to X^{(1)} \to X^{(2)} \to \cdots,$$

with cofiber sequences

$$X^{(n-1)} \to X^{(n)} \to S^{2k_n}.$$

Inductively assume that the MU-homology of $X^{(n-1)}$ is concentrated in even degrees. Since the MU-homology of S^{2k_n} is also concentrated in even degrees, we get a short exact sequence

$$0 \to MU_*X^{(n-1)} \to MU_*X^{(n)} \to MU_*S^{2k_n} \to 0.$$

Since the MU-homology of S^{2k_n} is furthermore free as an MU_*-module, the sequence splits. Inductively, we see that $MU_*X^{(n)}$ is free as an MU_*-module. Since MU_* commutes with filtered colimits, this follows for MU_*X as well. Finally, a basis x_{2k_n} for MU_*X gives rise to a map

$$\bigvee \Sigma^{2k_n} MU \to MU \wedge X,$$

which is an equivalence since it is an isomorphism on homotopy groups.

For the other statement, observe that

$$MU_*(X \wedge Y) = \pi_*(MU \wedge X \wedge Y) \simeq \bigoplus \pi_*(\Sigma^{2k_n} MU \wedge Y) \simeq \bigoplus MU_{*-2k_n} Y,$$

which is concentrated in even degrees.

The p-local variant follows from the same argument, with the obvious modifications. ∎

Remark 3.22. When Y is S^0, Lemma 3.21 shows that the MU-homology of a (p-local) bounded-below, finite type, even-cell complex is concentrated in even degrees.

Corollary 3.23. *Let*

$$X \to Y \to Z$$

be a cofiber sequence of (p-local) bounded-below, finite type, even-cell complexes, and let W be a spectrum whose MU-homology is concentrated in even degrees. Then

$$\Gamma_\star(X \wedge W) \to \Gamma_\star(Y \wedge W) \to \Gamma_\star(Z \wedge W)$$

is a cofiber sequence.

Proof. Lemma 3.21 establishes the hypothesis of Proposition 3.18. ∎

Remark 3.24. When W is S^0, Corollary 3.23 shows that $\Gamma_\star$ preserves cofiber sequences of (p-local) bounded-below, finite type, even-cell complexes.

Lemma 3.25. *Let*

$$\cdots \to X_i \to X_{i+1} \to \cdots$$

be a sequential diagram of uniformly bounded-below spectra such that the connectivity of the maps $X_i \to X_{i+1}$ tends to infinity. Let $X = \mathrm{colim}_i X_i$. Then

$$\operatorname*{hocolim}_i \Gamma_\star(X_i) \to \Gamma_\star(X)$$

is an equivalence.

Proof. We observe that formation of the cosimplicial object $\tau_{\geq 2w}(X_i \wedge MU^{\bullet+1})$ commutes with filtered colimits levelwise. We have to check that in our given situation, the filtered colimit also commutes with totalization. To see this, we first recall that filtered colimits always commute with finite limits. Thus, in the diagram

$$\begin{tikzcd}
\operatorname{hocolim} \operatorname{Tot}(\tau_{\geq 2w}(X_i \wedge MU^{\bullet+1})) \arrow[r] \arrow[d] & \operatorname{Tot}(\tau_{\geq 2w}(X \wedge MU^{\bullet+1})) \arrow[d] \\
\operatorname{hocolim} \operatorname{Tot}^n(\tau_{\geq 2w}(X_i \wedge MU^{\bullet+1})) \arrow[r] & \operatorname{Tot}^n(\tau_{\geq 2w}(X_i \wedge MU^{\bullet+1}))
\end{tikzcd}$$

the bottom map is an equivalence for any n. Since X_i and X are bounded below, the vertical maps are isomorphisms on homotopy groups through a range increasing with n. Since the upper horizontal map is independent of n, it follows that it induces an isomorphism on all homotopy groups, and thus is an equivalence. ∎

Proposition 3.26. *Let X be a (p-local) bounded-below, finite type, even-cell complex, and let Y have MU-homology concentrated in even degrees. Then*

$$\Gamma_\star X \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Y \to \Gamma_\star(X \wedge Y)$$

is an equivalence.

Proof. Let X be the homotopy colimit of

$$X^{(0)} \to X^{(1)} \to X^{(2)} \to \cdots,$$

with cofiber sequences

$$X^{(n-1)} \to X^{(n)} \to S^{2k_n}$$

(or their p-local variant).

Corollary 3.23 shows that

$$\Gamma_\star(X^{(n-1)} \wedge Y) \to \Gamma_\star(X^{(n)} \wedge Y) \to \Gamma_\star(S^{2k_n} \wedge Y)$$

is a cofiber sequence.

We have a diagram

$$\begin{tikzcd}
\Gamma_\star X^{(n-1)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y \arrow[r] \arrow[d] & \Gamma_\star X^{(n)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y \arrow[r] \arrow[d] & \Gamma_\star S^{2k_n} \wedge_{\Gamma_\star S^0} \Gamma_\star Y \arrow[d] \\
\Gamma_\star(X^{(n-1)} \wedge Y) \arrow[r] & \Gamma_\star(X^{(n)} \wedge Y) \arrow[r] & \Gamma_\star(S^{2k_n} \wedge Y)
\end{tikzcd}$$

in which the rows are cofiber sequences. The right vertical map is an equivalence by Lemma 3.13 and Example 3.14. The left vertical map is an equivalence by an induction assumption. Therefore, the middle vertical map is also an equivalence.

Apply homotopy colimits to obtain the equivalence

$$\operatorname*{hocolim}_n \left(\Gamma_\star X^{(n)} \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Y\right) \to \operatorname*{hocolim}_n (\Gamma_\star(X^{(n)} \wedge Y)).$$

The source of this map is equivalent to $\Gamma_\star X \wedge_{\Gamma_\star S^0} \Gamma_\star Y$ because the homotopy colimit commutes with $\Gamma_\star$ by Lemma 3.25, and also with smash products. The target is equivalent to $\Gamma_\star(X \wedge Y)$ by Lemma 3.25. ∎

4. The Steenrod algebra

Consider the $\Gamma_\star S^0$-module $\Gamma_\star H\mathbb{F}_2$. The Adams–Novikov spectral sequence for $H\mathbb{F}_2$ collapses. It follows by inspection of the definition of $\Gamma_\star$ that $\Gamma_\star H\mathbb{F}_2$ is equal to the filtered spectrum

$$\cdots * \to * \to H\mathbb{F}_2 \to H\mathbb{F}_2 \to \cdots,$$

where the values are $*$ in filtrations greater than zero, and the values are $H\mathbb{F}_2$ in filtrations less than or equal to zero. In particular, $\pi_{*,\star}\Gamma_\star H\mathbb{F}_2$ is isomorphic to $\mathbb{F}_2[\tau]$, where τ has degree $(0,-1)$. Not coincidentally, these homotopy groups are isomorphic to the motivic stable homotopy groups of the $\mathbb{C}$-motivic Eilenberg–Mac Lane spectrum.

The goal of this section is to compute the Hopf algebra of co-operations on $\Gamma_\star H\mathbb{F}_2$. This Hopf algebra is the dual Steenrod algebra in the context of $\Gamma_\star S^0$-modules.

Definition 4.1. Let $A_{*,\star}$ be the Hopf algebra $\pi_{*,\star}(\Gamma_\star H\mathbb{F}_2 \wedge_{\Gamma_\star S^0} \Gamma_\star H\mathbb{F}_2)_{*,\star}$.

We begin by studying $\Gamma_\star BP$ and related objects.

