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# Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness ☆

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ABSTRACT

This is the second in a trilogy of papers introducing and studying the notion of decomposition space as a general framework for incidence algebras and Möbius inversion, with coefficients in $\infty$-groupoids. A decomposition space is a simplicial $\infty$-groupoid satisfying an exactness condition weaker than the Segal condition. Just as the Segal condition expresses composition, the new condition expresses decomposition.

In this paper, we introduce various technical conditions on decomposition spaces. The first is a completeness condition (weaker than Rezk completeness), needed to control simplicial nondegeneracy. For complete decomposition spaces we establish a general Möbius inversion principle, expressed as an explicit equivalence of $\infty$-groupoids. Next we analyse two finiteness conditions on decomposition spaces. The first, that of locally finite length, guarantees the existence of the important length filtration for the associated incidence coalgebra. We show that a decomposition space of locally finite length is actually the left Kan extension of a semi-simplicial space. The second finiteness condition, local finiteness, en-

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sures we can take homotopy cardinality to pass from the level of $\infty$-groupoids to the level of $\mathbb{Q}$-vector spaces.
These three conditions — completeness, locally finite length, and local finiteness — together define our notion of *Möbius decomposition space*, which extends Leroux's notion of Möbius category (in turn a common generalisation of the locally finite posets of Rota et al. and of the finite decomposition monoids of Cartier–Foata), but which also covers many coalgebra constructions which do not arise from Möbius categories, such as the Faà di Bruno and Connes–Kreimer bialgebras.
Note: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [6] who call them unital 2-Segal spaces.



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## 0. Introduction

In the first paper of this trilogy [11], we introduced the notion of decomposition space as a general framework for incidence (co)algebras. It is equivalent to the notion of unital 2-Segal space of Dyckerhoff and Kapranov [6]. The relevant main results are recalled in Section 1 below. A decomposition space is a simplicial $\infty$-groupoid $X$ satisfying a certain exactness condition, weaker than the Segal condition. Just as the Segal condition expresses composition, the new condition expresses decomposition, and implies the existence of an incidence (co)algebra. There is a rich supply of examples in combinatorics [14]. An easy example is the decomposition space of graphs (yielding the chromatic Hopf algebra [31]), which will serve as a running example. In the present paper we proceed to establish a Möbius inversion principle for what we call *complete* decomposition spaces, and analyse the associated finiteness issues.

Classically [30], the Möbius inversion principle states that the zeta function of any incidence algebra (of a locally finite poset, say, or more generally a Möbius category in the sense of Leroux [25]) is invertible for the convolution product; its inverse is by definition the Möbius function. The Möbius inversion formula is a powerful and versatile counting device. Since it is an equality stated at the vector-space level in the incidence algebra, it belongs to algebraic combinatorics rather than bijective combinatorics. It is possible to give Möbius inversion a bijective meaning, by following the objective method, pioneered in this context by Lawvere and Menni [23], which seeks to lift algebraic identities to the 'objective level' of (finite) sets and bijections, working with certain categories spanned by the combinatorial objects instead of with vector spaces spanned by isoclasses of these objects. The algebraic identity then appears as the cardinality of the bijection established at the objective level.

To illustrate the objective viewpoint, observe that a vector in the free vector space on a set $B$ is just a collection of scalars indexed by (a finite subset of) $B$. The objective counterpart is a family of sets indexed by $B$, i.e. an object in the slice category $\boldsymbol{Set}_{/B}$. 'Linear maps' at this level are given by spans $A \leftarrow M \rightarrow B$. The Möbius inversion principle states an equality between certain linear maps (elements in the incidence algebra). At the objective level, such an equality can be expressed as a bijection between sets in the spans representing those linear functors. In this way, the algebraic identity is revealed to be just the cardinality of a bijection of sets, which carry much more structural information. As an example, the objective counterpart of the binomial algebra is the category of species with the Cauchy tensor product [14], a much richer structure, and at the objective level there are obstructions to cancellations in the Möbius function that take place at the numerical level only. The significance of these phenomena is not yet clear, and is under investigation [14]. Lawvere and Menni [23] established an objective version of the Möbius inversion principle for Möbius categories in the sense of Leroux [25].

Our discovery in [11] is that something considerably weaker than a category suffices to construct an incidence algebra, namely a decomposition space. This discovery is interesting even at the level of simplicial sets, but we work at the level of simplicial $\infty$-groupoids. Thus, the role of vector spaces is played by slices of the $\infty$-category of $\infty$-groupoids. In [10] we have developed the necessary 'homotopy linear algebra' and homotopy cardinality, extending and streamlining many results of Baez–Hoffnung–Walker [2] who worked with 1-groupoids.

The decomposition-space axiom on a simplicial $\infty$-groupoid $X$ is expressly the condition needed for a canonical coalgebra structure to be induced on the slice $\infty$-category $\mathcal{S}_{/X_1}$, (where $\mathcal{S}$ denotes the $\infty$-category of $\infty$-groupoids, also called spaces). The comultiplication is the linear functor

$$\Delta : \mathcal{S}_{/X_1} \rightarrow \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1}$$

given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

This can be read as saying that comultiplying an edge $f \in X_1$ returns the sum of all pairs of edges $(a, b)$ that are the short edges of a 2-simplex with long edge $f$. In the case that $X$ is the nerve of a category, this is the sum of all pairs $(a, b)$ of arrows with composite $b \circ a = f$.

The aims of this paper are to establish a Möbius inversion principle in the framework of *complete* decomposition spaces, and also to introduce the necessary *finiteness* conditions on a complete decomposition space to ensure that incidence (co)algebras and Möbius inversion descend to classical vector-space-level coalgebras on taking the homotopy cardinality of the objects involved. Along the way we also establish some auxiliary results of a more technical nature which are needed in the applications in the sequel papers [12–14].

We proceed to summarise the main results.

After briefly reviewing in Section 1 the notion of decomposition space and the notion of CULF maps between them — simplicial maps that induce coalgebra homomorphisms — we come to the notion of completeness in Section 2:

*Definition.* We say that a decomposition space $X$ is *complete* (2.1) when $s_0 : X_0 \to X_1$ is a monomorphism. It then follows that all degeneracy maps are monomorphisms (Lemma 2.5).

The motivating feature of this notion is that all issues concerning degeneracy can then be settled in terms of the canonical projection maps $X_r \to (X_1)^r$ sending a simplex to its principal edges: a simplex in a complete decomposition space is nondegenerate precisely when all its principal edges are nondegenerate (Corollary 2.16). Let $\vec{X}_r \subset X_r$ denote the subspace of these nondegenerate simplices.

For any decomposition space $X$, the comultiplication on $\mathcal{S}_{/X_1}$ yields a convolution product on the linear dual $\mathcal{S}^{X_1}$ (that is, the category of linear functors from $\mathcal{S}_{/X_1}$ to $\mathcal{S}$) called the *incidence algebra* of $X$. This contains, in particular, the *zeta functor* $\zeta$, given by the span $X_1 \xleftarrow{=} X_1 \to 1$, and the counit $\varepsilon$ (neutral for convolution) given by $X_1 \leftarrow X_0 \to 1$. In a complete decomposition space $X$ we can consider the spans $X_1 \leftarrow \vec{X}_r \to 1$ and the linear functors $\Phi_r$ they define in the incidence algebra of $X$. We can now establish the decomposition-space version of the Möbius inversion principle, in the spirit of [23]:

**Theorem 3.8.** *For a complete decomposition space, there are explicit equivalences*

$$\zeta * \Phi_{\text{even}} \;\simeq\; \varepsilon \;+\; \zeta * \Phi_{\text{odd}}, \qquad\qquad \Phi_{\text{even}} * \zeta \;\simeq\; \varepsilon \;+\; \Phi_{\text{odd}} * \zeta.$$

It is tempting to read this as saying that "$\Phi_{\text{even}} - \Phi_{\text{odd}}$" is the convolution inverse of $\zeta$, but the lack of additive inverses in $\mathcal{S}$ necessitates our sign-free formulation. Upon taking homotopy cardinality, as we will later, this yields the usual Möbius inversion formula $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$, valid in the incidence algebra with $\mathbb{Q}$-coefficients.

Having established the general Möbius inversion principle on the objective level, we proceed to analyse the finiteness conditions on complete decomposition spaces needed for this principle to descend to the vector-space level of $\mathbb{Q}$-algebras. There are two conditions: $X$ should be of locally finite length (Section 6), and $X$ should be locally finite (Section 7). The first is a numerical condition, like a chain condition; the second is a homotopy finiteness condition. Complete decomposition spaces satisfying both conditions are called *Möbius decomposition spaces* (Section 8). We analyse the two conditions separately.

*Definition.* (Cf. 6.1.) The *length* of an arrow $f$ is the greatest dimension of a nondegenerate simplex with long edge $f$. We say that a complete decomposition space is *of locally finite length* — we also say *tight* — when every arrow has finite length.

Although many examples coming from combinatorics do satisfy this condition, it is actually a rather strong condition, as witnessed by the following result, which is a consequence of Propositions 5.16 and 6.6:

*Every tight decomposition space is the left Kan extension of a semi-simplicial space.*

We can prove this result for more general simplicial spaces, and digress to establish this in Section 5: we say a complete simplicial space is *split* if all face maps preserve nondegenerate simplices. In Corollary 5.11 we show this is the analogue of the condition for categories that identities are indecomposable, enjoyed in particular by Möbius categories in the sense of Leroux [25]. We prove that a simplicial space is split if and only if it is the left Kan extension along $\mathbb{\Delta}_{\text{inj}} \subset \mathbb{\Delta}$ of a semi-simplicial space $\mathbb{\Delta}_{\text{inj}}^{\text{op}} \to \mathcal{S}$, and in fact we establish more precisely:

**Theorem 5.19.** *Left Kan extension along $\mathbb{\Delta}_{\text{inj}} \subset \mathbb{\Delta}$ induces an equivalence of $\infty$-categories*

$$\mathrm{Fun}(\mathbb{\Delta}_{\text{inj}}^{\text{op}}, \mathcal{S}) \simeq \textbf{\textit{Split}}^{cons},$$

*where the right-hand side is the $\infty$-category of split simplicial spaces and conservative maps.*

This has the following interesting corollary.

**Proposition 5.20.** *Left Kan extension along $\mathbb{\Delta}_{\text{inj}} \subset \mathbb{\Delta}$ induces an equivalence between the $\infty$-category of 2-Segal semi-simplicial spaces and ULF maps, and the $\infty$-category of split decomposition spaces and CULF maps.*

We show that a complete decomposition space $X$ is tight if and only if it has a filtration

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

of CULF monomorphisms, the so-called *length filtration*. This is precisely the structure needed to get a filtration of the incidence coalgebra (6.13).

In Section 7 we impose the finiteness condition needed to be able to take homotopy cardinality and obtain coalgebras and algebras at the numerical level of $\mathbb{Q}$-vector spaces (and profinite-dimensional $\mathbb{Q}$-vector spaces).

*Definition.* An $\infty$-groupoid $S$ is *locally finite* if at each base point $x$ the homotopy groups $\pi_i(S, x)$ are finite for $i \geq 1$ and are trivial for $i$ sufficiently large. It is called *finite* if furthermore it has only finitely many components. A map of $\infty$-groupoids is called a *finite map* if its fibres are finite $\infty$-groupoids.

A decomposition space $X$ is called *locally finite* (7.4) when $X_1$ is a locally finite $\infty$-groupoid and $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps.

The condition 'locally finite' extends the notion of locally finite for posets. The condition ensures that the coalgebra structure descends to coefficients in finite $\infty$-groupoids, and hence, via homotopy cardinality, to $\mathbb{Q}$-algebras. In Section 7.10 we calculate the section coefficients (structure constants for the (co)multiplication) in some easy cases.

Finally we introduce the Möbius condition:

*Definition.* A complete decomposition space is called *Möbius* (8.3) when it is locally finite and of locally finite length (i.e. is tight).

These are the conditions needed for the general Möbius inversion formula to descend to coefficients in finite $\infty$-groupoids and hence $\mathbb{Q}$-coefficients, giving the following formula for the Möbius function (convolution inverse to the zeta function):

$$|\mu| = |\Phi_{\text{even}}| - |\Phi_{\text{odd}}| .$$

We have strived throughout to distill the most natural conditions from the requirements imposed by applications of the theory, and we find it an attractive feature that all the conditions can be formulated categorically. Just as the decomposition-space axiom is an exactness condition (that certain 'active–inert' pushouts in $\mathbb{\Delta}$ are taken to pullbacks), it is noteworthy that further conditions we require — completeness, stiffness, indecomposable units, and splitness — are also exactness conditions (stipulating that certain other classes of pushouts are taken to pullbacks, cf. 2.7, 4.1, 5.5, and Corollary 5.10). This fact is both conceptually pleasing and facilitates efficient arguments.

**Related work.** The notion of decomposition space was discovered independently by Dyckerhoff and Kapranov [6], who call them unital 2-Segal spaces. While some of the basic results in [11] were also proved in [6], the present paper has no overlap with [6].

The results in this paper on Möbius inversion are in the tradition of Leroux et al. [25], [4], [26], Dür [5], and Lawvere–Menni [23]. There is a different notion of Möbius category, due to Haigh [15]. The two notions have been compared, and to some extent unified, by Leinster [24], who calls Leroux's Möbius inversion *fine* and Haigh's *coarse* (as it only depends on the underlying graph of the category). We should mention also the $K$-theoretic Möbius inversion for quasi-finite EI categories of Lück and collaborators [27], [7].

**Note.** This paper is the second in a series, originally posted on the arXiv as a single manuscript *Decomposition spaces, incidence algebras and Möbius inversion* [9] but split for publication into:

(0) Homotopy linear algebra [10]
(1) Decomposition spaces, incidence algebras and Möbius inversion I: basic theory [11]
(2) Decomposition spaces, incidence algebras and Möbius inversion II: completeness, length filtration, and finiteness [this paper]
(3) Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals [12]
(4) Decomposition spaces and restriction species [13]
(5) Decomposition spaces in combinatorics [14].

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## 1. Preliminaries on decomposition spaces

**1.1. $\infty$-Groupoids.** We work in the $\infty$-category $\mathcal{S}$ of $\infty$-groupoids, also called *spaces*, and in closely related $\infty$-categories such as its slices. By $\infty$-category we mean quasi-category in the sense of Joyal [18], [19], but follow rather the terminology of Lurie [28]. Most of our arguments are elementary, though, and for this reason we can get away with model-independent reasoning rather than working with the Joyal model structure on simplicial sets. In particular, when we refer to the $\infty$-category $\mathcal{S}_{/B}$ (whose objects are maps of $\infty$-groupoids $X \to B$), we only refer to an $\infty$-category determined up to equivalence by a certain universal property, and do not make any distinction between the specific models for this object exploited by Joyal and Lurie (normal slice and fat slice).

