Journal of Pure and Applied Algebra 2 (1972) 169 - 191. North-Holland Publishing Company

CATEGORIES OF CONTINUOUS FUNCTORS, I

P.J. FREYD
University of Pennsylvania, Philadelphia, Pa., 19104, USA

and

G.M. KELLY
University of New South Wales, N.S.W., Australia

Communicated by A. Heller
Received 12 June 1971

§ 1. Introduction

1.1. *The continuous functor problem.* By a *cylinder* in a category C we mean a *small* category K, two functors $P, Q : K \to C$, and a natural transformation $\alpha : P \to Q$; we denote the cylinder for short by α. When P is the *constant* functor at some object N of C, the cylinder α is called a *cone*, with vertex N.

A functor $T : C \to A$ is said to be *continuous with respect to* α if $\lim TP$ and $\lim TQ$ exist in A, and if $\lim T\alpha : \lim TP \to \lim TQ$ is an isomorphism. When α is a cone with vertex N, this continuity just means that $T\alpha : TN \to TQ$ is a limit of TQ in A. If T is continuous with respect to each α in a class Γ of cylinders in C, we say that T is *continuous with respect to* Γ.

Note that all of our categories are supposed to be locally small, and that a class is said to be *small* if it is a set, and *large* otherwise; completeness means small completeness.

If Γ is a class of cylinders in a small category C, the functors $T : C \to A$ continuous with respect to Γ form a full subcategory $[C, A]_\Gamma$ of the functor category $[C, A]$, which is easily seen to be closed under limits in $[C, A]$; the primary aim of this paper is to give sufficient conditions for it to be a reflective subcategory. Since the conditions we give include the completeness and the cocompleteness of A, they will also be sufficient for the completeness and the cocompleteness of $[C, A]_\Gamma$.

In a subsequent paper we shall give a different set of sufficient conditions. The two sets at least come close to being mutually incompatible, and we have no common proof of the two results.

There are many examples of categories of the form $[C, A]_\Gamma$; for instance the category of A-valued sheaves on a site C. Often Γ consists of cones that are themselves limits; then the functors in $[C, A]_\Gamma$ are said to be those that *preserve* these limits. One such example is that of the left-exact functors $C \to A$, where C and A are abelian; another is that of the C-algebras in A, where C is a theory with rank – in this case the limits to be preserved are the products of dimension less than the rank. In the case A = **Sets**, Lambek [10] considered the category of those *T* that preserve *all* small limits in C, and raised the question of its cocompleteness; this is a case in which the class Γ is not small.

As for the history of the problem, certain special cases are classical; for instance, the reflectivity of sheaves of abelian groups among presheaves; or the reflectivity of left exact functors, from a small abelian category to **Ab**, among all additive functors. The general assertion that $[C, A]_\Gamma$ is reflective in $[C, A]$ if A is **Sets**, **Ab**, $\mathbf{Sets}^{op}$ or $\mathbf{Ab}^{op}$, and if Γ consists of cones that are limits, was made by the first author ([15, pp. 118–119]). A proof in the case A = **Sets**, Γ = any *small* family of cones, was given in the unpublished but widely distributed notes of Gabriel. A proof in the case A = **Sets**, Γ = all cones in C that are limits, was given by Kennison [9]. During the writing-up of this paper, we have received a preprint of Ulmer's summary [14] of his forthcoming book with Gabriel [6]; they give a proof when Γ is small and A is what they call a *locally presentable* category. In view of their characterization of these categories (see §3.2 below), the problem for a locally presentable A reduces to the problem for A = **Sets** as originally considered by Gabriel.

In the present paper we prove the reflectiveness of $[C, A]_\Gamma$ for a class of categories A bigger than that of the locally presentable ones, containing such non-locally-presentable categories as topological spaces or compactly generated spaces. We also allow Γ to be large, insisting only that the class of cylinders in Γ which are *not* cones be small. The subsequent paper will deal with cases where A^{op} has the kinds of properties that A has here, and will include such cases as A = $\mathbf{Sets}^{op}$, $\mathbf{Rings}^{op}$ or $\mathbf{Top}^{op}$.

Our results seem to bear some relation, not too well understood, to those of Barr [1] and of Schubert [12] on the cocompleteness of the algebras over a monad (= triple).

1.2. *The orthogonal subcategory problem.* When suitably formulated in the functor category [C, A], our problem appears as a special case of a more general one, having nothing to do with functor categories as such.

We say that a morphism $k : M \to N$ and an object *B* in a category A are *orthogonal*, writing $B \perp k$ or $k \perp B$, if the function $A(k, B) : A(N, B) \to A(M, B)$ is bijective; that is, if every $f : M \to B$ is of the form $f = gk$ for a unique $g : N \to B$. If Δ is a class of morphisms of A, write $\Delta^\perp$ for $\{B \in A \mid B \perp k \text{ for every } k \in \Delta\}$, or equally for the full subcategory of A with this class of objects. Similarly, if B is a class of objects of A, write $\mathsf{B}^{\perp}$ for the class of morphisms k in A orthogonal to every $B \in \mathsf{B}$.

If $F : \mathsf{K} \to \mathsf{A}$ is a functor with limit L, and if $k : M \to N$ is orthogonal to FK for each $K \in \mathsf{K}$, it follows at once from the fact that $\mathsf{A}(N, -)$ and $\mathsf{A}(M, -) : \mathsf{A} \to \mathbf{Sets}$ preserve limits that k is orthogonal to L; in other words

Proposition 1.2.1. $\Delta^{\perp}$ *is closed under limits in* A.

It follows therefore that, when A is complete, $\Delta^{\perp}$ will be reflective in A if the solution-set condition is satisfied. We shall give in §4 sufficient conditions for this to be so.

1.3. *Reduction of the first problem to the second*. If X is a set and $A \in \mathsf{A}$, we write $X \otimes A$ for the coproduct of X copies of A; so that by definition $\mathsf{A}(X \otimes A, B) \cong \mathbf{Sets}(X, \mathsf{A}(A, B))$. If $H : \mathsf{C} \to \mathbf{Sets}$ is a functor and $A \in \mathsf{A}$, we also write $H \otimes A : \mathsf{C} \to \mathsf{A}$ for the functor sending C to $HC \otimes A$.

Let $\alpha : P \to Q : \mathsf{K} \to \mathsf{C}$ be a cylinder in the small category C. Consider the functor $\hat{P} : \mathsf{K}^{\mathrm{op}} \to [\mathsf{C}, \mathbf{Sets}]$ which sends K to $\mathsf{C}(PK, -)$, and let $\tilde{P}$ be its colimit in $[\mathsf{C}, \mathbf{Sets}]$. Define $\hat{Q}$ and $\tilde{Q}$ similarly. Then α gives rise to a natural transformation $\hat{\alpha} : \hat{Q} \to \hat{P}$ with components $\hat{\alpha}_K = \mathsf{C}(\alpha_K, -)$, and passage to the colimit gives a morphism $\tilde{\alpha} : \tilde{Q} \to \tilde{P}$ in $[\mathsf{C}, \mathbf{Sets}]$. If A is a category with copowers, we get for each $A \in \mathsf{A}$ a morphism $\tilde{\alpha} \otimes A : \tilde{Q} \otimes A \to \tilde{P} \otimes A$ in $[\mathsf{C}, \mathsf{A}]$.

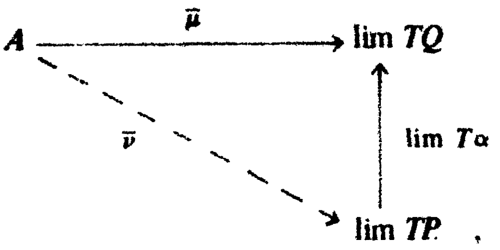
Proposition 1.3.1. *Let* A *be complete and cocomplete and let* Γ *be a class of cylinders in the small category* C. *Write* Δ *for the class of morphisms* $\tilde{\alpha} \otimes A : \tilde{Q} \otimes A \to \tilde{P} \otimes A$ *in* $[\mathsf{C}, \mathsf{A}]$, *where* $\alpha \in \Gamma$, $A \in \mathsf{A}$. *Then* $[\mathsf{C}, \mathsf{A}]_{\Gamma} = \Delta^{\perp}$.

Proof. To say that $T \in \Delta^{\perp}$ is to say that every $\mu : \tilde{Q} \otimes A \to T$ factorizes as follows for a unique ν:

$$
(1.1)\qquad
\begin{array}{ccc}
\tilde{Q} \otimes A & \xrightarrow{\ \mu\ } & T \\
{\scriptstyle \tilde{\alpha} \otimes A}\downarrow & \nearrow{\scriptstyle \nu} & \\
\tilde{P} \otimes A & &
\end{array}
$$

Such morphisms μ correspond bijectively to morphisms $\mu' : \tilde{Q} \to \mathsf{A}(A, T-)$; and these correspond bijectively to morphisms $\mu'' : \hat{Q}K \to \mathsf{A}(A, T-)$ which are natural in K (when $\mathsf{A}(A, T-)$ is considered as a functor of K which is independent of K).