Example 4.2. The Adams–Novikov spectral sequence for BP collapses. It follows from the definition that $\Gamma_\star BP$ is equal to the filtered spectrum

$$\cdots \to \tau_{\geq 4} BP \to \tau_{\geq 2} BP \to BP \to BP \to \cdots .$$

In particular, note that $\pi_{*,\star}\Gamma_\star BP$ is isomorphic to $\mathbb{Z}_{(p)}[\tau, v_1, v_2, \ldots]$, where τ has degree $(0,-1)$ and v_i has degree $(2^{i+1}-2, 2^i-1)$. Not coincidentally, these homotopy groups are isomorphic to the motivic stable homotopy groups of the $\mathbb{C}$-motivic Brown–Peterson spectrum $BPGL$.

Example 4.3. Generalizing both $\Gamma_\star H\mathbb{F}_2$ and Example 4.2, we find that the filtered spectrum $\Gamma_\star BP\langle n\rangle$ is

$$\cdots \to \tau_{\geq 4} BP\langle n\rangle \to \tau_{\geq 2} BP\langle n\rangle \to \tau_{\geq 0} BP\langle n\rangle \to \tau_{\geq -2} BP\langle n\rangle \to \cdots .$$

Moreover, $\pi_{*,\star}\Gamma_\star BP\langle n\rangle$ is isomorphic to $\mathbb{Z}_{(p)}[\tau, v_1, \ldots, v_n]$, where τ has degree $(0,-1)$ and v_i has degree $(2^{i+1}-2, 2^i-1)$.

If R is a ring spectrum and x is an indeterminant of degree n, then we write $R[x]$ for the ring spectrum $\bigvee_i \Sigma^{ni} R$, with the obvious multiplication corresponding to multiplication in a polynomial ring. We use the same notation for a filtered ring spectrum R and a bigraded indeterminant. The object $R[x_0, x_1, \ldots]$ with multiple indeterminants is defined analogously.

Recall that $BP \wedge BP$ is equivalent to $BP[t_1, t_2, \ldots]$, where t_i has degree $2^{i+1}-2$ [23, Theorem 4.1.18]. We now establish an analogous result for $\Gamma_\star BP$.

We observed in Proposition 3.8 that $\Gamma_\star$ is lax monoidal. Therefore, it takes ring objects to ring objects. In particular, $\Gamma_\star BP$ and $\Gamma_\star BP \wedge_{\Gamma_\star S^0} \Gamma_\star BP$ have ring structures induced from the usual ring structures on BP and $BP \wedge BP$.

Proposition 4.4. *As a ring object, the filtered spectrum $\Gamma_\star BP \wedge_{\Gamma_\star S^0} \Gamma_\star BP$ is equivalent to $\Gamma_\star BP[t_1, t_2, \ldots]$, where t_i has bidegree $(2^{i+1}-2, 2^i-1)$.*

Proof. We observed in Remark 3.20 that BP is a p-local bounded-below, finite type, even-cell complex. Therefore, Proposition 3.26 applies, and $\Gamma_\star BP \wedge_{\Gamma_\star S^0} \Gamma_\star BP$ is equivalent to $\Gamma_\star(BP \wedge BP)$, which is equivalent to $\Gamma_\star(BP[t_1, t_2, \ldots])$. An argument similar to the proof of Lemma 3.25 shows that $\Gamma_\star$ commutes with the infinite wedge that defines $BP[t_1, t_2, \ldots]$. Finally, use Lemma 3.13 to determine the bidegree of t_i. ■

Proposition 4.5. *The ring $\pi_{*,\star}(\Gamma_\star BP \wedge_{\Gamma_\star S^0} \Gamma_\star BP)$ is isomorphic to*

$$\mathbb{Z}_{(p)}[\tau][v_1, v_2, \ldots, t_1, t_2, \ldots],$$

where τ has degree $(0,-1)$, and v_i and t_i both have degree $(2^{i+1}-2, 2^i-1)$.

Proof. This follows from Example 4.2 and Proposition 4.4. ■

The computation of Proposition 4.5 is a bigraded version of the classical computation [23, Theorem 4.1.19]

$$BP_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, t_1, t_2, \ldots].$$

Definition 4.6. The $\Gamma_\star H\mathbb{F}_2$-homology of a $\Gamma_\star S^0$-module X is

$$H_{*,\star}(X) = \pi_{*,\star}\Big(X \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star H\mathbb{F}_2\Big).$$

Proposition 4.7. *The bigraded ring $H_{*,\star}(\Gamma_\star BP)$ is isomorphic to the free polynomial ring $\mathbb{F}_2[\tau][\xi_1, \xi_2, \ldots]$, where τ has degree $(0, -1)$ and ξ_n has degree $(2^{n+1} - 2, 2^n - 1)$.*

The computation of Proposition 4.7 is a bigraded version of the classical computation [3], [23, Theorem 4.1.12(b)]

$$H_*(BP) = \mathbb{F}_2[\xi_1, \xi_2, \ldots].$$

Here we are using the non-standard description

$$\frac{\mathbb{F}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 + \xi_{i+1}}$$

of the classical dual Steenrod algebra, since it aligns better with the notation for the dual Steenrod algebra $A_{*,\star}$ given below in Theorem 4.10.

Proof. First we determine the additive structure of $H_{*,\star}(\Gamma_\star BP)$. Let $X(-1)$ be $\Gamma_\star BP$, and define $X(n)$ inductively to be the cofiber of

$$v_n : \Sigma^{2^{n+1}-2, 2^n-1} X(n-1) \to X(n-1).$$

From the descriptions of $\Gamma_\star H\mathbb{F}_2$ and $\Gamma_\star BP$ at the beginning of Section 4, we see that $\Gamma_\star H\mathbb{F}_2$ is $\operatorname{hocolim}_n X(n)$. This mimics the standard construction of $H\mathbb{F}_2$ as $BP/(v_0, v_1, \ldots)$.

We have cofiber sequences

$$\Sigma^{2^{n+1}-2, 2^n-1} \Gamma_\star BP \underset{\Gamma_\star S^0}{\wedge} X(n-1) \xrightarrow{v_n} \Gamma_\star BP \underset{\Gamma_\star S^0}{\wedge} X(n-1) \to \Gamma_\star BP \underset{\Gamma_\star S^0}{\wedge} X(n).$$

Starting from Proposition 4.5, we can analyze the associated long exact sequences in homotopy groups. Inductively, we compute that $\pi_{*,\star}(\Gamma_\star BP \wedge_{\Gamma_\star S^0} X(n))$ is isomorphic to $\mathbb{F}_2[\tau][v_{n+1}, v_{n+2}, \ldots, \xi_1, \xi_2, \ldots]$, where each ξ_i corresponds to t_i in Proposition 4.5. In the limit, we obtain $\mathbb{F}_2[\tau][\xi_1, \xi_2, \ldots]$ as desired.

Finally, we determine the multiplicative structure of $H_{*,\star}(\Gamma_\star BP)$. Recall from Proposition 3.6 that $\operatorname{colim}_\star \Gamma_\star BP$ equals BP. Computationally, applying $\operatorname{colim}_\star$ has the effect of inverting τ, so $H_{*,\star}(\Gamma_\star BP)[\tau^{-1}]$ and $H_*(BP) \otimes \mathbb{F}_2[\tau^{\pm 1}]$ are isomorphic as rings. Now $H_*(BP)$ and $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$ are isomorphic as rings [3]. Therefore, $H_{*,\star}(\Gamma_\star BP)[\tau^{-1}]$ and $\mathbb{F}_2[\tau^{\pm 1}][\xi_1, \xi_2, \ldots]$ are isomorphic as rings. This determines the ring structure on $H_{*,\star}(\Gamma_\star BP)$ as well, since $H_{*,\star}(\Gamma_\star BP)$ has no τ-torsion. ∎

Remark 4.8. The last paragraph of the proof of Proposition 4.7 uses a technique that we shall rely on frequently. First, we determine an $\mathbb{F}_2[\tau]$-module M additively, and we note that it has no τ-torsion. Then we use classical information to recognize additional structure on $M[\tau^{-1}]$. Finally, we deduce additional structure on M itself since M has no τ-torsion.