**1.2. Pullbacks and fibres.** Pullbacks play an essential role in many of our arguments. By *pullback* we always mean pullback in the $\infty$-category $\mathcal{S}$. This notion enjoys a universal property which in the model-independent formulation is similar to the universal property of the pullback in ordinary categories (such as ***Set***). Again, we shall only ever need homotopy invariant properties, making it irrelevant which particular model is chosen for the notion of pullback in the Joyal model structure for quasi-categories. In particular, the *fibre* $X_b$ of a map $f : X \to B$ over a base point $b$ in $B$ is also a homotopy invariant notion: it is the pullback of $f$ along the map $\ulcorner b \urcorner : 1 \to B$ that picks out the base point.

One property which we shall use repeatedly is the following elementary lemma (a proof can be found in [28, 4.4.2.1]).

**Lemma 1.3.** *In any diagram of $\infty$-groupoids*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

*if the outer rectangle and the right-hand square are pullbacks, then the left-hand square is a pullback.*

**1.4. Monomorphisms.** The homotopy invariant notion of monomorphism of $\infty$-groupoids plays an important role throughout this paper, notably through the definition of complete decomposition space (2.1). A map of $\infty$-groupoids is a *monomorphism* when its (homotopy) fibres are $(-1)$-groupoids (i.e. are either empty or contractible).

(We warn against a potential point of confusion: in the Joyal model, $\infty$-groupoid means Kan complex, but the homotopy-invariant notion of monomorphism between $\infty$-groupoids is *not* the same as levelwise injective simplicial map between Kan complexes. For example, any equivalence of $\infty$-groupoids is a monomorphism, but not every equivalence of Kan complexes is levelwise injective. Conversely the inclusion $1 \to BG$ of a point into the classifying space of a group is not a monomorphism of $\infty$-groupoids, but it is injective levelwise in the sense of Kan complexes.)

In some respects, this notion of monomorphism does behave as for sets: for example, if $f : X \to Y$ is a monomorphism, then there is a complement $Z := Y \smallsetminus X$ such that $X + Z \simeq Y$. Hence a monomorphism is essentially an equivalence from $X$ onto some connected components of $Y$. On the other hand, a crucial difference between sets and $\infty$-groupoids is that diagonal maps of $\infty$-groupoids are not in general monomorphisms. In fact $X \to X \times X$ is a monomorphism if and only if $X$ is discrete (i.e. equivalent to a set).

**1.5. Linear algebra with coefficients in $\infty$-groupoids [10].** The slice $\infty$-categories of the form $\mathcal{S}_{/I}$ form the objects of a symmetric monoidal $\infty$-category ***LIN***, described in detail in [10]: the morphisms are the linear functors, meaning that they preserve homotopy sums, or equivalently indeed all colimits. Such functors are given by spans: the span

$$I \xleftarrow{p} M \xrightarrow{q} J$$

defines the linear functor

$$q_! \circ p^* : \mathcal{S}_{/S} \longrightarrow \mathcal{S}_{/T}$$

given by pullback along $p$ followed by composition with $q$. The $\infty$-category ***LIN*** can play the role of the category of vector spaces, although to be strict about that interpretation, finiteness conditions should be imposed, as we do later in this paper (Section 7).

The symmetric monoidal structure on ***LIN*** is easy to describe on objects: we have

$$\mathcal{S}_{/I} \otimes \mathcal{S}_{/J} := \mathcal{S}_{/I\times J},$$

just as the tensor product of vector spaces with bases indexed by sets $I$ and $J$ is the vector space with basis indexed by $I \times J$. The neutral object is $\mathcal{S}_{/1} \simeq \mathcal{S}$.

**1.6. Simplicial spaces.** Throughout, our main objects of study will be simplicial spaces $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, by which we mean objects in the functor $\infty$-category $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$. A simplicial space (synonym for simplicial $\infty$-groupoid) is thus a homotopy-coherent simplicial diagram of $\infty$-groupoids. Note that this means that the simplicial identities are squares that commute up to a homotopy, such as for example

$$\begin{tikzcd} X_3 \arrow[r, "d_3"] \arrow[d, "d_1"'] & X_2 \arrow[d, "d_1"] \\ X_2 \arrow[r, "d_2"'] & X_1, \end{tikzcd} \tag{1}$$

and it makes sense to ask whether such a square is a pullback. We shall never need to spell out the homotopies, as only their structural properties are needed.

By an *n-simplex* of $X$ we mean an object in the $\infty$-groupoid $X_n$, which in turn can be described (via the Yoneda lemma for $\infty$-groupoid-valued presheaves) as the mapping space $\mathrm{Map}(\Delta[n], X)$. If $\sigma$ is an object of $X_n$ then we write $\ulcorner\sigma\urcorner : 1 \to X_n$ for the corresponding map.

A *simplicial map* $f : X \to Y$ between simplicial spaces $X$ and $Y$ is by definition an object in the mapping space $\mathrm{Map}_{\mathcal{S}}(X, Y)$. It amounts to a sequence of maps $f_i : X_i \to Y_i$ commuting with the face and degeneracy maps up to specified coherent homotopies.

We briefly review the main notions and results from the first paper in the trilogy [11], and in particular the notion of decomposition space. This notion is equivalent to that of unital 2-Segal space, introduced by Dyckerhoff and Kapranov [6]. While Dyckerhoff and Kapranov formulate the condition in terms of triangulation of convex polygons, our formulation refers to the categorical notion of active and inert maps, which we recall next.

**1.7. Active and inert maps (generic and free maps).** The category $\mathbb{\Delta}$ of nonempty finite ordinals and monotone maps has an active–inert factorisation system. An arrow $a$ :

$[m] \to [n]$ in $\mathbb{\Delta}$ is *active* (also called *generic*) when it preserves end-points, $a(0) = 0$ and $a(m) = n$; and it is *inert* (also called *free*) if it is distance preserving, $a(i+1) = a(i) + 1$ for $0 \le i \le m-1$. A coface map $d^j : [m] \to [m+1]$ is active if and only if it is *inner*, i.e. $1 \le j \le m$. The active maps are generated by the codegeneracy maps and the inner coface maps, while the inert maps are generated by the outer coface maps. Every morphism in $\mathbb{\Delta}$ factors uniquely as an active map followed by an inert map.

The notions of generic and free maps are general notions in category theory, introduced by Weber [34,35], who extracted the notions from earlier work of Joyal [17]; a recommended entry point to the theory is Berger–Melliès–Weber [3]. We have adopted the more recent terminology 'active/inert' (due to Lurie [29]), which is more suggestive of the role the two classes of maps play.

**Lemma 1.8.** *Active and inert maps in $\mathbb{\Delta}$ admit pushouts along each other, and the resulting maps are again active and inert.*

**1.9. Decomposition spaces [11].** A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is called a *decomposition space* when it takes active–inert pushouts in $\mathbb{\Delta}$ to pullbacks. An example of such a square is (1) above.

Every Segal space is a decomposition space. For example, the nerve of a category or a poset is a decomposition space. In a Segal space $X$, all the information is contained in $X_0$ and $X_1$ and the composition map $d_1 : X_2 \to X_1$. This cannot be said for decomposition spaces in general, but we still have the following important property.

**Lemma 1.10.** *In a decomposition space $X$, every active face map is a pullback of $d_1 : X_2 \to X_1$, and every degeneracy map is a pullback of $s_0 : X_0 \to X_1$.*

**Proof.** If we consider the inert maps $f_j : [1] \to [m]$ given by $f_j(0) = j$ and $f_j(1) = j+1$ for $j = 0, \ldots, m-1$, then we have the following active–inert pushouts in $\mathbb{\Delta}$,

$$\begin{tikzcd} {[1]} \arrow[r, "f_j"] \arrow[d, "d^1"'] & {[m]} \arrow[d, "d^{j+1}"] \\ {[2]} \arrow[r] & {[m+1],} \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[r, "f_j"] \arrow[d, "s^0"'] & {[m]} \arrow[d, "s^j"] \\ {[0]} \arrow[r] & {[m-1],} \end{tikzcd}$$

which are sent to pullbacks by any decomposition space $X$. $\square$

As far as incidence coalgebras are concerned, the notion of decomposition space can be seen as an abstraction of that of poset: it is precisely the condition required to obtain a counital coassociative comultiplication on $\mathcal{S}_{/X_1}$. Precisely, the following is the main theorem of [11].

**Theorem 1.11.** *[11] For $X$ a decomposition space, the slice $\infty$-category $\mathcal{S}_{/X_1}$ has the structure of a strong homotopy comonoid in the symmetric monoidal $\infty$-category* ***LIN****, with the comultiplication $\Delta$ and counit $\varepsilon$ defined by the spans*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1, \qquad\qquad X_1 \xleftarrow{s_0} X_0 \longrightarrow 1.$$

If $X$ is the nerve of a locally finite category or poset, then $X_2$ is the set of composable pairs of arrows, and (after passing to $\mathbb{Q}$-vector spaces by taking homotopy cardinality as in 7.3 and [10]) the formula is the classical comultiplication formula

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b.$$

**1.12. CULF functors.** For the present purposes, the relevant notion of morphism between decomposition spaces is that of CULF functors, since these induce homomorphisms of the associated incidence coalgebras: a simplicial map (between arbitrary simplicial spaces) is called *ULF* (*unique lifting of factorisations*) if the naturality square for every inner coface map is a pullback, and it is called *conservative* if the naturality square for every codegeneracy map is a pullback. We write *CULF* for conservative and ULF, that is, the naturality square for every active map in $\mathbb{\Delta}$ is a pullback.

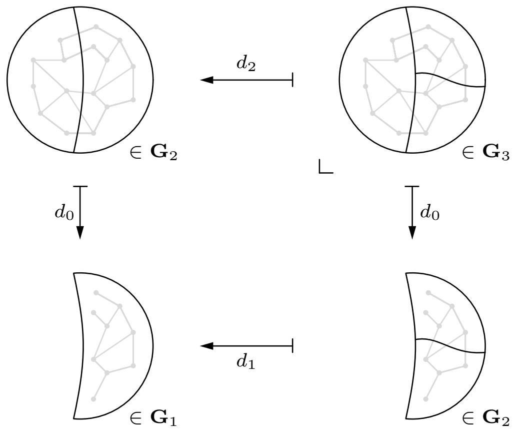
For maps between Rezk complete Segal groupoids, such as fat nerves of categories, the notion of conservative is the classical notion, i.e. only invertible maps are sent to invertible ones, and ULF is a homotopy version of the notion of unique lifting of factorisations.

**1.13. Example.** We describe a decomposition space $\mathbf{G}$ of finite graphs, whose incidence coalgebra is the chromatic Hopf algebra of Schmitt [31]. This will serve as a running example throughout the paper. For definiteness, by 'graph' we will mean simple non-directed graph, though other notions of graph would work too.

Let $\mathbf{G}_n$ be the groupoid of finite graphs with an *$n$-layering* (meaning an ordered partition of the vertex set into $n$ 'layers', which may be empty), and isomorphisms between them. In particular, $\mathbf{G}_0$ is the contractible groupoid consisting only of the empty graph (the only graph admitting a 0-layering), $\mathbf{G}_1$ is just the groupoid of all finite graphs, and $\mathbf{G}_2$ is the groupoid of finite graphs with vertex set partitioned into two. All the $\mathbf{G}_n$ assemble into a simplicial groupoid: the face maps join two adjacent layers, or project away the bottom or top layer; the degeneracy maps insert an empty layer. It is easy to see that this is not a Segal space: a 2-layered graph cannot be reconstructed from the graphs of its layers, since the information about edges joining the layers is missing. One can check that it *is* a decomposition space: that the square

$$\begin{tikzcd}
\mathbf{G}_2 \arrow[d, "d_0"'] & \mathbf{G}_3 \arrow[l, "d_2"'] \arrow[d, "d_0"] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbf{G}_1 & \mathbf{G}_2 \arrow[l, "d_1"]
\end{tikzcd}$$

is a pullback is to say that a graph with a 3-layering ($\in \mathbf{G}_3$) can be reconstructed uniquely from a pair of elements in $\mathbf{G}_2$ with common image in $\mathbf{G}_1$ (under the indicated face maps). The following picture represents elements corresponding to each other in the four groupoids.




The horizontal maps join the last two layers. The vertical maps forget the first layer. Clearly the diagram commutes. To reconstruct the graph with a 3-layering (upper right-hand corner), most of the information is already available in the upper left-hand corner, namely the underlying graph and all the subdivisions except the one between layer 2 and layer 3. But this information is precisely available in the lower right-hand corner, and their common image in $\mathbf{G}_1$ says precisely how this missing piece of information is to be implanted.

In the comultiplication formula, $d_1{}^*$ takes a graph $G$ to the groupoid of all possible 2-layerings on $G$, and $(d_2, d_0)_!$ returns the two layers, meaning the graphs induced by the two subsets of the vertex set $V$. After taking homotopy cardinality, this is precisely the comultiplication of the chromatic Hopf algebra of Schmitt [31]: it takes a basis element $G$ to the sum $\sum G|V_1 \otimes G|V_2$, the sum being over all 2-layerings $V = V_1 + V_2$.

There is a CULF functor from the decomposition space of graphs to the decomposition space of finite sets (defined similarly — its incidence coalgebra is the binomial coalgebra [14]), which to a graph associates its vertex set. The CULF condition simply says that the $n$-layerings on a graph are determined by the $n$-layerings of the vertex set. This CULF functor induces a coalgebra homomorphism from the chromatic coalgebra to the binomial coalgebra.

## 2. Complete decomposition spaces

In this section we introduce the notion of complete decomposition spaces, which is needed to talk about nondegenerate simplices in a meaningful way.

**2.1. Complete decomposition spaces.** A decomposition space $X$ is called *complete* if $s_0 : X_0 \to X_1$ is a monomorphism of $\infty$-groupoids.

**2.2. Discussion.** It is clear that a Rezk complete Segal space is complete in the sense of 2.1. While it makes sense to state the Rezk completeness condition for decomposition spaces too (cf. 5.13 below), our condition 2.1 covers some important examples which are not Rezk complete, such as the ordinary nerve of a group (cf. Example 2.3 below). The incidence algebra of the nerve of a group is the group algebra — certainly an example worth covering.

The completeness condition is necessary to define $\Phi_{\text{even}}$ and $\Phi_{\text{odd}}$ (the even and odd parts of the 'Möbius functor', see 3.4) and to establish the Möbius inversion principle at the objective level (Theorem 3.8). The completeness condition is also needed to make sense of the notion of length (6.1), and to define the length filtration (6.10), which is of independent interest, and is also required to be able to take homotopy cardinality of Möbius inversion.

**2.3. Examples.** If a decomposition space $X$ is *discrete*, meaning that each $X_i$ is a set, then it will be complete, because $s_0 : X_0 \to X_1$ is a section to $d_0 : X_1 \to X_0$ and is therefore an injection of sets. Slightly more generally, a decomposition space $X$ will be complete if $d_0 : X_1 \to X_0$ is discrete (that is, has discrete (homotopy) fibres) since a section to a discrete map is always a monomorphism.

