GERHARDT - GAP THEOREM

Note Title 10/26/2010

Goal: Prove the Gap Theorem

<u>Thm</u> $\pi_i \Omega = 0$ for $-4 < i < 0$

$G = C_{2^n}$

<u>Def.</u> An <u>isotropic</u> slice cell is one of the form $\Sigma^{\varepsilon}\hat{S}(m, K) = G_+ \wedge_K S^{m\rho_K}$ for $\varepsilon = 0, -1$ with $K < G$ nontrivial.

It is <u>regular</u> if $\varepsilon = 0$.

To a G-spectrum X we associate n-slices $P_n^n X$

Def An n-slice is perfect + isotropic if $M = \widehat{W} \wedge H\underline{\mathbb{Z}}$ where $\widehat{W}$ is a wedge of regular isotropic slice cells.

Def A G-spectrum is perfect + isotropic if all of its slices are.

Thm 1 If a G-spectrum X is perfect and isotropic then $\pi_i^G X = 0$ for $-4 < i < 0$.

Recall $\Omega = \tilde{\Omega}^{C_8}$ where $\quad P = P_{C_8}$

$\tilde{\Omega}$ is hty colim $\underrightarrow{\text{holim}}$ $S^{-j\ell\rho} \wedge MU^{((C_8))}$

$$\pi_1^{G} \widetilde{\Omega} = \varinjlim_{j} \pi_1^{G} S^{-j\rho} \wedge MU^{((8))}$$

We will assume

<u>Slice Thm</u> $MU^{((8))}$ is perfect + isotropic

Cor $S^{n\rho} \wedge MU^{((8))}$ is " "

<u>Pf</u> $S^{n\rho_G} \wedge \hat{S}(m, K) = S^{n\rho} \wedge (G_+ \underset{K}{\wedge} S^{m\rho_K})$

$$= G_+ \underset{K}{\wedge} S^{(m + n|G/K|)\rho_K}$$

$$= \hat{S}(m + n|G/K|, K).$$

This uses Frobenius reciprocity -

$$X \wedge (G_+ \wedge_K Y) = G_+ \wedge_K (i^* X \wedge Y)$$

QED.

Thm I $\Rightarrow \pi_i^G \, S^{-1} P \wedge MU^{((G))} = 0$ for $-4 < i < 0$.

$\Rightarrow \pi_i^G \, \tilde{\Omega} = 0$ "

$\Rightarrow \pi_i \, \tilde{\Omega}^G = \pi_i \, \Omega = 0$ "

Thus the Gap Thm follows from Thm I. It will follow from

<u>Cell Lemma</u> If $\hat{S}$ is an even dim'l isotropic slice cell then

$$\pi_i^G \, H\underline{\mathbb{Z}} \wedge \hat{S} = 0 \quad \text{for} \quad -4 < i < 0.$$

By the slice SS this implies Thm I.

$$\hat{S} = G_+ \underset{K}{\wedge} S^{m\rho_K}$$

$$\pi_i^G \left(H\underline{\mathbb{Z}} \wedge \hat{S} \right) = \left[S^i, H\underline{\mathbb{Z}} \wedge \left(G_+ \underset{K}{\wedge} S^{m\rho_K} \right) \right]_G$$

$$= \left[S^i, G_+ \underset{K}{\wedge} \left(H\underline{\mathbb{Z}} \wedge S^{m\rho_K} \right) \right]_G$$

$$= \left[S^i, F_K \left(G_+, H\underline{\mathbb{Z}} \wedge S^{m\rho_K} \right) \right]_K$$

$= \pi_1^K H\underline{\mathbb{Z}} \wedge S^{m\rho_K}$

We only need to consider $\hat{S} = S^{h\rho_G}$ for some h and $G \neq 1$

Want $\pi_1^G H\underline{\mathbb{Z}} \wedge S^{h\rho_G} = H_i^G(S^{h\rho_G}; \underline{\mathbb{Z}})$

If $h \geq 0$ then the above vanishes for all $i < 0$.