Proposition 4.9. *The bigraded ring $H_{*,\star}(\Gamma_\star BP\langle n\rangle)$ is isomorphic to*

$$\frac{\mathbb{F}_2[\tau][\tau_{n+1}, \tau_{n+2}, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 + \tau\xi_{i+1}},$$

where τ has degree $(0, -1)$, τ_i has degree $(2^{i+1} - 1, 2^i - 1)$ and ξ_i has degree $(2^{i+1} - 2, 2^i - 1)$.

Proof. We determine the additive structure of $H_{*,\star}(\Gamma_\star BP\langle n\rangle)$. The multiplicative structure then follows by comparison to the classical case, using the strategy described in Remark 4.8. In order for this strategy to work, it is essential to observe that all modules under consideration turn out to have no τ-torsion.

Let $Y(n)$ be $\Gamma_\star BP$, and define $Y(k)$ for $k > n$ inductively to be the cofiber of

$$v_k : \Sigma^{2^{k+1}-2, 2^k-1} Y(k-1) \to Y(k-1).$$

From the descriptions of $\Gamma_\star BP$ and $\Gamma_\star BP\langle n\rangle$ at the beginning of Section 4, we see that $\Gamma_\star BP\langle n\rangle$ is $\operatorname{hocolim}_n Y(n)$. This mimics the standard construction of $BP\langle n\rangle$ as $BP/(v_{n+1}, v_{n+2}, \ldots)$.

Suppose for induction that $H_{*,\star}(Y(k))$ is isomorphic to

$$\frac{\mathbb{F}_2[\tau][\tau_{n+1}, \tau_{n+2}, \ldots \tau_k, \xi_1, \xi_2, \ldots]}{\tau_i^2 + \tau\xi_{i+1}}.$$

Proposition 4.7 establishes the base case.

Consider the cofiber sequence

$$Y(k) \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star H\mathbb{F}_2 \to Y(k+1) \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star H\mathbb{F}_2 \\ \to \Sigma^{2^{k+2}-1, 2^{k+1}-1} Y(k) \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star H\mathbb{F}_2.$$

Using the strategy of Remark 4.8 and the analogous classical fact, we see that multiplication by v_{k+1} is zero on $H_{*,\star}(Y(k))$. Therefore, we have a short exact sequence

$$H_{*,\star}(Y(k)) \to H_{*,\star}(Y(k+1)) \to H_{*,\star}(\Sigma^{2^{k+2}-1, 2^{k+1}-1} Y(k)).$$

This establishes the additive structure of $H_{*,\star}(Y(k+1))$. ∎

Theorem 4.10. *The dual Steenrod algebra $A_{*,\star}$ is isomorphic to*

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 + \tau\xi_{i+1}},$$

where the comultiplication is given by the formulas

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \tau_k, \qquad \Delta(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k.$$

Proof. The additive structure is given by the $n = 0$ case of Proposition 4.9.

The formulas for the multiplication and comultiplication are deduced using the strategy of Remark 4.8 and the analogous classical formulas. ■

Remark 4.11. Having described the Steenrod algebra in Theorem 4.10, we can now return to the computations of $H_{*,\star}(\Gamma_* BP)$ and $H_{*,\star}(\Gamma_* BP\langle n\rangle)$ in Propositions 4.7 and 4.9. These objects have the evident $A_{*,\star}$-subcomodule structures reflected in the notation. This explains the use of ξ_i in Propositions 4.7 and 4.9, rather than the use of t_i as in Proposition 4.5. The computation of these comodule structures follows from the analogous classical result, using the strategy of Remark 4.8 since there is no τ-torsion.

5. Motivic modular forms

In this section, we study the $\Gamma_\star S^0$-module $\Gamma_\star \mathit{tmf}$. We will show that this object has the desired properties of a "motivic modular forms" spectrum. Note that $\Gamma_\star \mathit{tmf}$ is an E_∞-ring by Proposition 3.8.

In Section 4, we worked with $\Gamma_\star H\mathbb{F}_2$-homology because the dual Steenrod algebra is easier to describe than the Steenrod algebra. We now work with the dual $\Gamma_\star H\mathbb{F}_2$-cohomology because it is easier to state the specific computational results that we are pursuing.

Definition 5.1. The $\Gamma_\star H\mathbb{F}_2$-cohomology of a $\Gamma_\star S^0$-module X is

$$H^{*,*}(X) = \pi_{*,*} F_{\Gamma_\star S^0}(X, \Gamma_\star H\mathbb{F}_2),$$

where $F_{\Gamma_\star S^0}(-,-)$ is the internal function object in the ∞-category of $\Gamma_\star S^0$-modules.

Under suitable bounded-below, finite type assumptions on X, the $\Gamma_\star H\mathbb{F}_2$-cohomology of X and the $\Gamma_\star H\mathbb{F}_2$-homology of X are algebraic $\mathbb{F}_2[\tau]$-duals.

The main goal is to show that the $\Gamma_\star H\mathbb{F}_2$-cohomology of $\Gamma_\star \mathit{tmf}$ is isomorphic to $A/\!/A(2)$. This implies that the cohomology of $A(2)$ is the E_2-page of the $\Gamma_\star H\mathbb{F}_2$-based Adams spectral sequence for $\Gamma_\star \mathit{tmf}$.

Definition 5.2. Let A be the dual of $A_{*,\star}$. Using the monomial basis of $A_{*,\star}$, let

(1) Sq^1 be dual to τ_0;

(2) Q_i be dual to τ_i;

(3) Sq^{2^n} be dual to $\xi_1^{2^{n-1}}$ for $n \geq 1$;

(4) P_j^0 be dual to τ_{j-1} for $j \geq 1$;

(5) P_j^i be dual to $\xi_j^{2^{i-1}}$ for $i \geq 1$ and $j \geq 1$.

We will need to refer to some quotients of the dual Steenrod algebra $A_{*,\star}$ computed in Theorem 4.10.

Definition 5.3. (1) Let $A(n)_{*,\star}$ be the quotient

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \ldots, \tau_n, \xi_1, \xi_2, \ldots, \xi_n]}{\tau_i^2 + \tau\xi_{i+1}, \xi_1^{2^n}, \xi_2^{2^{n-1}}, \ldots, \xi_n^2, \tau_n^2}$$

of $A_{*,\star}$, and let $A(n)$ be the dual subalgebra of A.

(2) Let $E(n)_{*,\star}$ be the quotient

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \ldots, \tau_n]}{\tau_0^2, \tau_1^2, \ldots \tau_n^2}$$

of $A_{*,\star}$, and let $E(n)$ be the dual subalgebra of A.

It is straightforward to check that $E(n)$ is an exterior algebra on the elements Q_0, $Q_1, \ldots Q_n$, and $A(n)$ is the subalgebra of A generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n}$.

Definition 5.4. We define the finite even-cell complexes:

(1) X is the cofiber of $\nu : S^3 \to S^0$.

(2) Y is the cofiber of $\eta : \Sigma X \to X$.

(3) Z is the cofiber of $w_1 : \Sigma^5 Y \to Y$.

The best way to describe w_1 is in terms of the Adams spectral sequence for maps $\Sigma^5 Y \to Y$. It is detected by the element whose May spectral sequence name is h_{21}. This is essential below in Lemma 5.5 when we relate w_1 to the Steenrod operation P_2^1.

For our purposes, the essential property of Z is that $tmf_{(2)} \wedge Z$ is equivalent to $BP\langle 2\rangle$ [19]. We will start with the cohomology of $\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Z$, and then work backwards to obtain the cohomology of $\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y$, $\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star X$, and finally $\Gamma_\star tmf_{(2)}$. The basic idea is not original. See, for example, [28, Proposition 1.7] for an analogous argument that computes the homology of $BP\langle n\rangle$. See also [16, Section 5] for a motivic version of an argument that computes the cohomology of ko.