For the simplest example of a decomposition space which is not complete, let $G$ be a nontrivial group, and denote the corresponding one-object groupoid by $BG$. Consider the simplicial groupoid $X$ with $X_n = (BG)^n$. Here $s_0 : 1 \to BG$ is not a monomorphism (although it is a section of $BG \to 1$): its (homotopy) fibre is the set of elements of $G$.

**2.4. Example (continued from 1.13).** The decomposition space $\mathbf{G}$ of finite graphs is complete: indeed, $s_0 : \mathbf{G}_0 \to \mathbf{G}_1$ assigns to the empty graph with zero layers the empty graph with one layer. Clearly this has trivial automorphism group, so $s_0$ is a monomorphism.

The following basic result follows immediately from Lemma 1.10.

**Lemma 2.5.** *In a complete decomposition space, all degeneracy maps are monomorphisms.*

**2.6. Completeness for simplicial spaces.** We shall briefly need completeness also for general simplicial spaces, and the first batch of results holds in this generality. We shall say that $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is *complete* if all degeneracy maps are monomorphisms. In view of Lemma 2.5, this agrees with the previous definition when $X$ is a decomposition space.

**2.7. Completeness as an exactness condition.** It is interesting to note that completeness is an exactness condition, just as the decomposition-space axiom itself. Indeed, for $0 \leq i \leq n$ the squares

$$\begin{tikzcd} {[n]} \arrow[dr, phantom, "\lrcorner"] & {[n]} \arrow[l, "="'] \\ {[n]} \arrow[u, "="] & {[n+1]} \arrow[l, "s^i"] \arrow[u, "s^i"'] \end{tikzcd}$$

are pushouts in $\mathbb{\Delta}$, and the completeness condition on $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is precisely to send these pushouts to pullbacks (because a map is mono if and only if its pullback along itself is an equivalence). If $X$ is assumed to be a decomposition space, then the completeness condition can be expressed by the requirement that the following single square is a pullback.

$$\begin{tikzcd} X_0 \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "s_0"] \\ X_0 \arrow[r, "s_0"'] & X_1 . \end{tikzcd}$$

For the rest of this section, $X$ will denote a complete simplicial space, except where it is explicitly stated to be a complete decomposition space.

**2.8. Word notation.** Since $s_0 : X_0 \to X_1$ is mono, we can identify $X_0$ with a full sub-$\infty$-groupoid of $X_1$. We denote by $X_a$ its complement, the full sub-$\infty$-groupoid of *nondegenerate* 1*-simplices*:

$$X_1 = X_0 + X_a.$$

We extend this notation as follows. Consider the alphabet with three letters $\{0, 1, a\}$. Here 0 indicates degenerate edges $s_0(x) \in X_1$, the letter $a$ indicates edges which are nondegenerate, and 1 indicates edges which may be degenerate or nondegenerate. For $w$ a word in this alphabet $\{0, 1, a\}$, of length $|w| = n$, put

$$X^w := \prod_{i \in w} X_i \subset (X_1)^n.$$

This inclusion is full since $X_a \subset X_1$ is full by completeness. Denote by $X_w$ the $\infty$-groupoid of $n$-simplices whose principal edges have the types indicated in the word $w$, or more explicitly, the full sub-$\infty$-groupoid of $X_n$ given by the pullback diagram

$$\begin{tikzcd} X_w \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d] \\ X^w \arrow[r] & (X_1)^n. \end{tikzcd} \tag{2}$$

**Lemma 2.9.** *If $X$ and $Y$ are complete simplicial spaces and $f : Y \to X$ is conservative, then $Y_a$ maps to $X_a$, and the following square is a pullback:*

$$\begin{tikzcd} Y_1 \arrow[d] & Y_a \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ X_1 & X_a. \arrow[l] \end{tikzcd}$$

**Proof.** This square is the complement of the pullback saying what conservative means. But it is general in extensive $\infty$-categories such as $\mathcal{S}$, that in the situation

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A' + B' \arrow[d] & B' \arrow[l] \arrow[d] \\ A \arrow[r] & A + B & B, \arrow[l] \end{tikzcd}$$

one square is a pullback if and only if the other is. $\square$

**Corollary 2.10.** *If $X$ and $Y$ are complete simplicial spaces and $f : Y \to X$ is conservative, then for every word $w \in \{0, 1, a\}^*$, the following square is a pullback:*

$$\begin{tikzcd} Y_n \arrow[d] & Y_w \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ X_n & X_w. \arrow[l] \end{tikzcd} \tag{3}$$

**Proof.** The square is connected to

$$\begin{tikzcd} (Y_1)^n \arrow[d] & Y^w \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ (X_1)^n & X^w \arrow[l] \end{tikzcd} \tag{4}$$

by two instances of pullback-square (2), one for $Y$ and one for $X$. It follows from Lemma 2.9 that (4) is a pullback, hence also (3) is a pullback, by Lemma 1.3. $\square$

**Proposition 2.11.** *If $X$ and $Y$ are complete simplicial spaces and $f : Y \to X$ is CULF, then for any word $w \in \{0, 1, a\}^*$ the following square is a pullback:*

$$\begin{tikzcd} Y_1 \arrow[d, "f"'] & Y_n \arrow[l] & Y_w \arrow[l] \arrow[d, "f"] \arrow[dll, phantom, "\lrcorner", very near start] \\ X_1 & X_n \arrow[l] & X_w . \arrow[l] \end{tikzcd}$$

**Proof.** The required pullback square is a horizontal composite

$$\begin{tikzcd} Y_1 \arrow[d, "f"'] & Y_n \arrow[l] \arrow[d, "f"] \arrow[dl, phantom, "\llcorner", very near start] & Y_w \arrow[l] \arrow[d, "f"] \arrow[dl, phantom, "\llcorner", very near start] \\ X_1 & X_n \arrow[l] & X_w , \arrow[l] \end{tikzcd}$$

where the right-hand square is the pullback square (3) of Corollary 2.10. The horizontal arrows of the left-hand square are induced by the unique active map $[n] \to [1]$, and since $f$ is CULF this square is a pullback also. $\square$

**Lemma 2.12.** *Let $X$ be a complete simplicial space. Then for any words $v, v' \in \{0, 1, a\}^*$, we have*

$$X_{v1v'} = X_{v0v'} + X_{vav'},$$

*and hence*

$$X_n = \sum_{w \in \{0,a\}^n} X_w .$$

**Proof.** Consider the diagram

$$\begin{tikzcd} X_{v0v'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{v1v'} \arrow[d] & X_{vav'} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ X^{v0v'} \arrow[r] & X^{v1v'} & X^{vav'} \arrow[l] \end{tikzcd}$$

The two squares are pullbacks, by an application of Lemma 1.3, since horizontal composition of either with the pullback square (2) for $w = v1v'$ gives again the pullback square (2), for $w = v0v'$ or $w = vav'$.

Since the bottom row is a sum diagram, it follows that the top row is also (since the $\infty$-category of $\infty$-groupoids is extensive). $\square$

We now specialise to complete decomposition spaces, although the following result will be subsumed in Section 4 on *stiff simplicial spaces*.

**Proposition 2.13.** *Let $X$ be a complete decomposition space. Then for any words $v$, $v'$ in the alphabet $\{0, 1, a\}$ we have*

$$X_{v0v'} = \mathrm{Im}(s_{|v|} : X_{vv'} \to X_{v1v'}).$$

*That is, the $k$th principal edge of a simplex $\sigma$ is degenerate if and only if $\sigma = s_{k-1} d_k \sigma$.*

Recall that $|v|$ denotes the length of the word $v$ and, as always, the notation Im refers to the essential image.

**Proof.** From (2) we see that (independently of the decomposition-space axiom) $X_{v0v'}$ is characterised by the top pullback square in the diagram

$$\begin{tikzcd}
X_{v0v'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{v1v'} \arrow[d] \arrow[dd, bend left, "{d_\perp}^{|v|}\, {d_\top}^{|v'|}"] \\
X^{v0v'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{v1v'} \arrow[d] \\
X_0 \arrow[r, "s_0"'] & X_1
\end{tikzcd}$$

But the decomposition-space axiom applied to the exterior pullback diagram says that the top horizontal map is $s_{|v|}$, and hence identifies $X_{v0v'}$ with the image of $s_{|v|} : X_{vv'} \to X_{v1v'}$. For the final statement, note that if $\sigma = s_{k-1}\tau$ then $\tau = d_k\sigma$. $\square$

Combining this with Lemma 2.12 we obtain the following result.

**Corollary 2.14.** *Let $X$ be a complete decomposition space. For any words $v$, $v'$ in the alphabet $\{0, 1, a\}$ we have*

$$X_{v1v'} = s_{|v|}(X_{vv'}) + X_{vav'}.$$

**2.15. Effective simplices.** A simplex in a complete simplicial space $X$ is called *effective* when all its principal edges are nondegenerate. We put

$$\vec{X}_n := X_{a\cdots a} \subset X_n,$$

the full sub-$\infty$-groupoid of $X_n$ spanned by the effective simplices. (Every 0-simplex is effective by convention: $\vec{X}_0 = X_0$.) It is clear that outer face maps $d_\perp, d_\top : X_n \to X_{n-1}$ preserve effective simplices, and that every effective simplex is nondegenerate, i.e. is not in the image of any degeneracy map. It is a useful feature of complete *decomposition spaces* that the converse is true too:

**Corollary 2.16.** *In a complete decomposition space $X$, a simplex is effective if and only if it is nondegenerate:*

$$\vec{X}_n = X_n \setminus \bigcup_{i=0}^n \mathrm{Im}(s_i).$$

**Proof.** It is clear that $\vec{X}_n$ is the complement of $X_{01\cdots 1} \cup \cdots \cup X_{1\cdots 10}$ and by Proposition 2.13 we can identify each of these spaces with the image of a degeneracy map. □

In fact this feature is enjoyed by a more general class of complete simplicial spaces, called stiff, treated in Section 4.

Iterated use of Corollary 2.14 yields

**Corollary 2.17.** *For $X$ a complete decomposition space we have*

$$X_n = \sum s_{j_k} \dots s_{j_1}(\vec{X}_{n-k}),$$

*where the sum is over all subsets $\{j_1 < \cdots < j_k\}$ of $\{0, \dots, n-1\}$.*

**Lemma 2.18.** *If a complete decomposition space $X$ is a Segal space, then $\vec{X}_n \simeq \vec{X}_1 \times_{X_0} \cdots \times_{X_0} \vec{X}_1$, the $\infty$-groupoid of strings of $n$ composable nondegenerate arrows in $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$.*

This follows immediately from the pullback square (2). Note that if furthermore $X$ is Rezk complete, we can say non-invertible instead of nondegenerate.

## 3. Möbius inversion in the convolution algebra

In this section, we establish a Möbius inversion principle at the objective level for arbitrary complete decomposition spaces. (Later we shall impose the finiteness conditions necessary for taking (homotopy) cardinality to obtain the Möbius inversion principle also at the classical 'numerical' level.)

**3.1. Convolution.** In homotopy linear algebra [10], $\infty$-categories $\mathcal{S}_{/B}$ play the role of the vector spaces with basis $\pi_0 B$. Just as a linear functional is determined by its values on basis elements, linear functors $\mathcal{S}_{/B} \to \mathcal{S}$ correspond to arbitrary functors $B \to \mathcal{S}$, hence the $\infty$-category $\mathcal{S}^B$ can be considered the linear dual of the slice $\infty$-category $\mathcal{S}_{/B}$ (see [10] for the precise statements and proof).

If $X$ is a decomposition space, the coalgebra structure on $\mathcal{S}_{/X_1}$ therefore induces an algebra structure on $\mathcal{S}^{X_1}$. The convolution product of two linear functors

$$F, G : \mathcal{S}_{/X_1} \longrightarrow \mathcal{S},$$

given by spans $X_1 \leftarrow M \rightarrow 1$ and $X_1 \leftarrow N \rightarrow 1$, is the composite of their tensor product $F \otimes G$ and the comultiplication,

$$F * G : \quad \mathcal{S}_{/X_1} \xrightarrow{\Delta} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} \xrightarrow{F \otimes G} \mathcal{S} \otimes \mathcal{S} \xrightarrow{\sim} \mathcal{S}.$$

Thus the convolution product of $F$ and $G$ is given by the composite of spans

$$\begin{tikzcd}
X_1 & & \\
X_2 \arrow[u] \arrow[d] & M * N \arrow[l] \arrow[ul] \arrow[d] \arrow[dr] \arrow[dl, phantom, "\llcorner", very near start] & \\
X_1 \times X_1 & M \times N \arrow[l] \arrow[r] & 1.
\end{tikzcd}$$

The neutral element for convolution is $\varepsilon : \mathcal{S}_{/X_1} \rightarrow \mathcal{S}$ defined by the span

$$X_1 \xleftarrow{s_0} X_0 \rightarrow 1\,.$$

**3.2. The zeta functor.** The *zeta functor*

$$\zeta : \mathcal{S}_{/X_1} \rightarrow \mathcal{S}$$

is the linear functor defined by the span

$$X_1 \xleftarrow{=} X_1 \rightarrow 1\,.$$

We will see later in the locally finite situation (see 7.4) that on taking the homotopy cardinality of the zeta functor one obtains the constant function 1 on $\pi_0 X_1$, that is, the classical zeta function in the incidence algebra.

It is clear from the definition of the convolution product that the $k$th convolution power of the zeta functor is given by

$$\zeta^k : \; X_1 \xleftarrow{g} X_k \rightarrow 1,$$

where $g : [1] \rightarrow [k]$ is the unique active map in degree $k$.

Consider also the elements $\delta^a$ and $h^a$ of the incidence algebra given by the spans

$$\delta^a : \; X_1 \leftarrow (X_1)_{[a]} \rightarrow 1, \qquad h^a : \; X_1 \xleftarrow{\ulcorner a \urcorner} 1 \rightarrow 1$$

where $(X_1)_{[a]}$ denotes the component of $X_1$ containing $a \in X_1$. Then zeta is the sum of the elements $\delta^a$, or the homotopy sum of $h^a$

$$\zeta \ \simeq \sum_{a\in\pi_0 X_1} \delta^a \ \simeq \ \int^a h^a.$$

**3.3. The idea of Möbius inversion à la Leroux.** We are interested in the invertibility of the zeta functor under the convolution product. Unfortunately, at the objective level it can practically *never* be convolution invertible, because the inverse $\mu$ should always be given by an alternating sum (cf. Theorem 3.8)

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$$

(of the Phi functors defined below). We have no minus sign available, but following the idea of Content–Lemay–Leroux [4], developed further by Lawvere–Menni [23], we establish the sign-free equations

$$\zeta * \Phi_{\text{even}} \simeq \varepsilon + \zeta * \Phi_{\text{odd}}, \qquad \Phi_{\text{even}} * \zeta \simeq \varepsilon + \Phi_{\text{odd}} * \zeta.$$

In the category case (cf. [4] and [23]), $\Phi_{\text{even}}$ (resp. $\Phi_{\text{odd}}$) is given by even-length (resp. odd-length) chains of non-identity arrows. (We keep the $\Phi$-notation in honour of Content–Lemay–Leroux.) In the general setting of decomposition spaces we cannot talk about chains of arrows, but in the complete case we can still talk about effective simplices and their principal edges.