Since $\hat{Q}K = C(QK, -)$, the Yoneda lemma shows that such μ'' correspond bijectively to morphisms $\mu''' : A \to TQK$ that are natural in K, and thus finally to morphisms $\bar{\mu} : A \to \lim TQ$. Since the diagram (1.1) translates into the diagram



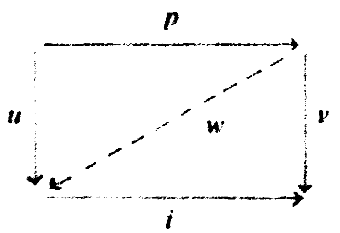
to say that $T \in \Delta^{\perp}$ is indeed to say that $T \in [C, A]_{\Gamma}$.

1.4. *Factorizations.* The formulation of our sufficient conditions involves in an essential way a chosen proper *factorization* (E, M) on the category A; this is the same thing as a *bicategorical structure* in the sense of Isbell [7]. Among the properties of these, we need the closure properties of E; these are largely the same as the closure properties of $B^{\perp}$, which we also need; and this is because these represent two special cases of something more general that we call a *prefactorization.* We give in § 2 a connected account of prefactorizations and of factorizations, in sufficient generality to be of use in other contexts.

Much of what we do in §2 can be found in one or other of two recent sources: the privately circulated manuscript of Barr [2], who deals with factorizations and generators; and that of Ringel [11], whose *regular D-pairs* coincide under mild completeness and cocompleteness hypotheses with our prefactorizations.

§ 2. Prefactorizations and factorizations

2.1. *Prefactorizations.* We work in a fixed category A. Given morphisms p and i of A we write $p \downarrow i$ if, for every pair of morphisms u, v with $vp = iu$, there is a *unique* diagonal w rendering commutative the diagram

(2.1)


If H is any class of morphisms we define two other classes as follows:

$$H^{\uparrow} = \{p \mid p \downarrow h \quad \text{for all } h \in H\},$$
$$H^{\downarrow} = \{i \mid h \downarrow i \quad \text{for all } h \in H\}.$$

By a *prefactorization* (E, M) in A we mean a pair of classes of morphisms of A such that $E = M^{\uparrow}$ and $M = E^{\downarrow}$.

By the usual arguments about Galois connexions, we get from any class H prefactorizations $(H^{\uparrow}, H^{\uparrow\downarrow})$ and $(H^{\downarrow\uparrow}, H^{\downarrow})$. If we order prefactorizations by setting $(E, M) \leqslant (E', M')$ whenever $E \subset E'$ (which is equivalent to $M \supset M'$), then they form a (possibly large) complete lattice: the supremum of (E_α, M_α) is $(M^{\uparrow}, M)$, where $M = \bigcap M_\alpha$.

If the colimit of the diagram $p_\alpha : A \to B_\alpha$ is $q_\alpha : B_\alpha \to C$, we call $q_\alpha p_\alpha$, which is independent of α, the *fibred coproduct* of the p_α; when all the p_α are epimorphisms, so that $q_\alpha p_\alpha$ is too, we substitute for "fibred coproduct" the word "cointersection".

Proposition 2.1.1. *Let (E, M) be a prefactorization. Then*

(a) *E contains the isomorphisms and is closed under composition;*
(b) *every push-out of an E is an E;*
(c) *$\Sigma p_\alpha : \Sigma A_\alpha \to \Sigma B_\alpha$ is an E if each $p_\alpha : A_\alpha \to B_\alpha$ is an E;*
(d) *the fibred coproduct of $p_\alpha : A \to B_\alpha$ is an E if each p_α is an E;*
(e) *if pq is an E so is p, provided that q is either an E or an epimorphism.*

Proof. All are easy consequences of the fact that E is of the form $H^{\uparrow}$ for some class H (cf. [11]).

Proposition 2.1.2. *If (E, M) is a prefactorization, $E \cap M$ is the class of isomorphisms.*

Proof. For the non-trivial part take $p = i \in E \cap M$, $u = 1$, $v = 1$ in diagram (2.1).

Proposition 2.1.3. *Let A have a terminal object and let B be any class of objects of A. Then $B^{\perp}$ is the class E of a prefactorization, and therefore has all the closure properties given in Proposition 2.1.1. Moreover, it has the following special properties:*

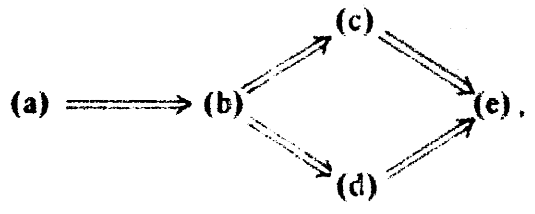
(a) *if $pq \in B^{\perp}$, then $q \in B^{\perp}$, provided that either $p \in B^{\perp}$ or q is an epimorphism;*
(b) *if q is an epimorphism, then pq is in $B^{\perp}$ if and only if p and q are both in $B^{\perp}$.*

Proof. For the first part observe that $B^{\perp} = H^{\uparrow}$, where H consists of all the morphisms $B \to Z$ with $B \in B$ and Z the terminal object. Assertion (a) follows easily from the definition of $B^{\perp}$, and (b) is immediate from (a) and Proposition 2.1.1 (e).

Proposition 2.1.4. *Let $(E, M) = (H^{\uparrow}, H^{\uparrow\downarrow})$ be a prefactorization in A and consider the following assertions:*

(a) *Every M is a monomorphism.*
(b) *Every H is a monomorphism.*
(c) *$pq \in E$ implies $p \in E$.*
(d) *Every coequalizer is in E.*
(e) *Every retraction is in E.*

Then



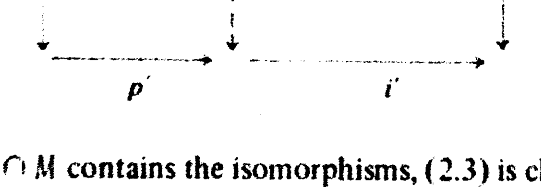
while (e) ⇒ (a) *if* A *admits either finite sums or weak kernel-pairs.*

Proof. (a) ⇒ (b), (c) ⇒ (e), (d) ⇒ (e) are trivial; (b) ⇒ (c) and (b) ⇒ (d) are easy consequences of the fact that $E = H^{\uparrow}$.

To prove (e) ⇒ (a), let $ix = iy$ where $i \in M$ and where $x, y : A \to B$. If we have finite sums, set $u = (1, x)$, $v = (1, y) : B + A \to B$. If we have weak kernel-pairs, let u, v be instead a weak kernel-pair of i. Then in both cases we have $1 = ut$, $1 = vt$ for some t; we have $x = us$, $y = vs$ for some s; and we have $iu = iv$. Since the retraction u is in E by hypothesis (e) and since i is in M, the square $iu = iv$ has a diagonal w with $v = wu$. But then $1 = vt = wut = w$; whence $u = v$ and therefore $x = y$, as desired. (cf. [2].)

2.2. *Factorizations.* A *factorization* (E, M) in A consists of two classes E, M of morphisms of A, each containing the isomorphisms and closed under composition, such that

(2.2) every morphism of A is of the form ip, where $i \in M$ and $p \in E$;
(2.3) if $vip = i'p'u'$, where $i, i' \in M$ and $p, p' \in E$, there is a unique w rendering commutative the diagram



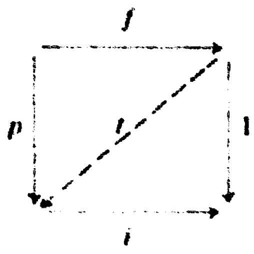
Since $E \cap M$ contains the isomorphisms, (2.3) is clearly equivalent to

(2.4) $E \subset M^{\uparrow}$ and $M \subset E^{\downarrow}$.

It is immediate that, if $u = 1$ and $v = 1$ in (2.3), then w is an isomorphism; so the factorization in (2.2) is essentially unique, and we call it the *canonical factorization* of the given morphism of A.