Lemma 5.5. *In the ∞-category of $\Gamma_\star S^0$-modules, there are cofiber sequences*

$$\Sigma^{3,2}\Gamma_\star S^0 \to \Gamma_\star S^0 \xrightarrow{\iota_X} \Gamma_\star X \xrightarrow{\pi_X} \Sigma^{4,2}\Gamma_\star S^0,$$

$$\Sigma^{1,1}\Gamma_\star X \to \Gamma_\star X \xrightarrow{\iota_Y} \Gamma_\star Y \xrightarrow{\pi_Y} \Sigma^{2,1}\Gamma_\star X,$$

$$\Sigma^{5,3}\Gamma_\star Y \to \Gamma_\star Y \xrightarrow{\iota_Z} \Gamma_\star Z \xrightarrow{\pi_Z} \Sigma^{6,3}\Gamma_\star Y.$$

In $\Gamma_\star H\mathbb{F}_2$-cohomology, the compositions

$$\Gamma_\star X \xrightarrow{\pi_X} \Sigma^{4,2}\Gamma_\star S^0 \xrightarrow{\Sigma^{4,2}\iota_X} \Sigma^{4,2}\Gamma_\star X,$$

$$\Gamma_\star Y \xrightarrow{\pi_Y} \Sigma^{2,1}\Gamma_\star X \xrightarrow{\Sigma^{2,1}\iota_Y} \Sigma^{2,1}\Gamma_\star Y,$$

$$\Gamma_\star Z \xrightarrow{\pi_Z} \Sigma^{6,3}\Gamma_\star Y \xrightarrow{\Sigma^{6,3}\iota_Z} \Sigma^{6,3}\Gamma_\star Z$$

are multiplication by $\mathrm{Sq}^4 = (\xi_1^2)^\vee$, $\mathrm{Sq}^2 = \xi_1^\vee$, *and* $P_2^1 = \xi_2^\vee$ *respectively.*

Proof. Apply Corollary 3.23 to the cofiber sequence

$$S^0 \to X \to S^4$$

to obtain the cofiber sequence

$$\Gamma_\star S^0 \to \Gamma_\star X \to \Gamma_\star S^4.$$

Lemma 3.13 identifies the third term to be $\Sigma^{4,2}\Gamma_\star S^0$, and then rotate to obtain the cofiber sequence in part (1).

The arguments for parts (2) and (3) are essentially identical. Apply Corollary 3.23 to the cofiber sequences

$$X \to Y \to \Sigma^2 X \quad \text{and} \quad Y \to Z \to \Sigma^6 Y$$

to obtain the cofiber sequences

$$\Gamma_\star X \to \Gamma_\star Y \to \Gamma_\star \Sigma^2 X \quad \text{and} \quad \Gamma_\star Y \to \Gamma_\star Z \to \Gamma_\star \Sigma^6 Y.$$

Lemma 3.13 identifies the third terms to be $\Sigma^{2,1} X$ and $\Sigma^{6,3} Y$ respectively, and then rotate to obtain the cofiber sequences in parts (2) and (3).

The assertion about the action in cohomology follows from the analogous classical facts, using the strategy of Remark 4.8. It is essential to observe that $H_{*,\star}(\Gamma_\star X)$, $H_{*,\star}(\Gamma_\star Y)$, and $H_{*,\star}(\Gamma_\star Z)$ have no τ-torsion. In fact, they are free $\mathbb{F}_2[\tau]$-modules of ranks 2, 4, and 8 respectively.

One way to understand the maps in classical cohomology is to observe that ν, η, and w_1 are represented in the Adams spectral sequence in filtration 1. They are detected by h_2, h_1, and h_{21}, which have cobar representatives $[\xi_1^2]$, $[\xi_1]$, and $[\xi_2]$ respectively. Here we are using the non-standard description

$$\frac{\mathbb{F}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 + \xi_{i+1}}$$

of the classical dual Steenrod algebra, since it aligns better with the notation for the dual Steenrod algebra $A_{*,\star}$ given in Theorem 4.10. ∎

Lemma 5.6. *$MU_* tmf_{(2)}$ is concentrated in even degrees.*

Compare Lemma 5.6 with [19, Corollary 5.2], which is stronger. The lemma is true without 2-localization, but we only need the 2-local statement here. At other primes, an analogous, but easier, variant of the given argument goes through.

Proof of Lemma 5.6. Consider the maps

$$\nu : S^3 \to S^0, \quad \eta : \Sigma X \to X, \quad w_1 : \Sigma^5 Y \to Y.$$

The targets of these maps are bounded-below, finite type, even-cell complexes, while the sources are suspensions of bounded-below, finite type, even-cell complexes. For degree reasons, these maps are zero in MU-homology. After smashing with MU, all three maps have sources and targets that are free MU-modules by Lemma 3.21. This means that they are null-homotopic after smashing with MU, since they are zero in MU-homology.

We get short exact sequences

$$\begin{aligned}
&0 \to MU_* tmf_{(2)} \to MU_*(tmf_{(2)} \wedge X) \to MU_*(\Sigma^4 tmf_{(2)}) \to 0,\\
&0 \to MU_*(tmf_{(2)} \wedge X) \to MU_*(tmf_{(2)} \wedge Y) \to MU_*(\Sigma^2 tmf_{(2)} \wedge X) \to 0,\\
&0 \to MU_*(tmf_{(2)} \wedge Y) \to MU_*(tmf_{(2)} \wedge Z) \to MU_*(\Sigma^6 tmf_{(2)} \wedge Y) \to 0.
\end{aligned}$$

Starting with the classical equivalence $BP\langle 2\rangle \simeq tmf_{(2)} \wedge Z$ (see [19]), we see that $MU_*(tmf_{(2)} \wedge Z)$ is concentrated in even degrees, since $MU_*(BP\langle 2\rangle)$ is. The exact sequences then imply that $MU_*(tmf_{(2)} \wedge Y)$, $MU_*(tmf_{(2)} \wedge X)$ and finally $MU_*(tmf_{(2)})$ are also concentrated in even degrees. ∎

Proposition 5.7. *$H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Z)$ is isomorphic to $A/\!/E(2)$.*

Proof. Start with the equivalence $BP\langle 2\rangle \simeq tmf_{(2)} \wedge Z$ (see [19]). Lemma 5.6 establishes the hypothesis of Proposition 3.26, so we get

$$\Gamma_\star BP\langle 2\rangle \simeq \Gamma_\star tmf_{(2)} \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Z.$$

Proposition 4.9 describes $H_{*,\star}(\Gamma_\star BP\langle 2\rangle)$, which is algebraically dual to what we want. By inspection of Proposition 4.9 and the description of $E(2)$ in Definition 5.3, dualization shows that $H^{*,\star}(\Gamma_\star BP\langle 2\rangle)$ is isomorphic to $A/\!/E(2)$. ∎

Proposition 5.8. *$H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y)$ is isomorphic to $A/\!/F$, where F is the subalgebra of A generated by Q_0, Q_1, Q_2, and P_2^1.*

Note that the dual $F_{*,\star}$ of F is

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \tau_2, \xi_2]}{\tau_0^2, \tau_1^2 = \tau\xi_2, \xi_2^2, \tau_2^2}.$$

Proof of Proposition 5.8. We shall establish a commutative diagram

$$\begin{tikzcd}
{} & 0 \arrow[d, leftarrow] \\
H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y) \arrow[u, "\delta"] & A/\!/F \arrow[l, "f"'] \\
H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Z) \arrow[u, "\iota_Z^*"] & A/\!/E(2) \arrow[l, "\cong"'] \arrow[u] \\
H^{*,\star}\Sigma^{6,3}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y) \arrow[u, "\pi_Z^*"] & \Sigma^{6,3}A/\!/F \arrow[l, "\Sigma^{6,3}f"'] \arrow[u] \\
{} \arrow[u, "\delta"] & 0 \arrow[u]
\end{tikzcd}$$

The left column is the long exact sequence associated to the third cofiber sequence of Lemma 5.5 smashed with $\Gamma_\star tmf_{(2)}$. The right column is short exact; one inspects the projection $A//E(2) \to A//F$ and recognizes the kernel to be $\Sigma^{6,3}A//F$, generated by P_2^1 as a right A-module. The middle horizontal arrow is the isomorphism of Proposition 5.7.