From now on we assume again that $X$ is a complete decomposition space.

**3.4. 'Phi' functors.** We define $\Phi_n$ to be the linear functor given by the span

$$X_1 \longleftarrow \vec{X}_n \longrightarrow 1,$$

where $\vec{X}_n$ is the full sub-$\infty$-groupoid of $X_n$ spanned by the effective simplices, which are the same as the non-degenerate simplices since $X$ is a complete decomposition space, see 2.15 and Corollary 2.16. If $n = 0$ then $\vec{X}_0 = X_0$ by convention, and $\Phi_0$ is given by the span

$$X_1 \longleftarrow X_0 \longrightarrow 1.$$

That is, $\Phi_0$ is the linear functor $\varepsilon$. Note that $\Phi_1 = \zeta - \varepsilon$. The minus sign makes sense here, since $X_0$ (representing $\varepsilon$) is really a full sub-$\infty$-groupoid of $X_1$ (representing $\zeta$).

To compute convolution with $\Phi_n$, a key ingredient is the following general lemma (with reference to the word notation of 2.8).

**Lemma 3.5.** *Let $X$ be a complete decomposition space. Then for any words $v$, $v'$ in the alphabet $\{0, 1, a\}$, the square*

$$\begin{tikzcd}
X_{vv'} \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_v \times X_{v'} \arrow[r] & X_1 \times X_1
\end{tikzcd}$$

*is a pullback.*

**Proof.** Let $m = |v|$ and $n = |v'|$. The square is the outer rectangle in the top row of the diagram

$$\begin{tikzcd}
X_{vv'} \arrow[r] \arrow[d] & X_{m+n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X_{1+n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X_2 \arrow[d] \\
X_v \times X_{v'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X_m \times X_n \arrow[r] \arrow[d] & X_1 \times X_n \arrow[r] & X_1 \times X_1 \\
X^v \times X^{v'} \arrow[r] & {X_1}^m \times {X_1}^n & &
\end{tikzcd}$$

The left-hand outer rectangle is a pullback by definition of $X_{vv'}$, and the bottom square is a pullback by definition of $X_v$ and $X_{v'}$. Hence the top-left square is a pullback. But the other squares in the top row are pullbacks because $X$ is a decomposition space (compare the square 1 of [11, 5.3]). ◻

**Lemma 3.6.** *We have*

$$\Phi_n \;\simeq\; (\Phi_1)^n \;=\; (\zeta - \varepsilon)^n,$$

*the nth convolution product of $\Phi_1$ with itself.*

**Proof.** This follows from the definitions and Lemma 3.5. ◻

**Proposition 3.7.** *The linear functors $\Phi_n$ satisfy*

$$\zeta * \Phi_n \;\simeq\; \Phi_n + \Phi_{n+1} \;\simeq\; \Phi_n * \zeta.$$

**Proof.** We can compute the convolution $\zeta * \Phi_n$ by Lemma 3.5 as

$$\begin{tikzcd}
X_1 & & \\
X_2 \arrow[u] \arrow[d] & X_{1a\cdots a} \arrow[l] \arrow[ul] \arrow[d] \arrow[dr] \arrow[dl, phantom, "\llcorner" very near start] & \\
X_1 \times X_1 & X_1 \times \vec{X}_n \arrow[l] \arrow[r] & 1
\end{tikzcd}$$

But Lemma 2.12 tells us that $X_{1a\cdots a} = X_{0a\cdots a} + X_{aa\cdots a} = \vec{X}_n + \vec{X}_{n+1}$, where the identification in the first summand is via $s_0$, in virtue of Proposition 2.13. This is an equivalence of $\infty$-groupoids over $X_1$ so the resulting span is $\Phi_n + \Phi_{n+1}$ as desired. The second identity claimed follows similarly. □

Put

$$\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n.$$

**Theorem 3.8.** *For a complete decomposition space, the following Möbius inversion principle holds:*

$$\begin{aligned} \zeta * \Phi_{\text{even}} &\simeq \varepsilon + \zeta * \Phi_{\text{odd}}, \\ \Phi_{\text{even}} * \zeta &\simeq \varepsilon + \Phi_{\text{odd}} * \zeta. \end{aligned}$$

*In fact, these four linear functors are all equivalent.*

**Proof.** This follows immediately from Proposition 3.7: all four linear functors are equivalent to $\sum_{r \geq 0} \Phi_r$. □

We note the following immediate corollary of Proposition 2.11, which can be read as saying 'Möbius inversion is preserved by CULF functors':

**Corollary 3.9.** *If $f : Y \to X$ is CULF, then $f^*\zeta \simeq \zeta$ and $f^*\Phi_n \simeq \Phi_n$ for all $n \geq 0$.*

## 4. Stiff simplicial spaces

We saw that in a complete decomposition space, degeneracy can be detected on principal edges. In Section 5 we shall come to split simplicial spaces, which share this property. A common generalisation is that of stiff complete simplicial spaces, which we now introduce.

**4.1. Stiffness.** A simplicial space $X : \mathbb{\Delta}^{\text{op}} \to \mathcal{S}$ is called *stiff* if it sends codegeneracy/inert pushouts in $\mathbb{\Delta}$ to pullbacks in $\mathcal{S}$. These pushouts are examples of active–inert pushouts, so in particular every decomposition space is stiff.

**Lemma 4.2.** *A simplicial space $X$ is stiff if and only if the following diagrams are pullbacks for all $0 \leq i \leq n$.*

$$\begin{tikzcd} X_n \arrow[r, "s_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d, "{d_\bot}^i\, {d_\top}^{n-i}"] \\ X_0 \arrow[r, "s_0"'] & X_1 \end{tikzcd}$$

**Proof.** The squares in the lemma are special cases of the degeneracy/inert squares. On the other hand, every degeneracy/inert square sits in between two of the squares of the lemma in such a way that Lemma 1.3 forces it to be a pullback too. □

The following two lemmas for stiff simplicial spaces are proved in the same way as for decomposition spaces [11, Lemmas 3.10 and 3.9 respectively].

**Lemma 4.3.** *('Bonus pullbacks') Let $X$ be a stiff simplicial space. For all $n \geq 3$ and all $0 < i < j < n$, the following squares of active face and degeneracy maps are pullbacks.*

$$\begin{tikzcd} X_{n-3} \arrow[r, "s_{i-1}"] \arrow[d, "s_{j-2}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-2} \arrow[d, "s_{j-1}"] \\ X_{n-2} \arrow[r, "s_{i-1}"'] & X_{n-1} \end{tikzcd} \qquad \begin{tikzcd} X_{n-1} \arrow[r, "d_i"] \arrow[d, "s_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-2} \arrow[d, "s_{j-1}"] \\ X_n \arrow[r, "d_i"'] & X_{n-1} \end{tikzcd} \qquad \begin{tikzcd} X_{n-1} \arrow[r, "d_{j-1}"] \arrow[d, "s_{i-1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-2} \arrow[d, "s_{i-1}"] \\ X_n \arrow[r, "d_j"'] & X_{n-1}. \end{tikzcd}$$

**Lemma 4.4.** *In a stiff simplicial space $X$, every degeneracy map is a pullback of $s_0 : X_0 \to X_1$. In particular, if just $s_0 : X_0 \to X_1$ is mono then all degeneracy maps are mono.*

**Corollary 4.5.** *A simplicial map $f : Y \to X$ between stiff simplicial spaces is conservative if and only if the naturality square for $s_0$ is a pullback:*

$$\begin{tikzcd} Y_0 \arrow[r, "s_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d] \\ X_0 \arrow[r, "s_0"'] & X_1. \end{tikzcd}$$

**Corollary 4.6.** *A stiff simplicial space $X$ is complete if and only if the canonical map from the constant simplicial space $X_0$ is conservative.*

**Proof.** This follows from the previous two lemmas and a standard pullback argument, exploiting the pullback characterisation of completeness 2.7. □

For complete simplicial spaces, stiffness can be characterised in terms of degeneracy:

**Proposition 4.7.** *The following are equivalent for a complete simplicial space $X$.*

(1) *$X$ is stiff.*
(2) *Outer face maps $d_\bot, d_\top : X_n \to X_{n-1}$ preserve nondegenerate simplices.*
(3) *Any nondegenerate simplex is effective. More precisely,*

$$\vec{X}_n = X_n \setminus \textstyle\bigcup_{i=0}^{n} \mathrm{Im}(s_{i-1}).$$

(4) *If the $i$th principal edge of $\sigma \in X_n$ is degenerate, then $\sigma = s_{i-1}d_{i-1}\sigma = s_{i-1}d_i\sigma$, that is*

$$X_{1\ldots101\ldots1} = \mathrm{Im}(s_{i-1} : X_{n-1} \to X_n)$$

(5) *For each word $w \in \{0, a\}^n$ we have*

$$X_w = \mathrm{Im}(s_{j_k-1} \ldots s_{j_1-1} : \vec{X}_{n-k} \to X_n).$$

*where $\{j_1 < \cdots < j_k\} = \{j : w_j = 0\}$.*

**Proof.** (1) ⇒ (2): Suppose $\sigma \in X_n$ and that $d_\top\sigma$ is degenerate. Then $d_\top\sigma$ is in the image of some $s_i : X_{n-2} \to X_{n-1}$, and hence by (1) already $\sigma$ is in the image of $s_i : X_{n-1} \to X_n$.

(2) ⇒ (3): The principal edges of a simplex are obtained by applying outer face maps, so nondegenerate simplices are also effective. For the more precise statement, just note that both subspaces are full, so are determined by the properties characterising their objects.

(3) ⇒ (4): As $\sigma$ is not effective, we have $\sigma = s_j\tau$. If $j > i-1$ then the $i$th principal edge of $\sigma$ is also that of $\tau$, so by induction $\tau \in \mathrm{Im}(s_{i-1})$. Therefore $\sigma \in \mathrm{Im}(s_{i-1})$ also, and $\sigma = s_{i-1}d_{i-1}\sigma = s_{i-1}d_i\sigma$ as required. If $j < i-1$ the argument is similar.

(4) ⇔ (1): To show that $X$ is stiff, by Lemma 4.2 it is enough to check that this is a pullback:

$$\begin{tikzcd}
X_n \arrow[r, "s_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+1} \arrow[d, "d_\bot{}^i\, d_\top{}^{n-i}"] \\
X_0 \arrow[r, "s_0"'] & X_1
\end{tikzcd}$$

But the pullback is by definition $X_{1\cdots101\cdots1} \subset X_{n+1}$, and by assumption this is canonically identified with the image of $s_i : X_n \to X_{n+1}$, establishing the required pullback.

(4) ⇔ (5): This is clear, using Lemma 2.12. □

In summary, an important feature of complete stiff simplicial spaces is that all information about degeneracy is encoded in the principal edges. We exploit this to characterise conservative maps between complete stiff simplicial spaces:

**Proposition 4.8.** *For $X$ and $Y$ complete stiff simplicial spaces, and $f : Y \to X$ a simplicial map, the following are equivalent.*

(1) *$f$ is conservative.*
(2) *$f$ preserves the word splitting, i.e. for every word $w \in \{0, a\}^*$, $f$ sends $Y_w$ to $X_w$.*
(3) *$f_1$ maps $Y_a$ to $X_a$.*

**Proof.** We already saw (Corollary 2.10) that conservative maps preserve the word splitting (independently of $X$ and $Y$ being stiff), which proves (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial. Finally assume that $f_1$ maps $Y_a$ to $X_a$. To check that $f$ is conservative, it is enough (by Corollary 4.5) to check that the square

$$\begin{tikzcd} Y_0 \arrow[r, "s_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d] \\ X_0 \arrow[r, "s_0"'] & X_1 \end{tikzcd}$$

is a pullback. But since $X$ and $Y$ are complete, this square is just

$$\begin{tikzcd} Y_0 \arrow[r, "s_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 + Y_a \arrow[d] \\ X_0 \arrow[r, "s_0"'] & X_0 + X_a, \end{tikzcd}$$

which is clearly a pullback when $f_1$ maps $Y_a$ to $X_a$. $\square$

This proposition can be stated more formally as follows. For $X$ and $Y$ stiff complete simplicial spaces, the space of conservative maps $\mathrm{Cons}(Y,X)$ is given as the pullback

$$\begin{tikzcd} \mathrm{Cons}(Y,X) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{n\in\mathbb{N}} \prod_{w\in\{0,a\}^n} \mathrm{Map}(Y_w, X_w) \arrow[d, tail] \\ \mathrm{Nat}(Y,X) \arrow[r] & \prod_{n\in\mathbb{N}} \mathrm{Map}(Y_n, X_n). \end{tikzcd}$$

The vertical arrow on the right is given as follows. We have

$$\mathrm{Map}(Y_n, X_n) \simeq \mathrm{Map}\Big(\sum_{w\in\{0,a\}^n} Y_w, \sum_{v\in\{0,a\}^n} X_v\Big) \simeq \prod_{w\in\{0,a\}^n} \mathrm{Map}\Big(Y_w, \sum_{v\in\{0,a\}^n} X_v\Big).$$

For fixed $w \in \{0,a\}^n$, the space $\mathrm{Map}(Y_w, \sum_{v\in\{0,a\}^n} X_v)$ has a distinguished subobject, namely consisting of those maps that map into $X_w$ for that same word $w$.

## 5. Split decomposition spaces

In this section, we digress to introduce split decomposition spaces, more general than the decomposition spaces of locally finite length of the following section. The interest in this notion is its relation to Kan extension of semi-simplicial spaces (Theorem 5.19).

**5.1. Split simplicial spaces.** In a complete simplicial space $X$, by definition all degeneracy maps are monomorphisms, so in particular it makes sense to talk about nondegenerate simplices in degree $n$: these form the full sub-$\infty$-groupoid of $X_n$ given as the complement of the degeneracy maps $s_i : X_{n-1} \to X_n$. A simplicial space is *split* if it is complete and the face maps preserve nondegenerate simplices.

**5.2. Example (continued).** The decomposition space $\mathbf{G}$ of finite graphs (1.13) is split. Indeed, the face maps join adjacent layers or project away the bottom or top layer. To be nondegenerate means having no empty layers, and this property is clearly preserved by the face maps.

By Proposition 4.7, a split simplicial space is stiff, so the results from the previous section are available for split simplicial spaces. In particular, nondegeneracy can be measured on principal edges, and we have

**Corollary 5.3.** *If $X$ is a split simplicial space, then the sum splitting*

$$X_n \;=\; \sum_{w \in \{0,a\}^n} X_w$$

*is realised by the degeneracy maps.*

**5.4. Non-example.** The strict nerve of any category with a nontrivial section-retraction pair of arrows, $r \circ s = \mathrm{id}$, constitutes an example of a complete decomposition space which is not split. Indeed, the nondegenerate simplices are the chains of composable non-identity arrows, but we have $d_1(s,r) = \mathrm{id}$.

In this way, splitness can be seen as an abstraction of the condition on a 1-category that its identity arrows be indecomposable. We proceed to formalise this, cf. Corollary 5.11 below.