Assume $h < 0$. Let $m = -h > 0$.

$$H_i^G(S^{h\rho_G}; \underline{\mathbb{Z}}) = H^{-i}_G(S^{m\rho_G}; \underline{\mathbb{Z}})$$

Thus the theorem will follow from

<u>Prop</u> If $m > 0$ then

$$H^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}) = 0 \text{ for } 0 < i < 4.$$

$$H^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}) = H_i\left(\mathrm{Hom}(C(m\rho_G), \underline{\mathbb{Z}})^G\right)$$

where $C(m\rho_G)$ is equiv cellular chain cx for $S^{m\rho_G}$.

Example $G = C_2$, $m = 1$. Compute $H^i_{C_2}(S^{\rho_{C_2}}, \underline{\mathbb{Z}})$

$\rho_{C_2} = 1 + \sigma$

$S^{\rho_{C_2}} = S^2$ with reflection thru equator.

$= S^1 \cup (C_{2+} \wedge e^2)$

Chain cx $\quad \underline{\mathbb{Z}} \longleftarrow \mathbb{Z}[C_2]$

The map is forced by the underlying homology of S^2

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}^2 \qquad \nabla = \text{fold}$$

Dualizing gives

$$\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^2 \qquad \Delta = \text{diagonal}$$

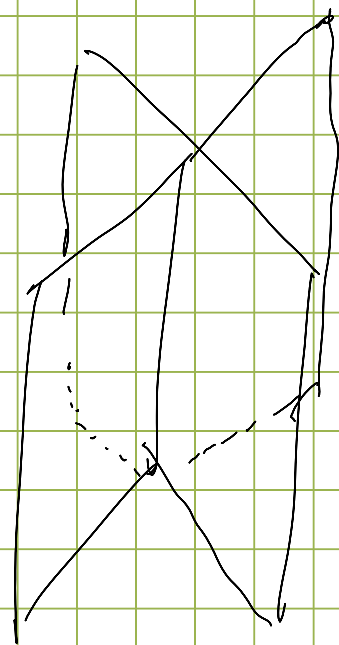
C_2-fixed pts

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z}$$

so $H^i_{C_2}(S^{\rho_{C_2}}; \underline{\mathbb{Z}}) = 0$ for all i.

<u>Next example</u> $G = C_4$, $m = 1$ $\qquad \lambda = \pi/4$ rotation

$$H^i_{C_4}(S^{\rho_G}; \underline{\mathbb{Z}}) \qquad \rho_G = 1 + \sigma + \lambda$$



$\sigma + \lambda$

$$S^{\sigma+\lambda} = S^0 \cup (C_{2+} \wedge e^1) \cup (C_{4+} \wedge e^2) \cup (C_{4+} \wedge e^3)$$

$$\mathbb{Z} \leftarrow \mathbb{Z}[C_4/C_2] \leftarrow \mathbb{Z}[C_4] \leftarrow \mathbb{Z}[C_4]$$

0 1 2 3

To get $\rho_{C_4} = 1+\sigma+\lambda$ we ~~suspend~~

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}[C_4/C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_4] \xleftarrow{1+\gamma} \mathbb{Z}[C_4]$$

Dualize + take fixed pts

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

$H^i_{G}(S^\rho; \mathbb{Z}) = 0$ for $0 < i < 4$

Example $G = C_4$, $m = 2$

$$2\rho = \underbrace{2}_{\text{fixed by } C_4} + \underbrace{2\sigma}_{\text{by } C_2} + \underbrace{2\lambda}_{\text{by } \{1\}}$$