Proposition 2.2.1. *Every factorization is a prefactorization.*

Proof. In view of (2.4) we need $E \supset M^{\uparrow}$ and its dual. Let $f \in M^{\uparrow}$ have canonical factorization $f = ip$. By the definition of $M^{\uparrow}$ there is a t rendering commutative



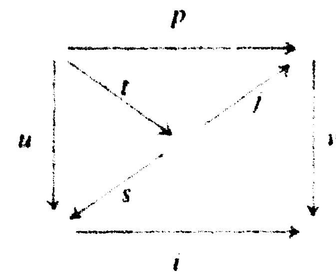
Let the canonical factorization of t be $t = jq$. By the uniqueness of canonical factorizations applied to $ij \cdot q = it = 1 = 1 \cdot 1$, we conclude that q is an isomorphism; whence $t \in M$. By the same uniqueness applied to $1 \cdot p = tf = ti \cdot p$, we conclude that ti is an isomorphism. Since $it = 1$ it follows that t is an isomorphism, whence $f = t^{-1}p \in E$. (cf. [2].)

Corollary 2.2.2. *Factorizations are just those prefactorizations that satisfy* (2.2).

2.3. *Proper factorizations.* A prefactorization, and in particular a factorization, is said to be *proper* if every E is an epimorphism and every M is a monomorphism.

Lemma 2.3.1. *Let* (E, M) *be a proper prefactorization in* A. *Suppose either that* (E, M) *is actually a factorization or else that* A *admits pull-backs. Then* $p \in E$ *if and only if, whenever* $p = jt$ *with* $j \in M$, j *is an isomorphism.*

Proof. For the "only if" part, $j \in E$ by Proposition 2.1.4 and then j is an isomorphism by Proposition 2.1.2. The "if" part is obvious for a factorization and it remains to prove it for a prefactorization. Let $iu = vp$ with $i \in M$; form the pull-back (s, j) of i and v getting



Then $j \in \mathcal{M}$ by Proposition 2.1.1 (b), whence j is an isomorphism by hypothesis. Then sj^{-1} is a diagonal for (2.1), unique since i is monomorphic; so $p \in \mathcal{M}^{\uparrow} = \mathcal{E}$.

Lemma 2.3.2. *A proper prefactorization* $(\mathcal{E}, \mathcal{M})$ *in* $\mathbf{A}$ *is actually a factorization if* $\mathbf{A}$ *admits pull-backs and admits intersections of arbitrary families of monomorphisms that lie in* $\mathcal{M}$.

Proof. Given $f : A \to B$ let $i_\alpha : C_\alpha \to B$ be the $\mathcal{M}$'s through which f factorizes and let $i : C \to B$ be their intersection. Then f factorizes through i, say as $f = ip$. We have $i \in \mathcal{M}$ by Proposition 2.1.1 (d) and $p \in \mathcal{E}$ by Lemma 2.3.1.

Write Epi, Mon for the classes of all epimorphisms and of all monomorphisms in $\mathbf{A}$. An epimorphism p in $\mathbf{A}$ is said to be *extremal* if, whenever $p = jt$ with j monomorphic, j is an isomorphism. Write Epi*, Mon* for the classes of extremal epimorphisms and of extremal monomorphisms.

Lemma 2.3.3. *If* $\mathbf{A}$ *is finitely complete or finitely cocomplete,* (Epi*, Mon) *is a proper pre-factorization.*

Proof. By Proposition 2.1.4, $\mathrm{Mon}^{\uparrow\downarrow} \subset \mathrm{Mon}$; whence $\mathrm{Mon}^{\uparrow\downarrow} = \mathrm{Mon}$, so that ($\mathrm{Mon}^{\uparrow}$, Mon) is a prefactorization. Since every coretraction is in Mon, $\mathrm{Mon}^{\uparrow} \subset \mathrm{Epi}$ by the dual of Proposition 2.1.4. It now follows from Lemma 2.3.1 that $\mathrm{Mon}^{\uparrow} = \mathrm{Epi}^*$.

Proposition 2.3.4. (Epi*, Mon) *is a proper factorization in* $\mathbf{A}$ *in each of the following cases:*

(i). $\mathbf{A}$ *is finitely complete and admits arbitrary intersections of monomorphisms.*

(ii). $\mathbf{A}$ *is finitely cocomplete and admits arbitrary cointersections of extremal epimorphisms.*

(iii). $\mathbf{A}$ *is finitely complete and finitely cocomplete, and composites of coequalizers are coequalizers.*

Proof. Cases (i) and (ii) follow from Lemmas 2.3.3 and 2.3.2. For case (iii), given f, let $f = ip$ where p is the coequalizer of the kernel-pair of f and let $i = jq$, where q is the coequalizer of the kernel-pair of i. Then f and p have the same kernel-pair, so that qp, which is intermediate between f and p, also has this kernel-pair. Since a coequalizer is the coequalizer of its kernel-pair, qp and p are coequalizers of the same thing, whence q is an isomorphism. This means that $i \in \mathrm{Mon}$, on the other hand $p \in \mathrm{Epi}^*$ by Lemma 2.3.3 and Proposition 2.1.4.

Thus any respectably complete or cocomplete category admits the proper factorizations (Epi*, Mon) and (Epi, Mon*), which may concide. These are clearly the least and the greatest, respectively, of the proper factorizations; there may be others between them. In Hausdorff topological spaces, Epi* = topological quotient maps, Mon = injections; Epi = maps onto a dense subset, Mon* = inclusions of a closed subspace. An intermediate proper factorization is given by $\mathcal{E}$ = surjections, $\mathcal{M}$ = inclusions of a subspace.

2.4. *Subobjects.* Let $\mathbf{A}$ be a category with a given proper factorization ($\mathcal{E}$, $\mathcal{M}$). We are going to use such words as "subobject", "well-powered", "union", "generator" in a sense relative to ($\mathcal{E}$, $\mathcal{M}$); if there is danger of confusion we can say "$\mathcal{M}$-subobject", etc.

As usual we say of monomorphisms $i : B \to A$ and $j : C \to A$ that $i \leqslant j$ if $i = jk$ for some (necessarily unique) $k : B \to C$; and we call i and j *equivalent* if $i \leqslant j$ and $j \leqslant i$. By a *subobject* of A we mean an equivalence class of monomorphisms $i : B \to A$, one (and therefore all) of which belong to $\mathcal{M}$. As usual we freely confuse the subobject with its representative monomorphism $i : B \to A$, and we often loosely call B itself the subobject. We order the subobjects as we do their representatives; thus they form a (possibly large) partially ordered class. We call $\mathbf{A}$ *well-powered* if for each $A \in \mathbf{A}$ the class of subobjects of A is small. We define *quotient object* and *co-well-powered* dually.

If $i_\alpha : B_\alpha \to A$ are subobjects of A and if the monomorphisms i_α admit an intersection $i : B \to A$, then i is again a subobject of A by Proposition 2.1.1(d); we call it the *intersection of the* i_α; clearly it is their infimum. We also write $B = \bigcap B_\alpha$.

Because there are highly respectable categories, like Spanier's quasi-topological spaces [13], that are not well-powered but admit intersections of arbitrary families of subobjects, we should define *union* for large families also. Let us therefore say that a *family* $q_\alpha : B_\alpha \to C\,(\alpha \in \Lambda)$ *is in* $\mathcal{E}$ if, whenever we have $k : D \to E$ in $\mathcal{M}$ and morphisms $u_\alpha : B_\alpha \to D\,(\alpha \in \Lambda)$ and $v : C \to E$ rendering commutative all the diagrams

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\;q_\alpha\;} & C \\ {\scriptstyle u_\alpha}\downarrow & & \downarrow{\scriptstyle v} \\ D & \xrightarrow[\;k\;]{} & E \end{array}$$

then there is a $w : C \to D$ with $kw = v$; that $wq_\alpha = u_\alpha$ and that w is unique follows because k is a monomorphism. If ΣB_α exists in $\mathbf{A}$, this is exactly to say that the morphism $q : \Sigma B_\alpha \to C$, with components q_α, is in $\mathcal{E}$.

A subobject $j : C \to A$ of A is now said to be the *union* of the subobjects $i_\alpha : B_\alpha \to A$ if $i_\alpha \leq j$ for all α and if, when we factorize the i_α through j as

$$B_\alpha \xrightarrow[q_\alpha]{} C \xrightarrow[j]{} A,$$

then the family q_α is in E. It is easy to see that a union is unique if it exists and that it is the supremum of the i_α. We also write $C = \bigcup B_\alpha$. Note that if the B_α are considered as subobjects of C rather than of A, then C is still their union.

The union of the i_α certainly exists if ΣB_α does; it is $j : C \to A$ where jq is the canonical factorization of the morphism $i : \Sigma B_\alpha \to A$ with components i_α. We leave the reader to prove that it also exists if A admits pull-backs and admits arbitrary intersections of subobjects.