**5.5. Indecomposable units.** A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is said to have *indecomposable units* when the following squares are pullbacks for all $0 \le i \le n$:

$$\begin{tikzcd} X_n \arrow[r, "="] \arrow[d, "s_i s_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "s_i"] \\ X_{n+2} \arrow[r, "d_{i+1}"'] & X_{n+1}. \end{tikzcd}$$

We note that having indecomposable units is an exactness condition: in $\mathbb{\Delta}$, the squares

$$\begin{tikzcd} {[n]} & {[n]} \arrow[l, "="'] \\ {[n+2]} \arrow[u, "s^i s^i"] \arrow[ur, phantom, "\lrcorner"] & {[n+1]} \arrow[l, "d^{i+1}"] \arrow[u, "s^i"'] \end{tikzcd}$$

are pushouts, and the condition stipulates that they be sent to pullbacks.

The first instance of the indecomposable-units condition,

$$\begin{tikzcd} X_0 \arrow[r, "="] \arrow[d, "s_0 s_0"'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd} \tag{5}$$

motivates the name, in view of the following important corollary.

**Corollary 5.6.** *For a simplicial space $X$ satisfying the pullback condition (5), if a 2-simplex $\sigma \in X_2$ has degenerate long edge $d_1\sigma$ then $\sigma$ itself is totally degenerate.*

For the nerve of a category, this is the classical notion of indecomposable identity arrows. Note that if $X$ is furthermore complete, then the statement of the corollary is actually 'if and only if'.

**Lemma 5.7.** *A stiff simplicial space $X$ satisfying the pullback condition (5) has indecomposable units.*

**Proof.** The pullback square for a general instance of the indecomposable-units condition can be connected to the first instance (5) by inert face maps, and the result follows from stiffness and the usual pullback argument. □

**Lemma 5.8.** *For $X$ stiff and complete, we have*

$$\begin{tikzcd} X_0 \arrow[r, "="] \arrow[d, "s_0 s_0"'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "s_0"] \\ \vec{X}_2 \arrow[r, "d_1"'] & X_1 . \end{tikzcd}$$

**Proof.** By completeness, we can write $X_2 = X_{00} + X_{0a} + X_{a0} + X_{aa}$. We compute the pullback of $s_0$ to each of these summands, exploiting that degenerate principal edges only arise from degeneracy maps, cf. Proposition 4.7. The first summand gives

$$
\begin{tikzcd}
X_0 \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[rr, "="] \arrow[d, "s_0"'] \arrow[dr, phantom, "\lrcorner"] & & X_0 \arrow[d, "s_0"] \\
X_0 \arrow[r, "s_0"'] & X_1 \arrow[r, "s_0"'] & X_2 \arrow[r, "d_1"'] & X_1,
\end{tikzcd}
$$

where the left-hand square is a pullback since $X$ is complete. The second summand gives

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[rr, "="] \arrow[d, "s_0"'] \arrow[dr, phantom, "\lrcorner"] & & X_0 \arrow[d, "s_0"] \\
X_a \arrow[r] & X_1 \arrow[r, "s_0"'] & X_2 \arrow[r, "d_1"'] & X_1,
\end{tikzcd}
$$

since $X_a$ and $X_0$ are disjoint in $X_1$. The third summand is analogous to the second. In conclusion, the total pullback gives $X_0$ if and only if and the fourth summand gives $\varnothing$. $\square$

**Proposition 5.9.** *A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is split if and only if it is stiff, complete and has indecomposable units.*

**Proof.** Suppose $X$ is split. Then it is complete, and it follows from Proposition 4.7 that it is stiff. By Lemmas 5.7 and 5.8, it remains just to check that the square

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "s_0"] \\
\vec{X}_2 \arrow[r, "d_1"'] & X_1
\end{tikzcd}
$$

is a pullback, but this follows from splitness: since $X$ is stiff and complete, nondegenerate is the same as effective (Proposition 4.7), so splitness implies that $d_1$ maps $\vec{X}_2$ into $\vec{X}_1$, and $\vec{X}_1$ is disjoint from $X_0$.

Suppose now that $X$ is stiff and has indecomposable units. Fix a simplex $\sigma \in X_{n+2}$. We must show that if $\sigma$ is nondegenerate then also $d_j\sigma$ is nondegenerate for all $0 \le j \le n+2$. By stiffness we already know that this is the case for $d_j$ inert, so it remains to treat the active case. The contrapositive statement is that if for some $0 < j < n+2$ we have that $d_j\sigma$ is degenerate then already $\sigma$ is degenerate. That is, if we have $d_j\sigma = s_i\tau$ for some indices $0 < j < n+2$ and $0 \le i \le n$, and some simplex $\tau \in X_{n+1}$, then there exists a simplex $\rho \in X_{n+1}$ and an index $k$ such that $\sigma = s_k\rho$. There are two cases: if $j = i+1$, then we have the pullback square expressing indecomposable units

$$\begin{tikzcd}
X_n \arrow[r, "="] \arrow[d, "s_i s_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "s_i"] \\
X_{n+2} \arrow[r, "d_{i+1}"'] & X_{n+1},
\end{tikzcd}$$

and we can take $\rho = s_i\tau$ and $k = i$. On the other hand for $j \neq i+1$, we have the 'bonus pullback' (cf. Lemma 4.3)

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "s_i"] \\
X_{n+2} \arrow[r, "d_j"'] & X_{n+1},
\end{tikzcd}$$

and we can take $\rho \in X_{n+1}$ to be the simplex corresponding to $(\sigma, \tau)$ in the pullback. In either case, we see that $\sigma$ is degenerate, as required. □

**Corollary 5.10.** *A complete simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is split if and only if it preserves pullbacks along degeneracy maps in $\mathbb{\Delta}^{\mathrm{op}}$. In other words, every degeneracy map forms pullbacks with any other face or degeneracy map.*

**Proof.** Since $X$ is stiff, Lemma 4.3 says that $s_i$ forms pullbacks with all $d_j$ except $d_{i+1}$, but this case is covered by having indecomposable units. On the other hand, again by bonus pullbacks, $s_i$ forms pullbacks against all $s_j$ except against itself, but this case is covered by being complete (cf. 2.7). □

**Corollary 5.11.** *A complete decomposition space is split if and only if it has indecomposable units.* □

The *long edge* of a simplex $\sigma \in X_n$ in a simplicial space is the element $g(\sigma) \in X_1$, where $g : X_n \to X_1$ is the unique active map.

**Proposition 5.12.** *In a split simplicial space $X$, if the long edge of a simplex $\sigma \in X_n$ is degenerate then the simplex is totally degenerate (that is, in the image of ${s_0}^n$).*

**Proof.** Induction on $n$. The case $n = 2$ is Corollary 5.6. Suppose the proposition is true in dimension $n$ and consider $\sigma' \in X_{n+1}$ with long edge $u' := {d_1}^n\sigma'$, assumed degenerate. Consider the 2-simplex $\tau := {d_1}^{n-1}\sigma'$ and the $n$-simplex $\sigma := d_{n+1}\sigma'$. Then the long edge of $\tau$ is $d_1\tau = {d_1}^n\sigma' = u'$, and the long edge of $\sigma$ is ${d_1}^{n-1}\sigma = {d_1}^{n-1}d_{n+1}\sigma' = d_2{d_1}^{n-1}\sigma' = d_2\tau$:

$$\sigma' : \quad \begin{tikzcd} & 1 \arrow[rr, dashed, bend left] & \sigma & n \arrow[dr, "d_0\tau"] & \\ 0 \arrow[ur] \arrow[urrr, "d_2\tau"'] \arrow[rrrr, "u'"'] & & & \tau & n{+}1 \end{tikzcd}$$

Since $u'$ is degenerate by assumption, $\tau$ is totally degenerate by induction, so in particular its principal edges $d_2\tau$ and $d_0\tau$ are degenerate. But $d_2\tau$ is the long edge of $\sigma$, so by induction $\sigma$ is totally degenerate. Since the principal edges of $\sigma'$ are those of $\sigma$ plus $d_0\tau$ in the end, we conclude that all principal edges of $\sigma'$ are degenerate, so $\sigma'$ is totally degenerate by Proposition 4.7 (as $X$ is stiff). $\square$

**5.13. Rezk complete simplicial spaces.** A simplicial space $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is called *Rezk complete* when $s_0 : X_0 \to X_1^{\mathrm{eq}}$ is an equivalence. Here $X_1^{\mathrm{eq}}$ is defined as the full sub-$\infty$-groupoid of $X_1$ spanned by those $f : x \to y$ for which there exists $\sigma, \tau \in X_2$ with $d_0\sigma \simeq f$ and $d_1\sigma \simeq s_0y$, and $d_2\tau \simeq f$ and $d_1\tau \simeq s_0x$. When $X$ is a Segal space, this definition agrees with the usual definition.

**Lemma 5.14.** *If a complete simplicial space has indecomposable units then it is Rezk complete.*

**Proof.** Since $X_1^{\mathrm{eq}} \to X_1$ is mono by construction, and $s_0 : X_0 \to X_1$ is mono by completeness, to show that the two sub-$\infty$-groupoids coincide, it is enough to show that every element $f : x \to y$ in $X_1^{\mathrm{eq}}$ is actually degenerate. But if $\sigma \in X_2$ exists with $d_1\sigma \simeq s_0y$ and $d_0\sigma \simeq f$, as in the definition of $X_1^{\mathrm{eq}}$, then indecomposability of units implies that $f$ is degenerate. $\square$

**5.15. Semi-decomposition spaces.** Let $\mathbb{\Delta}_{\mathrm{inj}} \subset \mathbb{\Delta}$ denote the subcategory consisting of all the objects and only the injective maps. A *semi-simplicial* space is an object in the functor $\infty$-category $\mathrm{Fun}(\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathcal{S})$. A *semi-decomposition* space is a semi-simplicial space preserving active–inert pullbacks in $\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$. Since there are no degeneracy maps in $\mathbb{\Delta}_{\mathrm{inj}}$, this means that we are concerned only with pullbacks between active face maps and inert face maps.

Every simplicial space has an underlying semi-simplicial space obtained by restriction along $\mathbb{\Delta}_{\mathrm{inj}} \subset \mathbb{\Delta}$. The forgetful functor $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathcal{S})$ has a left adjoint given by left Kan extension along $\mathbb{\Delta}_{\mathrm{inj}} \subset \mathbb{\Delta}$:

$$\begin{tikzcd} \mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \arrow[r, "Z"] \arrow[d] & \mathcal{S} \\ \mathbb{\Delta}^{\mathrm{op}} \arrow[ur, dotted, "\overline{Z}"'] & \end{tikzcd}$$

The left Kan extension has the following explicit description:

$$\begin{aligned} \overline{Z}_0 &= Z_0 \\ \overline{Z}_1 &= Z_1 + Z_0 \\ \overline{Z}_2 &= Z_2 + Z_1 + Z_1 + Z_0 \\ &\vdots \\ \overline{Z}_k &= \sum_{w \in \{0,a\}^k} Z_{|w|_a} \end{aligned}$$

For $w \in \{0,a\}^k$ and $\sigma \in Z_{|w|_a}$ the corresponding element of $\overline{Z}_k$ is denoted

$$s_{i_r} \dots s_{i_2} s_{i_1} \sigma$$

where $r = k - |w|_a$ and $i_1 < i_2 < \cdots < i_r$ with $w_{i_j} = 0$. The faces and degeneracies of such elements are defined in the obvious way.

**Proposition 5.16.** *A simplicial space is split if and only if it is the left Kan extension of a semi-simplicial space.*

**Proof.** Given $Z : \mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{S}$, it is clear from the construction that the new degeneracy maps in $\overline{Z}$ are monomorphisms. Hence $\overline{Z}$ is complete. On the other hand, to say that $\sigma \in \overline{Z}_n$ is nondegenerate is precisely to say that it belongs to the original component $Z_n$, and the face maps here are the original face maps, hence map $\sigma$ into $Z_{n-1}$ which is precisely the nondegenerate component of $\overline{Z}_{n-1}$. Hence $\overline{Z}$ is split.

For the other implication, given a split simplicial space $X$, since $X$ is stiff and complete, we know that nondegenerate is the same as effective (Proposition 4.7) and we have a sum splitting

$$X_n \;=\; \sum_{w \in \{0,a\}^n} X_w.$$

Now by assumption the face maps restrict to the nondegenerate simplices to give a semi-simplicial space $\vec{X} : \mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{S}$. It is now clear from the explicit description of the left Kan extension that $\overline{(\vec{X}_n)} = X_n$, from where it follows readily that $X$ is the left Kan extension of $\vec{X}$. $\square$

**Proposition 5.17.** *A simplicial space is a split decomposition space if and only if it is the left Kan extension of a semi-decomposition space.*

**Proof.** It is clear that if $X$ is a split decomposition space then $\vec{X}$ is a semi-decomposition space. Conversely, if $Z$ is a semi-decomposition space, then one can check by inspection that $\overline{Z}$ satisfies the four pullback conditions in [11, Proposition 3.3]: two of these diagrams concern only face maps, and they are essentially from $Z$, with degenerate stuff added. The two diagrams involving degeneracy maps are easily seen to be pullbacks since the degeneracy maps are sum inclusions. $\square$

**5.18. Example (continued).** The split decomposition space $\mathbf{G}$ of finite graphs (see Examples 1.13 and 5.2) is the left Kan extension of a semi-simplicial space $Z$ where $Z_n$ is the groupoid of $n$-layered graphs with no empty layers. Here $Z_0$ is still the contractible groupoid consisting of the 0-layered empty graph, and the left Kan extension freely adds all the degenerate $n$-layerings for $n > 0$.