Build $S^{2\rho}$ from S^2

$$S^2 \subset S^{2+\sigma} \subset S^{2+2\sigma} \subset S^{2+2\sigma+\lambda} \subset S^{2+2\sigma+2\lambda}$$

$$S^\sigma = S^0 \cup (C_4/C_{2+} \wedge e^1)$$

$$S^\lambda = S^0 \cup (C_{4+} \wedge e^1) \cup (C_{4+} \wedge e^2)$$

It follows that

$$S^{2+\sigma} = S^2 \wedge \left(S^0 \cup (C_4/C_{2+} \wedge e^1)\right) = S^2 \cup (C_4/C_{2+} \wedge e^3)$$

$$S^{2+26} = S^{2+6} \wedge \left(e^0 \vee C_{4+} \underset{C_{2+}}{\wedge} e^1\right)$$

$$= S^{2+6} \cup \left(C_{4+} \underset{C_2}{\wedge} e^4\right)$$

$$S^{2+26+\lambda} = S^{2+26} \wedge \left(S^0 \vee C_{4+} \wedge e^1 \vee C_{4+} \wedge e^2\right)$$

$$= S^{2+26} \cup C_{4+} \wedge e^5 \cup C_{4+} \wedge e^6$$

$$S^{2+26+2\lambda} = S^{2+26+\lambda} \cup C_{4+} \wedge e^7 \cup C_{4+} \wedge e^8$$

This is easy to generalize.

$$S^{\rho} = S^2 \cup \left(C_{4+} \underset{C_2}{\wedge} e^3\right) \cup \left(C_{4+} \underset{C_2}{\wedge} e^4\right) \cup \left(C_{4+} \wedge e^5\right) \cdots \left(C_{4+} \wedge e^8\right)$$

Chain cx

$$\begin{array}{cccccccccccccc} 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\ \mathbb{Z} & \xleftarrow{\nabla} & \mathbb{Z}[C_4/C_2] & \xleftarrow{1+\gamma} & \mathbb{Z}_2[C_4/C_2] & \xleftarrow{1+\gamma} & \mathbb{Z}[C_4] & \leftarrow & \mathbb{Z}[C_4] & \leftarrow & \mathbb{Z}[C_4] & \leftarrow & \mathbb{Z}[C_4] \end{array}$$

dualize + fix

$$\begin{array}{cccccccccccccc} 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\ \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

We find $H^i_{C_4}(S^{2\rho_4};\mathbb{Z}) = 0$ for $0 < i < 4$

Observe

1. $C_k(m\rho_{C_n}) = 0$ for $k < m$
2. Any time we have $\sigma \subset V$ then cx has to start with $\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}[C_{2^n}/C_{2^{n-1}}] \leftarrow \cdots$

Dualize + fix and get

$$\overset{m}{\mathbb{Z}} \xrightarrow{1} \overset{m+1}{\mathbb{Z}} \xrightarrow{0} \overset{m+2}{\mathbb{Z}} \cdots$$

③ More generally

$$C_k(m\rho_{2^n}) = \begin{cases} 0 & k < m \\ 0 & k > m\cdot 2^n \\ \mathbb{Z}[C_{2^n}/C_{2^{n-i}}] & \text{for } m\cdot 2^{n-i-1} < k \le m\cdot 2^{n-i} \end{cases}$$

Prop $H^i_{C_n}(S^{m\rho_{C_n}};\mathbb{Z}) = 0 \quad 0 \le i < 4 \quad$ for $m > 0$
and $i \neq 1$

Pf: For $m = 4$ it follows from ① above

by ②

$m=3$ we have

$$\overset{3}{\mathbb{Z}} \xrightarrow{1} \overset{4}{\mathbb{Z}} \longrightarrow \cdots$$

$m=2$

$$\overset{2}{\mathbb{Z}} \xrightarrow{1} \overset{3}{\mathbb{Z}} \longrightarrow \overset{4}{\mathbb{Z}} \cdots$$

$m=1$ $\quad G=C_2$ $\quad$ first example

$G=C_{2^n}$, $n>1$

cx starts with $\overset{1}{\mathbb{Z}} \longrightarrow \overset{2}{\mathbb{Z}} \xrightarrow{0} \overset{3}{\mathbb{Z}} \xrightarrow{2} \overset{4}{\mathbb{Z}} \longrightarrow \cdots$

$H^i(S^{\rho_{C_{2^n}}};\mathbb{Z})=0$ for $0<i<4$