It remains to construct the top and bottom horizontal maps, to show that they are the same up to suspension, and to show that both squares commute. The second part of Lemma 5.5 implies that the image of P_2^1 in $H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Z)$ is contained in the image of π_Z^*. Since the left column is exact, this shows that the image of P_2^1 maps to zero under ι_Z^*. In particular, the composition

$$A//E(2) \to H^{*,\star}\left(\Gamma_\star tmf_{(2)} \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Z\right) \to H^{*,\star}\left(\Gamma_\star tmf_{(2)} \underset{\Gamma_\star S^0}{\wedge} \Gamma_\star Y\right)$$

factors through $A//F$. This establishes the map f and the commutativity of the top square.

The existence of the bottom horizontal map, and the commutativity of the bottom square, now follow from a formal diagram chase, using the fact that the columns are exact. In fact, the top and bottom horizontal maps are the same map, up to suspension. This follows from the second part of Lemma 5.5, using the fact that $\Sigma^{6,3}A//F$ is generated by P_1^2.

Now we must show that f is an isomorphism. A diagram chase shows that $\Sigma^{6,3}f$ (and therefore also f) is an injection.

Finally, we use induction on the degree $*$ to show that f is a surjection. The base case is $* = -1$, since all objects are concentrated in degrees $* \geq 0$.

Let α be an element of $H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y)$. Apply the boundary map δ to obtain $\delta(\alpha)$ in $H^{*+7,\star+3}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y)$. By induction on $*$, we may assume that $\delta(\alpha) = f(\beta)$ for some β in $A//F$. If we chase β around the lower left corner of the bottom square, we get 0 since $\pi_Z^*\delta$ is the zero map. On the other hand, the composition around the upper right corner of the bottom square is an injection. Therefore, β must be zero, and $\delta(\alpha)$ is zero as well.

We have now shown that α equals $\iota_Z^*(\gamma)$ for some γ in $H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Z)$. Using the isomorphism in the middle row and the commutativity of the top square, we conclude that α lies in the image of f, as desired. ∎

Proposition 5.9. *$H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star X)$ is isomorphic to $A//G$, where G is the subalgebra of A generated by Q_0, Q_1, Q_2, P_2^1, and* Sq^2.

Note that the dual $G_{*,\star}$ of G is

$$\frac{\mathbb{F}_2[\tau][\tau_0, \tau_1, \tau_2, \xi_1, \xi_2]}{\tau_0^2 = \tau\xi_1, \xi_1^2, \tau_1^2 = \tau\xi_2, \xi_2^2, \tau_2^2}.$$

Proof. The proof is essentially the same as the proof of Proposition 5.8, using the diagram

$$\begin{tikzcd}
{} & 0 \\
H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star X) \arrow[u] & A//G \arrow[l] \arrow[u] \\
H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y) \arrow[u] & A//F \arrow[l, "\cong"'] \arrow[u] \\
H^{*,\star}\Sigma^{2,1}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star Y) \arrow[u] & \Sigma^{2,1}A//G \arrow[l] \arrow[u] \\
{} \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

∎

Theorem 5.10. *$H^{*,\star}(\Gamma_\star tmf)$ is isomorphic to $A//A(2)$.*

Equivalently, the $\Gamma_\star H\mathbb{F}_2$-homology of $\Gamma_\star tmf$ is $A_{*,\star}\Box_{A(2)_{*,\star}}\mathbb{F}_2[\tau]$.

Proof of Theorem 5.10. We first observe that the functor $\Gamma_\star$ is compatible with 2-localization, as one easily sees from its definition. In particular, $\Gamma_\star H\mathbb{F}_2$ is 2-local, and the $\Gamma_\star H\mathbb{F}_2$-cohomology of $\Gamma_\star tmf$ agrees with the $\Gamma_\star H\mathbb{F}_2$-cohomology of $\Gamma_\star tmf_{(2)}$.

The remainder of the proof is essentially the same as the proofs of Propositions 5.8 and 5.9, using the diagram

$$\begin{tikzcd}
{} & 0 \\
H^{*,\star}(\Gamma_\star tmf_{(2)}) \arrow[u] & A//A(2) \arrow[l] \arrow[u] \\
H^{*,\star}(\Gamma_\star tmf_{(2)} \wedge_{\Gamma_\star S^0} \Gamma_\star X) \arrow[u] & A//G \arrow[l, "\cong"'] \arrow[u] \\
H^{*,\star}\Sigma^{4,2}(\Gamma_\star tmf_{(2)}) \arrow[u] & \Sigma^{4,2}A//A(2) \arrow[l] \arrow[u] \\
{} \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

∎

Remark 5.11. The same technique can be used to compute that $H^{*,\star}(\Gamma_\star ko)$ equals $A//A(1)$. In this case, one uses the cofiber sequences

$$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \to H\mathbb{F}_2,$$

$$\Sigma^2 ku \xrightarrow{v_1} ku \to H\mathbb{Z},$$

$$\Sigma^1 ko \xrightarrow{\eta} ko \to ku.$$

See [16, Section 5] or [28, Proposition 1.7] for similar arguments.

6. Comparison to $\mathbb{C}$-motivic homotopy theory

In this section, we work with 2-complete versions of the categories $\mathbf{Sp}_{\mathbb{C}}$ of *cellular* motivic spectra, the ordinary category $\mathbf{Sp}$ of spectra, and our category $\mathrm{Mod}_{\Gamma_\star S^0}$ of $\Gamma_\star S^0$-modules, which we will denote by $\widehat{\mathbf{Sp}}_{\mathbb{C}}$, $\widehat{\mathbf{Sp}}$, and $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$ respectively. To avoid cumbersome notation, we use S^n and $S^{n,w}$ to refer to the 2-completions of the respective spheres throughout the section.

We will show that there is an equivalence between the stable ∞-categories $\widehat{\mathbf{Sp}}_{\mathbb{C}}$ and $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$. This means that both $\mathbf{Sp}_{\mathbb{C}}$ and $\mathrm{Mod}_{\Gamma_\star S^0}$ are uncompleted versions of the same category, but they do not agree before completion. For example, the completed $\mathbb{C}$-motivic element $\tau \in \pi_{0,-1}(S^{0,0})$ does not exist in the uncompleted $\mathbf{Sp}_{\mathbb{C}}$. On the other hand, τ does exist in $\mathrm{Mod}_{\Gamma_\star S^0}$ since it is just the structure map of the filtration.

Recall that $\mathbf{Sp}_{\mathbb{C}}$ is a stable ∞-category, and thus for any motivic spectra X and Y, there exists a mapping spectrum $F_s(X, Y)$ with the property that $\pi_k F_s(X, Y)$ equals $[\Sigma^{k,0} X, Y]$. The subscript s indicates that we are considering only the function object as a classical spectrum, not as a motivic spectrum.

We will rely on the Betti realization functor $B : \widehat{\mathbf{Sp}}_{\mathbb{C}} \to \widehat{\mathbf{Sp}}$ from the ∞-category of cellular 2-complete $\mathbb{C}$-motivic spectra to the ∞-category of ordinary 2-complete spectra. Recall that $B(S^{p,q})$ is S^p. Also, the map $\tau : S^{0,-1} \to S^{0,0}$ realizes to the identity $S^0 \to S^0$.