**Theorem 5.19.** *The left adjoint functor* $\mathrm{Fun}(\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$ *given by Kan extension along* $\mathbb{\Delta}_{\mathrm{inj}} \subset \mathbb{\Delta}$ *induces an equivalence of* $\infty$*-categories*

$$\mathrm{Fun}(\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathcal{S}) \simeq \textbf{\textit{Split}}^{cons},$$

*the* $\infty$*-category of split simplicial spaces and conservative maps.*

**Proof.** Let $X$ and $Y$ be split simplicial spaces, then $\vec{X}$ and $\vec{Y}$ are semi-simplicial spaces whose left Kan extensions are $X$ and $Y$ again. The claim is that

$$\mathrm{Cons}(Y, X) \simeq \mathrm{Nat}(\vec{Y}, \vec{X}).$$

Intuitively, the reason this is true can be seen in the first square as in the proof of Lemma 4.8: to give a pullback square

$$\begin{tikzcd} Y_0 \arrow[r, "s_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 + Y_a \arrow[d] \\ X_0 \arrow[r, "s_0"'] & X_0 + X_a, \end{tikzcd}$$

amounts to giving $Y_0 \to X_0$ and $Y_a \to X_a$ (and of course, in both cases this data is required to be natural in face maps), that is to give a natural transformation $\vec{Y} \to \vec{X}$. To formalise this idea, note first that $\mathrm{Nat}(\vec{Y}, \vec{X})$ can be described as a limit

$$\mathrm{Nat}(\vec{Y}, \vec{X}) \longrightarrow \prod_{n \in \mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n) \to \dots$$

where the rest of the diagram contains vertices indexed by all the face maps, expressing naturality. Similarly $\mathrm{Nat}(Y,X)$ is given as a limit

$$\mathrm{Nat}(Y,X) \longrightarrow \prod_{n\in\mathbb{N}} \mathrm{Map}(Y_n, X_n) \to \dots$$

where this time the rest of the diagram furthermore contains vertices corresponding to degeneracy maps. The full subspace of conservative maps is given instead as

$$\mathrm{Cons}(Y,X) \longrightarrow \prod_{w\in\{0,a\}^*} \mathrm{Map}(Y_w, X_w) \to \dots$$

as explained in connection with Lemma 4.8. Now for each degeneracy map $s_i : X_n \to X_{n+1}$, there is a vertex in the diagram. For ease of notation, let us consider $s_0 : X_n \to X_{n+1}$. The corresponding vertex sits in the limit diagram as follows: for each word $v \in \{0,a\}^n$, we have

$$\begin{tikzcd}
\prod\limits_{w\in\{0,a\}^*} \mathrm{Map}(Y_w, X_w) \arrow[r, "\mathrm{proj}"] \arrow[d, "\mathrm{proj}"'] & \mathrm{Map}(Y_{0v}, X_{0v}) \arrow[d, "\mathrm{pre}\ s_0"] \\
\mathrm{Map}(Y_v, X_v) \arrow[r, "\mathrm{post}\ s_0"'] & \mathrm{Map}(Y_n, X_{n+1}).
\end{tikzcd}$$

Now both the pre and post composition maps are monomorphisms with essential image $\mathrm{Map}(Y_v, X_{0v})$, so the two projections coincide, which is to say that the limit factors through the corresponding diagonal. Applying this argument for every degeneracy map $s_i : X_n \to X_{n+1}$, and for all words, we conclude that the limit factors through the product indexed only over the words without degeneracies,

$$\prod_{n\in\mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n).$$

Having thus eliminated all the vertices of the limit diagram that corresponded to degeneracy maps, the remaining diagram has precisely the shape of the diagram computing $\mathrm{Nat}(\vec{Y},\vec{X})$, and we have already seen that the 'starting vertex' is the same, $\prod_{n\in\mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n)$. For the remaining vertices, those corresponding to face maps, it is readily seen that in each case the space is that of the $\mathrm{Nat}(\vec{Y},\vec{X})$ diagram, modulo some constant factors that do not play any role in the limit calculation. In conclusion, the diagram calculating $\mathrm{Cons}(Y,X)$ as a limit is naturally identified with the diagram calculating $\mathrm{Nat}(\vec{Y},\vec{X})$ as a limit. $\square$

**Proposition 5.20.** *The equivalence of Theorem 5.19 restricts to an equivalence between semi-decomposition spaces and all maps and split decomposition spaces and conservative* *maps, and it restricts further to an equivalence between semi-decomposition spaces and ULF maps and split decomposition spaces and CULF maps.*

**5.21. Dyckerhoff–Kapranov 2-Segal semi-simplicial spaces.** Dyckerhoff and Kapranov's notion of 2-Segal space [6] does not refer to degeneracy maps at all, and can be formulated already for semi-simplicial spaces: a 2-Segal space is precisely a simplicial space whose underlying semi-simplicial space is a semi-decomposition space. We get the following corollary to the results above.

**Corollary 5.22.** *Every split decomposition space is the left Kan extension of a 2-Segal semi-simplicial space.*

## 6. The length filtration

In this section we introduce the notion of length of an edge (1-simplex) in a complete decomposition space, and the corresponding notion of locally finite length, which endows the resulting coalgebra with an important filtration. Locally finite length is one of two finiteness conditions in the notion of Möbius decomposition space that we are building up to.

**6.1. Length.** Let $a$ be an edge in a complete decomposition space $X$. The *length* of $a$ is defined to be the largest dimension of an effective simplex (that is, of a nondegenerate simplex, see 2.15 and Corollary 2.16) with long edge $a$:

$$\ell(a) := \max\{\dim \sigma \mid \sigma \in \vec{X}, g(\sigma) = a\},$$

where as usual $g : X_r \to X_1$ denotes the unique active map. More formally: the length is the greatest $r$ such that the pullback

$$\begin{tikzcd} (\vec{X}_r)_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \vec{X}_r \arrow[d, "g"] \\ 1 \arrow[r, "\ulcorner a \urcorner"'] & X_1 \end{tikzcd}$$

is nonempty (or $\infty$ if there is no such greatest $r$). Length zero can happen only for degenerate edges.

**6.2. Decomposition spaces of locally finite length.** A complete decomposition space $X$ is said to have *locally finite length* when every edge $a \in X_1$ has finite length. That is, the pullback

$$\begin{tikzcd} (\vec{X}_r)_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \vec{X}_r \arrow[d, "g"] \\ 1 \arrow[r, "\ulcorner a \urcorner"'] & X_1 \end{tikzcd}$$

is empty for $r \gg 0$. We shall also use the word *tight* as synonym for 'complete and of locally finite length', to avoid confusion with the notion of 'locally finite' introduced in Section 7.

**Example 6.3.** For posets, the notion of locally finite length coincides with the classical notion (see for example Stern [33]), namely that for every $x \leq y$, there is an upper bound on the possible lengths of chains from $x$ to $y$. When $X$ is the strict (resp. fat) nerve of a category, locally finite length means that for each arrow $a$, there is an upper bound on the length of factorisations of $a$ containing no identity (resp. invertible) arrows.

A paradigmatic non-example is given by the strict nerve of a category containing an idempotent non-identity endo-arrow, $e = e \circ e$: clearly $e$ admits arbitrarily long decompositions $e = e \circ \cdots \circ e$.

**6.4. Example (continued).** The decomposition space $\mathbf{G}$ of finite graphs (Example 1.13) is of locally finite length, since a graph $G \in \mathbf{G}_1$ with $k$ vertices can have at most $k$ nonempty layers. (For similar reasons, many other examples of decomposition spaces of combinatorial nature have locally finite length [14].)

**Proposition 6.5.** *If $f : Y \to X$ is CULF and $X$ is a tight decomposition space, then also $Y$ is tight.*

**Proof.** Since $X$ is a decomposition space and since $f$ is CULF, also $Y$ is a decomposition space ([11, Lemma 4.6]), and the CULF condition ensures that $Y$ is furthermore complete, because the $s_0$ of $Y$ is the pullback of the $s_0$ of $X$. Finally, $Y$ is also tight by Proposition 2.11. □

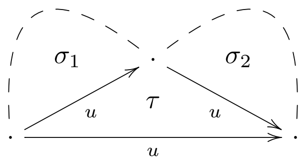
**Proposition 6.6.** *A tight decomposition space is split.*

**Proof.** A tight decomposition space $X$ is in particular complete and stiff, so by Lemmas 5.7 and 5.8 it is enough to prove that for $r = 2$ we have a pullback square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d, "s_0"] \\ \vec{X}_r \arrow[r, "g"'] & X_1, \end{tikzcd}$$

where $g : X_r \to X_1$ is the unique active map (or equivalently, the long-edge map $g$ preserves nondegenerate simplices). We actually prove this for $r \geq 2$. Suppose that

$\sigma \in \vec{X}_r$ has degenerate long edge $u = g\sigma$. The idea is to exploit the decomposition-space axiom to glue together two copies of $\sigma$, called $\sigma_1$ and $\sigma_2$, to get a bigger nondegenerate simplex $\sigma_1 \# \sigma_2 \in \vec{X}_{r+r}$ again with long edge $u$. By repeating this construction we obtain a contradiction to the finite length of $u$. It is essential for this construction that $u$ is degenerate, say $u = s_0 x$, because we glue along the 2-simplex $\tau = s_0 u = s_1 u = s_0 s_0 x$ which has the property that all three edges are $u$. Here is a picture of the gluing:




To formalise this, consider the diagram

$$\begin{tikzcd}
X_{r+r} \arrow[r, "{d_1}^{r-1}"'] \arrow[rr, bend left, "{d_\perp}^r"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, "{d_\top}^r"'] & X_{r+1} \arrow[r, "d_\perp"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[d, "{d_2}^{r-1}"] \arrow[dd, bend right, "{d_\top}^r"'] & X_r \arrow[d, "g={d_1}^{r-1}"] \\
& X_2 \arrow[r, "d_\perp"'] \arrow[d, "d_\top"] & X_1 \\
X_r \arrow[r, "g={d_1}^{r-1}"'] & X_1 .
\end{tikzcd}$$

The two squares are pullbacks since $X$ is a decomposition space, and the triangles are simplicial identities. In the right-hand square we have $\tau \in X_2$ and $\sigma_2 \in X_r$, with $d_\perp \tau = u = g\sigma_2$. Hence we get a simplex $\rho \in X_{r+1}$. This simplex has ${d_\top}^r \rho = d_\top \tau = u$, which means that in the left-hand square it matches $\sigma_1 \in X_r$, to produce altogether the desired simplex $\sigma_1 \# \sigma_2 \in X_{r+r}$. By construction, this simplex belongs to $\vec{X}_{r+r}$: indeed, its first $r$ principal edges are the principal edges of $\sigma_1$, and its last $r$ principal edges are those of $\sigma_2$. Its long edge is clearly the long edge of $\tau$, namely $u$ again, so we have produced a longer decomposition of $u$ than the one given by $\sigma$, thus contradicting the finite length of $u$. $\square$

Alternative characterisations of the length of an edge in a tight decomposition space can now be given:

**Proposition 6.7.** *Let $X$ be a tight decomposition space, and $f \in X_1$. Then the following conditions on $r \in \mathbb{N}$ are equivalent:*

(1) *For all words $w$ in the alphabet $\{0, a\}$ with $|w|_a \geq r+1$ (that is, the letter $a$ occurs at least $r+1$ times in $w$), the fibre $(X_w)_f$ is empty,*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_w \arrow[d] \\ 1 \arrow[r, "\ulcorner f \urcorner"'] & X_1. \end{tikzcd}$$

(2) *For all $k \geq r+1$, the fibre $(\vec{X}_k)_f$ is empty.*
(3) *The fibre $(\vec{X}_{r+1})_f$ is empty.*

*The length $\ell(f)$ of an edge in a tight decomposition space is the least $r \in \mathbb{N}$ satisfying these equivalent conditions.*

**Proof.** Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and, by definition, the length of $f$ is the least integer $r$ satisfying (2). It remains to show that (3) implies (1). Suppose (1) is false, that is, we have $w \in \{0, a\}^n$ with $k \geq r+1$ occurrences of $a$ and an element $\sigma \in X_w$ with $g(\sigma) = f$. Then by Corollary 2.17 we know that $\sigma$ is an $(n-k)$-fold degeneracy of some $\tau \in \vec{X}_k$, and $\sigma$ and $\tau$ will have the same long edge $f$. Finally we see that (3) is false by considering the element ${d_1}^{k-r-1}\tau \in X_{r+1}$, which has long edge $f$, and is nondegenerate (and hence effective) since $\tau$ is and face maps preserve nondegenerate simplices (as $X$ is split by Proposition 6.6). $\square$

**6.8. The length filtration of the space of 1-simplices.** Let $X$ be a tight decomposition space. We define the $k$th stage of the *length filtration* for 1-simplices to consist of all the edges of length at most $k$:

$$X_1^{(k)} := \{f \in X_1 \mid \ell(f) \leq k\}.$$

Then $X_1^{(k)}$ is the full sub-$\infty$-groupoid of $X_1$ given by any of the following equivalent definitions:

(1) the complement of $\mathrm{Im}(\vec{X}_{k+1} \to X_1)$.
(2) the complement of $\mathrm{Im}(\coprod_{|w|_a > k} X_w \to X_1)$.
(3) the full sub-$\infty$-groupoid of $X_1$ whose objects $f$ satisfy $(X_{k+1})_f \subset \bigcup s_i X_k$
(4) the full sub-$\infty$-groupoid of $X_1$ whose objects $f$ satisfy $(\vec{X}_{k+1})_f = \varnothing$
(5) the full sub-$\infty$-groupoid of $X_1$ whose objects $f$ satisfy $(X_w)_f = \varnothing$ for all $w \in \{0, a\}^r$ such that $|w|_a > k$

It is clear from the definition of length that we have a sequence of monomorphisms

$$X_1^{(0)} \hookrightarrow X_1^{(1)} \hookrightarrow X_1^{(2)} \hookrightarrow \ldots \hookrightarrow X_1.$$

The following is now clear.

**Proposition 6.9.** *A complete decomposition space is tight if and only if the* $X_1^{(k)}$ *constitute a filtration, i.e.*

$$X_1 = \bigcup_{k=0}^{\infty} X_1^{(k)}.$$

**6.10. Length filtration of a tight decomposition space.** Now define the length filtration for all of $X$: the length of a simplex $\sigma$ with longest edge $g\sigma = a$ is defined to be the length of $a$:

$$\ell(\sigma) := \ell(a).$$

In other words, we are defining the filtration in $X_r$ by pulling it back from $X_1$ along the unique active map $X_r \to X_1$. This automatically defines the active maps in each filtration degree, yielding an active-map complex

$$X_\bullet^{(k)} : \mathbb{\Delta}^{\text{op}}_{\text{active}} \to \mathcal{S}.$$

To get the outer face maps, the idea is simply to restrict (since by construction all the maps $X_1^{(k)} \hookrightarrow X_1^{(k+1)}$ are monos). We need to check that an outer face map applied to a simplex in $X_n^{(k)}$ again belongs to $X_{n-1}^{(k)}$. This will be the content of Proposition 6.11 below. Once we have done that, it is clear that we have a sequence of CULF maps

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

and we shall see that $X_\bullet^{(0)}$ is the constant simplicial space $X_0$.

**Proposition 6.11.** *In a tight decomposition space $X$, face maps preserve length: precisely, for any face map $d : X_{n+1} \to X_n$, if $\sigma \in X_{n+1}^{(k)}$, then $d\sigma \in X_n^{(k)}$.*

**Proof.** Since the length of a simplex only refers only to its long edge, and since an active face map does not alter the long edge, it is enough to treat the case of outer face maps, and by symmetry it is enough to treat the case of $d_\top$. Let $f$ denote the long edge of $\sigma$. Let $\tau$ denote the triangle ${d_1}^{n-1}\sigma$. It has long edge $f$ again. Let $u$ and $v$ denote the short edges of $\tau$,

$$\begin{tikzcd} & \cdot \arrow[dr, "v"] & \\ \cdot \arrow[ur, "u"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=4, "\tau"] & & \cdot \end{tikzcd}$$

that is $v = d_\perp \tau = {d_\perp}^n \sigma$ and $u = d_\top \tau$, the long edge of $d_\top \sigma$. The claim is that if $\ell(f) \le k$, then $\ell(u) \le k$. If we were in the category case, this would be true since any decomposition of $u$ could be turned into a decomposition of $f$ of at least the same length, simply by postcomposing with $v$. In the general case, we have to invoke the decomposition-space condition to glue with $\tau$ along $u$. Precisely, for any simplex $\kappa \in X_w$ with long edge $u$ we can obtain a simplex $\kappa \#_u \tau \in X_{w1}$ with long edge $f$: since $X$ is a decomposition space, we have a pullback square

$$\begin{tikzcd} \kappa\#_u\tau \in X_{w1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_w \ni \kappa \arrow[d, "g"] \\ \tau \in X_2 \arrow[r, "d_\top"'] \arrow[d, "d_1"'] & X_1 \ni u \\ f \in X_1 & \end{tikzcd}$$

and $d_\top \tau = u = g(\kappa)$, giving us the desired simplex in $X_{w1}$. With this construction, any simplex $\kappa$ of length $> k$ violating $\ell(u) = k$ (cf. the characterisation of length given in (1) of Proposition 6.7) would also yield a simplex $\kappa\#_u\tau$ (of at least the same length) violating $\ell(f) = k$. $\square$

The following is an immediate consequence of Proposition 5.12.