A morphism $f : A \to B$ induces a map from the subobjects of A to those of B; it assigns to a subobject $i : C \to A$ of A the subobject $j : D \to B$ of B where the canonical factorization of fi is jq. We write $D = fC$ and call it the *direct image* of C. In particular we have the case where $i = 1_A$; if $f : A \to B$ has the canonical factorization $A \to C \xrightarrow[k]{} B$, we call $k : C \to B$ the *image* of f, and write $C = \operatorname{im} f = fA$. It is immediate that $f(\bigcup C_\alpha) = \bigcup fC_\alpha$ if the left side exists.

Suppose that A admits pull-backs. Then a morphism $f : A \to B$ also induces a map from the subobjects of B to those of A. It assigns to a subobject $j : D \to B$ of B the subobject $i : C \to A$ of A given by the pull-back diagram

$$\begin{array}{ccc} C & \xrightarrow{\;i\;} & A \\ \downarrow & & \downarrow f \\ D & \xrightarrow[\;j\;]{} & B \end{array}$$

Note that $i \in \mathsf{M}$ by Proposition 2.1.1 (b). We write $C = f^{-1}D$ and call it the *inverse image* of D. It is immediate that $f^{-1}(\bigcap D_\alpha) = \bigcap f^{-1}D_\alpha$ if the left side exists.

2.5. *Generators.* Let A again have a given proper factorization (E, M). A *generator* of A is a *small* full subcategory G of A such that, for each $A \in \mathsf{A}$, the family of all morphisms $G \to A$ with domain $G \in \mathsf{G}$ is in E. If A admits coproducts, there is a canonical morphism

$$\kappa_A : \Sigma_{G \in \mathsf{G}}\, \mathsf{A}(G, A) \otimes G \to A,$$

whose (G, f)-component, for $G \in \mathsf{G}$ and $f \in \mathsf{A}(G, A)$, is f; and to say that G is a generator is to say that $\kappa_A \in \mathsf{E}$ for each $A \in \mathsf{A}$. Note that G remains a generator if (E, M) is replaced by a larger proper factorization $(\mathsf{E}', \mathsf{M}')$.

Proposition 2.5.1. *Consider the statements:*

(a) *G is a generator.*

(b) *For any proper subobject $i : B \to A$, there is a $G \in$ G and a morphism $f : G \to A$ that does not factorize through i.*

Then (a) ⇒ (b), *while* (b) ⇒ (a) *if* A *admits coproducts or pull-backs.*

Proof. (a) ⇒ (b) is immediate from the definition of a generator; (b) ⇒ (a) follows from Lemma 2.3.1 if A admits coproducts, and is left to the reader when A admits pull-backs.

Corollary 2.5.2. *If* A *has a generator and admits finite intersections of subobjects, it is well-powered.*

Proof. Of two different subobjects of A at least one differs from their intersection, whence there is a $G \in$ G and a morphism $G \to A$ factorizing through one but not through the other.

Proposition 2.5.3. *Let* A *admit coproducts or be finitely complete. Consider the following statements:*

(a) *G is a small dense (= adequate) subcategory of* A.

(b) *If* $A(G, f) : A(G, A) \to A(G, B)$ *is an isomorphism for all* $G \in$ G, *then f is an isomorphism.*

(c) *G is a generator.*

(d) *Whenever* $f \neq g : A \to B$ *there is a* $G \in$ G *and an* $h : G \to A$ *such that* $fh \neq gh$.

Then (a) ⇒ (b) ⇒ (c) ⇒ (d); *moreover* (d) ⇒ (c) *if the factorization* (E, M) *is* (Epi, Mon*), *and* (c) ⇒ (b) *if* (E, M) *is* (Epi*, Mon).

Proof. Density of G means that every family $\lambda_G : A(G, A) \to A(G, B)$ that is natural in G is of the form $A(G, f)$ for a unique $f : A \to B$; this clearly implies (b). The latter implies (c) by Proposition 2.5.1.

That (c) ⇒ (d), and the converse when E = Epi, is evident if coproducts exist; the proofs in the other case are left to the reader. To see that (c) ⇒ (b) when M = Mon, let f satisfy the hypothesis of (b). It follows easily from the fact that (c) ⇒ (d) that $f \in$ Mon; Proposition 2.5.1 then shows that f is an isomorphism.

§ 3. Boundedness

3.1. *Definition of boundedness.* We suppose henceforth that A, besides having a given proper factorization (E, M), is complete and cocomplete.

We recall that a *regular* cardinal is an infinite one that is not the sum of a lesser number of lesser cardinals, and that there are arbitrarily large regular cardinals – for any infinite cardinal, the next greater one is regular.

An ordered set J is *σ-directed*, for a regular cardinal σ, if every subset of J of cardinal $< \sigma$ has an upper bound in J. By a *σ-directed family of subobjects of B* we mean a family $i_\alpha : C_\alpha \to B$ $(\alpha \in J)$ of subobjects of B where J is σ-directed and where $i_\alpha \leqslant i_\beta$ whenever $\alpha \leqslant \beta$. We then call $\bigcup C_\alpha$ a *σ-directed union.*

Let σ be a regular cardinal. An object $A \in \mathbb{A}$ is said to be *bounded by σ* if

(3.1) Any morphism from A to a set-indexed union $\bigcup_{\alpha \in I} C_\alpha$ factorizes through $\bigcup_{\alpha \in K} C_\alpha$ for some subset K of I of cardinal $< \sigma$.

This clearly implies

(3.2) Any morphism from A to a σ-directed union $\bigcup_{\alpha \in J} C_\alpha$ factorizes through some C_α.

In fact, (3.1) and (3.2) are equivalent, for

$$\bigcup_{\alpha \in I} C_\alpha = \bigcup_{K \in J} \bigcup_{\alpha \in K} C_\alpha ,$$

where J is the set of subsets of I of cardinal $< \sigma$; and this set is σ-directed.

We say that $A \in \mathbb{A}$ is *bounded* if it is bounded by some regular cardinal σ; and we call the least such σ the *bound* of A. We say that $\mathbb{A}$ is *bounded* if each $A \in \mathbb{A}$ is bounded.

We can call $\mathbb{A}$ *ordinally bounded* if (3.2) holds whenever J, in addition to being σ-directed, is well-ordered. This is equivalent to (3.2) if $\sigma = \omega$ (we identify cardinals with the corresponding initial ordinals), but is presumably strictly weaker for a general σ. Everything we say below remains true if we replace "bounded" by "ordinally bounded" and "σ-directed union" by "well-ordered σ-directed union". Ordinal boundedness suffices for our applications; but all the examples we give are actually bounded.

We say that *pull-backs preserve σ-directed unions* if whenever $f : A \to B$ and whenever $C_\alpha \to B$ is a σ-directed family of subobjects of B, then $\bigcup f^{-1} C_\alpha = f^{-1} \bigcup C_\alpha$. Clearly it makes no difference if we require this only in the case when B is itself $\bigcup C_\alpha$. If we require it only for $f \in \mathbb{M}$, we say that *intersections preserve σ-directed unions*; we may then write the condition as $\bigcup (A \cap C_\alpha) = A \cap \bigcup C_\alpha$.

Proposition 3.1.1. *$\mathbb{A}$ is bounded if either*

(i). *$\mathbb{A}$ is well-powered and, for some σ, pull-backs preserve σ-directed unions, or*

(ii). *$\mathbb{A}$ is well-powered and co-well-powered and, for some σ, intersections preserve σ-directed unions.*

Proof. (i). Given $A \in \mathbb{A}$, let τ be a regular cardinal $\geqslant \sigma$ and $>$ the number of subobjects of A. Then for any morphism $f : A \to \bigcup C_\alpha$ into a τ-directed union we have

$A = \bigcup f^{-1}C_\alpha$ by (i). Since there are fewer than τ different $f^{-1}C_\alpha$, they are all contained in some $f^{-1}C_\beta$. Thus $A = f^{-1}C_\beta$ and f factorizes as $A = f^{-1}C_\beta \to C_\beta \to \bigcup C_\alpha$.

(ii). This time choose $\tau \geqslant \sigma$ and $>$ the number of subobjects of every quotient object of A. If $f = ip$ is the canonical factorization of a morphism $f : A \to \bigcup C_\alpha$ into a τ-directed union, then i factorizes through some C_β as in the proof of (i), so f factorizes through C_β.

Proposition 3.1.2. *If* $\mathbb{A}$ *has a generator* $\mathcal{G}$, *the following are equivalent:*

(a). $\mathbb{A}$ *is bounded.*

(b). G *is bounded for each* $G \in \mathcal{G}$.

(c). *For some* σ, *pull-backs preserve* σ*-directed unions.*

Proof. (a) ⇒ (b) trivially, and (c) ⇒ (a) by Proposition 3.1.1 and Corollary 2.5.2.