Lemma 6.1. *If $p \le q$, then Betti realization induces an equivalence*

$$\mathrm{Map}_{\widehat{\mathbf{Sp}}_{\mathbb{C}}}(S^{0,p}, S^{0,q}) \xrightarrow{\simeq} \mathrm{Map}_{\widehat{\mathbf{Sp}}}(S^0, S^0).$$

Proof. This corresponds to the observation in [7] that Betti realization induces an isomorphism $\pi_{s,w} \to \pi_s$ when $w \le 0$, which in turn follows from naive considerations of the $\mathbb{C}$-motivic Adams–Novikov spectral sequence. ∎

The key point of Lemma 6.1 is that when $p \le q$, a map $S^{0,p} \to S^{0,q}$ is uniquely determined up to homotopy by its Betti realization.

Now consider $\mathbb{Z}$ as a poset category with respect to $\le$, with symmetric-monoidal structure obtained from addition.

Lemma 6.2. *Up to contractible choice, there is a unique symmetric-monoidal ∞-functor $S^{0,\star} : \mathbb{Z} \to \widehat{\mathbf{Sp}}_{\mathbb{C}}$ that sends $n \mapsto S^{0,n}$ such that $B(S^{0,\star})$ is the constant functor $\mathbb{Z} \to \widehat{\mathbf{Sp}}$ with value S^0. The induced maps $S^{0,n} \to S^{0,n+1}$ are homotopic to τ.*

Proof. A lax symmetric-monoidal functor between symmetric-monoidal ∞-categories $\mathcal{C} \to \mathcal{D}$ is the same as a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ of the associated colored operads. Thus the data of a lax symmetric-monoidal ∞-functor $F : \mathbb{Z} \to \mathcal{C}$ gives, for each tuple $(i_1, \ldots, i_k; n)$ with $i_1 + \cdots + i_k \le n$, a point in the associated multi-mapping space $\mathrm{Map}_{\mathcal{C}}(F(i_1) \otimes \cdots \otimes F(i_k), F(n))$, with coherence homotopies between the various ways to compose and permute the domain factors.

For our desired functor $S^{0,\star}$, the relevant multi-mapping spaces in $\widehat{\mathbf{Sp}}_{\mathbb{C}}$ are of the form $\mathrm{Map}_{\widehat{\mathbf{Sp}}_{\mathbb{C}}}(S^{0,i_1+\cdots+i_k}, S^{0,n})$. Betti realization induces an equivalence

$$\mathrm{Map}_{\widehat{\mathbf{Sp}}_{\mathbb{C}}}(S^{0,i_1+\cdots+i_k}, S^{0,n}) \xrightarrow{\simeq} \mathrm{Map}_{\widehat{\mathbf{Sp}}}(S^0, S^0) \tag{3}$$

by Lemma 6.1.

The constant functor $\mathbb{Z} \to \widehat{\mathbf{Sp}}$ with value S^0 is lax symmetric-monoidal, since S^0 is an E_∞-ring spectrum. The corresponding coherence data on multi-mapping spaces lifts uniquely (up to contractible choice) along the equivalences (3), giving us the desired lax symmetric-monoidal functor $S^{0,\star}$. To see that it is symmetric-monoidal, it is enough to observe that the lax structure maps

$$S^{0,i_1} \otimes \cdots \otimes S^{0,i_k} \to S^{0,i_1+\ldots+i_k}$$

are in fact equivalences by Lemma 6.1 because their Betti realizations are the equivalences $S^0 \otimes \cdots \otimes S^0 \simeq S^0$.

Finally, we must identify the maps $S^{0,n} \to S^{0,n+1}$. The Betti realizations of these maps are the identity on S^0, so they must be homotopic to τ by Lemma 6.1. ∎

Definition 6.3. Define $-\otimes S^{0,\star} : \widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}} \to \widehat{\mathbf{Sp}}_{\mathbb{C}}$ by the formula

$$Y_\star \otimes S^{0,\star} = \operatorname*{hocolim}_{i\geq j} Y_i \otimes S^{0,j}.$$

Definition 6.3 is a type of Day convolution, using that motivic spectra are tensored over classical spectra, and using the fixed motivic object $S^{0,\star}$.

Lemma 6.4. *The functor $-\otimes S^{0,\star} : \widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}} \to \widehat{\mathbf{Sp}}_{\mathbb{C}}$ is symmetric-monoidal.*

Proof. This follows by direct computation with the definitions. Let $X_\star$ and $Y_\star$ be two objects of $\widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}}$. Then $(X_\star \otimes Y_\star) \otimes S^{0,\star}$ equals

$$\operatorname*{hocolim}_{i+j\geq p}(X_i \wedge Y_j) \otimes S^{0,p}. \tag{4}$$

On the other hand, $(X_\star \otimes S^{0,\star}) \wedge (Y_\star \otimes S^{0,\star})$ equals

$$\Big(\operatorname*{hocolim}_{i\geq p} X_i \otimes S^{0,p}\Big) \wedge \Big(\operatorname*{hocolim}_{j\geq q} Y_j \otimes S^{0,q}\Big),$$

which equals

$$\operatorname*{hocolim}_{i\geq p, j\geq q}(X_i \wedge Y_j) \otimes (S^{0,p} \wedge S^{0,q}).$$

We may identify $S^{0,p} \wedge S^{0,q}$ with $S^{0,p+q}$ by Lemma 6.2. We then obtain the same expression as in (4). ∎

Definition 6.5. Define $\Omega^{0,\star}_s : \widehat{\mathbf{Sp}}_{\mathbb{C}} \to \widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}}$ to be the right adjoint of $-\otimes S^{0,\star} : \widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}} \to \widehat{\mathbf{Sp}}_{\mathbb{C}}$.

More explicitly, $\Omega_s^{0,\star} X$ is the filtered spectrum

$$\cdots \to F_s(S^{0,n+1}, X) \to F_s(S^{0,n}, X) \to F_s(S^{0,n-1}, X) \to \cdots,$$

where the structure maps are determined by the maps of $S^{0,\star}$ from Lemma 6.2. The notation reflects the fact that the definition of $\Omega_s^{0,\star}$ is similar to the usual desuspension functor, except that we use the spectrum-valued function object instead of the internal function object. Another possible description of $\Omega_s^{0,\star}$ is that it corresponds under the equivalence between functors $\widehat{\mathbf{Sp}}_{\mathbb{C}} \to \widehat{\mathbf{Sp}}^{\mathbb{Z}^{\mathrm{op}}}$ and functors $\widehat{\mathbf{Sp}}_{\mathbb{C}} \times \mathbb{Z}^{\mathrm{op}} \to \widehat{\mathbf{Sp}}$ to

$$\widehat{\mathbf{Sp}}_{\mathbb{C}} \times \mathbb{Z}^{\mathrm{op}} \to \widehat{\mathbf{Sp}} : (X, n) \mapsto F_s(S^{0,n}, X).$$

Remark 6.6. Recall from Section 2 that the bigraded homotopy group $\pi_{i,j}(Y_\star)$ of a filtered spectrum $Y_\star$ is equal to $\pi_i Y_j$. For any motivic spectrum X, the group $\pi_{i,j}(\Omega_s^{0,\star} X)$ is equal to $\pi_i F_s(S^{0,j}, X)$, which equals the motivic stable homotopy group $\pi_{i,j} X$. This observation is precisely the point of the construction of $\Omega_s^{0,\star}$; it records the motivic stable homotopy groups of X in a filtered spectrum.

The functor $\Omega_s^{0,\star}$ is automatically lax symmetric-monoidal with respect to Day convolution, since its left adjoint is symmetric-monoidal. One consequence is that $\Omega_s^{0,\star} X$ is an $\Omega_s^{0,\star}(S^{0,0})$-module for all X. Our next goal is to identify $\Omega_s^{0,\star}(S^{0,0})$.