**Corollary 6.12.** *For a tight decomposition space $X$ we have $X_n^{(0)} = X_0$ for all $n$.* $\square$

**6.13. Coalgebra filtration.** If $X$ is a tight decomposition space, the sequence of CULF maps

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

defines coalgebra homomorphisms

$$\mathcal{S}_{/X_1^{(0)}} \to \mathcal{S}_{/X_1^{(1)}} \to \cdots \to \mathcal{S}_{/X_1}$$

which clearly define a coalgebra filtration of $\mathcal{S}_{/X_1}$.

Recall that a filtered coalgebra is called connected if its 0-stage coalgebra is the trivial coalgebra (the ground ring). In the present situation the 0-stage is $\mathcal{S}_{/X_1^{(0)}} \simeq \mathcal{S}_{/X_0}$, so we see that $\mathcal{S}_{/X_1}$ is connected if and only if $X_0$ is contractible.

On the other hand, the 0-stage elements are precisely the degenerate edges, which almost tautologically are group-like. Hence the incidence coalgebra of a tight decomposition space will always have the property that the 0-stage is spanned by group-like elements. For some purposes, this property is nearly as good as being connected (cf. [21], [22] for this viewpoint in the context of renormalisation).

**6.14. Grading.** Given a 2-simplex $\sigma \in X_2$ in a complete decomposition space $X$, it is clear that we have

$$\ell(d_2\sigma) + \ell(d_0\sigma) \leq \ell(d_1\sigma)$$

generalising the case of a category, where $f = ab$ implies $\ell(a)+\ell(b) \leq \ell(f)$. In particular, the following configuration of edges illustrates that one does not in general have equality:

$$\begin{tikzcd}
& \cdot \arrow[rr] & & \cdot \arrow[dr] & \\
\cdot \arrow[ur] \arrow[rrrr, "f"] \arrow[drr, "a"'] & & & & \cdot \\
& & \cdot \arrow[urr, "b"'] & &
\end{tikzcd}$$

Provided none of the edges can be decomposed further, we have $\ell(f) = 3$, but $\ell(a) = \ell(b) = 1$. For the same reason, the length filtration is not in general a grading: $\Delta(f)$ contains the term $a \otimes b$ of degree splitting $1+1 < 3$. Nevertheless, it is actually common in examples of interest to have a grading: this happens when all maximal chains composing to a given edge $f$ have the same length, $\ell(f)$. Many examples from combinatorics have this property [14].

The abstract formulation of the condition for the length filtration to be a grading is this: For every $k$-simplex $\sigma \in X_k$ with long edge $a$ and principal edges $e_1, \ldots, e_k$, we have

$$\ell(a) = \ell(e_1) + \cdots + \ell(e_k).$$

Equivalently, for every 2-simplex $\sigma \in X_2$ with long edge $a$ and short edges $e_1$, $e_2$, we have

$$\ell(a) = \ell(e_1) + \ell(e_2).$$

The length filtration is a grading if and only if the functor $\ell : X_1 \to \mathbb{N}$ extends to a simplicial map to the nerve of the monoid $(\mathbb{N}, +)$ (this map is rarely CULF, though).

If $X$ is the nerve of a poset $P$, then the length filtration is a grading if and only if $P$ is *ranked*, i.e. for any $x, y \in P$, every maximal chain from $x$ to $y$ has the same length [32].

**6.15. Example (continued).** The decomposition space $\mathbf{G}$ of finite graphs (Example 1.13) is graded by the number of vertices.

## 7. Locally finite decomposition spaces

In order to be able to take homotopy cardinality of the $\mathcal{S}$-coalgebra obtained from a decomposition space $X$ to get a coalgebra at the numerical level (vector spaces), we need to impose certain finiteness conditions on $X$. Firstly, just for the coalgebra structure to have a homotopy cardinality, we need $X$ to be *locally finite* (7.4) but it is not necessary that $X$ be complete. Secondly, in order for Möbius inversion to admit a homotopy cardinality, what we need in addition is precisely the filtration condition (which in turn assumes completeness). We shall define a *Möbius decomposition space* to be a locally finite tight decomposition space (8.3).

We begin with a few reminders on finiteness of $\infty$-groupoids.

**7.1. Finiteness conditions for $\infty$-groupoids.** (Cf. [10].) An $\infty$-groupoid $B$ is *locally finite* if at each base point $b$ the homotopy groups $\pi_i(B, b)$ are finite for $i \geq 1$ and are trivial for $i$ sufficiently large. It is called *finite* if furthermore it has only finitely many components. We denote by $\mathcal{F}$ the $\infty$-category of finite $\infty$-groupoids. A map of $\infty$-groupoids is *finite* if its fibres are [10, §3]. The role of vector spaces is played by finite-$\infty$-groupoid slices $\mathcal{F}_{/B}$ (where $B$ is a locally finite $\infty$-groupoid), while the role of profinite-dimensional vector spaces is played by finite-presheaf $\infty$-categories $\mathcal{F}^B$. Linear maps are given by spans of *finite type*, meaning $A \xleftarrow{p} M \xrightarrow{q} B$ in which $p$ is a finite map. Prolinear maps are given by spans of *profinite type*, where $q$ is a finite map. Inside the $\infty$-category ***LIN***, we have two $\infty$-categories: $\underrightarrow{\boldsymbol{lin}}$ whose objects are the finite-$\infty$-groupoid slices $\mathcal{F}_{/B}$ and whose mapping spaces are $\infty$-groupoids of finite-type spans, and the $\infty$-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are finite-presheaf $\infty$-categories $\mathcal{F}^B$, and whose mapping spaces are $\infty$-groupoids of profinite-type spans.

We shall also need $\mathcal{S}_{/B}^{\text{rel.fin.}}$, the full subcategory of $\mathcal{S}_{/B}$ spanned by the finite maps $p : X \to B$, and $\mathcal{F}^B_{\text{fin.sup.}}$, the full subcategory of $\mathcal{S}^B$ spanned by presheaves with finite values and finite support. By the support of a presheaf $F : B \to \mathcal{S}$ we mean the full sub-$\infty$-groupoid of $B$ spanned by the objects $b$ for which $F(b) \neq \varnothing$.

**Proposition 7.2** *(Cf. [10, Proposition 4.3]). For a span $A \xleftarrow{p} M \xrightarrow{q} B$ of locally finite $\infty$-groupoids, the following are equivalent.*

(1) *$p$ is finite.*
(2) *The linear functor $F := q_! \circ p^* : \mathcal{S}_{/A} \to \mathcal{S}_{/B}$ restricts to*
$$\mathcal{F}_{/A} \xrightarrow{p^*} \mathcal{F}_{/M} \xrightarrow{q_!} \mathcal{F}_{/B}.$$
(3) *The transpose $F^t := p_! \circ q^* : \mathcal{S}_{/B} \to \mathcal{S}_{/A}$ restricts to*
$$\mathcal{S}_{/B}^{\text{rel.fin.}} \xrightarrow{q^*} \mathcal{S}_{/M}^{\text{rel.fin.}} \xrightarrow{p_!} \mathcal{S}_{/A}^{\text{rel.fin.}}.$$
(4) *The dual functor $F^\vee : \mathcal{S}^B \to \mathcal{S}^A$ restricts to*
$$\mathcal{F}^B \to \mathcal{F}^A.$$
(5) *The dual of the transpose, $F^{t\vee} : \mathcal{S}^A \to \mathcal{S}^B$ restricts to*
$$\mathcal{F}^A_{\text{fin.sup.}} \to \mathcal{F}^B_{\text{fin.sup.}}.$$

**7.3. (Homotopy) cardinality.** (Cf. [1,10]) The homotopy cardinality of a finite $\infty$-groupoid $B$ is by definition

$$|B| := \sum_{b \in \pi_0 B} \prod_{i>0} |\pi_i(B,b)|^{(-1)^i}.$$

Here the norm signs on the right refer to order of homotopy groups. From now on we will just say *cardinality* for homotopy cardinality.

For each locally finite $\infty$-groupoid $B$, there is a 'relative' notion of cardinality

$$|\ | : \mathcal{F}_{/B} \longrightarrow \mathbb{Q}_{\pi_0 B},$$

sending a basis element $\ulcorner b \urcorner$ to the basis element $|\ulcorner b \urcorner| := \delta_b$ corresponding to $b \in \pi_0 B$. The delta notation for these basis elements is useful to keep track of the level of discourse.

Dually, there is a notion of cardinality $|\ | : \mathcal{F}^B \to \mathbb{Q}^{\pi_0 B}$. The profinite-dimensional vector space $\mathbb{Q}^{\pi_0 B}$ is spanned by the characteristic functions $\delta^b = \frac{|h^b|}{|\Omega(B,b)|}$, the cardinality of the representable functor $h^b$ divided by the cardinality of the loop space.

**7.4. Locally finite decomposition spaces.** A decomposition space $X : \mathbb{\Delta}^{\text{op}} \to \mathcal{S}$ is called *locally finite* if $X_1$ is locally finite and both $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps.

**Lemma 7.5.** *Let $X$ be a decomposition space.*

(1) *If $s_0 : X_0 \to X_1$ is finite then so are all degeneracy maps $s_i : X_n \to X_{n+1}$.*
(2) *If $d_1 : X_2 \to X_1$ is finite then so are all active face maps $d_j : X_n \to X_{n-1}$, $j \neq 0, n$.*
(3) *$X$ is locally finite if and only if $X_n$ is locally finite for every $n$ and $g : X_m \to X_n$ is finite for every active map $g : [n] \to [m]$ in $\mathbb{\Delta}$.*

**Proof.** Since finite maps are stable under pullback [10, Lemma 3.13], both (1) and (2) follow from Lemma 1.10.

Re (3): If $X$ is locally finite, then by definition $X_1$ is locally finite, and for each $n \in \mathbb{N}$ the unique active map $X_n \to X_1$ is finite by (1) or (2). It follows that $X_n$ is locally finite [10, Lemma 3.15]. The converse implication is trivial. $\square$

**7.6. Remark.** If $X$ is the nerve of a poset $P$, then it is locally finite in the above sense if and only if it is locally finite in the usual sense of posets [32], viz. for every $x, y \in P$, the interval $[x, y]$ is finite. The points in this interval parametrise precisely the two-step factorisations of the unique arrow $x \to y$, so this condition amounts to $X_2 \to X_1$ having finite fibre over $x \to y$. (The condition $X_1$ locally finite is void in this case, as any discrete set is locally finite; the condition on $s_0 : X_0 \to X_1$ is also void in this case, as it is always just an inclusion.)

For posets, 'locally finite' implies 'locally finite length'. (The converse is not true: take an infinite set, considered as a discrete poset, and adjoin a top and a bottom element: the result is of locally finite length but not locally finite.) Already for categories, it is not true that locally finite implies locally finite length: for example the strict nerve of a finite group is locally finite but not of locally finite length.

**7.7. Example (continued).** The decomposition space $\mathbf{G}$ of finite graphs (Example 1.13) is locally finite. Indeed, $\mathbf{G}_1$ is locally finite since a finite graph has finite automorphism group; the map $s_0 : \mathbf{G}_0 \to \mathbf{G}_1$ is finite since it is a monomorphism (see Example 2.4), and $d_1 : \mathbf{G}_2 \to \mathbf{G}_1$ is finite since a given graph admits only finitely many different 2-layerings. (For similar reasons, many other examples of decomposition spaces of combinatorial nature are locally finite [14].)

**7.8. Numerical incidence algebra.** It follows from Proposition 7.2 that, for any locally finite decomposition space $X$, the comultiplication maps

$$\Delta_n : \mathcal{S}_{/X_1} \longrightarrow \mathcal{S}_{/X_1\times X_1\times\cdots\times X_1}$$

given for $n \geq 0$ by the spans

$$X_1 \xleftarrow{\;m\;} X_n \xrightarrow{\;p\;} X_1 \times X_1 \times \cdots \times X_1$$

restrict to linear functors

$$\Delta_n : \mathcal{F}_{/X_1} \longrightarrow \mathcal{F}_{/X_1\times X_1\times\cdots\times X_1}.$$

The linear functors $\Delta_2$ and $\Delta_0$ are just the comultiplication $\Delta$ and the counit $\varepsilon$ of Theorem 1.11,

$$\mathcal{F}_{/X_1} \xrightarrow{\;\Delta\;} \mathcal{F}_{/X_1\times X_1}, \qquad\qquad \mathcal{F}_{/X_1} \xrightarrow{\;\varepsilon\;} \mathcal{F}.$$

and we can take their cardinality to obtain a coalgebra structure,

$$\mathbb{Q}_{\pi_0 X_1} \xrightarrow{\;|\Delta|\;} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1}, \qquad\qquad \mathbb{Q}_{\pi_0 X_1} \xrightarrow{\;|\varepsilon|\;} \mathbb{Q}$$

termed the *numerical incidence coalgebra* of $X$.

**7.9. Morphisms.** It is worth noticing that for *any* CULF functor $F : Y \to X$ between locally finite decomposition spaces, the induced coalgebra homomorphism $F_! : \mathcal{S}_{/Y_1} \to \mathcal{S}_{/X_1}$ restricts to a functor $\mathcal{F}_{/Y_1} \to \mathcal{F}_{/X_1}$. In other words, there are no further finiteness conditions to impose on CULF functors.

**7.10. Numerical convolution product.** By duality, if $X$ is locally finite, the convolution product descends to the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$ obtained by taking cardinality of $\mathcal{F}^{X_1}$. It follows from the general theory of homotopy linear algebra (see [10]) that the cardinality of the convolution product is the linear dual of the cardinality of the comultiplication. Since it is the same span that defines the comultiplication and the convolution product, it is also the exact same matrix that defines the cardinalities of these two maps. It follows that the structure constants for the convolution product (with respect to the pro-basis $\{\delta^x\}$) are the same as the structure constants for the comultiplication (with respect to the basis $\{\delta_x\}$). These are classically called the section coefficients, and we proceed to derive formulae for them in simple cases.