To prove (b) ⇒ (c) let $f : A \to \bigcup C_\alpha$ be a morphism into a σ-directed union, where σ is the least regular ordinal bounding each $G \in \mathcal{G}$. For each $G \in \mathcal{G}$ and each $h : G \to A$, the composite fh factorizes through some C_α, whence h factorizes through $f^{-1}C_\alpha$ and *a fortiori* through $\bigcup f^{-1}C_\alpha$. Thus $\bigcup f^{-1}C_\alpha = A$ by Proposition 2.5.1.

3.2. *Comparison with definitions of Barr and of Gabriel-Ulmer.* Barr in [2] defines $A \in \mathbb{A}$ as ***having rank*** $\leqslant \sigma$ if, whenever $i_\alpha : C_\alpha \to B$ is a σ-directed family of subobjects, every morphism $f : A \to \operatorname{colim} C_\alpha$ factorizes through some C_α. He considers in particular categories with a generator $\mathcal{G}$ such that each $G \in \mathcal{G}$ has rank; in such categories every object has a rank.

Given the above σ-directed family of subobjects let the canonical factorization of the induced morphism $\operatorname{colim} C_\alpha \to B$ be

(3.3) $$\operatorname{colim} C_\alpha \xrightarrow[p]{} D \xrightarrow[k]{} B .$$

The coprojection $j_\alpha : C_\alpha \to \operatorname{colim} C_\alpha$ is in $\mathcal{M}$ by Proposition 2.1.4, because $i_\alpha = kpj_\alpha$ is in $\mathcal{M}$. Since $j : \Sigma C_\alpha \to \operatorname{colim} C_\alpha$ is a coequalizer, it is in $\mathcal{E}$ by Proposition 2.1.4. Therefore $\operatorname{colim} C_\alpha$ is the union of the $j_\alpha : C_\alpha \to \operatorname{colim} C_\alpha$. Since pj is also in $\mathcal{E}$, D is the union of the $i_\alpha : C_\alpha \to B$.

To say that A has rank $\leqslant \sigma$, then, is to say that every $f : A \to \bigcup C_\alpha$ factorizes through some C_α if $\bigcup C_\alpha$ is a σ-directed union *that coincides with* $\operatorname{colim} C_\alpha$. Thus if A is bounded it certainly has rank. The converse is false: in the category of Hausdorff spaces with the factorization (Epi, Mon*), the one-point space is a generator with rank but is not bounded; see Example 3.3.5 below. The strictly stronger condition of boundedness, or at least ordinal boundedness, seems to be necessary for our results.

If $\mathcal{G}$ is a generator with rank for the proper factorization $(\mathcal{E}, \mathcal{M})$, it is obviously also one for any larger proper factorization (E', M'); but boundedness is highly sensitive to the factorization and may disappear if we make it either smaller or larger; see Examples 3.3.4 and 3.3.5 below.

Lemma 3.2.1. *Let* A *have a generator* G *each object of which has rank* $\leqslant \sigma$. *Then the* p *of* (3.3) *is a monomorphism.*

Proof. Let $px = py$, where $x, y : G \to \operatorname{colim} C_\alpha$ with $G \in G$. Then x and y both factorize through i_α for some α, say as $x = i_\alpha u$, $y = i_\alpha v$. Since $px = py$, we have $i_\alpha u = i_\alpha v$, whence $u = v$ and $x = y$. From the (c) $\Rightarrow$ (d) part of Proposition 2.5.3 it follows that p is a monomorphism.

Proposition 3.2.2. *Let the factorization be* (Epi*, Mon) *and let* A *have a generator* G *with rank. Then* A *is bounded.*

Proof. With σ as in Lemma 3.2.1 we have $p \in$ Mon; but $p \in E =$ Epi*, so that p is an isomorphism; hence every $G \in G$ actually is bounded by σ.

The forthcoming paper by Gabriel and Ulmer [6], summarized by Ulmer in [14] calls an object $A \in A$ *σ-presentable* if the representable functor $A(A, -) : A \to$ **Sets** preserves all σ-directed colimits; that is, colimits of functors from a σ-directed small category, of which a σ-directed ordered set is a special case. These authors call A *locally presentable* if it is cocomplete and has a generator G for the factorization (Epi*, Mon) such that each $G \in G$ is σ-presentable for some σ. Such a category is complete, and is co-well-powered even for the factorization (Epi, Mon*); moreover every $A \in A$ is τ-presentable for some τ depending upon A. The categories A and A^{op} cannot both be locally presentable unless A is a small complete lattice.

The authors give various characterizations of locally presentable categories. They are (in the language of our § 1) those categories of the form $[C, \mathbf{Sets}]_\Gamma$, where Γ is a *small* class of limits in C; and here we can choose C to have all limits of size $<$ some σ and Γ to consist of these limits. They are also those full subcategories of functor categories $[C, \mathbf{Sets}]$ that are of the form $\Delta^\perp$ for a *small* class Δ of morphisms in $[C, \mathbf{Sets}]$.

Since a σ-presentable object certainly has rank $\leqslant \sigma$ in Barr's sense, Proposition 3.2.2 gives

Proposition 3.2.3. *Locally presentable categories are bounded for the factorization* (Epi*, Mon).

We exhibit in Example 5.2.3 below a bounded category with a generator, for the factorization (Epi*, Mon), that is not locally presentable.

3.3. *Examples of bounded categories*

Example 3.3.1. A complete and cocomplete, well-powered and co-well-powered, abelian AB5 category, with its unique proper factorization (Epi, Mon), is bounded by Proposition 3.1.1. Since the first author has given ([5, p. 131]) an example of such a category without a generator, it follows that a bounded category need not have a generator. All of our remaining examples will however have one.

Example 3.3.2. If we use the factorization (Epi*, Mon) all the locally presentable categories are examples of bounded ones with a generator. Some examples of locally presentable categories, mostly quoted from [14], are: the category **Sets**; the algebras in **Sets** over a theory with rank; the category of ordered sets; the category **Cat** of small categories; the dual $\mathbf{Comp}^{\mathrm{op}}$ of the category of compact (= compact Hausdorff) spaces; the category of sheaves of sets on a Grothendieck topology; a cocomplete abelian AB5 category with a generator; the category of torsion-free abelian groups; the category of those abelian groups in which $4x = 0$ implies $2x = 0$.

In all of these cases, except perhaps $\mathbf{Comp}^{\mathrm{op}}$, the boundedness is immediately evident; for example in the algebras over a theory with rank σ, the free algebra on one element is a generator and a σ-directed union of subalgebras is just their set-theoretical union.

The example of **Cat** shows ([8, p. 139]) that coequalizers need not be closed under composition in a locally presentable category, and hence (see [8]) that the pull-back of a coequalizer need not be an epimorphism. Thus in a bounded category with a generator the pull-bck of an E need not be an E.

Example 3.3.3. The category **Top** of topological spaces is not locally presentable. In fact it is not even bounded for the factorization (Epi*, Mon); here Epi* = the topological quotient maps and Mon = the injections. To see this, let A be the two-point space $\{0, 1\}$ with the trivial topology and let σ be any regular cardinal, identified with the corresponding initial ordinal. Let B be the set of ordinals $\leqslant \sigma$ with the trivial topology. For each $\alpha < \sigma$ set $V_\alpha = \{\beta \in B \mid \beta \geqslant \alpha\}$, and take C_α to be the same set as B but with the topology in which the open sets are the empty set, B, and the sets V_β for $\beta \geqslant \alpha$. Taking $i_\alpha : C_\alpha \to B$ as the identity map, we have a σ-directed family of subobjects of B, and clearly $\bigcup C_\alpha = B$. Yet the map $f : A \to B$ given by $f(0) = 0$, $f(1) = \sigma$, factorizes through no C_α.

Example 3.3.4. With the factorization (Epi, Mon*) the category **Top** has the one-point space as a generator and is bounded; for Epi = the surjections, Mon* = the inclusions of subspaces, and every union of subspaces is their set-theoretical union. This shows that boundedness for a given factorization does not imply boundedness for a smaller one.

Example 3.3.5. For exactly the same reasons the category of Hausdorff spaces is bounded with a generator if we use the factorization in which E = the surjections and M = the inclusions of subspaces. For the factorization (Epi, Mon*), however, the category is no longer bounded although the one-point space is still a generator. To see this, recall that Mon* now consists of the inclusions of closed subspaces. Let A be the one-point space, let σ be any regular cardinal, let B be the set of ordinals $\leq \sigma$ with the order topology, and let C_α for $\alpha < \sigma$ be the closed subspace $\{\beta \in B \mid \beta \leq \alpha\}$. Then for this factorization $\bigcup C_\alpha$ is B, but the map $A \to B$ sending A to σ factorizes through no C_α. This shows that boundedness for a given factorization does not imply boundedness for a larger one.