Betti realization induces a map

$$F_s(S^{0,w}, \widehat{MGL}^{n+1}) \to F(S^0, \widehat{MU}^{n+1}) = \widehat{MU}^{n+1},$$

where the object on the right is the usual function object for classical spectra. Lemma 6.7 determines this map more explicitly.

Lemma 6.7. *For all w and n, the map $F_s(S^{0,w}, \widehat{MGL}^{n+1}) \to \widehat{MU}^{n+1}$ is the canonical map*

$$\tau_{\geq 2w}(\widehat{MU}^{n+1}) \to \widehat{MU}^{n+1}.$$

Proof. Recall that $\pi_* \widehat{MU}$ is isomorphic to the 2-completed Lazard ring $\mathbb{Z}_2[x_1, x_2, \ldots]$, where x_i has degree $2i$ [20]. Also, $\pi_{*,*}\widehat{MGL}$ is isomorphic to $\mathbb{Z}_2[\tau][x_1, x_2, \ldots]$, where x_i has degree $(2i, i)$ [12, Theorem 7]. Moreover, Betti realization takes x_i to x_i.

The classical homotopy groups of $F_s(S^{0,w}, \widehat{MGL})$ are equal to the motivic homotopy groups $\pi_{*,w}(\widehat{MGL})$. From the description in the previous paragraph, we see that these groups are the same as the homotopy groups of $\tau_{\geq 2w}(\widehat{MU})$, and that Betti realization

$$F_s(S^{0,w}, \widehat{MGL}) \to \widehat{MU}$$

induces an isomorphism on homotopy groups in degrees $2w$ and above. This establishes the result for the case $n = 0$.

The proof for $n > 0$ is similar, using the fact that MU^{n+1} splits as a wedge of even suspensions of MU [1, p. 87], and $\widehat{MGL}^{n+1}$ splits correspondingly as a wedge of $\Sigma^{2k,k}$-suspensions of $\widehat{MGL}$ [21]. ∎

Proposition 6.8. *The filtered spectrum $\Omega_s^{0,\star} S^{0,0}$ is equivalent to $\widehat{\Gamma}_\star S^0$.*

Proof. Using the fact that the motivic Adams–Novikov spectral sequence converges strongly, [12, Theorem 8], we find that $S^{0,0}$ is equivalent to $\mathrm{Tot}(\widehat{MGL}^{\bullet+1})$, where $\widehat{MGL}^{\bullet+1}$ is the standard cosimplicial $\widehat{MGL}$-resolution of $S^{0,0}$. Therefore, for each w, the spectrum $\Omega_s^{0,w} S^{0,0}$ is equivalent to $\Omega_s^{0,w}\,\mathrm{Tot}(\widehat{MGL}^{\bullet+1})$. Using the fact that homotopy limits commute with function spectra, this is equivalent to $\mathrm{Tot}(F_s(S^{0,w}, \widehat{MGL}^{\bullet+1}))$. Lemma 6.7 identifies this spectrum with $\mathrm{Tot}(\tau_{\geq 2w}\widehat{MU}^{\bullet+1})$, which is precisely the 2-completion of $\Gamma_w S^0$. ■

Since $\Omega_s^{0,\star}$ is lax symmetric-monoidal, Proposition 6.8 shows that the functor $\Omega_s^{0,\star}$ takes values in 2-complete $\Gamma_\star S^0$-modules, rather than just filtered spectra. From this perspective, we can define its left adjoint as a functor from $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$ to $\widehat{\mathbf{Sp}}_{\mathbb{C}}$.

Definition 6.9. Let

$$\widehat{\mathrm{Mod}}_{\Gamma_\star S^0} \to \widehat{\mathbf{Sp}}_{\mathbb{C}} : Y_\star \mapsto Y_\star \underset{\Gamma_\star S^0}{\otimes} S^{0,\star}$$

be the left adjoint to the functor $\Omega_s^{0,\star}\colon \widehat{\mathbf{Sp}}_{\mathbb{C}} \to \widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$.

We can describe $Y_\star \otimes_{\Gamma_\star S^0} S^{0,\star}$ in more concrete terms, although this description is not strictly necessary. It is the geometric realization of the simplicial bar construction

$$\cdots \to Y_\star \otimes \widehat{\Gamma}_\star S^0 \otimes S^{0,\star} \rightrightarrows Y_\star \otimes S^{0,*}.$$

In each degree, all but the last tensor symbol indicate Day convolution of two filtered spectra, while the last tensor symbol represents the functor of Definition 6.3.

Recall the suspension functor $\Sigma^{i,j}$ for filtered spectra described in Section 2.

Lemma 6.10. *The motivic spectrum $(\Sigma^{i,j}\widehat{\Gamma}_\star S^0) \otimes_{\Gamma_\star S^0} S^{0,\star}$ is equivalent to $S^{i,j}$.*

Proof. By adjointness, maps $(\Sigma^{i,j}\Gamma_\star S^0) \otimes_{\Gamma_\star S^0} S^{0,\star} \to Y$ correspond to maps $S^{i,j} \to \Omega_s^{0,\star} Y$ of filtered spectra. From the definition of $S^{i,j}$ in Definition 2.2, such maps correspond to maps $S^i \to F_s(S^{0,j}, Y)$ in spectra, which in turn correspond to maps $S^{i,j} \to Y$ in motivic spectra. ■

Proposition 6.11. *For any 2-complete $\Gamma_\star S^0$-module $Y_\star$, the functor $- \otimes_{\Gamma_\star S^0} S^{0,\star}$ induces an isomorphism*

$$\pi_{i,j}(Y_\star) = [\Sigma^{i,j}\widehat{\Gamma}_\star S^0, Y_\star] \to \left[S^{i,j}, Y_\star \underset{\Gamma_\star S^0}{\otimes} S^{0,\star}\right] = \pi_{i,j}\left(Y_\star \underset{\Gamma_\star S^0}{\otimes} S^{0,\star}\right).$$

Proof. First suppose that $Y_\star$ is of the form $\Sigma^{p,q}\widehat{\Gamma}_\star S^0$. Then $Y_\star \otimes_{\Gamma_\star S^0} S^{0,\star}$ is equal to $S^{p,q}$ by Lemma 6.10, so $\pi_{i,j}(Y_\star \otimes_{\Gamma_\star S^0} S^{0,\star})$ equals the motivic stable homotopy group $\pi_{i-p,j-q}$. On the other hand, $\pi_{i,j}(Y_\star)$ is equal to $\pi_{i-p}(\widehat{\Gamma}_{j-q} S^0)$, which equals $\pi_{i-p} F_s(S^{0,j-q}, S^{0,0})$ by Proposition 6.8. Finally, this last group also equals the motivic stable homotopy group $\pi_{i-p,j-q}$. Thus, the proposition holds when $Y_\star$ is of the form $\Sigma^{p,q}\widehat{\Gamma}_\star S^0$.

Let $\mathcal{C}$ be the class of all 2-complete $\Gamma_\star S^0$-modules $Y_\star$ for which the proposition is true. We have just shown that $\mathcal{C}$ contains $\Sigma^{p,q}\widehat{\Gamma}_\star S^0$. Moreover, $\mathcal{C}$ is closed under arbitrary coproducts and filtered colimits by compactness of the completed spheres in $\widehat{\mathbf{Sp}}$ and $\widehat{\mathbf{Sp}}_{\mathbb{C}}$.

Finally, if $X_\star \to Y_\star \to Z_\star$ is a cofiber sequence of $\Gamma_\star S^0$-modules and any two of $X_\star$, $Y_\star$, and $Z_\star$ belong to $\mathcal{C}$, then the third belongs to $\mathcal{C}$ as well by the five lemma applied to long exact sequences of homotopy groups.