Let $X$ be a locally finite decomposition space. The comultiplication at the objective level

$$\begin{aligned} \mathcal{F}_{/X_1} &\longrightarrow \mathcal{F}_{/X_1 \times X_1} \\ \ulcorner f \urcorner &\longmapsto \big[ R_f : (X_2)_f \to X_2 \to X_1 \times X_1 \big] \end{aligned}$$

yields a comultiplication of vector spaces by taking cardinality (remembering that $|\ulcorner f \urcorner| = \delta_f$):

$$\begin{aligned} \mathbb{Q}_{\pi_0 X_1} &\longrightarrow \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \delta_f &\longmapsto |R_f| \\ &= \int^{(a,b) \in X_1 \times X_1} |(X_2)_{f,a,b}| \, \delta_a \otimes \delta_b \\ &= \sum_{a,b} \big|(X_1)_{[a]}\big| \, \big|(X_1)_{[b]}\big| \, |(X_2)_{f,a,b}| \, \delta_a \otimes \delta_b, \end{aligned}$$

where $(X_2)_{f,a,b}$ is the fibre over the three face maps. The integral sign is a sum weighted by homotopy groups. These weights together with the cardinality of the triple fibre are called the *section coefficients*, denoted

$$c^f_{a,b} := |(X_2)_{f,a,b}| \cdot \big|(X_1)_{[a]}\big| \, \big|(X_1)_{[b]}\big| \, .$$

In the case where $X$ is a Segal space (and even more, when $X_0$ is a 1-groupoid), we can be very explicit about the section coefficients. For a Segal space we have $X_2 \simeq X_1 \times_{X_0} X_1$, which helps to compute the fibre of $X_2 \to X_1 \times X_1$:

**Lemma 7.11.** *The pullback*

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & X_1 \times_{X_0} X_1 \arrow[d] \\ 1 \arrow[r, "\ulcorner (a,b) \urcorner"'] & X_1 \times X_1 \end{tikzcd}$$

*is given by*

$$S \;\simeq\; \mathrm{Map}_{X_0}(d_0 a, d_1 b) \;\simeq\; \begin{cases} \Omega(X_0, y) & \textit{if } d_0 a \simeq y \simeq d_1 b \\ \varnothing & \textit{else.} \end{cases}$$

**Proof.** We can compute the pullback as

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & X_1 \times_{X_0} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & X_0 \arrow[d, "\mathrm{diag}"] \\ 1 \arrow[r, "\ulcorner (a,b) \urcorner"'] & X_1 \times X_1 \arrow[r, "d_0 \times d_1"'] & X_0 \times X_0, \end{tikzcd}$$

and the result follows since the fibre of the diagonal is the mapping space. □

**Corollary 7.12.** *Suppose $X$ is a Segal space, and that $X_0$ is a 1-groupoid. Given $a, b, f \in X_1$ such that $d_0 a \cong y \cong d_1 b$ and $ab = f$, then we have*

$$(X_2)_{f,a,b} \simeq \Omega(X_0, y) \times \Omega(X_1, f).$$

**Proof.** In this case, since $X_0$ is a 1-groupoid, the fibres of the diagonal map $X_0 \to X_0 \times X_0$ are 0-groupoids. Thus the fibre of the previous lemma is the discrete space $\Omega(X_0, y)$. When now computing the fibre over $f$, we are taking that many copies of the loop space of $f$. □

**Corollary 7.13.** *With notation as above, the section coefficients for a locally finite Segal 1-groupoid are*

$$c^{ab}_{a,b} = \frac{|\mathrm{Aut}(y)|\,|\mathrm{Aut}(ab)|}{|\mathrm{Aut}(a)|\,|\mathrm{Aut}(b)|}.$$

Coassociativity of the incidence coalgebra says that the section coefficients $\{c^{ab}_{a,b}\}$ form a *2-cocycle*,

$$c^{ab}_{a,b} c^{abc}_{ab,c} = c^{bc}_{b,c} c^{abc}_{a,bc}.$$

In fact this cocycle is cohomologically trivial, given by the coboundary of a 1-cochain,

$$c_{a,b}^{ab} = \partial(\phi)(a,b) = \phi(a)\phi(ab)^{-1}\phi(b).$$

In fact, if one fixes $s$, $t$ such that $s+t=1$, the 1-cochain may be taken to be

$$\phi(x \xrightarrow{a} y) = \frac{|\operatorname{Aut}(x)|^s |\operatorname{Aut}(y)|^t}{|\operatorname{Aut}(a)|}.$$

**7.14. 'Zeroth section coefficients': the counit.** Let us also say a word about the zeroth section coefficients, i.e. the computation of the counit: the main case is when $X$ is complete (in the sense that $s_0$ is a monomorphism). In this case, clearly we have

$$\varepsilon(f) = \begin{cases} 1 & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

If $X$ is Rezk complete, the first condition is equivalent to being invertible.

The other easy case is when $X_0 \simeq 1$. In this case

$$\varepsilon(f) = \begin{cases} |\Omega(X_1, f)| & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

**7.15. Example.** The strict nerve of a 1-category $\mathcal{C}$ is a decomposition space which is discrete in each degree. The resulting coalgebra at the numerical level (assuming the due finiteness conditions) is the coalgebra of Content–Lemay–Leroux [4], and if the category is just a poset, that of Rota et al. [16].

For the fat nerve $X$ of $\mathcal{C}$, we find

$$h^a * h^b \simeq \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ \varnothing & \text{else,} \end{cases}$$

as follows from Lemma 7.11. Note that the cardinality of the representable $h^a$ is generally different from the canonical basis element $\delta^a$.

**7.16. Finite support.** It is also interesting to consider the subalgebra of the incidence algebra consisting of functions with finite support, i.e. the full subcategory $\mathcal{F}^{X_1}_{\text{fin.sup.}} \subset \mathcal{F}^{X_1}$, and numerically $\mathbb{Q}^{\pi_0 X_1}_{\text{fin.sup.}} \subset \mathbb{Q}^{\pi_0 X_1}$. Of course we have canonical identifications $\mathcal{F}^{X_1}_{\text{fin.sup.}} \simeq \mathcal{F}_{/X_1}$, as well as $\mathbb{Q}^{\pi_0 X_1}_{\text{fin.sup.}} \simeq \mathbb{Q}_{\pi_0 X_1}$, but it is important to keep track of which side of duality we are on.

That the decomposition space is locally finite is not the appropriate condition for these subalgebras to exist. Instead, for the convolution product to descend to functors with finite support, the requirement is that $X_1$ be locally finite and the functor

$$X_2 \to X_1 \times X_1$$

be finite. (This is always the case for a locally finite Segal 1-groupoid, by Lemma 7.11.) Similarly, one can ask for the convolution unit to have finite support, which is to require $X_0 \to 1$ to be a finite map.

Dually, the same conditions ensure that comultiplication and counit extend from $\mathcal{F}_{/X_1}$ to $\mathcal{S}^{\text{rel.fin.}}_{/X_1}$, which numerically is some sort of vector space of summable infinite linear combinations [10, 6.8]. An example of this situation is given by the bialgebra of $P$-trees (actually $P$-forests) [20], whose comultiplication does extend to $\mathcal{S}^{\text{rel.fin.}}_{/X_1}$, but whose counit does not (as there are infinitely many $P$-forests without nodes). Importantly, this non-counital coalgebra is the home for the so-called Green function, an infinite (homotopy) sum of trees, and for the Faà di Bruno formula it satisfies, which does not hold for any finite truncation. See [8] for these results.

**7.17. Examples.** If $X$ is the strict nerve of a 1-category $\mathcal{C}$, then the finite-support convolution algebra is precisely the *category algebra* of $\mathcal{C}$. (For a finite category, of course the two notions coincide.)

Note that the convolution unit is

$$\varepsilon = \sum_x \delta^{\mathrm{id}_x} = \begin{cases} 1 & \text{for id arrows} \\ 0 & \text{else,} \end{cases}$$

the sum of all indicator functions of identity arrows, so it will be finite if and only if the category has only finitely many objects.

In the case of the fat nerve of a 1-category, the finiteness condition for comultiplication is implied by the condition that every object has a finite automorphism group (a condition implied by local finiteness). On the other hand, the convolution unit has finite support precisely when there is only a finite number of isoclasses of objects, already a more drastic condition. Note the 'category algebra' interpretation: compared to the usual category algebra there is a symmetry factor (cf. Lemma 7.15):

$$h^a * h^b \simeq \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ \varnothing & \text{else.} \end{cases}$$

Finally, the finite-support incidence algebras are important in the case of the Waldhausen $S$-construction: they are the Hall algebras (see [11]). The finiteness conditions are then homological, namely finite $\mathrm{Ext}^0$ and $\mathrm{Ext}^1$.

## 8. Möbius decomposition spaces

We finally come to the Möbius condition, which ensures the Möbius inversion principle descends to the numerical level.

Recall that $\mathcal{F}$ denotes the $\infty$-category of finite $\infty$-groupoids, as defined in 7.1.

**Lemma 8.1.** *If $X$ is a complete decomposition space then the following conditions are equivalent*

(1) $d_1 : X_2 \to X_1$ *is finite.*
(2) $d_1 : \vec{X}_2 \to X_1$ *is finite.*
(3) $d_1^{r-1} : \vec{X}_r \to X_1$ *is finite for all $r \geq 2$.*

**Proof.** We show the first two conditions are equivalent; the third is similar. Using the word notation of 2.8 we consider the map

$$\vec{X}_2 + \vec{X}_1 + \vec{X}_1 + X_0 \xrightarrow{\simeq} \vec{X}_2 + X_{0a} + X_{a0} + X_{00} \xrightarrow{=} X_2 \xrightarrow{d_1} X_1$$

Thus $d_1 : X_2 \to X_1$ is finite if and only if the restriction of this map to the first component, $d_1 : \vec{X}_2 \to X_1$, is finite. By completeness the restrictions to the other components are finite (in fact, mono). $\square$

**Corollary 8.2.** *A complete decomposition space $X$ is locally finite if and only if $X_1$ is locally finite and $d_1^{r-1} : \vec{X}_r \to X_1$ is finite for all $r \geq 2$.*

**8.3. Möbius condition.** A complete decomposition space $X$ is called *Möbius* if it is locally finite and tight (i.e. of locally finite length). It then follows that the restricted composition map

$$\sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$$

is finite. In other words, the spans defining $\Phi_{\text{even}}$ and $\Phi_{\text{odd}}$ are of finite type, and hence descend to the finite slices $\mathcal{F}_{/X_1}$. In fact we have:

**Lemma 8.4.** *A complete decomposition space $X$ is Möbius if and only if $X_1$ is locally finite and the restricted composition map*

$$\sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$$

*is finite.*

**Proof.** 'Only if' is clear. Conversely, if the map $m : \sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$ is finite, in particular for each individual $r$ the map $\vec{X}_r \to X_1$ is finite, and then also $X_r \to X_1$ is finite, by Lemma 8.1. Hence $X$ is altogether locally finite. But it also follows from finiteness of $m$ that for each $a \in X_1$, the fibre $(\vec{X}_r)_a$ must be empty for big enough $r$, so the filtration condition is satisfied, so altogether $X$ is Möbius. $\square$

**Remark 8.5.** If $X$ is a Segal space, the Möbius condition says that for each arrow $a \in X_1$, the factorisations of $a$ into nondegenerate $a_i \in \vec{X}_1$ have bounded length. In particular, if $X$ is the strict nerve of a 1-category, then it is Möbius in the sense of the previous definition if and only if it is Möbius in the sense of Leroux [25]. (Note however that this would also have been true if we had not included the condition that $X_1$ be locally finite (as obviously this is automatic for any discrete set). We insist on including the condition $X_1$ locally finite because it is needed in order to have a well-defined cardinality.)

**8.6. Filtered coalgebras in vector spaces.** A Möbius decomposition space is in particular length-filtered. The coalgebra filtration (6.13) at the objective level

$$\mathcal{S}_{/X_1^{(0)}} \to \mathcal{S}_{/X_1^{(1)}} \to \cdots \to \mathcal{S}_{/X_1}$$

is easily seen to descend to $\mathcal{F}$-coefficients (finite $\infty$-groupoids):

$$\mathcal{F}_{/X_1^{(0)}} \to \mathcal{F}_{/X_1^{(1)}} \to \cdots \to \mathcal{F}_{/X_1},$$

and taking cardinality then yields a coalgebra filtration at the numerical level too. From the arguments in 6.13, it follows that this coalgebra filtration

$$C_0 \hookrightarrow C_1 \hookrightarrow \cdots \hookrightarrow C$$

has the property that $C_0$ is generated by group-like elements. (This property has been found useful in the context of perturbative renormalisation [21], [22], where it serves as a basis for recursive arguments, as an alternative to the more common assumption of connectedness.) Finally, if $X$ is a graded Möbius decomposition space, then the resulting coalgebra at the algebraic level is furthermore a graded coalgebra.

The following is an immediate corollary to Lemma 5.14. It extends the classical fact that a Möbius category in the sense of Leroux does not have non-identity invertible arrows [23, Lemma 2.4].

**Corollary 8.7.** *Every Möbius decomposition space $X$ is Rezk complete.*

**8.8. Möbius inversion at the algebraic level.** Assume $X$ is a locally finite complete decomposition space. The span $X_1 \xleftarrow{=} X_1 \longrightarrow 1$ defines the zeta functor (cf. 3.2), which as a presheaf is $\zeta = \int^t h^t$, the homotopy sum of the representables. Its cardinality is the usual zeta function in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$.

The spans $X_1 \longleftarrow \vec{X}_r \longrightarrow 1$ define the Phi functors

$$\Phi_r : \mathcal{S}_{/X_1} \longrightarrow \mathcal{S},$$

with $\Phi_0 = \varepsilon$. By Lemma 8.1, these functors descend to

$$\Phi_r : \mathcal{F}_{/X_1} \longrightarrow \mathcal{F},$$

and we can take cardinality to obtain functions $|\zeta| : \pi_0(X_1) \to \mathbb{Q}$ and $|\Phi_r| : \pi_0(X_1) \to \mathbb{Q}$, elements in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$.

Finally, when $X$ is furthermore assumed to be Möbius, we can take cardinality of the abstract Möbius inversion formula of Theorem 3.8:

**Theorem 8.9.** *If $X$ is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|$:*

$$|\zeta| * |\mu| = |\varepsilon| = |\mu| * |\zeta| .$$

**8.10. Example (continued).** We have seen that the decomposition space $\mathbf{G}$ of finite graphs of Example 1.13 is complete, tight, and locally finite, (Examples 2.4, 6.4, and 7.7, respectively). Hence it is a Möbius decomposition space. The general Möbius inversion formula $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$ yields a Möbius inversion formula in the chromatic Hopf algebra, but at the numerical level this is not the most economical. The well-known cancellation-free formula $\mu(G) = (-1)^n$, where $n$ is the number of vertices of $G$, can be established by exploiting the CULF functor to the decomposition space of finite sets mentioned at the end of Example 1.13. This argument works more generally, for any decomposition space arising as a restriction species [13].

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