Note that the same example shows that **Comp**, with its unique proper factorization, is not bounded and so not locally presentable.

Example 3.3.6. The same arguments show that compactly-generated spaces, whether Hausdorff or not (see [3] for the definition of the latter), and the quasi-topological spaces of Spanier [13], are bounded with a generator when M consists of the inclusions of the appropriate sub-structures. Since there are a proper class of quasi-topologies on the two-point set $\{0, 1\}$ (for each cardinal σ define one by taking the admissible maps $f : C \to \{0, 1\}$, for $C \in$ **Comp**, to be those for which $f^{-1}(0)$ is the intersection of $\leq \sigma$ open sets), we see that a bounded category with a generator need not be co-well-powered.

Example 3.3.7. We know that **Sets**^{op}, with its unique proper factorization, is not locally presentable (since **Sets** is). In fact it is not bounded either. For let $A = \{0, 1\}$ and let σ be any regular cardinal. Let B be the set of ordinals $< \sigma$ and let C_α for $\alpha < \sigma$ be the set of ordinals $\leq \alpha$. Define $i_\alpha : B \to C_\alpha$ by $i_\alpha(\beta) = \beta$ for $\beta \leq \alpha$, $i_\alpha(\beta) = 0$ for $\beta > \alpha$. Since $i : B \to \prod C_\alpha$ is injective, B is the union in **Sets**^{op} of the σ-directed family of subobjects i_α. Yet $f : B \to A$ given by $f(\beta) = 0$ if β is even, $f(\beta) = 1$ if β is odd, factorizes through no C_α.

The same example, with everything given the trivial topology, shows that **Top**^{op} is not bounded for any proper factorization.

Example 3.3.8. Let A be bounded and let C be any small category. The factorization (E, M) on A gives rise to one on the functor category $[C, A]$ if we define $f : T \to S$ in $[C, A]$ to be in E or in M if and only if each component $f_C : TC \to SC$ is in E or in M. Since colimits are formed pointwise, so are unions. It follows at once that $[C, A]$ is bounded, the bound of T being the least regular cardinal bounding TC for all $C \in C$. Moreover, if A has a generator G, the set of "generalized representable functors" $C(C, -) \otimes G$, for $G \in G$, forms a generator for $[C, A]$.

Example 3.3.9. Further examples of bounded categories are provided by Remark 5.2.2 below, and in particular by Example 5.2.3.

§4. The orthogonal subcategory theorem

4.1. *Proof of the theorem.* We now return to the problem of §1.2 and give sufficient conditions for the reflectiveness of $\Delta^\perp$.

Lemma 4.1.1. *Let $i : C \to B$ be in $\mathcal{M}$ where $B \in \Delta^\perp$. Then*
(a) *if $k : M \to N$ is in Δ and if $f, g : N \to C$ satisfy $fk = gk$ then $f = g$;*
(b) *if $\Delta \subset \mathcal{E}$ then $C \in \Delta^\perp$.*

Proof. (a). From $ifk = igk$ we have $if = ig$ since $B \in \Delta^\perp$ and then $f = g$ since i is monomorphic.

(b). For any $k : M \to N$ in Δ and any $f : M \to C$ we have $if = hk$ for some $h : N \to B$ since $B \in \Delta^\perp$. Because $i \in \mathcal{M}$ and $k \in \mathcal{E}$ the square $if = hk$ has a diagonal $g : N \to C$ with $gk = f$. If also $g'k = f$ then $g' = g$ by (a).

Lemma 4.1.2. *Let the typical morphism k of Δ have canonical factorization $k'k''$; write Δ' for the class of all such k', and write Δ'' for the class of all such k''. Then $\Delta^\perp = (\Delta' \cup \Delta'')^\perp$.*

Proof. By Proposition 2.1.3 (b) we have $B \perp k$ if and only if $B \perp k'$ and $B \perp k''$.

Theorem 4.1.3. *Let $\mathcal{A}$ be a complete and cocomplete category with a given proper factorization $(\mathcal{E}, \mathcal{M})$. Let $\mathcal{A}$ be bounded and co-well-powered. Let the class $\Delta = \Phi \cup \Psi$ where Φ is small and where $\Psi \in \mathcal{E}$. Then $\Delta^\perp$ is a reflective subcategory of $\mathcal{A}$.*

Proof. By Proposition 1.2.1 we have only to verify the solution-set condition for $\Delta^\perp$.

Let $k : M_k \to N_k$ be the typical element of Φ, and let σ be a regular cardinal bounding M_k for all $k \in \Phi$.

We now produce a solution-set for a given $A \in \mathcal{A}$. For each ordinal α we define inductively a set S_α of objects of $\mathcal{A}$:

S_0 = the set of quotient objects of A,
$S_{\alpha+1}$ = the set of quotient objects of objects of the form
$C + \Sigma_{k \in \Phi} \mathcal{A}(M_k, C) \otimes N_k$, where $C \in S_\alpha$,

and for a limit ordinal β,

S_β = the set of quotient objects of objects of the form $\Sigma_{\alpha<\beta} C_\alpha$, where $C_\alpha \in S_\alpha$ for each α.

We assert that $S_\sigma \cap \Delta^\perp$ is a solution-set for A.

Suppose given, then, $f : A \to B$ with $B \in \Delta^\perp$. We are to show that f factorizes through some object of $S_\sigma \cap \Delta^\perp$. We define inductively for each ordinal α a subobject $i_\alpha : A_\alpha \to B$ through which f factorizes, with $i_\alpha \geqslant i_\beta$ for $\alpha \geqslant \beta$. Recall from §2.4 the definition of *image*.

(i). We take $i_0 : A_0 \to B$ to be the image of $f : A \to B$.

(ii). Suppose that $i_\alpha : A_\alpha \to B$ has been constructed. For each $k \in \Phi$ and each $x \in \mathcal{A}(M_k, A_\alpha)$ there is, because $B \in \Delta^\perp$, a unique $y : N_k \to B$ rendering commutative the diagram

$$
\begin{array}{ccc}
M_k & \xrightarrow{\quad k \quad} & N_k \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\
A_\alpha & \xrightarrow[\quad i_\alpha \quad]{} & B
\end{array}
\tag{4.1}
$$

Write $y_{k,x}$ for this y. We define $i_{\alpha+1} : A_{\alpha+1} \to B$ to be the union of A_α and of the images of all the $y_{k,x}$ for $k \in \Phi$ and $x \in \mathcal{A}(M_k, A_\alpha)$. In other words, $i_{\alpha+1}$ is the image of the morphism $A_\alpha + \Sigma_{k \in \Phi} \mathcal{A}(M_k, A_\alpha) \otimes N_k \to B$ whose first component is i_α and whose (k, x)-component is $y_{k,x}$.

(iii). For a limit ordinal β, we take $i_\beta : A_\beta \to B$ to be the union of the i_α for $\alpha < \beta$; that is, to be the image of $\Sigma_{\alpha<\beta} A_\alpha \to B$.

It is evident that $A_\alpha \in S_\alpha$ for each α; in particular $A_\sigma \in S_\sigma$. Since f factorizes through A_σ as $A \to A_0 \to A_\sigma \to B$, the proof will be complete if we show that $A_\sigma \in \Delta^\perp$.

Let $k \in \Phi$. Since A_σ is a σ-directed union, any $g : M_k \to A_\sigma$ factorizes as $M_k \to A_\alpha \to A_\sigma$ for some $\alpha < \sigma$. If we write x for this morphism $M_k \to A_\alpha$, we have $y = y_{k,x}$ as in (4.1). By the definition of $A_{\alpha+1}$, y factorizes through $A_{\alpha+1}$ and *a fortiori* through A_σ. If this morphism $N_k \to A_\sigma$ is h, we have $i_\sigma hk = i_\sigma g$, whence $hk = g$ since i_σ is monomorphic. If also $h'k = g$ then $h' = h$ by Lemma 4.1.1 (a); thus $A_\sigma \in \Phi^\perp$. By Lemma 4.1.1 (b) we have $A_\sigma \in \Psi^\perp$, whence $A_\sigma \in \Delta^\perp$ and the proof is complete.

4.2. ***The boundedness of*** $\Delta^\perp$. We shall show that $\Delta^\perp$ is again bounded, but first we must give it a proper factorization.