Finally, Proposition 3.12 implies that $\mathcal{C}$ equals the entire ∞-category $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$. ■

Theorem 6.12. *The functors* $-\otimes_{\Gamma_\star S^0} S^{0,\star}$ *and* $\Omega_s^{0,\star}$ *are inverse equivalences between* $\widehat{\mathbf{Sp}}_{\mathbb{C}}$ *and* $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$.

Proof. Since equivalences in $\widehat{\mathbf{Sp}}_{\mathbb{C}}$ and in $\widehat{\mathrm{Mod}}_{\Gamma_\star S^0}$ are both detected by bigraded homotopy groups, it suffices to show that the counit map $(\Omega_s^{0,\star} X) \otimes_{\Gamma_\star S^0} S^{0,\star} \to X$ induces an isomorphism of motivic stable homotopy groups, and that the unit map $Y_\star \to \Omega^{0,\star}(Y_\star \otimes_{\Gamma_\star S^0} S^{0,\star})$ induces an isomorphism of bigraded homotopy groups of filtered spectra. These claims follow from Remark 6.6 and Proposition 6.11. ■

Remark 6.13. In order to prove Theorem 6.12, we used formal categorical tools, together with the following non-formal properties of the cellular 2-complete $\mathbb{C}$-motivic stable ∞-category $\widehat{\mathbf{Sp}}_{\mathbb{C}}$:

(1) $\mathrm{Map}(S^{0,p}, S^{0,q}) \simeq \mathrm{Map}(S^0, S^0)$ if $p \le q$ (see Lemma 6.1).

(2) The bigraded $\mathbb{C}$-motivic homotopy groups detect equivalences; equivalently, $\widehat{\mathbf{Sp}}_{\mathbb{C}}$ is generated by the bigraded spheres.

(3) $F_s(S^{0,w}, \widehat{MGL}^{n+1}) \simeq \tau_{\ge 2w}(\widehat{MU}^{n+1})$ (see Lemma 6.7).

The same formal categorical tools can be applied to establish an equivalence between $\mathrm{Mod}_{\Gamma_\star S^0}$ and the ∞-category of even MU-based synthetic spectra [22], even without completing. The required non-formal properties of synthetic spectra are immediate consequences of [22, Theorems 1.5, 6.2, and Proposition 4.60].

Acknowledgements. We thank Saul Glasman for some helpful hints about E_∞-rings in stable ∞-categories. We thank Piotr Pstrągowski for suggesting the structure of some of the computational arguments in Section 4. We thank Ben Antieau for Remark 3.7.

Funding. The second author was supported by NSF grant DMS-1606290.

References

[1] Adams, J. F.: Stable Homotopy and Generalised Homology. Chicago Lectures in Math., Univ. of Chicago Press, Chicago, IL (1974) Zbl 0309.55016 MR 0402720

[2] Antieau, B.: Periodic cyclic homology and derived de Rham cohomology. Ann. K-Theory **4**, 505–519 (2019) Zbl 1451.14061 MR 4043467

[3] Brown, E. H., Jr., Peterson, F. P.: A spectrum whose Z_p cohomology is the algebra of reduced p^{th} powers. Topology **5**, 149–154 (1966) Zbl 0168.44001 MR 192494

[4] Douglas, C. L., Francis, J., Henriques, A. G., Hill, M. A. (eds.): Topological Modular Forms. Math. Surveys Monogr. 201, Amer. Math. Soc., Providence, RI (2014) Zbl 1304.55002 MR 3223024

[5] Gheorghe, B.: Exotic motivic periodicities. arXiv:1709.00915 (2017)

[6] Gheorghe, B.: The motivic cofiber of τ. Doc. Math. **23**, 1077–1127 (2018) Zbl 1407.55007 MR 3874951

[7] Gheorghe, B., Isaksen, D. C.: The structure of motivic homotopy groups. Bol. Soc. Mat. Mex. (3) **23**, 389–397 (2017) Zbl 1375.14084 MR 3633139

[8] Gheorghe, B., Wang, G., Xu, Z.: The special fiber of the motivic deformation of the stable homotopy category is algebraic. Acta Math. **226**, 319–407 (2021) Zbl 07378147 MR 4281382

[9] Glasman, S.: Day convolution for ∞-categories. Math. Res. Lett. **23**, 1369–1385 (2016) Zbl 1375.18041 MR 3601070

[10] Heine, H.: A characterization of cellular motivic spectra. arXiv:1712.00521 (2017)

[11] Hovey, M., Palmieri, J. H., Strickland, N. P.: Axiomatic stable homotopy theory. Mem. Amer. Math. Soc. **128**, no. 610, x+114 pp. (1997) Zbl 0881.55001 MR 1388895

[12] Hu, P., Kriz, I., Ormsby, K.: Remarks on motivic homotopy theory over algebraically closed fields. J. K-Theory **7**, 55–89 (2011) Zbl 1248.14026 MR 2774158

[13] Isaksen, D. C.: The cohomology of motivic $A(2)$. Homology Homotopy Appl. **11**, 251–274 (2009) Zbl 1193.55009 MR 2591921

[14] Isaksen, D. C.: Stable stems. Mem. Amer. Math. Soc. **262**, no. 1269, viii+159 pp. (2019) Zbl 1454.55001 MR 4046815

[15] Isaksen, D. C., Wang, G., Xu, Z.: More stable stems. arXiv:2001.04511 (2020)

[16] Isaksen, D. C., Shkembi, A.: Motivic connective K-theories and the cohomology of A(1). J. K-Theory **7**, 619–661 (2011) Zbl 1266.14015 MR 2811718

[17] Krause, A.: Periodicity in motivic homotopy theory and over BP_*BP. Ph.D. thesis, Universität Bonn (2018)

[18] Lurie, J.: Higher algebra. https://www.math.ias.edu/~lurie/papers/HA.pdf (2017)

[19] Mathew, A.: The homology of tmf. Homology Homotopy Appl. **18**, 1–29 (2016) Zbl 1357.55002 MR 3515195

[20] Milnor, J.: On the cobordism ring Ω^* and a complex analogue. I. Amer. J. Math. **82**, 505–521 (1960) Zbl 0095.16702 MR 119209

[21] Naumann, N., Spitzweck, M., Østvær, P. A.: Existence and uniqueness of E_∞ structures on motivic K-theory spectra. J. Homotopy Related Struct. **10**, 333–346 (2015) Zbl 1326.14054 MR 3385689

[22] Pstrągowski, P.: Synthetic spectra and the cellular motivic category. arXiv:1803.01804 (2018)

[23] Ravenel, D. C.: Complex Cobordism and Stable Homotopy Groups of Spheres. Pure Appl. Math. 121, Academic Press, Orlando, FL (1986) Zbl 0608.55001 MR 860042

[24] Stahn, S.-T.: The motivic Adams–Novikov spectral sequence at odd primes over $\mathbb{C}$ and $\mathbb{R}$. arXiv:1606.06085 (2016)

[25] Voevodsky, V.: Reduced power operations in motivic cohomology. Publ. Math. Inst. Hautes Études Sci. **98**, 1–57 (2003) Zbl 1057.14027 MR 2031198

[26] Voevodsky, V.: Motivic cohomology with $\mathbf{Z}/2$-coefficients. Publ. Math. Inst. Hautes Études Sci. **98**, 59–104 (2003) Zbl 1057.14028 MR 2031199

[27] Voevodsky, V.: Motivic Eilenberg–MacLane spaces. Publ. Math. Inst. Hautes Études Sci. **122**, 1–99 (2010) Zbl 1227.14025 MR 2737977

[28] Wilson, W. S.: The Ω-spectrum for Brown–Peterson cohomology. II. Amer. J. Math. **97**, 101–123 (1975) Zbl 0303.55003 MR 383390