Lemma 4.2.1. *Let $S : \mathsf{A} \to \mathsf{B}$ be left adjoint to $T : \mathsf{B} \to \mathsf{A}$, and let A have a prefactorization (E, M). Let $S\mathsf{E}$ be the class $\{Sp \mid p \in \mathsf{E}\}$ in B, and let $(\mathsf{E}', \mathsf{M}')$ be the prefactorization $((S\mathsf{E})^{\downarrow\uparrow}, (S\mathsf{E})^{\downarrow})$ on B. Then $i \in \mathsf{M}'$ if and only if $Ti \in \mathsf{M}$.*

If moreover T is faithful and B admits finite products, then $(\mathsf{E}', \mathsf{M}')$ is proper if (E, M) is proper.

Proof. The first assertion follows at once from the observation that $Sf \downarrow g$ if and only if $f \downarrow Tg$. Now let (E, M) be proper. Since T is faithful it reflects monomorphisms and hence every M' is a monomorphism. If i is an equalizer in B then Ti is an equalizer in A; hence M' contains all equalizers, and therefore every E' is epimorphic by Proposition 2.1.4.

Proposition 4.2.2. *Let A and Δ satisfy all the hypotheses of Theorem 4.1.3, and in addition let A admit intersections of arbitrary families of subobjects. Then $\Delta^\perp$ has a proper factorization $(\mathsf{E}', \mathsf{M}')$ in which M' consists of those $i \in \mathsf{M}$ that lie in $\Delta^\perp$; and $\Delta^\perp$ is complete and cocomplete, admits intersections of arbitrary families of subobjects, is co-well-powered, and is bounded. Moreover if A admits a generator so does $\Delta^\perp$.*

Proof. $\Delta^\perp$ has a proper prefactorization with the given M' by Lemma 4.2.1, and it is actually a factorization by Lemma 2.3.2. By the reflectivity of $\Delta^\perp$ in A, it is complete and cocomplete and admits intersections of arbitrary families of subobjects. If A admits a generator G, and if S is the reflection of A into $\Delta^\perp$, the SG for $G \in \mathsf{G}$ clearly form a generator for $\Delta^\perp$.

To see that $\Delta^\perp$ is co-well-powered, let $f : A \to B$ belong to E', and let $A \to A_\sigma \to B$ be the factorization of f constructed in the proof of Theorem 4.1.3, with $A_\sigma \in S_\sigma \cap \Delta^\perp$ and $A_\sigma \to B$ in M. Then since $A_\sigma \to B$ is in M' it is an isomorphism by Lemma 2.3.1; which gives the desired co-well-poweredness since $S_\sigma \cap \Delta^\perp$ is a set depending only upon A.

Finally we show that $\Delta^\perp$ is bounded. Let σ be the cardinal in the proof of Theorem 4.1.3. We first show that if $i_\alpha : C_\alpha \to B$ is a σ-directed family of subobjects in $\Delta^\perp$ then the union $\bigcup C_\alpha$ in A of the C_α in fact lies in $\Delta^\perp$ and is therefore also their union in $\Delta^\perp$. If $k \in \Phi$ and $g : M_k \to \bigcup C_\alpha$ then g factorizes through some C_α by the definition of σ; the morphism $M_k \to C_\alpha$ is nk for some n, since $C_\alpha \in \Delta^\perp$, and therefore g is hk for some h. That h is unique follows from Lemma 4.1.1 (a), whence $\bigcup C_\alpha \in \Phi^\perp$; and $\bigcup C_\alpha \in \Psi^\perp$ by Lemma 4.1.1 (b). Thus $\bigcup C_\alpha \in \Delta^\perp$.

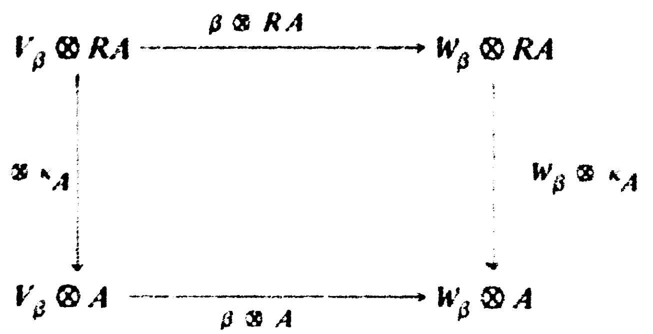
Now if $A \in \Delta^\perp$ is bounded in A by τ, it is clearly bounded in $\Delta^\perp$ by $\max(\sigma, \tau)$.

§ 5. The continuous functor theorem

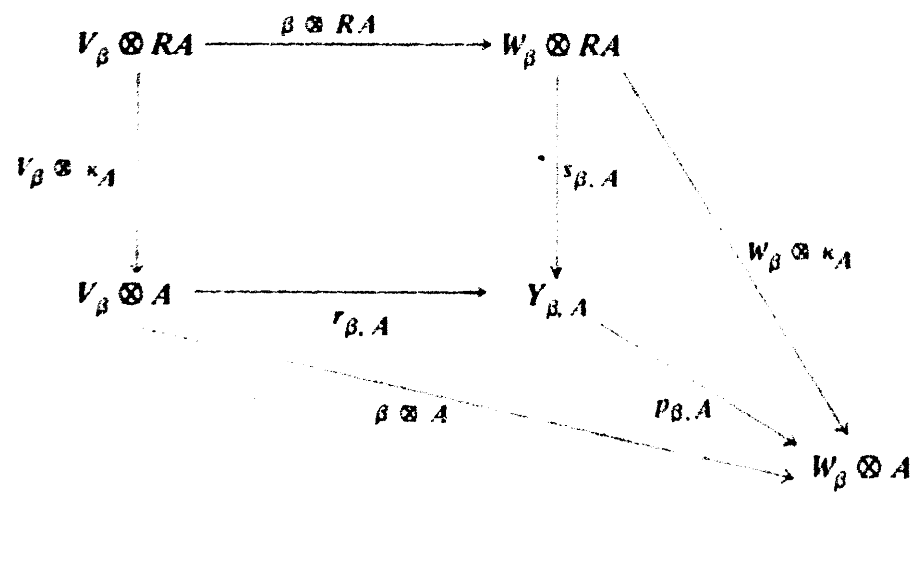
5.1. *Reduction of the size of* Δ. We return finally to the problem of § 1.1, reduced to that of § 1.2 by Proposition 1.3.1. The Δ of the latter proposition is very large, and we must reduce its size if we hope to apply Theorem 4.1.3. We do so by supposing that A has a generator. Recall that a proper factorization (E, M) on A gives a proper factorization (E, M) on $[C, A]$ as in §3.3.8.

Lemma 5.1.1. *Let* A *be a cocomplete category with a given proper factorization* (E, M) *and with a generator* G. *Let* Θ *be a class of morphisms* $\beta : V_\beta \to W_\beta$ *in* $[C, \mathbf{Sets}]$; *write* Δ *for the class of morphisms* $\beta \otimes A : V_\beta \otimes A \to W_\beta \otimes A$, $\beta \in \Theta$, $A \in$ A, *in* $[C, A]$; *and write* Δ_1 *for the subclass of* Δ *consisting of the* $\beta \otimes G$ *with* $\beta \in \Theta$ *and* $G \in$ G. *Then there is a class* Ω *of morphisms in* $[C, A]$ *with* $\Omega \subset E$ *such that* $\Delta^{\perp} = (\Delta_1 \cup \Omega)^{\perp}$.

Proof. Write RA for $\Sigma_{G \in G}\, A(G, A) \otimes G$, and $\kappa_A : RA \to A$ for the canonical morphism of §2.5. From the commutative diagram



by pushing out from the top left corner form a commutative diagram



Let $T \in [C, A]$ be in $\Delta_1^\perp$. Since

$$V_\beta \otimes RA \cong \Sigma_{G \in G} A(G, A) \otimes (V_\beta \otimes G),$$

with similar expressions for $W_\beta \otimes RA$ and $\beta \otimes RA$, it follows from Propositions 2.1.3 and 2.1.1 (c) that T is orthogonal to $\beta \otimes RA$ for each $\beta \in \Theta$, $A \in A$. From Propositions 2.1.3 and 2.1.1 (b) it then follows that T is orthogonal to the push-out $r_{\beta, A}$ of $\beta \otimes RA$. From Propositions 2.1.3 and 2.1.1 (a) and (e) it follows that T is orthogonal to $\beta \otimes A$ if and only if it is orthogonal to $p_{\beta, A}$. Thus $\Delta^\perp = (\Delta_1 \cup \Omega)^\perp$ where Ω is the class of all $p_{\beta, A}$, $\beta \in \Theta$, $A \in A$.

Since $\kappa_A \in E$, each component of $W_\beta \otimes \kappa_A$ is in E by Proposition 2.1.1 (c); therefore $W_\beta \otimes \kappa_A$ is in E, and so $p_{\beta, A}$ is in E by Proposition 2.1.4. This completes the proof.

Remark 5.1.2. If Θ is small so is Δ_1, but Ω is still large. If, however, A is complete, and if G is a generator not only for the factorization (E, M) but also for the factorization (Epi*, Mon) – which is the case in the work of Gabriel and Ulmer [6] – then the result of Lemma 5.1.1 can be improved to: $\Delta^\perp = \Delta_1^\perp$.

To see this, write $[X, A]$ for the product of X copies of A, where $X \in$ **Sets** and $A \in A$. Then for functors $V: C \to$ **Sets** and $T: C \to A$ write $[V, T]$ for $\int_{C \in C} [VC, TC]$ (see [4] for the explanation of this notation). For $A \in A$, we have

$$A(A, [V, T]) \cong [C, A] (V \otimes A, T).$$

Now $T \in \Delta^\perp$ if and only if $[C, A] (\beta \otimes A, T)$ is an isomorphism for each $\beta \in \Theta$, $A \in A$; which is to say that each $A(A, [\beta, T])$ is an isomorphism; or simply that $[\beta, T]$ is an isomorphism for each $\beta \in \Theta$. By Proposition 2.5.3 this will be so if $A(G, [\beta, T])$ is an isomorphism for each $G \in G$, that is if each $[C, A] (\beta \otimes G, T)$ is an isomorphism, that is if $T \in \Delta_1^\perp$.

(For the application of this remark to the situation of Proposition 1.3.1 we do not need the above generality; it suffices to rewrite the proof of the proposition replacing $A \in A$ by $G \in G$.)

5.2. *Proof of the theorem*

Theorem 5.2.1. *Let A be a complete and cocomplete category with a given proper factorization (E, M). Let A be bounded with a generator, and co-well-powered. Let Γ be a class of cylinders in the small category C, and let all but a set of these cylinders be cones. Then $[C, A]_\Gamma$ is a reflective subcategory of $[C, A]$.*

Proof. By Proposition 1.3.1 and Lemma 5.1.1, $[C, A]_\Gamma = (\Delta_1 \cup \Omega)^\perp$, where

$\Omega \subset E$ and where Δ_1 consists of the $\tilde{\alpha} \otimes G$, $\alpha \in \Gamma$, $G \in$ the generator G. Here $\tilde{\alpha} : \tilde{Q}_\alpha \to \tilde{P}_\alpha$ in the language of § 1.3, where $\alpha : P_\alpha \to Q_\alpha$.

We claim that there are only a set of different $\tilde{P}_\alpha$. By hypothesis there are only a set for which α is not a cone. But if α is a cone of vertex N, P_α is the constant functor at N, $\hat{P}_\alpha$ is the constant functor at $C(N, -)$, and its colimit $\tilde{P}_\alpha$ is just $C(N, -)$; there are no more of these than C has objects.

So the class of codomains of the elements of Δ_1 is small. Now by Lemma 4.1.2 we have $\Delta_1^\perp = (\Delta_1' \cup \Delta_1'')^\perp$, where Δ_1' consists of the M's, and Δ_1'' of the E's, in the canonical factorizations $k = k'k''$ of the elements k of Δ_1. But since $[C, A]$ has a generator by § 3.3.8, it is well-powered by Corollary 2.5.2, whence the domains of the k' form a set since the codomains do. Thus Δ_1' is small. Since $[C, A]_\Gamma = (\Delta_1' \cup \Delta_1'' \cup \Omega)^\perp$ and since $\Delta_1'' \cup \Omega \subset E$, the reflectivity of $[C, A]_\Gamma$ follows from Theorem 4.1.3.

Remark 5.2.2. From the above together with Proposition 4.2.2, it follows that $[C, A]_\Gamma$ with a suitable proper factorization (E', M') is bounded with a generator. Note that if $(E, M) = (\text{Epi}^*, \text{Mon})$ for A then $(E', M') = (\text{Epi}^*, \text{Mon})$ for $[C, A]_\Gamma$, since the inclusion $[C, A]_\Gamma \to [C, A]$ both preserves and reflects monomorphisms. More generally the same is true for any subcategory $\Delta^\perp$ of $[C, A]$ where Δ is any class of morphisms in $[C, A]$ all but a set of which are in E.

This gives new examples of bounded categories: the algebras over a theory-with-rank in **Top**, with a suitable (E, M); the category of all limit-preserving functors in $[C, \textbf{Sets}]$, with (Epi*, Mon) as the factorization.

Moreover, we can now exhibit a bounded category with a generator, for (Epi*, Mon), which is not locally presentable; at least if we suppose that there are no measurable cardinals. The example is the following, for which we are indebted to John Isbell.

Example 5.2.3. Let B be the category of boolean σ-algebras and maps preserving countable meets, countable joins and 0. Let Δ be the class of maps of the form $2^S/C \to 1$, where S is a set, 2^S the boolean algebra of subsets of S, C the ideal of countable sets, 1 the terminal object. Define $M = \Delta^\perp$. The non-existence of measurable cardinals is equivalent with $2 \in M$. M is easily seen to be bounded for (Epi*, Mon) with 2^2 as generator. Because M is closed with respect to products, $2^S \in M$ for any set S.

2^2 as an object in M is not presentable for any cardinal. The functor represented by 2^2 is the forgetful functor. $B(2^2, -)$ reflects and preserves all limits. It suffices to show that $M \to B$ fails to preserve $\aleph_\alpha$-directed limits for every α.

Let S be a set of cardinality $\aleph_{\alpha+1}$. For $A \subset B \subset S$ there is a natural map $2^{S \setminus A} \to 2^{S \setminus B}$. Letting A and B range through the subsets of cardinality $\aleph_\alpha$ we obtain an $\aleph_\alpha$-directed diagram whose colimit in $\mathcal{B}$ is $2^S/D$ where D is the ideal of all subsets of cardinality $\leq \aleph_\alpha$. But $2^S/D \notin M$.

Isbell produced this example for a better reason: there exists small category $\mathcal{C}$ and class Γ of cones such that $M \approx [\mathcal{C}^{op}, \mathbf{Sets}]_\Gamma$, but because M is not locally presentable, there is no small Γ for this purpose. Define $\mathcal{C}$ to be the full subcategory in $\mathcal{B}$ of all free algebras of finite or countable rank and the terminal object 1. Let Γ_1 be the set of cones necessary to display each free algebra as a copower of the rank-1 free algebra and one more cone to display 1 as the coequalizer of the two maps from 2^2 to 2. Then $[\mathcal{C}^{op}, \mathbf{Sets}]_{\Gamma_1} \approx \mathcal{B}$. Now for each set S take the canonical diagram in $\mathcal{C}$ whose colimit is $2^S/D$ and define Γ as Γ_1 together with all such diagrams turned into cones by making 1 (not $2^S/D$) their vertices. Then $[\mathcal{C}^{op}, \mathbf{Sets}]_\Gamma \approx M$.

Added in proof, August 1, 1972. A simpler example of a bounded category with a generator, for (Epi*, Mon), which is not locally presentable, still under the hypothesis that there are no measurable cardinals, is given on page 104 of [6] which has now appeared.

References

[1] M. Barr, Coequalizers and free triples, Math. Z. 116 (1970) 307–322.
[2] M. Barr, Factorizations, generators and rank, privately circulated manuscript.
[3] B.J. Day, A reflection theorem for closed categories, J. Pure Appl. Algebra 2 (1972) 1–11.
[4] B.J. Day and G.M. Kelly, Enriched functor categories, Lecture Notes in Math. 106 (Springer, Berlin, 1969).
[5] P.J. Freyd, Abelian categories, 1st ed. (Harper and Row, New York, 1964).
[6] P. Gabriel and F. Ulmer, Lecture Notes in Math. 221 (Springer, Berlin, 1971).
[7] J. Isbell, Subobjects, adequacy, completeness and categories of algebras, Rozprawy Mat. 36 (1964) 1–32.
[8] G.M. Kelly, Monomorphisms, epimorphisms, and pull-backs, J. Austr. Math. Soc. 9 (1969) 124–142.
[9] J.F. Kennison, On limit-preserving functors, Illinois J. Math. 12 (1968) 616–619.
[10] J. Lambek, Completions of categories, Lecture Notes in Math. 24 (Springer, Berlin, 1966) 1–70.
[11] C.M. Ringel, Diagonalisierungspaare I, Math. Z. 117 (1970) 249–266.
[12] H. Schubert, Kategorien III (Springer, Berlin, 1971).
[13] E. Spanier, Quasi-topologies, Duke Math. J. 30 (1963) 1–14.
[14] F. Ulmer, Locally α-presentable and locally α-generated categories, Rept. Midwest Category Seminar 5, Lecture Notes in Math. 195 (Springer, Berlin, 1971).